Interactions Between Teacher, Student, Software and Mathematics: Getting a Purchase on Learning with Technology

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Abstract In this chapter three examples of teacher-guided use of ICT stimuli for learning mathematics (screencast, animation and applet) are critically examined using a range of distinctions derived from a complex framework. Six modes of interaction between teacher, student and mathematics are used to distinguish different affordances and constraints; five different structured forms of attention are used to refine the grain size of analysis; four aspects of activity are used to highlight the importance of balance between resources and motivation; and the triadic structure of the human psyche (cognition, affect and enaction, or intellect, emotion and behaviour) is used to shed light on how affordances may or may not be manifested, and on how constraints may or may not be effective, depending on the attunements of teachers and students. The conclusion is that what matters is the way of working within an established milieu. The same stimulus can be used in multiple modes according to the teacher's awareness and aims, the classroom ethos and according to the students' commitment to learning/thinking. The analytic frameworks used can provide teachers with structured ways of informing their choices of pedagogic strategies.

Keywords Interaction • Teacher-guided • Ways of working • e-screens • Screencast • Animation • Activity

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Introduction

What roles can teachers play in using e-screens[1] to support interactions with students and mathematics? How might teachers' pedagogic choices be informed? The questions being explored in this chapter concern the affordances of teacher-guided use of ICT for stimulating interactions between teacher, student and mathematics. Attention is restricted to interactions which begin with the teacher taking initiative, either because the applet itself is a screencast of a tutor, or because the use of the applet is directed by the teacher.

Ordinarily one expects to find a description of theoretical constructs before being told the method undertaken for the collection of data and the theoretical frame(s) for its subsequent analysis. However my approach is fundamentally experiential, which means that the data being offered are what arises in the reader through what they notice (what comes-to-mind) while reading and undertaking task-exercises. The analysis consists of a narrative to account for observations that I have made which may resonate with what others have observed, or as in the case here, to give an account of affordances based on experience in multiple settings. The empirical aspect of these studies lies not in my presenting my data here, but in generating recent experience in the reader. The analysis is informed by experience. Note the parallel with teaching and learning mathematics: experience can inform action-in-the-moment without being used to try to convince others using extra-spective data collected in some other situation.

The following assumptions provide an overview of the theoretical constructs being used, but these are only elaborated after you have had some exposure to the specific stimuli being considered.

A0: The human psyche involves cognition, affect, behaviour and attention-will.
A1: Teaching takes place in time and learning takes place over time.
A2: Action requires three roles to be filled: initiating, responding and mediating, and each of these roles can be played by the teacher, the student and the content.
A3: Effective activity requires a balance between motivation and resources.
A4: One thing that we do not seem to learn from experience is that we do not often learn from experience alone. Tasks are provided for students to initiate activity, which provides experience and, in order to learn effectively from experience, it helps to adopt a reflexive stance.
A5: Aligning teacher and student attention improves communication.

The affordances of the three forms of e-screen stimuli arise from the form of relations amongst discerned details in what is experienced. These relations are suggestive of general properties, which apply to many situations, being instantiated in the particular. Validity of these general properties can be tested by considering whether the proposed narrative fits or resonates with recent personal experience; whether the distinctions made help make sense of personal past experience; and most importantly, whether this articulated experience informs future practice through being sensitised to notice opportunities to act freshly and more effectively (Mason 2004).

[1] I use 'e-screens' to refer to electronic screens, as distinct from the mental 'screen' which is the domain of mental imagery.

Three Studies

The studies offered here all involve the use of an e-screen to initiate activity, in the form of a screencast, an animation, and an applet. Little will be achieved simply by reading the accounts however, since it is necessary to experience the stimuli for yourself, perhaps sensitised by assumptions A0 through A5.

ScreenCasts

With Jing and related software it is easy to record short videos showing work on mathematical problems or conceptual animations. There is a set of them at www.maths-screencasts.org.uk (set up July 2011; accessed Feb 2012) or Khan Academy (www.khanacademy.org accessed Mar 2012).

> Pick one of the screencasts, say the one on Lagrange multipliers:
> http://www.maths-screencasts.org.uk/scast/LagrangeMult.html
> What are you attending to as the screencast proceeds?
> What learning is afforded by watching the screencast?
> What would a student have to do to learn something from the screencast?

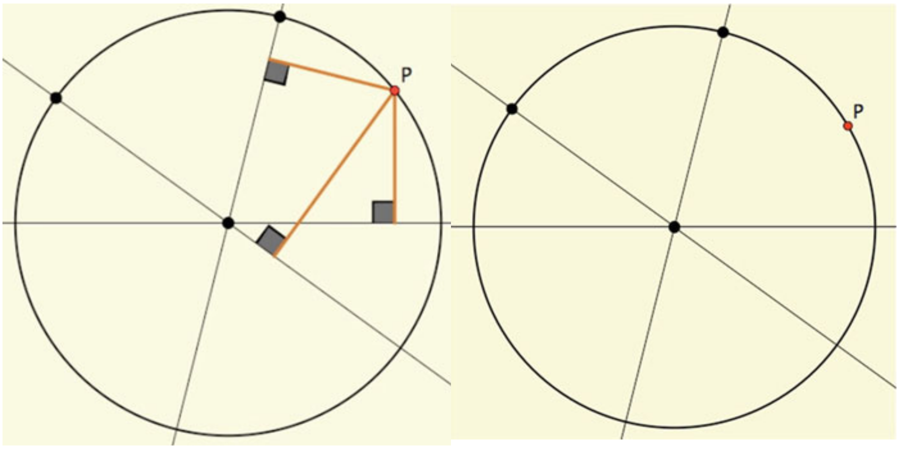
On the surface, the task for students is to make mathematical sense of what is presented, and to increase their confidence that they can tackle a similar problem effectively in the future. The question of what constitutes a 'similar problem' might need to be discussed explicitly. During the screencast your attention may have shifted between what was on the screen and what was being said, and drawn to the symbols being written and spoken at the same time. Would a student watching this know how the presenter knew to perform the actions she does?

Rolling Polygons

At the heart of this task is an animation, however the presentation begins with setting the scene by inviting the use of mental imagery.

> Imagine a point P moving in a circle centred at point C. Imagine a finite number of lines (at least 3) being drawn through C. From P drop perpendiculars onto your lines and mark their feet as $F1$, $F2$, $F3$, … . Now join $F1$, $F2$, … in sequence to form a polygon. What happens to the polygon as P moves around the circle?

Changing *P* mentally is pretty difficult, and so the task proper begins with an animation involving a triangle (downloadable from ref http://extras.springer.com*; double click on right hand figure to see animation).



What role is played by the initial mental imagery?
What are you attending to as the animation proceeds?
What actions are stimulated by watching the animation?
What would a student have to do to learn something from the animation and the applet?

On the surface, the initial task for students is to become aware of and be at least somewhat surprised by a phenomenon, and then to begin to seek an explanation for that phenomenon. This in turn is likely to call upon students' powers to make deductions about angles using previously encountered fact such as the effects on angles of rotating lines through 90°, or angles in a quadrilateral with two right angles and angles subtended at a circle on the same side of a chord, etc..

