Meta-Didactical Transposition: A Theoretical Model for Teacher Education Programmes

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Abstract We propose a new model for framing teacher education projects that takes both the research and the institutional dimensions into account. The model, which we call *Meta-didactical Transposition*, is based on Chevallard's anthropological theory and is complemented by relevant elements that focus on the specificity of both researchers' and teachers' roles, while enabling a description of the evolution of their praxeologies over time. The model is illustrated with examples from different Italian projects, and it is discussed in light of current major research studies in mathematics teacher education.

Keywords Meta-Didactical Transposition • Communities of inquiry • Research for innovation within institutions • Teacher education practices • Meta-didactical praxeologies • Mathematics laboratory

Introduction

The education of teachers is a relevant issue in the evolution of a society and is even more significant at particular historical moments of social or political change. Since the 1960s, with the progressive diffusion of socio-constructivism as a cognitive model, social interaction in the classroom came to the fore, resulting in an increased attention to the social dynamics of learning. This progressive change of attention, from the individual to the social construction of meaning, along with an increasing

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A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*,
Mathematics Education in the Digital Era 2, DOI 10.1007/978-94-007-4638-1_15,

use of technological artefacts, led to a corresponding interest in teacher education. Particularly in the last decade, attention to teacher education has increased (Ball and Bass 2003; Ball et al. 2008; Clark and Hollingsworth 2002; Even and Ball 2009; Wood 2008) and digital technologies have an increasing relevance in this context (Drijvers et al. 2010; Hoyles and Lagrange 2009; Lagrange et al. 2003). In Italy, we have witnessed the multiplication of teacher education programmes involving digital technology at the European,[1] national, regional, and local levels. As researchers, we are involved both at the level of teacher education programme development and management, and in studying teaching and learning processes in the classroom. This has prompted the emergence of a deeper reflection on the resulting complexity.

We began to recognise the importance that institutions play in the school context, including the national curriculum, national assessment tools and the constraints of teachers' time and space, and textbooks. Our attention was directed toward the theoretical elements that could adequately frame these, which we found in Chevallard's (1985, 1992, 1999) Anthropological Theory of Didactics (ATD), particularly with respect to his notion of *didactical transposition*.

The complexity arising from the intertwining of the processes involved during a teacher education programme has led us to introduce a descriptive and interpretative model, which considers some of the main variables in teacher education (the community of teachers, the researchers, the role of the institutions), and accounts for their mutual relationships and evolution over time. We call the overall resulting process *Meta-didactical Transposition*. We offer the model as a tool for studying the complexity of teacher education as a research problem that involves a transposition from the practice of research to that of teaching.

In the following sections, after some theoretical background on teacher education, we present the Italian context in which our research is situated. Then we present the *Meta-Didactical Transposition* model (in short, MDT). We use this model to analyse the different variables listed above and their dynamic relationship, contextualised within three Italian teacher education programmes that use digital technologies. The three programmes are used as 'generic examples' that we hope will find resonance within other international contexts. Finally, we discuss the results of our analysis, pointing to the model's potential with respect to current research in the field.

Teacher Education and the Italian Context

In 2000, the International Commission on Mathematics Instruction (ICMI) commissioned a study that was coordinated by Anna Sfard on the relationships between research and teaching practice in mathematics education. The results of this study were presented at ICME in Copenhagen, 2004. It highlights three main periods in

[1] One of the European projects in which we have been involved is the EU funded project *EdUmatics* (*50324-UK-2009-COMENIUS-CMP; European Development for the Use of Mathematics Technology in Classrooms)*, http://www.edumatics.eu.

the evolution of issues addressed by mathematics education research: the *era of the curriculum*, mainly focused on the study of education programmes; the *era of the learner*, focused on student's learning and difficulties; and, the *era of the teacher*, focused on teachers and teacher education.

Sfard (2005) stresses that the advent of *the era of the teacher* has brought about a re-conceptualisation of the relationship between the teacher and the researcher, which constitutes "a big leap toward research that plays a genuine role in shaping and improving practice" (p. 405). She argues that in most of the international research studies, the question is not *what* is taught in classrooms, but *how* it is taught: "rather than trying to arrive at a mechanistic view of 'what works in classrooms', I focus on how things work and try to make myself aware of alternative possibilities" (p. 406). This shift of attention to teaching practices is due in part to international comparative tests (TIMSS, PISA), which often show poor results, despite the quantity of resources devoted to curricular changes.

In the last years, many publications have focused on teacher education. They have been concerned with teachers of different school levels, addressing issues such as the relationship between teachers and both curricular or methodological innovation and technology integration. In particular, research on teacher education programmes has intensified, gradually changing the focus from pre-service to in-service education, with an emphasis on the role played by specific tools and methods on the professional development of teachers. An overview on this wide-ranging research can be found in the 15th ICMI study on teacher education (Even and Ball 2009) and the four volumes of the *International Handbook of Mathematics Teacher Education* (2008).

Much of the research on teacher education has focused on identifying the knowledge that is necessary for the teaching of mathematics. Researchers generally agree that this knowledge consists of three main components, which progressively interrelate to each other: knowledge about mathematics content; general pedagogical knowledge; and the mathematical-didactical knowledge. These components can be related to those introduced by Shulman (1986), who was the first to identify the notion of *pedagogical content knowledge* (PCK) as the particular knowledge for teaching: "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (p. 9). In the case of the teaching of mathematics, PCK concerns the intertwining of mathematics and pedagogy in relation to the different conditions for and ways of teaching and learning specific content.

Taking Shulman's studies as a starting point, Ball and Bass (2003) propose a finer and more effective characterisation of what they refer to as the *mathematical knowledge for teaching* (MKT), which Bass (2005) defines as "the mathematical knowledge, skills, habits of mind, and sensibilities that are entailed by the actual work of teaching" (p. 429), that is "the daily tasks in which teachers engage, and the responsibilities they have to teach mathematics, both inside and outside the classroom". Ball et al. (2008) highlight the fundamental difference between mathematics and mathematics for teaching. While the former has the capability of compressing the information into abstract forms, the latter requires a sort of decompression, in that the main ideas pertaining to the mathematical content is

made more explicit. These authors choose to characterise MKT through the analysis of the daily practice of teachers:

> Instead of starting with the curriculum, or with standards for students learning, we study the work that teaching entails. […] We seek to unearth the ways in which mathematics is involved in contending with the regular day-to-day, moment-to-moment demands of teaching. Our analyses lay the foundation for a practice-based theory of mathematical knowledge for teaching. (p. 395)

They thus analyse the typical features of mathematics that are involved in teaching and identify the main components of MKT in relation to Shulman's subject matter knowledge (SMK) and pedagogical content knowledge (PCK). They distinguish three sub-domains of PCK: (a) *knowledge of content and students*; (*b) knowledge of content and teaching; (c) knowledge of content and curriculum.* Referring to SMK they identify *specialised content knowledge* (SCK) as an important sub-domain of mathematical knowledge. Bass (2005) stresses that SCK

> is strictly mathematical knowledge (not about students or about pedagogy) that proficient teachers need and use, yet is not known by many other mathematically trained professionals, for example, research mathematicians. Contrary to popular belief, the purely mathematical part of MKT is not a diminutive subset of what mathematicians know. It is something distinct, and, without dedicated attention, it is not something likely to be part of the instruction in content courses for teachers situated in mathematics departments. (p. 429).

