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Data Handling and Statistics Teaching and Learning

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Keywords

Statistics; Data handling; Exploratory data analysis; Teaching and learning statistics; Research on teaching and learning statistics; Statistical reasoning; Statistical literacy; Technological tools in statistics learning

Definition

Over the past several decades, changes in perspective as to what constitute statistics and how statistics should be taught have occurred, which resulted in new content, pedagogy and technology, and extension of teaching to school level. At the same time, statistics education has emerged as a distinct discipline with its own research base, professional publications, and conferences. There seems to be a large measure of agreement on what content to emphasize in statistics education: exploring data (describing patterns and departures from patterns), sampling and experimentation (planning and conducting a study), anticipating patterns (exploring random phenomena using probability and simulation), and statistical inference (estimating population parameters and testing hypotheses) (Scheaffer 2001). Teaching and learning statistics can differ widely across countries due to cultural, pedagogical, and curricular differences and the availability of skilled teachers, resources, and technology.

Changing Views on Teaching Statistics Over the Years

By the 1960s statistics began to make its way from being a subject taught for a narrow group of future scientists into the broader tertiary and school curriculum but still with a heavy reliance on probability. In the 1970s, the reinterpretation of statistics into separate practices comprising exploratory data analysis (EDA) and confirmatory data analysis (CDA, inferential statistics) (Tukey 1977) freed certain kinds of data analysis from ties to probability-based models, so that the analysis of data could begin to acquire status as an independent intellectual activity. The introduction of simple data tools, such as stem and leaf plots and boxplots, paved the way for students at all levels to analyze real data interactively without having to spend hours on the underlying theory, calculations, and complicated procedures. Computers and new pedagogies would later complete the "data revolution" in statistics education.

In the 1990s, there was an increasingly strong call for statistics education to focus more on statistical literacy, reasoning, and thinking. Wild and Pfannkuch (1999) provided an

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empirically based comprehensive description of the processes involved in the statisticians' practice of data-based inquiry from problem formulation to conclusions. One of the main arguments presented is that traditional approaches to teaching statistics focus on skills, procedures, and computations, which do not lead students to reason or think statistically.

These changes are implicated in a process of democratization that has broadened and diversified the backgrounds and motivations of those who learn statistics at many levels with very diverse interests and goals. There is a growing recognition that the teaching of statistics is an essential part of sound education since the use of data is increasingly common in science, society, media, everyday life, and almost any profession.

A Focus on Statistical Literacy and Reasoning

The goal of teaching statistics is to produce statistically educated students who develop statistical literacy and the ability to reason statistically. Statistical literacy is the ability to interpret, critically evaluate, and communicate about statistical information and messages. Statistically literate behavior is predicated on the joint activation of five interrelated knowledge bases – literacy, statistical, mathematical, context, and critical – together with a cluster of supporting dispositions and enabling beliefs (Gal 2002). Statistical reasoning is the way people reason with the "big statistical ideas" and make sense of statistical information during a data-based activity. Statistical reasoning may involve connecting one concept to another (e.g., center and spread) or may combine ideas about data and chance. Statistical reasoning also means understanding and being able to explain statistical processes and being able to interpret statistical results.

The "big ideas" of statistics that are most important for students to understand and use are data, statistical models, distribution, center, variability, comparing groups, sampling and sampling distributions, statistical inference, and covariation. Additional important underlying concepts are uncertainty, randomness, evidence strength, significance, and data production (e.g., experiment design). In the past few years, researchers have been developing ideas of *informal* statistical reasoning in students as a way to build their conceptual understanding of the foundations of more formal ideas of statistics (Garfield and Ben-Zvi 2008).

What Does Research Tell Us About Teaching and Learning Statistics?

Research on teaching and learning statistics has been conducted by researchers from different disciplines and focused on students at all levels. Common faulty heuristics, biases, and misconceptions were found in adults when they make judgments and decisions under uncertainty, e.g., the representativeness heuristic, law of small numbers, and gambler's fallacy (Kahneman et al. 1982). Recognizing these persistent errors, researchers have explored ways to help people correctly use statistical reasoning, sometimes using specific methods to overcome or correct these types of problems.

Another line of inquiry has focused on how to develop good statistical reasoning and understanding, as part of instruction in elementary and secondary mathematics classes. These studies revealed many difficulties students have with concepts that were believed to be fairly elementary such as data, distribution, center, and variability. The focus of these studies was to investigate how students begin to understand these ideas and how their reasoning develops when using carefully designed activities assisted by technological tools (Shaughnessy 2007).

A newer line of research is the study of preservice or practicing teachers' knowledge of statistics and probability and how that understanding develops in different contexts. The research related to teachers' statistical pedagogical content knowledge suggests that this knowledge is in many cases weak. Many teachers do not consider themselves well prepared to teach statistics nor face their students' difficulties (Batanero et al. 2011).

The studies that focus on teaching and learning statistics at the college level continue to point out the many difficulties tertiary students have in learning, remembering, and using statistics and point to some modest successes. These studies also serve to illustrate the many practical problems faced by college statistics instructors such as how to incorporate active or collaborative learning in a large class, whether or not to use an online or "hybrid" course, or how to select one type of software tool as more effective than another. While teachers would like research studies to convince them that a particular teaching method or instructional tool leads to significantly improved student outcomes, that kind of evidence is not actually available in the research literature. However, recent classroom research studies suggest some practical implications for teachers. For example, developing a deep understanding of statistics concepts is quite challenging and should not be underestimated; it takes time, a well thought-out learning trajectory, and appropriate technological tools, activities, and discussion questions.

Teaching and Learning

As more and more students study statistics, teachers are faced with many challenges in helping these students succeed in learning and appreciating statistics. The main sources of students' difficulties were identified as: facing statistical ideas and rules that are complex, difficult, and/or counterintuitive, difficulty with the underlying mathematics, the context in many statistical problems may mislead the students, and being uncomfortable with the messiness of data, the different possible interpretations based on different assumptions, and the extensive use of writing and communication skills (Ben-Zvi and Garfield 2004).

The study of statistics should provide students with tools and ideas to use in order to react intelligently to quantitative information in the world around them. Reflecting this need to improve students' ability to reason statistically, teachers of statistics are urged to emphasize statistical reasoning by providing explicit attention to the basic ideas of statistics (such as the need for data, the importance of data production, the omnipresence of variability); focus more on data and concepts, less on theory, and fewer recipes; and foster active learning (Cobb 1992). These recommendations require changes of teaching statistics in *content* (more data analysis, less probability), *pedagogy* (fewer lectures, more active learning), and *technology* (for data analysis and simulations) (Moore 1997).

Statistics at school is usually part of the mathematics curriculum. New K–12 curricular programs set ambitious goals for statistics education, including promoting students' statistical literacy, reasoning, and understanding (e.g., NCTM 2000). These reform curricula weave a strand of *data handling* into the traditional school mathematical strands (number and operations, geometry, algebra). Detailed guidelines for teaching and assessing statistics at different age levels complement these standards. However, school mathematics teachers, which are often not versed in statistics, find it challenging to teach data handling in accordance with these recommendations.

In order to face this challenge and promote statistical reasoning, good instructional practice consists of implementing inquiry or project-based learning environments that stimulate students to construct meaningful knowledge. Garfield and Ben-Zvi (2009) suggest several design principles to develop students' statistical reasoning: focus on developing central statistical ideas rather than on presenting set of tools and procedures; use real and motivating data sets to engage students in making and testing conjectures; use classroom activities to support the development of students' reasoning; integrate the use of appropriate technological tools that allow students to test their conjectures, explore and analyze data, and develop their statistical reasoning; promote classroom discourse that includes statistical arguments and sustained exchanges that focus on significant statistical ideas; and use assessment to learn what students know and to monitor the development of their statistical learning, as well as to evaluate instructional plans and progress.

Technology has changed the way statisticians work and has therefore been changing what and how statistics is taught. Interactive data visualizations allow for the creation of novel representations of data. It opens up innovative possibilities for students to make sense of data but also place new demands on teachers to assess the validity of the arguments that students are making with these representations and to facilitate conversations in productive ways. Several types of technological tools are currently used in statistics education to help students understand and reason about important statistical ideas. However, using technological tools and how to avoid common pitfalls are challenging open issues (Biehler et al. 2013).

These changes in the learning goals of statistics have led to a corresponding rethinking of how to assess students. It is becoming more common to use alternative assessments such as student projects, reports, and oral presentations than in the past. Much attention has been paid to assess student learning, examine outcomes of courses, align assessment with learning goals, and alternative methods of assessment.

For Further Research

Research in statistics education has made significant progress in understanding students' difficulties in learning statistics and in offering and evaluating a variety of useful instructional strategies, learning environments, and tools. However, many challenges are still ahead of statistics education, mostly in transforming research results to practice, evaluating new programs, planning and disseminating high-quality assessments, and providing attractive and effective professional development to mathematics teachers (Garfield and Ben-Zvi 2007). The ongoing efforts to reform statistics instruction and content have the potential to both make the learning of statistics more engaging and prepare a generation of future citizens that deeply understand the rationale, perspective, and key ideas of statistics. These are skills and knowledge that are crucial in the current information age of data.

Cross-References

- ▶ Inquiry-Based Mathematics Education
- ▶ Mathematical Literacy
- ▶ Probability Teaching and Learning

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Deaf Children, Special Needs, and Mathematics Learning

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Keywords

Deaf children; Special needs; Mathematics difficulty

Characteristics

The aim of mathematics instruction in primary school is to provide a basis for thinking mathematically about the world. This is as basic a skill as literacy in today's world. Mathematical knowledge is also a means to achieve better employment and to enter higher education. For all these reasons, it is of great importance that deaf children have adequate access to mathematical thinking, but unfortunately most deaf children show a severe delay in mathematics learning. This delay has been persistent over many years. The average score in mathematics achievement tests for deaf children in the age range 8–15 in a study carried out in 1965 showed that they were one standard deviation below the average for hearing children, a result replicated about three decades later. This means that about 50 % of the deaf pupils perform similarly to the weakest 15 % of the hearing pupils. Later results continue to confirm this weak performance. In the UK, deaf students aged 16–17 years, at the end of compulsory school, were found to have a mathematical age between 10 and 12.5 years. In the USA, the mathematical ability of 80 % of the deaf 14-year-olds was described as "below basic" in problem solving and knowledge of mathematical procedures. A recent systematic review confirmed these findings (Gottardis et al. 2011) and analyzed individual differences among deaf children.

This serious and persistent difficulty is not universal among children who are deaf; approximately 15 % perform at age appropriate levels. The successful minority indicates that deafness is not a direct cause of difficulty in mathematics learning (see Nunes 2004, for a discussion). This article considers what is involved in learning mathematics in primary school, why deaf children may be at a disadvantage, and how schools can support their learning of mathematics.

Learning Mathematics in Primary School

In order to think mathematically, people need to learn to represent quantities, relations, and space using culturally developed and transmitted thinking tools, such as oral and written number systems, graphs, and calculators.

Some researchers argue that numerical concepts have a neurological basis that is independent of language learning, without which learning mathematics is extremely difficult. In view of the pervasiveness of deaf children's mathematical difficulties, it could be hypothesized that they have an inadequate development of such concepts. Basic numerical cognition has been studied in research with young deaf children as well as adults, and the hypothesis has been discarded. Deaf children and adults performed at least as well as their hearing counterparts in such tasks.

The possible consequences of delays in the acquisition of other language-based numerical concepts have also been explored. Two examples are knowledge of counting and understanding of arithmetic operations.

Counting

Deaf children lag behind hearing children in learning to count, independently of whether they are learning to count orally or in sign (Leybaert and Van Cutsem 2002). Consequently, they perform less well than hearing children on school-entry numeracy tests, which typically include tasks that require counting (e.g., "show me 5 blocks"; "tell me which number is bigger"). This delay could be related to the well-established finding that deaf people perform less well than hearing people on serial learning tasks, in which words or gestures must be learned in an exact sequence, just as the number string. However, they perform better if the tasks are presented differently and use spatial cues to organize the information. These findings are provocative rather than conclusive. First, they raise the possibility that deaf children could learn to count more easily if appropriate visual and spatial methods were used for teaching rather than serial learning instruction. Second, serial learning is not an appropriate description of counting skills beyond a certain number (about 20 or 30 in English but this may differ depending on the counting system). Research with hearing and deaf children shows that counting is a structured activity: for example, errors are more likely to occur at the boundaries between decades (e.g., . . .38, 39, 50, 51, 52. . .) than within decades. Therefore, in principle deaf children's initial disadvantage in counting could be overcome with appropriate teaching methods and with support for mastery of the structure of the system. However, it is possible that their initial struggle with learning to count lowers adults' expectations about what they can learn in mathematics, resulting in less stimulation on mathematical tasks, and that it also interferes with the children's own discoveries in the domain of mathematical reasoning.

Early Mathematical Reasoning and Arithmetic Operations

The development of mathematical reasoning starts before school, when children solve practical problems using actions, which they learn to combine with counting. When most children start school (at age 5 or 6), they can already solve simple addition and subtraction problems by putting together or separating objects and counting, and some can also solve multiplication and division problems. By counting, children use explicit numerical representation both for thinking and communicating. When numbers are small and the children can use objects, deaf children do as well as hearing children in solving these problems, but if the numbers go above 10 or 20, most deaf children fall behind. When they are compared with hearing children of the same counting ability, they are just as competent in solving numerical tasks (Leybaert and Van Cutsem 2002), but their disadvantage in counting is reflected in their problem-solving skills when they are compared to same-age hearing peers. Thus, it is possible that, not knowing number words well enough to support their mathematical reasoning, they do not discover how to use counting to solve simple arithmetic problems or important ideas for their later success, such as the inverse relation between addition and subtraction. However, Nunes and colleagues (2008a, b) have shown that relatively small amounts of teaching can effectively improve young deaf children's performance in the mathematical reasoning and arithmetic tasks, with which they were struggling before the teaching.

Conclusion

There is little doubt that many deaf children show severe and persistent difficulties in learning mathematics. Evidence suggests that there is no direct connection between deafness and problems with basic number concepts that precede language. However, deaf children lag behind hearing children in learning to count, whether orally or in sign, and at school entry they are behind their hearing counterparts in mathematical knowledge. It is possible that falling behind in counting places deaf children at a disadvantage from the adults' perspective and that they end up receiving less stimulation to solve mathematical problems early on. It is also possible that their own informal mathematical knowledge is limited by their difficulty in representing quantities explicitly with number words. These findings and conclusions suggest that, if parents and preschool teachers could find visual and spatial ways to teach counting to deaf children, one would see positive changes in the average achievement of deaf children in mathematics in the future.

Cross-References

- ▶ Blind Students, Special Needs, and Mathematics Learning
- ▶ Concept Development in Mathematics Education
- ▶ Inclusive Mathematics Classrooms
- ▶ Language Disorders, Special Needs and Mathematics Learning

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Deductive Reasoning in Mathematics Education

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Keywords

Logic; Reasoning; Proof; History of proof; Proof schemes

Definition

This entry examines the different facets of deductive reasoning with respect to the learning and teaching of mathematical proof. Deductive reasoning may be defined as a formal way of reasoning, usually top-down [from the general to the particular] with adherence to logical consistency.

Characteristics of Deductive Reasoning

The examinations of the learning and teaching of proof are multifaceted. They address a broad range of factors: mathematical, historical-epistemological, cognitive, sociological, and instructional. Research questions involving these factors include the following:

Mathematical and Historical-Epistemological Factors

1. What is proof and what are its functions?
2. How are proofs constructed, verified, and accepted in the mathematics community?
3. What are some of the critical phases in the development of proof in the history of mathematics?

Cognitive Factors

4. What are students' current conceptions of proof?
5. What are students' difficulties with proof?
6. What accounts for these difficulties?

Instructional-Sociocultural Factors

7. Why teach proof?
8. How should proof be taught?
9. How are proofs constructed, verified, and accepted in the classroom?
10. What are the critical phases in the development of proof with the individual student and within the classroom as a community of learners?
11. What classroom environment is conducive to the development of the concept of proof with students?
12. What form of interactions among the students and between the students and the teacher can foster students' conception of proof?
13. What mathematical activities – possibly with the use of technology – can enhance students' conceptions of proof?
14. How is proof currently being taught?
15. What do teachers need to know in order to teach proof effectively?

Theoretical Factors

16. What theoretical tools seem suitable for investigating and advancing students' conceptions of proof?

One's investigation of these questions is greatly influenced by her or his philosophical orientation to the processes of learning and teaching and would reflect her or his conclusion to questions such as the following: What bearing, if any, does the epistemology of proof in the history of mathematics have on the conceptual development of proof with students? What bearing, if any, does the way mathematicians construct proofs have on instructional treatments of proof? What bearing, if any, does everyday justification and argumentation have on students' proving behaviors in mathematical contexts?