The initial mental imagery is intended to contribute to the 'reality' of the task through exercising a fundamental human power and evoking curiosity as to what might happen. It sets the scene. This is a pedagogic strategy that can be used in many situations, because imagining ourselves doing something in the future is the basis for planning.

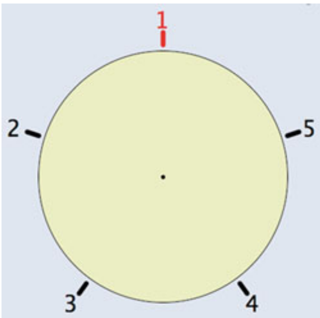
Secret Places

This task and its applet support is intended for teacher-led exploration, though it can be used by individuals or small groups working without the teacher.

*Log in with ISBN 978-94-007-4638-1.

Initially there are five places around a table, and one of them has been selected as a 'secret place'. You can probe any place, but all you will be told is whether it is 'hot' (meaning either it or one of the adjacent places on either side of it is the secret place), or 'cold'. How can you most efficiently (least number of probes) discover the secret place?

In the applet (http://extras.springer.com*), if you click on a place, it will show either 'red' or 'blue'. Red signals that the secret place is either the one chosen, adjacent to it, whereas blue signals that this is not the case.



What are you attending to as you explore the effect of clicking on places?
What actions are stimulated by predicting, justifying and then clicking?
What would a student have to do to learn something from a teacher-led search for a strategy?

Applet available for download at
http://mcs.open.ac.uk/jhm3/Applets%20&%20Animations/Reasoning/Secret%20Places/Secret%20Places%201D.html (Set up Feb 2011; accessed Oct 2012).

On the surface, the task for students is to locate the 'secret place' as efficiently as possible, thus drawing on their natural powers to imagine (what could happen if they click somewhere) and to reason (possible consequences of clicking and whether that would be helpful).

For all three stimuli, what matters is what students do next, having encountered the stimulus; what ways of working have been established; and what sort of atmosphere students are used to.

Elaboration of Assumptions

The assumptions that follow make no direct reference to the use of technology. However, later in the chapter the descriptions of interactions with the e-screens include this necessary elaboration.

*Log in with ISBN 978-94-007-4638-1.

A0: The Human Psyche Involves Cognition, Affect, Behaviour and Attention-Will

This first assumption is implicit in Western psychology, but has its roots in ancient Eastern philosophy-psychology (Ravindra 2009; Mason 1994b). Despite this, it is all too easy to forget to engage the whole of students' psyche.

Consequences

Gattegno (1970) placed the notion of *awareness* at the core of his *science of education.* By *awareness* he meant 'that which enables action', which includes the somatic (eg. control of breathing, heart-rate, perspiration etc.) and the automated or internalised, as well as both the subconscious (eg. Freudian and other impulses) and the conscious. Gattegno claimed that it is awareness that can be educated, and indeed that that is all that can be 'educated': *only awareness is educable*. With the sample ICT uses, this raises the question of what awarenesses are available for educating due to the affordances of the ways of working and the medium used.

By contrast, *only behaviour is trainable*. This conforms with an image found in several of the Upanishads (Rhadakrishnan 1953, p. 623; Mason 1994), in which the human psyche is seen as a chariot. The chariot itself is seen as a metaphor for the body and hence for behaviour.

from website: http://members.ozemail.com.au/~ancientpersia/page8a.html

The horses drawing the psyche-chariot represent the emotions (affect). These are the source of energies which are made available to the psyche. Thus *only emotion is harnessable*. Emotion is the way that we access energy which acts through the disposition of various selves that take charge in the individual. All of these contribute to the setting in which attention acts, which many philosophers equate with the will, since as William James (1890, p. 424) observed, "each of us literally *chooses*, by his way of attending to things, what sort of a universe he shall appear to himself to inhabit". It is not simply what you are attending to, but how you are attending to it that matters (Mason 1998, see A5). A mixture of surprise and curiosity draws on psychic energy accessed through the emotional-affective charge channelled according to the habits of the active 'self'. The issue is developing a milieu in which mastering the mathematical aspect of a situation matters to students.

Trained behaviour is essential, being the manifestation of automated functioning and habits, but on its own it can be limiting and inflexible, whereas coupled with awareness the two together can stimulate and exploit the energy that is often called creativity. None of the studies offered here are directly intended to train behaviour concerning the carrying out of a mathematical procedure, though they could be used to train behaviour concerning collective and individual mathematical thinking.

Assumption $A0$ can be used to probe implications of the adage 'practice makes perfect' which is the foundation stone of behaviourist theories of how learning takes place. Certainly it is necessary to integrate behaviours into psycho-somatic functioning. However, repetition alone is no more likely to lead to internalisation than is constant exposure to the same idea. Even stimulus–response (Skinner 1954) is only effective in certain circumstances. As Piaget (1970) pointed out under the label *genetic epistemology*, the individual is an active agent, constructing her own narrative. From a Vygotskian perspective, narrative construction is based in and takes place within socio-cultural milieu. The role of a teacher is to direct attention towards appropriate narratives which constitute conceptual understanding (Bruner 1990; Norretranders 1998), and to provoke students to integrate appropriate action or functioning through subordinating attention (Gattegno 1970; Hewitt 1994, 1996). 'Integration through subordination' is achieved by withdrawing the attention initially required to carry out an action so that the action can be carried out in future while absorbing a minimum of attention.

Henri Poincaré (1956) expressed surprise that people find mathematics difficult to learn, because from his perspective mathematics is entirely rational, and humans are rational beings. Jonathon Swift (1726) had already challenged this notion, proposing that human beings are at best 'animals capable of reason'. If rational reasoning is not activated, mathematical thinking is likely to be experienced as mysterious. The success of behaviourist strategies based on stimulus–response combinations shows that Swift was correct: people can be trained and enculturated into certain types of behaviours and this can be partly conscious and partly unwitting on their part. They can be successful in routine situations such as tests, but their success is short-term unless routines are frequently rehearsed. Such training only takes you so far. Once will is activated, attention wanders, different selves with different energy flows and dispositions come into play, and learning becomes much more complex. Hence the need to educate awareness as well as to train behaviour.

To be responsible for your own learning is a commonplace sentiment that fits with Western democratic values. The word *responsible* has roots in parallel with the Italian *spondere* which means 'to be able to justify actions' (to respond). Jürgen Habermas (1998) began from a similar position to Poincaré's assumption of rationality, but he focused on responsibility, which he cast in terms of justification:

> "The rationality of a person is proportionate to his expressing himself rationally and to his ability to give account for his expressions in a reflexive stance. A person expresses himself rationally insofar as he is oriented performatively toward validity claims: we say that he not only behaves rationally but is himself rational if he can give account for his orientation toward validity claims. We also call this kind of rationality *accountability* (Zurechnungsfähigkeit)." (Habermas 1998, p. 310 emphasis in the original, quoted in Ascari 2011, p. 83)

He delineated three different domains or types of rational justification:

Epistemic: factual; assertive (epistemic rationality of knowledge)
Teleological: intentions behind actions (teleological rationality of action)
Communicative: attempts to convince, requiring listener acquiescence (communicative rationality of convincing), with both a *weak* and a *strong* form.

