Chapter 22
Dynamic Behaviour of a Travelling Viscoelastic Band in Contact with Rollers

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Abstract The dynamic behaviour of an axially moving viscoelastic band, in contact with supporting rollers, is studied. A model of a thin, viscoelastic beam (panel) subjected to bending and centrifugal forces is used. An initial-boundary value problem for a fifth-order partial differential equation describing the movement of the band is formulated in detail. In this paper, five boundary conditions in total are used for the first time within the present model. An external force describing the normal force of the roller supports is included. Combining this viscoelastic model with the roller contact simulation is a new approach among moving band behaviour studies. The initial-boundary value problem is solved numerically using the fourth-order Runge-Kutta method and the central finite differences, and the band behaviour is illustrated for different band velocities and degrees of viscosity. It is found that the damping effect of viscoelasticity increases when the band velocity increases, and that the roller contact has a greater effect on the elastic panel behaviour than on the viscoelastic panel behaviour.

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22.1 Introduction

The behaviour of systems, in which some material travels axially at a fast speed between two supports, has been studied widely. Interest in these studies arises from the extensive amount of applications in industry, e.g., in paper making processes.

In paper machines, the radius of supporting rollers is usually large compared to the length of an open draw, see Fig. 22.1. However, in the often studied models, the effect of the rollers on the behaviour of the moving web has been neglected.

Vibrations of travelling strings, beams, and bands were first studied by Sack [28], Archibald and Emslie [1], Miranker [22], Swope and Ames [30], Mote [24–26], Simpson [29], Ulsoy and Mote [31], Chonan [9], and Wickert and Mote [33]. These studies focused on one-dimensional free and forced vibrations including the nature of wave propagation in moving media and the effects of axial motion on the eigenfrequencies and eigenmodes. Stability of travelling two-dimensional rectangular membranes and plates was studied, e.g., by Ulsoy and Mote [32], Lin and Mote [19], Lin [18], and Banichuk et al. [2].

Archibald and Emslie [1] and Simpson [29] studied effects of the axial motion on the eigenfrequencies and eigenfunctions. It was shown that the natural frequency of each mode decreases when the transport speed increases, and that both the travelling string and beam experience divergence instability at a sufficiently high speed.

Wet paper and many other materials have viscoelastic properties. The first study on transverse vibrations of a travelling viscoelastic material was carried out by Fung et al. [11], who used a string model. They investigated numerically the effects of material parameters and transport velocity on the transient amplitudes. Extending their work, they studied the material damping effect in their later research [12]. Fung et al. used a standard linear solid model to describe the viscoelasticity of material.

String and beam models have been widely used models in the studies concerning travelling viscoelastic materials. Oh et al. [27] and Lee and Oh [17] studied critical speeds, eigenvalues, and natural modes of axially moving viscoelastic beams using a spectral element model.

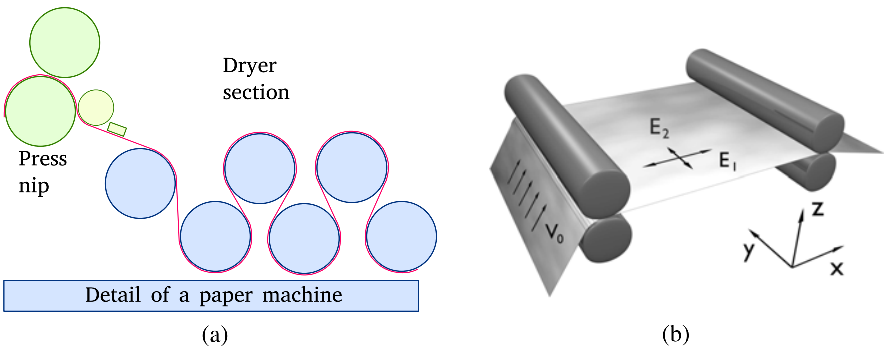


Fig. 22.1 An overview. (**a**) Paper machine cross-section. (**b**) A qualitative drawing of an open draw

Chen and Zhao [8] represented a modified finite difference method to simplify a non-linear model of an axially moving string. They studied numerically the free transverse vibrations of elastic and viscoelastic strings.

Yang and Chen [7, 34] studied vibrations and stability of axially moving viscoelastic beams with periodic parametric excitations. Yang and Chen [34] studied dynamic stability of axially moving viscoelastic beams with a time-pulsating speed. They found that the viscoelastic damping decreases the instability region of subharmonic resonance. Chen and Yang [7] studied free vibrations of a viscoelastic beam travelling between simple supports with torsion strings. They studied the viscoelastic effect by perturbing the similar elastic problem and using the method of multiple scales.

Marynowski and Kapitaniak [20] studied the difference between the Kelvin-Voigt model and the Bürgers model in internal damping and found out that both models give accurate results with a small damping coefficient, but with a large damping coefficient, the Bürgers model is more accurate. In 2007, they compared the models with the Zener model studying the dynamic behavior of an axially moving viscoelastic beam [21]. They found out that the Bürgers and Zener models gave similar results for the critical transport speed whereas the Kelvin-Voigt model gave significantly greater transport speed compared to the other two models.