Another important element that characterises the main studies on teacher education is their involvement of teachers in the joint analysis and reflection on the main features of the didactical projects being researched. Within the research literature, this involvement is described in terms of communities of practice, communities of inquiry, adaptive systems, collective participation, sustained conversation and egalitarian dialogue. The cornerstone of these studies is the notion of critical reflection, conceived not only as a fundamental attitude to be instilled in teachers but also as a professional responsibility. Drawing on Schön's studies (1987), many researchers stress the value of critical reflection as well as the importance of sharing reflections amongst teachers and between teachers and researchers (e.g. Mason 1998, 2002; Jaworski 1998, 2003; Schoenfeld 1998). These studies suggest that teachers should share their interpretations of teaching and that observing different ways of acting can lead them to re-conceiving their ideas about their role in the classroom as well as the nature of their profession. As we will show, this philosophy permeates the practice developed by Italian mathematics education research since the 1980s.

With respect to the evolution of the research on teacher education, another essential aspect is its strict interrelation with the research on the integration of new technologies in the teaching of mathematics. The focus of this research has shifted from the study of new programming languages for the implementation of algorithms (in the 1980s and 1990s), to the exploration of didactical software expressly conceived for education (in the 1990s and later), to the more recent use of new technologies not only for the teaching of mathematics but also as tools for communication and education in general, which led to the constitution of a specific research area on *educational technology* (Guin et al. 2005).

Arzarello and Bartolini Bussi (1998) provide a synthesis of the Italian research in the 1960s–1990s, which reflects the different dynamics and the changes that occurred. The authors identify four different trends, the fourth of which represents the dominant Italian research paradigm of *research for innovation*. According to this paradigm, the main features that characterise the work carried out during teacher education programmes by the teachers and the researchers are collaboration, mutually supportive and integrative of knowledge and skills. This collaboration links theory and practice, and is fundamental for the professional development both of teachers and researchers in mathematics education. A peculiar feature of *research for innovation* is the important role played by the 'teacher-researchers', that is, teachers that are deeply involved in all phases of the research process, from planning to implementation to data analysis to dissemination (Malara and Zan 2002). Whereas only a relatively small number of teachers become teacher-researchers, a greater number of them have been involved in institutions (e.g. Ministry of Education), in research communities within pre-service and in-service teacher education programmes, or as tutors or trainers for other teachers.

The model we present is strongly culturally framed in the Italian context, from which we identify the main variables. However, we are confident that it is possible to extend this model to other contexts, because of its flexibility in describing teacher education as a complex system and in highlighting the interaction between its variables.

A New Paradigm: Meta-Didactical Transposition

The model we propose, which takes into consideration the practices of mathematics educators (researchers) and those of teachers, when both communities are engaged in teachers' education activities, is based on the Chevallard's Anthropological Theory of Didactics (Chevallard 1985, 1992, 1999; Bosch and Chevallard 1999) It adapts and extends ATD to the context of teacher education. This model, called *Meta-Didactical Transposition,* considers:

(i) the complex dynamic interplay, which develops in activities involving different communities (e.g. between the teachers and the mathematics educators);
(ii) the constraints imposed by the institutions that promote such activities (including schools and Ministry of Education) in view of some specific goals (e.g. promoting teachers' knowledge of new curricula or of new technologies);
(iii) other 'institutional' constrains, including the tradition of the school(s), the related (intended, implemented, attained) curricula and the textbooks used by the teachers.

Meta-Didactical Transposition involves five intertwined features: the *institutional aspects*, the *meta-didactical praxeologies*, the *double dialectics*, the *brokering* processes and the dynamics between *internal and external components*. We describe each aspect in the next sections. Our model thus complements the MKT

model described above insofar that it focuses on these main aspects of teachers' education programmes: their dynamicity; the dialectic between the communities[2,3] of teachers and those of the researchers who coach them; and the influence of the institutional components and their relationships to the communities.

Institutional Aspects

ATD focuses on the institutional dimension of mathematical knowledge, placing mathematical activity, hence the activity of studying in mathematics, within the bulk of the human activities and of the social institutions (Chevallard 1999). In our view, it is important to consider such an institutional dimension in teacher education activities since these activities are fully situated within and constrained by the context of social institutions (research communities, schools, the Ministry of Education, the policy makers, the teachers associations, etc.). In Italy, as in many other European countries, the whole educational system (from kindergarten to university) is public and is governed by several institutions at different levels (national, regional, local). Within this context, the importance of the institutional dimension is also at play within the politics of the European Union. As lifelong education is considered a strategic element for development in Europe, community programmes are promoted for prospective or in-service teacher education. These programmes assume a clear cooperation between the research world and the institutional-political world (see http://ec.europa.eu/education/llp/official-documents-on-the-llp_en.htm).

Chevallard (1992) stresses the fact that the very nature of mathematical objects in school is dependent on the person or the institution with which it is related: "An object exists since a person, or an institution acknowledges that it exists (for it itself)" (p. 9). With respect to teacher education, our model focuses on two types of communities, which sometimes intertwine: (a) the *communities of the researchers*, who design and coach the educational programmes, generally as an official task commissioned by the responsible authorities (e.g., School administration, Ministry of Education);

[2] We refer to this term in tune with the following characterisation of communities of inquiry proposed by Jaworski (2008): "*In terms of Wenger's (1998) theory, that belonging to a community of practice involves engagement, imagination and alignment, we might see the normal desirable state as engaging students and teachers in forms of practice and ways of being in practice with which they align their actions and conform to expectations…In an inquiry community, we are not satisfied with the normal (desirable) state, but we approach our practice with a questioning attitude, not to change everything overnight, but to start to explore what else is possible; to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to provide answers to them. In this activity, if our questioning is systematic and we set out purposefully to inquire into our practices, we become researchers.*"

[3] It derives from the Chevallard's notion of didactical transposition (Chevallard 1985), which roughly speaking, consists in the relationships between the production, the use and the teaching of the scientific knowledge and in the ways, according to which it adapts itself in order to 'work' in different types of institutions (compare for example a theorem as expressed in the Journal where it is proved by a mathematician, what Chevallard calls "le savoir savant", with the same theorem as it is written in a textbook, "le savoir enseigné").

(b) the c*ommunities of the teachers*, who participate within the projects, either on a voluntary basis or because of an official duty. Both of these communities are in relationship with the school: the actual schools where the teachers teach, and the School as an institution with its curricula, its teaching traditions, the textbooks used, etc.

