Chapter 8
On the Advantages and Drawbacks of A Posteriori Error Estimation for Fourth-Order Elliptic Problems

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Abstract In this survey contribution, we present and compare, from the viewpoint of adaptive computation, several recently published error estimation procedures for the numerical solution of biharmonic and some further fourth order elliptic problems mostly in 2D. In the hp-adaptive finite element method, there are two possibilities to assess the error of the computed solution a posteriori: to construct a classical analytical error estimate or to obtain a more accurate reference solution by the same procedure as the approximate solution and, from it, the computational error estimate. For the lack of space, we sometimes only refer to the notation introduced in the papers quoted. The complete hypotheses and statements of the theorems presented should also be looked for there.

8.1 Introduction

Numerical computation has always been connected with some control procedures. It means that the approximate result is of primary importance, but also the error of this computed result, i.e. some norm of the difference between the exact and approximate solution brings important information. The exact solution is usually not known. This means that we can get only some estimates of the error.

The development of numerical procedures has been accompanied with *a priori error estimates* that are very useful in theory but usually include constants that are completely unknown, in better cases can be estimated. In particular, the development of the finite element method, and its h-version and hp-version required reliable and computable estimates of the error that depend only on the approximate solution just computed, if possible. This is the means for the local mesh refinement in the h-version and, moreover, also for the increase of the polynomial degree in the p-version.

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We employ a quantity called the *a posteriori error indicator* η_T for all triangles T of the triangulation $\mathscr{T}_h$ and, if not defined otherwise, the *error estimator*

$$\varepsilon = \sqrt{\sum_{T \in \mathscr{T}_h} \eta_T^2},$$

see [5], in each of the estimation strategies that follow to assess the error of the approximate solution. The quality of an a posteriori error estimator is often measured by its *effectivity index*, i.e. the ratio of some norm of the error estimate and the true error. An error estimator is called *effective* if both its effectivity index and the inverse of the index remain bounded for all meshsizes of triangulations. It is called *asymptotically exact* if its effectivity index converges to 1 as the meshsize tends to 0.

Undoubtedly, obtaining efficient and computable a posteriori error estimates is not easy. (Note that *computable* means, among others, that the degree of piecewise polynomials approximating the solution is high enough.) The papers [2, 3] by Babuška and Rheinboldt represent the pioneering work in this field. The books [1, 4] are surveys of the state of the art some time ago while [17] is an attempt to compare some a posteriori error estimators.

There are several classes of a posteriori error indicators and estimators based on different approaches and their names slightly vary in the literature. We consider residual or recovery a posteriori error indicators for the solution of the biharmonic equation in the classical weak formulation [19, 20] and in the Ciarlet-Raviart formulation [8, 12] in Sect. 8.3. We further present recovery or residual a posteriori error indicators for the solution of a more general 4th order equation [6, 14] and, in particular, functional error estimators [11, 13, 16] in Sect. 8.4. Section 8.5 is devoted to a brief conclusion.

8.2 Notation and Preliminaries

A common notation is introduced in this section. We write $C(S)$ for the space of all functions continuous on the set S, $C_m(S)$ for that of all functions continuous together with their m derivatives.

Let $\Omega \subset R^n$, $n \geq 1$, be a bounded domain (i.e. a bounded connected open set) with the boundary Γ. We use the obvious notation for the $L_2(\Omega)$, $L_\infty(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$ norms, and for the $H^k(\Omega)$ seminorm. Let $\Phi = [\varphi_{ik}]$ and $\Psi = [\psi_{ik}]$ be $n \times n$ matrices, $\Phi, \Psi \in R^{n \times n}$. We introduce their *elementwise matrix product* $\Phi \odot \Psi \in R$ and the *Frobenius* or *Schur norm* of the matrix Φ as $\|\Phi\|_{\mathrm{F}} = \sqrt{\Phi \odot \Phi}$.

The norm or seminorm may be restricted to any open set $\omega \subset \Omega$ with the Lipschitz boundary γ. We thus write, e.g., $\|\cdot\|_{0;\omega}$ for the $L_2(\omega)$ norm. We also employ the spaces $H_0^1(\Omega)$, $H_0^2(\Omega)$, etc. and the adjoint spaces $H^{-k}(\Omega)$, $k > 0$, of linear functionals. We often omit the symbol Ω if Ω is the domain concerned.