In all discussed studies above, a partial time derivative has been used instead of a material derivative in the viscoelastic constitutive relations. Mockensturm and Guo [23] suggested that the material derivative should be used. They studied nonlinear vibrations and the dynamic response of axially moving viscoelastic strings, and found significant discrepancy in the frequencies at which non-trivial limit cycles exist comparing the models with the partial time derivative and the material time derivative. In Chen et al. [4], Ding and Chen [10], Chen and Wang [6], and Chen and Ding [5], the material derivative was also used in the viscoelastic constitutive relations. Ding and Chen [10] studied stability of axially accelerating viscoelastic beams using the method of multiple scales and parametric resonance. Chen and Wang [6] studied the stability of axially accelerating viscoelastic beams using the asymptotic perturbation analysis. In a recent research by Chen and Ding [5], the steady-state response of transverse vibrations for axially moving viscoelastic beams was studied. Kurki and Lehtinen [16] suggested, separately, that the material derivative in the constitutive relations should be used in their study concerning the in-plane displacement field of a travelling viscoelastic plate.

Using the material derivative in the viscoelastic constitutive relations for a beam model leads to a partial differential equation that is fifth-order with respect to the space coordinate. In Ding and Chen [10], Chen and Wang [6], and Chen and Ding [5], the fifth-order dynamic equation is attained but only four boundary conditions (in space) are used. However, the amount of boundary (initial) conditions should coincide with the order of the equation with respect to each variable.

We also mention studies by Guan (et al.) [13–15]. They used a different (from the references mentioned above) kind of approach in modelling of viscoelastic effects in moving web-handling systems applying the White–Metzner rheological equation. In those studies, permanent web deformations and web tension behaviour as a function of time were investigated.

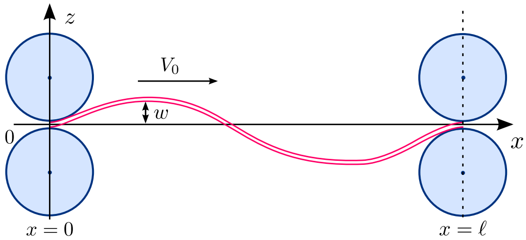


Fig. 22.2 An assumption of cylindrical deformation

In this study, we investigate the transverse displacement of a viscoelastic panel travelling between and in contact with two supports. Using a linear Kirchhoff plate model and a Kelvin-Voigt viscoelasticity model, a fifth-order partial differential equation for the transverse displacement of the panel is derived in detail. Simply supported boundary conditions are used at both edges and, at the in-flow edge, an additional boundary condition corresponding to the travelling angle is used. That is, five boundary conditions in total are used. The contact with the supporting rollers is modelled by a nonlinear spring force between the rollers and the panel. Numerical simulations of the behaviour of the panel are presented. A comparison of the behaviour between the model including the contact effect and the classic model with no contact is made.

22.2 Problem Setup

Consider a viscoelastic band travelling at a constant axial velocity V_0 (in the x direction) in a span. The domain of this study is the span between two rollers located at $x=0$ and $x=\ell$. We investigate the transverse displacement w of the band as a dynamic problem taking into account the contact with the rollers. We assume that the transverse displacements are small to make the linear theory justifiable. We also assume that the displacement w is cylindrical, that is, the displacement does not vary in the cross direction to the movement, see Fig. 22.2. The thickness of the band is assumed to be constant, h. The tension at the edges is supposed to be constant, T_0. The plate is assumed to have a constant bending rigidity, D, and a constant viscous bending rigidity D^{v}. The mass per area of the band is m.

The equation describing the transverse displacement $w=w(x,t)$ of the panel (a plate with cylindrical deformation) is derived using the Kirchhoff plate model and the Kelvin-Voigt model for the viscoelasticity.

We first write the equilibrium equation for the bending forces affecting the panel, which is

$$\frac{\partial^2 M}{\partial x^2} + T_0 \frac{\partial^2 w}{\partial x^2} + q = 0, \quad x \in (0,\ell), \tag{22.1}$$

where T_0 is the tension force in the x direction, M the bending moment, and q the intensity of external load distributed over the upper surface of the panel.

Let σ denote the flexural stress. This stress depends on a strain that is defined by

$$\varepsilon = -z\,\frac{\partial^2 w}{\partial x^2}. \tag{22.2}$$

The bending moment is related to the flexural stress by

$$M = \int_{-h/2}^{h/2} z\sigma\,\mathrm{d}z. \tag{22.3}$$

The stress depends on the strain by the relation

$$\sigma = C\varepsilon + \Gamma\,\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}, \tag{22.4}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + V_0\frac{\partial}{\partial x},$$

and

$$C = \frac{E}{1-\nu^2}, \qquad \Gamma = \frac{\eta}{1-\varphi^2}.$$

Here, E is the Young modulus, ν the Poisson ratio, and η and φ are the corresponding viscous material constants.

For the balance equation (22.1), we calculate, first, the bending moment. By inserting (22.4) into (22.3) and (22.2) into (22.4), we obtain

$$M = \int_{-h/2}^{h/2} z\sigma\,\mathrm{d}z = \int_{-h/2}^{h/2} z\left(C\varepsilon + \Gamma\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right)\mathrm{d}z = -\frac{h^3}{12}\left(C\,\frac{\partial^2 w}{\partial x^2} + \Gamma\,\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial^2 w}{\partial x^2}\right). \tag{22.5}$$

We calculate the second space derivative of the bending moment (22.5). We obtain

$$\frac{\partial^2 M}{\partial x^2} = -\frac{h^3}{12}\left(C\,\frac{\partial^4 w}{\partial x^4} + \Gamma\,\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial^4 w}{\partial x^4}\right). \tag{22.6}$$

Substituting (22.6) into (22.1), we obtain

$$-\frac{h^3}{12}\left(C\,\frac{\partial^4 w}{\partial x^4} + \Gamma\,\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial^4 w}{\partial x^4}\right) + T_0\,\frac{\partial^2 w}{\partial x^2} + q = 0. \tag{22.7}$$