The point is that justification is an essential core component of mathematical thinking, as well as involvement in society. A successful practitioner without a narrative by means of which to justify choices on the basis of explicit criteria is at the mercy of habits in the face of changing conditions (Mason 1998).

The psyche operates within a socio-cultural-historical milieu with its undoubtedly important influences, most especially the atmosphere or ethos developed in the classroom or other setting and the social pressures from peers and from institutional norms (Brousseau 1997). One of the difficult things about online activity is that it is much harder to influence from a distance the atmosphere in which students are working than it is in face-to-face interactions.

A1: Teaching Takes Place in Time; Learning Takes Place Over Time

Despite the desire by government to have inspectors witnessing learning, learning is a maturation process. It requires time (Piet Hein 1966). Gattegno (1987) went so far as to suggest that learning actually takes place during sleep, when our brains choose what sense-impressions from the day to let go of. Thus memory is not about storing but about making and breaking links. Learning is reinforced not simply through re-encountering similar actions, activities and experience in fresh contexts, in what Bruner (1966) referred to as a spiral approach to the curriculum, but through developing an increasingly complex narrative to accompany the developing richness of connections.

Development

At the Open University (1981) we used the trio of *see–experience–master* (SEM) to emphasise that 'learning' does not take place on first encounter, nor even after some further experience. The triple can act as a reminder that encountering new ideas is a bit like being in a train station. An initial encounter is like seeing an express train go by when details are hard to make out and it all seems off-putting or complicated. With continued encounters there is growing familiarity, a bit like seeing a freight train rumble by. Eventually there is a degree of confidence and mastery, as when a passenger train stops and you get on and go-with the idea.

We translated Bruner's three modes of (re)presentation (enactive-iconic-symbolic) into a spiral of *manipulating-the-familiar, getting-a-sense-of,* and *articulating* that sense (*MGA*) as a reminder that it is natural to use what is familiar in order to get a sense of underlying relationships which, when articulated more and more succinctly eventually become confidence-inspiring and familiar for use in further manipulation.

MGA fits well with the principle of variation (Marton and Booth 1997; see also Watson and Mason 2005, 2006): learning a concept is becoming aware of what aspects of an example can be varied, and over what range, while remaining an instance of the concept. What is available to be learned is what is varied in a succession of experiences in contiguous space and time. Spiral learning and exposure to variation in key aspects is sometimes replaced by frequent repetition of nearly identical tasks in an attempt to train behaviour. However, if attention is not drawn (explicitly or implicitly) to carefully engineered variation of key aspects, the result may be successful performance on routine exercises, without educating awareness. It may also all too easily have a negative influence on disposition to engage, with students only willing to undertake what they know they can already succeed at.

Consequences

Learning, seen as educating awareness, training behaviour and harnessing emotion within a particular milieu, can be cast in terms of developing dispositions to attend in appropriate ways. A teacher cannot 'do the learning' for students. Indeed, the more they try to indicate to students the behaviour being sought as evidence of learning, the easier it is for students to display that behaviour without actually generating it for themselves, without educating their awareness (this is the *didactic tension* first articulated by Brousseau: see Brousseau 1997). What a teacher *can* do is participate in the various possible modes of interaction with students, without looking for evidence of 'learning' in too short a term (Piet Hein 1966). It often takes time to integrate a way of acting into your own functioning, even when this is stimulated by efficient and effective pedagogy (integration through subordination of attention).

Implications for Teaching

When choosing or designing task-sequences SEM and MGA can act as reminders when choosing or designing task-sequences to arrange for multiple encounters, and within each encounter, multiple instances with relevant variation. Learning is seen as a maturation process, like baking bread or brewing beer. It takes time. When rushed, the tendency is to revert to superficial success through routine exercises carried out using templates based on 'worked examples'.

SEM and MGA can also act as reminders that ‘responsible learning’, that is, having access to justifications for actions initiated is a gradual process, as complex narratives take time and multiple encounters with phenomena in order to come to articulation. As the articulation becomes more succinct and familiar, it becomes available as a component in yet further development.

A2: Action Requires Three Roles to be Filled: Initiating, Responding and Mediating

Following Bennett (1966, 1993) who developed a framework called *Systematics*, based on the quality of numbers, action has the quality of three-foldedness. Action requires an initiating impulse, a responding impulse and a mediating impulse. Without the mediator, there is nothing to bring or hold the initiating and responding together. Put another way, any action takes place within a context or milieu (Brousseau 1997) that enables the action to take place.

Consequences

From this perspective, interaction between a teacher-tutor, a student, and mathematics can take place in one of the six combinatorially distinct ways of arranging these three components in the three roles (Mason 1979). For convenience these six modes are known as the six ex’s: Expounding, Explaining, Exploring, Examining, Expressing, Exercising, all within a milieu consisting of institutional affordances and constraints (including classroom and institutional social norms and demands). The milieu also includes the focal world(s) or spaces of the participants. Usually this consists of the mental worlds in which people dwell and from which they express their insights, but the presence of virtual screen-worlds provides a more explicitly taken-as-shared world of experience, namely the world of phenomena acted out on, and interacted with, a screen (Mason 2007).

The key feature for consideration here is the mediating or reconciling contribution of one of these roles so as to bring the other two into relation, and so as to sustain that relation for long enough for the action to reach fruition, leading to a result that can partake in further actions. The use of electronic screens associated with the tasks suggested above centres on the teacher as initiating impulse, and so draws particularly on the interactions summarised as *expounding* and *explaining*, although there are plenty of opportunities to shift into other modes from time to time.

Expounding is characterised by the presence (actual or virtual) of students bringing the teacher into contact with the mathematics in a special way. The term *pedagogic content knowledge* (Shulman 1986) has been used to describe what is needed in order to carry through this action effectively, while others try to capture it by describing the *knowledge needed for teaching* (Davis and Simmt 2006). Here the focus is more on the experience of the action as the teacher crafts tasks for students through contacting the didactic peculiarities of the topic, calling upon relevant pedagogic strategies. The effect of the action is to draw the students into the world and mind-set of the teacher. When the micro world or software plays the role of teacher, the quality of the action depends on the quality of the preparation of the software, which requires sensitivity to student experience and deep knowledge about didactic tactics and pedagogic strategies in relation to classic misunderstandings and misapprehensions within the particular topic.

Explaining is used in this way of thinking with a non-standard meaning. It is characterised by the teacher making contact with the thinking of the student, entering the student's world, centred on, made possible by, and hence mediated by the particular mathematical content. As soon as the teacher experiences "Ah that is where the difficulty lies", there is likely to be a shift into expounding. Staying with the world of the student involves 'teaching by asking' and 'teaching by listening' (Davis 1996) rather than teaching by telling. The more usual sense of *explain* as 'to make plain' is highly idiosyncratic, because what is 'plain' to the speaker may not be 'plain' to the audience. Thus the usual meaning of *explaining* is usually an instance of the action of *expounding*.

