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## Early Algebra Teaching and Learning

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### What is Early Algebra?

Early algebra refers to a program of research, instructional approaches, and teacher education that highlights the importance of algebraic reasoning throughout K-12 mathematics education. The program stresses that elementary arithmetic rests on ideas and principles of algebra that merit a place in the early curriculum. Early algebra focuses on principles and representations of algebra that can be and presumably need to be mastered by young students as the foundations for later learning.

In some countries, preparation for algebra is implicitly integrated into the early mathematics curriculum. This can be assessed by analysis of curricula implemented in different countries, a task that goes beyond the scope of this account of research on early algebra. For now, it suffices to state that the goal of introducing algebra in elementary school is far from being universally embraced, despite promising results of classroom intervention studies of early algebra.

As early algebra developed as an area of research, different proposals for introducing algebra into the existing K-12 curriculum emerged (see Carraher and Schliemann 2007). Intervention studies based on these perspectives have consistently shown that, well before adolescence, students' demonstrate algebraic reasoning, use conventional algebraic forms for expressing such reasoning, and make mathematical generalizations that have an algebraic character.

### What Is Algebraic Reasoning?

Algebraic reasoning is generally understood as some combination of (a) operating on unknowns; (b) thinking in terms of variables and their relations (where variables have a domain and co-domain containing many, possibly an infinite number of, elements); and (c) acknowledging algebraic structure. Students may be engaged in algebraic reasoning, regardless of whether they are using algebraic notation.

#### Operating on Unknowns

A variable is a symbol or placeholder (typically a letter but sometimes a simple figure or other token) that stands for an element of a set of possible values. The set typically contains numbers or measures (i.e., numbers along with units of measure), but it may be defined over any sorts of objects, mathematical or not.

Although mathematics tends not to distinguish an unknown from a variable, in mathematics education, an unknown is often taken to refer to a fixed number. As a result, the term *unknown* leaves open the issue of whether the variable is

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employed in the former or latter sense. Given this ambiguity, variability (the idea that a variable can take on multiple values) is generally treated as a distinct feature of algebraic reasoning.

Operating on unknowns entails being able to express the relationship among quantities (variable or not) in a novel way. The statement, "Michael had some marbles, then won 8 marbles, finishing with 14 marbles," is a natural language representation of what might be expressed through algebraic notation as "$x + 8 = 14$." A student who realizes that the answer can be found by subtracting 8 from 14 has reconfigured the description of the relationship among known and unknown values such that the answer can be directly calculated from the givens without having to resort to trial and error. This rudimentary form of algebraic reasoning through inverting or "undoing" is significantly different from solving a problem through recall of number facts or adding counting numbers to 8 in order to obtain the sum of 14. Algebraic reasoning is entailed whenever one validly expresses the relationship among givens and unknowns in an alternative form.

Early algebra research (see Kaput et al. 2008; Schliemann et al. 2007) shows that children as young as 8 and 9 years of age can learn to use letters to represent unknown values, to operate on those representations, and to draw new inferences. They can do so without assigning specific values to variables. This brings us to the second characteristic of algebraic reasoning.

### Thinking About Variables

Algebraic reasoning can take place in the absence of algebraic notation. Variables can be represented through expressions such as *amount of money*, *elapsed time*, *number of children*, *distance* (from school to home), etc. Young students may use simplified drawings to represent variables (e.g., a wallet to represent the amount of money in a wallet). It is important to distinguish such cases from literal drawings depicting one single value or unknown.

Students are engaged in algebraic reasoning whenever they are thinking about variables and relations among variables.

### Acknowledging Algebraic Structure

Algebraic structure is primarily captured in the Rules of Arithmetic (the field axioms) and in the principles for transforming equations (the original techniques which gave rise to the subject known as algebra).

In the early grades, students can focus on the algebraic structure of simple equations to the extent that they treat the letters as generalized numbers (e.g., when $2n + 2 = 2 \times (n + 1)$, for *all* n in the domain) and, thereby, treat the operations as having validity over a particular set of numbers.

## Approaches to Early Algebra Instruction

Early algebra proponents have adopted three general complementary approaches, each showing some success in developing students' algebraic reasoning. They focus on students' reasoning about (a) physical quantities and measures, (b) the properties of the number system, and/or (c) functions.

### Reasoning About Physical Quantities and Measures

In this approach, students are encouraged from early on to use letter notation for comparing unknown magnitudes (e.g., a displayed distance or a distance expressed as a magnitude of a unit of measure). For example, they learn to express the length of a line segment, A, as greater than the length of another line segment, B, by the inequality $A > B$ (or $B < A$) or through equations such as $A = B + C$, $B = A - C$. Furthermore, they use multiple forms of representation (diagrams of line segments, tables of values, and algebraic notation) to express relations among givens and unknown magnitudes.

For example, research by Davydov's (1991) group, in the former Soviet Union, shows that quantitative reasoning in concert with multiple forms of representation can support the emergence of algebraic reasoning among second to fourth graders who solve problems like: "In the kindergarten, there were 17 more hard chairs than soft ones. When 43 more hard chairs were added, there were five times more hard chairs than soft ones. How many hard and soft chairs were there?"

### The Properties of Number Systems: Generalized Arithmetic

A generalized arithmetic approach emphasizes algebraic structure early on. For example, the equation 8 + 7 = 9 + ❐ sets the stage for a discussion about the equal sign as meaning something different from the idea of "makes" or "yields"; rewriting the number sentence as 8 + 7 = 8 + (1+ ❐) may evoke the insight that 1 + ❐ equals 7, making use of the associative property of addition.

Authors whose work falls under this general approach (e.g., Bastable and Schifter 2007; Carpenter et al. 2003) find that elementary school children come to display implicit algebraic reasoning and generalizations supported by intuitive arguments, discuss the truth or falsity of number sentences, and think about the structural relations among the numbers, considering them as placeholders or as variables.

### Functions Approaches to Early Algebra

Functions approaches subordinate many arithmetic topics to more abstract ideas and concepts. Multiplication by 3 is viewed as a subset of the integer function, $3n$, that maps a set of input values to unique output values, thus preparing the ground for the continuous function, $f(x) = 3x$, over the real numbers and its representation in the Cartesian plane. Functions approaches often rely on multiple representations of mathematical functions: descriptions in natural language, function tables, number lines, Cartesian graphs, and algebraic notation. Students are encouraged to treat what might initially appear to be a single value (e.g., "John and Mary each have a box containing the same number of candies. Mary has three additional candies. What can you say about how many candies they each have?"), as a set of possible values.

Results of classroom studies using a functions approach to early algebra are consistently positive. Moss and Beatty (2006) show that, after working with patterns where the position or step is explicitly treated as an independent variable, while the count of some property (e.g., points in a triangular figure) is treated as a dependent variable, students in grades 2–4 can learn to formulate rules that are consistent with a closed form representation of the function such as $3x + 7$. Blanton and Kaput (2005) found that children come to represent additive and multiplicative relations, transitioning from iconic and natural language registers at grades PreK-1 to use of t-charts and algebraic notational systems by grade 3. Students from grades 3 to 5 who participated in a longitudinal study of early algebra, focused on variables, functions, and their multiple representations (Carraher et al. 2008) have been found to perform better than their control peers in the project's written assessment problems related to algebraic notation, graphs, and equations, as well as in algebra problems included in state mandated tests. The benefits of the intervention persisted 2–3 years later, when treatment students were, again, compared to a control group (Schliemann et al. 2012).

## In Summary

Early algebra highlights the algebraic character of time-honored topics of early mathematics. The successful adoption of early algebra depends upon the fluidity with which teachers are able to move back and forth between algebraic representations and those expressed through natural language, diagrams, tables of values, and Cartesian graphs. There are robust examples of how this can be done in the research literature. The next step is to prepare teachers to interweave these activities into their regular curriculum.

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# Early Childhood Mathematics Education

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## Keywords

Early childhood; Mathematics education; History of early childhood education; Numerical and geometric content domains; Informal knowledge; Play; Picture books; Information and communication technology

## What Is Meant by Early Childhood Mathematics Education?

Early childhood mathematics education includes providing activities or creating learning environments by professionals such as teachers and care takers in order to offer young children experiences aimed at stimulating the development of mathematical skills and concepts. In general, early childhood mathematics education involves children who are 3–6 years old. Depending on the age of the children and the educational system of their country, early childhood education takes place in preschool care centers or in kindergarten classes. Children's mathematical development can also be stimulated by encounters and events that take place outside an educational setting, that is, in the children's home environment, in which, among other things, children can develop some basic notions about number by playing games with their siblings. These family-based activities are highly esteemed as the foundation on which mathematics education in the early years can build.

## History

Teaching mathematics to young children has already a long history. Saracho and Spodek (2009a, b) gave in two articles an overview of it. According to them we can consider the beginning of early mathematics education in 1631 when Comenius, who was at that time a teacher in Poland, published his book *School of Infancy*. In this book, Comenius described the education of children in their first 6 years. By emphasizing the observation and manipulation of objects as the main source for children's learning, Comenius stimulated the creation of mathematics programs for young children which heavily rely on the use of concrete materials. Two centuries later, in the nineteenth century, Comenius' approach was reflected in the educational method of Pestalozzi in Switzerland which also focused on observing and manipulating physical objects.

A further landmark in the development of mathematics education for young children was the foundation of the *Infant School* by Owen in Scotland in 1816. The method of this school for teaching arithmetic was aimed at developing understanding of different arithmetic operations for which, like by Pestalozzi, concrete materials were used. Similarly, in the United States, Goodrich introduced in 1818 in his book *The Children's Arithmetic* the idea that young children can discover arithmetic rules when they manipulate concrete objects such as counters and bead frames. This innovative approach rejected the view that arithmetic is learned through memorization. Later, in the United States, Colburn used Goodrich's and Pestalozzi's work to develop a method which he called "mental arithmetic." The book *First Lessons*, which he published in 1821, was meant for 4- and 5-year-old children and started with simple levels of numerical reasoning elicited by word problems and naturally progressed to more complex levels. Colburn attached much value to children having pleasure in their solutions because this contributes to their learning and the integration of concepts. Moreover, he emphasized the inductive approach, which has many similarities to the constructivist view on learning.

In the second half of the nineteenth century, early childhood mathematics education was influenced by Fröbel who in 1837 established the first kindergarten in Germany and developed an educational program for young children. A central component in this program were the so-called *gifts*, small manipulative materials by which children could be made aware of numerical and geometric relationships and which could provide them experiences with respect to, for example, patterns, symmetry, counting, measurement, addition, division, fractions, and properties of shapes. One of the *gifts* consisted of a series of cubes made out of wood, divided into smaller parts, and followed by square and triangular tablets. The *gifts* were offered to the children in a prescribed sequence, and the children were expected to build precise forms with them. Although children in the Fröbelian kindergarten might have acquired a substantial amount of mathematical knowledge, attained incidentally and instinctively through play, the ultimate goal of Fröbel was not to teach children mathematics, but help 3- to 6-year-olds to understand the relationship between nature, God, and humanity.

At the turn of the twentieth century, many from the kindergarten community began to question the appropriateness of Fröbel's curriculum and his methods. For example, Dewey considered the Fröbelian activities as mindless copying and manipulation of artificial objects. These concerns led to the so-called "child-centered approach," which originated from the eighteenth century philosopher Rousseau. In this approach there was no specific program for mathematics instruction, but children were engaged in activities based on their interests, which would incidentally help children prepare for the later learning of formal mathematics. This approach also applied to the nursery school which was established firstly in England in the beginning of the twentieth century. The educational program was predominantly focused on children's play and ignored academic subjects which would be taught later when the children are older.



A different approach was reflected by Montessori, who at the beginning of the twentieth century introduced a method for teaching young children that was deeply mathematical. Most of the activities she suggested were requiring, for example, working with patterns and exploring the properties of geometric shapes, numbers and operations. Her approach included working with sensory materials and was based on the idea that children use their senses to acquire information about the world. For example, children felt the shape of numerals made of sandpaper before writing these numerals.

Halfway the twentieth century, the ideas of Piaget influenced the teaching of mathematics to young children. He related the construction of number concepts to the development of children's logical thinking and focused on understanding common properties of quantities like conservation, seriation, and class inclusion rather than on counting. Piaget emphasized that there is a relationship between the basic structures of modern mathematics and the mental structures developed in children. Although these and other ideas of Piaget were questioned, Piaget, together with other pioneers since Comenius, has contributed to the present awareness of the importance of mathematics education for young children.