Historical-Epistemological Developments

Deductive reasoning is a mode of thought commonly characterized as a sequence of propositions where one must accept any of the propositions to be true if he or she has accepted the truth of those that preceded it in the sequence. This mode of thought was conceived by the Greeks more than twentieth centuries ago and is still dominant in the mathematics of our days. So remarkable is the Greeks' achievement that their mathematics became a historical mark to which other kinds of mathematics are compared. The nature of deductive reasoning varies throughout history (Kleiner 1991). Of particular contrast is Greek mathematics versus modern mathematics. In Greek mathematics, the particular entities under investigation are idealizations of experiential spatial realities and so also are the propositions on the relationships among these entities. Logical deduction came to be central in the reasoning process, and it alone necessitated and cemented the geometric edifice they created. In constructing their geometry, as is depicted in Euclid's Elements, the Greeks had only one model in mind – that of imageries of idealized physical reality. From the vantage point of modern mathematics, neither the primitive terms nor the axioms in Greek mathematics were variables, but constants referring to a single spatial model (Klein 1968; Wilder 1967), as is expressed in the ideal world of Plato's philosophy. In modern mathematics, on the other hand, primary terms and axioms are open to different possible realizations. An important manifestation of this revolution is the distinction between Euclid's Elements and Hilbert's Grundlagen. The latter characterizes a structure that fits different models, that is, in an abstraction of numerous models, such as the Euclidean space, the surface of a half-sphere and the ordered pairs and triples of real numbers, including the interpretation that the axioms, are meaningless formulas.

Considerations of historical-epistemological developments led to new research questions with direct bearing on the learning and teaching of proofs. For example, to what extent and in what ways is the nature of the content intertwined with the nature of proving? In geometry, for example, does students' ability to construct an image of a point as a dimensionless geometric entity impact their ability to develop the Greek conception of proof? What is the cognitive or social mechanism by which deductive proving can be necessitated for the students? The Greek's construction of their geometric edifice seems to have been a result of their desire to create a consistent system that was free from paradoxes. Would paradoxes of the same nature create a similar intellectual need with students? Students encounter difficulties in moving empirical reasoning to deductive reasoning, particularly from the Greek's conception of proof to the modern conception of proof. Exactly what are these difficulties? What role does the emphasis on form rather than content in modern mathematics (as opposed to Greek mathematics, where content is more prominent) play in this transition?

Classifications of Conceptualizations of Proof

Harel and Sowder (1998) call these conceptualizations proof schemes, which they classify into a system of subcategories. Their taxonomy is organized around three main classes of categories: the *external conviction proof schemes* class, the *empirical proof schemes* class, and the *deductive proof schemes* class. A partial description of these classes follows.

External Conviction Proof Schemes

Proving within the *external conviction* proof schemes class depends either (a) on an authority such as a teacher or a book, (b) on strictly the appearance of the argument (e.g., proofs in geometry must have a two-column format), or (c) on symbol manipulations, with the symbols or the manipulations having no potential coherent system of referents (e.g., quantitative and spatial) in the eyes of the student. Accordingly, the *external conviction* proof schemes class consists of three categories: the *authoritarian proof scheme* category, the *ritual proof scheme* category, and the *non-referential symbolic proof scheme* category.

Empirical Proof Schemes

Schemes in the *empirical proof scheme* class are marked by their reliance on either (a) an evidence from examples (sometimes just one example) of direct measurements of quantities, substitutions of specific numbers in algebraic expressions, etc., or (b) perceptions. Accordingly, this class consists of two categories: the *inductive proof scheme* category and the *perceptual proof scheme category*.

Deductive Proof Schemes

The *deductive proof schemes* class consists of two subcategories, each consisting of various proof schemes: the *transformational proof scheme* category and the *axiomatic proof scheme* category.

Classifications of Functions of Proof

In general, the empirical proof schemes and the deductive proof schemes categories correspond to what Bell (1976) calls *empirical justification* and *deductive justification* and Balacheff (1988) calls *pragmatic* justifications and *conceptual justifications*, respectively. *Pragmatic justification* is further divided into three categories: *naïve empiricism* (justification by a few random examples), *crucial experiment* (justification by carefully selected examples), and *generic example* (justification by an example representing salient characteristics of a whole class of cases). *Conceptual justification* is divided into two categories: *thought experiment*, where the justification is disassociated from specific examples, and *symbolic calculation*, where the justification is based solely on transformation of symbols.

These taxonomies are not explicit enough about many critical functions of proof within mathematics. There is a need to point to these functions due to their importance in mathematics in general and to their instructional implications in particular. The work by Hanna (1990), Balacheff (1998), Bell (1976), Hersh (1993), and de Villiers (1999) explicitly address these functions. De Villiers, who built on the work of the others scholars mentioned here, raises two important questions about the role of proof: (a) "What different functions does proof have within mathematics itself?" and (b) "how can these functions be effectively utilized in the classroom to make proof a more meaningful activity?" According to de Villiers, mathematical proof has six not mutually exclusive roles: ***Verification*** refers to the role of proof as a means to demonstrate the truth of an assertion according to a predetermined set of rules of logic and premises – the *axiomatic proof scheme*. ***Explanation*** is different from verification in that for a mathematician it is usually insufficient to know only that a statement is true. He or she is likely to seek insight into why the assertion is true. "Mathematicians routinely distinguish proofs that merely demonstrate from proofs which explain" (Steiner 1978, p. 135). For many, the role of mathematical proofs goes beyond achieving certainty – to show that something is true; rather, "they're there to show... why [an assertion] is true," as Gleason, one of solvers of the solver of Hilbert's Fifth Problem (Yandell 2002, p. 150), points out. Two millennia before him, Aristotle, in his *Posterior Analytic*, asserted, "... We suppose ourselves to possess unqualified scientific knowledge of a thing, as opposed to knowing it in the accidental way in which the sophist knows, when we think that we know the cause on which the fact depends as the cause of the fact and of no other" (p. 4). ***Discovery*** refers to the situations where through the process of proving, new results may be discovered. For example, one might realize that some of the statement conditions can be relaxed, thereby generalizing the statement to a larger class of cases. Or, conversely, through the proving process, one might discover counterexamples to the assertion, which, in turn, would lead to a refinement of the assertion by adding necessary restrictions that would eliminate counterexamples. ***Systematization*** refers to the presentation of verifications in organized forms, where each result is derived sequentially from previously established results, definitions, axioms, and primary terms. ***Communication*** refers to the social interaction about the meaning, validity, and importance of the mathematical knowledge offered by the proof produced. ***Intellectual Challenge*** refers to the mental state of self-realization and fulfillment one can derive from constructing a proof.

Students' Proof Schemes

Status studies on students' conceptualization of proof show the absence of the deductive proof scheme and the pervasiveness of the empirical proof scheme among students. Students base their responses on the appearances in drawings, and mental pictures alone constitute the meaning of geometric terms. They justify mathematical statements by providing specific examples, not able to distinguish between inductive and deductive arguments. Even more able students may not understand that no further examples are needed, once a proof has been given. Students' preference for proof is ritualistically and authoritatively based. For example, when the stated purpose was to get the best mark, they often felt that more formal – e.g., algebraic – arguments might be preferable to their first choices. These studies also show a lack of understanding of the functions of proof in mathematics, often even among students who had taken geometry and among students for whom the curriculum pays special attention to conjecturing and explaining or justifying conclusions in both algebra and geometry. They believe proofs are used only to verify facts that they already know and have no sense of a purpose of proof or of its meaning. Students have difficulty understanding the role of counterexamples; many do not understand that one counterexample is sufficient to disprove a conjecture. Students do not see any need to prove a mathematical proposition, especially those they considered to be intuitively obvious. This is the case even in a country like Japan where the official curriculum emphasizes proof. They view proof as the method to examine and verify a later particular case. Finally, the studies show that students have difficulty writing valid simple proofs and constructing, or even starting, simple proofs. They have difficulty with indirect proofs, and only a few can complete an indirect proof that has been started.

Impact of Instruction

Students who receive more instructional time on developing analytical reasoning by solving unique problems fare noticeably better on overall test scores. Likewise, students who have been expected to write proofs and who have had classes that emphasized proof were somewhat better than other students. It also seems possible to establish desirable sociomathematical norms relevant to proof, through careful instruction, often featuring the student role in proof-giving. There has been the concern that the ease with which technology can generate a large number of examples naturally could undercut any student-felt need for deductive proof schemes. Several studies have shown that with careful, nontrivial planning and instruction over a period of time, progress toward deductive proof schemes is possible in technology environments, where such desiderata as making conjectures and definitions occur.

Cross-References

▶ Abstraction in Mathematics Education
▶ Argumentation in Mathematics
▶ Argumentation in Mathematics Education
▶ Logic in Mathematics Education

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22q11.2 Deletion Syndrome, Special Needs, and Mathematics Learning

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Keywords

Genetic disorder; Mathematics difficulties; Cognitive impairment

Characteristics

Chromosome 22q11.2 deletion syndrome (22q) is the most common genetic deletion syndrome with an estimated prevalence of between one in 3,000 and 6,000 births (e.g., Kobrynski and Sullivan 2007). It has only been detectable with 100 % accuracy since 1992 using techniques such as the FISH test (fluorescence in situ hybridization). Prior to identification of a single associated deletion, the syndrome had been given a number of different labels according to the primary medical condition, for example, velocardiofacial syndrome, DiGeorge syndrome, Cayler syndrome, Shprintzen syndrome, and Catch 22.

The majority of individuals with 22q experience some degree of learning difficulty and generally show a marked imbalance in performance across different subtests within IQ batteries. Verbal IQ scores are usually significantly higher than performance IQ scores (e.g., Moss et al. 1999; Wang et al. 2007). The majority of children will receive some form of support at school although some individuals experience no difficulties at all. Indeed a very wide level of individual differences in attainment in individuals with 22q is noted in all studies to date.

There is consistent evidence that mathematics skills are weaker than literacy skills in the majority children with 22q. This profile is unusual as children with mathematics difficulties are often reported to have comorbid reading difficulties. Typically, performance on standardized tests of reading and spelling is within the normal range, but performance on mathematical reasoning and arithmetic tasks is at least one standard deviation below age norms in children with 22q. Children with 22q specifically selected to have full scale IQ of at least 70 also demonstrate this profile, thereby suggesting that it is associated with 22q per se rather than low general ability.

There are very few studies examining number skills in detail in children with 22q. De Smedt et al. (2006, 2007a, b) tested children, selected to have an IQ of more than 70, on a series of computerized tests assessing performance in number reading and writing, number comparison, counting, and single and multi-digit arithmetic. A mathematical word-solving task was also included and reading ability was measured. Children were individually matched with typically developing children from the same class at school for gender, age, and parental education level. Consistent with their hypotheses, De Smedt et al. (2007a, b) report group differences on multi-digit operations involving a carry, word-solving problems, and speed in judging the relative value of two digits. There was no difference in reading, number reading and writing, single digit addition, or verbal and dot counting accuracy. The difficulties with multi-digit operations are unsurprising given the visuospatial requirements of operations such as borrowing and carrying. Previous researches suggest that multi-digit arithmetic is an area of particular difficulty in children with visuospatial learning disability as well as arithmetic difficulties (Venneri et al. 2003). More research is needed to further uncover the nature of the mathematical difficulties experienced by children with 22q and to aim to uncover best practice methods for teaching number skills in 22q as so far, certainly in the UK, no consensus has been reached.

Cross-References

- ▶ Autism, Special Needs, and Mathematics Learning
- ▶ Deaf Children, Special Needs, and Mathematics Learning
- ▶ Down Syndrome, Special Needs, and Mathematics Learning
- ▶ Inclusive Mathematics Classrooms
- ▶ Language Disorders, Special Needs and Mathematics Learning
- ▶ Learning Difficulties, Special Needs, and Mathematics Learning

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Design Research in Mathematics Education

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Keywords

Engineering research; Design experiments; Design research

Definition

Design-based research is a formative approach to research, in which a product or process (or "tool") is envisaged, designed, developed, and refined through cycles of enactment, observation, analysis, and redesign, with systematic feedback from end users. In education, such tools might, for example, include innovative teaching methods, materials, professional development programs, and/or assessment tasks. Educational theory is used to inform the design and refinement of the tools and is itself refined during the research process. Its goals are to create innovative tools for others to use, describe and explain how these tools function, account for the range of implementations that occur, and develop principles and theories that may guide future designs. Ultimately, the goal is *transformative*; we seek to create new teaching and learning possibilities and study their impact on teachers, children, and other end users.

The Origins and Need for Design Research

Educational research may broadly be categorized into three groups: the *humanities* approach, scholarly study that generates fresh insights through critical commentary, the *scientific* approach that analyzes phenomena empirically to better understand how the world works, and the *engineering* approach that not only seeks to understand the status quo but also attempts to use existing knowledge to systematically develop "high-quality solutions to practical problems" (Burkhardt and Schoenfeld 2003). Design research falls into this "engineering" category and, as such, seeks to provide the tools and processes that enable the end users of mathematics education (teachers and students, administrators and politicians) to tackle practical problems in authentic settings.

Design research is an unsettled construct and the field is in its youth. It is only at the beginning of the last two decades that we see design research as an emerging paradigm for the study of learning through the systematic design of teaching strategies and tools. The beginnings of this movement, at least in the USA, are usually attributed to Brown (1992) and Collins (1992), though in a sense, it was an idea waiting to be named (Schoenfeld 2004). In Europe there have long been traditions of principled design-based research under other guises, such as curriculum development and didactical engineering (e.g., Bell 1993; Brousseau 1997; Wittmann 1995).

Prior to the 1990s, much educational and psychological research had relied heavily on quasi-experimental studies that had been developed successfully in other fields such as agriculture. These involved experimental and control treatments to evaluate whether or not particular variables were associated with particular outcomes. In mathematics education, for example, one might design a novel approach to teaching a particular area of content, assign students to an experimental or control group, and assess their performance on some defined measures, using pre- and posttesting. Though sounding straightforward, this practice proved highly problematic (Schoenfeld 2004): the goals of education are more complex than the mastery of specific skills; the control of variables in naturalistic settings is often impossible, undesirable, and sometimes even unethical; and much of the theory is "emergent," only becoming apparent as one engages in the research.

In the early 1990s, a number of researchers began to question the limitations of traditional experimental psychology as a paradigm for educational research. Brown's paper on "design experiments" was seminal (Brown 1992). Brown recounts how her own research moved away from laboratory settings towards naturalistic ones in which she attempted to transform classrooms from "worksites under the management of teachers into communities of learning." She vividly recounts her own struggles in reconceptualizing her focus and methodology, deconstructing methodological criticisms against it (such as the Hawthorne effect). Interestingly Brown still saw the need for lab-based research, both to precede and stimulate work in naturalistic settings and also for the closer study of phenomena that had arisen in those settings. At about the same time, Collins (1992, pp. 290–293) began to argue for a design science in education, distinguishing *analytic* sciences (such as physics or biology) as where research is conducted in order to *explain* phenomena from *design* sciences (such as aeronautics or acoustics) where the goal is to determine how designed artifacts (such as airplanes or concert halls) behave under different conditions. He argued strongly for the need of the latter in education. In mathematics education, such designed artifacts might include, for example, new teaching methods, materials, professional development programs, assessment tasks, or any combination of these.

Since that time, "design research" has become more widespread and respectable in education. However it must be said that not all so-called "design research" studies satisfy the definition described above. Some, for example, do not satisfy the requirement that the designs should be theory-based and develop theory, while others do not move beyond the early stages and test their designs in the hands of others not involved in the development process.

Characterizing Design-Based Research

There have been many attempts to characterize design-based research (Barab and Squire 2004; Bereiter 2002; Cobb et al. 2003; DBRC 2003, p. 5; Kelly 2003; Lesh and Sriraman 2010;

Swan 2006, 2011; van den Akker et al. 2006). While design research is still in its infancy and its characterization is far from settled, most researchers do seem to agree that design-based research is:

Creative and Visionary

The researcher identifies a problem in a defined context and, drawing on prior research, envisions a tool that might help end users to tackle it. A draft design is developed, possibly with the assistance of end users. For example, the researcher identifies a particular student learning need and uses research to design a series of lessons. The ultimate aim is to produce an effective design, an account of the theory and principles underpinning the design, and an analysis of the range of ways in which the design functions in the hands of a typical sample of the target population of teachers and students.

Ecologically Valid

The researcher studies and refines the design in authentic settings, such as classrooms. This precludes the prior manipulation of variables in the study. It is important, therefore, to distinguish those aspects of the design that are being studied from those that are extraneous.

Interventionist and Iterative

The role of the researcher evolves as the research proceeds. During early iterations, the design is usually sketchy and the researcher needs to intervene to make it work. With teaching materials, for example, this phase may be conducted with small samples of students. Later, as the design evolves, the researcher holds back, in order to see how the design functions in the hands of end users. Early iterations are often conducted in a few favorable contexts. Early drafts of teaching materials, for example, may be tested in carefully chosen classrooms with confident teachers, in order to gain insights into what is possible with faithful implementation. Later iterations aim to study how the design functions in a wider range of authentic contexts, with teachers who have not been involved in the design process. Under these conditions, "design mutations" invariably occur. Rather than viewing these as negative, interfering factors, the designs and theories evolve to explain these mutations. With each cycle of the process, the sample size is increased and becomes more typical of the target population. From time to time, a particular issue may arise that the researcher wants to study closely. In such a case, it is possible to go back to the small-scale study of that isolated issue.

Theory-Driven

The outputs of design research include developing theories about learning, interventions, and tools. Rather than focusing on learning outcomes, using pre- and posttests, the research seeks to understand *how* designs function under different conditions and in different classroom contexts. The theories that evolve in this way are *local* and *humble* in scope and should not be judged by their claims to "truth" but rather their claims to be useful (Cobb et al. 2003). Theory in design research usually focuses on an explanation of how and why a particular design feature works in a particular way. It is both specific and generative in that it can be used to predict ways in which future designs will function if they embody this feature.

Some Issues and Challenges

Design research done well requires great skill on the part of researchers. Indeed, the combination of skills required is not usually found in individuals but in teams. A design research team will typically involve people with knowledge of the literature (researchers), an understanding of pedagogy (teachers), creative "care and flair" (designers), and facility with "delivering" the design (publishers IT technicians).