Introducing the parameters

$$D = \frac{h^3}{12}\,C, \qquad D^{\mathrm{v}} = \frac{h^3}{12}\,\Gamma,$$

and adding dynamical components into (22.7), we obtain

$$-D\,\frac{\partial^4 w}{\partial x^4} - D^{\mathrm{v}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial^4 w}{\partial x^4} + T_0\,\frac{\partial^2 w}{\partial x^2} + q = m\,\frac{\mathrm{d}^2 w}{\mathrm{d}t^2}. \tag{22.8}$$

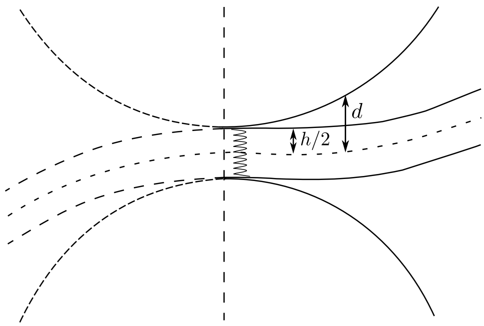


Fig. 22.3 A spring model in the cross direction of the plate. A detail near one end of the span

Expanding the expressions in (22.8) and re-organizing the terms, the dynamic equation for $w = w(x,t)$ reads

$$\frac{\partial^2 w}{\partial t^2} + \left(2V_0\frac{\partial}{\partial x} + \frac{D^{\mathrm{v}}}{m}\frac{\partial^4}{\partial x^4}\right)\frac{\partial w}{\partial t} + \left[\left(V_0^2 - \frac{T_0}{m}\right)\frac{\partial^2}{\partial x^2} + \frac{D}{m}\frac{\partial^4}{\partial x^4} + V_0\frac{D^{\mathrm{v}}}{m}\frac{\partial^5}{\partial x^5}\right]w = \frac{q}{m}, \tag{22.9}$$

where $x \in (0, \ell)$, $t \in (0, t_{\mathrm{f}})$, and t_{f} is the end point of the time domain. We use classical simply supported boundary conditions at both edges, and an additional condition at the in-flow edge. The boundary conditions read

$$w(0,t) = w(\ell,t) = 0, \qquad \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(\ell,t) = 0, \tag{22.10}$$

and

$$\frac{\partial w}{\partial x}(0,t) = \theta, \tag{22.11}$$

where θ is a given constant describing the angle between the panel and the x axis at the in-flow edge. The angle θ represents the feeding angle of the web, and in a multi-span system it could be predicted (calculated) for one span from the behaviour of the panel on the preceding span. However, in this study we concentrate only on one isolated span and assume that the feeding angle is known. The initial conditions for the dynamic problem are

$$w(x,0) = g_1(x), \qquad \frac{\partial w}{\partial t}(x,0) = g_2(x), \tag{22.12}$$

where g_1 and g_2 are some given functions.

The contact force between the moving panel and the supporting rollers is now to be included in the panel model. The transverse direction of the panel is modelled as a non-linear spring such that the maximum compression of the panel is one half of its thickness, see Figs. 22.3 and 22.4. The force function depending on the distance

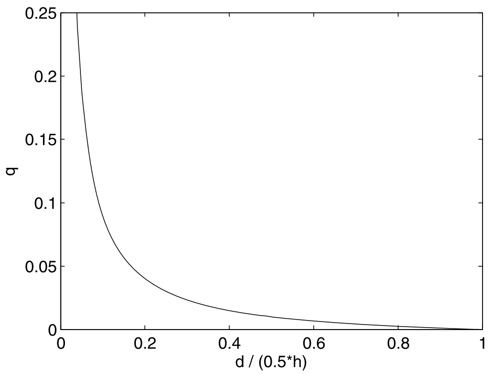


Fig. 22.4 Contact force as a function of the distance between the plate center and the roller

d between the panel center and the roller surface is given by

$$q(d) = a\left(\frac{h}{2d} - 1\right), \quad 0 < d \leq h/2. \tag{22.13}$$

The parameter a is a constant describing the strength of the force. Inside the rollers ($d \leq 0$) the force is not defined, and if there is no contact ($d > h/2$), then the force q is zero.

22.3 Numerical Investigation

We use central difference formulae and the fourth-order Runge-Kutta for the space and time discretisations, respectively. In the central differences, the higher order derivatives need node values from a distance of three nodes of the node being computed. We neglect the fifth-order derivatives at the boundary. The interval $[0, \ell]$ is divided to $n+1$ subintervals equal in length. The end points of the subintervals are labeled as $0 = x_0, x_1, \ldots, x_n, x_{n+1} = \ell$. We need one virtual point from the boundary conditions for both edges. From the boundary conditions, we get $w(x_0) = 0$, $w(x_{n+1}) = 0$, $w(x_{-1}) = -w(x_1)$ and $w(x_{n+2}) = -w(x_n)$. In boundary condition (22.11), we choose $\theta = 0$, which leads to $w(x_{-1}) = w(x_1)$ and finally $w(x_1) = 0$.

In Fig. 22.5, it is illustrated how the rollers and computation nodes are connected by simple geometry. It must be noticed that we are considering the transverse displacements merely, and therefore, the contact force effects are considered in the z direction only.