## Recent Interest in Early Childhood Mathematics Education

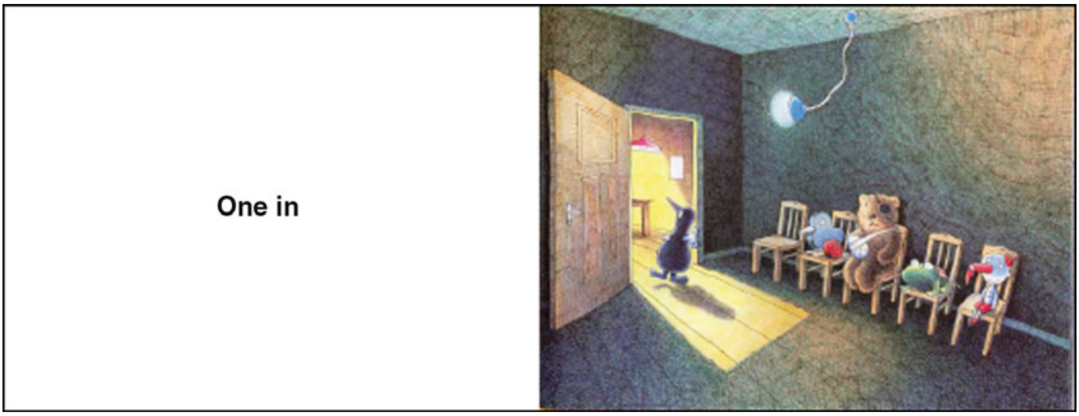
Currently, early childhood education has risen to the top of the national policy agenda with recognition that ensuring educational success and attainment must begin in the earliest years of schooling (National Research Council 2009). An important reason for this is that research has shown that the amount of mathematical knowledge children bring with them when they start in grade 1 has large, long-term consequences for their further learning of mathematics (Duncan et al. 2007).

For example, in the United States, the recent awareness of mathematics as a key aspect of early childhood education was boosted in 2000 when the National Council of Teachers of Mathematics published their revised 1989 standards for elementary and secondary school mathematics and included prekindergarten for the first time in their description of standards. A further step was a joint position statement titled Early Childhood Mathematics: Promoting Good Beginnings by the National Association for the Education of Young Children and the National Council of Teachers of Mathematics (NAEYC and NCTM 2002) that was aimed at achieving high-quality mathematics education in child care and other early education settings. The book resulting from the Conference on Standards for Early Childhood Mathematics Education (Clements et al. 2004) and the Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics (NCTM 2006) were other breakthroughs for early childhood mathematics education. Similar documents for teaching mathematics in the early years of schooling were also released in other countries, for example, in the United Kingdom (Department for Children, Schools and Families 2008), France (Ministère de l'Education Nationale 2002), Australia (Australian Association of Mathematics Teachers and Early Childhood Australia 2006), and the Netherlands (Van den Heuvel-Panhuizen and Buys 2008).

Another indication for the new prominent position of early childhood mathematics education is reflected by the establishment, in 2009, of the working group on Early Years Mathematics in the Congress of the European Society for Research in Mathematics Education (CERME), which focuses into research on learning and teaching mathematics to children aged 3–8. The work of this group in the last two meetings of CERME has shown that investigating mathematics education during the early years is a rather complex and multidimensional endeavor. The specificities of early childhood education in different countries and educational systems, e.g., the differences in the conception of schooling and early years mathematics and in the transition ages from preprimary to primary school and the differences in the education and development of prospective preschool and kindergarten teachers regarding the didactics of mathematics as well as the constraints in the ability of young children to articulate their mathematical thinking and understanding, are only some of the factors that contribute to this complexity.

## Mathematics Taught in Early Childhood

Although in the past, early childhood mathematics education was often restricted to teaching arithmetic, several early pioneers such as Fröbel and Montessori as well as Piaget offered a wider program to children. Presently, there is expert consensus (see National Research Council 2009) that two content areas of mathematics are particularly important for young children to learn, namely, (1) numerical and quantitative ideas and skills and (2) geometric and spatial ideas and skills. Moreover, according to Clements and Sarama (2007), these ideas and skills are permeated by mathematical activities such as dealing with patterns, analyzing data, and sorting and sequencing.




**Early Childhood Mathematics Education, Fig. 1** Page 3 of the picture book Vijfde zijn [Being Fifth], Left side: Text "One in", Right side: Illustration of five broken toys in a doctor's waiting room (Jandl and Junge 2000)

## Ways of Teaching Mathematics to Young Children

There is also general agreement that "teaching" mathematics to young children should have many characteristics of the informal learning as it takes place in the family setting where children come along with mathematics in a natural way and "mathematical ideas permeate children's play" (Ginsburg and Amit 2008, p. 275). Young children develop mathematical ideas and skills primarily in informal ways which make sense to them. Thus a major part of early mathematics education needs to be organized in informal contexts which are meaningful for the young children.

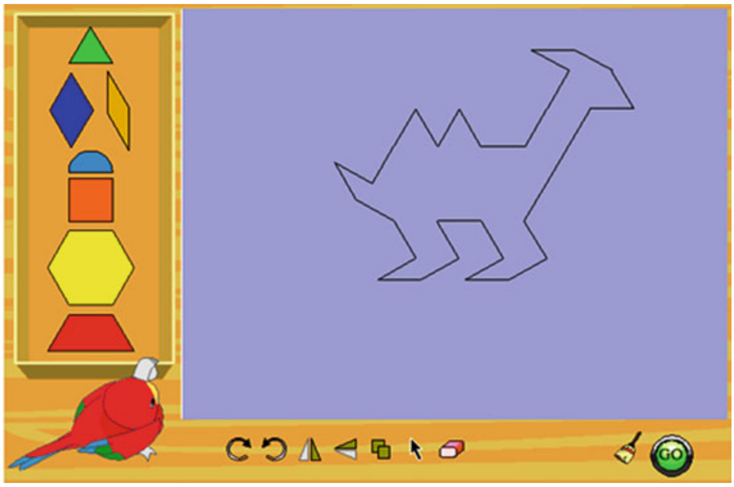
### Play

Such learning opportunities can be provided in kindergarten through play (Pramling-Samuelsson and Fleer 2009). By offering playful activities such as free play, sensorimotor play, making constructions, and role playing, children can know the world mathematically. They can spontaneously deal, for example, with counting up to large numbers, comparing the height of their towers of blocks, creating and extending patterns when jumping up and down, and connecting movements to verbal expressions, investigating shapes, and exploring symmetry and spatial relations.

According to Vygotsky play in early childhood becomes the leading activity of development. The challenges the children encounter during play and the help they receive from more knowledgeable others, such as teachers, who assign mathematical meaning to their play actions, enable the children to move a step forward in their abilities. In this way they enter the zone of proximal development.

### Picture Books

Another way of offering children meaningful contexts in which they can encounter mathematics-related problems, situations, and phenomena that can support the learning of mathematics is by reading them picture books (Van den Heuvel-Panhuizen and Elia 2012). From a Vygotskian and action-psychological approach to learning (Van Oers 1996), picture books can contribute to forming, exchanging, and negotiating all kinds of personal meanings within everyday practices and to acquiring mathematics as an activity involving historically developed and approved meanings. Furthermore, they can offer cognitive hooks to explore mathematical concepts and skills. An example concerns the book *Vijfde zijn* [Being Fifth] (Jandl and Junge 2000), which is about a doctor's waiting room in which five broken toys are waiting for their turn (see Fig. 1). Even though the book was not written for the purpose of teaching mathematics,




**Early Childhood Mathematics Education, Fig. 2** Geometric puzzle in a Building Blocks' software tool

it implicitly touches on counting backwards and spatial orientation as part of the narrative and has the power to offer children a rich environment for eliciting mathematical thinking (Van den Heuvel-Panhuizen and Van den Boogaard 2008).

## Information and Communication Technology

Although there is still debate about whether Information and Communication Technology (ICT) is appropriate for teaching young children, there is ample evidence from research that computer use can be meaningful, motivating, and beneficial for children 3 years of age and above (e.g., Haugland 2000; Clements et al. 2004). The use of computers in early years' mathematics can support young children's mathematical thinking in various ways. One of the most powerful affordances of the use of computers in early childhood mathematics education is that they embody the processes children need to develop and mentally use. Computers can also help children connect concrete and symbolic representations of the same mathematical concept, e.g., by providing a dynamic link between base-ten blocks and numerical symbols. Using mathematical computer games enables children to explore mathematical concepts, such as geometric figures, in ways that they cannot with physical manipulatives. For example, they can modify the size of geometric shapes, without changing their critical attributes. Furthermore, the use of computers can support children in bringing mathematical processes and ideas, such as shape transformations, in an explicit level of awareness. The Building Blocks program (Clements et al. 2004), for example, uses computer software tools (see Fig. 2) to help preschoolers acquire geometric and numerical ideas and skills.

In sum, the computers can provide valuable opportunities for learning in early childhood mathematics education. However, realizing the full potential of technology requires comprehensive, meaningful, and well-planned instructional settings. The development and organization of such settings strongly depends on the curriculum and the teacher (Clements 2002). Thus, effectively integrating technology in the early childhood mathematics curriculum and appropriate professional development of kindergarten teachers should be vitally important concerns in relation to computer use in mathematics education in the early years.

## Future Perspectives in Early Childhood Mathematics Education

Presently there is broad diversity of theories of learning mathematics ranging from cognitivist theories including a Piagetian approach, situated cognition, and semiotic approaches to various constructivist theories and social-cultural theories. A recent research direction in mathematics education is the theory of embodied learning in mathematics which claims on the basis of knowledge from neuroscience that cognition and concepts are strongly founded on bodily experiences. Although this new approach to learning is closely related to how young children explore and make sense of their environment, not much research has been carried out in how ideas from embodiment theory can be used to acquire a better understanding of young children's mathematical development and how early childhood education can contribute to this development.

## Cross-References


- ▶ Constructivism in Mathematics Education
- ▶ Informal Learning in Mathematics Education
- ▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
- ▶ Mathematical Games in Learning and Teaching
- ▶ Semiotics in Mathematics Education
- ▶ Situated Cognition in Mathematics Education
- ▶ Technology and Curricula in Mathematics Education
- ▶ Theories of Learning Mathematics
- ▶ Zone of Proximal Development in Mathematics Education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## Education of Facilitators (for Educators of Practicing Teachers)

▶ Education of Mathematics Teacher Educators

## Education of Mathematics Teacher Educators

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### Keywords

Education of teacher educators; Knowledge of teacher educators; Professional development; Reflection; Formal education

### Synonyms

Education of facilitators (for educators of practicing teachers)

### Definition

Education of teacher educators refers to the preparation, professional development, teaching, or facilitating of teacher educators. It is understood as a goal-directed intervention in order to promote teacher educators' learning and further development of beliefs, knowledge, and practice, including formal as well as informal activities. Nowadays, the term "teacher educators" commonly refers to both those who educate prospective teachers and those who educate practicing teachers, that is, to those who initiate, guide, and support teacher learning across the lifespan (Even 2008; Krainer and Llinares 2010). Yet, sometimes the term "teacher educators" refers only to educators of prospective teachers, that is, to those who teach future teachers and not to those who provide professional development for practicing teachers.

### Theoretical Background

There is general recognition and agreement today that the education and professional development of teachers is key to improving students' opportunities to learn (Even and Ball 2009; Krainer 2011). Accordingly, the focus and nature of the education of prospective and practicing teachers have received immense international attention in recent years, and the past decades have seen substantial increase in scholarship on mathematics teacher education. A significant issue identified recently as crucial for improving the education and professional development of mathematics teachers is the education and development of teacher educators and related research (Adler et al. 2005; Even and Ball 2009; Jaworski and Wood 2008).

In different countries around the world, various professionals are responsible for initiating, guiding, and supporting teachers' learning: university faculty with disciplinary expertise and those who specialize in education; school teachers, teacher mentors, and staff of curriculum implementation projects; educators whose major occupation is to work with teachers and those who do it only as an add-on part-time temporary activity; those who work with both prospective and practicing teachers; and those whose role is to educate solely prospective or practicing teachers, but not both. Yet, this vast range of teacher educators has little formal preparation for their work. Most become teacher educators through practice with little institutional and professional support. With the expanding current interest in the issue of professional education and development of teacher educators in different countries, pioneering formal programs to prepare educators to educate teachers started to emerge. These include, for example, the Pedagogy and Subject-Didactics for Teachers (PFL) Program in Austria, the MANOR Program in Israel for educating educators of practicing mathematics teachers, the School for Research and Development of Education Programs for Teacher College

Faculty (MOFET Institute) in Israel, a special M.Ed. program in Pakistan, and the Leadership Curriculum for Mathematics Professional Development (LCMPD) Project in the USA.

## Important Scientific Research and Open Questions

The education of teacher educators has only recently become of interest to the international community. Thus, not much is known about the development of teacher educators and about effective ways to educate educators to initiate, guide, and support teacher learning (Even 2008).

Research studies that center on issues pertaining to professional education and development of teacher educators in a specific subject area are rare. Mathematics is among the subjects where efforts in investigating the education of teacher educators have become visible recently (Even 2005; Jaworski and Wood 2008; Nardi 2008; Oikkonen 2009). Most research on the professional education and development of mathematics teacher educators includes reflections of teacher educators on their own personal development (e.g., Cochran-Smith 2003; Jaworski and Wood 2008). This research suggests that reflective inquiry has a central role in learning to teach teachers and in developing as teacher educators. Yet, this line of research provides information mainly on the professional development of university-based teacher educators with research interest in teacher education, but not on that of the wide range of professionals responsible for supporting prospective and practicing teachers' learning.

