Chapter 9
On Modal Logics Defining Jaśkowski's D_2-Consequence

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9.1 Introduction

The basic features of notions of deductive and deductive discussive systems used by Jaśkowski are as follows (see Jaśkowski 1948, p. 61 and Jaśkowski 1999, pp. 37–38).

- By theses of a deductive system Jaśkowski meant all expressions asserted within it, i.e. axioms and theorems deduced from them or proved in a specific way for a given system.
- A deductive system is based on a certain logic iff the set of its theses is closed under *modus ponens* rule with respect to theorems of the logic.
- A deductive system is overcomplete iff the set of its theses is equal to the set of all meaningful expressions of the language.
- A deductive system is inconsistent iff among its theses there are two theses such that one of them is the negation of the other.
- Usually, theses of a deductive system are formally expressed theorems of some consistent theory.
- If there is no assumption that theses of a deductive system express opinions which do not contradict each other, then such a system is called *discussive*.

Jaśkowski's aim was to formulate a logic, which when applied to inconsistent systems would not generally entail overcompleteness.

Jaśkowski gave an example of the way in which theses of discussive systems can be generated by referring to a discussion. Decisive for such a choice was the fact that during a discussion inconsistent voices can appear, however, we are not inclined to deduce every thesis from them.

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One can treat voices appearing explicitly in the discussion as preceded by the following restriction: “according to the opinion of one of the participants of the discussion”, which formally one can express by preceding the given statement with: ‘it is possible that’. If we take a position of an external observer (i.e. someone that does not take part in a discussion) all voices appearing in a discussion are *only possible*. It is so, since a person who is not involved in the discussion has every right to treat particular voices in disbelief or to dissociate from discussants’ statements. For the same reason, also conclusions following from explicitly expressed statements in a discussion are *only possible*. Conclusions one can treat as implicitly included into a discussion, since a given discussion consists not only of voices explicitly expressed, but also statements concluded from them. Summarizing, explicit voices, as well as their conclusions, are treated as theses of a discussive system.

Since the above pattern requires use of a modal language, one has to choose some specific modal logic. Jaśkowski himself chose the logic **S5**.

It is obvious that one needs to consider the language of full sentential logic, since otherwise one would have to treat all sentences as atomic ones, and it would not be possible to analyze logical deducibility based on the meaning of logical sentential constants.

In the present paper, ‘p’ and ‘q’ are propositional letters, used to built formulae (both discussive and modal). Capital Latin letters ‘A’, ‘B’ and ‘C’ (with or without subscripts) are metavariables for formulae, a Greek letter ‘Π’ is a metavariable for sets of formulae, while small Latin latter ‘a’ is a metavariable for propositional letters. Besides, following Jaśkowski’s custom, Gothic letters are used to denote instances of concrete sentences of the natural language.

Jaśkowski observed that while formulating a discussive system one can not treat the implication ‘$\rightarrow$’ as a material one, since sets of theses of discussive systems would not be closed under the *modus ponens* rule:

$$\frac{\begin{array}{c}\mathfrak{P} \rightarrow \mathfrak{Q}\\ \mathfrak{P}\end{array}}{\mathfrak{Q}}$$

> [...] out of the two theses one of which is
>
> $$\mathfrak{P} \rightarrow \mathfrak{Q}\,,$$
>
> and thus states: “it is possible that if $\mathfrak{P}$, then $\mathfrak{Q}$”, and the other is
>
> $$\mathfrak{P}\,,$$
>
> and thus states: “it is possible that $\mathfrak{P}$”, it does not follow that “it is possible that $\mathfrak{Q}$”, so that the thesis
>
> $$\mathfrak{Q}\,,$$
>
> does not follow intuitively, as the rule of *modus ponens* requires. (Jaśkowski 1999, p. 43, see also Jaśkowski 1948, p. 66)

Jaśkowski meant that the formula:

$$\Diamond(p \to q) \to (\Diamond p \to \Diamond q)$$

is not a thesis of **S5**. Thus, not for all sentences $\mathfrak{P}$ and $\mathfrak{Q}$, the following sentence

$$\Diamond(\mathfrak{P} \to \mathfrak{Q}) \to (\Diamond\mathfrak{P} \to \Diamond\mathfrak{Q})$$

is a substitution of a logical thesis.