Meta-Didactical Praxeologies

ATD proposes a general epistemological model of mathematical knowledge, conceived as a human activity developed for the purpose of addressing specific families of tasks. Its main theoretical tool is the notion of *praxeology* (or mathematical organisation), which is structured in terms of two main levels (García et al. 2006): (a) The 'know how' (*praxis*), which includes a family of similar *problems* to be studied, as well as the *techniques* available to solve them (e.g. 2nd degree equations and the formulae for their solution); (b) The 'knowledge' (*logos*), which is the 'discourses' that describe, explain and justify the techniques that are used within a more or less sophisticated frame and may even produce new techniques (e.g. the justification of the formula for 2nd degree equations through the completion of squares or even the theory of algebraic equations and how it encompasses 2nd degree equations).[4] A praxeology consists of a task, a technique, and a more or less structured argument that justifies or frames the technique for that task. Hence, it encompasses both the *know-how* and the *knowledge*, with respect to a family of tasks.

In constructing our model, we consider the *meta-didactical praxeologies*, which consist of the tasks, techniques, and justifying discourses that develop during the process of teacher education. For example, consider the teacher training course described by Sullivan (2008), in which he used the question "which is bigger, 2/3 or 201/301?" (p. 3) in order to prompt teachers for ideas that might be used as the basis of a lesson. The discussion with the teachers made evident at least three points of view, according to which one can answer the question: the mathematics knowledge, the knowledge specific for teaching and the pedagogical knowledge. According to such knowledge, specific interventions could be designed to introduce the students to the task, e.g. to think of baseball statistics: if a player passes from 200/300 to 201/301 his score increases. All of this can be considered as an example of a *meta-didactical praxeology* in that the task is stimulating the teachers' reflection, and the techniques are those that Sullivan used in the course to promote discussion. During this discussion, it is possible that the two communities of mathematics educators and teachers, respectively, shared a common theoretical framework, which would justify the techniques being discussed. For example, based on one's professional experience, the teachers might discuss why the initial question presents difficulties for many students and why the baseball example makes sense in a classroom and thus help overcome these difficulties. Moreover, the teachers may scaffold their

[4] The 'knowledge level' can be further decomposed in two components, i.e. *Technologies* and *Theories*. The provided description is enough for our purposes.

arguments within specific pedagogical discourses, for example stressing the necessity to foster the transition from everyday to scientific and formal concepts, according to a Vygotskian approach. The theoretical side of the *meta-didactical praxeology* also includes the reflection made by Sullivan on the possible reasons why the activity was a good illustration of the way teachers can become aware of MKT, an aspect that may have been highlighted within Sullivan's exposition.

Within *meta-didactical praxeologies*, what is under scrutiny is not the didactics in the classroom but the practices and the theoretical reflections developed in teacher education activities. Of course, they are the result of the interaction between the reflections of the community of researchers about the didactic praxeologies previously designed and developed, and the concrete practices used by the teachers in their professional activities.[5]

We now have the basic ingredients that allow us to introduce the core of our model. Looking at teacher education processes from a dynamic point of view, we initially identify two communities: that of researchers, who design and coach the activities, and that of the teachers, who are engaged in an education process. For the modeling purpose, let us distinguish two kinds of praxeologies: the *researcher praxeologies*[6] and the *teacher praxeologies*. The researchers and teachers praxeologies in some cases may be shared, but we assume that in general, when the teachers encounter the researchers for the first time at the beginning of the education process, they are not. Teacher education programme aim to develop teachers' existing praxeologies towards new ones, which consist of a blending of the two initial praxeologies. This evolution is the result of an interaction with the community of researchers and, for this reason, we call it a s*hared praxeology*. For example, from the discussion of different techniques to address a problem, new ones can be acquired by the teachers, with a suitable theoretical justification, thus replacing or integrating old techniques and so as to change the nature of the teacher's MKT. Also within this dynamic evolution are some external components, which may play a crucial role. A typical example is when the activity is developed in response to changes in the official curriculum or in external assessment expectations for students.

The community of researchers generally reflects upon the nature of, and reasons for, the changes produced by the teacher education programme and possibly shares such reflections with the community of teachers. This can result in *new researcher praxeologies*. Also the teacher praxeologies may change, and develop into *new teacher praxeologies*, a process that can repeat and further refine itself. A global illustration of this is provided in Fig. 1.

[5] This is true for activities with in-service teachers; in the case of prospective teachers, the second component may be missing but their beliefs are active and still constitute a powerful part of the component.

[6] Of course there may be more than one praxeology referring to researchers, as well as referring to teachers: in the text we will use either singular or plural (researchers praxeologies; teachers praxeologies). In particular the researchers have their own praxeologies as researchers, which concern the praxis and the logos of their researches; but they have also their praxeologies as teachers' educators, where the praxis and the logos concern the concrete way they coach these activities, because of their theories about teachers' educational processes.

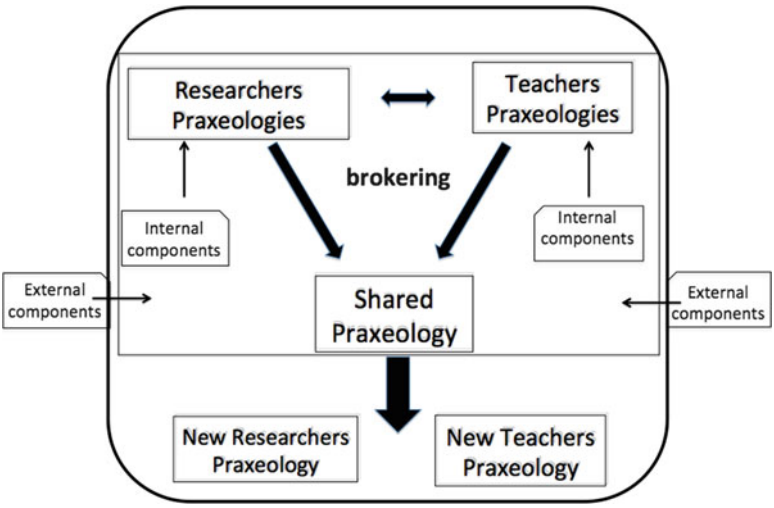


Fig. 1 The Meta-didactical Transposition model

Meta-didactical Transposition consists of a dynamic process through which, thanks to the dialectical interactions between two communities, both the didactic praxeologies of the community of researchers and of the teachers' community change within the institutional environment in which the two communities reside. This dialectical interaction leads to the development of a shared praxeology, which represents the core of our model. One of the main results of the dialectical interaction is the teachers' development of both a new awareness (on the cultural level) and new competences (on the methodological-didactical level, i.e. that of teaching practice), which lead them to activate, in their classrooms, a didactical transposition in line with recent educational trends. Therefore, the term 'meta-didactical' refers to the fact that important issues related to the didactical transposition of knowledge are faced at a meta-level.

Internal and External Components

An important feature of Meta-didactical Transposition is that some of the components of the two communities' praxeologies change their status over time. Typically they move from being *external* to becoming *internal* with respect to the community under scrutiny. To clarify this crucial point, which will be further discussed in the following sections, we give a brief example. Consider a community of teachers that starts an educational programme in which, due to some institutional situation (e.g. curriculum changes), a community of researchers introduces a specific ICT tool (e.g. a dynamic geometry software). Initially, the tool is an external component for the teachers (and possibly also for the researchers). However, at the end of the

educational programme, it has become an internal component in their praxeologies, albeit possibly at different levels. Such an internalisation process, which happens via a Meta-didactical Transposition, defines a *meta-didactical trajectory*,[7] that is, the dynamic evolution of the teachers' education programme. For example, a technique (and the theory that justifies it) is initially in the hands of the researchers. Their aim is to make it shared within the community of inquiry as a technique and possibly, in addition, build an understanding of the theoretical arguments that justify its use. At the end of the process, the initial techniques (and possibly also the theoretical part) has become a new a set of shared techniques, as a result of the actions taken by the researchers and teachers.[8] As we will point out in the next section, this evolution is fostered by a dialectic interaction between these components.