Let V be a real Hilbert space and $a : V \times V \to R$ a bounded symmetric coercive bilinear form. The energy norm induced by this bilinear form is denoted by

$$|||v||| = \sqrt{a(v, v)}. \tag{8.1}$$

We use the notation

$$\operatorname{div} A = \nabla \cdot A = \sum_{s=1}^{n} \frac{\partial a_s}{\partial x_s} \in R$$

for the divergence of a differentiable vector-valued function $A = [a_1, \ldots, a_n]$. We put $\nabla A = \nabla \otimes A \in R^{n \times n}$, where $\otimes$ is the tensor product, for the vector-valued function A and $\nabla b = \operatorname{grad} b \in R^n$ for the gradient of a differentiable scalar-valued function b. Furthermore, for a differentiable matrix-valued function $\Theta = [\vartheta_{ij}]_{i,j=1}^{n}$ we introduce its divergence as a vector-valued function

$$\operatorname{Div} \Theta = \nabla \cdot \Theta = \sum_{j=1}^{n} \frac{\partial \vartheta_{ij}}{\partial x_j} \in R^n.$$

Let $R_{\mathrm{s}}^{n \times n}$ be the space of real symmetric $n \times n$ matrices. We consider also the space $H(\operatorname{div}, \Omega) = \{Y \in L_2(\Omega, R^n) \mid \operatorname{div} Y \in L_2(\Omega)\}$ of vector-valued functions Y and the space $H(\operatorname{Div}, \Omega) = \{\Theta \in L_2(\Omega, R_{\mathrm{s}}^{n \times n}) \mid \operatorname{Div} \Theta \in L_2(\Omega, R^n)\}$ of symmetric matrix-valued functions Θ.

For a matrix-valued function $\Phi : \Omega \to R^{n \times n}$, $\Phi = [\varphi_{ik}]$, we put

$$\operatorname{div}^2 \Phi = \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 \varphi_{ik}}{\partial x_i \partial x_k} \in R$$

provided these derivatives exist.

Finally, let

$$H\big(\operatorname{div}^2, \Omega\big) = \big\{\Phi \in L_2\big(\Omega, R^{n \times n}\big) \mid \operatorname{div}^2 \Phi \in L_2(\Omega)\big\},$$
$$H(\operatorname{div}\operatorname{Div}, \Omega) = \big\{\Phi \in L_2\big(\Omega, R_{\mathrm{s}}^{n \times n}\big) \mid \operatorname{div}\operatorname{Div} \Phi \in L_2(\Omega)\big\}$$

be the spaces of matrix-valued and symmetric matrix-valued functions, respectively.

Symbols $c, c_1, \ldots$ are generic. They may represent different quantities (depending possibly on other different quantities) at different occurrences.

8.2.1 Finite Element Mesh Notation

Let $\mathscr{F} = \{\mathscr{T}_h \mid h > 0\}$ be a family of triangulations $\mathscr{T}_h$ of Ω. For any triangle $T \in \mathscr{T}_h$ we denote by h_T its diameter, while h indicates the maximum size of all the triangles in the mesh. We further denote by ϱ_T the diameter of the largest ball inscribed into T. Let $\mathscr{E}(T)$ be the set of all edges and $\mathscr{N}(T)$ the set of all nodes of T. We set

$$\mathscr{E}_h = \bigcup_{T \in \mathscr{T}_h} \mathscr{E}(T), \qquad \mathscr{N}_h = \bigcup_{T \in \mathscr{T}_h} \mathscr{N}(T).$$

We split $\mathscr{E}_h$ in the form $\mathscr{E}_h = \mathscr{E}_{h,\Omega} \cup \mathscr{E}_{h,\Gamma}$ with

$$\mathscr{E}_{h,\Omega} = \{E \in \mathscr{E}_h \mid E \subset \Omega\}, \qquad \mathscr{E}_{h,\Gamma} = \{E \in \mathscr{E}_h \mid E \subset \Gamma\}.$$

For $T \in \mathscr{T}_h$ we define

$$\omega_T = \bigcup_{\mathscr{E}(T)\cap\mathscr{E}(T')\neq\emptyset} T'.$$

The length of $E \in \mathscr{E}_h$ is denoted by h_E. Finally, with every edge $E \in \mathscr{E}_h$ we associate a unit normal vector n_E. The choice of the outer direction of n_E is arbitrary but fixed.