In relation to the previous axiom concerning teaching taking place in time, calling upon modes of interaction in which students play the initiating role, and those in which the content plays this role, can at least balance the student experience of modes of interaction, and can provide opportunity for *exploring* the ideas (teacher mediates between content and student); *expressing* (students feeling the need to construct their own narrative, so the student mediates between the content and the teacher); *exercising* through practising what needs to be practised (the teacher mediates between the student and the content by providing exercises); all in preparation for *examining*, in which students' own developing criteria for whether they are understanding and appreciating appropriately are tested against the expert's criteria (the content mediates between student and teacher).

Implications for Teaching

Perceiving actions in which one participates as involving three impulses within a milieu can transform teaching by altering what a teacher attends to, and how, and also how they see their contribution. Arranging the energies of the classroom so that as teacher you can dwell in mediating or in responding can be exhilarating as well as liberating for students. Provoking students into experiencing the desire to express promotes the maturation of their understanding and their appreciation of what they are integrating into their functioning, that is, the education of their awareness.

A3: Effective Activity Requires a Balance Between Motivation and Resources

Following Bennett (*op. cit.*), activity involves two axes: motivation and operation within a world of attention. Motivation in an activity has to do with the perceived

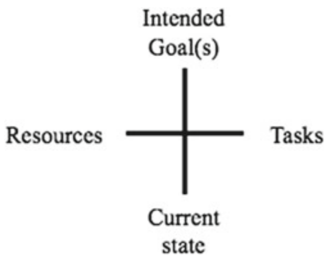


Fig. 1 Structure of activity in Systematics

gap between goals or aims and current state. It is 'what matters' to the student (Fig. 1). This conforms with a Vygotskian perception of activity but with the addition of a tension or gap between current state and goal, which plays the role of disturbance identified by Heidegger (1962) and many others (e.g. Festinger 1957; Piaget 1971) as what activates learning (leading to assimilation and accommodation).

The second axis concerns the resources available (both those brought by the students and those provided by the environment) and the tasks provided. If the resources available are inadequate for the gap between current state and goal, or if the tasks do not actually provide sufficient stimulus to reach the goal, then the activity will be ineffective.

Resources include student propensities and dispositions, and learner access to their natural powers such as stressing and ignoring, imagining and expressing etc. Where student powers are usurped by textbooks or modes of interaction with the teacher, students soon learn to park their powers at the door as not being required, and so become dependent on the teacher to initiate mathematical actions.

Tasks are inherently multiple by nature: as conceived by the author; as intended by the teacher; as construed by the student(s); as enacted by the students; and as recalled in retrospect by the student(s). Tahta (1981) pointed out that there are different aspects of a task: the outer task is what the task states (and is interpreted as by students), whereas the inner task is implicit, and has to do with mathematical concepts and themes that may be encountered, powers that may be used, and propensities that may come to the surface, all contributing to educating awareness.

In the language of affordances, constraints and attunements (Gibson 1979), affordances arise from the relationship between resources and tasks. The constraints are usually imposed from the tasks, for as is well known, creativity only takes place when there are constraints. Both student attunement and teacher attunement contribute to the motivational and the operational axes.

Ainley and Pratt (2002) distinguish between *purpose* of a task as the local context which gives learners a purpose in undertaking it, and the *utility* of a task or a technique in terms of the range of situations in which it can be used in the future. Both contribute to the development of positive or negative dispositions and propensities.

The two-axis structure of activity provides a richer structure than that provided by the adage 'start where the learners are'. Indeed, calling upon the whole psyche, and mindful of Vygotsky's distinction between *natural* and *scientific* knowledge, the effective teacher ‘starts’ where the learners could be rather than where they are, by invoking their energies through surprise or a sense of a gap so that they strive to move along the motivational axis, supported by access to appropriate resources and well judged tasks.

A4: One Thing We Don’t Seem to Learn from Experience, Is that We Don’t Often Learn from Experience Alone

Evidence for this is widely available, as you try to remember what you have read in the newspaper, what you saw on television, even what you set out to accomplish when you went into another room. What students get from engaging in an activity is highly variable, as Jaworski (1994) found when she asked students what a lesson had been about in which the task as set had been to draw and cut out copies of quadrilaterals and see if they would tessellate. Many students reported that the lesson was about ‘cutting out quadrilaterals’, ‘using scissors’, etc., and only a few mentioned tessellation. This reinforces the observation that different students attend to different things, stressing some things and ignoring others, and that even when they are attending to what the teacher intends, they may be attending in different ways (Mason 2003).

The student’s stance towards learning, delineated by Marton and Saljö (1976) as a mixture of *surface*, *deep* and *strategic* approaches, colours all of learners’ actions, and the closer they are to the strategic–surface, the more likely it is that task-completion characterises their epistemological stance. Even participation in suitable activity may not lead to the intended learning. Many students act as if their role is to attempt the tasks they are set, and that somehow those attempts will be sufficient to produce the expected learning. This epistemological stance is the basis of the *didactic contract* (Brousseau 1997). However tasks are supposed to generate activity, through which learners gain experience. Yet “one thing we don’t seem to learn from experience, is that we don’t often learn from experience alone” (Mason 1994a). A reflective stance, a withdrawing from the action in order to become aware *of* the action can make learning much more efficient than without it. To paraphrase William James (1890) “a succession of experiences does not add up to an experience of that succession”. More is required. This is particularly hard to arrange when students are studying at a distance.

Evidence of learning is informed action in the future, which is what some call an *enactivist* stance (Varela et al. 1991) in which *knowing* is the same as *(en)acting*. This requires having an appropriate action come-to-mind (be-enacted) when needed, which brings us back to the education of awareness.

Implications for Teaching

To promote learning, including learning how to learn, it is useful to get learners to withdraw from activity and to reflect not only on which actions were successful and which were not, but on why. A further step is to prompt them to identify actions they would like to have come-to-mind in the future. This is how people most often learn from experience. Construction tasks (Watson and Mason 2005) are very useful for this purpose because they enrich personal example spaces while at the same time exercising techniques.

Stimulating effective reflection involves creativity and sensitivity, because the same prompts used over and over can lead to learners becoming dependent on the teacher rather than developing independence (Baird and Northfield 1992). In order not to train students to depend on the teacher to indicate appropriate behaviour, it is necessary to use both scaffolding and fading (Brown et al. 1989). Another way to express this is to say that the teacher needs to be alert to moving from directing behaviour (instruction) to increasingly indirect prompting as required, until students are spontaneously initiating that action themselves. This is what van der Veer and Valsiner (1991) suggest was intended by Vygotsky's notion of *zone of proximal development* (Mason et al. 2007).

A5: Aligning Teacher and Student Attention Improves Communication and Hence Affordances

Bringing what the teacher and what the students are attending to into alignment is only the beginning of effective teaching; alignment in how the teacher and the students are attending also matters. Different forms of attention include:

Holding Wholes: gazing in an unfocused manner, absorbing the overall, placing oneself in a state of receptivity towards a situation;
Discerning Details: distinguishing entities (which can then be held as 'wholes');
Recognising Relationships between discerned details in the situation;
Perceiving Properties as being instantiated as recognised relationships between discerned details; and,
Reasoning on the basis of agreed properties.