The internal/external distinction is adapted from Clark & Hollingsworth (2002). They distinguish an external domain, located outside the teacher's personal world, from the internal domains, which "constitute the individual teacher's professional world of practice, encompassing the teacher's professional actions, the inferred consequences of those actions, and the knowledge and beliefs that prompted and responded to those actions" (, p. 951). Compared with their approach, our model emphasises the process of the teachers' professional evolution, according to which some of the external components become internal as a result of the process of Meta-didactical Transposition.

A Meta-didactical Transposition produces a dynamic change in the praxeologies of the community of teachers. Some components of the praxeologies of the community of researchers enter the praxeologies of the community of teachers as an outcome of the Meta-didactical Transposition. Presumably, also, the researcher praxeologies change as well, as a result of their encounters with the community of teachers. It is possible that some of these components may be external to both communities and it is the educational process that produces their transformation into internal components of the communities.

We will see below that this change is only one of the possible transformations that Meta-Didactical Transposition can produce within the praxeologies of the two interacting communities.

Brokering

The Meta-didactical Transposition model integrates the ideas of ATD with elements coming from other frameworks. The notion of brokering is an example; it is introduced because it describes the role that teachers and researchers often find

[7] The choice of this term to refer to teachers' education programmes is in tune with Simon definition of Learning Trajectory: "The Hypothetical learning trajectory consists of the goal for the students' learning, the mathematical tasks that will be used to promote students' learning and hypothesis about the process of the students' learning" (Simon 1995).

[8] This process has a common feature with the processes of instrumental genesis, as described by Trouche (2005). Space does not allow us to develop this issue.

themselves playing within the different communities. According to Rasmussen et al. (2009), a *broker* belongs to more than one community. Typically a teacher belongs to the community of mathematics experts, to that of her/his school teachers and to her/his classroom community:

> Brokers […] are able to make new connections across communities of practice, enable coordination, and – if they are good brokers – open new possibilities for meaning (p. 109).

Brokers facilitate the transition of mathematical concepts from one community to the other (*boundary crossing*), which is accomplished by drawing on *boundary objects*:

> boundary objects are those objects that both inhabit several communities of practice and satisfy the informational requirements of each of them. (Bowker and Star 1999, p. 297)

Within Meta-didactical Transposition, brokering is a common habit and, frequently, researchers play a brokering role between the two communities involved. A good example of a typical boundary object is the baseball score used by Clark (cited in Sullivan 2008). Teachers can use such a boundary object to move students' thinking from the usual meaning of the score to a more mathematical comparison between two fractions (2/3 and 201/301). At the same time, used within an episode of teacher education, this example is a boundary object used by the researcher to move the teachers from the standard mathematical meaning of fractions to an everyday contextualised meaning that is useful for teaching. In this sense, the researcher makes a brokering action with respect to the teachers.

Double Dialectic

Another important element of our model is the double dialectic involved in the *Meta-Didactical Transposition.* The first dialectic is at the *didactic level* in the classroom in that it is between the personal meanings that students attach to a didactic situation, to which they are exposed in the didactic activity, and its scientific, shared sense (Vygotsky 1978). The second dialectic is at the *meta-didactic level*, which lies between, on the one hand, the interpretation that the teachers give to the first dialectic as a result of their personal meaning, which is a result of their praxeology and, on the other, the meaning that the first dialectic has according to the community of researchers, which results from researcher praxeology. The second dialectic corresponds to the *scientific shared meaning* of the first dialectic.

Typically, the second (meta-didactical) dialectic arises from a contrast between researcher praxeologies and teacher praxeologies and the first dichotomy engenders the second one as an outcome of a suitable meta-didactical trajectory, which is designed by the researchers. It is through this double dialectic that teacher praxeologies can change and align with the praxeologies of the researchers, which may cause a significant evolution of the teacher professional competences.

Examples from the Italian National and Regional Programmes

We will now discuss three different Italian teacher education programmes, which will show how the *Meta-didactical Transposition* model can help describe and analyse some important aspects of these programmes that that have not been adequately addressed in existing approaches. These aspects relate to the relationships between the Research, the Institutions, and Mathematics Education research. In particular, the three examples are meant to highlight different aspects of the model.

The first example aims to illustrate the various components of the model and the relationships between them. It concerns an ongoing Italian programme for teacher education called M@t.abel, which is based on an extensive use of ICT and, in particular, a purposeful, dedicated internet-based platform. We will show how the *Meta-Didactical Transposition* model allows the role of ICT to be adequately framed within teacher education. In particular, we will highlight the dynamics between the internal and external components and the brokering function of the tutors within the transposition.

The second example, MMLab-ER (Laboratories of Mathematical Machines for Emilia Romagna), shows how to exploit the potential of the *Meta-Didactical Transposition* in order to analyse the *development* of the project. In particular, the model allows us to identify and describe when, how and why the different components of the praxeologies changed during a teacher education programme.

The third example, which refers to the teacher education programme within the ArAl Project, is aimed at showing how, through the planning of an appropriate meta-didactical trajectory, it is possible to both highlight a first-level dialectic (didactical dialectic) and engender a second-level dialectic (meta-didactical dialectic), which enables teachers to develop a new awareness of their role in the classroom.

The three examples have been chosen being very different in scope, activities, and modalities of action. Considering the specific aspects that each example highlights can give a taste of the potential value of the MTD model in objectifying complex and different situations of teacher education projects.

A National Example of Meta-Didactical Transposition: The M@t.abel Project

The M@t.abel Project is a national teacher education programme for in-service mathematics teachers supported by the Ministry of Education. It started in 2006 and, to date, it has involved more than 10,000 secondary school teachers distributed across the whole of Italy. M@t.abel has its roots in the Italian *research for innovation* paradigm and, in particular, a previous project called '*Matematica per il cittadino*' (Mathematics for the citizen,[9] 2001–2005), which was elaborated within an

[9] http://www.umi-ciim.it/in_italia--28.html.

innovative curriculum in mathematics, from primary to secondary school (Anichini et al. 2004). The *Matematica per il cittadino* curriculum is based on the idea of the *mathematics laboratory*, intended as a methodology based on varied and structured activities. These activities aim toward the construction of meanings, in which the students can learn by doing, seeing, imitating and communicating with each other, under the guidance of the teacher, as in a Renaissance workshop. This methodology fosters close interaction between novices and experts, in the context of *cognitive apprenticeship*. This phrase "refers to the fact that the focus of the learning-through-guided-experiences is on cognitive and meta-cognitive, rather than on physical, skills and processes" (Collins et al. 1989, p. 458).