Because formal preparation for mathematics teacher educators scarcely exists, research that examines formal programs and activities intended to educate mathematics teacher educators is sparse. Pioneering work in this direction addresses various aspects of curriculum (What should teacher educators learn?) and pedagogy (How should teacher educators be taught?). It suggests several areas of professional knowledge base for mathematics teacher educators (Jaworski and Wood 2008); two relate to knowledge shared by teacher educators and teachers: pedagogical knowledge and disciplinary knowledge. A third area of professional knowledge base for educating teacher educators relates to knowledge specific to the mathematics teacher educator: knowledge of teaching teachers and of teachers' learning. In addition to professional knowledge base, research suggests the need to purposely teach practices of educating teachers, giving explicit attention to the nature of work in which mathematics teacher educators engage. These practices may be general, such as teaching courses, supervising student teachers, and facilitating seminars (Cochran-Smith 2003), or subject matter specific, such as planning, conducting, and assessing activities, workshops, and courses for mathematics teachers (Even 2005). This line of research also suggests that inquiry is central to learning to teach teachers and to developing as mathematics teacher educators. Additionally, it shows the importance of attending to the relationships of knowledge and practice.

Thus far, it is not known whether, or in what ways, formal education of mathematics teacher educators needs to be responsive to the wide range of professionals responsible for supporting teachers' learning or may be common to all, for example, whether the professional education of educators of practicing mathematics teachers needs to be different from the education of educators of prospective mathematics teachers, as the education of prospective and that of practicing mathematics teachers are commonly of different nature, often occurring in different settings, and not necessarily conducted by the same people.

## Cross-References

▶ Inquiry-Based Mathematics Education
▶ Mathematics Teacher Educator as Learner

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# Elkonin and Davydov Curriculum in Mathematics Education

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## Keywords

Developmental learning; Activity theory; Sociocultural theory; Vygotsky; Curriculum; Measurement

## Definition

The Elkonin-Davydov mathematics curriculum was an elementary mathematics curriculum developed in Russia based on Russian activity theory. In recent years, the original Russian curriculum has been expanded to include grades K–8 and has been refined into several different curricula. In addition, research projects in other countries (e.g., USA) have investigated applications with local populations.

## Characteristics and Origin

In 1959, Daniil Borissowitsch Elkonin (1904–1984) and Vasily Vasil'evich Davydov (1930–1998), Russian psychologists and students of Lev Vygotsky, developed an elementary mathematics curriculum. Their work was initially situated in experimental school #91 in Moscow where their team functioned as researchers and teachers. The project was grounded in Russian activity theory, which grew out of the cultural-historical theory of Vygotsky.

Davydov was critical of the existing schooling system and argued that traditional pedagogy failed to develop a general concept of number that could support the learning of numbers of all types. Students were forced to learn a new concept of number each time they focused on a different number domain (e.g., integers, rational numbers, irrational numbers, imaginary numbers). Elkonin and Davydov believed that developmental learning coupled with Vygotsky's description of the development of scientific concepts (Vygotsky 1987) could overcome the restrictions of a traditional approach.

The E-D approach is characterized by two essential principles within developmental learning. The first is dialectical logic, which can be thought of as diametrically opposed to empirical thinking in which learning is based on accumulation of cases (Davydov 1990). To support dialectical logic, the E-D approach aims at the learning of more general ideas and then builds on those general ideas to develop advanced concepts that incorporate those ideas. Thus, in the E-D curriculum a general concept of number is developed and then built on as different number domains are explored.

Elkonin and Davydov believed that thinking about conceptual and abstract ideas should lead to a child's ability to analyze, reflect, and plan. Explicitly, analysis is the child's ability to isolate the critical and essential relation in a problem. Reflection is the child's understanding of the bases of his/her own activity. Planning is the child's ability to construct ways to solve a problem based on systems of activities.

The second principle of developmental learning is learning through one's own activity (Leont'ev 1978). In the E-D approach, this is characterized by students' activities in which they reconstruct mathematical ideas from their origin. That is, the mathematics is presented so that students see how ideas build, one on another. There is a specific learning goal toward which the instructional tasks are structured. In their work on the tasks, students interact with specific tools that help them see the mathematics in particular ways during the learning process.

In order to foster a general understanding of number that can support learning related to all types of number, the E-D curriculum (Davydov et. al 1999) starts with a prenumeric stage rather than counting and builds on a foundation of measurement concepts. In the prenumeric stage, children first identify the attributes of objects that can be compared and engage in direct comparison. For example, two bottles can be compared in multiple ways such as their height, the area of their bases, the volume of water they can hold, and their masses. These four attributes are considered to be generalized, nonspecific continuous quantities. Continuous quantities, in contrast with discrete quantities, can be subdivided a limitless number of times and each part of the subdivisions is of the same type. The quantities are generalized and nonspecific because they have no number (as determined by measure or count) associated with them.

By using the attributes of length, area, volume, and mass, children explore equality and inequality including creating an equal relationship from one that is unequal (by adding or subtracting the difference). The fundamental properties of arithmetic (such as commutativity and associativity) naturally arise from these explorations – all without numbers. Reasoning about generalized quantities is supported by introducing letters to represent the quantities and arrow diagrams and equations to represent the relationship between quantities.

The prenumeric work, in which students examine relationships among physical quantities, forms the basis for the E-D curriculum. Number is not a primitive idea as it is in curricula that begin with counting. Number is the result of measuring a quantity with a unit. The need for measurement is introduced in order to compare quantities that cannot be compared directly (e.g., two lengths that cannot be laid side by side). To measure a quantity, one needs to determine a unit that can measure the quantity. If a quantity and a unit exist, then to find the count, the unit is iterated until the quantity has been fully measured. The counting of the iterations drives the introduction of number. Thus a *number* is defined as the result of measuring a quantity with a unit. Note that neither the quantity nor the unit has numbers associated with them. Numbers are produced through measuring one with the other.

In each of the grades, however, the E-D curriculum consistently begins a topic of study with learning problems that lead to a system of activities. Learning problems are situations that significantly change students' thinking. The change occurs within children's activity and thus the material chosen for the learning problem is ultimately an important consideration. It must support the acquisition of constructing a general way to view the activity itself.

For example, initially in grade 1, students use direct comparison to find the relationship between two quantities. In a new learning problem, students are then given the challenge to determine how two quantities compare when they cannot be moved to perform a direct comparison. This motivates students to consider how the direct comparison method can be changed so that it will fit the new parameters of the problem.

In the above example, the inability to perform a direct comparison requires children to consider the use of a tool that mediates the situation. From this task, the need for a portable representation of

$$E \xrightarrow{n} Q \qquad \frac{Q}{E} = n$$

**Elkonin and Davydov Curriculum in Mathematics Education, Fig. 1** Example of two ways to express relationship of quantity, unit, and count

at least one of the quantities is created. Children must now negotiate a tool and find a systematic way to use it. Additionally, if they construct the tool to be only some part of the whole quantity, it becomes the introduction to counting as they measure the quantity through iterations. By changing the task ever so slightly, children are beginning the generalization of the process of measuring. Since the task represented above can occur in any of the four continuous quantities, children come to view this as a generalized model for any measurement, even those associated with discrete sets.

The outcome of this approach is that children see "unit" as the basis of all number. The relationship of the unit to a quantity and its measure is critical in determining how each component relates one to another. The relationship is expressed in multiple forms that reflect the action used to determine the count and show the relationship across the unit ($E$), the quantity ($Q$), and the count ($n$) (See Fig. 1).

From these representations, children generalize that as the unit ($E$) gets larger, the count ($n$) gets smaller. Even though this is introduced in grade 1, it is an important concept for the development of rational number. Subsequent instruction builds on these initial concepts of quantities, units, measurement, and number. Place value is taught as relationship between different size units in a system of units in which each larger unit is n times larger than the prior unit. Multiplication is taught as the use of an intermediate unit to find the number of units in a quantity. For example, a meter could be used as an intermediate unit to find out how many centimeters are in a quantity. Multiplication is the relationship between the number of centimeters in a meter and the number of meters in the quantity that gives the number of centimeters in the quantity. Fractions are taught by introducing partial units, initially by reversing the process that created larger place values.

## Implementation and Adaptation

The E-D elementary mathematics curriculum has been implemented in about 10 % of elementary schools throughout the Russian Federation since the collapse of the Soviet educational system in 1991. Evaluation studies consistently demonstrate that students in E-D elementary classrooms do better overall than students in other elementary classrooms (Nezhnov et al. 2009; Vysotskaia and Pavlova 2007; Zuckerman 2005). In a comparative study of E-D (Davydov et. al 1999) and six other curricula in Russia, Vysotskaia and Pavlova (2007) found that the E-D students were better able to solve a variety of problems than those in other curricula. Similarly, Zuckerman (2005) compared the E-D curriculum to two other curricula using selected problems from the PISA international mathematics tests. She found that 15-year-old students who had been taught through the E-D curriculum demonstrated a higher ability to use diagrams, graphs, and other representations for solving problems.

There are at least two significant adaptations of the E-D curriculum outside of Russia. One adaptation focused on grades 1–3 only in one school in the USA. The results, however, are compelling in that the findings from multiyear implementations indicate the use of E-D curriculum supported computational competency as well as the development of algebraic concepts (Schmittau 2005).

On a larger scale, in 2001, the Curriculum Research & Development Group, University of Hawaii, entered into a collaborative arrangement with the Elkonin-Davydov group to create an adaptation of the E-D curriculum for grades 1–5. The adapted curriculum, Measure Up (Dougherty 2008), closely followed the E-D approach but revised the instructional approaches to include significant language components (reading, writing, speaking, and critical listening). Additionally, some contents, such as fractions, were introduced in a slightly different way even though the focus on quantitative reasoning and measurement was maintained. The resulting curriculum (Dougherty et al. 2004) was implemented and tested in two sites in Hawaii with significant results. A study (Slovin and Venenciano 2008) used the Chelsea Diagnostic Mathematics Test: Algebra (Hart et al. 1985) (originally designed for 13–15-year-old students) to determine how well 5th and 6th grade students who had engaged in the Measure Up curriculum were prepared for algebra. Measure Up students performed disproportionately better than students who had not experienced Measure Up on a subset of items focused on concept of variable.

Even though studies both in the USA and Russia have indicated that students learn significant mathematics, the issue of broader dissemination remains problematic for at least three reasons. First, the approach to mathematics is unique in that it does not follow the conventional approach we have come to expect in elementary mathematics where we begin with counting and number. Second, content knowledge that is expected in teacher preparation courses is not sufficient for teaching the E-D or Measure Up curricula. Finally, high-stakes assessments are often based on a conventional approach and sequence to elementary mathematics. Thus children are learning concepts and skills in a different sequence.

## Cross-References


- ▶ Activity Theory in Mathematics Education
- ▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
- ▶ Number Teaching and Learning


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# Embodied Cognition


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## Keywords


Abstraction; Cognition; Dienes; Embodiment; Situated cognition


## Definition

Embodied cognition is a subdomain of cognitive psychology that focuses on the interaction between an individual and the environment (social, environmental, instructional). It moves beyond the traditional distinctions between mind and body in the sense that actions or interactions embody projections of the mind and vice versa.

### Some Definitional Differences in Mathematics Education

Many embodied ideas eventually are represented symbolically in mathematics. Examples of these are enumeration systems which are abstractions of human gestures for counting, pointing, and measuring. Freudenthal (1973) claimed that geometry is based upon our experiences with our bodies in the world. This suggests that the only mathematics we are able to know is the mathematics that our bodies and brains allow us to know (Lakoff and Nunez 2000; Fyhn 2010). Freudenthal (1973) also claimed that geometry is about grasping space. Fyhn (2010) interprets "Space" according to this definition as that "in which the child lives, breathes and moves" (p. 296). The idea of a "grounding metaphor" is used to connect different mathematical ideas such as arithmetic, the Cartesian coordinate system, functions (Bazzini 2001), and even calculus (Lakoff and Nunez 2000) to everyday activities. One should note that there is a difference between micro-embodied experiences such as gestures and macro-embodied experiences such as throwing an object, climbing stairs, or climbing a wall.

## Embodied Cognition in Mathematics Education

Nunez et al. (1999) claim that learning and using mathematics are closely associated with the social, cultural, historical, and contextual factors (p. 45). These have also been labeled as "situated" learning (Lave 1988). Mathematics is conceived as a product of human activities in the process of adapting to the external environment and needs, and shared and made meaningful through language and other means, but based ultimately on biological and bodily experiences. The creation of mathematics through "situated" cognition and sensemaking is not arbitrary, rather is bodily grounded (Lakoff and Nunez 2000). From an embodied cognition perspective, the learning of mathematical knowledge occurs in naturally situated, often unconscious, everyday thoughts. The implication of embodied cognition in the pedagogy of mathematics education is that rather than teaching students to learn "rigorous" definitions/theorems of the pre-given mathematical ideas, one needs to focus on the understanding and sensemaking that students need to develop. It is daily experiences that provide the initial grounds for the abstractions that constitute mathematics. This view has been suggested earlier since the early 1960s by Zoltan Paul Dienes (Sriraman and Lesh 2007).