Secondly, design research often takes a great deal longer than other forms of research. There is often a significant "entry fee" in terms of time and energy taken up with producing a prototype before any study of it can begin. This is particularly true if the design involves creating new software. Then, each cycle of design, implementation, analysis, and redesign can each occupy weeks, if not months.

Thirdly, design research is data rich. A mixture of qualitative and quantitative methods is used to develop a rich description of the way the design works as well as the kinds of learning outcomes that may be expected. This often results in a proliferation of data. Brown, for example, found that she "had no room to store all the data, let alone time to score it" (Brown 1992, p. 152). Data may include lesson observations, videos of the designs in use, and questionnaires and interviews with users. In early iterations, observation plays a dominant role. Later, however, more indirect means are also needed as the sample size grows. Reliability may be improved through the use of triangulation from multiple data sources and repetition of analyses across cycles of implementation and through the use of standardized measures.

Fourthly, design research requires discipline. It is all too tempting to turn a "good idea" into a draft design and then ask someone to try it out to "see what happens." Good design-based research is more than formative evaluation, however; it is theory-driven. In preparation for a design-based research study, one must try to articulate the theory and draw clear lines of connection between this and the design itself. This may be done by eliciting "principles" to direct the design. The research involves putting these principles in "harms way" (Cobb et al. 2003). Then, the focus of the research needs to be articulated. For early iterations this may be on the potential impact of the faithful use of the design, while on later iterations, we may be more interested in refining the design by studying end users' interpretations and mutations.

Finally, writing up design research is problematic. Most designs are too extensive to be described and analyzed in traditional journal articles that emphasize methods and results over tools. Recently e-journals have begun to appear that allow for a much clearer articulation of design-based research. These, for example, allow extensive extracts of teaching and professional development materials to be displayed, along with videos of the designs in use (see for example, http://www.educationaldesigner.org).

Cross-References

▶ Curriculum Resources and Textbooks in Mathematics Education
▶ Mathematics Curriculum Evaluation
▶ Didactic Engineering in Mathematics Education
▶ Didactic Situations in Mathematics Education

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Dialogic Teaching and Learning in Mathematics Education

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Keywords

Dialogue; Literacy; Mathemacy; Investigation; Inquiry; Inquiry cooperation model; Critical mathematics education

Definition

Dialogic teaching and learning refers to certain qualities in the interaction between teachers and students and among students. The qualities concern possibilities for the students' involvement in the educational process, for establishing enquiry processes, and for developing critical competencies.

Characteristics

Sources of Inspiration

There are different sources of inspiration for bringing dialogue into the mathematics classroom, and let me just refer to two rather different.

The notion of dialogue plays a particular role in the pedagogy of Paulo Freire. He sees dialogue as crucial for developing *literacy*, which refers to a capacity in reading and writing the world: reading it, in the sense that one can interpret sociopolitical phenomena, and writing it, in the sense that one becomes able to make changes.

With explicit reference to mathematics, the crucial role of dialogue can be argued with allusion to Imre Lakatos' presentation in *Proof and Refutations* (Lakatos 1976). Here Lakatos shows that a process of mathematical discovery is of dialogic nature, characterized by proofs and refutations.

Critical mathematics education and social constructivism have developed dialogic teaching and learning through a range of examples and studies. It has been emphasized that dialogue is principal for establishing critical perspectives on mathematics and for a shared construction of mathematical notions and ideas. In fact dialogic teaching and dialogic learning represents two aspects of the same process.

Marilyn Frankenstein (1983) has emphasized the importance of Freire's ideas for developing critical mathematics education, and Paul Ernest (1998) has opened the broader perspective of social constructivism, also acknowledging the importance of Lakatos work.

The Inquiry Cooperation Model

The notion of dialogue appears to be completely open. As a consequence, it becomes important to try to characterize what a dialogue could mean. The *Inquiry Cooperation Model* as presented in Alrø and Skovsmose (2002) provides such a specification with particular references to mathematics.

This model characterizes different dialogic acts: *Getting in contact* refers to the act of tuning in at each other. *Locating* and *identifying* refer to forms of grasping perspectives, ideas, and arguments of the other. *Advocating* means providing arguments for a certain point of view – although not necessary one's own. Thinking aloud means making public details of one's thinking, for instance, through gestures and diagrams. *Reformulating* refers to particular attempts in grasping other ideas by rethinking, rephrasing, and reworking them. *Challenging* means questioning certain ideas, which is an important way of sharpening mathematical arguments. *Evaluating* refers to reflexive questioning, like: What insight might we have reached? What new questions have we encountered?

Dialogic teaching and learning can be characterized as a process rich of such dialogic acts.

New Qualities in Teaching and Learning

The idea of dialogic teaching and learning is to promote an education with new qualities. Let me refer to just a few having to do with the students' interest, making investigations, and developing a mathemacy.

Students' Interest. It has been emphasized that dialogic teaching and learning includes a sensitivity to the students' perspectives and possible interests for learning. This sensitivity has not only to do with the dialogic act of "getting in contact" but with all the acts represented by the Inquiry Cooperation Model. A principal point of dialogic teaching is to invite students into the learning process as active learners.

Making Investigations. Dialogic teaching and learning can be characterized in terms of investigative approaches, where both teacher and students participate in the same inquiry process. Barbara Jaworski (2006) makes a particular emphasis on establishing communities of inquiry, and in any such communities, dialogue plays a defining role. Landscapes of investigations (Skovsmose 2011) might also provide environments that facilitate dialogic teaching and learning.

Similar to literacy, *mathemacy* refers not only to a capacity in dealing with mathematical notions and ideas but also to a capacity in interpreting sociopolitical phenomena and acting in a mathematized society. Thus, mathemacy combines a capacity in reading and writing *mathematics* with a capacity in reading and writing *the world* (see Gutstein 2006). Dialogue teaching and learning is in hectic development, both in theory and in practice. A range of new studies and new classroom initiatives are being developed. In particular, the very notion of dialogue is in need of further development; see, for instance, Alrø and Johnsen-Høines (2012).

Cross-References

▶ Critical Mathematics Education
▶ Mathematization as Social Process

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Didactic Contract in Mathematics Education

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Keywords

Didactical situations; Mathematical situations; Didactical contract; Didactique; Milieu; Devolution; Institutionalization

Introduction

Teachers manage *didactical situations* that create and exploit *mathematical situations* where practices are exercised and students' mathematical knowledge is developed. The study of the *didactical contract* concerns the compatibility on this precise subject of the aspirations and requirements of the students, the teachers, the parents, and the society.

Definition

A "*didactical contract*" is an interpretation of the commitments, the expectations, the beliefs, the means, the results, and the penalties envisaged by one of the protagonists of a *didactical situation* (student, teacher, parents, society) for him- or herself and for each of the others, *à propos of the mathematical knowledge being taught* (Brousseau and Otte 1989; Brousseau 1997). The objective of these interpretations is to account for the actions and reactions of the partners in a didactical situation.

The didactical contract can be broken down into two parts: a contract of devolution – the teacher organizes the mathematical activity (see ► Didactic Situations in Mathematics Education) of the student who in response commits him- or herself to it – and a contract of institutionalization – the students propose their results and the teacher vouches for the part of their results that conforms to reference knowledge.

Customary practices (Balacheff 1988), whether explicit or tacit, leave the hope that divergences are accidental and reducible and that there exist real contracts, whether or not they can be made explicit, that are compatible and satisfactory. This is not so, owing to various paradoxes that became apparent in the course of teaching in a way that is based on mathematical situations. This gave rise to many questions, among them are as follows:

How could students commit themselves to the subject of knowledge that they have not yet learned?

What are the respective roles of what is inexpressible, of what is said, of what is not said or cannot be said to the other in the teaching relationship?

Does there exist knowledge that ought not to be made explicit before being learned?

The study of these questions was the origin of the theory of didactical situations.

Characteristics

Background: Illustrative Examples

These questions arose in the course of research at the COREM (Center for Observation and Research on Mathematics Education, entity formed of a laboratory and a school establishment by the IREM of the University of Bordeaux (1973–1999)) on the possibility of assigning to *mathematical situations* the job of managing what the teacher cannot say or the student cannot yet understand from a text, and in the clinical observation of students failing selectively in mathematics:

(a) *The Case of Gaël*. Gaël (8 years old) always responded in the manner of a very young child. It was not a developmental delay, but rather a posture. By replacing some lessons with "games" in which he could take a chance and see the effects of his decisions and by getting him to make bets – without too much risk – on whether his answers were right, the experimenters saw his attitude changes radically and his difficulties disappear. A new "didactical contract" with him had been constructed (Brousseau and Warfield 1998).

(b) *The Age of the Captain*. Researchers at the Institute for Research on the Teaching of Mathematics (IREM of Grenoble) offered students at age 8 the following problem: "On a boat there are 26 sheep and 10 goats. How old is the captain?" 76 of the 97 students answered, "36 years old."

This experiment produced a scandal. Some accused the teachers of stupefying their students; others reproached the researchers for "laying stupid traps for the children." In a letter to the experimenters, G. Brousseau indicated to them that it was a matter of an "effect of the contract" for which neither the students nor the teachers were responsible. So the researchers asked the students: “What do you think of this problem?” The students responded: “It is stupid!” The researchers ask: “Then why did you answer it?” The students answered: “Because the teacher asked for it!” The researchers ask: “And if the captain was 50 years old?” The students made a response: “The teacher didn’t give the right numbers.” A similar experiment done with established teachers produced the same behavior: for various reasons (such as the hope of an explanation that the teacher wanted to hear) the subjects produce the answer least incompatible with their knowledge, even when they see very well that it is false: the obligation of answering is stronger than that of answering correctly. Despite these explanations, for years the initial observation elicited strong criticisms of the work of the teachers (Sarrazy 1996).

Didactical and Ethical Responsibility

The teacher has the responsibility of supporting the collective and individual activity of the students, of attesting in the end to the truth of the mathematics that has been done, of confirming it or giving proofs, of organizing it in the standard way, of identifying errors that have been or might be made and passing judgment on them (without passing judgment on their authors), and of providing the students with a moderate amount of individual help (as with the natural learning of a language.) Occasional individual help conforms to the collective process of mathematical communities. If the teacher finds himself acceding to an institutional function, he may be subject to obligations of equity and of means for which the responsibility is shared with the institution. Decisions made about the teacher and the students based on individual and isolated results are a dangerous absurdity. Experts, parents, and society share the responsibility for the effects of such decisions.

Paradoxes of the Didactical Contract

The teacher wants to teach what she knows to a student who does not know it. This has many consequences, among them are as follows:

(a) Custom can determine pedagogical and psychological relationships, but not those proper to new knowledge, because new knowledge is a specific unexpected adventure that consists of a modification and an augmentation of old knowledge and of its implications. Thus, it cannot be known in advance by the student: the teacher can only commit himself to general procedures, and for her part the student cannot commit herself to a project of which she does not know the main part.

(b) Paradox of devolution: the knowledge and will of the teacher need to become those of the student, but what the student knows or does by the will of the teacher is not done or decided by his own judgment. The didactical contract can only succeed by being broken: the student takes the risk of taking on a responsibility from which he already releases the teacher (a paradox similar to that of Husserl).

(c) Paradox of the said and unsaid (consequence of the preceding): it is in what the teacher does not say that the student finds what she can say herself.

(d) Paradox of the actor: the teacher must pretend to discover with his students knowledge that is well known to him. The lesson is a stage production.

(e) The paradox of uncertainty: knowledge manifests itself and is learned by the reduction of uncertainty that it brings to a given situation. Without uncertainty or with too much uncertainty, there is neither adaptation nor learning. The result is that the optimal progression of normal individual or collective learning is accompanied by a normal optimal rate of errors. Artificially reducing it damages both individual and collective learning. It is useful to arrange things so that it is not always the same students who are condemned to supply the necessary errors.

(f) As in the case of learning, excessive or premature adaptation of complex knowledge to conditions that are too particular leads it to be replaced by a simplified and specific knowledge. This can then constitute an *epistemological* or *didactical obstacle* to its later adaptation to new conditions. (For example, division of natural numbers is associated with a meaning, sharing, which becomes an obstacle to understanding it in the case where a decimal number needs to be divided by a larger decimal number, e.g., 0.3/0.8.)

(g) The paradox of rhetoric and mathematics. To construct the students' mathematical knowledge and its logical organization, the teacher uses various rhetorical means, designed to capture their attention. The culture, pedagogical procedures, and even mathematical discourse (commentaries on mathematics) overflow with metaphors, analogies, metonyms, substitutions, word pictures, etc. The mathematical concepts are often constructed against these procedures (e.g., "correlation is not causation"). The teacher should thus at the same time as an educator teach the culture with its historical mistakes and as a specialist cause the rejection of the parts that science has disqualified.

These paradoxes can only be unraveled by specific situations and processes carefully planned out in the light of well-shared knowledge of mathematical and scientific *didactique* (Brousseau and Otte 1989; Brousseau 2005).

Observations of Reactions of Teachers to Difficulties

These observations and the experimental and theoretical studies of the *didactical contract* make it possible to understand and predict the cumulative effects of teachers' decisions.

The contract manifests itself essentially in its ruptures. These are revealed by the reactions of the students or by the interventions of the teachers, and they can be classified as follows:

(a) *Abandonment*. The teacher does not react to an error made by the students (e.g., because it would be too complicated to explain it), or she repeats the question identically or she gives the complete solution.

(b) The progressive *reduction* or manipulation of the students' uncertainty, using a great variety of means:
   - Bringing in mathematical, technical, or methodological information
   - Decomposition of the problem into intermediate questions (decomposition of the objectives)
   - Use of various extra-mathematical rhetorical means: analogies, metaphors, metonyms, or mnemonic minders (the "Topaze effect")

(c) *Critical commentary on the errors*, the question, the knowledge, or the material

(d) *A trial of the student* and its consequences: penalties, discrimination, and individualization

In case of failure, the contract obligates the teacher to try again. The new attempt either replaces the preceding one or criticizes and corrects it, making of it a new teaching object (a meta-process).

For each of these types of response, there are conditions under which it is the most appropriate response; *thus there is no universal response*.

For example, Novotná and Hošpesová (2007) identify and classify the behaviors whose systematic repetition generates Topaze effects:

1. Explicitly, the teacher
   (a) Gives the steps of the solution and transforms it into the execution of a sequence of tasks
   (b) Asks questions in a sequence that mandates the procedures of the solution
   (c) Gives warnings about a possible error
   (d) Enumerates previous experiences or knowledge, pointing out analogies with problems that have previously been resolved or are obvious or well known
2. Implicitly, he
   (a) Reformulates students' propositions or his own
   (b) Uses "guide" words
   (c) Pronounces the first syllable of words
   (d) Poses new questions that orient the student towards the solution
   (e) Shows doubt about dubious initiatives

Their research confirms that the resulting Topaze effects go unnoticed but have a high cost. The students, apparently active, become dependent on this aid and lose their confidence in themselves. An error is understood to be a transgression of the didactical contract and proof itself, badly supported, becomes something to be learned rather than understood.

By using jointly the notions of *milieu*, of situation, and of the didactical contract, Perrin-Glorian and Hersant (2003) were able to show in numerous examples on the one hand what the student and the *milieu* are in charge of and thus the occasions for learning that are their responsibility, and on the other hand the help brought in by the teacher.

Predicting and Explaining Certain Long-Term Effects

The uncontrolled recursive resumption of the same type of response leads to drifting and inevitable failures. For example, for the students studying the procedure for solving problems by the same pedagogical methods, studying theorems is just as costly, less sure, and less useful.

As another example, a sequence of meta-slippages contributed to the failure of the reform of "modern math": the foundations of mathematics were interpreted by "naïve" set theory, which was itself formalized into algebra. This was metaphorically represented by "graphs," which were finally interpreted in vernacular language. Each representation betrayed the preceding one slightly and supported new conventions, and the slippages were ultimately uncontrollable. In the absence of didactical situations and proven epistemological processes, varying the types of response seems to be the best strategy.

Enforcing requirements based on the results of individuals leads to a mincing up of the objectives, to the abandonment of high-level objectives, and to addressing the objectives by painful behaviorist methods. These slow the learning and lead to an individualization that slows it yet further. Each of these tends to destroy the role of provisional knowledge and to augment mechanically the time for teaching and learning without positive impact on the results.

Specifying the means of teaching a subject involves precise and specific protocols for performances that are known and accepted by the population. Specifying required results for the teachers as for the students has absolutely no scientific basis. Its disastrous effects, predicted since 1978, have been observable for 40 years.

The mean rate of success is a "regulated" variable of the system. Otherwise stated, the global progress of *all* the students is less rapid if one requires at every stage a 100 % rate of success. The conception of mathematical activity as an adventure and a collective practice makes it possible to mitigate the effects of difference in rhythms of learning.

It seems that today the requirements of the different partners of teaching towards one another are less and less compatible with each other, perhaps because of the variety of possibilities, of offers, and of perspectives provided by numerous ill-coordinated sciences.

The experiments on teaching rational and decimal numbers (Brousseau 1997) or statistics and probability (Brousseau et al. 2002) prove that it is possible to organize efficient and communicable processes with the help of didactical contracts based on the nature of the knowledge to be acquired.