The parameters used are as follows:

$$\begin{array}{llll} \ell = 0.25 \text{ m}, & T_0 = 500 \text{ N/m}, & h = 10^{-4} \text{ m}, & m = 0.08 \text{ kg/m}^2, \\ E = 10^9 \text{ Pa}, & \nu = 0.3, & h_s = 0.5 \cdot h, & r_s = 0.12 \text{ m}. \end{array} \tag{22.14}$$

Here, ℓ is the length of the open draw, T_0 is constant tension applied at the ends of the panel, h is the thickness of the panel, m is mass per unit area, E is the Young

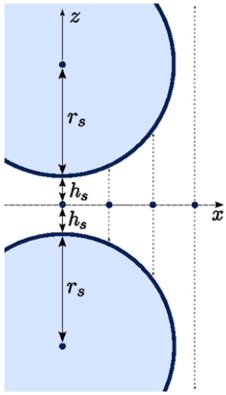


Fig. 22.5 Nodes between the rollers. A detail near one end of the span

modulus, ν is the Poisson ratio, r_s is the radius of the rollers, and h_s is one half of the distance between the pressing rollers, see Fig. 22.5. We define

$$D^{\mathrm{v}} = \alpha_{\mathrm{v}} D.$$

The multiplier α_{v} is here called the relative viscosity, for which the values $\alpha_{\mathrm{v}} = 0.0008, 0.08$ were used. We studied the dynamic behaviour of the panel for the first 0.05 seconds, for three different velocities $V_0 = 0,\ 30,\ 60$ m/s. The strength of the force (22.13) was $a = 0.01$. The used number of the computation nodes was $n = 150$.

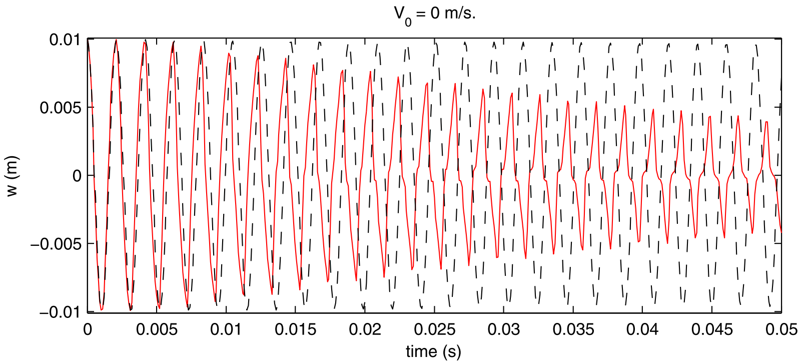
The used initial conditions were

$$w(x,0) = 0.01 \sin\left(\frac{\pi x}{\ell}\right),$$

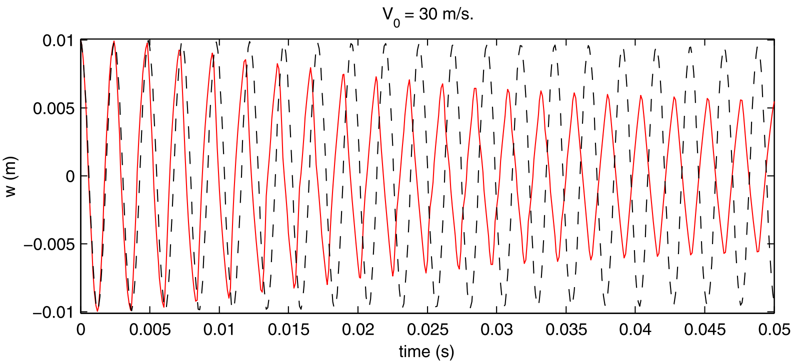
$$\frac{\partial w}{\partial t}(x,0) = 0.$$

The investigated cases include the behaviour of the midpoint of the panel (Figs. 22.6 and 22.7), from which the frequency and amplitude of the vibrations can be analysed, and the space-time behaviour of the panel (Figs. 22.8 and 22.10). The results for the stationary panel are shown in Figs. 22.6a, 22.8 (almost elastic material), and Figs. 22.7(a), 22.9 (viscoelastic material). The results for the moving panel are shown in Figs. 22.6(b), 22.6(c), 22.10 (almost elastic material), and Figs. 22.7(b), 22.7(c), 22.11 (viscoelastic material).

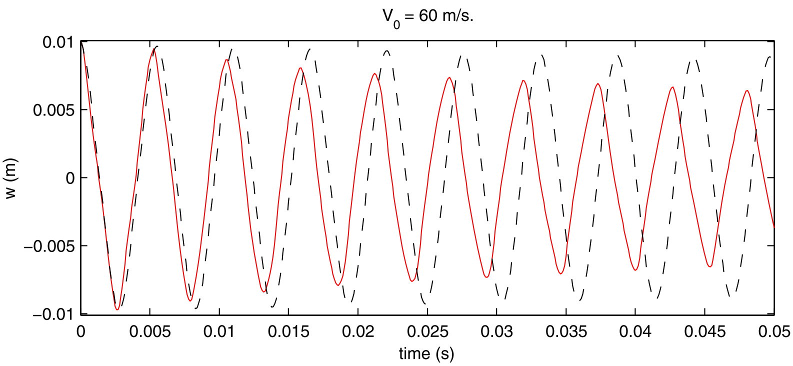
In Figs. 22.6 and 22.7, the behaviour of the panel center is shown for a panel travelling at different velocities for both viscoelastic and almost elastic materials. From Figs. 22.6(a) and 22.7(a), it can be seen that the contact with the rollers is decreasing the amplitude of the vibrations in the case of an almost elastic panel and increasing the amplitude in the case of a viscoelastic panel compared to the case with no roller contact. In both cases, the frequency of the vibrations is increased. Also, when the panel is moving at a constant velocity (Figs. 22.6(b), 22.6(c), 22.7(b), and 22.7(c)), the frequency of vibrations in the case with roller contact is greater



(a) $V_0 = 0$, $\alpha_v = 0.0008$.