As an appropriate implication to be used in the formulation of a discussive logic Jaśkowski chose a discussive one. We will denote it by '$\to^{\mathrm{d}}$'. In the formal language Jaśkowski defined a formula

$$p \to^{\mathrm{d}} q$$

by

$$\Diamond p \to q\,.$$

Jaśkowski intuitively understood it in the following way: "if anyone states that p, then q" (Jaśkowski 1999, p. 44, see also Jaśkowski 1948, p. 67).

In the same fragment, Jaśkowski pointed to the fact that:

> In every discussive system two theses, one of the form:
>
> $$\mathfrak{P} \to^{\mathrm{d}} \mathfrak{Q}\,,$$
>
> and the other of the form:
>
> $$\mathfrak{P}\,,$$
>
> entail the thesis
>
> $$\mathfrak{Q}\,,$$
>
> and that on the strength of the theorem
>
> $$\Diamond\,(\Diamond p \to q) \to (\Diamond p \to \Diamond q). \qquad (\mathrm{M}_2 1)$$

Thus, such an understanding of the implication ensures that sets of theses of deductive systems are closed under the *modus ponens* rule.

A discussive equivalence (notation: '$p \leftrightarrow^{\mathrm{d}} q$') Jaśkowski defined as:

$$(\Diamond p \to q) \wedge (\Diamond q \to \Diamond p).$$

In Jaśkowski (1948) (see also Jaśkowski (1969)), three classical connectives are used: negation ('$\neg$'), disjunction ('$\vee$') and conjunction ('$\wedge$'). Moreover, a discussive conjunction '$\wedge^{\mathrm{d}}$', was introduced in Jaśkowski (1949). Any sentence of the form '$p \wedge^{\mathrm{d}} q$' expresses a statement: "p and it is possible that q", i.e. formally: '$p \wedge \Diamond q$'. Notice that in Jaśkowski (1949) the classical conjunction was not dropped from the language of discussive systems.

> Dwuwartościowy dyskusyjny rachunek zdań oznaczony jako D_2 można wzbogacić definiując koniunkcję dyskusyjną *Kd*. [*In English*: The two-valued discussive propositional calculus denoted as D_2 can be enriched with a definition of the discussive conjunction $\wedge^{d}$]. (Jaśkowski 1949, p. 171, Jaśkowski 1999a, p. 57)

The question arises: *what is the natural interpretation of the classical conjunction in the context of discussive systems?* It seems that the classical conjunction can be used to "glue" particular statements of a given participant of the discussion. For example, if a given participant expresses two statements $\mathfrak{P}$ and $\mathfrak{Q}$ then she/he asserts $\ulcorner\mathfrak{P} \wedge \mathfrak{Q}\urcorner$, i.e. taking the external point of view we have in the modal language $\ulcorner\Diamond(\mathfrak{P}^{\bullet} \wedge \mathfrak{Q}^{\bullet})\urcorner$, where $(-)^{\bullet}$ is the appropriate translation of discussive connectives which can appear within $\mathfrak{P}$ and $\mathfrak{Q}$. On the other hand discussive conjunction is usually meant as a tool adequate to express the status of a given discussion from the point of view of a given participant of the discussion. Thus, if we have assertions $\mathfrak{P}$ and $\mathfrak{Q}$ made by two participants, then the appearance of these two statements—taking the point of view of the first participant—can be expressed as follows: $\ulcorner\mathfrak{P} \wedge^{d} \mathfrak{Q}\urcorner$. From the external point of view such a statement becomes $\ulcorner\Diamond(\mathfrak{P}^{\bullet} \wedge \Diamond\mathfrak{Q}^{\bullet})\urcorner$, which in the logic **S5** is equivalent to $\ulcorner\Diamond\mathfrak{P}^{\bullet} \wedge \Diamond\mathfrak{Q}^{\bullet}\urcorner$. We obtain the same formula if we start with the consideration of the point of view of the second participant. Indeed, we have the discussive record of the discussion from the point of view of the second participant: $\ulcorner\mathfrak{Q} \wedge^{d} \mathfrak{P}\urcorner$, while the external point of view of this statement becomes: $\ulcorner\Diamond(\mathfrak{Q}^{\bullet} \wedge \Diamond\mathfrak{P}^{\bullet})\urcorner$, equivalently on the basis of **S5** we have $\ulcorner\Diamond\mathfrak{Q}^{\bullet} \wedge \Diamond\mathfrak{P}^{\bullet}\urcorner$.