Let T_+ and T_- be any two triangles with a common edge $E \in \mathscr{E}_{h,\Omega}$, the subscripts $+$ and $-$ being chosen in such a way that the unit outer normal to T_- at E corresponds to n_E. Given a piecewise continuous scalar-valued function w on Ω, call w^+ or w^- its trace $w|_{T_+}$ or $w|_{T_-}$ on E. The jump of w across E in the direction of n_E is given by

$$[w]_E = w^+ - w^-.$$

The jump across an edge from $\mathscr{E}_{h,\Gamma}$ is simply given by the trace of the function w on the edge (i.e., the value of w outside Ω is assumed to be zero). For a vector-valued function, the jump is defined componentwise.

We further write $P_l(T)$ for the space of polynomials of degree at most l on T, $l \geq 0$ fixed. In the sequel, $\pi_{l,T}$ denotes the L_2 orthogonal projection of $L_1(T)$ onto $P_l(T)$.

Finally, let f_h be an approximation of a function $f \in L_2(\Omega)$ on a triangle $T \in \mathscr{T}_h$. We then put

$$e_T = \|f - f_h\|_{0;T}. \tag{8.2}$$

8.3 Dirichlet and Second Problems for Biharmonic Equation

8.3.1 Dirichlet Problem for Biharmonic Equation

Let the domain $\Omega \subset R^2$ have a polygonal boundary Γ. We consider the two dimensional biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{8.3}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \tag{8.4}$$

with $f \in L_2(\Omega)$ that models, e.g., the vertical displacement of the mid-surface of a clamped plate subject to bending.

Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Let $\mathscr{L}(X,Y)$ denote the Banach space of continuous linear maps of X on Y and $\mathrm{Isom}(X,Y) \subset$

$\mathscr{L}(X, Y)$ an open subset of linear homeomorphisms of X onto Y. Let $Y^* = \mathscr{L}(Y, R)$ be the dual space of Y and $\langle \cdot, \cdot \rangle$ the corresponding duality pairing.

Let us put, in particular,

$$X = Y = H_0^2(\Omega), \qquad \|\cdot\|_X = \|\cdot\|_Y = \|\cdot\|_2,$$
$$\big\langle F(u), v \big\rangle = \int_\Omega \Delta u \Delta v - \int_\Omega f v. \tag{8.5}$$

We then say that $u \in X$ is the weak solution of the problem (8.3), (8.4) if

$$\big\langle F(u), v \big\rangle = 0 \tag{8.6}$$

for all $v \in Y$.

Since the bilinear form

$$\{u, v\} \to \int_\Omega \Delta u \Delta v$$

is continuous and coercive on X (cf. [9]), we have $\mathrm{d}F(u) \in \mathrm{Isom}(X, Y^*)$ for all $u \in X$, where $\mathrm{d}F$ is the derivative.

Let $\mathscr{F} = \{\mathscr{T}_h \mid h > 0\}$ be a regular family of triangulations $\mathscr{T}_h$ of Ω (see, e.g., [9]). For the discretization of the problem (8.3), (8.4) we assume that $X_h \subset X$ and $Y_h \subset Y$ are finite element spaces corresponding to $\mathscr{T}_h$ and consisting of piecewise polynomials. These conditions imply in particular that the functions in X_h and Y_h are of class C_1. Denote by k, $k \geq 1$, the maximum polynomial degree of the functions in X_h. Further, put $f_h = \pi_{l,T} f$ on T for a fixed $l \geq 0$.

Replacing f in the definition (8.5) by f_h to get the functional F_h, we say that $u_h \in X_h$ is the approximate solution of the problem (8.3), (8.4) if

$$\big\langle F_h(u_h), v_h \big\rangle = 0 \tag{8.7}$$

for all $v_h \in Y_h$.

Using the notation (8.2) for e_T and defining the *local residual a posteriori error indicator*

$$\eta_{\mathrm{V},T} = \Bigg(h_T^4 \big\| \Delta^2 u_h - f_h \big\|_{0;T}^2 + \sum_{E \in \mathscr{E}(T) \cap \mathscr{E}_{h,\Omega}} \big(h_E \big\| [\Delta u_h]_E \big\|_{0;E}^2 + h_E^3 \big\| [n_E \cdot \nabla \Delta u_h]_E \big\|_{0;E}^2 \big) \Bigg)^{1/2}$$

for all $T \in \mathscr{T}_h$, we have the following theorem [19].