Extensions

Sarrazy (1996, 1997) studied the pitfalls of these meta-didactical slippages and more particularly those that are consequences of a teaching that aims at making the contractual expectations explicit, frequently taking the form of the teaching of metacognitive or heuristic procedures – or even of algorithms for solving problems. Complementing the work engaged in by Schubauer-Léoni (1986) in a psychosocial approach to the didactical contract, Sarrazy radicalized the paradox of the consubstantial rule (A rule does not contain in itself its conditions for use) of the contract at the intersection of the theory of situations and Wittgensteinian anthropology (Wittgenstein 1953). Contrary to the psychological or linguistic interpretations of the contract (such as that of "the age of the captain"), he showed how these slippages lead to a veritable demathematization of teaching by a displacement of the goals of the contract. These works also made it possible to establish the primacy of the role of situations and that of school cultures (Sarrazy 2002; Clanché and Sarrazy 2002; Novotná and Sarrazy 2011) and family habits conceived as *backgrounds* (Searle 1979) of the didactical contract. These backgrounds make it possible to explain the differences *in sensitivity to the didactical contract*, that is, the objective differences of the various positions of the students with regard to the implicit elements of the contract and thus of their spontaneous (and not necessarily conscious or thought-out) "representations" of the division of responsibilities in the contract (e.g., some of the students answer that the captain is 36 years old, others refrain from giving an answer, still others finally say that they do not know, and some of them authorize themselves to declare that this problem is absurd). These results reaffirm the importance of the Theory of Situations and notably the explicative power of the contract, but also underline the interest of considering the pedagogical ideologies of the teachers and the cultures of the students in the interpretation of contractual phenomena. These works together lead into a perspective of study baptized "anthropo-didactique," situating the phenomena of the didactical contract in the double perspective mentioned above. This theoretical current has made it possible to reinterpret in a fertile way a certain number of phenomena of teaching (*lato sensu*), as much on the micro-didactical level as the macro-didactical, and of their interactions, such as school inequities (Sarrazy 2002), school difficulties (Clanché and Sarrazy 2002; Sarrazy and Novotna´ 2005) heterogeneities, didactical time and didactical visibility (Chopin 2011), student teacher interactions, and the effects of the *genre*. These themes have traditionally been studied by connected disciplines (psychology, sociology, anthropology, etc.) but independently of the didactical dimensions which in fact are necessarily involved in these phenomena. This approach thus realized the study of what Brousseau designated in 1991 "didactical conversions": "The causes of phenomena of a non-didactical nature can only influence didactical phenomena by the intermediary of elements having their origin in didactical theory." This "reinterpretation" of a non-didactical phenomenon in didactical terms is a didactical conversion (Brousseau and Centeno 1991, p. 186).

Cross-References

▶ Didactic Situations in Mathematics Education

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Didactic Engineering in Mathematics Education

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Keywords

Didactical engineering; Theory of didactical situations; A priori and a posteriori analysis; Research methodology; Classroom design; Development activities

Definition

In mathematics education, there exists a tradition of research giving a central role to the design of teaching sessions and their experimentation in classrooms. Didactical engineering, which emerged in the early 1980s and continuously developed since that time, is an important form taken by this tradition. In the educational community, it mainly denotes today a research methodology based on the controlled design and experimentation of teaching sequences and adopting an internal mode of validation based on the comparison between the a priori and a posteriori analyses of these. However, since its emergence, the expression didactical engineering has also been used for denoting development activities, referring to the design of educational resources based on research results or constructions and to the work of didactical engineers.

History

From its emergence as an academic field of study, mathematics education has been associated with the design and experimentation of innovative teaching practices, in terms of both mathematical content and pedagogy. The importance to be attached to design was early stressed by researchers as Brousseau and Wittman, for instance, who very early considered that mathematics education was a genuine field of research that should develop its own frameworks and practices and not just a field of application for other sciences such as mathematics and psychology.

The idea of didactical engineering (DE), which emerged in French didactics in the early 1980s, contributed to firmly establish the place of design in mathematics education research. Foundational texts regarding DE such as Chevallard (1982) make clear that the ambition of didactic research of understanding and improving the functioning of didactic systems where the teaching and learning of mathematics takes place cannot be achieved without considering these systems in their concrete functioning, paying the necessary attention to the different constraints and forces acting on them. Controlled realizations in classrooms should thus be given a prominent role in research methodologies for identifying, producing, and reproducing didactic phenomena, for testing didactic constructions. As a research methodology, DE emerged with this ambition, relying on the conceptual tools provided by the Theory of Didactical Situations (TDS), and conversely contributing to its consolidation and evolution (Brousseau 1997). It quickly became a well-defined and privileged methodology in the French didactic community, accompanying the development of research from elementary school up to university level as evidenced in the synthesis proposed at the 1989 Summer School of Didactics of Mathematics (Artigue 1990, 1992).

From the 1990s, DE migrated outside its original habitat, being extended to the design of teacher preparation and professional development sessions, used by didacticians from other disciplines, for instance, physical sciences or sports, and also by researchers in mathematics education in different countries. Simultaneously, the progressive shift of research attention towards teachers increased the use of methodologies based on naturalistic observations of classrooms, leading to theoretical developments and results that, in turn, affected DE. Moreover, design-based research perspectives emerged in other contexts, independently of DE (Design-Based Research Collaborative 2003). These evolutions and the resulting challenges are analyzed in Margolinas et al. (2011).

DE as a Research Methodology

As a research methodology, DE is classically structured into four different phases: preliminary analyses; design and a priori analysis; realization, observation, and data collection; and a posteriori analysis and validation (Artigue 1990, 2009).

Preliminary analyses usually include three main dimensions: an epistemological analysis of the mathematical content at stake, an analysis of the conditions and constraints that the DE will face, and an analysis of what educational research has to offer for supporting the design.

In the second phase, design and a priori analysis, research hypotheses are engaged in the process. Design requires a number of choices, from global to local. They determine *didactic variables*, which condition the interactions between students and knowledge, between students and between students and teachers, thus the opportunities that students have to learn. In line with TDS, in design, particular importance is attached:

To the search for *fundamental situations*, i.e., mathematical situations encapsulating the epistemological essence of the concepts

To the characteristics of the *milieu* with which the students will interact in order to maximize the potential it offers for autonomous action and productive feedback

To the organization of *devolution* and *institutionalization* processes by which the teacher, on the one hand, makes students accept the mathematical responsibility of solving the task and, on the other hand, connects the knowledge they produce to the scholarly knowledge aimed at

The a priori analysis makes clear these choices and their relation to the research hypotheses. Conjectures are made regarding the possible dynamic of the situation, students' interaction with the *milieu*, students' strategies, their evolution and their outcomes, about teacher's necessary input and role. Such conjectures regard not individuals but a *generic and epistemic student* entering the mathematical situation with some supposed knowledge background and accepting to enter the mathematical game proposed to her. The actual realization will involve students with their personal specificities and history, but the goal of the a priori analysis is not to anticipate all these personal behavior; it is to build a reference with which classroom realizations will be contrasted in the a posteriori analysis.

During the phase of realization, data are collected for the analysis a posteriori. The nature of these data depends on the precise goals of the DE, the hypotheses tested, and the conjectures made in the a priori analysis. The realization can lead to some adaptation of the design in *itinere*, especially when the DE is of substantial size. These adaptations are documented and taken into account in the a posteriori analysis.

A posteriori analysis is organized in terms of contrast with the a priori analysis. Up to what point the data collected during the realization support the a priori analysis? What are the significant convergences and divergences and how to interpret them? The hypotheses underlying the design are put to the test in this contrast. There are always differences between the reference provided by the a priori analysis and the contingence analyzed in the a posteriori analysis. The validation of the hypotheses underlying the design does not, thus impose perfect match between the two analyses. Moreover, the validation of the research hypotheses may require the collection of complementary data to those collected during the classroom, especially for appreciating the learning outcomes of the process. Statistical tools can be used, but what is essential is that validation is internal, not in terms of external comparison between control and experimental groups.

These are the characteristics of DE as research methodology when associated with the conception of a sequence of classroom sessions having a precise mathematical aim. However, as shown in Margolinas et al. (2011), this methodology has been extended to other contexts such as teacher education, more open activities such as project work or modeling activities, and even mathematical activities carried out in informal settings. In these last cases, the content of preliminary analyses must be adapted; what the design ambitions to control in terms of learning trajectories and the reference provided by the a priori analysis cannot exactly have the same nature.

Realizations

The first exemplars of DE research regarded elementary school. Paradigmatic examples are the long-term designs produced by Brousseau, on the one hand, and by Douady, in the other hand, for extending the field of numbers from whole numbers to rational numbers and decimals (Brousseau et al. 2014; Douady 1986). The two constructions were different, but they proved both to be successful in the experimental settings where they were tested, and they significantly contributed to the state of the art regarding the learning and teaching of numbers. Beyond that, they had theoretical implications. The development of the *tool-object dialectics* and the identification of the learning potential offered by the organization of *games between mathematical settings* by Douady are intrinsically linked to her DE for the extension of the number field; the idea of *obsolescence of didactic situations* emerged from the attempts made at reproducing Brousseau's DE year after year. These are only two examples among the many we could mention. DEs were progressively developed at all levels of schooling, covering a diversity of mathematical domains and addressing a diversity of research issues. At university level, for instance, paradigmatic examples remain the construction developed by Artigue and Rogalski for the study of differential equations, combining qualitative, algebraic, and numerical approaches to this topic (Artigue 1993) and that developed by Legrand for the teaching of Riemann integral within the theoretical framework of the scientific debate (Legrand 2001). Both were experimented with first year students and showed their resistance to students' diversity. Constraints met at more advanced levels of schooling contributed to the deepening of the reflection on an optimized organization of the sharing of mathematical responsibilities between students and teacher in DE and to the softening of the conditions and structures often imposed to design at more elementary levels. DE was also enriched by its use in other domains than mathematics and by researchers trained in other cultural traditions. A good example of it is provided by its use in sports, already mentioned, and by the elaboration of DE combining the theoretical support of TDS and that of semiotic approaches (cf. for instance, (Falcade et al. 2007; Maschietto 2008) using such combination for studying the educational potential of digital technologies). More globally, ICT has always been a privileged domain for DE, for exploring and testing the potential of new technologies, and for supporting technological development as well as theoretical advances in that area. Another interesting example is the use of DE within the socio-epistemological framework in mathematics education (Farfán 1997; Cantoral and Farfán 2003).

Challenges and Perspectives

DE developed as a research methodology, but DE from the beginning had also the ambition of providing a model for productive interaction between fundamental research and action on didactic systems. DEs produced by research were natural candidates for supporting such a productive interaction. Quite soon, researchers however experienced the fact that the DEs they had developed and successfully tested in experimental setting did not resist to the usual dissemination processes. This problem partly motivated the shift of interest towards teachers' representations and practices. Addressing it requires to clearly differentiate research DE (RDE) and development DE (DDE), acknowledging that these cannot obey the same levels of control. In Margolinas (2011), this issue is especially addressed by Perrin-Glorian through the idea of DE of second generation, in which the progressive loss of control that the elaboration of a DDE requires is co-organized in collaboration with teachers and illustrated by an example. Such a strategy implies a renewed conception of dissemination of research results, in line with the current evolution of vision of relationships between researchers and teachers.

Another challenge is the issue of relationships between the tradition of DE described above and the different forms of design which are developing in mathematics education under the umbrella of design-based research, reflecting the increased interest for design in the field, or the vision of design introduced in the Anthropological Theory of Didactics (ATD) in the last decade in terms of Activities of Study and Research (ASR) and Courses of Study and Research (CSR) (Chevallard 2006). Despite de fact that ATD and TSD emerged in the same culture, the visions of design they propose today present substantial differences. Establishing productive connections between the two approaches without losing the coherence proper to each of them is a problem not fully solved but also addressed in Margolinas (2011).

Cross-References

▶ Anthropological Approaches in Mathematics Education, French Perspectives
▶ Design Research in Mathematics Education
▶ Didactic Situations in Mathematics Education

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Didactic Situations in Mathematics Education

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Keywords

Didactical situation; A-didactical situation; Mathematical situation; Acculturation; Didactique

Didactical Situation

A *didactical situation* in mathematics is a project organized so as to cause one or some students to *appropriate* some piece of mathematical reference knowledge. (The organizer and the student may be individuals, a population, institutions, and so on.)

Components

Every didactical process is a sequence of situations, each pertaining to one of the following three types:

A "**situation of devolution**" in which the teacher sets the students up:

- to accept boldly and confidently the challenge of an engaging and instructive mathematical situation whose instructions he gives in advance: conditions, rules, goal, and above all the criterion for success
- and to do it without his help, on their own responsibility (Brousseau 1997, pp. 230–235)

A "**mathematical situation**" that supports the students in autonomous mathematical activities, both individual and collective, that represent those in use by mathematicians. Rather than looking to gain credit for themselves, the students are engaged in:

- Producing "new" statements and discussing their validity
- Making decisions, formulating hypotheses, predicting and judging their consequences, attempting to communicate information, producing and organizing models, arguments and proofs, etc., adequate for certain precise projects
- and evaluating and correcting by themselves the consequences of their choices

It is thus not the students who are in question, but some conjectures and some knowledge (Brousseau 1997, pp. 230–235).

A "**situation of institutionalization**" in which the teacher:

- Takes note of the progress of the mathematical situation, of the questions and answers that have been obtained or studied from it, and of those that have emerged, and places them within the perspective of the curriculum
- Distinguishes among the pieces of knowledge (*connaissances*) that have appeared those that have revealed themselves to be false and those that are correct, and among the latter those that will serve as references, presenting in that case the canonical way of formulating them
- And draws conclusions for the organization of further sequences (exercises, problems, etc.) (Brousseau 1997, pp. 235–243).

Teaching methods

Teaching methods can be distinguished first by the interpretation, the role, and the importance assigned to each of the components. Here are two very different examples of this:

Example 1: In certain methods, devolution consists of a prerequisite teaching of new knowledge (a lecture), followed by examples and exercises, and followed by the presentation of problems whose autonomous solution by the students constitutes the mathematical situation. Institutionalization consists of correction, evaluation, and the conclusions that the teacher draws from them. Sometimes the mathematical situation is considered only as a means of verifying the individual learning produced by the lecture.

Example 2: In other methods, devolution is reduced to the organization, presentation, and staging of an individual or collective *mathematical situation* aimed at provoking activities and processes like those of mathematicians: a search for solutions or proofs but also production of questions, hypotheses or conjectures, reformulations, definitions and study of objects, sorting, debates, challenges, etc. Learning is the means and the product of this activity. Institutionalization then consists of identifying and organizing, among the correct pieces of knowledge produced by the students, those consistent with common usage and with *accepted mathematical knowledge*, and among those the ones that are sufficiently "acquired" by all of the students so that the teacher and students can refer to them with each other in future mathematical situations. The "lecture" consists of a conclusion and of putting things in order. Exercises are a means of training available to the students (Margolinas et al. 2005; Illustrative examples in Warfield 2007).

Origin and Necessity of the Concept of "Didactical Mathematical Situation"

The Reform of the Foundations (1907–1980)

The term "didactical situation" appeared in the 1960s with the meaning "mathematical situation for teaching."

The new mathematical concepts on which teaching was to be rebuilt were communicated by formalized texts in a symbolic language unintelligible to students and/or by reformulations, metaphorical representations, and ambiguous commentaries. On the other hand, they referred necessarily to examples taken from the classical mathematics that they were reorganizing. The "fundamental" concepts were thereby postponed to the end of the studies.

The challenge was thus to imagine conditions, situations, that could induce in the students the geneses of fundamental mathematical concepts, in a form and by processes comparable to those put into operation by mathematicians *before* the final presentation of their results, in the process mathematical development. This idea found justification in the work of the period: the acquisition of language does not follow the classic formulation of its grammar, and Piaget identified certain mathematical structures in the genesis of logical thought in children.

Conceiving of similar geneses, and especially imagining conditions capable of inducing them, could only arise from the competence of the mathematical community. It did so through a gigantic effort of its researchers and of its teachers, realizing as it did so the aspirations of pedagogues like Dewey, Montessori, or Freinet. But diffusing these conceptions more widely, *against* the traditional culture of teaching, posed yet more redoubtable problems, which have not at this point been surmounted.

Learning Mathematics by Doing It Reverses the Classic Pedagogical Order

The teaching of mathematics is based on a text or some texts that express it in a canonical way (i.e., in the order: definitions, properties and theorems, and finally proofs). The classical conception consists of teaching using the texts first, so that a student could never argue that he or she is being required to use a piece of knowledge that was not first revealed and taught. Teaching pieces of knowledge before needing to use them gives the appearance of being a "rational" method, but it introduces a disassociation (learn with metamethods that have no relationship with the object and its use), an inversion (learn terms before understanding them and doing anything with them), and finally teleological requirements: the student is blamed in the course of learning for not having first learned what is in fact the goal of the teaching that is going on. This epistemological error greatly limits the field of application, the age of learning, and the degree of success of the classical method.

Conversely, direct *acculturation* to specific mathematical practices that can produce these texts brings their learning closer to that of vernacular language or natural thought. Everything then rests on the power of the situations to induce in the children the "process of mathematization."

It would be absurd and detrimental to want to exclude some method or to uniformly recommend it over some other. The conditions to which each is best adapted must be scientifically studied and their advantages combined. For example, situations of cooperative discovery and collective adventures create homogeneity and motivation and make it possible to acquire the classical practices by use. Exercises can help in doing well and rapidly what is worthwhile and has been understood (Brousseau 1992).