Of course we are not interested only in the ≪external description≫ of a given discussion, but also whether $\mathfrak{Q}$ discussively follows from given statements $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ of n participants ($n > 0$). Using modal translations and the usual understanding of deduction in modal logics we inquire whether the following statements (equivalent by the positive logic):

(a) $(\Diamond\mathfrak{P}_1^{\bullet} \wedge \cdots \wedge \Diamond\mathfrak{P}_n^{\bullet}) \to \Diamond\mathfrak{Q}^{\bullet}$,
(b) $\Diamond\mathfrak{P}_1^{\bullet} \to (\ldots \to (\Diamond\mathfrak{P}_n^{\bullet} \to \Diamond\mathfrak{Q}^{\bullet})\ldots)$

are valid in **S5**.[1] Equivalently we can look into the problem of validity of the following sentences in the discussive logic:

(a)^{d} $(\mathfrak{P}_1 \wedge^{d} \cdots \wedge^{d} \mathfrak{P}_n) \to^{d} \mathfrak{Q}$,
(b)^{d} $\mathfrak{P}_1 \to^{d} (\ldots \to^{d} (\mathfrak{P}_n \to^{d} \mathfrak{Q})\ldots)$.[2]

In both cases (a)^{d} and (b)^{d}—using the logic **S5**—we obtain the equivalent translations of sentences into the modal language. We have to remember that in the case of validity in the discussive logic the translation obtained has to be preceded by '$\Diamond$',

[1] For $n = 0$ we inquire whether the sentence $\mathfrak{Q}$ is valid in the discussive logic, i.e. whether the modal sentence $\Diamond\mathfrak{Q}^{\bullet}$ is valid in **S5**.

[2] Notice that for $n = 1$ and any $m > 0$ a sentence $\ulcorner(\mathfrak{p}_1 \wedge \cdots \wedge \mathfrak{p}_m) \to^{d} \mathfrak{Q}\urcorner$ has a form (a)^{d} as well as a form (b)^{d}, for $\mathfrak{P}_1 := \ulcorner\mathfrak{p}_1 \wedge \cdots \wedge \mathfrak{p}_m\urcorner$. Thus, it can be treated as expressing the external point of view where only one participant is considered.

since from the point of view of an external observer the sentences $(a)^{d}$ and $(b)^{d}$ are *only possible*. Thus indeed (a) and (b) are the modal counterparts of $(a)^{d}$ and $(b)^{d}$, respectively.

As it is known the formula (a), resp. (b), is valid in **S5** iff there is a finite sequence beginning with sentences $\ulcorner\Diamond\mathfrak{P}_1^{\bullet}\urcorner, \ldots, \ulcorner\Diamond\mathfrak{P}_n^{\bullet}\urcorner$, and ending with $\ulcorner\Diamond\mathfrak{Q}^{\bullet}\urcorner$, where the other elements (as well as $\ulcorner\Diamond\mathfrak{Q}^{\bullet}\urcorner$) are either theses of **S5** and/or are sentences obtained from some sentences preceding in the sequence obtained by *modus ponens*.

The main aim of our paper is to find the smallest normal logic and the smallest regular logic which could be used instead of **S5**. For these logics it is not enough to have the same theses beginning with '$\Diamond$' as **S5**; since we consider here the discussive deducibility relation, thus these logics have to include also ($M_2$1).

Remark 9.1. In the case of a sentence of the form (a), resp. (b), for $n = 0$ we only try to find out whether a wanted logic has the same thesis beginning with '$\Diamond$'. This problem has already been solved in the case of normal and regular classes of logics (Furmanowski 1975; Perzanowski 1975; Nasieniewski and Pietruszczak 2008). □

Nowadays in the considerations concerning the logic $\mathbf{D_2}$ the classical conjunction is usually omitted. It is justified by the functional completeness obtained by classical connectives of '$\neg$' and '$\vee$'. Thus, we also do not include the classical conjunction in the discussive language.