Although the current Italian National Curriculum mirrors in some respects the influence of the project *Matematica per il cittadino*, the school reality is quite far from being broadly influenced by the new perspectives, and the innovation is restricted to isolated cases (teachers, schools, or networks of schools) and to primary or middle, rather than secondary schools. For this reason, the M@t.abel project aims to improve school mathematics education at the secondary level, through the wide-scale dissemination of the ideas and didactic activities (i.e. the didactic praxeologies) of the *Matematica per il cittadino* curriculum. To reach this aim, a fundamental part of the M@t.abel project requires that teachers try out activities in their own classrooms that involve new didactic praxeologies (using a problem solving approach, tasks that involve discovering-conjecturing-arguing and proving, group work and discussions, and digital technologies).

In the M@t.abel Project, the institutional aspects are fundamental, because the Ministry of Education (MIUR), along with the Agency of School (Indire), is responsible for the project, which also includes researchers as members of the Scientific Committee. Researchers are called upon to plan all of the components of the teacher education programme as described above, to implement the educational meetings for the tutors and to prepare materials for the teachers.

In the project, the praxeologies of the researchers encounter teacher praxeologies by means of a two-step process. Each step can be considered as a Meta-Didactical Transposition process, where the first step concerns the tutors' education and the second one the teachers' education. The *tutors* are a small number of expert teachers who take part in research projects with University researchers. In many cases, tutors have previously participated in the *Matematica per il cittadino* project. In some cases, they may be teacher-researchers. The whole tutor community is formed at the beginning of their involvement in the project. In this first Meta-Didactical Transposition, the researchers play the role of *brokers* between the two communities (see Fig. 2a). In the second and far-reaching steps of the project, tutors themselves play the role of brokers in the Meta-Didactical Transposition that is directed toward a large number of teachers (see Fig. 2b).

Due to the limitations of space, in this Chapter we only describe this second process in more detail.

In order to develop shared praxeologies, the teachers are organised into *communities of inquiry*, composed of 15–20 teachers and supervised by tutors. Within this context, the tutors act as *brokers* between the two communities of teachers (involved

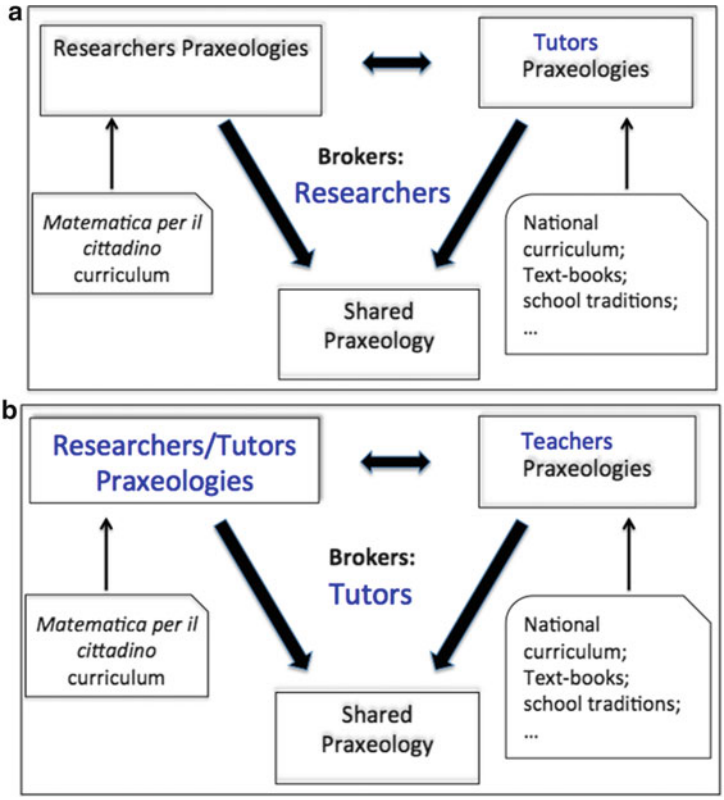


Fig. 2 (**a**, **b**) The two Meta-didactical Transpositions of the M@t.abel project

as learners in the educational programme) and of researchers (involved as designers of the programme). The tutors are confident with the innovative paradigms that emanate from research,[10] and they share with teachers their experience.

The communities of teachers work both remotely through an e-learning platform and during face-to-face meetings with tutors. Initially, the tutor outlines the spirit of the project, presents the activities during some meetings and asks the teachers to analyse them from a didactical point of view. Then, the tutor coordinates the groups of teachers remotely through synchronous meetings (using screen sharing) and asynchronous discussions (emails, forums). Having shared activities and methods, the teachers choose four activities and experiment with them in their own classrooms. These trials are a fundamental part of the teacher education programme and,

[10] According to the Italian paradigm of 'research for innovation', in this second step the tutors praxeologies may be assimilated to the researchers ones: as said above, in many cases the tutors are teachers-researchers, i.e. are experienced with research studies and methodologies, having been part of research teams in mathematics education for many years. Of course this is not always the case. For the purpose of the paper, we privilege clarity, taking the risk of over-simplification.

during the experimentations, the tutor asks the teachers to carefully observe their students' processes,[11] and to record their notes in a logbook, which is uploaded on the platform. More precisely, in the logbook the teacher is asked to:

- Make explicit the principal conceptual points of the activity;
- Describe the classroom experience and the methodology followed (worksheets, groupwork, software, …);
- Monitor how the students participate in the activities and appreciate them;
- Signal the main student difficulties;
- Comment on the evaluation of the tasks and their conclusions.

In the researcher praxeologies, the logbook is meant to be a tool that helps the teachers plan, monitor, and control their own work and, in particular, organise the observation of what happens in the classroom, focusing their attention on processes rather than on products. In this sense, it can enable the teachers to orient or re-orient their didactic practice, and contribute to improving their teaching practice by means of self-reflection. Furthermore, logbooks can be a valuable means of exchange between teachers working around the same mathematics topic, or at the same school level, and may provide information tools for external observers. As an institutional constraint, the project enables the participating teachers to gain certification that is useful within their career progression; also, the completion of the logbook is a required element of this accreditation.

Over their years of practice, each teacher has developed her/his own individual praxeologies comprising tasks, techniques and theoretical discourse. Depending on the individual teachers, the initial praxeologies of the researchers and the teachers can be far apart. For instance, some of the teachers involved in the programme often used quite traditional tasks and techniques, consisting of lectures, exercises and applications. Their (often implicit) theoretical discourse that justified these choices was based on traditional textbooks and an old curriculum. During the educational programme, the praxeologies could evolve and change through meta-didactical trajectories, and develop toward shared praxeologies (e.g. laboratory practices and use of ICT).

Figure 3 contains a portion of a teacher's logbook, in which the teacher presents a synthesis documenting her consciousness about the changes in her praxeologies related to the teaching of geometry. The choice of the tabular format for the logbook is made to this particular teacher. In the third column we can get some evidence of a shared praxeology. We can notice technical terms originating from the researcher/tutor praxeology, which are also expressed in the *Matematica per il cittadino* curriculum, such as 'guiding students in discovering properties' (see the reference to cognitive apprenticeship above), 'discussing with them about descriptions, definitions, properties', 'institutionalising knowledge', and so on.