The Project of a Mathematical Science: *Didactique*

The organization of these mathematical situations and their succession obey various reasons: mathematical, epistemological, rational, empirical, ideological, etc. Their scientific study combines:

1. The (anthropological) observation and the analysis (semiological) of the practices and conceptions of the teachers and of the students
2. The conception, realization, and experimental study of original mathematical situations appropriate to each of the pieces of mathematical knowledge aimed for (▶ Didactic engineering in mathematics education)
3. The inventory of possible choices, their modeling in the form of situations, the experimental and theoretical study of their conditions and of their properties, and the creation of appropriate instruments of analysis (theory of didactical situations)

The conception of these situations requires prior and specific mathematical study of the *knowledge to be taught*, along with that of its historical genesis, of its epistemological properties, and of its possible didactical geneses and their properties. But the scientific confrontation of these speculations with actual teaching is fundamental.

The theory of situations, its concepts, and its research methods is one of the most ambitious among the numerous scientific approaches to the phenomenon of *didactique*.

But well before being able to offer teachers, in the name of mathematicians, an aid, or some ready-to-implement solutions for teaching mathematics, *didactique* must describe, understand, and explain in a scientific manner mathematical activity and its possible *didactical transpositions*.

Didactique plays a role in the reorganization and transformation of mathematical knowledge. Its results are thus first addressed to the community of mathematicians, to whom falls – for good reasons – the responsibility towards society of the reference in teaching materials to the established knowledge of its specialty. *Didactique* of mathematics requires specific concepts and methods of study. It thus joins logic, computer science, epistemology, history of mathematics, and so on as one of the mathematical sciences. It takes charge of the knowledge of everything that is specific to the discovery, the diffusion, or the appropriation of each piece of mathematical knowledge, new or not, that results from the adventures specific to it. It extends, enriches, and puts to the test the general contributions of classical social sciences, which are indispensible but insufficient for clarifying all the facets of this teaching.

Mathematical Situations

Definition

Every mathematical concept is the solution of at least one specific system of mathematical conditions, which itself can be interpreted by at least one situation, for example, a game, whose solution (decision, message, argument) is one of the typical manifestations of the concept. A situation is composed of a milieu and a project. The duration of the life of a mathematical situation (the time of studying it) can vary from a few seconds to several centuries for humanity or several months for teaching.

Examples

Example 1: Children 4–5 years old. From a collection of thirty or so familiar objects, 5 or 6 are hidden in a box by a child in the morning. In the afternoon, she is supposed to enumerate them to another child, who confirms the presence or absence of the objects she names. The solution of this game is the creation, enumeration, and use of lists. Knowing neither how to read nor how to write, the children represent the objects in their own way (pictograms) to distinguish them, first individually and then collectively. The lists of symbols represent sets; belonging or not, conjunctions and disjunctions of characters are used, corrected, understood, and formulated in vernacular language (Pérès 1984; Digneau 1980).

Example 2: Children 10–11 years old. To be certain of the number of white marbles contained in a firmly closed opaque bottle with a known number of marbles, some white and some black, students invent hypothesis testing and the measure of events (33 short sessions) (Brousseau et al. 2002).

A great many researchers have imagined and studied various types of situations destined for all sorts of notions, for all levels of school and even university. See, for example, Bessot (2000), Laborde and Perrin-Glorian (2005), Bloch (2003).

Types of Mathematical Knowledge, Reference Knowledge (*Savoirs*)

Classical methods forbid the teacher from tolerating without immediate correction, the manifestation of anything contrary to written established mathematics. A genuine mathematical activity necessarily gives rise to all sorts of knowledge. Some is knowledge sought for – these are the references, recognized as correct, true and known: they are professed and expected. But there also necessarily appear pieces of knowledge that are ill made, ill formed, incomplete, doubtful, false, or even inexpressible. They are "knowledge" in the sense of "the trace of an encounter." Their presence, whether or not firmly nailed down, is indispensible to thought. For example, a theorem that the student knows very well (savoir), but about whose usefulness in a situation is unsure, functions provisionally as a simple piece of nonestablished knowledge (connaissance).

The teacher cannot intervene in this flow of activities without blocking its functioning and must therefore delegate the responsibility for exercising a pragmatic penalty to the initiatives of the students that result from their knowledge. He entrusts it to a *milieu* that is clearly stripped of teleological or pedagogical intentions [its reactions depend neither on the intended goal nor on the individuals].

The *milieu* of a situation is what the students exercise their actions on and what gives them objective responses. The teacher thus entrusts to the *milieu* the job of showing the students' errors by their effects, without using an argument of authority or revealing any intentions. The milieu may comprise informative texts; material objects; other students, cooperating or concurrent; and so on. To this must be added the established knowledge of the student as well as her memories of relevant previous events, and objective conditions, that may not be known to the student but that intervene in her choices and in the effects of her decisions. The *cognitive variables* of the situation are those whose value has an influence on the issue of the situation or on the knowledge developed. These variables are didactical if their value can be chosen by the teacher (the sex of the students may influence the progress of a situation, but it is not a didactical variable). The *milieu* can be interpreted metaphorically by games that present some states that are permissible and some that are excluded, rules of action, and issues of which one would be the goal sought (Warfield 2007).

Examples of *Milieux*

1. *Cabri geometry* permits the student to realize, in the context of geometrical objects and transformations, which of her projects are constructible, that is, compatible with the axioms (Laborde et al. 1995). The projects lead the students to gain knowledge of, formulate, and test what the *milieu* permits them to glimpse.
2. Analysis of a situation. The reader will find an example of the analysis of a didactical situation (the Race to 20), of its *milieu*, of the strategies used by students, of the theorems in action that support them, and of the didactical methods that make it possible to lead them to a complete proof and then to extend it so as to have them reinvent an algorithm: the search for the remainder of a Euclidean division, in Brousseau (1997, pp. 3–18). This work also includes numerous other examples.

The project is an objective, a final state of the *milieu*, the response to a question, or even a pretext for exploration. It is what explains, justifies, or condemns after the fact the choices that have been chosen or ventured by the subject.

The resolution is the occasion to put to trial not the student, but a way of knowing.

Remarks: The *milieu* of a situation is not a natural *milieu* and does not turn mathematics into a sort of experimental science. The project is essential, and its goal is to establish the consistency of certain statements.

Different branches of mathematics developed in different *milieux*: geometry in the knowledge of space, probability in the statistics of games, algebra in arithmetic, arithmetic in the measurement of amounts, etc.

In elementary teaching, knowledge of these *milieux* is neither spontaneous nor contained in their mathematical interpretation. For example, the knowledge that is useful for finding one's way around a big city merits specific work that cannot be reduced to some geometry.

Types of Mathematical Situations Characteristic of Activities, of Pieces of Knowledge, and of Pieces of Mathematical Learning

The mathematical knowledge of a student manifests itself in her interactions with a milieu, as a means of attaining or maintaining a desired state. These interactions are grouped in four types of situations which are, in the order of didactical necessity, inverse to the ordinary chronological order:

1. Situation of *reference*: A person (student or teacher) refers the person asking to a piece of mathematical knowledge (a proof, a theorem, a definition, etc.) that belongs to their common repertoire (Perrin-Glorian 1993).
2. Situation of *argumentation* (of *proof*): A proposer communicates to an opponent an argument, an element of proof. He makes use for that of their common repertoire which his message tends to augment. The argument makes reference to a *milieu* and a (mathematical) project in common that gives it its meaning and its value. The two speakers have the same information, in particular, on the *milieu*, the same rights of refutation, and the same interest in arriving at a consistent agreement (for an action on the *milieu*).
3. Situation of *information (communication)*: The transmitter and receiver cooperate on an action on the *milieu*, in whose success they are interested and which depends on their joint action. Neither of the two has at the same time all of the information and all of the necessary means of action. They exchange messages in order to realize a common mathematical project.
4. Situation of *action*: A subject intervenes on the *milieu* to modify it with a determined aim. She observes the effect of her actions and attempts to anticipate them by constructing pieces of knowledge, conscious and explainable or not. This situation encompasses all of the others, but it extends beyond them by stimulating the existence of inexpressible and possibly even unconscious models of action.

Each of these types of situation creates *distinct typical motivations* (modify a *milieu*, communicate some information, debate the validity of a declaration, establish a reference) that mobilize and expand the *corresponding repertoires* (implicit models of action, semiological or linguistic repertoires, logical repertoires, mathematics or metamathematics, established knowledge and theory) which are themselves acquired according to specific different *modes of learning or acculturation.*

The *actual situations* are, every one of them, specific to a precise piece of knowledge.

This is the level which must be appealed to in order to judge the relevance of the contributions of other scientific domains (pedagogy, psychology, sociology, etc.).

The Processes

Different modes of composition and articulation of these elementary situations make it possible to create composite situations and sequences of situations that form processes:

1. *Process of mathematization*: A sequence of autonomous mathematical situations that are introduced by didactical interventions of the teacher and that work together towards

the construction of the same complex knowledge (e.g., rational and decimal numbers (Brousseau et al. 2004, 2007, 2008, 2009)).
2. *Genetic situation*: It introduces and without other external intervention generates the sequence of situations that lead to the acquisition of a concept (e.g., how many white marbles [article cited]).

The didactical work of the teacher then consists of maintaining the intensity and the relevance of the exchanges and implementing their progress and their conclusion. Examples of process: on areas, Perrin-Glorian M.J. (1992), and on geometry, Salin M.H., Berthelot R. (1998).

Some of the Results of Research on Didactical Situation

The notion of didactical situation was used in many research projects. It gave rise to numerous reflections and, with modifications, was expanded in more work than it is possible to summarize or cite here:

1. One of its first results was to establish that adaptation to certain conditions tends to render it more difficult to adapt to others and thus creates the phenomenon of didactical obstacles, then to show that the history of mathematics presents phenomena similar to the epistemological obstacles detected by G. Bachelard, and finally, to take advantage of this phenomenon in teaching by use of situations presenting "jumps in informational complexity" (► Epistemological Obstacles in Mathematics Education)
2. Research on situations had the goal of furnishing alternatives to the classical conceptions that showed their limitations in the face of the influx of knowledge to be taught and of the fundamental reorganizations necessitated by that influx. This research showed the importance of the role of the *unsayable* in mathematical situations and of the *unsaid* in the didactical relationship.

Rather than imagining teaching and producing learning of the *texts* that resulted from real mathematical activity by universal, that is nonspecific, nonmathematical teaching methods, it appeared that it would be preferable to have the students themselves produce this knowledge and these texts, thanks to specific mathematical activities that best stimulated the real activity of mathematicians.

The many didactical situations realized showed that this project was realizable. Experiments proved it. Curricula were conceived, experimented with, and reproduced for all the branches of mathematics and for all the basic levels of teaching (3–12 years old) in an establishment conceived for the purpose (the COREM).

Currently they cannot be developed because of the complexity of knowledge necessary for the teachers to conduct them and for the public to accept them.

This research produced counterexamples to most of the "universal principles," explicit or implicit, of classical didactics, for behaviorist methods as well as radical constructivism. It showed, for example, that in the classical conception, errors can have no status other than that of being far from some norm. They are interpreted as a failure of the student and/or the teacher that involves their responsibility and ultimately their guilt for a failure of their will. This absurd process generates very bad working conditions for the students and for the teachers.

Among many other results, The classical conception led to seeking out individualization of teaching, but this individualization did not improve the results, because mathematical knowledge is produced by the cooperation of numerous individuals operating in the same community, and no isolated brain can produce the exact form that history has given it. For a large portion of the students, the real use of communication and mathematical debates is indispensible.

The concept of situation has been the object and has been illustrated in a great deal of research of different types:

1. *Empirical*, so as to identify the observables of a given teaching episode and analyze them a priori and a posteriori
2. *Experimental*, to conceive of either a precise teaching project (engineering) or a teaching design (of cognitive psychology, of sociology, of *didactique*, etc.)

3. *Theoretical*, to study their properties (economic, ergonomic, etc.) on appropriate models, possibly mathematics (automata, games, various systems), or conceive of modes or specific indices for these studies (implicative statistical analysis for the study of dependencies) (Artigue and Perrin-Glorian 1991)

The results of these studies were used in many research projects more particularly centered on students, teachers, or school knowledge and the didactical transposition (Mercier et al. 2000).

Research Perspectives

1. The study of the optimal conditions for articulation of mathematical situations and of institutionalization is a necessity. Pieces of "knowledge" proposed in mathematical situations, whether erroneous or valid, must evolve sufficiently rapidly to arrive at established knowledge. Making these pieces of provisional knowledge the object of classical teaching, on the pretext that they were produced by the students themselves, is a major error. On the contrary, the reorganization of spontaneous knowledge around established knowledge with a complement of information (a lecture) is a mathematical necessity that offers an indispensible time gain. *Didactique* is a science of dynamic equilibrium of situations.
2. What are the relationships between the teaching of mathematics (*microdidactique)* and the explicit or latent mathematical or didactical conceptions held by the various social, economic, cultural, and scientific components of a society (*macrodidactique*)?
3. What are the factors in the failure of the reform of modern mathematics?

Cross-References

▶ Didactic Contract in Mathematics Education
▶ Didactic Engineering in Mathematics Education
▶ Epistemological Obstacles in Mathematics Education

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Didactic Transposition in Mathematics Education

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Keywords

Anthropological theory of the didactic; Scholarly knowledge; Knowledge to be taught; Institutional transposition; Noosphere; Ecology of knowledge; Reference epistemological models

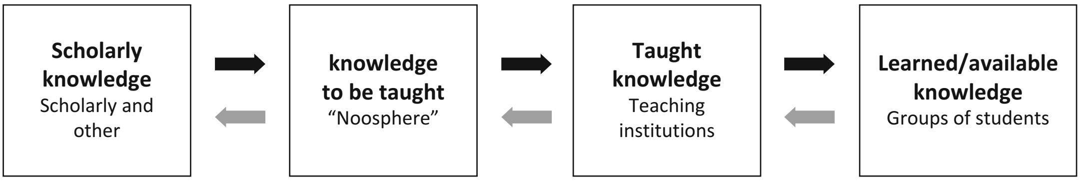
Definition

The process of didactic transposition refers to the transformations an object or a body of knowledge undergoes from the moment it is produced, put into use, selected, and designed to be taught until it is actually taught in a given educational institution. The notion was introduced in the field of didactics of mathematics by Yves Chevallard (1985, 1992b). It highlights the fact that what is taught at school is originated in other institutions, constructed in concrete practices, and organized in particular sets of objects. In the case of mathematics or any other subject, the *taught knowledge*, the concrete practices and bodies of knowledge proposed to be learned at school, originates from what is called the *scholarly knowledge*, generally produced at universities and other scholarly institutions, also integrating elements taken from a variety of related social practices. When one wishes to "transpose" a body of knowledge from its original habitat to school, specific work should be carried out to rebuild an appropriate environment with activities aimed at making this knowledge "teachable," meaningful, and useful.

Different actors participate in this *transpositive work* (see Fig. 1): producers of knowledge, teachers, curriculum designers, etc. They belong to what is called the *noosphere*, the sphere of those who "think" about teaching, an intermediary between the teaching system and society. Its main role is to negotiate and cope with the demands made by society on the teaching system while preserving the illusion of "authenticity" of the knowledge taught at school, thus possibly denying the existence of the process of didactic transposition itself. It must appear that *taught knowledge* is not an invention of school. Although it cannot be a reproduction of *scholarly knowledge*, it should look like preserving its main elements. For instance, the body of knowledge taught at school under the label of "geometry" (or "mechanics," "music," etc.) has to appear as genuine. It is thus important to understand the choices made in the designation of the *knowledge to be taught* and the construction of the *taught knowledge* to analyze what is transposed and why and what mechanisms explain its final organization and to understand what aspects are omitted and will therefore not be diffused.

Scope

Besides mathematics, research on didactic transposition processes has been carried out in many other educational fields, such as the natural



Didactic Transposition in Mathematics Education, Fig. 1 Diagram of the process of didactic transposition

sciences, philosophy, music, language, technology, and physical education. These investigations have spread faster in the French- and Spanish-speaking communities (Arsac 1992; Arsac et al. 1994; Bosch and Gascón 2006) than in the English-speaking ones, although some prominent figures soon contributed to develop the first transpositive analyses (Kang and Kilpatrick 1992). The notion of didactic transposition has been generalized to *institutional transposition* (Chevallard 1989, 1992a; Artaud 1995) when knowledge is transposed from one social institution to another. Because of social needs, bodies of knowledge originated and developed in different "places" or institutions of society need to "live" in other institutions where they should be transposed. They have to be transformed, deconstructed, and reconstructed in order to adapt to their new institutional setting. For instance, the mathematical objects used by economists, geographers, or musicians need to be integrated in other practices commonly ignored by the mathematicians who produced them. It is clear from the history of science that institutional transpositions – including didactic ones – do not necessarily produce degraded versions of the initial bodies of knowledge. Sometimes the transpositive work *improves* the organization of knowledge and makes it more understandable, structured, and accurate to the point that the knowledge originally transposed is itself bettered. The organization of knowledge in fields and disciplines as it exists today is the fruit of complex and changing historical interactional processes of institutional and didactic transpositions that are not well known yet.

An Emancipatory Tool

In a field of research, new notions are not only introduced to describe reality but to provide new ways of questioning and new possibilities to modify it. The notion of didactic transposition is conceived, first of all, as an analytical instrument to avoid the "illusion of transparency" concerning educational phenomena and, more particularly, the nature of the knowledge involved, that is, to emancipate research from the viewpoint of the scholarly and the teaching institutions about the knowledge involved in educational processes.