9.2 Basic Notions

Let For^{d} be the set of all formulae of the discussive language with constants: '$\neg$', '$\vee$', '$\wedge^{d}$', '$\rightarrow^{d}$', and '$\leftrightarrow^{d}$'. Let $\mathrm{For_m}$ be the set of all modal formulae.[3] *Jaśkowski's transformation* is the function $-^{\bullet}$ from For^{d} into $\mathrm{For_m}$ such that:

1. $(a)^{\bullet} = a$, for any propositional letter a,
2. and for any $A, B \in \mathrm{For}^{d}$:
   (a) $(\neg A)^{\bullet} = \ulcorner\neg A^{\bullet}\urcorner$,
   (b) $(A \vee B)^{\bullet} = \ulcorner A^{\bullet} \vee B^{\bullet}\urcorner$,
   (c) $(A \wedge^{d} B)^{\bullet} = \ulcorner A^{\bullet} \wedge \Diamond B^{\bullet}\urcorner$,
   (d) $(A \rightarrow^{d} B)^{\bullet} = \ulcorner\Diamond A^{\bullet} \rightarrow B^{\bullet}\urcorner$
   (e) $(A \leftrightarrow^{d} B)^{\bullet} = \ulcorner(\Diamond A^{\bullet} \rightarrow B^{\bullet}) \wedge \Diamond(\Diamond B^{\bullet} \rightarrow A^{\bullet})\urcorner$.[4]

Assume that voices in a discussion are written formally by schemes: $A_1, \ldots, A_n$. We consider a possible conclusion B. Since formulae $A_1, \ldots, A_n$ and B may contain

[3] In Appendix we recall some chosen basic facts and notions concerning modal logic.

[4] If the classical conjunction were considered, one would have to add the following condition: $(A \wedge B)^{\bullet} = \ulcorner A^{\bullet} \wedge B^{\bullet}\urcorner$.

logical constants thus, instead of $\Diamond A_1, \ldots, \Diamond A_n$ and $\Diamond B$ we have to consider their discussive versions: $\Diamond A_1^\bullet, \ldots, \Diamond A_n^\bullet$ and $\Diamond B^\bullet$. Taking into account examples given by Jaśkowski we see that he used the following definition of a discussive relation: *B follows discussively from* $A_1, \ldots, A_n$ iff the following formula

$$\Diamond A_1^\bullet \to (\ldots \to (\Diamond A_n^\bullet \to \Diamond B^\bullet) \ldots)$$

belongs to **S5**.[5]

To conclude, discussive deductive systems are to be based on a certain logic connected with the following consequence relation for formulae from $\mathrm{For}^{\mathrm{d}}$.

Definition 9.1 *For any* $\Pi \subseteq \mathrm{For}^{\mathrm{d}}$ *and* $B \in \mathrm{For}^{\mathrm{d}}$: $\Pi \vdash_{\mathbf{D_2}} B$ *iff for some* $n \geqslant 0$ *and for some* $A_1, \ldots, A_n \in \Pi$ *we have*

$$\ulcorner \Diamond A_1^\bullet \to (\ldots \to (\Diamond A_n^\bullet \to \Diamond B^\bullet) \ldots) \urcorner \in \mathbf{S5}\,.$$

In other words,

$$\Pi \vdash_{\mathbf{D_2}} B \qquad \textit{iff} \qquad \{\Diamond A^\bullet : A \in \Pi\} \vdash_{\mathbf{S5}} \Diamond B^\bullet,$$

where $\vdash_{\mathbf{S5}}$ *is the pure modus-ponens-style inference relation based on* **S5** *(see Definition 9.A.1 and Fact 9.A.1 in Appendix).*

Jaśkowski used notation '$\mathbf{D_2}$' referring to a logic, i.e. a certain set of formulae.

Definition 9.2 $\mathbf{D_2} := \{ A \in \mathrm{For}^{\mathrm{d}} : \ulcorner \Diamond A^\bullet \urcorner \in \mathbf{S5} \}$.

Thus, on the basis of $\mathbf{D_2}$ one can characterize the consequence relation for discussive systems in the following way:

Fact 9.1 *For any* $n \geqslant 0$, $A_1, \ldots, A_n, B \in \mathrm{For}^{\mathrm{d}}$:

$$\begin{aligned} A_1, \ldots, A_n \vdash_{\mathbf{D_2}} B \quad &\textit{iff} \quad \ulcorner(\Diamond A_1^\bullet \wedge \cdots \wedge \Diamond A_n^\bullet) \to \Diamond B^\bullet \urcorner \in \mathbf{S5} \\ &\textit{iff} \quad \ulcorner \Diamond A_1^\bullet \to (\ldots \to (\Diamond A_n^\bullet \to \Diamond B^\bullet) \ldots) \urcorner \in \mathbf{S5} \\ &\textit{iff} \quad \ulcorner \Diamond (A_1 \to^{\mathrm{d}} (\ldots \to^{\mathrm{d}} (A_n \to^{\mathrm{d}} B) \ldots))^\bullet \urcorner \in \mathbf{S5} \\ &\textit{iff} \quad \ulcorner A_1 \to^{\mathrm{d}} (\ldots \to^{\mathrm{d}} (A_n \to^{\mathrm{d}} B) \ldots) \urcorner \in \mathbf{D_2} \\ &\textit{iff} \quad \ulcorner (A_1 \wedge^{\mathrm{d}} \cdots \wedge^{\mathrm{d}} A_n) \to^{\mathrm{d}} B \urcorner \in \mathbf{D_2}\,. \end{aligned}$$

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[5]In da Costa and Doria (1995) a similar relation was used, yet not for $\mathrm{For}^{\mathrm{d}}$, but for a modal language enriched with some discussive connectives. However, in this modal language the discussive conjunction was defined as follows: $\ulcorner (A \wedge^{\mathrm{d}} B) \leftrightarrow (\Diamond A \wedge B) \urcorner$. But, as it was proved in Ciuciura (2005), for a new transformation $-^*$ such that $(A \wedge^{\mathrm{d}} B)^* = \ulcorner \Diamond A^* \wedge B^* \urcorner$, we obtain another discussive logic $\mathbf{D_2^*}$ which differs from $\mathbf{D_2}$.

Proof. By **PL**, $(5^{\Diamond}!)$, $(R^{\Diamond\Box})$, and definitions of the relation $\vdash_{\mathbf{D_2}}$, the function $-^{\bullet}$, and the logic $\mathbf{D_2}$.

Notice that, by the above fact, we can express the relation $\vdash_{\mathbf{D_2}}$ as the pure *modus-ponens*-style inference relation based on $\mathbf{D_2}$.

Fact 9.2 *For any* $\Pi \subseteq \mathrm{For}^{\mathrm{d}}$ *and* $B \in \mathrm{For}^{\mathrm{d}}$:

$\Pi \vdash_{\mathbf{D_2}} B$ *iff* *there exists a sequence* $A_1, \ldots, A_n = B$ *in which for any* $i \leqslant n$, *either* $A_i \in \Pi \cup \mathbf{D_2}$ *or there are* $j, k < i$ *such that* $A_k = \ulcorner A_j \rightarrow^{\mathrm{d}} A_i \urcorner$.

Proof. Because ($M_2$1) belongs to **S5**, so $\mathbf{D_2}$ is closed under *modus ponens* for '$\rightarrow^{\mathrm{d}}$', i.e., for any $A, B \in \mathrm{For}^{\mathrm{d}}$, if $A, \ulcorner A \rightarrow^{\mathrm{d}} B \urcorner \in \mathbf{D_2}$, then $B \in \mathbf{D_2}$. Moreover, $\mathbf{D_2}$ contains for any $A, B, C \in \mathrm{For}^{\mathrm{d}}$ the following formulae:

$$A \rightarrow^{\mathrm{d}} (B \rightarrow^{\mathrm{d}} A)$$

$$(A \rightarrow^{\mathrm{d}} (B \rightarrow^{\mathrm{d}} C)) \rightarrow^{\mathrm{d}} ((A \rightarrow^{\mathrm{d}} B) \rightarrow^{\mathrm{d}} (A \rightarrow^{\mathrm{d}} C))$$

So the condition from the fact is equivalent to the following condition: for some $n \geqslant 0$ and for some $A_1, \ldots, A_n \in \Pi$ we have $\ulcorner A_1 \rightarrow^{\mathrm{d}} (\ldots \rightarrow^{\mathrm{d}} (A_n \rightarrow^{\mathrm{d}} B)\ldots)\urcorner \in \mathbf{D_2}$.[6] Thus, by Fact 9.1, it is equivalent to $\Pi \vdash_{\mathbf{D_2}} B$.