These five 'states' or structures of attention correspond closely with the 'levels' distinguished by Dina van Hiele-Geldof and Pierre van Hiele (van Hiele 1986) with the notable difference that rather than being seen as levels in a progression of development, attention is experienced as shifting rapidly between these states in no specific order.

Implications for Teaching

In order to be helpful to students it is necessary for teachers to be aware not only of what they are attending to in the moment, but how they are attending to it. This enables them to make use of an appropriate mode of interaction and to direct learner attention (however subtly or explicitly) so that either it comes into alignment with their own (cf. *exposition*) or it brings theirs into alignment with that of learners (cf. *explaining)*.

Drawing Threads Together

Tasks are offered to students so that they engage in activity. The activity itself is not sufficient to generate learning. Rather, students need to participate in transformative action in which they experience shifts in the focus and structure of their attention. It is not simply a matter of agentiveness, of converting assenting into asserting (Mason 2009) but of relationship (Wan Kang and Kilpatrick 1992; Handa 2011), of playing various roles in different modes of action. Experience alone is not sufficient, and for most students, especially in order to stimulate the education of awareness as accompaniment to training of behaviour, an explicitly reflexive stance is required, as students become explicitly aware of actions that have proved fruitful and of actions that have not. Imagining themselves in the future initiating those actions can improve the chances that a relevant action will come-to-mind when needed, and this is how development takes place. This is what Vygotsky was getting at with the *zone of proximal development*: the actions that can be used when cued become actions that can be initiated by the student without explicit cues (van der Veer and Valsiner 1991; Mason et al. 2007).

Analytic Narrative Concerning the Three Studies

ScreenCasts

Background

The design of Open University Mathematics Summer Schools in the 1970s was based on a framework known as *Systematics* (Bennett *op cit.*). The format of one type of session introduced was called *Technique Bashing*: a tutor would publicly tackle an examination question, revealing as much as possible of their inner monologue and procedural incantations as they went. The idea was to draw the student into the world experienced by the tutor (a form of expounding). This was a real-time version of a mode of interaction based on tape-frames used in our distance taught courses, in which students listened to a tutor talking through a concept or a technique while directing their attention to a series of printed frames containing key phrases and whatever else needed to be written, together with space for students' own work. There were lots of stop instructions for students to switch modes and take initiative, either *exercising* or *expressing* but also *exploring*, in ways that are not possible in a face-to-face tutorial.

The idea of tape-frames was to have the tutor's voice in the student's head through the use of earphones, and we made use of BBC expertise to develop rules of thumb for linking the audio with the text so that students always knew what to be attending to, so we did not simply read the text out loud. Emphasis was placed on how the tutor knew what to do next, not just on what they did next, and this conforms with a plethora of subsequent research on worked examples indicating that what students want most is to know how the expert knows what to do next (see for example Renkl 1997, 2002).

With readily-available online video (Redmond 2012), students can now access screencasts of tutors displaying worked examples of concepts and techniques in this technique-bashing mode. What is not so clear is how, when recording a tutor's performance, student attention can be provoked to shift from dwelling in the particular (recognising relationships in the particular) so as to see the general through the particular (perceiving properties as being instantiated). Emphasis is on factual (A0) rather than teleological rationality; the person gaining most from bringing to articulation is the performing tutor (communicative rationality).

Questions

Students almost always ask for more examples, as if somehow exposure to sufficient examples will mean that they internalise or learn what is intended. This is a manifestation of the epistemological stance mentioned earlier. Having someone taking me sensitively through the steps, where I can stop and rewind whenever I want, looks like a powerful resource. Thus screencasts of a tutor 'working' typical problems are likely to be popular with students, as any teacher will surmise on the basis of what students ask them for. But what do students actually do with them, and what do students need to do so as to use them effectively and efficiently? How can initiative be shifted back to the student? These are important questions at any time when planning a lesson, but particularly when preparing a self-study resource such as a screencast.

Affordances

One question to be asked is what the student is attending to, and whether the resources required (student background, disposition and concern, and powers) are available. For example, what does the student think is 'typical' or generic about the particular example whose working is displayed in the screencast (A4)? Unless either the students have become used to asking this for themselves, or the tutor is explicit about it in the screencast, many students are likely to recognise at best a limited range of permissible change in the salient aspects that can be varied, and may even overlook some of those 'dimensions' (Marton and Tsui 2004).

Clearly some of the affordances are that the student can pause and back-up at will, as with a tape-frame but unlike a live lecture or tutorial. Constraints are that the examples worked are determined by the screencast. Even if students could choose the example, and a CAS could display the workings step by step, it would be difficult to insert the tutor commentary, especially the inner-incantations, which is what students appreciate (Jordan et al. 2011, p. 13). Of course students would also like to be able to stop and ask questions, but that involves a two-way interaction in real time, at least with current technology.

Here the action involves the tutor-screencast, the mathematical content and the student. The student begins the interaction when experiencing a sense of disturbance at not fully grasping or understanding a concept or the use of a technique (A5). They then choose to run the screencast seeking specific assistance. They may subsequently initiate a change of action by pausing, stopping or rerunning. However once running, the initiative immediately switches to the tutor-screencast in terms of the tutor's words and actions. The student attempts to follow. They need extra energy or initiative to shift from assenting to what they see and hear to asserting (trying their own version).

Commentary

If the student stance is 'watching and listening', then the interaction is typical of expounding (A2): the tutor has, by virtue of imagining the students watching and listening, been brought into contact with the content in a particular manner, presumably with awareness of typical stumbling blocks and sticking points experienced by students. Both mathematical and pedagogical content knowledge are required in order to be effective. Sensitivity to the nature and scope of one's own attention is necessary in order to be effective in aligning student attention with the tutor's attention (A5). One reason for not showing the tutor's face is to reduce distraction, to approximate the sense of the 'tutor in your head' being shown what to do. Even so students may be distracted by unfamiliar accent, turns of phrase, and a possible gap between them wanting 'the answer' and the tutor 'expounding'.

If the student is trying to make contact with the mathematical content, then there may be periods of time when the student is initiating and, if the tutor has focused on what the student seeks to find out, the tutor-screencast can act as the intermediary or mediating force to bring the student into contact with the mathematics concerning relevant issues. Typical of the interaction mode of *explaining* is the teacher trying to enter the world of the student; here the student enters the world of the tutor who is trying to act like a student, a form of pseudo-explaining. The use of short tightly focused screencasts is likely to contribute to their usefulness because students can pick and choose which ones might meet their needs most effectively. This leads to the need for an appropriate organisation of screencasts so that users can find what they are looking for and know what each contains without excessive effort, otherwise they will not be used.