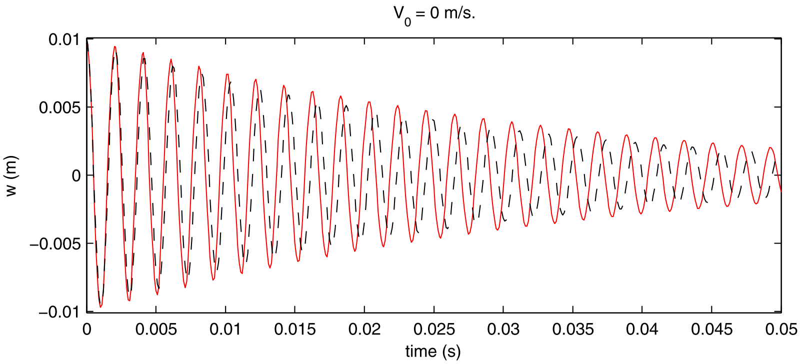


(b) $V_0 = 30$ m/s, $\alpha_v = 0.0008$.

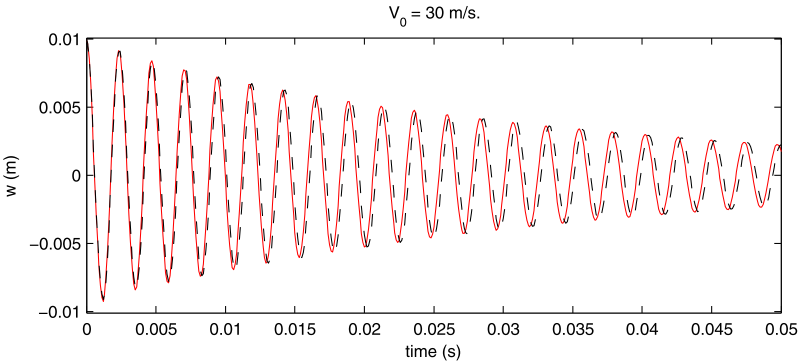


(c) $V_0 = 60$ m/s, $\alpha_v = 0.0008$.

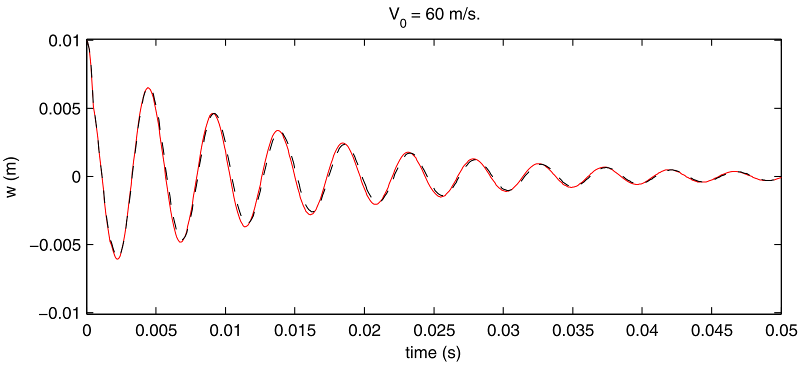
Fig. 22.6 Behaviour of the midpoint of the panel during the first 0.05 seconds for an almost elastic material. The *solid line* shows the case with roller contact, and the *dashed line* shows the case without contact. V_0 is the panel velocity and α_v is the relative viscosity



(a) $V_0 = 0$, $\alpha_\text{v} = 0.08$.

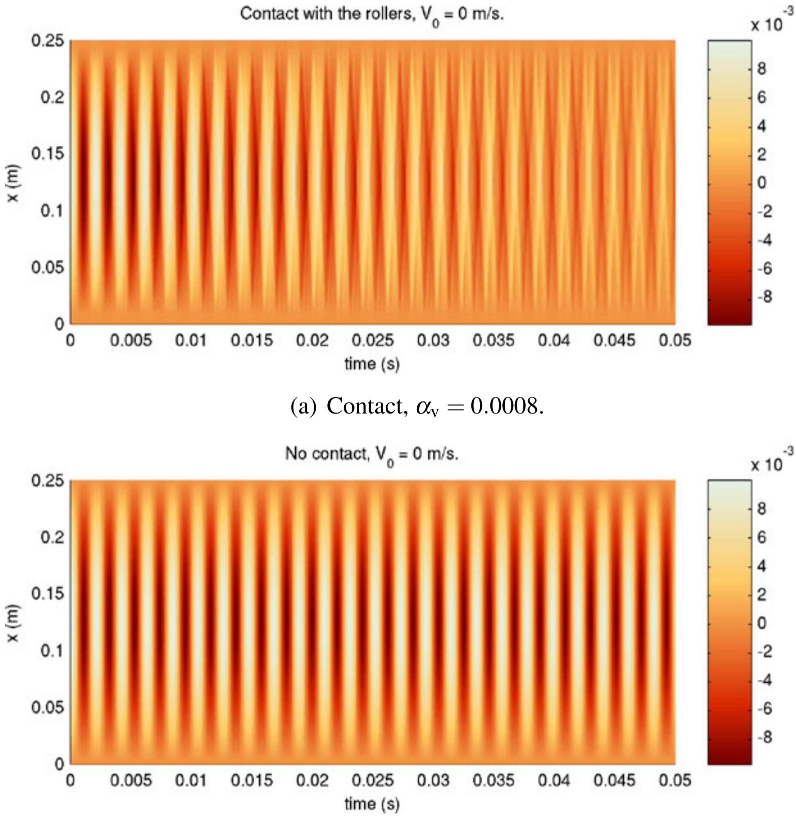


(b) $V_0 = 30$ m/s, $\alpha_\text{v} = 0.08$.