9.3 Other Logics Defining $\mathbf{D_2}$

Definition 9.3 *Let* $\boldsymbol{L}$ *be any modal logic.*

(i) We say that $\boldsymbol{L}$ *defines* $\mathbf{D_2}$ iff $\mathbf{D_2} = \{ A \in \mathrm{For}^{\mathrm{d}} : \ulcorner \Diamond A^{\bullet} \urcorner \in \boldsymbol{L} \}$.
(ii) Let $\mathbf{S5}_{\Diamond}$ be the set of all modal logics which have the same theses beginning with '$\Diamond$' as **S5**, i.e., $\boldsymbol{L} \in \mathbf{S5}_{\Diamond}$ iff $\forall_{A \in \mathrm{For_m}} (\ulcorner \Diamond A \urcorner \in \boldsymbol{L} \iff \ulcorner \Diamond A \urcorner \in \mathbf{S5})$.

Fact 9.3 *Nasieniewski and Pietruszczak (2008). For any classical modal logic* $\boldsymbol{L}$: $\boldsymbol{L}$ *defines* $\mathbf{D_2}$ *iff* $\boldsymbol{L} \in \mathbf{S5}_{\Diamond}$.

In Furmanowski (1975), it was shown that **S4** and **S5** have the same members beginning with '$\Diamond$'—thus, one can use weaker modal logics to define $\mathbf{D_2}$. In Perzanowski (1975), the smallest normal modal logic (denoted by $\mathbf{S5^M}$) possessing this property was indicated.

[6] So notice that for the logic $\mathbf{D_2}$ we have an analogous fact to Fact 9.A.1.

In Perzanowski (1975) $\mathbf{S5^M}$ was defined as the smallest normal logic containing $\ulcorner\Diamond\top\urcorner$,[7]

$$\Diamond\Box(\Diamond\Box p \to \Box p) \tag{ML5}$$

$$\Diamond\Box(\Box p \to p) \tag{MLT}$$

and closed under the following rule:

$$\text{if } \ulcorner\Diamond\Diamond A\urcorner \in \mathbf{S5^M} \text{ then } \ulcorner\Diamond A\urcorner \in \mathbf{S5^M}. \tag{RM^2_1}$$

Let $\mathbf{NS5}_\Diamond$ and $\mathbf{RS5}_\Diamond$ be respectively the sets of all normal and regular logics from $\mathbf{S5}_\Diamond$.

Fact 9.4 *Perzanowski (1975).* $\mathbf{S5^M}$ *is the smallest logic in* $\mathbf{NS5}_\Diamond$.

Notice that one can drop two out of the three axioms of the original formulation of $\mathbf{S5^M}$ (see also Fact 9.8ii).

Fact 9.5 *Nasieniewski and Pietruszczak (2008).* $\mathbf{S5^M}$ *is the smallest normal logic which contains* (MLT) *and is closed under* (RM^2_1).

Besides, it was proved in Błaszczuk and Dziobiak (1977) that one can define the logic $\mathbf{S5^M}$ without the rule (RM^2_1), using instead—as an additional axiom—the following formula ("semi-4"):

$$\Box p \to \Diamond\Box\Box p \tag{4_s}$$

Fact 9.6 *Błaszczuk and Dziobiak (1977).* $\mathbf{S5^M}$ *is the smallest normal logic containing* (4_s) *and* (MLT), *i.e.* $\mathbf{S5^M} = \mathbf{K4_s(MLT)}$.[8]

Additionally, in Nasieniewski (2002) another axiomatisation of the logic $\mathbf{S5^M}$ without the rule (RM^2_1) was given.

Fact 9.7 *Nasieniewski (2002).* $\mathbf{S5^M}$ *is the smallest normal logic which contains* (4_s) *and the converse of* (5)

$$\Box p \to \Diamond\Box p \tag{5_c}$$

i.e. $\mathbf{S5^M} = \mathbf{K4_s5_c}$.

In Nasieniewski and Pietruszczak (2008) a regular version of the logic $\mathbf{S5^M}$ was considered. It was proved that while defining the logic $\mathbf{D_2}$ one can use a weaker modal logic than $\mathbf{S5^M}$.

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[7] As it is well known, in all regular logics (and so in normal ones) the formula $\ulcorner\Diamond\top\urcorner$ is equivalent to the formula (D) (see Lemma 9.A.7). The smallest normal logic containing (D) (equivalently $\ulcorner\Diamond\top\urcorner$) is denoted by '**KD**' or simply by '**D**'. We have, $\mathbf{D} \subsetneq \mathbf{S5^M}$.