There is a difficult issue of milieu-at-a-distance. It is hard enough to persuade live students that making and later modifying conjectures is preferable to keeping silent until you are certain that you are correct. On a screencast a tutor can display this behaviour, but always at the risk of students losing confidence in the tutor who, for example, might keep correcting themselves (explicitly and intentionally modifying previous conjectures). The tutor in a screencast is a role model for the doing of mathematics. If correct and clear mathematics flows out of a pen on screen then students will imagine that unless this happens for them, they are failing or deficient in some way.

Any explicit support the tutor offers in the screencast may not be detected by the student *as* support, because student attention is likely to be on making sense of the mathematical content. Consequently explicit attention to fading of any scaffolding support is going to be required, either through prompting reflection (A4) or by being increasingly indirect about the prompts and commentary in the screencast.

Ways of Working

There is a curious phenomenon with all screen-based activity, namely, "what does the student do when the show is over" (Mason 1985). The cessation of movement and sound creates a hiatus, not unlike the moment when you finish reading an engrossing novel. In that moment, attention shifts to the concerns of the material world, to what is to be done next; insights, relations and properties experienced during the session can evaporate all too readily. In order to learn from the experience of using a screencast (*A*4), students may need to be trained to pause at or near the end and to ask themselves what they have now understood that they did not before, and what they would now like to do in the future that they might otherwise not have done before. It is tempting to suggest that each screencast needs a linked set of exercises on which the student might be advised to work. However, it is not the doing of multiple exercises that leads to effective and efficient learning (*A*1/*MGA*), but rather the bringing to articulation for oneself of what makes a task belong to the space of exercises (Sangwin 2005) coped with by the technique, and the space of examples (Watson and Mason 2005) associated with a concept. The most powerful study strategy a student can use is to construct their own exercises and their own examples of concepts. Effective learning involves training students to 'learn how to learn' (Shah 1978; Claxton 1984).

It is less than clear how watching a screencast, however often, contributes to learning in the sense of the student having an appropriate action come-to-mind in the future as a consequence of interacting with the screencast. It seems that what matters is what activity the student engages in using the screencast as stimulus. Screencasts begin as the tutor *expounding* the use of a technique to solve a particular exercise (*A*2). The tutor is of course aware of the specific exercise as an example of a class of similar exercises. They see the particular as an instance of the general (*perceiving properties A5*). The students, however, see the particular. They may need extra stimulus to see the general through the particular, perhaps in the form of explicit meta-comments by the tutor who draws their own attention, and that of the students, out of the immediate activity so as to become aware of the actions being employed. There is a vast literature on the effectiveness of worked examples (see Atkinson et al. 2000) which could inform the way in which worked examples are presented on screencasts so as to maximise their usefulness and effectiveness for students.

For students who know what they do not know, screencasts could be very effective in clarifying the components of a technique, enriching a concept, or alerting students to mathematical powers, themes and heuristics. Their effectiveness will depend as much on the disposition and agency of the student as on the quality of the awareness exhibited in the screencast.

In some commercial collections such as the Khan Academy (*op cit.*) there is evident lack of sensitivity to classic student misapprehensions, such as for example, confusing the name of a person with their age when working on age related word problems (Word Problems 3). What the student encounters from the screencast is some measure of excitement-concern but manifested as behaviour without access to the thinking that brought that behaviour to mind as being appropriate to the situation.

Extensions

It would be useful to develop screen casts that display other aspects of learning and doing mathematics such as:

The mathematical use of various human powers (imagining and expressing, specialising and generalising, conjecturing and convincing: see Polya 1962 or Mason et al. 1982/2010) in a multitude of contexts;
The recognition of mathematical themes (such as doing and undoing, invariance in the midst of change, freedom and constraint see Gardner 1992, 1993a, b, c); and,
Example construction, including counter-example construction (see Watson and Mason 2005; Mason and Klymchuk 2009).

Rolling Polygons

Background

Mathematical animations have been used for over 50 years to introduce topics, to stimulate exploration and to provide a context for applying ideas to new contexts (Salomon 1979; Tahta 1981). A particularly effective way of working with animations, posters and mental imagery was developed by a group called Leapfrogs (1982) and involves watching (on an actual or a mental screen), then reconstructing what was seen, leading to mathematical interpretation and seeking justification for conjectures about relationships that were articulated.

The Rolling Polygon animation was made in order to offer experience of a range of ‘ways of working’ including a ‘silent start’ to a lesson or task, reconstruction, discussion, conjecturing, reasoning and justifying, and reflection (A1, A2, A3). These can all be used in many different contexts beyond animations.

Here the factual rationality is of little import, although one affordance is to bring to attention the way in which mathematical thinking depends on recognising factual relationships encountered in the past as being present. Put another way, relationships recognised in the current situation may be perceived as instances of more general properties.

Narrative

Attention is at first, naturally enough, directed towards the point *P* moving around the circle. Watching an animation and then reconstructing it proves to be an effective way of aligning student attention. The film invites the conjecture that the triangle remains the same shape independently of the position of *P* on the circle. This may or may not be experienced as the more technical description ‘congruent’; it may not emerge until reconstruction of what was seen. The film also invites the conjecture that a point on the triangle traces an ellipse as *P* moves around the circle. There are implicit generalities which, if expressed as conjectures, give substance to conjectured relationships as properties of a whole class of phenomena. Thus the size of the circle, the angles between the lines and the position of the point on the triangle could all be varied.

In terms of variation theory, what is likely to stand out for most people is the invariance of the shape of the triangle. Astute observation may reveal that the angles of the triangle are the angles between the lines. Such an observation, treated as a conjecture, might lead to a shift in what is attended to, and how. The presence of the right-angles, for instance, could trigger the possibility of cyclic quadrilaterals or of diameters of a single circle. Choosing between alternative relationships to pursue is an important feature of mathematical problem solving.

Commentary

This task is typical of *phenomenal mathematics* (Mason 2004, 2008) in which a mathematical theorem or technique is introduced by displaying a phenomenon. When the phenomenon is surprising, many students are moved to want to explain it, to make sense of it and to explore possible variations which leave the phenomenon invariant (A0). At first the fact that the triangle appears to remain invariant in shape but not location is a surprise. The fact that a point on the triangle follows an ellipse is equally surprising, and leads to questions such as predicting the positions of the foci from the shape of the triangle, or determining under what conditions the locus will be a circle. If the triangle shape remains invariant, then it must be a rotation of the original, so one possibility is to seek the centre of that rotation, which could then lead to a justification of the first conjecture.

As with any challenging geometrical relationships, there are opportunities to catch shifts in both what is being attended to and what is being stressed (A5). Familiarity with stressing and consequent ignoring (Gattegno 1970) could open up questions about what is being ignored (and that might fruitfully be stressed!).

Affordances

The ‘silent presentation’ of the task, coupled with its surprise offers, students the opportunity to pose themselves problems as a way of making sense. It provides a task in which everyone can participate because it draws upon known resources (A3). Beginning with an invitation to imagine, before seeing the diagram, affords an opportunity to work on strengthening the power to form mental images (which may be pictorial, verbal, kinaesthetic or some combination of all three). The imagery instructions are in expounding mode, but as soon as surprise is experienced, there can be shifts to other modes such as exploring and expressing (A2). Describing how the film unfolded without recourse to a diagram or the film itself provides an opportunity to express what is being imagined or re-imagined (A5). Various possible approaches may begin to come to mind, so there is an opportunity to park ideas as they emerge so that an efficient and insight-generating approach can be selected.