Theorem 8.1 *Let $u \in X$ be the unique weak solution of the problem* (8.3), (8.4), *i.e. of* (8.6), *and let $u_h \in X_h$ be an approximate solution of the corresponding discrete* *problem* (8.7). *Then we have the a posteriori estimates*

$$\|u-u_h\|_2 \le c_1\varepsilon_{\mathrm{V}} + c_2\Bigg(\sum_{T\in\mathscr{T}_h} h_T^4 e_T^2\Bigg)^{1/2} + c_3\big\|F(u_h)-F_h(u_h)\big\|_{Y_h^*} + c_4\big\|F_h(u_h)\big\|_{Y_h^*}$$

and

$$\eta_{\mathrm{V},T} \le c_5\|u-u_h\|_{2;\omega_T} + c_6\Bigg(\sum_{T'\subset\omega_T} h_{T'}^4\varepsilon_{T'}^2\Bigg)^{1/2}$$

for all $T\in\mathscr{T}_h$. *The quantities* $\|F(u_h)-F_h(u_h)\|_{Y_h^*}$ *and* $\|F_h(u_h)\|_{Y_h^*}$ *represent the consistency error of the discretization and the residual of the discrete problem, and the quantities* $c_1,\dots,c_6$ *may depend only on* h_T/ϱ_T, *and the integers* k *and* l.

The proof is given in [19]. It seems that this is the first a posteriori error estimate for 4th order problems published.

Let us now consider a nonconforming approximate solution. We say that the family $\mathscr{F}=\{\mathscr{T}_h \mid h>0\}$ of triangulations $\mathscr{T}_h$ is *shape regular* if there are positive constants r_1 and r_2 such that for each triangle $T\in\mathscr{T}_h$ we may inscribe a ball of radius r_1h_T in T and inscribe T in a ball of radius r_2h_T. Thus, let $\mathscr{F}$ be a shape regular family of triangulations $\mathscr{T}_h$ of Ω. Letting T_x be an arbitrary triangle containing the point x, we denote by $h(x)$ the diameter of the triangle T_x.

Let (T, P_T, Φ_T) be the Zienkiewicz element with the triangle $T\in\mathscr{T}_h$, the shape function space P_T, and the set of nodal parameters Φ_T consisting of the function values and two values of first-order derivatives at the three vertices of T [9]. This element is sometimes called the TQC9 element and the corresponding finite element approximation of the fourth-order problem (8.3), (8.4) is nonconforming.

Corresponding to $\mathscr{T}_h$, denote by V_h and V_{h0} the above introduced Zienkiewicz element spaces with respect to H^2 and H_0^2, respectively. For $u_h\in V_h$ and $T\in\mathscr{T}_h$, we define the *local residual a posteriori error indicators* $\eta_{\mathrm{W},T}$ and $\widetilde{\eta}_{\mathrm{W},T}$ like in [20]. The corresponding statement proven there yields two a posteriori error estimates that contain unknown positive constants C_1 and C_2.

8.3.2 Dirichlet and Second Problems for Biharmonic Equation in Mixed Finite Element Formulation

Let $\Omega\subset R^2$ be a convex polygonal domain with the boundary Γ. We consider the two-dimensional biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{8.8}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \tag{8.9}$$

with $f \in H^{-1}(\Omega)$ that is used both for linear plate analysis and incompressible flow simulation.

Put $V = H_0^1(\Omega)$ and $X = H^1(\Omega)$ and define the continuous bilinear forms

$$a(w,z) = \int_\Omega wz \quad \text{on } X \times X \quad \text{and} \quad b(z,u) = \int_\Omega \nabla z \cdot \nabla u \quad \text{on } X \times V \tag{8.10}$$

with scalar-valued functions u, w, and z.

The Ciarlet-Raviart weak formulation [10] of (8.8) and (8.9) then reads: Find $\{w,u\} \in X \times V$ such that

$$a(w,z) + b(z,u) = 0 \quad \text{for all } z \in X, \tag{8.11}$$

$$b(w,v) + \int_\Omega fv = 0 \quad \text{for all } v \in V. \tag{8.12}$$

The existence and uniqueness of the solution $\{w = \Delta u, u\}$ of the problem (8.11) and (8.12) are proven in [7].