## Cognitive Science of Embodied Cognition

Lakoff and Nunez (2000) discussed the cognitive science of mathematics based on the key concept of embodied cognition. The basic assumption is that mathematics is not mind-free. There are claims such as newborn babies aged 3 or 4 days old having the innate arithmetic abilities to discriminate between collections of two and three items (Antell and Keating 1983) which are supported by other studies beyond the scope of this entry. Basic arithmetic uses various capacities of our brain such as subitizing, perception of simple arithmetic relationships, estimate and approximation, and the ability to use symbols (Dehaene 1997). Mathematical cognition often occurs unconsciously (Lakoff and Nunez 2000). This is because the general cognitive mechanisms that use everyday nonmathematical thoughts can create mathematical understanding and structure mathematical ideas (p. 29). Again Lakoff and Nunez (2000) claim that there are two types of conceptual metaphors that play an important role in the development of mathematical ideas, i.e., grounding metaphors and linking metaphors. The interested reader should examine chapters from *Where Mathematics Comes From* that focus on these ideas. In a nutshell a grounding metaphor refers to basic, direct mathematical ideas. For example, multiplication as repeated addition sets as containers and elements of a set as objects in a container. Linking metaphor refers to abstraction, which produces sophisticated ideas. For instance, geometric figures as algebraic equations (Lakoff and Nunez 2000, p. 53).

### Dienes' Contributions to Embodied Mathematics

Based on a survey of prior studies in mathematics education, Sriraman and Lesh (2007) claimed that Dienes not only studied a phenomenon that later cognitive scientists have come to call embodied knowledge and situated cognition but he also emphasized the *multiple embodiment principle* whereby students need to make predictions from one structured situation to another. And he also emphasized the fact that, when conceptual systems are partly off-loaded from the mind using a variety of interacting representational systems (including not only spoken language written symbols, and diagrams but also manipulatives and stories based on experience-based metaphors), every such model is, at best, a useful oversimplification of both the underlying conceptual systems being expressed and the external systems that are being described or explained. Thus, Dienes' notion of *embodied knowledge* presaged other cognitive scientists who eventually came to recognize the importance of *embodied knowledge* and *situated cognition* – where knowledge and abilities are organized around experience as much as they are organized around abstractions (as Piaget, e.g., would have led us to believe) and where knowledge is distributed across a variety of tools and communities of practice.

## Cross-References

▶ Early Childhood Mathematics Education
▶ Enactivist Theories

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# Enactivist Theories

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## Keywords

Autopoiesis; Structural coupling; Structural determination; Triggering; Co-emergence

## Definition

Enactivist theories assert that cognition is a process that occurs through feedback loops within the interaction of complex dynamical organisms/systems.

## Characteristics

Closely related and often conflated with enactivist theory is embodied cognition. The distinction taken here is made on the basis of the roots of the two theories. Enactivism has biological roots, for example, in the writing of Maturana and Varela (1992) and others, whereas embodied mathematics has linguistic roots (see Embodied cognition).

Enactivist theory is a development of biological and evolutionary science and complexity theory and addresses, among other things, the critique of Cartesian dualistic notions of object/subject. In enactivist theory it is argued that cognition is a process that occurs through the interaction between the living organism and its environment (*autopoiesis*).

We propose as a name the term *enactive* to emphasize the growing conviction that cognition is not the representation of a pregiven world by a pregiven mind but is rather the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs (Varela et al. 1991, p. 9).

From an enactivist perspective learning is seen as a process of restructuring that is *triggered* by interaction that occurs within the complex dynamic system of coupling (*structural coupling*) between person and environment.

We speak of structural coupling whenever there is a history of recurrent interactions leading to the structural congruence between two (or more) systems (Maturana and Varela 1992, p. 75).

Restructuring within the person, however, is determined by the (biological) structural properties of the person (*structural determination*), not by the properties of the environment within which the restructuring occurs. The interaction also triggers changes in the environment, which is also consequently determined by the structure of the environment; this is referred to as coevolution/coadaptation, or *co-emergence*. As can be deduced from the above quotation from Varela et al. enactivism also challenges theories that require some form of mental knowledge representation structures in which perception and reflection are actions upon mental representations of the world constructed independently by the perceiving subject. Cognition and knowing are explained within enactivist theory as active processes that occur directly through the interaction between the cognizing subject and the environment, rather than as a construction of representations of the environment by the cognizing subject.

Knowing is effective action, that is, operating effectively in the domain of existence of living beings (Maturana and Varela 1992, p. 29).

Enactivist theories have roots in biological sciences (Maturana and Varela 1992; Varela et al. 1991) and Darwinian theory of evolution and thus might be viewed as a development of Piaget's constructivism. However, Proulx (2008a) draws attention to some ontological and epistemological differences between enactivism and constructivism. Philosophical antecedents of enactivist theories are shared with closely related "embodied" theory, and more generally situated cognition, these theories refer to seminal philosophical contributions by Edmund Husserl, Maurice Merleau-Ponty, and Ludwig Wittgenstein (Reid 1996).

Autopoiesis: Complex dynamic systems can be defined at many levels, from complex molecular structures within a single cell to solar systems within a galaxy. Autopoiesis is asserted by Maturana and Varela to be the process that distinguishes living beings.

Our proposition is that living beings are characterized in that, literally, they are continually self-producing. We indicate this process when we call the organization that defines them an *autopoietic organization* (Maturana and Varela 1992, p. 43).

Cognition and knowing is one part of autopoietic organization.

Thus a learner within a mathematics classroom constitutes a dynamic system; alternatively one, or a group of, teacher(s) within a professional development setting constitute a system. The learner is a *distinct unity* (Maturana and Varela 1992, p. 40) within the environment of a mathematics class comprising other learners, teacher, and resources. The learner is *structurally coupled* with the classroom environment. Disturbances within the environment *trigger* changes within the learner as she/he adapts herself/himself to the environment. However, the adaptation of the learner is determined by the "structure" (prior experiences and learning and affective characteristics) of the learner, not by the interaction with the environment. The interaction merely "triggers" the change. Thus enactivist theory asserts that cognition is *structurally determined* by the organization of the learner (Maturana and Varela 1992, p. 96).

Enactivist theories began to emerge within the research field of mathematics education in the 1980s, especially following the publication of Maturana and Varela's book *Tree of Knowledge* (1992). A group of Canadian mathematics education researchers established themselves as a center of interest in enactivist theories forming an "Enactivist Research Group" (Reid 1996). However, research within enactivist theories as a framework and methodology is now actively pursued throughout the world, as can be seen from the account below. The account indicates how enactivist theories have entered into the discourse of mathematics over three decades, 1982–2012, thematically, geographically, and through publication in the major scientific journals and conferences in the field.

Tom Kieren and Daiyo Sawada (Canada) became interested in the work of Maturana and Varela in 1982, and later Kieren and Sawada introduced enactivist theory to the mathematics education group at the University of Alberta, Canada (Proulx et al. 2009). The first edition of Humberto Maturana and Francisco Varela's book *The Tree of Knowledge* was published in 1987 (Maturana and Varela 1992). Then around 1993 The Enactivist Research Group was established in Canada (Reid 1996).

Maturana and Varela's theory entered the international discourse of mathematics education through the annual conferences of the International Group for the Psychology of Mathematics Education (PME) during the period 1994–1996. In 1994 at the 18th PME conference held in Lisbon, John Mason (UK) made reference to Maturana and Varela's work in his plenary lecture "Researching from the inside in mathematics education." One year later at the 19th PME conference in 1995 held in Recife, Rafael Núñez and Laurie Edwards (USA) convened a discussion group that focused on embodied cognition; the participants included David Reid (Canada) and Laurinda Brown (UK) who later became significant contributors to the development and application of enactivist theory within mathematics education research and practice. At the same PME conference Edwards and Núñez presented a theoretical paper in which enactivism was identified as one of the several nonobjectivist theories within the compass of new paradigms in cognitive science. A year later David Reid presented a research report at the 20th PME conference held in Valencia in 1996; in this Reid set out enactivism as a methodology. He described research from an enactivist perspective in terms of autopoietic relationships, between researcher and data: between researchers as they engage with each other and the co-emergence of ideas between researchers and the "coemergent autopoetic (sic) ideas which live in the medium of our minds and of which we are emergent phenomena (as the herd is of the antelope)" (Reid 1996, p. 205). The report included a brief review of enactivist theory and its roots.

Also in 1995 Brent Davis (Canada) published a paper in the journal *For the Learning of Mathematics* that set out an enactivist rationale for learning mathematics; the paper included a brief account of the nature of mathematical activity from an enactivist perspective. In this paper Davis applies an enactivist argument to emphasize the inseparability of process and product in mathematical activity (Davis 1995). In 1997 Davis suggested that enactivism provides "a framework for interpreting the phenomenon of mathematics teaching . . . that might allow us to embrace the insights of constructivism without losing the substance of the social critics' arguments," in a report published by *Journal for Research in Mathematics Education* (Davis 1997, p. 355).

During the following decade (1998–2007) interest in enactivist theory developed internationally and in its application to various domains of research within mathematics education. In 1998 Markku Hannula (Finland) applied enactivist theory to research into affect and learning mathematics. He later published more extensively, for example, in the journals *Educational Studies in Mathematics* and *Research in Mathematics Education* (see Hannula 2012 for references). A year later in 1999, Andy Begg

(New Zealand) presented a paper introducing enactivist theory at the annual conference of the Mathematics Education Research Group of Australasia (MERGA-22) (Begg 1999). In the same year, Laurinda Brown and Alf Coles (UK) explained how enactivism informs their research at the November day conference of the British Society for Research into the Learning of Mathematics.

In 2000 the journal *Mathematics Thinking and Learning* published a paper by Edward Drodge and David Reid (Canada) that considers emotional orientation through the lens of embodied cognition. Drodge and Reid take an enactivist perspective to explore the role of decision making in learning mathematics and use illustrations from an episode in which a group of boys engaged in a geometry problem solving task (Drodge and Reid 2000). Later, David Reid, in 2002, adopted an enactivist perspective of learning to describe "clearly one pattern of reasoning observed in the mathematical activity of students in a Grade 5 class" and explore and clarify the characteristics of mathematical reasoning. Reports from this study are published in *Journal for Research in Mathematics Education* and *Journal of Mathematical Behavior* (Reid 2002).

In 2003 Davis and Simmt (Canada) focused on the application of complexity science and how this might contribute "to discussions of mathematics learning and teaching" (Davis and Simmt 2003, p. 138); complexity theory is deeply embedded in the notion of autopoiesis.

In 2005 Elena Nardi, Barbara Jaworski, and Stephen Hegedus (UK) published enactivist framed research into teaching mathematics at university level in *Journal for Research in Mathematics Education* (Nardi et al. 2005). The following year, 2006 Laurinda Brown and David Reid (UK & Canada) applied enactivist theory to explore learner's "non-conscious" decision making processes that occur prior to conscious awareness of making choices and how emotions subsequently structure events (Brown and Reid 2006). The first, nonconscious decisions might be explained as a feature of "structural determinism," and the latter, restructuring of events, explained as "coemergence" as the environment is shaped by the learner.

Maria Trigueros and Maria-Dolores Lozano (Mexico) reported in 2007 on the use of an enactivist approach in the design of resources for teaching and learning mathematics with digital technologies in the journal *For the Learning of Mathematics* (Trigueros and Lozano 2007). A year later, 2008, Lozano reported an enactivist analysis and interpretation of students algebra learning from a longitudinal study of grade 6 (elementary school) through grades 7 and 8 (first years at secondary school) (Lozano 2008). In the same year Jérôme Proulx (Canada) published his use of the enactivist notion of structural determinism to explain characteristics of mathematics teachers' learning (Proulx 2008b). Proulx (2008a) also argues that there are ontological and epistemological differences between constructivist and enactivist theories of cognition, such that enactivism "should not be (mis) interpreted as another form of constructivism" (p. 24).