Unfortunately, many teachers involved in the project did not or could not annotate daily their logbook (Rapporto PON M@t.abel 2009–10). These teachers only wrote up their logbook at the end of their experience in the project, as a sort of compulsory homework that was required by the system (Institutional constraint).

[11] Contrasted with their products. e.g. students' reasoning, arguments, difficulties, and so on.

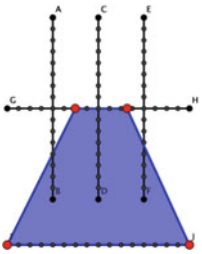
	What I thought before the teaching experiment on quadrilaterals	What I think now
Working modalities with students	Work in pairs with concrete materials	Work in pairs in laboratory with GeoGebra software
Teacher's role	Teaching, explaining, exemplifying	Guiding students in discovering properties of quadrilaterals Discussing with them about description, definition, properties, Coordinating discussions giving stimuli, ordering conjectures Institutionalising knowledge
Tools and their functions	Concrete materials (paper and pencil) as a model where constructing quadrilaterals according to their symmetry properties	Software GeoGebra for constructing the same model of concrete materials (Fig 4)  Paper sheets forevery activity Instruction for constructing quadrilaterals in GeoGebra

Fig. 3 Excerpt from a teacher's logbook

For some, time pressures seem to have prevented thoughtful writings. They describe concisely the experiment as a finished product. Consequently, there is no important information in the log books about the real processes occurring during the development of the teaching experiment and there are only few reflections about the difficulties encountered and the planned changes. For these teachers, the interaction with the tutor was less dialogic and limited to the use of forums, email, and online resources in the platform.

As mentioned above, the Meta-didactical Transposition in the case of the M@t.abel Project has its strength in the use of a *platform* for synchronous and asynchronous activities among teachers. The platform is the environment that gives new techniques to teachers, influencing and supporting them in changing their praxeologies. In particular, if a teacher has worked for many years in a traditional and isolated way, now she is forced to discuss new methodological issues through ICT. For instance, the GeoGebra software (and the figure constructed with the software, like the one in Fig. 3) can be used as a boundary object between the community of tutors and that of teachers.

The platform, together with the brokering function of the tutors, aims to build a community of teachers with *shared praxeologies*. Besides being a *communication infrastructure*, allowing synchronous and asynchronous interactions, for sharing ideas, materials and methods, the platform works also as a *representational infrastructure* (Hegedus and Moreno-Armella 2009), fostering the use of a shared

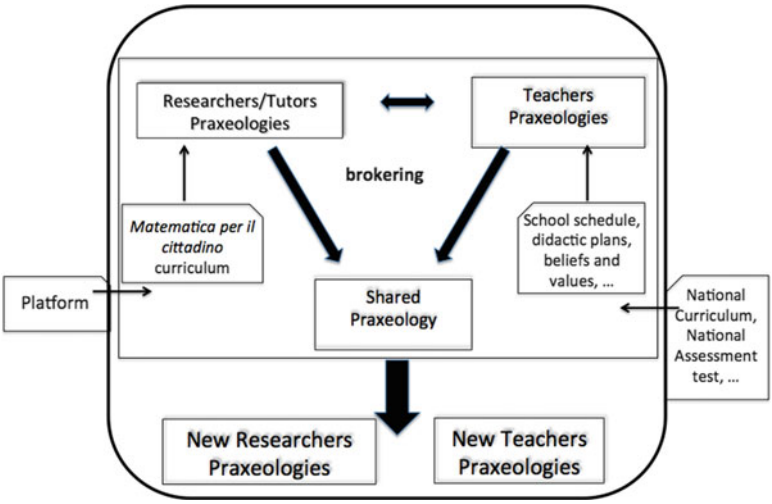


Fig. 4 Internal and external components in the Meta-didactical Transposition of M@t.abel

desktop where the teachers can work together on-line on the same topic or mathematical object. In the Meta-didactical Transposition model, the platform constitutes an example of an *external component* (for both researchers/tutors' and teachers' initial Praxeologies) that becomes an *internal* one over the course of the Project. Finally, the model includes two important effects of Meta-Didactical Transposition, which are the changes brought about by the project both in the researcher praxeologies and in teacher praxeologies. The change in the researcher praxeologies occurs through the researchers' ongoing reflection, which is prompted by considering the evolution of the system over time and the analysis of the internal/external components. Figure 4 provides an overall picture of the Meta-didactical Transposition within the M@t.abel project (the second step):

As a picture of a dynamic process, Fig. 4 cannot capture temporal evolution. If we imagine the model evolving over time, we can focus our attention on *occurred* and *not-yet-occurred evolutions*. Concerning the *occurred evolutions* in M@t.abel, we find two external components that become internal ones: the platform (as described above), and the *Matematica per il Cittadino* curriculum (which was internal for researchers/tutors' praxeologies but not for those of the teachers). The first component is a technical component and the second is a part of the theoretical discourse, which justifies certain tasks and techniques.

As mentioned above, unfortunately the logbook often constitutes a case of *not-yet-occurred* evolution on a large scale. For many teachers in fact the logbook did not function as a helpful day-to day observation tool, which means that it did not become an internal component in their praxeology. Instead, for a small number of them, it was a component that became internal thanks to their participation in the project.

In general, by looking at the evolutions in terms of the internal and external components, researchers can identify those features of the teacher education process that

are in need of further reflection and work. For instance, the logbook tool is currently under investigation by the researchers, in order to understand why it did not work as expected and to set up suitable changes. This kind of consideration is part of a *new researchers' praxeology*. Further considerations of this aspect of the model will be presented in the next examples.

The Evolution of Researcher Praxeologies Over Time: The Example of the MMLab-ER Project

In this second example of a teacher education programme, we show the potential of the *Meta-didactical Transposition* model for studying the *evolution of praxeologies over time*. Specifically, we use the model to identify and describe when, how and why some different components of praxeologies changed during the MMLab-ER teacher education programme. We also use the model to plan and control the variation in praxeologies.

MMLab-ER (http://www.mmlab.unimore.it/site/home/progetto-regionale-emilia-romagna.html, Martignone 2010) is a regional project that responds to national and international standards on IBSE (Inquiry Based Science Education; see Rocard et al. 2007). It aims to construct a network of competent practising teachers using the *mathematics laboratory* method (in the sense introduced in the previous section). In this project, old and new tools are involved (for example, reconstructions of historical mathematical tools and technologies). The compass is a well-known mathematical tool but, in the history of mathematics, several other mathematical machines (e.g. pantographs for geometric transformations, curve creators, perspectographs) have been designed and used for theoretical and practical purposes (Bartolini Bussi et al. 2010). In the MMLab-ER project, teachers and students work both with real mathematical machines and with virtual reconstructions of them, created by means of dynamic geometry software.