We construct the conforming second order discretization according to [15]. Let $\mathscr{F} = \{\mathscr{T}_h \mid h > 0\}$ be a regular family of triangulations $\mathscr{T}_h$ of Ω. For the sake of simplicity, we also assume that the family is uniformly regular [9] to guarantee that the inequality (8.13) holds, even though it is not easy to satisfy this condition in the presence of mesh refinements.

The finite element spaces X_h and V_h are then

$$X_h = \left\{x_h \in X \mid x_h|_T \in P_2(T) \text{ for all } T \in \mathscr{T}_h\right\},$$

$$V_h = \left\{v_h \in V \mid v_h|_T \in P_2(T) \text{ for all } T \in \mathscr{T}_h\right\}.$$

Our assumption of uniform regularity of the family $\mathscr{F}$ implies that there is a positive constant c such that the *inverse inequality*

$$|x_h|_{m;T} \leq ch^{l-m}|x_h|_{l;T} \tag{8.13}$$

holds for all integers l and m, $l \leq m$, and all $x_h \in X_h$ and $T \in \mathscr{T}_h$.

The discrete formulation of the problem (8.11) and (8.12) now reads: Find $\{w_h, u_h\} \in X_h \times V_h$ such that

$$a(w_h, z_h) + b(z_h, u_h) = 0 \quad \text{for all } z_h \in X_h, \tag{8.14}$$

$$b(w_h, v_h) + \int_\Omega f v_h = 0 \quad \text{for all } v_h \in V_h. \tag{8.15}$$

We introduce the *local residual a posteriori error indicators* $\eta_{C,T}$ and $\widetilde{\eta}_{C,T}$ based on local residuals like in [8]. Then the following theorem holds.

Theorem 8.2 *Let* $\{w,u\} \in X \times V$ *be the unique weak solution of the problem* (8.8) *and* (8.9), *i.e. of* (8.11) *and* (8.12), *and let* $\{w_h, u_h\} \in X_h \times V_h$ *be an approximate* *solution of the corresponding discrete problem* (8.14) *and* (8.15)*. Then we have the a posteriori estimates*

$$\|u - u_h\|_1 + h\|w - w_h\|_0 \le C_1\big(\varepsilon_{\mathrm{C}} + h^2\widetilde{\varepsilon}_{\mathrm{C}}\big)$$

with some positive constant C_1 *independent of* h *and*

$$\eta_{\mathrm{C},T} + h_T^2\widetilde{\eta}_{\mathrm{C},T} \le C_2\Bigg(|u - u_h|_{1;\omega_T} + h_T\|w - w_h\|_{0;\omega_T} + h_T^3 \sum_{T'\subset\omega_T} e_{T'}\Bigg)$$

for $T \in \mathscr{T}_h$ *with some positive constant* C_2 *independent of* h *and* e_T *given by* (8.2).

The proof is given in [8].

On the convex polygonal domain $\Omega \subset R^2$ with the boundary Γ we now consider the two dimensional second biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{8.16}$$

$$u = \Delta u = 0 \quad \text{on } \Gamma \tag{8.17}$$

with $f \in L_2(\Omega)$ that models the deformation of a simply supported thin elastic plate. Putting $w = \Delta u$, we can rewrite the problem (8.16), (8.17) as the system of two Poisson equations, both with the homogeneous Dirichlet boundary condition.

Define the continuous bilinear forms $a(w,z)$ and $b(z,u)$ by (8.10) but with all the scalar-valued functions u, w, and z from $V = H_0^1(\Omega)$. The Ciarlet-Raviart weak formulation [10] of (8.16) and (8.17) then reads: Find $\{w,u\} \in V \times V$ such that (8.11) and (8.12) hold for all $z, v \in V$.

Let $\mathscr{F} = \{\mathscr{T}_h \mid h > 0\}$ be a quasiuniform family of triangular or rectangular partitions $\mathscr{T}_h$ of Ω [1]. Put

$$V_h = \big\{z \in C(\overline{\Omega}) \mid z|_T \in P_k(T),\ k \ge 1,\ \text{for all } T \in \mathscr{T}_h\big\} \cap H_0^1(\Omega).$$

The discrete weak formulation of the problem (8.16) and (8.17) now reads: Find $\{w_h, u_h\} \in V_h \times V_h$ such that (8.14) and (8.15) hold for all $z_h, v_h \in V_h$.