The period 2009–2012 reveals both consolidation of international effort and maturation of research conducted within enactivist theory. In 2009 the 33rd annual conference of PME held in Mexico included a Research Forum on enactivist theory of cognition (Proulx et al. 2009). The "forum" included brief papers by many researchers and groups (from Canada, Emirates, New Zealand, Mexico, the UK, the USA) that were applying enactivist theory in their research. The report offered a "state of the art" (in 2009) account of enactivism in mathematics education from an international perspective. Proulx concludes the report by suggesting a number of outstanding questions related to learning and teaching mathematics that might focus further research from an enactivist perspective. In 2010 Duncan Samson (South Africa) reported at MERGA-33 the application of enactivism as a theoretical framework and research methodology to inquire into the sense students make of the visual clues held within the figural patterns of algebraic generalization tasks (Samson 2010). Then in 2011 Brown and Coles (UK) reported their application of enactivist theory to teacher learning in professional development settings, and they draw links with the notion of co-learning of teachers and researchers/developers in communities of inquiry. In a paper published in *ZDM*, they explain how an enactive approach is taken to "reframe" teacher education at the University of Bristol. Attention is given to the links between perception and action emphasized with enactivist theory and how this is worked out in terms of experience as the basis of working approaches, discussions, and focusing attention in teacher education (Brown and Coles 2011). In 2012 Hannula (Finland) reported in the journal *Research in Mathematics Education* how enactivist theory can be used to explain a dimension of a "metatheoretical foundation for relating different branches of research on mathematics-related affect to each other" (Hannula 2012). In the same year Brown and Coles (2012) published research in the journal *Educational Studies in Mathematics* that takes an enactivist stance to analyze "how we do reflection" (p. 222) in the processes of learning to teach mathematics.

Enactivist theories have been used within mathematics education including theoretical reflections and studies about the nature of mathematics and the rationale for learning mathematics (Davis 1995), issues of learning topics within mathematics (geometry, Drodge and Reid 2000; reasoning, Reid 2002; algebra, Lozano 2008; and algebraic generalization, Samson 2010), teacher knowledge and teacher learning (Proulx 2008b), teacher education (Brown and Coles 2011), mathematics teaching at university level (Nardi et al. 2005), affective issues in teaching and learning mathematics (Brown and Reid 2006; Hannula 2012), design research (Trigueros and Lozano 2007), and as a research methodology (Reid 1996).

## Cross-References

- ▶ Complexity in Mathematics Education
- ▶ Constructivism in Mathematics Education
- ▶ Embodied Cognition
- ▶ Situated Cognition in Mathematics Education

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# Epistemological Obstacles in Mathematics Education

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The concept of epistemological obstacle emerges in philosophy of science in the works of Bachelard (1938) who is the first to interpret the genesis of scientific knowledge with the support of this concept: "It is in terms of obstacles that one must pose the problem of scientific knowledge [. . .] it is in the very act of knowing that we will show causes of stagnation and even of regression, this is where we will distinguish causes of inertia that we will call epistemological obstacles."

The examples given by Bachelard are typical of the prescientific thinking and connect to what he calls the obstacle of primary experience. In this, the substantialist obstacle consists in referring to a substance equipped with quasi magic properties in order to explain the observed phenomena: as an example, the attraction of dust by an electrically charged surface will be explained by the existence of an electric fluid. Bachelard rightly explains that the obstacle arises from the fact that this is not a metaphor but indeed an explanation of the situation created by what our senses tell us: "We think as we see, we think what we see: dust sticks to the electrically charged surface, so electricity is an adhesive, is a glue. One is then taking a wrong way where false problems will generate worthless experiments, the negative result of which will fail in their role of warning, so blinding is the first image [. . .]."

Brousseau (1976, 1983) is the first to transpose the concept of epistemological obstacle to the didactics of mathematics by highlighting the change in status for the error, that this notion generates: it is not a "result of ignorance [. . .] or chance" but rather an "effect of prior knowledge that was relevant and had its success, but which now proves to be false, or simply inadequate" (Brousseau 1983). Among the obstacles to learning, Brousseau distinguishes indeed the ontogenic obstacles, related to the genetic development of intelligence, the didactical obstacles, that seem to only depend on the choice of a didactic system, and the epistemological obstacles from which there is no escape due to the fact that they play a constitutive role in the construction of knowledge. At one and the same time, the concept of epistemological obstacle extends to the didactics of experimental science (Giordan et al. 1983).

The pioneering works in didactics deal with, among others, obstacles related to extensions to sets of numbers – relative numbers in Glaeser (1981), rational and decimal numbers in Brousseau (1983) – with obstacles related to the absolute value in the research from Duroux (1983), with those that tend to hide the concept of limit, as studied by Cornu (1983) and Sierpinska (1985), with obstacles related to learning the laws of classical mechanics according to Viennot (1979) and with those arising from a sequential reasoning in solving electrical circuits, of which Closset (1983) shows the excessive strength. From these works and others, Artigue (1991) conducts an analysis in which several questions arise, that are subject to debate when trying to characterize the concept of epistemological obstacle: can we talk about epistemological obstacles when there is no identification of errors and but simply of difficulties? Should we look for their appearance and their resistance in the history of mathematics? Look for their unavoidable character in the students' learning process? What does their status of knowledge consist of, having its domain of validity? Can we talk, in certain cases, about a reinforcement of epistemological obstacles due to didactical obstacles?

Other studies also ask the question of the scale at which it is appropriate to look at the epistemological obstacles, as well as that of their cultural character. The works of Schneider (1988) raise these two questions in an articulated manner by showing that the same epistemological position, namely, empirical positivism, can account for multiple difficulties in the learning of calculus: errors when calculating areas and volumes in relation with misleading subdivisions of surfaces into lines and of solid surfaces into surface slices, a "geometric" conception of limits leading students to think of segments as being "limits" of rectangles, and of the tangent line as being "limit" of secants without reference to any topology whatsoever, and their reluctance to accept that the concept of derivative will provide the exact value for an instantaneous velocity. This empirical positivism which, *mutatis mutandis*, converges with the primary experience from Bachelard in the sense of "experience placed before and above criticism" goes well beyond learning calculus (Schneider 2011). This example illustrates indeed, on the one hand, an obstacle considered at a large scale, with its interpretive scope covering errors or multiple difficulties and, on the other hand, its cultural aspect which can be considered as a pure product of Western modernity. It also shows that, despite the opinion of Bachelard, the notion of epistemological obstacle applies to mathematical thinking, at least on a first level.

The debate on the scope and cultural character of epistemological obstacles, of which the examples above illustrate the probable dependence, is animated and most probably not closed. Regarding the first aspect, Artigue insists on the interest in considering what she calls "obstacle-generating processes," including "undue formal regularization" that, as an example, leads students to the misapplication of linearization processes such as "distributing" an exponent on the terms of a sum, or "fixing on a familiar contextualization or modeling," such as the excessive attachment to the additive model of losses and gains when considering relative numbers. About the second aspect, Sierpinska (1989) puts back in a theory of culture some sayings of Bachelard who thinks that, if empirical knowledge of reality is an obstacle to scientific knowledge, it is because the first acts as an unquestioned "preconception" or as an "opinion" based on the authority of the person who professes it. Johsua (1996) continues to believe that some spontaneous reasonings, like those transgressing the laws of classical mechanics, have a cross-cultural character, while Radford (1997) argues that the so-called epistemological obstacle refers more to local and cultural conceptions that one develops on mathematics and science in general. And presumably, we cannot settle this debate without specifying it, example after example, as cautiously proposed by Brousseau 20 years earlier: "The notion of obstacle itself is beginning to diversify: it is not that easy to propose relevant generalizations on this topic, it is better to perform studies on a case by case basis." All this without yielding to the temptation of qualifying as epistemological obstacle whatever is related to recurring errors for which we did not analyze the origins (Schneider 2011).

The identification of epistemological obstacles brings forward the question of their didactical treatment: should we have students to bypass them or, on the contrary, should we let them clear the obstacle and what does that mean? Let us first turn to "educator" Bachelard (as described by Fabre 1995). It is the intellectual distancing that Bachelard emphasizes as major learning issue, when he writes that "an educator will always think of detaching the observer from his object, to defend the student against the mass of affectivity which focuses on certain phenomena being too quickly symbolized [. . .]" (1949). An echo hereof is the psychological shift of perspective ("décentration") of Piaget that, among children, the interpretation of an experience assumes: as such, it "does not obviously make sense" that sugar dissolved in water has disappeared on the account that one cannot see it anymore! One of the primary goals of education would thus be to promote, among students, the detachment from "false empirical objects" born from the illusion that the facts and observations are given things, and not constructed, that is to say to get them to pass from world 1 of physical realities, in the sense of Popper (1973), to world 2 of states of consciousness and to world 3 of concepts that contain "more than what we did put in them." It is presumably those connections that lead Astolfi and Develay (1989) to place Piaget, Bachelard, and Vygotski at the origin of the constructivist movement in didactics of science, the first explaining "how it works,", the second "why it resists," and the third pointing out "how far one can go." Brousseau (1983), as for him, provides clear-cut answers to the questions above: "an epistemological obstacle is constitutive of achieved knowledge in the sense that its rejection must ultimately be mandatorily justified." There resides, according to him, the interest of "adidactical situations" whose fundamental nature with respect to the target knowledge will allow invalidating an old knowledge that proves to be an obstacle to new knowledge, by highlighting the limits of the scope of operation of the former. Martinand (1986) goes further by making obstacles – be these from the works of Bachelard, Piaget, or Wallon – a selection mode for objectives: the concept of "objective-obstacle" appears then in opposition to the usual idea of blocking point. One can think today, together with Sierpinska (1997), that an equivalent coupling may have been too systematic or even normative at a given time in didactics of mathematics, but it is probably advisable that the teacher should manage, at least by a vigorous heuristic discourse, the epistemological obstacles identified on a large scale (Schneider 2011).

The notion of epistemological obstacle has some kinship with that of conception or more precisely that of misconception, but also with that of cognitive or socio-cognitive conflict as illustrated in the acts of an international symposium on knowledge construction (Bednarz and Garnier 1989). The concept of misconception itself may be related to the mental object from Freudenthal (1973) or to the image-concept in Tall and Vinner (1981) who, despite some differences, indicate that the mind of students being taught is not in a virgin state but is a host of intuitions keen to facilitate learning but also to hinder it. In some examples, misconceptions converge with epistemological obstacles in an obvious manner. As such, some of the probabilistic misconceptions identified by Lecoutre and Fischbein (1988) are explained by causal and chronologist conceptions of the notion of conditional probability which, according to Gras and Totohasina (1995), are obstacles of epistemological nature. As for the concepts of cognitive or socio-cognitive conflicts that underpin the Piagetian and Vygotskian theories, they also rely on the assumption that learning is motivated, on the one hand, by an imbalance between the reality and the image that an individual makes up of it and, on the other hand, by confronting his opinion with that of others or with a contradictory social representation. The transfer of the concept of epistemological obstacle to the didactics of mathematics is then bringing a new contribution to the theories mentioned above, in terms of close dependency between the evolution of conceptions among students and the didactical situations they are confronted with: "[. . .] the crossing of an obstacle barrier requires work of the same nature as the setting up of knowledge, that is to say, repeated interactions, dialectics of the student with the object of his knowledge. This remark is fundamental to distinguish what a real problem is; it is a situation that allows this dialectic and that motivates it" (Brousseau 1983). And this is indeed what makes the link between didactics and epistemology to be so tight.

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# Equity and Access in Mathematics Education

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## Keywords

Equity; Social justice; Bourdieu; Social class; SES (Socioeconomic status); Gender inequality; Group placement

## Definition

*Social class background* – social class is best understood through a Marxist orientation as the groupings people fall into as a result of explicit economic forces within society. These groupings are a direct result of similarities with and differences between people, particularly through the resources to which they have access, but also to their tastes and dispositions, which ultimately position them within educational systems.

## Characteristics

Usually an encyclopedia entry will begin with some definitions. With both "equity" and "access," that's not possible. Each of these terms is politically loaded and reflects political and ideological dispositions both in the pedagogical arena of the classroom and in the intellectual arena of the academy. One problem of defining equity is due to it being assumed to be a universal good; surely everyone wants equity? Actually that's far from the case, and at least there will be little agreement on how we define and more importantly operationalize the terms. Equity is not a key driving force for those who sit on the political right. There, meritocracy and individual endeavor are markers of a democratic society, providing a way out of poverty for those who work hard. For those on the political left, the economic superstructure itself, and the education system which serves that system, hides structural inequality and merely perpetuates that structural inequality based on accumulated wealth. For the left, equity itself is a key feature of a democratic society.

One cannot therefore assume a single perspective on equity and access but needs to look for the relationship to political orientation (Gates and Jorgensen 2009). A first, moderate or conservative, stance on equity focuses on individual responsibility. Here there is a recognition of unfairness but a rejection of the social structural underpinnings of that unfairness. A second, more liberal, stance does recognize structural inequalities but locates itself largely within the classroom looking at what classroom practices might alleviate the disparities between pupils. Finally there is a radical stance that recognizes structural inequality but goes further and examines how social inequality is built into existing classroom practices. This stance sees groups of individuals as subject to vastly different sets of experiences and opportunities such that many choices are restricted. But furthermore, these arbitrary barriers become internalized through school and subject cultures. Consequently pupils develop identities which accept their location in the hierarchy.