Considering what can be changed while preserving the phenomenon is further opportunity to imagine and to express, and to conjecture various generalisations. Seeking a justification for the initial phenomenon may lead to recognition of relationships that are expressed as properties in some standard geometrical theorems. Reflecting on that reasoning can lead to increasing the scope of generality of the phenomenon itself.

Trapping the intentions (teleological rationality A0) behind approaches taken is really only possible by intentional withdrawal from action and reflection upon that action.

Ways of Working

Animations lend themselves to a way of working in which individuals collectively experience a phenomenon, then mentally re-play it for themselves, before joining others to try to reconstruct the ‘plot’, the sequence of images. This in turn alerts attention to critical details that can be examined on a second viewing. Thus shifts of mode of interaction can be rapid and multiple, providing a range of roles for students, teacher and mathematics (A2). Once a reasonable account of what was seen begins to develop, people naturally want to account-for the phenomenon, but it is particularly valuable to try to separate accounts-of and accounting-for, if only because that is vital when interpreting classroom video (Mason 2004) or when cooperating in a collaborative peer group.

Extensions

After thinking about the problem, students might feel moved to use a dynamic geometry package to explore for themselves. Alternatively, an applet (available on the website) can be provided which enables you to vary different constraints, such as the number of lines and the angles between the lines. You can also release the moving point from being confined to a circle to being confined to an ellipse, or even allow it to be completely free in the plane. There is also the question of what role the perpendiculars play: it they were replaced with lines of given slope, perhaps parallel to some given lines, would the triangle remain invariant, and would the locus remain an ellipse?

Implications for Teaching

Seeing this task as an instance of a class of tasks under the general heading *phenomenal mathematics* could transform tasks used with students. A reasonable conjecture is that every topic and every technique in school mathematics and in at least the first few years of undergraduate mathematics could be introduced through generating a phenomenon that surprises many people and invites or invokes attempts to explain what lies behind the phenomenon.

Secret Places

Background

Tom O'Brien (2006) demonstrated that children as young as 9 and 10 are capable of reasoning mathematically when number calculations are not required. The applet was produced to enable primary teachers and teacher educators to experience their own use of mathematical reasoning, in order to sensitise them to possibilities for children. The applet is designed to be used in a tutor-led mode rather than individuals by themselves.

Most people with whom this has been used rise immediately to the challenge. There is an initial sense that it should not be too difficult, however people often discover that they need to re-think what the blue and red information is telling them. Despite several decades of human computer interaction there is still some emotional arousal due to the machine responding to probes (as distinct, say, from a person playing the role of the computer).

Some people display a propensity to want to start clicking without thinking, so the role of the tutor is to act as a brake, getting people to park their first impulse and think more deeply. Participants find themselves imagining what will happen one or more steps ahead, with some resorting to notation in order to keep track of the possibilities. This could provide an instance of 'reasoning by cases' and of being systematic. Attention tends to be on resolving the particular at first, rather than developing a general strategy, so again the role of the teacher is to promote movement to the general.

Narrative

People seem to respond to the challenge very quickly, despite an absence of 'purpose' or evident 'utility' (A3). It seems that the challenge appears tractable, and the dissonance of not-knowing but finding out stimulates emotions which are then harnessed (A0). One or more initial forays with the applet involving rapid clicking develops discernment of pertinent screen details and a sense of the task. Attention then shifts to what information is revealed by different choices, which invokes relationships. There are of course differences in subsequent actions depending on the result of the first click (A5).

People quickly work out that it does not matter which place you try first, so it becomes a practice to click on place 1 to start with. However there is often a split of opinion about what the colour actually means.

Some people want to know how they will know if/when they get the correct place. However the software never confirms the location of the secret place(s). Under most conditions (sufficient places given the number of secret places chosen) there is no need for the applet to validate secret locations, since that 'knowing', with certainty, is the result of reasoning. Even after several 'games', confusion comes to the surface regarding what a blue place tells you about the adjacent places. This is amplified where people work in groups of two or more in an ethos which values conjecturing and justification (communicative rationality A0).

After a few random trials to *get-a-sense-of* what is going on (*A*1), people usually want to shift into individual or small group work. The initiating impulse has changed, either into an *exploratory* mode in which the teacher and presence of the software introduce and maintain the students in contact with the mathematical reasoning, or into an *exercising* mode in which the desire to try examples initiates student activity (*A*2). Integral to Pólya's advice (*op cit.*) but unfortunately sometimes overlooked, is the role of specialising (manipulating, exercising) not simply to collect data, but in order to get a sense of underlying structure, leading to a conjectured generality. Put succinctly, *doing* ≠ *construing*; something more is required (A4).

Working individually or in small groups, people usually recognise the need for case by case analysis. Sometimes it takes a while to realise that the number of clicks you have to make before you can be certain (for one secret place) depends on what colours show up when you make choices. For many this is an unexpected situation. Bringing to articulation a method for locating the secret places most efficiently can take some time, even when it can be done in practice: *doing* is not the same as *saying* and that again is not the same as *recording* succinctly (*A*4). Communicating with yourself, then a friend, then a sceptic (Mason et al. 1982/2010) is useful for prompting clarification and experience of locating and distilling the underlying essential relationships forming the structure of the situation.

While the initial or outer task is to 'find the secret place' the implicit inner cognitive task is to develop an efficient method or algorithm for succeeding given what happens with a specified number of places and what is revealed in successive clicks, and to justify this as the best possible strategy in dealing with all possible situations for that number of places. A great deal depends on how teachers prompt reasoning by requiring justifications for choices of places to click, and all that depends on past experience the class has had of mathematical thinking, conjecturing, justifying, etc..

In order to be able to support desired shifts, for example between resolving the particular and seeking a general strategy, it is useful for the teacher to be aware of differences in goals (A3) and, over time, to direct student attention into alignment with the larger educational goal. Emotional commitment (harnessed emotion) may be so strong that students are locked into a simplistic version of the didactic contract (doing what is required will produce expected learning), whereas the teacher is aware that although the outer task is to find the secret location, the inner task (A3) is for the students to educate their awareness about the use of reasoning, becoming aware of possible actions (clicking, deducing, anticipating, conjecturing, …), analysis of cases, ruling out ineffective actions, and developing a general strategy. This is the teacher's teleological rationality, but needs to be picked up by students if they are to gain substantially from the activity. Clearly the factual rationality is of little import in itself.

Adding a second secret place among five places produces an ambiguity because in some configurations there is not enough information to locate them. This can lead to seeking the minimum number of places for which a given number of secret places can be located, or what is equivalent, the maximum number of secret places among a given number of places for which the secret places can always be located.