(c) $V_0 = 60$ m/s, $\alpha_\text{v} = 0.08$.

Fig. 22.7 Behaviour of the midpoint of the panel during the first 0.05 seconds for a viscoelastic material. *Solid line* shows the case with roller contact, and the *dashed line* shows the case without contact. V_0 is the panel velocity and α_v is the relative viscosity



(a) Contact, $\alpha_v = 0.0008$.

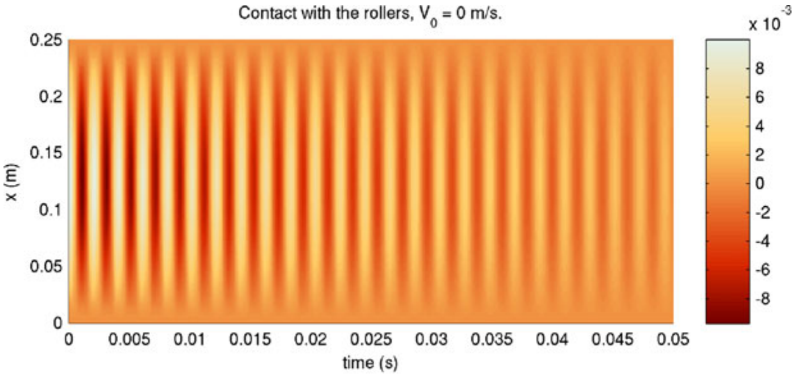
(b) No contact, $\alpha_v = 0.0008$.

Fig. 22.8 Behaviour of the panel during the first 0.05 seconds, when the panel is not moving ($V_0 = 0$). Almost elastic material, $\alpha_v = 0.0008$

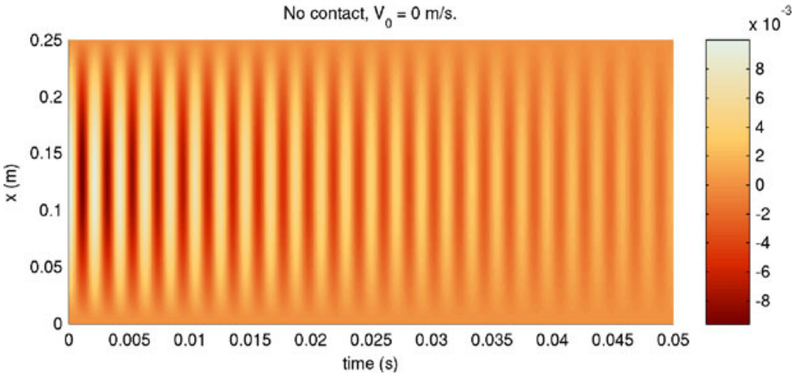
to the case with no roller contact. When the viscoelastic panel is moving fast (see Fig. 22.7(c)), the viscous damping is so fast that the effect of contact cannot be noticed.

In Figs. 22.8, 22.9, 22.10 and 22.11, coloursheets of the panel behaviour are provided for different panel velocities ($V_0 = 0, 30, 60$ m/s) for both almost elastic and viscoelastic materials. For a stationary panel, also the cases with no contact are drawn as reference cases (Figs. 22.8(b) and 22.9(b)). It can be seen that the viscous damping depends on the panel velocity and the relative viscosity. For a stationary panel (Figs. 22.8 and 22.9), the effect of the contact can be seen clearly for the almost elastic panel but the effect is very slight for the viscoelastic panel.

For a moving panel (Figs. 22.10 and 22.11), it can be noted that the upper-x half of the panel experiences its maximum or minimum amplitude before the lower-x half does. Similar behaviour was reported in [3]. The effect of viscous damping increases if the panel velocity is increased. Note that the viscoelastic panel (beam)



(a) Contact, $\alpha_v = 0.08$.



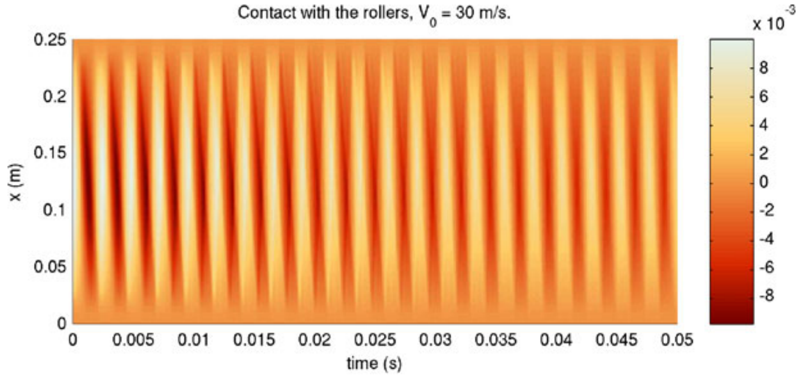
(b) No contact, $\alpha_v = 0.08$.