Any *taught* field or discipline is the product of an intricate process the singularity of which should never be underrated. As a consequence, one should not take for granted the current, observable organization of a field or discipline taught at school, as if it were the only possible one. Instead one should see it against the (fuzzy) set of organizations that *could* have existed, some of which may someday turn into reality. Considering the "scholarly knowledge" as part of the object of study of research in didactics is part of this emancipatory movement of detachment. Although school teaching has to be legitimized by external entities that guarantee the pertinence and epistemological relevance of the knowledge taught (in a complex process of negotiations which includes crises and disagreements), researchers do not have to consider these institutional perspectives as the true or correct viewpoints or as the wrong ones; they just need to know them and integrate them in the analysis of educational phenomena.

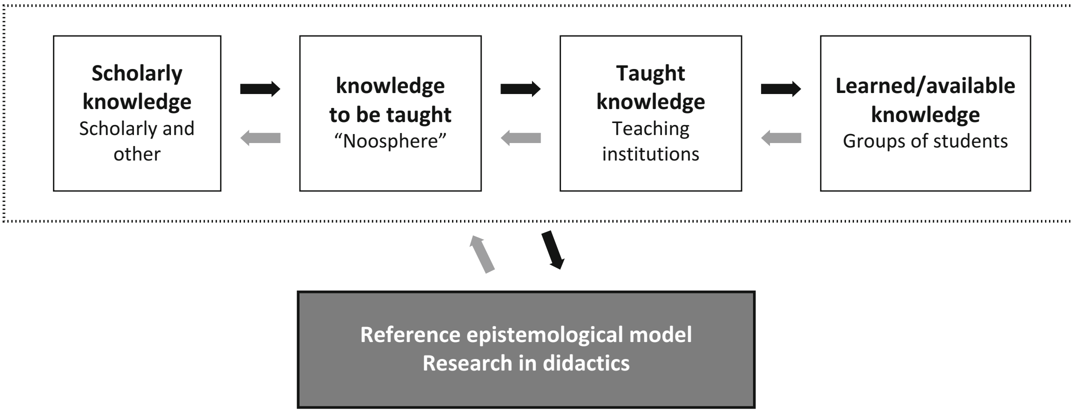
In some cases, the "scholarly legitimation" of school knowledge can be questioned by the noosphere, on behalf of its cultural relevance: "Is this the geometry citizens need?" Such a conflict situation can change significantly the conditions of teaching and learning, by allowing a self-referential, epistemologically confined teaching. Moreover, there are certain teaching processes in which the scholarly body of knowledge is created afterwards because of the need to teach a given content that has to be organized, labeled, and recognized as something relevant (an illustrative example is the case of accounting and its corresponding body of knowledge, accountancy). It is also possible that something that is not even commonly recognized as a proper body of knowledge may appear as "scholarly knowledge" for the role it assumes in a given educational process. For instance, in the teaching of sports, the scholarly knowledge, albeit not academically tailored, includes that of high-level sport players, even if they are a far cry from what we normally consider "scholars" to be!

Enlargement of the Object of Study

The second consequence of the detachment process introduced by the notion of didactic transposition is the evolution of the object of study of didactics as a research discipline. Besides studying students' learning processes and how to improve them through new teaching strategies, the notion of didactic transposition points at the object of the learning and teaching itself, the "subject matter," as well as its possible different ways of living – its diverse *ecologies* – in the institutions involved in the transposition process.

Let us take an example on negative numbers. Regarding the transpositive process, the first issue is to consider what the *taught knowledge* is made of (what concrete activities that are proposed to the students, their organization, the domain or block of contents they belong to, etc.) and how official guidelines and *noospherian* discourses present and justify these choices (the *knowledge to be taught*). Today, at most schools, negative numbers are officially related to the measure of quantities with opposite directions and introduced in the context of real-life situations. Where does this school organization come from? It results from different scholar ("new mathematics") or social ("back-to-basics") pressures, canalized by the noosphere, that cannot be presented here but that delimit the kind of mathematical practices our students learn (or fail to learn) about this body of knowledge. If we look at scholarly knowledge, the environment is different: negative numbers are defined as an extension of the set of natural numbers **N** and form the ring of integers **Z**, without any specific discussion (http://www.encyclopediaofmath.org/index.php/Integer). This has not always been the case: it is very well known that until the mid-nineteenth century, the possibility of "quantities less than zero" was still denied by many scholars. Their final acceptation was strongly related to the needs of algebraic work, which explains why, for a long time, integers were called "algebraic numbers." It also explains why the introduction of negative numbers was considered one of the main differences between arithmetic and algebra. This relationship to elementary algebraic work has now completely disappeared from the scholar's and school's conception of negative numbers, despite the fact that some practices of calculation – for instance, those involving the product of integers – acquire their full sense when interpreted in this context.

Various other analyses have brought similar results regarding how the transposition process has affect other different mathematical contents (school algebra, linear algebra, limits of functions, proportionality, geometry, irrational numbers, functions, arithmetic, statistics, proof, modeling, etc.): more generally speaking, there is no such thing as an eternal, context-free notion or technique, the matter taught being always shaped by institutional forces that may vary from place to place and time to time. These investigations underline the institutional relativity of knowledge and show to what extend most of the phenomena related to the teaching and learning of mathematics are strongly affected by constraints coming from the different steps of the didactic transposition process. Consequently, the empirical unit of analysis of research in didactics becomes clearly enlarged, far beyond the relationships between teachers and students and their individual characteristics.



Didactic Transposition in Mathematics Education, Fig. 2 The external position of researchers

The Need for Researchers' Own Epistemological Models

Taking into consideration *transpositive phenomena* means moving away from the classroom and being provided with notions and elements to describe the bodies of knowledge and practices involved in the different institutions at different moments of time. To do so, the epistemological emancipation from scholarly and school institutions requires researchers to create their own perspective on the different kinds of knowledge intervening in the didactic transposition process, including their own way of describing knowledge and cognitive practices, their own epistemology. In a sense, there is no privileged reference system from which to observe the phenomena occurring in the different institutions involved in the teaching process: the scholarly one, the noosphere, the school, and the classroom. Researchers should build their own *reference epistemological models* (Barbé et al. 2005) concerning the bodies of knowledge involved in the reality they wish to approach (see Fig. 2). The term "model" is used to emphasize the fact that any perspective provided by researchers (what mathematics is, what algebra is, what measuring is, what negative numbers are, etc.) always constitutes a methodological proposal for the analysis; as such, it should constantly be questioned and submitted to empirical confrontation.

From Didactic Transposition to the Anthropological Approach

When knowledge is considered a changing reality embodied in human practices taking place in social institutions, one cannot think about teaching and learning in individualistic terms. The evolution of the research perspective towards a systematic epistemological analysis of knowledge activities explicitly appears at the foundation of the anthropological theory of the didactic (Chevallard 1992a, 2007; Winslow 2011). It is approached through the study of the conditions enabling and the constraints hindering the production, development, and diffusion of knowledge and, more generally, of any kind of human activity in social institutions.

Cross-References

- ▶ Anthropological Approaches in Mathematics Education, French Perspectives
- ▶ Curriculum Resources and Textbooks in Mathematics Education
- ▶ Didactic Engineering in Mathematics Education
- ▶ Didactic Situations in Mathematics Education

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Didactical Phenomenology (Freudenthal)

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Keywords

Phenomena in reality; Mathematical thought objects; Didactics; Realistic mathematics education; Analyses of subject matter

What Is Meant by Didactical Phenomenology?

The term *didactical phenomenology* was coined by Hans Freudenthal. Although his initial ideas for it date from the late 1940s, he likely first used the term in a German article in 1974. A few years later, the term appeared in English in his book *Weeding and Sowing – Preface to a Science of Mathematical Education* (Freudenthal 1978). Understanding the term requires comprehending Freudenthal's notion of a *phenomenology* of mathematics, which refers to describing mathematical concepts, structures, or ideas, as thought objects (*nooumena*) in their relation to the phenomena (*phainomena*) of the physical, social, and mental world that can be organized by these thought objects.

The term *didactical* is used by Freudenthal in the European continental tradition referring to the way we teach students and the organization of teaching processes. This definition of didactics goes back to Comenius' (1592–1670) *Didactica Magna* (Great Didactics) that contains a well-founded view on what and how students should be taught. As such, this meaning of didactics contrasts with the Anglo-Saxon tradition in which it merely has a superficial meaning involving a set of instructional tricks.

Combining the two terms into *didactical phenomenology* implies considering the *phenomenology* of mathematics from a *didactical* perspective.

Merit of a Didactical Phenomenology for Mathematics Education

In Freudenthal's words (1983, p. ix), a didactical phenomenology of mathematics can "show the teacher the places where the learner might step into the learning process of mankind." In other words, a didactical phenomenology informs us on how to teach mathematics, including how mathematical thought objects can help organizing and structuring phenomena in reality, which phenomena may contribute to the development of particular mathematical concepts, how students can come in contact with these phenomena, how these phenomena beg to be organized by the mathematics intended to be taught, and how students can be brought to higher levels of understanding. As such, Freudenthal's didactical phenomenologies are landmarks for developing teaching outlines.

Relation with Realistic Mathematics Education

By disclosing the sources of mathematics in reality, a didactical phenomenology is strongly related to Realistic Mathematics Education (RME), the domain-specific instruction theory for mathematics, which has been developed in the Netherlands and in which Freudenthal was heavily involved (Freudenthal 1991). In RME, rich, realistic situations have a prominent position in the learning process. These situations serve as sources for initiating the development of mathematical concepts, tools, and procedures. What situations can serve as contexts for this development is revealed by a didactical phenomenology. By tracing phenomena in reality that can elicit mathematical thoughts, the students are given access to the sources of mathematics in everyday experiences. Building on these sources offers them an orientation basis they experience as real and opens the possibility of personal engagement and solving problems in a way they find meaningful. This attachment of meaning to mathematical constructs students have to develop touches on a main principle of RME.

Examples of Didactical Phenomenology

In *Weeding and Sowing*, Freudenthal exemplified his idea of a didactical phenomenology by providing an analysis of the topic of ratio and proportion. Furthermore, he announced to deal comprehensively with didactical phenomenology in a following book. That book was *Didactical phenomenology of mathematical structures* (Freudenthal 1983). In this book, he gave more examples of didactical phenomenologies, including those of length, natural numbers, fractions, geometry and topology, negative numbers and directed magnitudes, algebraic language, and functions.

Remarkably, these examples did not just deal with connecting mathematical thought objects to phenomena in reality to find starting points for learning mathematics. In fact, these examples were profoundly scrutinized analyses of subject matter in which the key concepts of a particular mathematical topic were disclosed together with contexts which have a model character and with significant landmarks in students' learning pathways.

The Method

Unfortunately, in *Didactical phenomenology of mathematical structures*, Freudenthal did not elaborate much on how to establish these didactical phenomenologies. Although the book contains a short chapter titled *The method*, this did not reveal how to generate such phenomenologies. Nevertheless, a corner of the veil was lifted when Freudenthal (1983, p. 29) considered the material he needed to write this book:

> I have profited from my knowledge of mathematics, its application, and its history. I know how mathematical ideas have come or could have come into being. From an analysis of textbooks

> I know how didacticians judge that they can support the development of such ideas in the minds of learners. Finally, by observing learning processes I have succeeded in understanding a bit about the actual process of constitution of mathematical structures and the attainment of mathematical concepts

This statement and the provided examples show how a didactical phenomenology results from a number of analyses, each taking a different perspective: didactical, phenomenological, epistemological, and historical-cultural.

Mathematics-Related Analyses Constituting the Didactics of Mathematics

These analyses have in common that they all take mathematics as their starting point. Didactical analyses examine the nature of the mathematical content as a basis for teaching this content. By identifying the determining aspects of mathematical concepts and their relationships, knowledge is gathered about didactical models that can help students to understand these concepts. Phenomenological analyses disclose possible manifestations of these mathematical concepts in reality and can suggest contexts for students to meet these concepts. Epistemological analyses focus on students' learning processes and can uncover how the mathematical understanding of students in a classroom interaction may shift. Finally, in historical-cultural analyses, we may encounter current and past approaches to teaching mathematics through which we can gain a better understanding of learning mathematics and how education can contribute to it.

These analyses are all included in Freudenthal's didactical phenomenology and surpass its narrow literal meaning, which would certainly have his approval, as in *Weeding and Sowing* Freudenthal (1978, p. 116) already stated: "[T]he name does not matter; nor is that activity [didactical phenomenology] an invention of mine; more or less consciously it has been practiced by didacticians of mathematics for a long time" (Freudenthal 1978, p. 116). Indeed, the name is not essential, but these analyses are. In Freudenthal's view, they form the heart of researching and developing mathematics education.

Cross-References

▶ Realistic Mathematics Education

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Discourse Analytic Approaches in Mathematics Education

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Keywords

Discourse; Discourse analysis; Foucault; Identity; Language; Linguistics; Methodology; Social practice

Introduction

The first challenge in addressing this topic is the multiplicity of ways in which the term *discourse* is used and defined – or not defined – within mathematics education (see Ryve 2011). It is frequently found, especially in discussions within the context of curriculum reform, simply to signify student engagement in talk in the classroom. Without denying the value of the development of such engagement, the approaches to discourse and discourse analysis considered in this article all take rather more complex and theoretically shaped views of the nature of discourse – views that influence the focus of research and the analytic methods. An important component of the ways these approaches conceive of discourse is a concern with the relationship between language (and other modes of communication), the social context in which it is used, and the meanings that are produced in this context (Howarth 2000). It is this concern and the fundamental assumption that studying the way language is used can provide insight into the activity or practice (mathematics or mathematics education) in which it is used that leads researchers to adopt discourse analytic approaches. Of course, a very high proportion of the data used in studies across many branches of mathematics education research is primarily linguistic or textual: interviews, written responses to questionnaires, classroom transcripts, written texts produced by students, etc. Increasingly it has also been recognized by researchers using a wide range of approaches that the language produced by students or other research subjects is not a transparent medium through which it is easy to decipher an underlying truth. What distinguishes research that adopts a discourse analytic approach is the assumption that the language is itself an inextricable part (or, for some researchers, even the whole) of the object of study. This assumption is shared with another analytic approach, conversation analysis, and some discourse analysts make use of methods developed in conversation analysis. However, whereas discourse analysis is generally interested in characterizing the practices within which language plays a role, conversation analysis focuses primarily on how linguistic interactions themselves are organized to achieve social actions (see Wooffitt 2005, for an introduction to the two approaches from a conversation analytic perspective).

Gee (1996) makes a useful distinction between *discourse*, defined as instances of communication, and *Discourses*, the conjunctions of ways of speaking, subject positions, values, etc. that characterize and structure particular social practices. The notion of *Discourses* has its origin in the thinking of the French philosopher Foucault (e.g., 1972) whose work includes studies of the construction of "regimes of truth" about notions such as madness or sexuality. Though not all discourse analytic research in mathematics education comes from this tradition, it can generally be characterized as tending either towards analysis of *discourse*, focusing on communication events and the local social practices within which they arise, or towards analysis of *Discourse*, taking larger scale social practices and structures as the object of research. Of course, some approaches move between the two, generating interpretation of specific communication events by applying knowledge of wider social practices and structures or building a picture of a significant social practice through analysis of local communication events. Discourse analytic approaches thus vary in two dimensions: the extent to which they make use of detailed linguistic analysis and the extent of their focus on social practices, structures, and institutions.

The adoption and development of discourse analytic approaches in mathematics education research largely coincided with what Lerman termed the "social turn" (Lerman 2000). Increased recognition of the importance of studying and taking account of the social nature of mathematics education practices as well as of individual cognition demanded the development of theoretical ways of conceiving of social practices and methodological approaches to studying them. Discourse analytic approaches provided one way of addressing this demand. This development within the field of mathematics education reflected a much wider development of theories of discourse and discourse analytic methods within social science and the humanities. As researchers have begun to draw on theories and methods originating outside the field of mathematics education, they have faced the challenge of ensuring that both theory and methods take account of the specialized nature of mathematical communication and practices and that they have the power to illuminate issues of interest to mathematics education. Facing this challenge is a continuing project; notable contributions have come from within mathematics education (e.g., Morgan 1998; Sfard 2008) and from linguistics (e.g., O'Halloran 2005).

With a few exceptions, notably the work of Walkerdine (1988) who used analyses of *Discourses*, including Discourses of gender and of child-centered education, in order to understand how differences between various social groups are constructed in mathematics education practices, early interest in discourse analytic approaches, such as that represented in the Special Issue of *Educational Studies in Mathematics* edited by Kieren et al. (2001), was dominated by analysis of communication events (*discourse*), focusing on understanding classroom interaction and the development of mathematical thinking in interaction. At a time when the majority of research in mathematics education focused on the mathematical thinking of individuals, this application of discourse analysis may be seen as an incremental manifestation of the "social turn," addressing the same interest in mathematical thinking but reconceptualizing it as a phenomenon that is evident (and, for some researchers, produced) in social interaction. More recently, the issues addressed by the mathematics education research community have expanded, incorporating a wider conceptualization of mathematics and mathematics education as social practices. Thus more research has addressed, *inter alia*, identity, power relationships, and social justice – issues that lend themselves to study using approaches that focus on *Discourses*. Some of this research has adopted approaches that may be characterized as structuralist, drawing on sociological accounts of social structures such as the work of Basil Bernstein (e.g., 2000) to describe and interpret discursive phenomena. Others have adopted poststructural approaches, in which the communicative action itself constructs the "reality" of which it speaks. A recent edited book entitled *Equity in Discourse for Mathematics Education* (Herbel-Eisenmann et al. 2012) reflects this range of approaches and interpretations, combining detailed analyses of classroom interactions with concern for how these interactions and broader social practices affect the possibilities for participation in mathematics of students from different social groups.