Affordances

The initial task offers opportunity to encounter and use the notion of symmetry, to realise the importance of considering different possible cases, to break the situation down into all possible distinct cases and to embark on a systematic examination of them all in turn. It also offers opportunity to imagine an action and its consequences, to make conjectures and to modify them in the face of contrary evidence, and to reason about what information is provided by discovering a 'hot' or a 'cold' place. Finding a 'method' which works with a minimal number of clicks is one form of generality (over all choices of location of the secret place).

Maintaining a plenary mode interspersed with individual and small group reconstruction and exploration allows for multiple modes of interaction, and exposure to aspects of mathematical thinking that can be called upon in the future when working on core curriculum topics, informed by the teacher's awareness of the affordances, inner tasks and goals of the activity (A3).

Effectiveness depends greatly on the working ethos and atmosphere of the social setting. It can work well in generating mathematical reasoning in a conjecturing atmosphere in which everything asserted is treated as a conjecture and expected to be modified unless and until it is satisfactorily justified, and in which those who are confident question and support those who are not so confident. It does not work well in an ethos of striving to get the right answer.

The extended task promotes a sense of generality through relating the number of places with the number of clicks required (with one secret place) and then to extend this further to deal with several secret places. It also offers repeated exposure to similar forms of reasoning in multiple situations which can contribute to students integrating these actions into their repertoire of available actions (exercising).

Ways of Working

The applet was designed to be used in plenary so that the teacher is in charge of when buttons get pressed. Ever since electronic screens came into use in classrooms, it has been appreciated that requiring agreement as to what buttons to press next is a powerful stimulus to communication and reasoning, and some projects are based almost solely on this idea (Dawes et al. 2004). In fact the applet has a 'locking feature' to restrict what users can access if it is to be used in small group mode by students. The resistance to acting upon the first idea that comes to mind is one of the contributions that a teacher-led plenary mode can contribute to the education of student awareness (A1), by blocking the first impulse and calling upon more considered thinking. Learning to 'park' an idea and look for a different or better one is an important contribution and part of the potential 'inner task'(A3).

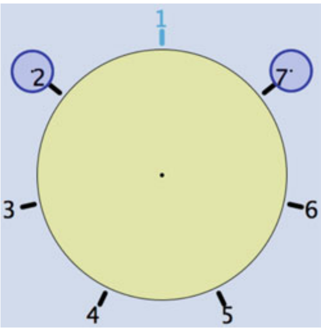
The point of the applet is not actually to find the secret place but to convince yourself and others (friends and sceptics) that your method will always find the secret place(s) in no more than the number of clicks that you claim. Satisfaction and other effective rewards arise from personal use of reasoning powers, and agreement from peers and an expert (teacher). Note however that there is no 'purpose' offered apart from the arising of curiosity, the activating of desire to find the location, and an initial sense that it cannot be too difficult. No one has ever dismissed the task as "well just click all the places … who cares?".

In order to bring justification through reasoning (reasoning on the basis of agreed properties) to the fore, the teacher needs to manage the discussion, creating and maintaining a conjecturing atmosphere, providing thinking time as well as time and space for expressing ideas and insights, and for rehearsing and challenging the conjectures of others. Opportunities abound for constructing configurations for which a conjectured 'method' does not always find the secret place in the minimum number of clicks.

The applet itself at best provides an introduction to or on-going experience of reasoning by considering and eliminating cases. Unless it is used as part of a programme of experience of activities involving similar types of reasoning, with appropriate drawing of attention to effective and ineffective actions, use of the applet would be mere entertainment.

Extensions

The applet permits changes to the number of places at the table (numbers from 4 to about 25 are distinguishable), the number of secret places, and the spread of the 'hot' information (default value is 1 place each side of the secret place).



Here position 1 has been clicked and found to be 'cold'. Deductions have been made that positions 2 or 7 could not be the secret place, and have been marked 'cold' by the users to assist their reasoning.

But there is potential ambiguity in this additional notation: interesting things happen when it emerges that some people interpret the cold-marker to mean that clicking there would necessarily give a 'cold' response!

There is a second version that takes the same idea (locating a secret place) into two dimensions, where the space of activity is ostensibly a finite grid of squares. ‘Hot’ means adjacent horizontally or vertically, but the grid can be turned into a cylinder, torus, Mobius band or Klein bottle, and some of these can have displacements. The intention is to introduce ‘as a matter of course’ rather than as an object of explicit attention, different surfaces that can be constructed by identifying edges of a rectangle, as multiple contexts in which to exercise similar reasoning. The 2D version offers opportunity to encounter and explore topological notions of ‘nearness’ on familiar and unfamiliar surfaces all generated in the same manner (identifying some edges). Reflecting on what is the same and what different about the 1D and the various 2D contexts could reinforce awarenesses that students have begun to educate in themselves.

Reflection

Focusing on the use of applets by a teacher as stimulus to activity by students, and using the framework of six modes of interaction, combined with distinguishing various human powers which can be used and developed mathematically, and with distinctions drawn concerning different ways of attending, it emerges that even these apparently simple ways of using software with students are both complex and demanding. The complexity arises from recognition of the need to vary the modes of interaction so as to keep the whole of the psyche involved, and to prompt a reflexive stance in order to learn from experience. The demanding nature of these pressures arises from the need to have come-to-mind appropriate pedagogic strategies in order to maximise the learning potential for students.

The three studies are representative of only a restricted range of stimuli to mathematical thinking afforded by software. The stance taken here is that even taking one mode of interaction as the initial activity, different modes of interaction between stimulus (teacher-applet), student and mathematics are possible and desirable. It is not so much the stimulus that is ‘rich’ but the ways of working with that stimulus that can be pedagogically rich or impoverished. The narratives offered based on the case studies suggest general observations about what applets can provide:

A means of initiating enquiry and exploration (producing a phenomenon to be explained as in the case of *Secret*, *Rolling Polygons*);
An environment in which to work (at least some of the time, as in all three studies);
A means of stimulating continued study of a topic;
A means of testing conceptual grasp and manipulative proficiency (as in the case of *Secret Places*) or of reinforcing and clarifying techniques and concept images (as in the case of *Screencasts*);
An environment in which to make use of what has been learned about a topic in further exploration (as in the case of *Secret Places, Rolling Polygons*).

It seems clear that a screencast can, if well constructed, initiate and support conceptual understanding and appreciation, and the details of techniques or procedures. A screencast can even activate desire to master a technique or appreciate a concept. However, screencasts are not well placed to provoke the kind of activity that leads to effective integration, educated awareness that can initiate an action in the future when required. If used in conjunction with routine exercises, then the integration will be only as effective as the structure of the exercises (Mason and Watson 2005; Watson and Mason 2006).

The addition of software into the educational milieu affords both potential and complexity:

Pedagogical complexity arises from the need to develop fresh ways of working effectively, both when students work for themselves or in a small group to make sense of a screencast, and when activity is directed by a teacher using an applet as the focus;

Mathematical complexity arises from the greater scope for a mismatch between the mathematical potential and the teacher's grasp of the topic or concepts; and,

Learning complexity arises from the demands made on students' commitment to learning deeply and effectively.

It may be that within this complexity lay some of the obstacles to greater use within the mathematics classroom.

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