Let the basis function $v_{h,N}$ from V_h be associated with the node $N \in \mathscr{N}_{h,\Omega} = \mathscr{N}_h \cap \Omega$. Put $\omega_N = \operatorname{supp} v_{h,N}$. We introduce the *gradient recovery operator* $Gv_h : V_h \to V_h \times V_h$ in the following way [12]. Assume that

$$v_h(x) = \sum_{N\in\mathscr{N}_{h,\Omega}} \beta_N v_{h,N}(x), \quad x \in \Omega,$$

with some coefficients β_N and put

$$\widetilde{G}v_{h,N} = \sum_{T\cap\omega_N\neq\emptyset} \alpha_N^T (\nabla v_{h,N})|_T, \quad \text{where} \sum_{T\cap\omega_N\neq\emptyset} \alpha_N^T = 1$$

and $0 \le \alpha_N^T \le 1$ are chosen weights. Note that the vector $\nabla v_{h,N}$ is constant on each triangle. Finally, we set

$$Gv_h(x) = \sum_{N \in \mathcal{N}_{h,\Omega}} \widetilde{G}v_{h,N}\, v_{h,N}(x), \quad x \in \Omega.$$

For $u_h, w_h \in V_h$ and $T \in \mathcal{T}_h$, define a *local recovery a posteriori error indicator* $\eta_{\mathrm{L},T}$ like in [12]. The corresponding statement proven there yields a lower as well as an upper a posteriori error estimate that both contain unknown positive constants c, C, C_1, and C_2 independent of h. In the paper, the authors further claim that the global error estimator ε_{L} is asymptotically exact if the mesh is uniform and the solution is smooth enough.

8.4 Dirichlet Problem for Fourth Order Elliptic Equation

8.4.1 Some Recovery and Residual Error Indicators

Put $\Omega = (0, 1) \subset R^1$. Let all the functions concerned be scalar-valued functions of a single variable. We consider the one dimensional boundary value problem for the ordinary fourth-order elliptic equation

$$\left(au''\right)'' = f \quad \text{in } \Omega$$

with the boundary conditions

$$u(0) = u'(0) = 0, \qquad u(1) = u'(1) = 0.$$

The weak solution $u \in H_0^2(\Omega)$ and the approximate solution $u_h \in V_h$ are defined in the usual way [14]. V_h is a finite element space consisting of piecewise Hermite cubic polynomials.

We introduce a *recovery operator* Gv_h for the second derivative of $v_h \in V_h$ and, for $u_h \in V_h$ and $T \in \mathcal{T}_h$, define a *local recovery a posteriori error indicator* $\eta_{\mathrm{P},T}$ like in [14]. The corresponding statement proven there yields an upper estimate for the difference of the global error estimator ε_{P} and the energy norm of the true error [14]. The global error estimator is asymptotically exact.

Consider the bending problem of an isotropic linearly elastic plate. The bilinear form for the problem is

$$a(u, v) = \left(\gamma\varepsilon(\nabla u), \varepsilon(\nabla v)\right)_0, \quad u, v \in H_0^2,$$

where γ is the fourth-order positive definite *elasticity tensor* and ε the *small strain tensor* [6]. We employ the *discrete Morley space* W_h that is nonconforming for the finite element solution of the problem, see, e.g., [9].

With the help of the bilinear form $a_h(u_h, v_h)$, $v_h \in W_h$, defined in an obvious way we introduce the approximate solution $u_h \in W_h$. The bilinear form a_h is positive definite on the space W_h, therefore there is a unique solution $u_h \in W_h$ to the problem, cf. [6].

For $u_h \in W_h$ and $T \in \mathscr{T}_h$, define a *local residual a posteriori error indicator* $\eta_{\mathrm{B},T}$ like in [6]. The corresponding statement proven there yields lower as well as upper a posteriori error estimates in a discrete norm introduced there. Both these estimates contain an unknown positive constant C independent of h.

8.4.2 Dirichlet Problem for Fourth Order Partial Differential Equation

Let $\Omega \in R^n$ be a bounded connected domain and Γ its Lipschitz continuous boundary. We consider the 4th order elliptic problem for a scalar-valued function u,

$$\operatorname{div}\operatorname{Div}(\gamma\nabla\nabla u) = f \quad \text{in } \Omega, \tag{8.18}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma, \tag{8.19}$$

where $f \in L_2(\Omega)$, $\gamma = [\gamma_{ijkl}]^n_{i,j,k,l=1}$ and $\gamma_{ijkl} = \gamma_{jikl} = \gamma_{klij} \in L_\infty(\Omega)$.