Mathematics therefore plays a significant, if often hidden part in the politics of education as the sociologist Pierre Bourdieu claims:

> Often with a psychological brutality that nothing can *attenuate*, the school institution lays down its final judgements and its verdicts, from which there is no appeal, ranking all students in a unique hierarchy of all forms of *excellence*, nowadays dominated by a single discipline, mathematics. (Bourdieu 1998, p. 28)

Indeed if equity was not an important issue, this encyclopedia entry would not have been written. The philosopher of mathematics education Paul Ernest takes this a step further by suggesting mathematics as a social filter:

> Mathematics has been remarked upon as playing a special role in *sorting out* students and preparing them for and directing them to different social stations. . . .. Thus, the teaching and learning of mathematics seems to occupy a special place in the provision of social justice—or its obstruction—within the education system. (Ernest 2007, p. 3)

Here is the argument that if mathematics serves a purpose of filtering and directing people into diverse levels in society, equity – how it does this – ought to be a key concern for those charged with teaching mathematics, the schools. The first question then is can schools help foster equity or can they only perpetuate existing inequality. This is a central consideration and one which differentiates academics.

In order to understand the place of equity in mathematics education, one has to grasp the divergence between individual accounts and collective accounts; meritocracy and individual endeavor contrasted with social influences and restricted opportunities.

Of course it is not a coincidence, as evidence from around the world indicates, that achievement and engagement in mathematics vary according to the social class background of the learners. One argument would suggest that social class is the largest influence in pupil underachievement, whereas others would argue schools can make a difference. Evidence for these claims can be found in every school around the world. Whereas it is well known that individual pupils can succeed against the odds, the reality of many mathematics classrooms is reflected in the following comment from a teacher:

> You know, a lot of my bottom group really struggle with maths – and I've noticed they all come from the same part of town, and they have got similar family backgrounds. Surely that can't be a coincidence? (Cited in Gates 2006, p. 367)

There is now widespread focus in the academic literature on the systematic traditional failure to educate students from disenfranchised groups (Secada 1989), and attempts to understand the "systematic" nature of the patterns of achievement have looked at the schools themselves as playing a fundamental role in the furtherance of structured inequality.

> The vast majority of schooling for children . . . of poor and working class, girls and boys of colour and so many others is not neutral, not its means and certainly not its outcomes . . . but who controls the economic, social and educational conditions that make it so? Whose vision of schooling, whose vision of what counts as real knowledge organises the lives in classrooms? (Apple 1995, p. 330)

Historically, a focus on equity in mathematics education developed out of concerns over the achievement of girls (Burton 1990). While early thinking looked at biological differences, this approach soon became discredited, with a recognition that "girls and boys make choices throughout their education and professional careers, and there are *systematic differences* in these choices" (Herman et al. 2010, p. 3). The previous relative underachievement of girls in mathematics is structurally similar to achievement differences resulting from other social characteristics. For example, both ethnicity and social class have a substantial research literature testifying to the unrepresentative levels of underachievement of young people from disadvantaged and working class backgrounds and from ethnic minority groups, including young people from black, Caribbean, indigenous, and Latino communities.

One of the arguments for a systemic underachievement by certain groups of young people in mathematics is that they do not share the advantages of dominant, more affluent groups. Their culture and histories can be different, their languages and relationships are different, and their economic conditions force a rather different set of priorities to those experienced by more comfortable middle-class communities (see Zevenbergen 2000). As a result, choices are forced on families because they do not have credible alternatives and as a result "*the social world of school operates by different rules or norms than the social world these children live in*" (Pellino 2007). The literature on the effects of poverty draws our attention to some of the characteristics of children in poverty. They experience high mobility, hunger, repeated failure, low expectations, undeveloped language, clinical depression, poor health, emotional insecurity, low self-esteem, poor relationships, difficult home environment, and a focus on survival. A strand of research, often termed critical mathematics education, has examined the conditions of such pupils whose backgrounds are obscured and ignored by both schools and the academic research community. For example, the hungry, the homeless, and those children in care all have particular needs – yet because they do not fit the ideal are placed outside the norms (Skovsmose 2011).

To claim there are systematic differences in the choices individuals can make is fairly controversial on two counts. First it assumes that we are free to make choices. Second, there is the assumption that schools, through the energizing of these choices, can make a difference to outcomes. The first of these assumptions overlooks the structural accumulated history that young people carry with them: expectations, identity, self-efficacy, language fluency, etc., all of which place learners at different starting points. One strand in the literature here assumes that if choices are influenced and limited by misinformation and low expectations, then it is entirely possible for schools to overcome these barriers by providing an environment that redresses those limitations – the second assumption.

Between 1980 and 2010, research in mathematics education has seen a noticeable shift in what some have seen as a sociocultural turn in research agendas (Lerman 2000), placing an emphasis on an understanding through the exploration of sociocultural factors – recognizing the importance of the social context upon one's action and choices. But this has also recognized that we need to look and think beyond the individual level of cognition to see how different responses to mathematics might be explained. How do we explain, for example, that earlier comment by a teacher, that achievement at mathematics is very highly correlated to the pupils' home background? Do we believe it is because some people are not as intelligent as others? Or do we believe some children are held back in order for some others to progress? Where one stands on that will largely influence how you personally think about equity.

One way in which children can be held back is through restriction of the curriculum and a further strand in equity and access to mathematics education is the access afforded by the school curriculum to mathematics itself – and to the powerful ideas it allows us to use. In mathematics education in some – but not all – countries, access to the curriculum is organized around structured grouping usually claimed to be on some measure of ability. In some countries (UK, USA, Australia, etc.), it is an almost universal practice, and teachers seem to be unable to conceive of how it might be otherwise given a claimed hierarchical nature of mathematics. However, in other countries (Denmark, Finland, etc.) the practice of ability discrimination is outlawed.

In the literature, group placement is a highly controversial and contested practice, and much research has indicated the effect it has upon young people who do not fit an ideal model of successful learner – usually pupils from working class homes and some ethnic minorities. Such pupils are systematically more likely to be placed in lower groups than others even when performance is taken into account (Zevenbergen 2003). Various studies have shown "*that placement in ability groups increases the gap between students at different group levels*" (Cahan et al. 1996, p. 37). In other words, the very placement of pupils in a group influences their outcomes.

A lack of equitable practices leads to restricted access by schools and teachers through the provision of a restricted curriculum to lower achieving pupils. The pedagogical jump here made by teachers is to assume that pupils who are doing less well are not (cap)able of *higher-order thinking*. In a series of studies, this has been explored (Zohar 1999; Zohar et al. 2001; Zohar and Dori 2003) with the conclusion that teachers do not really believe weak pupils (invariably pupils from poor backgrounds) can think in higher-order ways.

Studies of pupils' mathematical experiences that take account of social backgrounds (Lubienski 2000a, b, 2007) have found very specific differences in two main areas – *whole class discussion* and *open-ended problem solving* – and these can throw some light onto the way in which equitable practices are compromised and access to big ideas is restricted. These are two well-researched pedagogical strategies and classroom practices which at least in professional discourse are held in some esteem. Discussion-based activities were perceived differently by pupils from different social backgrounds. Pupils from high socioeconomic status (SES) backgrounds thought discussion activities were for them to analyze different ideas while those pupils from lower social groups thought it was about getting right answers. The two groups had different levels of confidence in their own type of contributions with the low SES pupils wanting more teacher direction. Higher SES pupils felt they could sort things out for themselves – as their parents do in life presumably.

The second area was that of *open-ended problem solving* – a mainstay of recent reform agendas in mathematics. The high level of ambiguity in such problems caused frustration in low SES pupils which in turn caused them to give up. High-SES pupils just thought harder and engaged more deeply. It is well known that middle-class pupils come to school armed with a set of dispositions and forms of language which gives them an advantage because these dispositions and language use are exactly the behaviors that schools and teachers are expecting and prioritize (Zevenbergen 2000). High-SES pupils have a level of self-confidence very common in middle-class discourses, while working class discourses tend to be located in more subservient dependency modes, accepting conformity and obedience (Jorgensen et al. 2013).

Equity and access then are both key issues in the provision of mathematics education but are both controversial and deeply political.

## Cross-References

- ▶ Cultural Diversity in Mathematics Education
- ▶ Gender in Mathematics Education
- ▶ Immigrant Students in Mathematics Education
- ▶ Inclusive Mathematics Classrooms
- ▶ Indigenous Students in Mathematics Education
- ▶ Language Background in Mathematics Education
- ▶ Political Perspectives in Mathematics Education
- ▶ Socioeconomic Class in Mathematics Education

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# Ethnomathematics

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## Introduction

In this entry, I will refer mainly to my views and my participation in the emergence of this field as a research area and the benefits acquired in understanding and interpreting the cultural, political, material, and even economic forces recognized in building up these strategies. A basic reference is my 1985 basilar paper in *For the Learning of Mathematics*, which has been republished since then in various handbooks (D'Ambrosio 1985).

I will discuss mainly the theoretical basis of Ethnomathematics and its values as part of a culture. I present Ethnomathematics as a research program in the History and Philosophy of Mathematics with societal and pedagogical implications (D'Ambrosio 1992). The program depends on theories that explain human knowledge and behavior.

In considering Ethnomathematics a research program, it is recognized as a broader focus than simply the recognition of mathematical ideas and practices of different cultural groups. Of course, the Ethnomathematics of different cultural groups is the main source for this research program. But the major objective of the Program Ethnomathematics is to propose a broader vision of knowledge and of human behavior, by making sense of how different communities, societies, and civilizations faced their struggle for survival and transcendence in their environmental, cultural, economic, and social contexts.

The concern with other cultures and with other forms of knowledge has been always present in the History of Ideas and goes back in history to all civilizations. Others may have a different approach and base their reflection on other scenarios, thus showing another picture of the field. Throughout this entry, there are traces of many different approaches to the theme, but there are few explicit references to them.

A basilar question is the reason to look into non-Western cultures and civilizations for a research into the History and Philosophy of Science and Mathematics, which are Western constructs. I paraphrase Brian Fay in the introductory essay in the issue of *History and Theory* devoted to *Unconventional History*, and claim that learning about other cultures and civilizations is, at the very same time, learning about our civilization, its strengths, and limitations (Fay 2002).

## Definition

The word Ethnomathematics may be misleading. It is easily confused with ethnic-mathematics. Although ethnic groups are contemplated, I consider Ethnomathematics a much broader concept, focusing on cultural and environmental identities. The name also suggests Mathematics. Again, I use it in a much broader concept than Mathematics, which is a late Western concept. Indeed, in the sense we use the word "mathematics" today, it goes back to about the fifteenth century. Former uses of the word mathematics have a different meaning. Today, historians opt for using the word "mathematics" also when they refer to some practices and theories of the Antiquity and the Middle Ages, which bare some common objectives, concepts, and techniques with Mathematics. This option is convenient for historical narratives. But it is misleading. A similar, also misleading convenience is adopted by ethnographers and cultural anthropologists, when describing and analyzing other cultures.

There is a very natural question: "Why to use the word Ethnomathematics for my research on the strategies developed by different communities, societies, and civilizations to face the struggle for survival and transcendence in their environmental, cultural, economic, social contexts?" I will try to explain my choice, which is indeed an etymological construction. The word Ethnomathematics is obviously, not new, and it has been used mainly with an ethnographical focus for decades.

The main concern that guides my research is to identify the ways, modes, styles, arts, and techniques, generated and organized by different cultural groups for learning, explaining, understanding, doing, and coping with their natural, social, cultural, and imaginary environment. This is a long explanation, and I tried to synthesize it with the resource of an etymological exercise. I looked for words with meanings that convey this long explanation and I found Greek roots that can do it. The root *techne* means, roughly, the arts and techniques, the ways and modes, the styles; *mathema* is a difficult root, which generally means learning, explaining, understanding, doing, and coping with some reality; and *ethno* means a natural, social, cultural, and imaginary environment. Thus, I may synthesize the long phrase "ways, modes, styles, arts and techniques to learn, explain, understand, doing and coping with distinct natural, social, cultural, imaginary environment" as the *technes* of *mathema* in distinct *ethnos*. Thus, using *tics* as a simplified spelling for *techne*, the long phrase became *tics* of *mathema* in distinct *ethnos*, or making an obvious rearrangement, *ethno* ± *mathema* ± *tics* or *ethnomathematics*. Thus, I started to use the word *Ethnomathematics* as a result of this etymological exercise (D'Ambrosio 1998, 2006).

It is noticeable that Mathematics diverted from the concept of the *mathema*. In the words of Oswald Spengler "The present-day sign-language of mathematics perverts its real content" (Spengler 1962). Ethnomathematics is particularly concerned with real contents. For educational purpose, the restoration of this concept is the major support of my proposal for a modern *trivium* in education: literacy, matheracy, and technoracy (D'Ambrosio 1999).

It should not be surprising at all that Mathematics, as we know it, is a special Ethnomathematics, the same as are the theories and practices of Pharmacology, of Cardio-Surgery, of Dance, of Algebra, and, indeed, any form of knowledge. All these disciplines are the concern of specific professional groups [*ethno*] to develop ways, modes, styles, arts, and techniques [*tics*] for learning, explaining, understanding, doing, and coping with [*mathema*] with specific and related facts, phenomena, and problems. They rely on their natural, social, cultural, and imaginary environments.