In this article there is no space to provide a detailed review of the full range of approaches taken to discourse analysis. Instead, we provide a small number of contrasting cases, exemplifying the scope of discourse analytic methods and the problems in mathematics education that they may be used to address.

Critical Discourse Analysis

Critical discourse analysis (CDA) comprises a group of analytic approaches, all of which make strong analytic connections between forms of language use, social practices, and social structures. The label "critical" indicates a concern of the researchers to make use of the knowledge achieved through the analysis in order to enable critique and transformation of the social practices and/or structures. Research using CDA approaches thus tends to produce analyses that not only describe existing practices but also critique the ways these practices position students and/or teachers and the kinds of mathematics and mathematical identities that are valued and made possible.

CDA studies generally involve detailed analyses of texts, including oral and written texts produced and used by students and teachers in the classroom but also including texts such as the curriculum and policy documents that structure and regulate these educational practices and thus affect the interpretation of classroom texts. Within mathematics education, probably the most widely used type of CDA is based on the approach of Norman Fairclough (2003), using linguistic tools drawn from systemic functional linguistics (SFL). This approach has been used to investigate specific practices such as the assessment of student reports of mathematical investigation (Morgan 1998) or the use of "real-world problems" in an undergraduate mathematics course (Le Roux 2008). Research adopting a CDA approach may also use a range of other methods to address textual data, including corpus analysis of large data sets (e.g., Herbel-Eisenmann et al. 2010).

Whatever the linguistic tools used to describe the data, the interpretative stage of CDA involves considering how the features identified in the data function to construe the "reality" of the practice being studied and the social positionings and relations of the participants. As Fairlough argues, such interpretation requires explicit use of "insider knowledge" of the social practices studied (Fairclough 2003). This means that researchers in mathematics education need to bring knowledge of broader mathematics education practices and knowledge of mathematical practices to bear on their analyses. For example, Morgan's study of teachers' assessment practices is informed by an analysis of the constructs and values found in the associated curriculum documents, policy, and professional literature, while Le Roux draws on Sfard's (2008) characterization of mathematical discourse (discussed further below) to enable her analysis to address the nature of the mathematical activity involved in the use of real-world problems.

Poststructural Approaches

The approaches to discourse analysis discussed under the heading of postmodern or poststructural share with CDA approaches a concern with issues such as power and subjectivity that arise in considering relationships between individuals and social practices and structures. There are, however, both philosophical and methodological differences between the approaches. There is a range of philosophical positions associated with postmodern and poststructural thought; however, a shared foundation is a rejection of the notions of an objective world and of the fixed subjectivity of a rational knowing subject. These philosophical assumptions are shared by some but certainly not by all those employing CDA approaches, though there is a common interest in characterizing the key entities that play a role in a Discourse and the possibilities for individual subjectivities, identities, or positioning.

The major distinction drawn here between the approaches to discourse analysis discussed in this section and those identified under the heading CDA is methodological. While CDA involves close analysis of specific texts, usually employing analytical tools and methods drawn from linguistics, the starting point for postmodern/poststructural researchers tends to be at the level of the major functions of discourse. For example, Hardy (2004) uses the Foucauldian constructs of *power as production* and *normalization* as her analytical tools for interrogating a teacher training video produced as part of the English National Numeracy Strategy to demonstrate "effective teaching" of mathematics in a primary classroom. Rather than focus on detailed characteristics of the discourse of this video, Hardy uses these constructs to provide an alternative perspective on the data as a whole. This enables her to tell a story of what the Discourse of the National Numeracy Strategy achieves – how it produces assumptions about what is normal and what is desirable – a story that runs counter to the "common sense" stories about effective teaching.

A rather different approach is taken by Epstein et al. (2010), though again founded in Foucauldian theory. They first characterize the ways in which mathematics and mathematicians are represented in popular media – as hard, logical, and ultrarational but also as eccentric or even insane. Having identified different and in some cases apparently contradictory Discourses about mathematics, Epstein et al. then use these to analyze interviews with students, identifying how individual students deploy the various discursive resources in order to produce their own identities as mathematicians or as nonmathematicians and their relationships to mathematics as a field of study.

Mathematical Discourse, Thinking, and Learning

The main discourse analytic theories mentioned so far have their origins outside mathematics education, drawing on fields such as linguistics, ethnography, sociology, and philosophy. For mathematics education researchers, this raises the important theoretical and methodological problem of the extent to which the specifically mathematical aspects of the practices being studied may be captured and accounted for. In order to address this problem, an increasing number of researchers, including some of those working with CDA or other discourse analytic approaches, are turning to the work of Anna Sfard (2008). While Sfard draws on a number of sources, including Wittgenstein's notion of language game, her own theory of mathematical discourse has been developed within the field of mathematics education and is designed to address the problems arising in this field. Her communicative theory of cognition identifies thinking mathematically as participating in mathematical discursive practices, that is, as communicating (with oneself or with others) using the forms of discourse characteristic of mathematics. Sfard identifies four aspects of mathematical discourse that form the basis for her analytic method: specialized mathematical *vocabulary and syntax* (what may be considered the "language" of mathematics), *visual mediators* (nonlinguistic forms of communication such as algebraic notation, graphs, or diagrams), *routines* (well-defined repetitive patterns, e.g., routines for performing a calculation, solving an equation, or demonstrating the congruence of two triangles), and *endorsed narratives* (the sets of propositions accepted as true within a given mathematical community). Scrutinizing how these four aspects are manifested in discourse provides a means of describing mathematical thinking and hence allows one to address questions such as the following: How does children's thinking about a mathematical topic vary from that expected by their teacher or by an academic mathematical community? How does children's thinking develop (i.e., how does their use of a mathematical form of discourse change over time)? What kinds of mathematical thinking are expected of students taking an examination or using a textbook?

As may be seen from the research topics and questions illustrated in this article, discursive approaches can address a wide range of issues of concern within the field of mathematics education, bridging, as indicated in the title of Kieran et al.'s (2001) Special Issue of *Educational Studies in Mathematics*, the individual and the social. While the various approaches share a basic assumption that language and social practices play a role in the ways that individuals make sense of mathematical activity, they differ in the ways they conceptualize this role (and, indeed, in how they conceptualize language, social practice, and mathematics). Hence they also differ in the research questions they pose and the methodological tools they employ. It can be argued that discourse analytic approaches allow us to see through *what is said* to reveal *what is achieved* by using language. The challenge for researchers and for the readers of research is to clarify how the theoretical and methodological tools employed enable this and to distinguish which kinds of actions and achievements are made visible by the different approaches.

Cross-References

- ▶ Discursive Approaches to Learning Mathematics
- ▶ Mathematical Language
- ▶ Poststructuralist and Psychoanalytic Approaches in Mathematics Education
- ▶ Sociological Approaches in Mathematics Education

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Discrete Mathematics Teaching and Learning

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Keywords

Discrete mathematics; Discrete; Continuous; Reasoning; Proof; Mathematical experience

Definition

The teaching of "discrete mathematics" is not always clearly delimited in the curricula and can be diffuse. In fact, the meaning of "discrete mathematics teaching and learning" is twofold. Indeed, it includes the teaching and learning of discrete concepts (considered as defined objects inscribed in a mathematical theory), but it also includes skills regarding reasoning, modeling, and proving (such skills are specific to discrete mathematics or transversal to mathematics).

What Is Discrete Mathematics?

Discrete mathematics is a comparatively young branch of mathematics with no agreed-on definition (Maurer 1997): only in the last 30 years did it develop as a specific field in mathematics with new ways of reasoning and generating concepts. Nevertheless, the roots of discrete mathematics are older: some emblematic historical discrete problems are now well known, also in education where they are often introduced as enigma, such as the Four Color Theorem (map coloring problem), the Königsberg's bridges (traveling problem), and other problems coming from the number theory for instance.

There is no exact definition of *discrete mathematics*. The main idea is that *discrete mathematics* is the study of mathematical structures that are "discrete" in contrast with "continuous" ones. Discrete structures are configurations that can be characterized with a finite or *countable* set of relations. (A *countable* set is a set with the same cardinality (number of elements) as some subset of the set of natural numbers. The word "countable" was introduced by Georg Cantor.) And discrete objects are those that can be described by finite or countable elements. It is strongly connected to number theory, graph theory, combinatorics, cryptography, game theory, information theory, algorithmics, discrete probability but also group theory, algebraic structures, topology, and geometry (discrete geometry and modeling of traditional geometry with discrete structures).

Furthermore, discrete mathematics represents a mathematical field that takes on growing importance in our society. For example, discrete mathematics brings with it the mathematical contents of computer science and deals with algorithms, cryptography, and automated theorem proving (with an underlying philosophical and mathematical question: is an automated proof a mathematical proof?).

The aims of discrete mathematics are to explore discrete structures, but also to give a specific modeling of continuous structures, as well as to bring the opportunity to consider mathematical objects in a new manner. Then new mathematical questions can emerge, as well as new ways of reasoning, which implies a challenge for mathematicians.

Some famous problems of discrete mathematics have inspired mathematics educators. That is the case of a combinatorial game: the game of Nim, played since ancient times with many variants. The regular game of Nim is between two players. It is played with three heaps of any number of objects. The two players alternatively take any number of objects from any single one of the heaps. The goal is to be the last one to take an object. Brousseau (1997) explicitly refers to the game theory to conceptualize the theory of didactical situations. The game of Nim is the background of the generic example of Brousseau's theory, "the Race to 20."

Why Teach and Learn Discrete Mathematics? New Context, Concepts, and Ways of Reasoning: A New Realm of Experience for the Classroom

To Integrate Discrete Mathematics into the School Curriculum: A Current Challenge

More and more fields of mathematics use results from discrete mathematics (topology, algebraic geometry, statistics, among others). Moreover, discrete mathematics is an active branch of contemporary mathematics. New needs for teaching are identified: they are linked to the evolution of the society and also other disciplines such as computer science, engineering, business, chemistry, biology, and economics, where discrete mathematics appears as a tool as well as an object. Then discrete mathematics should be an integral part of the school curriculum: the concepts and the ways of reasoning that should be taught in a specific field labeled "discrete mathematics" still should be more precisely identified. A dialog between mathematicians and mathematics educators can help for this delimitation.

However, the place of discrete mathematics in curricula is today very variable depending on the countries and on the levels. In a few countries, there has been a long tradition to introduce graph theory in the secondary level among other components of discrete mathematics. This place is strengthened and attested by the contents at the university level. In other countries, only a very small number of discrete mathematics concepts are taught, especially those involved in the fields of combinatorics and number theory. Things are changing; the reader can refer to Rosenstein et al. (1997), and DIMACS (2001) contributions to go into details regarding the challenge of introducing discrete mathematics in curricula (especially the example of the NCTM standards [*National Council of Teachers of Mathematics*] which focus on discrete mathematics as a field of teaching). The following arguments summarize the main ideas of these contributions, emphasizing the interests and the potential ways to implement discrete mathematics in the curricula:

- Proof and abstraction are involved in discrete mathematics (for instance, in number theory, induction, etc.).
- It allows an introduction to modeling and proving processes, but also to optimization and operational research, as well as experimental mathematics.
- Problems are accessible and can be explored without an extensive background in school mathematics.
- The results in discrete mathematics can be applied to real-world situations.
- Discrete mathematics brings a specific work on algorithms and recursion.
- The main problems in discrete mathematics are still unsolved in ongoing mathematical research: a challenge for pupils and students to be involved in a solving process close to the one of mathematicians and to promote cooperative learning (in a specific and suitable context: in particular, teachers should be trained to discrete problems and also to their teaching and management).

Benefits from Teaching and Learning Discrete Mathematics: Some Examples

Learning discrete mathematics clearly means learning new advantageous concepts but also new ways of reasoning, making room for a mathematical experience.

Many variants exist of the following famous problems that are developed below. Some of them are presented and analyzed for instance in Rosenstein et al. (1997) and on the website http://mathsamodeler.ujf-grenoble.fr/.

Accessible Problems and Concepts

Discrete concepts are easily graspable, applicable, accessible, and also neutral when not yet included in the curricula: this last argument implies that the way students deal with discrete concepts is quite new and different from the way they usually consider mathematics.

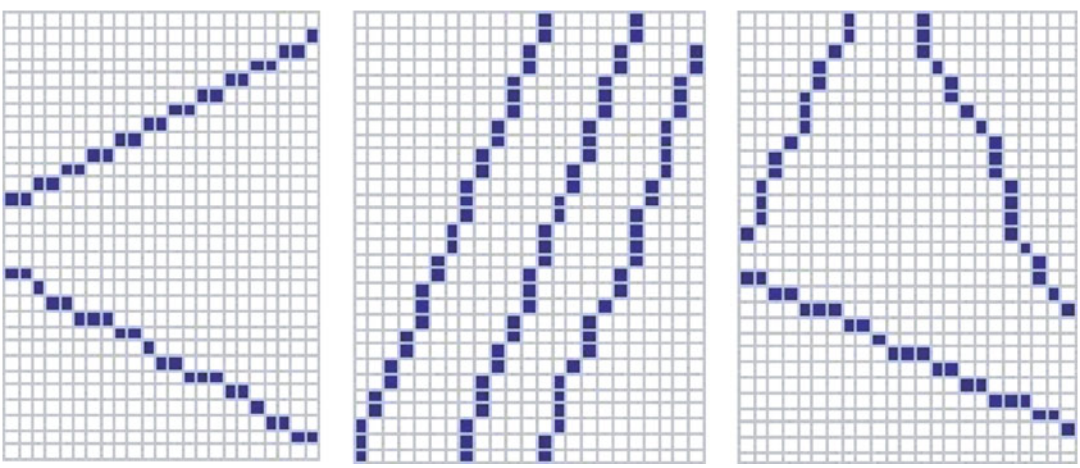
Traveling salesperson problem: the problem is to find the best route that a salesperson could take if he/she would begin at the home base, visit each customer, and return to the home base ("best" was defined as minimizing the total distance).

Map coloring problem (combinatorial optimization problem): a map coloring problem consists in discovering the minimum number of colors needed to *properly* color a map (or a graph). A map is *properly* colored if no two countries sharing a border have the same color. The proof of the minimum number of colors is also required. Similar coloring problems exist in graph theory. Such map and graph coloring problems are very useful to explore what discrete mathematical modeling is.

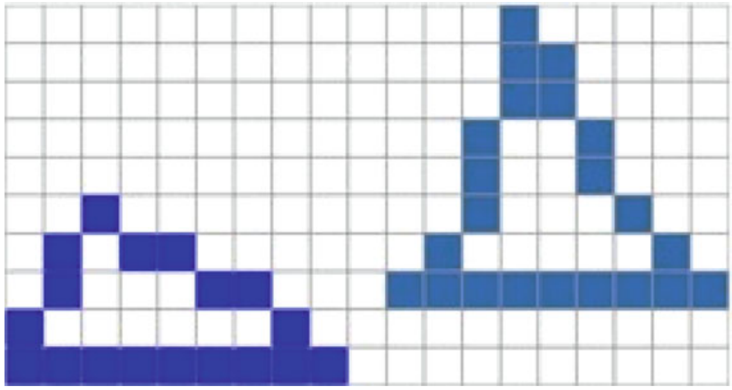
Richness of Discrete Concepts, A Way to Deal with the Construction of Axiomatic Theory

A certain amount of discrete objects can be defined in several ways, with different characterizations. The modeling of continuous concepts in the discrete case raises the problem of the construction of a mathematical consistent theory from an axiomatic point of view. It is illustrated with the following example of discrete geometry.

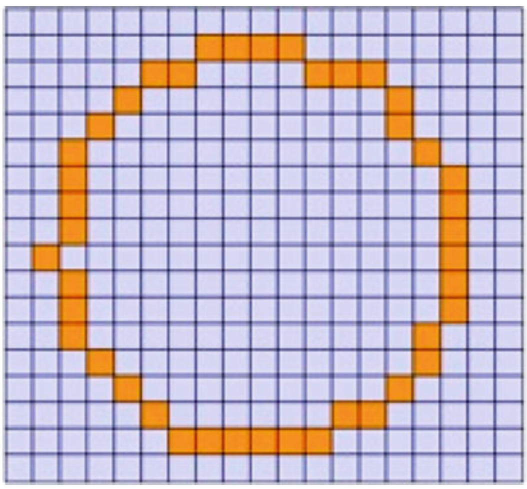
Discrete Geometry: Example of Discrete Straight Lines Discrete straight lines form a concept accessible by its representation. It is noninstitutionalized (an institutionalized concept is a "curriculum" concept, i.e., a concept that has a place in the classic taught content). Delimiting what a straight line can be in a discrete context proves to be quite a challenge. Professional researchers have several definitions of it at their disposal, but the proof of the equivalence of these definitions remains worth considering. Research on a discrete axiomatic theory is still in progress (it implies, for instance, the following questions: what is the intersection of two discrete straight lines? What does it mean to be parallel in the discrete case? etc.): the question of a "good" definition of a discrete straight line is currently an



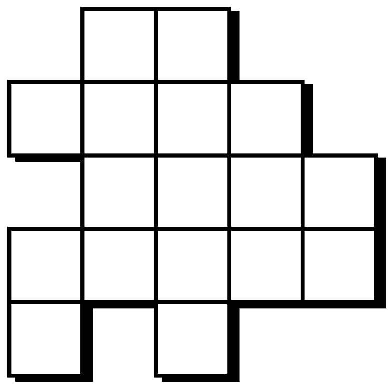
Discrete Mathematics Teaching and Learning, Fig. 1 Are these lines straight lines?