Fig. 22.9 Behaviour of the panel during the first 0.05 seconds, when the panel is not moving ($V_0 = 0$). Viscoelastic material, $\alpha_v = 0.08$

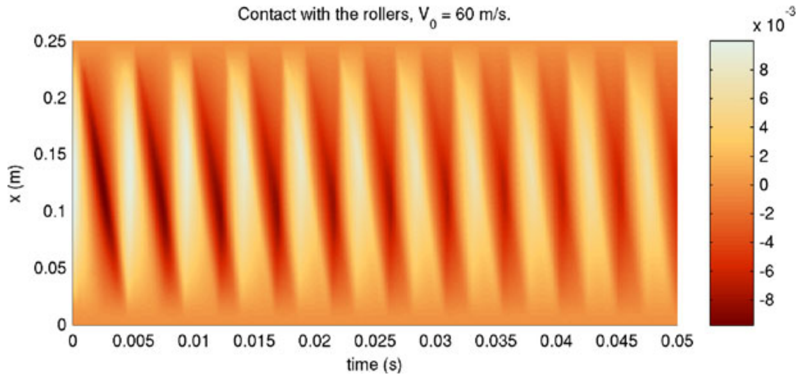
is expected to experience divergence instability at a sufficiently high speed, and the divergence velocity is expected to be close to the one of an elastic panel (beam) [17]. The critical velocity of a travelling elastic panel can be determined analytically by $(V_0)_{cr} = \sqrt{T_0/m + (\pi/\ell)^2 D/m} \approx 79.0581$ m/s [33].

22.4 Conclusions

In this study, the dynamical behaviour of an axially moving viscoelastic panel in contact with supporting rollers was investigated. The combination of the contact model with this kind of viscoelastic model was done for the first time. The dynamical equation describing the panel vibrations was derived and an initial-boundary value problem was formulated. The continuum equation was discretised via central finite differences in space and by the fourth-order Runge-Kutta method in time,



(a) Velocity $V_0 = 30$ m/s, $\alpha_v = 0.0008$.



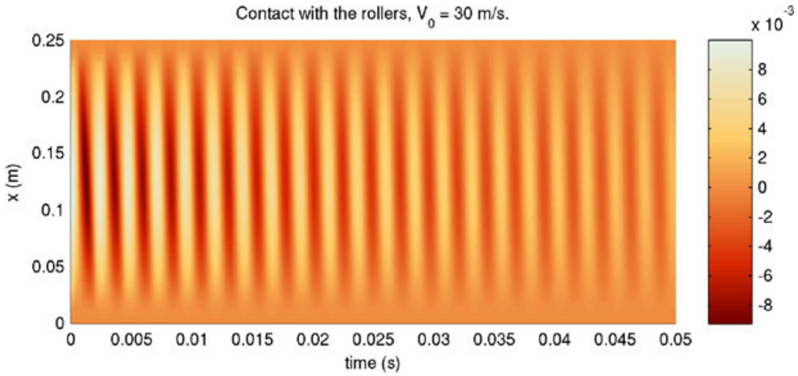
(b) Velocity $V_0 = 60$ m/s, $\alpha_v = 0.0008$.

Fig. 22.10 Behaviour of the panel during the first 0.05 seconds, when the panel is moving at a constant velocity. Almost elastic material, $\alpha_v = 0.0008$

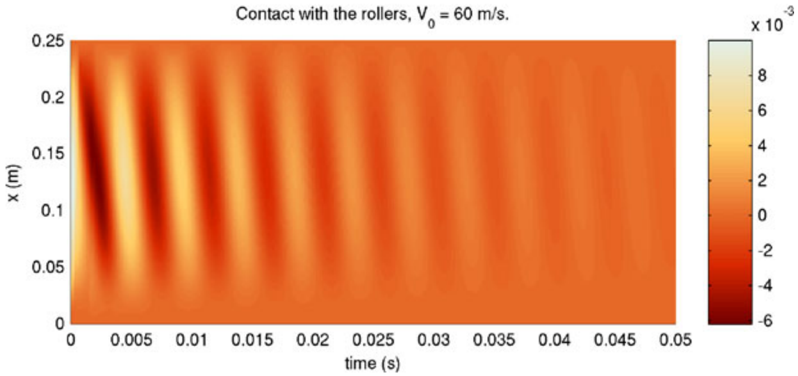
and solved numerically. Dynamics of the panel was studied for different relative viscosities and for different panel velocities, and the effect of roller contact was investigated by comparing the behaviour including contact with the behaviour with no contact.

In this study, it was noted that in the partial differential equation, describing the dynamics of an viscoelastic panel or beam and which is fifth-order in space, the amount of boundary conditions must be five in total.

From numerical investigations, it was seen that the contact force may decrease the amplitude of vibrations in the case of an almost elastic panel and increase the amplitude in the case of a viscoelastic panel compared to the case with no roller contact. The decrease in viscous damping introduced by the roller contact was surprising. It was also noted that the viscous damping increases a lot when the panel velocity is increased.



(a) Velocity $V_0 = 30$ m/s, $\alpha_v = 0.08$.



(b) Velocity $V_0 = 60$ m/s, $\alpha_v = 0.08$.

Fig. 22.11 Behaviour of the panel during the first 0.05 seconds, when the panel is moving at a constant velocity. Viscoelastic material, $\alpha_v = 0.08$

Note that, in this study concerning moving viscoelastic panels, the effects of surrounding fluid were excluded to investigate solely the role of material viscoelasticity in the panel dynamics. The presence of fluid is known to considerably affect the panel behaviour [3], and thus the present study should primarily be seen as academic basic research. The behaviour of a moving viscoelastic panel submerged in fluid remains a topic of future research.