It is not surprising that the word Ethnomathematics suggests Mathematics. After all, Mathematics is the dorsal spine of Modern Civilization. Indeed, throughout history, Mathematics has been well integrated into the technological, industrial, military, economic, and political systems and Mathematics has been relying on these systems for the material bases of its continuing progress. The same for Science and Technology and Philosophy as well. Hence for models of society.

The issues are essentially political. There has been reluctance among mathematicians, to a certain extent among scientists in general, to recognize the symbiotic development of mathematical ideas and models of society. Mathematics has grown parallel to the elaboration of what we call Modern Civilization. Historians amply recognize this.

Modern World Civilization sprang out of Europe as the result of 500 years of conquest and colonization. Modern Civilization is a body supported by a dorsal spine, recognized by philosophers, historians, scientists, and just about everyone, as Mathematics.

Mathematics as the dorsal spine of Modern Civilization, is beautiful, rigorous, and perfect, so respected by everyone, even feared, particularly by children and students. But, paradoxically, Modern Civilization, is ugly, plagued with inequity, arrogance, and bigotry.

What went wrong with Modern Civilization? How is it possible that a perfect dorsal spine supports such an ugly body?

To understand this paradoxical discord has been a guiding quest in my research and in proposing the Program Ethnomathematics.

## Knowledge, Behavior, and Culture

How did everything begin? The myths of creation are present in every civilization. The founding myths and traditions of Western civilization leads to the history of monotheistic religions (Judaism, Christianity, Islamism) and the emergence of techniques and the arts and links to understanding how Mathematics permeates all this. A great insight is gained in trying to identify and to understand what happened in the founding myths and traditions of non-Western civilizations.

The main difficulty I encounter, and this is true for every one doing cultural studies, is the difficulty of understanding and interpreting other cultures with the categories and analytic instruments other than those that are part of my cultural heritage. I have been trying to avoid, at least to minimize, this difficulty. We rely on informants, and there is a difficulty in building up trust.

The goal is to develop a generic comprehensive theory of knowledge and behavior. I base my research on universal forms of knowledge (communications, languages, religions, arts, techniques, explanations, or sciences) and in a theoretical/methodological model of knowledge and behavior that I call the "cycle of knowledge."

The aim of research in the Program Ethnomathematics is the recognition of practices and its relation to theories. Thus, I focus history of science (and, of course, of mathematics) trying to understand the role of technology as a consequence of science, but also as an essential element for furthering scientific ideas and theories. I guide my investigation on three basic questions:

1. How do ad hoc practices and solution of problems develop into methods?
2. How do methods develop into theories?
3. How do theories develop into scientific invention?

## Current Work in Ethnomathematics

The Program Ethnomathematics was initially inspired by recognizing ideas and ways of doing that reminds us of Western mathematics. What we call mathematics in the academia is a Western construct. Although dealing with space, time, classifying, and comparing, which are proper to the human species, the codes and techniques to express and communicate the reflections on these behaviors are undeniably contextual. Thus came my approach to Cultural Anthropology (curiously, my first book on Ethnomathematics was placed by the publishers in a collection of Anthropology).

Much work is going on in many countries. Many national, regional, and international meetings are held. An overall account of the progress of the field is seen in the site of the *International Study group on Ethnomathematics/ISGEm*, with links to the most relevant works in the area. Access the links at http://isgem.rpi.edu/pl/ethnomathematics-web.

Although a new field, there are important publications revealing the strength of the area of Ethnomathematics. It would be difficult to produce a bibliography. There are innumerous pioneers and active researchers in this field. In attempting to give a full bibliography I would surely leave important references. I mention three basic works:

- *Native American Mathematics*, Michael Closs editor, University of Texas Press, Austin, 1986.
- *Ethnomathematics. Challenging Eurocentrism in Mathematics Education*, Arthur B. Powell and Marilyn Frankenstein, editors, State University of New York Press, Albany, 1997.
- *Mathematics Across Cultures. The History of Non-Western Mathematics*, Helaine Selin editor, Kluwer Academic Publishing, Dordrecht, 2000.

Besides many references, after having put together the bibliographies of each chapter, we have a comprehensive relevant bibliography for the area.

The *International Conferences on Ethnomathematics/ICEm* are well attended events. The *Fourth ICEm* took place in Towson, Maryland, in 2011. Most of the papers presented in this conference are published in the *Journal of Mathematics and Culture* volume 6 Number 1 Focus Issue ICEM4, a free access publication linked to the site of the *ISGEm* indicated above.

It is appropriate to say that the Program Ethnomathematics is a promising emerging research field.

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# External Assessment in Mathematics Education

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## Keywords

Assessment; Assessment design; Assessment for learning; Assessment in education; Assessment of complex systems; Complex thinking; Design-based assessment; Evaluation; Higher-order thinking

## Characteristics

Much of the discussion about measurement in education in the past half century has revolved around the need to move beyond the application of psychometrics to a broader model of educational assessment that supports learning (Flanagan 1951; Ebel 1962; Glaser 1963). Historically, significant attention has been given to the differences between norm-referenced and criterion-referenced tests (Glaser 1963; Hambleton 1994), focusing on relative versus absolute standards of quality that are more or less appropriate to measure abilities or achievements. However, briefly, we will describe why neither of these approaches to assessment allows us to assess higher-order understandings in mathematics that the field is mostly interested in studying nor do they consider latest advancements in what we now know about how students learn mathematics, as they interact with teachers, schools, and curricular innovations. Furthermore, we propose a new challenge and purpose for assessment: *How can assessments of complex mathematical achievements be achieved in a way that provides useful information for relevant decision makers?* After presenting an overview of the dichotomy between norm-referenced and criterion-referenced approaches to assessment, we describe the characteristics of assessment designs that are needed to assess the complexity in the continually adaptive development of high-order mathematical thinking that mostly interests the field of mathematics education.

Norm-referenced tests grow from the psychometric tradition, based on the measurement of general intelligence (g) as an inheritable characteristic of an individual that is fixed over time. This psychometric tradition has its roots in the mid-1800s with the work of Galton and Pearson and Spearman's contributions in the beginning of the 1900s (Gipps 1999; Gardner et al. 1996). Usually associated with the measurement of aptitude (as opposed to achievement), norm-referenced tests are constructed with the purpose of comparing respondents on attributes which presumably (although seldom in reality) do not depend on instruction. Thus, each item is assumed to have a difficulty level relative to other items; again, this level of difficulty is assumed to be independent of individual's experiences. So, items are selected "to discriminate among those tested in order to spread scores along the normal distribution" (Gipps 2012, p. 70), and items that have a low discrimination index are discarded from the test (e.g., items in which most students score correctly and items in which most students score incorrectly). However, items were selected to be those that are not influenced by learning experiences are not likely to provide important information about what students learned or didn't learn. Consequently, their elimination from the test leads to one of the most noted limitations of norm-referenced tests, which is their insensitivity to instruction (Popham 1987; Carmona et al. 2011).

Rather than focusing on relative measures, leading psychometricians have argued that criterion referenced should be used which are dependent upon an "absolute standard of quality"

(Glaser 1963, p. 519) in relation to specific objectives (Popham 1987). Thus, criterion-referenced tests are considered to be more appropriate to measure achievement and determine current levels of student performance. These tests assume a continuum of knowledge acquisition from no proficiency to perfect performance, and the reference criteria are expected to include a representative sample of important achievements in relevant domains, regardless of their discrimination index. So, scores are determined by calculating the proportion of these tasks to determine mastery or nonmastery for an individual.

Supported in behaviorism (e.g., Skinner 1968), and as a rational approach to evaluation through determining individual's learning gains after instruction, Glaser (1963) associates criterion-referenced tests with measuring student attainment of explicit criteria as indicators of behavioral objectives (Popham 1987; Gardner et al. 1996). This learning perspective views the mind as inaccessible and, therefore, studies learning as the way behaviors, which are observable, are acquired. All behaviors are considered to be a result of chained reactions to events in the environments called stimuli, and mental activity is defined in terms of observable and measurable stimuli-response patterns. Learning of complex ideas is formulated as a partitioning into smaller behaviors, or pieces, that are organized along a one-dimensional continuum of increasing level of difficulty, assuming mastery of a lower-level behavior as a prerequisite to achieve higher-level understanding. Behavioral objectives are generally stated in the form of statements as follows: *Given situation **S**, the student will be able to do **D**, to level of proficiency **P***. However, in recently developed curriculum standard documents, it is clear that in fields such as mathematics education, many of the most important goals of instruction cannot be reduced to lists of declarative statements (i.e., facts) or condition-action rules (i.e., skills). To address these shortcomings, Lesh and Clarke (2000) present another type of instructional goal defined as *cognitive objectives*, which are found more relevant in mathematics and science education than their counterparts, because cognitive objectives focus on students' *interpretations* of situations, rather than on their *actions* in these situations. Examples of relevant cognitive objectives in mathematics and science education include models, metaphors, and complex conceptual systems, to mention a few. In order to operationally define what it means to "understand" such cognitive objectives, it is important to include (a) **situations** that optimize the probability that the targeted construct will be elicited in an observable form, (b) **observation tools** that allow observers to identify the construct from other irrelevant information that might also be elicited, and (c) **quality assessment criteria** that allow for meaningful comparisons to be made among alternative possible solutions.

Lesh, Lamon, Lester, and Behr (1992) argue the need for an entire paradigm shift to rethink assessment issues in mathematics education. Rather than focusing on behavioral (or other types of) objectives, they identify conceptual objectives as those we are mostly interested in assessing and which cannot be examined neither from a norm reference nor criterion reference perspectives. Lesh and Lamon (1992) highlight the need to provide well-articulated operational definitions that focus less on value judgments about students (good/bad) and instead focus on providing useful documentation for the decision makers to be able to make a better-informed decision based on specific purposes (Carmona 2012).

This paradigm shift evidences significant changes on assessment-related topics such as data collection, data interpretation, data analysis, and the nature of reports. It involves "new decision makers, new decision-making issues, new sources of assessment information and new understandings about the nature of mathematics, mathematics instruction, and mathematics learning and problem solving" (p. 380). In addition, this new perspective requires a revision on what it means for assessments to be valid, reliable, and generalizable (Pellegrino, Chudowsky, and Glaser 2001), focusing assessment on an increased authenticity of tests and an increase on the credibility and fairness of the inferences made based on test results (Messick 1994). Consistent with these views, Chudowsky and Pellegrino (2003) emphasize the need to generate new situations in a way in which assessments are designed to support and measure learning and elicit student thinking in its complexity (Lesh et al. 2000). The following section provides an overview outlining the main components of this new perspective into assessment design we call *design-based assessment*.

### Design-Based Assessment

During the past 30 years, mathematics educators have pioneered a new class of research methodologies, which have become known as *design research studies*. These design research studies have been proposed to encourage the relevance of research to practice (Brown 1992) and to highlight the importance of incorporating practitioners' wisdom to theory development (Collins 1992; Collins et al. 2004). But, most of all, in mathematics education, where most researchers are also practitioners (e.g., teachers, teacher educators, curriculum developers), the main reasons why design research methodologies have been useful are because (a) like engineers, mathematics education researchers tend to be trying to design and develop the same "subjects" that they are trying to understand and explain and (b) like engineers, the kinds of complex and continually adapting subjects that mathematics educators are trying to understand usually cannot be explained by drawing on only a single theory. Instead, it should be expected that useful conceptual frameworks (or models) will need to integrate ideas and procedures drawn from a variety of relevant theories (and disciplines). One reason why single-theory ways of thinking seldom work is that solutions to realistically complex problems usually involve competing and partly conflicting factors and trade-offs – such as those involving high quality and low costs.

When *design research methodologies* emphasize the measurement of complex and continually adapting subjects, they can be called *design-based assessment methodologies*. And, assessing curriculum innovations can be thought of as being similar to the methodologies that are needed to assess complex artifacts such as space shuttles or transportation systems. Some relevant assumptions include the following.

- For the kinds of complex and continually adapting systems and situations that need to be understood and explained, it generally must be assumed that no two situations are ever exactly alike – and that the exact same thing never happens twice. Furthermore, for most such systems, many of their most important attributes can only be "observed" by documenting their effects on other things, and (like neutrinos or other subatomic particles in physics) to measure them often involves changing them.
- In general, complex systems and complex achievements cannot be understood by breaking them into tiny pieces – and additively combining measurements of the pieces. For example, even if it is true that developing some higher-order *conceptual understanding* (C) implies that a list of lower-order *behavioral objectives* ($B_1, B_2, B_3, \ldots B_n$) should have been mastered, it does not follow that mastering each, $B_1, B_2, B_3, \ldots B_n$, implies that C has been achieved. Yet, this *fragmentation fallacy* is an assumption underlying psychometric conceptions of knowledge development. One of the many things that mathematics educators can learn from engineers and other design scientists is that as the complexity of designed constructs (such as space shuttles) increases, a far greater percentage of assessment activities need to focus on relationships and connections among parts and relatively less time focuses on assessments of isolated pieces.
- Why is it impossible to assess most conceptual understandings using tests that are based on psychometric theory? As stated above, psychometric theory was developed originally to measure aptitude (i.e., general intelligence – where performance is not influenced by teaching and learning). Whereas, tests that are designed to measure the results of learning and instruction are called achievement tests. In particular, in intelligence testing, items are discarded as being "unreliable" if student performance increases in the course of

responding to them. That is, to be reliable, a students' performance should not change for a sequence of tasks which are all designed to test the same attribute.