The laboratory sessions with mathematical machines, guided by a specific methodology and particular tasks (Bartolini Bussi et al. 2011; Martignone 2011), are suitable environments for the development of fundamental mathematical activities such as problem solving, production of conjectures, argumentation processes and generation of proofs within Euclidean geometry. This is one of the underlying assumptions of the MMLab-ER project. To date, approximately 200 teachers have participated in the first cycle (mainly grades 4–8) and the second cycle (grades 9–12, in high schools and vocational schools). The researchers worked as teacher educators with small groups of teachers (15–25 teachers from each of the eight Italian provinces involved) in both face-to-face sessions (28 h) and through an e-learning platform and email. The project began in 2008 with a 2-year period of regional financial support and then, at the beginning of 2012, as a result of new financial support, it recommenced with other teachers and schools.

This development over 3 years, which includes a 1-year break during which time the researchers analysed the project results, enables study of the evolution of

praxeologies in the MMLab-ER project. By means of the *Meta-Didactical Transposition* model, we analysed how the different components of the *researcher praxeologies* have developed from the beginning of the project until now. The MMLab research group analysed the experiences carried out both during the teacher education programmes and the teaching experiments, and have modified some of their praxeologies. The *researcher praxeologies* arose from a dynamic evolution of the relation between research and teaching that characterises the *Italian Research for Innovation* (Arzarello and Bartolini Bussi 1998). At the beginning of the Meta-Didactical Transposition process, the researchers had their own praxeologies linked to their studies of students' activities in the classroom and their experiences with and studies of teacher education. In the former, the *task* is to study the educational potential of the laboratory activities with mathematical machines and the *techniques* involve the design and analysis of activities for primary and secondary school students. In the latter, the *task* is to design activities that shift teachers' attentions to the processes of exploration, the resulting conjectures and the constructions of proof by means of laboratory sessions involving the mathematical machines. The *techniques* concern the development of tasks for teachers that include, for example, the selection of suitable educational paths to be discussed and the analysis of different teaching experiments. The *theoretical discourse* that describes, explains and justifies the techniques of these praxeologies is based on studies of mathematics teaching and learning by means of laboratory activities with mathematical machines (within the theoretical framework of *semiotic mediation* (Bartolini Bussi and Mariotti 2008). The *Meta-didactical Transposition* model is useful in describing and analysing the evolution over time of the different components of these researcher praxeologies. We can identify the aspects that do not change and describe how the levels of praxis and logos are modified. Concerning the level of praxis, some *techniques* were improved, such as: how new activities were introduced and elaborated with the teachers; the modification of some tasks for teachers; modification of the classroom tasks, taking into account what was discussed during the teacher education programme (s*hared praxeologies*); and refining the tools for analysing teachers' and students' worksheets, logbooks, video, etc. In addition, the *theoretical discourse* was improved by refining some theoretical tools. After the first year, in order to study the exploration of the mathematical machines carried out by teachers and students in more depth, cognitive studies concerning mathematical machines were developed that identified and analysed the argumentation processes involved (Antonini and Martignone 2011). After 2 years of the project, the researchers analysed all of the documentation, which included the videos, worksheets about the laboratory activities carried out by teachers and students, teachers' reflections collected in the logbooks, and the final reports of the teaching experiments. New research was carried out in order to identify, study and characterise the main features of the MMLab-ER teacher education programme (Garuti and Martignone 2010). These studies showed that the project's main results were not only the dissemination of innovative teaching methods, but also the design and testing of activities that seemed to develop teachers' skills in analysing the cultural aspects involved in the laboratory activities with mathematical machines. In order to interpret the kind of *teacher knowledge* the

MMLab-ER programme had developed, the researchers referred to the aforementioned studies about *Mathematical Knowledge for Teaching* (MKT) (described earlier). This construct was used to identify specialised teacher knowledge related to the non-pedagogical content. In particular, we found that the teachers' Specialised Content Knowledge (SCK), which is linked to the *cultural analysis of contents* (Boero and Guala 2008), improved through the development of new praxeologies involving reflection on teaching and learning activities. A teacher has (or can acquire) SCK linked to the cultural analysis of contents if s/he come to appreciate the potential of, and can analyse the cultural aspects (e.g. attention and analysis of historical-epistemological and cognitive aspects) related to some specific mathematical content (Garuti and Martignone 2010). The identification of this specialised content knowledge modified the *logos* level of *researcher praxeologies,* enriching them with a new theoretical construct. Today, these new praxeologies, with the praxeologies of the new teachers involved, are the starting point of the *Meta-didactical Transposition processes* that are going on in the second part of the MMLab-ER teacher education programme.

The Double-Level Dialectic: The Teacher Education Programme Within the ArAl Project

As we stated before, this third section is aimed at exemplifying how the *Meta-didactical Transposition* model highlights a typical aspect of teacher education programmes, which is the activation, through appropriate *meta-didactical trajectories*, of a *first-level dialectic* and, at the same time, the engendering of a *second-level dialectic* that enables teachers to acquire a new awareness of their role in the classroom.

The teacher education programme we present here is the *ArAl Project*, whose main objective is to foster a linguistic and constructive approach to early algebra (Malara and Navarra 2003) within an integrated teacher education programme (Cusi et al. 2010). The model for teacher education developed within the ArAl project resonates particularly with research carried out by Mason (1998, 2002) and Jaworski (1998, 2003, 2006). This programme is based on the hypothesis that observation and critical-reflective study of class processes, activated both individually and among communities of inquiry, is a necessary condition to foster teachers' development of awareness (Mason 2008) about the 'subtle sensitivities' that could guide their future choices and determine their effective action in the classroom. Another fundamental hypothesis is that giving teachers the possibility to analyse and interpret the activities they conduct in their classrooms, referring to specific theoretical lenses for the observation of the role they play, can foster a shift of attention for teachers as they reflect on their own practice, enabling them to focus not only on students' difficulties, but also on the interrelation between the attitudes and behaviours of teachers and students.

For this reason, the chosen methodology for our work with the teachers, which reflects the chosen *meta-didactical trajectory*, is characterised by these main three aspects: (1) in order to foster the development of a *community of inquiry*, the teachers involved in the same teaching experiment are associated with a mentor-researcher with whom the teachers engage in face-to-face work as well as email exchanges, which becomes the starting point of a *dialectic interaction between the two communities*; (2) teachers are involved in activities of theoretical study, aimed at providing them with theoretical and methodological tools useful for interpreting, through new lenses, their own actions in the classroom (in this way, teachers and researchers start to refer to a *shared praxeology*); (3) teachers are involved in a complex activity of critical analysis of the transcripts of audio-recordings concerning classroom processes and associated reflections, which is carried out by developing what we call *Multi-commented transcripts* (MT). The experimenters-teachers send the transcripts, together with their own comments and reflections, to mentors-researchers, who make their own comments and send them back to the authors, to other teachers involved in similar activities, and sometimes to other researchers (at this level, the dialectic interaction between the two communities is particularly intense). Often, the authors make further interventions in this cycle, commenting on comments or inserting new ones (see Malara 2008; Cusi et al. 2010; Malara and Navarra 2011).

The MT methodology, which helps teachers reflect on the activities they carried out in their classrooms, reveals their attitudes and behaviours as well as the effects of their interventions on their students. Thus, they highlight: the contrast/interaction between the personal sense their students attribute to the activities and the institutional meaning of the same activities (the *first-level dialectic*); and the role they play in fostering (or not) students' development of a personal sense, which is in tune with the institutional meaning of the activities.