Discrete Mathematics Teaching and Learning, Fig. 2 Are these shapes triangles?



Discrete Mathematics Teaching and Learning, Fig. 3 Is it a circle?



Discrete Mathematics Teaching and Learning, Fig. 4 A garden

Discrete Mathematics Teaching and Learning, Fig. 5 A beast (a beast can be rotated or reversed)

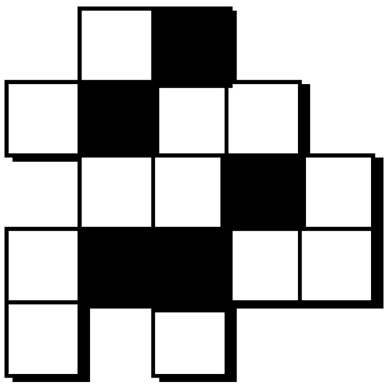
open and interesting problem. So are the questions of the definitions of other discrete geometrical concepts (Figs. 1–3).

Several Ways of Questioning, Proving, and Modeling

Besides, discrete mathematics arouses interest because it offers a new field for the learning and teaching of proofs (Grenier and Payan 1999; Heinze et al. 2004; http://mathsamodeler.ujf-grenoble.fr/). Some discrete problems fruitfully bring different ways to consider proof and proving processes. How can discrete mathematics contribute to make students acquire the fundamental skills involved in defining, modeling, and proving, at various levels of knowledge? It is still a fundamental question in mathematics education. The following example brings an opportunity to deal with an optimization problem which involves several kinds of reasoning. Besides, this problem is close to the contemporary research in discrete mathematics.

Hunting the beast. Your garden is a collection of adjacent squares (see Fig. 4) and a beast is itself a collection of squares (like the one drawn in Fig. 5). Your goal is to prevent a beast from entering your garden. To do this, you can buy traps. A trap is represented by a single black

Discrete Mathematics Teaching and Learning, Fig. 6 Not a solution



Discrete Mathematics Teaching and Learning, Fig. 7 A solution with 5 traps

square that can be placed on any square of the garden. The question we ask is the following: what is the minimum number of traps you need to place so that no beast can land on your garden?

On Fig. 6, the disposition of the traps does not provide a solution to the problem, since a beast can be placed. On Fig. 7, a solution with five traps is suggested. Is it an optimal one for this configuration?

In the literature, the problem *Hunting the beast* can be seen as a variation of the *Pentomino Exclusion Problem* introduced by Golomb (1994).

A Mathematical Experience

Discrete mathematics then brings the opportunity for students to be involved in a mathematical experience. Harel (2009) points out the following principle:

> The ultimate goal of instruction in mathematics is to help students develop ways of **understanding** and ways of **thinking** that are compatible with those practiced by contemporary mathematicians. (p. 91)

The "doing mathematics as a professional" component is clearly a new direction for the educational research in the problem solving area, and discrete mathematics offers promising nonroutine potentialities to develop powerful heuristic processes, as underscored by Goldin (2009).

Bearing in mind the aforesaid arguments, discrete mathematics provides a mathematical experience and is a field of experiments that questions concepts involved in other mathematical branches as well. Nevertheless, if the discrete problems are sometimes (and even often) easier to grasp than the continuous ones, the mathematics behind can be quite advanced. That is the reason why the didactic should analyze both the discrete mathematics for itself and the discrete mathematics helping the teaching of other concepts.

Interesting Perspectives for Research in Mathematics Education

Discrete mathematics is a relatively young science, still in progress with accessible and graspable concepts and ongoing questionings; hence the questions regarding the introduction of it in the curricula and in the classroom concern both mathematics educators and mathematicians.

Two separated but linked perspectives for the educational research emerge:

- The didactical study of teaching and learning discrete mathematics
- The didactical study of the teaching of concepts and skills (such as proof and modeling) with the help of discrete problems

Besides, discrete mathematics can be introduced either as a mathematical theory or as a set of tools to solve problems. The links between discrete mathematics as a tool and discrete mathematics as an object in teaching and learning should also be analyzed in depth, as well as the proof dimension involved in dealing with discrete concepts and structures. The didactic transposition of discrete concepts and ways of reasoning is still a current problem for mathematics education. It can raise the question of the development of models for teaching and learning discrete mathematics. Some epistemological models do exist (around transversal concepts such as implication, definition, and proof (see, for instance, Ouvrier-Buffet 2006) and specific contents such as the teaching of graph theory (see the work of Cartier 2008)) but the work is still in progress. Note that it involves the same questionings for mathematics education as the introduction of algorithmics in the curricula.

Furthermore, the introduction of discrete mathematics in the curricula clearly offers an opportunity to infuse new instructional techniques. In this perspective, the teacher training should be rethinked.

Cross-References

- ▶ Algorithmics
- ▶ Argumentation in Mathematics
- ▶ Inquiry-Based Mathematics Education
- ▶ Logic in Mathematics Education
- ▶ Mathematical Games in Learning and Teaching
- ▶ Mathematical Modelling and Applications in Education
- ▶ Mathematical Proof, Argumentation, and Reasoning
- ▶ Mathematics Teachers and Curricula
- ▶ Problem Solving in Mathematics Education
- ▶ Word Problems in Mathematics Education

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Discursive Approaches to Learning Mathematics

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Keywords

Learning; Mathematics; Communication; Discourse; Language

Definition

Discursive approach to learning is a research framework grounded in the view that learning such subjects as mathematics, physics, or history is a communicational activity and should be studied as such. Learning scientists who adopt this approach treat discourse and its development as the primary object of exploration rather than as mere means to the study of something else (e.g., development of mental schemes). The term *discourse* is to be understood here as referring to a well-defined type of multimodal (not just verbal) communicational activity, which does not have to be audible or synchronous.

Background

Ever since human learning became an object of systematic study, researchers have been aware of its intimate relationship with language and, more generally, with the activity of communicating. The basic agreement on the importance of discourse notwithstanding, a range of widely differing opinions have been proposed regarding the way these two activities, learning and communicating, are related. At one end of the spectrum, there is the view that language-related activities play only the secondary role of means to learning; the other extreme belongs to those who look upon discourse as the object of learning. It is this latter position, the one that practically equates mathematic with a certain well-defined form of communicational activity, that can be said to fully reflect a discursive approach to learning.

Several interrelated developments in philosophy, sociology, and psychology combined together to produce this approach. It is probably the postmodern rejection of the notion of "absolute truth," the promise of which fuelled the positivist science, that put human studies on the path toward the "discursive turn." Rather than seeing human knowledge as originating in the nature itself, postmodern thinkers began picturing it as "a kind of discourse" (Lyotard 1979, p. 3) or as a collection of narratives gradually evolving in the "conversation of mankind" (Rorty 1979, p. 389).

Following this foundational overhaul, the interest in discourses began crossing disciplinary boundaries and established itself gradually as a unifying motif of all human sciences, from sociology to anthropology, to psychology, and so forth. Throughout human sciences, the discursivity – the fact that all human activities are either purely discursive or imbued with and shaped by discourses – has been recognized as a hallmark of humanity. Nowhere was this realization more evident than in the relatively young brand of psychology known as "discursive" (Edwards 2005) and defined as "one that takes language and other forms of communication as critical in the possibility of an individual becoming a human being" (Lerman 2001, p. 93). As evidenced by the steadily increasing number of studies dealing with interactions in mathematics classroom, the discursive turn has been taking place also in mathematics education research (Ryve 2011).

Foundations

For many discursively minded researchers, even if not for all, the shift to discourse means that some of those human activities that, so far, were considered as merely "mediated" or "helped" by concomitant discursive actions may now be rethought as being communicational in nature. For example, as an immediate entailment of viewing research as a communicational practice, one can now say that the research discipline known as mathematics is a type of discourse, and thus learning mathematics is a discursive activity as well.

Recognition of the discursive nature of mathematics and its learning, if followed all the way down to its inevitable entailments, inflicts a lethal blow to the famous "Cartesian split," the strict ontological divide between what is going on "inside" the human mind and what is happening "outside." Once thinking, mathematical or any other, is recognized as a discursive activity, mental phenomena lose their ontological distinctiveness and discourse becomes the superordinate category for the "cognitive" and the "communicational." This non-dualist position, which began establishing itself in learning sciences only quite recently, has been implicitly present already in Lev Vygotsky's denial of the separateness of human thought and speech and in Ludwig Wittgenstein's rejection of the idea of "pure thought," the amorphous entity supposed to preserve its identity through a variety of verbal and nonverbal expressions (Wittgenstein 1953).

In spite of the fact that the announcement of the ontological unity of thinking and communicating has been heralded by some writers as the beginning of the "second cognitive revolution" (Harré and Gillett 1995), non-dualism has not become, as yet, a general feature of discursive research. More often than not, discursivist researchers eschew explicit ontological commitments (Ryve 2012), whereas their occasional use of hybrid languages brings confusing messages about the nature of the objects of their study. One can therefore speak about weaker and stronger discursive approaches, with the adjective "strong" signaling the explicit adoption of non-dualist stance (Sfard 2008).

The ontological heterogeneity notwithstanding, all discursively oriented researchers seem to endorse Vygotsky's (1978) famous statement that uniquely human learning originates on the "social plane" rather than directly in the world. Consequently, they also view learning as a collective endeavor and recognize the need to always consider its broad social, historical, cultural, and situational context. Strong discursivists, in addition, are likely to claim that objects of discourses – *numbers* or *functions* in the case of mathematics and *conceptions* or *meanings* in the case of researcher's own discourse – grow out of communication rather than signifying any self-sustained entities preexisting the discourse about them. As a consequence, the researchers always keep in mind that any statement on the existence or nature of these entities is a matter of personal interpretation and must be presented as such. Moreover, since the protagonists of researchers' stories are themselves active storytellers, researchers must always inquire about the status of their own narratives vis-à-vis those offered by the participants of their study.

Strands

The current discursive research on learning at large and on mathematics learning in particular may be roughly divided into three main strands, according to perspectives adopted, aspects considered, and questions asked. The first two of these distinct lines of research are concerned with different features of the discourse under investigation and can thus be called intra-discursive or inward looking. The third one deals with the question of what happens between discourses or, more precisely, how inter-discursive relations impact learning.

The first intra-discursively oriented strand of research on mathematics learning focuses on learning-teaching interactions, whereas its main interest is in the impact of these interactions on the course and outcomes of learning. Today, when *inquiry learning*, *collaborative learning*, *computer-supported collaborative learning*, and other conversation-intensive pedagogies (also known as "dialogical") become increasingly popular, one of the main questions asked by researchers is that of what features of small group and whole-class interactions make these interactions conducive to high-quality learning. *Participation structure*, *mediation*, *scaffolding*, and *social norms* are among the most frequently used terms in which researchers formulate their responses. Whereas there is no doubt about theoretical and practical importance of this strand of research, some critics warn against the tendency of this kind of studies for being unhelpfully generic, which is what happens when findings regarding patterns of learning-teaching interactions are presented as if they were independent of their topic.

This criticism is no longer in force in the second intra-discursively oriented line of research on mathematics learning, which inquires about the development of mathematical discourse and thus looks on those of its features that make it into distinctly mathematical: the use of specialized mathematical words and visual mediators, specifically mathematical routines, and narratives about mathematical objects that the participants endorse as "true." Comparable in its aims to research conducted within the tradition of conceptual change, this relatively new type of study on learning is made distinct by its use of methods of discourse analysis, and this means, among others, its attention to contextual issues, its sensitivity to the inherent situatedness of learning, and its treatment of the discourse in its entirety as the unit of analysis, rather than restricting the focus to a single concept. Questions asked within this strand include queries about ways in which learners construct *mathematical objects*, develop *sociomathematical norms*, engage in *argumentation*, or cope with uneasy transitions to *incommensurable discourses*. Methods of systemic functional linguistics (Halliday 2003) are often employed in this kind of study. One of the main tasks yet to be dealt with is to forge subject-specific methods of discourse analysis, tailored according to the distinct needs of the discourse under study. Another is to explore the possibility of improving school learning by overcoming its situatedness. Yet another regards the question of how mathematical learning occurring as if of itself while people are dealing with their daily affairs differs from the one that takes place in schools and results from teaching.

Finally, the inter-discursively oriented studies inquire about interactions between discourses and their impact on learning. This type of research is grounded in the recognition of the fact that one's participation in mathematics discourse may be supported or inhibited by other discourses. Of particular significance among these learning-shaping aspects of communication are those that pertain to specific cultural norms and values or to distinct ideologies. Studies belonging to this strand are often concerned with issues of *power*, *oppression*, *equity*, *social justice*, and *race*, whereas the majority of researchers whom this research brings together do not hesitate to openly admit their ideological involvement. The notion of *identity* is frequently used here as the conceptual device with which to describe the way cultural, political, and historical narratives impinge upon individual learning. Methods of critical discourse analysis (Fairclough 2010) are particularly useful in this kind of study.

Methods

As different as these three lines of research on learning may be in terms of their focus and goals, their methods have some important features in common. In all three cases, the basic type of data is the carefully transcribed communicational event. A number of widely shared principles guide the processes of collection, documentation, and analysis of such data. Above all, researchers need to keep in mind that different people may be using the same linguistic means differently and that in order to be able to interpret other person's communicational actions, the analysts have to alternate between being insiders and outsiders to their own discourse: they must sometimes look "through" the word to what they usually mean by it, and they also must be able to ignore the word's familiar use, trying to consider alternative interpretations. For the same reason, events under study have to be recorded and documented in their entirety, with transcriptions being as accurate and complete records of participants' verbal and nonverbal actions as possible. Finally, to be able to generalize their findings in a cogent way, researchers should try to support qualitative discourse analysis with quantitative data regarding relative frequencies of different discursive phenomena.

The admittedly demanding methods of discourse analysis, when at their best, allow the analyst to see what inevitably escapes one's attention in real-time conversations. The resulting picture of learning is characterized by high resolution: one can now see as different things or situations that, so far, seemed to be identical and is able to perceive as rational those discursive actions that in real-time exchange appeared as nonsensical.

Cross-References

▶ Argumentation in Mathematics
▶ Collaborative Learning in Mathematics Education
▶ Equity and Access in Mathematics Education
▶ Inquiry-Based Mathematics Education
▶ Mathematics Teacher Identity
▶ Scaffolding in Mathematics Education
▶ Sociomathematical Norms in Mathematics Education

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Down Syndrome, Special Needs, and Mathematics Learning

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Keywords

Genetic disorder; Mathematics difficulties; Number difficulties; Cognitive impairment

Characteristics

Down syndrome is a genetic disorder which has serious consequences for cognitive development. Most children with Down syndrome show mild to moderate cognitive impairments with language skills typically being more severely impaired than nonverbal abilities (Næss et al. 2011). Children with Down syndrome are frequently reported to have problems with short-term and working memory. While a relatively large number of studies have investigated the language and reading skills (Hulme et al. 2011) of children with Down syndrome, much less is known about the development of number skills in this group.

Early case studies and studies using highly selected samples have reported some relatively high levels of arithmetic achievement in individuals with Down syndrome. However, for the majority of individuals with Down syndrome, simple single digit calculations and even counting represent a significant challenge (Gelman and Cohen 1988). Carr (1988) reported that more than half of her sample of 41 individuals aged 21 years could only recognize numbers and count on the Vernon's arithmetic-mathematics test. Buckley and Sacks (1997) surveyed 90 secondary school-age children with Down syndrome in the and found that only 18 % could count beyond 20 and only half of the sample could solve simple addition problems.

Studies conducted on larger samples consistently report low arithmetic achievement in individuals with Down syndrome relative to other scholastic skills such as reading accuracy (Hulme et al. 2010; Buckley and Sacks 1987; Carr 1988). Age equivalents on standardized number tests are typically reported to lag age equivalent reading scores by around 2 years in children with Down syndrome (e.g., Carr 1998).

Arithmetic performance is reported to improve with chronological age in children with Down syndrome, but this varies widely within IQ levels and is not true for all children (e.g., Carr 1988). It seems highly plausible that a relationship might exist between IQ level and arithmetic performance level, but thus far, there is no consensus in the literature. Education has a positive influence on arithmetic performance as might be expected, and individuals in mainstream school are reported to achieve higher levels of mathematical attainment compared to special school (e.g., Carr 1988).

Individual differences in response to intervention are primarily determined by quality and quantity of teaching (Nye et al. 2005). In the UK, Jo Nye has written a book on adapting Numicon for use with children with Down syndrome, "Teaching Number Skills to Children with Down Syndrome Using the Numicon Foundation Kit." In the USA, DeAnna Horstmeier has written a book titled "Teaching Math to People with Down Syndrome and Other Hands-On Learners: Basic Survival Skills." More research is needed to determine the origin of the difficulties that individuals with Down syndrome before a theory driven intervention program can be designed.

Cross-References

- ▶ 22q11.2 Deletion Syndrome, Special Needs, and Mathematics Learning
- ▶ Autism, Special Needs, and Mathematics Learning
- ▶ Deaf Children, Special Needs, and Mathematics Learning
- ▶ Inclusive Mathematics Classrooms
- ▶ Language Disorders, Special Needs and Mathematics Learning
- ▶ Learning Difficulties, Special Needs, and Mathematics Learning

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