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References

1. Archibald FR, Emslie AG (1958) The vibration of a string having a uniform motion along its length. J Appl Mech 25:347–348

2. Banichuk N, Jeronen J, Kurki M, Neittaanmäki P, Saksa T, Tuovinen T (2011) On the limit velocity and buckling phenomena of axially moving orthotropic membranes and plates. Int J Solids Struct 48(13):2015–2025
3. Banichuk N, Jeronen J, Neittaanmäki P, Tuovinen T (2011) Dynamic behaviour of an axially moving plate undergoing small cylindrical deformation submerged in axially flowing ideal fluid. J Fluids Struct 27(7):986–1005
4. Chen L-Q, Chen H, Lim CW (2008) Asymptotic analysis of axially accelerating viscoelastic strings. Int J Eng Sci 46(10):976–985. doi:10.1016/j.ijengsci.2008.03.009
5. Chen L-Q, Ding H (2010) Steady-state transverse response in coupled planar vibration of axially moving viscoelastic beams. J Vib Acoust 132:011009. doi:10.1115/1.4000468
6. Chen L-Q, Wang B (2009) Stability of axially accelerating viscoelastic beams: asymptotic perturbation analysis and differential quadrature validation. Eur J Mech A, Solids 28(4):786–791. doi:10.1016/j.euromechsol.2008.12.002
7. Chen L-Q, Yang X-D (2006) Vibration and stability of an axially moving viscoelastic beam with hybrid supports. Eur J Mech A, Solids 25(6):996–1008. doi:10.1016/j.euromechsol.2005.11.010
8. Chen L-Q, Zhao W-J (2005) A numerical method for simulating transverse vibrations of an axially moving string. Appl Math Comput 160(2):411–422. doi:10.1016/j.amc.2003.11.012
9. Chonan S (1986) Steady state response of an axially moving strip subjected to a stationary lateral load. J Sound Vib 107:155–165
10. Ding H, Chen L-Q (2008) Stability of axially accelerating viscoelastic beams: multi-scale analysis with numerical confirmations. Eur J Mech A, Solids 27(6):1108–1120. doi:10.1016/j.euromechsol.2007.11.014
11. Fung R-F, Huang J-S, Chen Y-C (1997) The transient amplitude of the viscoelastic travelling string: an integral constitutive law. J Sound Vib 201(2):153–167. doi:10.1006/jsvi.1996.0776
12. Fung R-F, Huang J-S, Chen Y-C, Yao C-M (1998) Nonlinear dynamic analysis of the viscoelastic string with a harmonically varying transport speed. Comput Struct 66(6):777–784. doi:10.1016/S0045-7949(98)00001-7
13. Guan X (1995) Modeling of viscoelastic effects on web handling system behavior. PhD thesis, Oklahoma State University
14. Guan X, High MS, Tree DA (1995) Viscoelastic effects in modeling web handling systems: steady-state analysis. J Appl Mech 62(4):908–914. doi:10.1115/1.2789031
15. Guan X, High MS, Tree DA (1998) Viscoelastic effects in modeling web handling systems: unsteady-state analysis. J Appl Mech 65(1):234–241. doi:10.1115/1.2789031
16. Kurki M, Lehtinen A (2009) In-plane strain field theory for 2-d moving viscoelastic webs. In: Papermaking research symposium 2009, Kuopio, Finland. PRS
17. Lee U, Oh H (2005) Dynamics of an axially moving viscoelastic beam subject to axial tension. Int J Solids Struct 42(8):2381–2398
18. Lin CC (1997) Stability and vibration characteristics of axially moving plates. Int J Solids Struct 34(24):3179–3190
19. Lin CC, Mote CD (1995) Equilibrium displacement and stress distribution in a two-dimensional, axially moving web under transverse loading. J Appl Mech 62:772–779
20. Marynowski K, Kapitaniak T (2002) Kelvin-Voigt versus Bürgers internal damping in modeling of axially moving viscoelastic web. Int J Non-Linear Mech 37(7):1147–1161. doi:10.1016/S0020-7462(01)00142-1
21. Marynowski K, Kapitaniak T (2007) Zener internal damping in modelling of axially moving viscoelastic beam with time-dependent tension. Int J Non-Linear Mech 42(1):118–131. doi:10.1016/j.ijnonlinmec.2006.09.006
22. Miranker WL (1960) The wave equation in a medium in motion. IBM J Res Dev 4:36–42
23. Mockensturm EM, Guo J (2005) Nonlinear vibration of parametrically excited, viscoelastic, axially moving strings. J Appl Mech 72(3):374–380. doi:10.1115/1.1827248
24. Mote CD (1968) Divergence buckling of an edge-loaded axially moving band. Int J Mech Sci 10:281–295
25. Mote CD (1972) Dynamic stability of axially moving materials. Shock Vib Dig 4(4):2–11

26. Mote CD (1975) Stability of systems transporting accelerating axially moving materials. J Dyn Syst Meas Control 97:96–98
27. Oh H, Cho J, Lee U (2004) Spectral element analysis for an axially moving viscoelastic beam. J Mech Sci Technol 18(7):1159–1168. doi:10.1007/BF02983290
28. Sack RA (1954) Transverse oscillations in traveling strings. Br J Appl Phys 5:224–226
29. Simpson A (1973) Transverse modes and frequencies of beams translating between fixed end supports. J Mech Eng Sci 15:159–164
30. Swope RD, Ames WF (1963) Vibrations of a moving threadline. J Franklin Inst 275:36–55
31. Ulsoy AG, Mote CD (1980) Analysis of bandsaw vibration. Wood Sci 13:1–10
32. Ulsoy AG, Mote CD (1982) Vibration of wide band saw blades. J Eng Ind 104:71–78
33. Wickert JA, Mote CD (1990) Classical vibration analysis of axially moving continua. J Appl Mech 57:738–744
34. Yang X-D, Chen L-Q (2005) Dynamic stability of axially moving viscoelastic beams with pulsating speed. Appl Math Mech 26(8):905–910