At the same time, the researchers' analyses of the reflections proposed by teachers in the MT and the identification of a possible contraposition between teachers' and researchers' comments, enable a *second-level dialectic* to be highlighted in relation to both: (a) the possible different interpretations of the dynamics realised during class activities and (b) the possible different uses of the same theoretical lenses made by teachers and researchers in their analysis of classroom processes. Moreover, through the *a posteriori* analysis of the different comments proposed on the MT, the teachers can also become aware of this second-level dialectic.

The reflection carried out with the teachers involved in the project provides an opportunity to notice how the tension developed as a result of this *double-level dialectic* produces an evolution in the interrelations between the different components of the praxeologies involved within the process of *Meta-didactical Transposition*. In particular: it fosters the development of *new teachers' praxeologies*, related both to roles they should activate in their classrooms and to ways of pursuing their professional development; it enables researchers to hypothesise a possible refinement of the theoretical lenses for the observation of class processes and the possibility for further evolutions of the methodology to be adopted in the work with teachers, therefore fostering an enhancement of the chosen *meta-didactical trajectory*.

Discussion

In this chapter, we introduced the *Meta-didactical Transposition* model as a theoretical tool to objectify and describe the complex dialectic between research and institutional dimensions of teacher education programmes. This model is based on Chevallard's ATD, but was complemented by additional constructs in order to account for some dynamic features occurring when teachers and researchers are engaged in teacher education activities. We outlined and analysed three examples from the Italian context, in order to illustrate how this model can be productively used to analyse diverse types of teacher education programmes.

The first example discussed is the M@t.abel Project. By means of the Meta-didactical Transposition model, we were able to observe the dynamics between external and internal components (Fig. 2), and to identify strong and weak points of the project that we had not noticed as clearly before. We found two relevant *occurred evolutions* from external to internal components in the Meta-didactical Transposition: the *Matematica per il cittadino* curriculum and the platform. The curriculum was at first internal to the researcher praxeologies, and external to the teacher ones. The passage from external to internal was fostered by the brokering actions of the tutors. The platform is a technological device, which is initially external to both teacher and researcher praxeologies. This platform not only enabled the communication of ideas, feelings and didactic plans between teachers and tutors, but it also opened up a concrete space for the development of didactic activities that involved the use of software. The logbook constitutes a more delicate element. It too was initially an external component to the teacher praxeologies. Throughout the project, for some teachers, the logbook became an internal component of the shared praxeology as they used it to organise their classroom observation and plan their work better. This dynamic transformation from an external to an internal component did not occur for all teachers. Many of them, in fact, wrote their logbook at the end of the whole project, despite the constant prompts of their tutors, and for these teachers the logbook remained an external component that did not alter their praxeologies.

In the second example, we highlighted how the Meta-didactical Transposition model offers a framework that enabled us to analyse the evolution of praxeologies over time in the MMLab-ER project. In particular, we focused on the researcher praxeologies related to the changing of logos and praxis. The Meta-didactical Transposition model was useful in analysing not only how the praxeologies changed, but also why they were modified in relation to the teacher education programme development. Furthermore, the model allowed us to objectify, through the identification of the researcher and teacher praxeologies, their evolutions over time, while also maintaining a systemic view. At the end of the Meta-didactical Transposition process, both researchers and teachers developed new praxeologies, changing some of their techniques as well as their ways of explaining and justifying these techniques. This could become the new starting point of a fresh Meta-didactical Transposition process.

The third example revealed the essential role played by an appropriate 'meta-didactical trajectory' in: helping teachers become aware of the first-level dialectic

related to the contrast/interaction between the personal sense their students attribute to the activities and the institutional meaning of the same activities; and in enabling researchers and teachers highlight the second-level dialectic which is related to the contrast/interaction between the different interpretations of the dynamics realised in the classrooms, given by teachers and researchers, in relation to specific theoretical lenses. Moreover, this example showed that the tension developed out of this to the double-level dialectic could foster the evolution of both researcher and teacher praxeologies. In particular, it highlighted the strict interrelation between this evolution and the chosen methodology of work with teachers. Involving teachers in the critical-reflective study of class processes, in fact, means, as Jaworski (2003) states, making them shift from a context of perpetuation of existing practices (the communities of teachers within a school) to a new context, typical of communities of inquiry, characterised by "the importance attached to meta-knowing through reflecting on what is being or has been constructed and on the tools and practices involved in the process" (p. 256).

Globally, the three examples must be conceived as pieces of a puzzle, which shed light on specific aspects of the model. They illustrated how the Meta-didactical Transposition model responds to the challenge of studying "how different approaches to teacher development have different effects on particular aspects of teachers' pedagogical content knowledge" (Ball et al. 2008, p. 405). We have already noticed that a similar construct was used by Clark & Hollingsworth (2002) to underline that teacher education programmes can produce changes in teachers' teaching strategies, "that represented in themselves new pedagogical knowledge" (*ibid.,* p. 953) for those teachers and that "were subsequently put into practice" (*ibid.,* p. 954). In other words, teacher education programmes can produce changes in teacher praxeologies. In fact, our model is similar but not identical to that of Clark & Hollingsworth, since ours underscores the interdependence of such changes with the institutions (according to the ATD approach), and focuses on the Meta-didactical components of the processes, which remain more implicit in Clark & Hollingsworth's approach.

The Meta-didactical Transposition model is deeply related also to the MKT construct. Both models focus on the intertwining of the theoretical knowledge and the common practices needed by teachers in their work, but each stresses different aspects of this intertwining. The MKT focuses on the structure of the mathematical knowledge for teaching while the Meta-didactical Transposition stresses more the dynamic evolution of its components. In particular, as illustrated in the examples above, it shows the relevance of the double-level dialectic and of the evolution from external to internal components in promoting and supporting the processes of teacher education.

As the MKT model refines Shulman's PCK model (Pedagogical Content Knowledge, 1986), so the Meta-didactical Transposition model enriches the MKT one. In fact, it essentially adds dynamicity to its description, allowing a transition from the snapshot illustrated by the fixed categories of MKT to the film of the Meta-didactical Transposition model as shown in Fig. 1. More precisely, our model introduces the temporal dimension, the double level dialectic and the internal-external dynamics, which are all elements that allow us to focus on the dynamic evolution of teachers'

educational programmes, which eventually produce the specificity of the different "domains of mathematical knowledge for teaching", to use Ball's terminology (Ball et al. 2008, p. 403). The lens of the meta-didactical praxeology allows the dynamicity of the process to be made evident. In fact, the model helps reveal the evolution of MKT, which can be hard to see because it is so imbedded in its particular institutional context. In the researchers' and teachers' hands, the MDT model can become a conscious tool in order to plan, develop and accomplish teacher educational programmes taking into account the complex interplay and dynamics between their components.

The study of this potential efficacy also introduces a fresh and promising strand of future investigation, which could produce further results concerning the nature of the domains of mathematical knowledge for teaching and the underlying processes of teachers' education.

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