We assume the existence of constants $0 < m \le M$ such that

$$m\|\Phi\|^2_{\mathrm{F}} \le (\gamma\Phi) \odot \Phi \le M\|\Phi\|^2_{\mathrm{F}} \quad \text{for all } \Phi \in R^{n\times n}_{\mathrm{s}}. \tag{8.20}$$

Then the inverse tensor γ^{-1} exists and we define for any matrix-valued function $\Phi \in L_2(\Omega, R^{n\times n})$, analogically to (8.1), the norms

$$|||\Phi|||^2 = \int_\Omega (\gamma\Phi) \odot \Phi \quad \text{and} \quad |||\Phi|||^2_* = \int_\Omega \left(\gamma^{-1}\Phi\right) \odot \Phi.$$

A function $u \in H^2_0(\Omega)$ is now said to be the weak solution of the problem (8.18), (8.19) if it satisfies the identity

$$\int_\Omega (\gamma\nabla\nabla u) \odot (\nabla\nabla v) = \int_\Omega fv$$

for all test functions $v \in H^2_0(\Omega)$.

Let $\bar{u}$ be a function from $H^2_0(\Omega)$ considered as an approximation of the weak solution u. In [16], no specification of the way $\bar{u}$ has been computed is required, it is just an arbitrary function of the admissible class.

Define the *global functional a posteriori error estimator*

$$\varepsilon_{\mathrm{R}}(\beta, \Phi, \bar{u}) = (1+\beta)|||\gamma\nabla\nabla\bar{u} - \Phi|||^2_* + \left(1 + \frac{1}{\beta}\right)C^2_{1\Omega}\|\operatorname{div}\operatorname{Div}\Phi - f\|^2_0,$$

where β is an arbitrary positive real number, Φ an arbitrary symmetric matrix-valued function from $H(\operatorname{div}\operatorname{Div},\Omega)$, and $C_{1\Omega}$ the constant from the Friedrichs inequality

$$\|w\|_0 \le C_{1\Omega} |||\nabla\nabla w||| \tag{8.21}$$

valid for all $w \in H_0^2(\Omega)$. Then the following theorem holds [16].

Theorem 8.3 *Let $u \in H_0^2(\Omega)$ be the weak solution of the problem (8.18), (8.19) and $\bar{u} \in H_0^2(\Omega)$ an arbitrary function. Then*

$$\left|\left|\left|\nabla\nabla(\bar{u}-u)\right|\right|\right|^2 \le \varepsilon_{\mathrm{R}}(\beta,\Phi,\bar{u}) \tag{8.22}$$

for any symmetric matrix-valued function $\Phi \in H(\operatorname{div}\operatorname{Div},\Omega)$ and any positive number β.

The proof of the theorem is based on a more general statement proven in [16]. The estimate (8.22) corresponds to the decomposition $\operatorname{div}\operatorname{Div}\Theta = f$, $\Theta = \gamma\nabla\nabla u$ of Eq. (8.18). However, the condition $\operatorname{div}\operatorname{Div}\Theta \in L_2(\Omega)$ is rather demanding.

To avoid possible difficulties of this kind, we can derive another error estimate if we introduce a further global functional error estimator,

$$\begin{aligned}\widetilde{\varepsilon}_{\mathrm{R}}(\beta,\Phi,Y,\bar{u}) = (1+\beta)|||\gamma\nabla\nabla\bar{u}-\Phi|||_*^2 \\ + \frac{1+\beta}{\beta}(C_{1\Omega}\|\operatorname{div}Y - f\|_0 + C_{2\Omega}\|\operatorname{Div}\Phi - Y\|_0)^2,\end{aligned}$$

where β is a positive real number, Φ an arbitrary symmetric matrix-valued function from $H(\operatorname{Div},\Omega)$, $C_{2\Omega}$ the constant from the Friedrichs inequality

$$\|\nabla w\|_0 \le C_{2\Omega}|||\gamma\nabla\nabla w||| \tag{8.23}$$

valid for all $w \in H_0^2(\Omega)$, and Y an arbitrary vector-valued function from $H(\operatorname{div},\Omega)$. Then we get the same statement as in Theorem 8.3 but with $\widetilde{\varepsilon}_{\mathrm{R}}(\beta,\Phi,Y,\bar{u})$ on the right-hand part of (8.22) (cf. [16], where the proof is given). The estimate corresponds to the decomposition $\operatorname{div}Y = f$, $\operatorname{Div}\Theta = Y$, $\Theta = \gamma\nabla\nabla u$ of Eq. (8.18).