*Design-based assessment* focuses on three interacting and continually adapting "subjects" of assessment studies – students, teachers, and curriculum innovations (i.e., programs). Space limitations preclude considering other important "subjects" – such as administrators, home environments, or classroom learning environments – even though it is well known that these latter factors often strongly influence the ways that students and teachers interact and adapt in response to curriculum innovations. For example, the impact of a curriculum innovation may vary significantly if the classroom norms that govern student-teacher and student-student discussions emphasize the practice of requiring students to accept procedures and claims based on appeals to authority – rather than requiring them to justify and explain things based on students' mathematical sense making. By focusing on students, teachers, and programs, we hope that readers will find it easy to generalize to other relevant subjects.

Notice that, in our descriptions of assessment practices, we also emphasize the importance of documenting and assessing two-way interactions among "subjects" – rather than restricting attention to one-way/cause-and-effect relationships. For example, teachers don't just influence students' thinking about the meanings of the mathematical concepts and processes that they are expected to develop, but, students also influence teachers' thinking about what it means to "understand" these concepts and processes. So, even in situations where a single teacher teaches two groups of students with comparable abilities, the personae that an excellent teacher adopts for one group of students may need to be significantly different than for another group of students. This is because groups as a whole often develop significantly different group personalities.

Next, notice that our descriptions of assessment practices also emphasize developmental perspectives about "subjects" who are assumed to be complex and dynamically adaptive systems – not at all like widgets being created using machine-like processes. Consequently, regardless whether attention focuses on the continually adapting conceptual systems that are developed by students or teachers or whether attention focuses on the systems of learning experiences that are intended to promote student and teacher development, we recognize that when these systems are acted on, they react. Furthermore, based on results from research involving very simple aptitude-treatment interaction studies, we know that, when such feedback loops occur, second- and third-order effects are often far more significant that first-order effects. So, for realistically large and complex curriculum innovations, entry-level teachers' first-year implementations generally should be expected to be significantly different than second-, third-, or fourth-year implementations (when increasingly more experienced teachers are likely to be available).

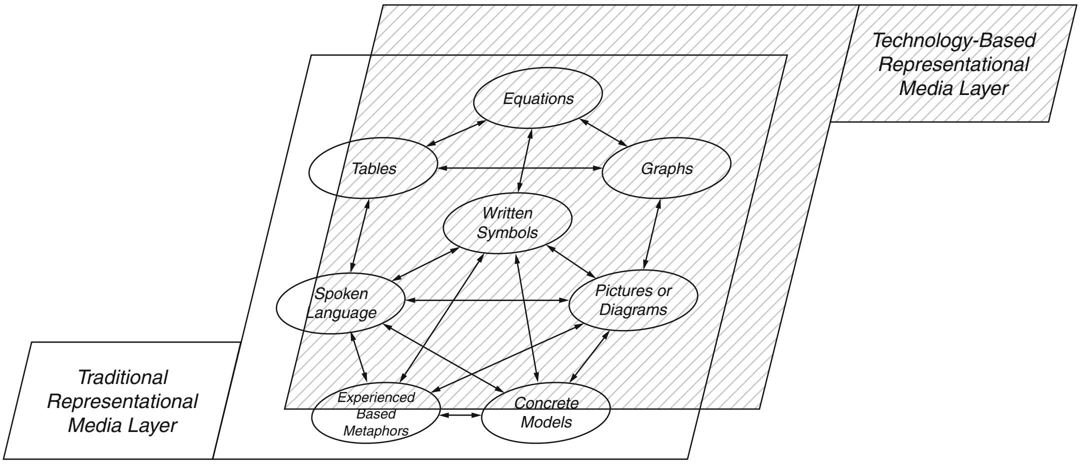
Finally, notice that our attention focuses on *assessment* rather than simply *evaluation*. Whereas evaluation only involves assigning a value to various subjects, assessment involves generating useful descriptions of where various "subjects" are, and where they need to develop in some landscape of possibilities. In general, both assessment and evaluation are intended to provide useful information for decision makers – who may range from students, to teachers, to administrators. So, to assess the quality of a given assessment or evaluation, it is important to consider the following questions: *Who are the intended decision makers?* (because the information that is useful to a teacher may be quite different than the information that is useful to an administrator or politician). *What decisions are priorities for these decision makers to make? What kind of information is most useful for these decision-making purposes?*

For example, low-stakes-but-rapid-turn-around assessments that are intended to help teachers provide individualized attention to students tend to be quite different than high-stakes-and-slow-turn-around assessments that are intended to screen students or limit future opportunities. Sometimes, the former types of assessments are referred to as summative assessments, and the latter are referred to as formative assessments. But, these summative and formative functions often get muddled when (a) summative assessments are used explicitly to change the nature of what is taught and how it is taught and (b) modern statistical procedures often make it possible to use patterns or trends to generate highly reliable summaries of achievement based on collections of documentation.

When analyses of assessment practices begin by asking *who the decision makers are and what decisions are priorities for them to make*, then it tends to become clear that in modern technology-based societies, most decision makers tend to have ready access to computer-based tools which are capable of easily generating interactive graphics-based displays of information that are both simple to understand and easy to customize to fit the purposes and prejudices of individual decision makers and decision-making issues. For nearly any of the "subjects" that are important in educational decision making, single number characterizations are virtually useless and essentially remove decision makers from the decision-making process – by proclaiming, for example, that subject #1 is better than subject #2 regardless of what decisions are being made or what factors are important to consider. Answers may be different for different decision makers.

In educational research and assessment, there is no such thing as a tool or methodology that is "most scientific" (for all subjects, for all decision makers, and for all decision-making issues). Every assessment tool is based on assumptions which may or may not be appropriate for the subjects or purposes of a given study. And, a "scientific methodology" or a "scientific tool" is one whose assumptions are, insofar as possible, consistent with those associated with the subjects, decision makers, and decision-making purposes of the study. For example, when assessing the achievements of students, teachers, or curriculum innovations, the following kinds of questions are important to ask:

- *Do the tools or methodologies emphasize achievements that are well-aligned with the goals of the project, teacher, or students?* For example, even the most recently developed curriculum standards documents, such as the USA's *Common Core State Standards*, none of the higher-order achievements are operationally defined in ways that are measurable. Furthermore, when tests such as the Educational Testing Services' Scholastic Achievement Test were originally designed to be *Scholastic Aptitude Tests*, then the entire *psychometric theory*, which was created to provide development standards, can be expected to emphasize student attributes intended to be unchangeable due to instruction. *Can tests which are explicitly being used to change what is taught and how it is taught be thought of as not being among the most powerful parts of the educational "treatments" being assessed?*
- *Do methodologies which claim to randomly assign students to "treatment groups" and "control groups" really succeed in creating situations which factor out the influences of all but a small number of variables?* (Notice that similar methods have failed even in the case of very small and simple aptitude-treatment interaction studies.) *Can the most important factors really be thought of as being "controlled" when the parallel development of students, teachers, and program implementations interact in ways that usually lead to second-order effects which are as powerful as first-order effects – and when influences due to factors such as administrators, classroom learning environments, and students' home environments tend to be ignored?*
- *Are mixed-methods methodologies adequate to assess students' and teachers' knowledge or content of curriculum innovations?* Quantitative research produces quantitative statements or quantitative answers to questions, whereas qualitative research produces qualitative statements or qualitative answers to questions. But, design-based assessment research is about knowledge development, and very little of what we are studying consists of declarative statements (i.e., facts) or answered questions (i.e., rules). For example, some of the most important kinds of




**External Assessment in Mathematics Education, Fig. 1** A merged Kaput-Lesh diagram for thinking about representational fluency

knowledge that we develop consist of models for describing, explaining, designing, or developing complex systems. So, models (often embedded in purposeful artifacts or tools) are among the most important kinds of knowledge that we need to develop and assess. Consequently, the question we must ask is as follows: *How do we validate models?* And, the answer is that both qualitative and quantitative methods are useful for validating models. But the product isn't simply a quantitative or qualitative claim. It's a validated model – and trends and patterns involving development.

- *Is the unbiased objectivity of an assessment really assured by using "outside" specialists whose only familiarity with the relevant subjects come from pre-fabricated off-the-shelf tests, questionnaires, interviews, and observation protocols which are not modified to emphasize the distinctive characteristics of the subjects and their interactions?* And, if these "outside measures" are used for purposes of accountability, *can they really avoid having powerful influences on the treatments themselves?*
- *Can comparability of treatments really be guaranteed by taking strong steps aimed at trying to ensure that all teachers and all students do exactly the same things, in exactly the same ways, and at exactly the same times?* Notice that, in the literature on the diffusion of innovations, complex systems tend to evolve best when measurable goals are clear to all relevant subjects – and when strong steps are taken to encourage diversity (of interactions), selection (of successful interactions), communication (about successful interactions), and accumulation (of successful interactions).

In mathematics education, many of the most important and powerful types of conceptual understandings occur in one of two closely related forms. The first focuses on students' abilities to mathematize (e.g., quantify, dimensionalize, coordinate) situations which do not occur in pre-mathematized forms and the second focuses on representational fluency – or abilities that are needed to translate from one type of description to another. For example, in the case of representational fluency, Kaput's (1989) research on early algebra and calculus concepts emphasized the importance of translations within and among the three types of representations which are designated in the three ovals shown at the top of Fig. 1 (i.e., equations, tables, and graphs), and in a series of research studies known collectively as *The Rational Number Project*, Lesh, Post, and Behr (1987) emphasized the importance of translations within and among the five types of representations which are designated in the five ovals shown at the bottom of Fig. 1 (i.e., written symbols, spoken language, pictures or diagrams, concrete models, and experience-based metaphors).

From the perspective of psychometric theory, two of the main difficulties with test items that involve representational fluency result from the fact that when tasks involve description of situations (a) there always exist a variety of different levels and types of descriptions and (b) responding to one such task often leads to improvements of similar tasks. So, according to psychometric theory, where tasks are considered to have a single-level of difficulty which is unaffected by instruction and where the relative difficulty of two tasks also is considered to be unaffected by instruction, such tasks are discarded as being unreliable. Similarly, when tasks focus on students' abilities to conceptualize situations mathematically, there once again exist a variety of different levels and types of mathematical descriptions, explanations, or interpretations that can be given. So, once again, the same two difficulties occur as for representational fluency.

Especially when tests are used for accountability purposes and teachers are pressured to teach to these tests, it is important for such tests to include tasks that involve actual work samples of desired outcomes of learning – instead of restricting attention to indirect indicators of desired achievements. For example, if the development of a given concept implies that a student should be able to do skill-level tasks $T_1$, $T_2$, ... $T_n$, then tasks $T_1$, $T_2$, ... $T_n$ tend to be *indicators* similar to wrist watches or thermometers – in the sense that it is possible to change the readings on wrist watches or thermometers without in any way influencing the time or the weather. But, *how can assessments of complex achievements be achieved inexpensively, during brief periods of time, and in a timely fashion that provides useful information for relevant decision makers?* In modern businesses where continuous adaption is necessary, and especially in knowledge industries or in academic institutions, decision makers seldom use multiple-choice tests or questionnaires to assess the quality of the kinds of complex work that constitute the most important activities of their employees. So, how do specialists (or teams of specialists) get recognized and rewarded for the quality of their work? For example, how do professors validate their work? Or, how do doctoral students validate the work on their Ph.D. dissertations? Answers to these questions should provide guidelines for the assessment of development related to students, teachers, curriculum innovations, and other "subjects" in mathematics education research. Space limitations do not allow detailed answers to such questions to be given here. But, when attention focuses on the systems of knowledge being developed by students, teachers, and curriculum innovations, (a) it's important to focus on the half-dozen-to-a-dozen "big ideas" which the subjects are intended to develop, (b) it's often useful to recognize that a large part of what it means to "understand" these big ideas tends to involve the development of models (or interpretation systems) for making sense of relevant experiences, (c) these models often are embodied and function within purposeful tools and artifacts, and (d) these tools and artifacts often can be assessed in ways that simultaneously allow the underlying models to be assessed. Procedures for achieving these goals have been described in a variety of recent publications about design research (e.g., Lesh and Kelly 2000; Lesh et al. 2007; Kelly et al. 2008), and it is straightforward to adapt most of these procedures to apply to assessment purposes.

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