Theorem 8.3 is equivalent to the statements proven in [13, Sect. 6.6]. Moreover, in [13] the authors use another global functional a posteriori error estimator to prove a lower estimate for the error.

The constants $C_{1\Omega}$ and $C_{2\Omega}$ can be estimated from above by $m^{-1}C_{1\square}$ and $m^{-1}C_{2\square}$, where m is the constant from (8.20), and $C_{1\square}$ and $C_{2\square}$ appear in the Friedrichs inequalities (8.21), (8.23) that hold for any $w \in H_0^2(\Omega)$ on a rectangular domain $\square$ containing Ω [16].

A posteriori error estimates for Eq. (8.18) with other boundary conditions can be derived, too. Instead of $C_{1\Omega}$ and $C_{2\Omega}$ they involve constants appearing in inequalities analogous to (8.21) and (8.23).

The biharmonic equation

$$\Delta^2 u = f \quad \text{in } \Omega$$

is a particular case of Eq. (8.18). Considering it with the Dirichlet boundary condition (8.19) and introducing a particular error estimator, we obtain a statement analogous to Theorem 8.3, see [16].

Consider another Dirichlet problem. Let $\mathrm{d}^2 u$ denote the Hessian matrix of a function $u : \Omega \to R$, $u \in H^2(\Omega)$. Let the matrix-valued function $\Lambda = [\lambda_{ik}]$, $\Lambda : \Omega \times R^{n\times n} \to R^{n\times n}$ be measurable and bounded with respect to the variable $x \in \Omega$ and of class C_2 with respect to the matrix variable $\Theta \in R^{n\times n}$.

Let the domain $\Omega \subset R^n$ have a piecewise C_1 boundary. We consider the fourth-order elliptic problem

$$\operatorname{div}^2 \Lambda\big(x, \mathrm{d}^2 u\big) = f \quad \text{in } \Omega, \tag{8.24}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \tag{8.25}$$

with $f \in L_2(\Omega)$.

Making proper assumptions on the Jacobian arrays $\Lambda'(x, \Theta)$, we get the existence of Λ^{-1}, the inverse of Λ with respect to $\Theta \in R^{n\times n}$ [11].

The problem (8.24), (8.25) has a unique weak solution $u \in H_0^2(\Omega)$ that satisfies

$$\int_\Omega \Lambda\big(x, \mathrm{d}^2 u\big) \odot \mathrm{d}^2 v - \int_\Omega f v = 0 \quad \text{for all } v \in H_0^2(\Omega).$$

Let $\bar{u}$ be a function from $H_0^2(\Omega)$ considered as an approximation of the weak solution u. In [11], no specification of the way $\bar{u}$ has been computed is required, it is just an arbitrary function of the admissible class.

We measure the error of the approximate solution $\bar{u}$ by a functional $E(\bar{u})$ introduced in [11]. For $\bar{u} \in H_0^2(\Omega)$, an arbitrary matrix-valued function $\Psi \in H(\operatorname{div}^2, \Omega) \cap L_\infty(\Omega, R^{n\times n})$ and an arbitrary scalar-valued function $w \in H_0^2(\Omega)$, define the *global functional a posteriori error estimator* $\varepsilon_{\mathrm{K}}(\Psi, w, \bar{u})$ like in [11]. It contains four generally unknown positive constants. The corresponding statement proven there yields an upper a posteriori error estimate. To avoid the computation of Λ^{-1} we can introduce another global functional a posteriori error estimator and reformulate the above mentioned statement. Moreover, the authors prove in [11] that the global estimator $\varepsilon_{\mathrm{K}}(\Psi, w, \bar{u})$ is sharp for a sufficiently smooth weak solution.

8.5 Conclusion

The quantitative properties of the indicators and estimators cannot be easily assessed and compared analytically. There are, however, analytical error estimators for some classes of problems (see, e.g., [11, 13, 18]) that require as few unknown constants as possible. The a posteriori estimates with unknown constants, however, are not optimal for the practical computation. They can be efficient if they are asymptotically exact.

The computation of the reference solution is rather time-consuming. Nevertheless, we use reference solutions as robust error indicators with no unknown constants to control the adaptive strategies in the most complex finite element computations.

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