[8] For an explanation of the *Lemmon code* $\mathbf{KX}_1 \ldots \mathbf{X}_n$ or $\mathbf{CX}_1 \ldots \mathbf{X}_n$ see page 160.

Definition 9.4 *Let* $\mathbf{rS5^M}$ *denote the smallest regular logic which contains* (MLT) *and is closed under the rule* (RM^2_1).

Fact 9.8 *Nasieniewski and Pietruszczak (2008).*

(i) The logic $\mathbf{rS5^M}$ is not normal. In other words, $\mathbf{rS5^M}$ has no thesis of the form $\ulcorner \Box B \urcorner$. Thus, $\mathbf{rS5^M} \subsetneq \mathbf{S5^M}$.
(ii) (D), (ML5) $\in \mathbf{rS5^M}$.
(iii) $\mathbf{rS5^M}$ is the smallest logic in $\mathsf{RS5}_\diamond$; so $\mathbf{rS5^M}$ is the smallest regular logic defining $\mathbf{D_2}$.

From Fact 9.8(iii) we obtain:

Corollary 9.1. *For any modal logic* L: *if* $\mathbf{rS5^M} \subseteq L \subseteq \mathbf{S5}$, *then* $L \in \mathsf{S5}_\diamond$.

In Nasieniewski and Pietruszczak (2009) three axiomatisations of $\mathbf{rS5^M}$ where given: two of them were formulated without (RM^2_1) rule, while one was using (RM^2_1). Axiomatisations of $\mathbf{rS5^M}$ correspond to axiomatisations of the logic $\mathbf{S5^M}$. These results have been summarized below.

Fact 9.9 *Nasieniewski and Pietruszczak (2009).*
$\mathbf{rS5^M}$ *is the smallest regular logic which*:

(i) Contains (MLT) and (4_s), i.e. $\mathbf{rS5^M} = \mathbf{C4_s(MLT)}$;
(ii) Contains (5_c) and (4_s), i.e. $\mathbf{rS5^M} = \mathbf{C4_s5_c}$;
(iii) Contains (5_c) and is closed under (RM^2_1).

Besides, we have the upward analogue of the result from Fact 9.8(iii).

Fact 9.10 *Nasieniewski and Pietruszczak (2008).*
If L *is a regular logic defining* $\mathbf{D_2}$, *then* $L \subseteq \mathbf{S5}$.[9]

9.4 KD45 in the Formulation of D_2-Consequence

It appears that the consequence relation $\vdash_{\mathbf{D_2}}$ is closely related to the normal logic **KD45** (= **K5!** = $\mathbf{K55_c}$; see Lemma 9.A.8(v)). To start an investigation of this relationship, we will prove the following lemma.

Lemma 9.1.

(i) (4_s) $\in \mathbf{CD4} \subsetneq \mathbf{KD4}$.
(ii) (4), (5) $\notin \mathbf{K4_s5_c} = \mathbf{S5^M}$.
(iii) $\mathbf{S5^M} \subsetneq \mathbf{KD4} \subsetneq \mathbf{KD45}$.

Proof. (i) By (4), (US) and **PL**, the formula '$\Box p \to \Box\Box\Box p$' belongs to **C4**. Moreover, by (D), (US) and **PL**, we obtain that (4_s) $\in$ **CD4**.

[9] It was proved in Błaszczuk and Dziobiak (1975) that if $L \in \mathsf{NS5}_\diamond$, then $L \subseteq \mathbf{S5}$.

(ii) By "the corresponding Hintikka condition" from Segerberg (1971), Theorem 6.5 (see also Błaszczuk and Dziobiak 1977; Nasieniewski 2002) we know that normal logics defined by (5_c), and (4_s) are determined by frames $\langle W, R\rangle$ fulfilling, respectively, the following conditions:

$$\forall_u \exists_x \big(u\,R\,x \;\&\; \forall_v (x\,R\,v \Longrightarrow u\,R\,v)\big) \tag{h5$_c$}$$

$$\forall_u \exists_x \big(u\,R\,x \;\&\; \forall_v (x\,R^2\,v \Longrightarrow u\,R\,v)\big) \tag{h4$_s$}$$

We can indicate a model whose frame fulfils this conditions in which (4) and (5) are falsified. Thus, (5), (4) $\notin$ $\mathbf{K4_s5_c}$. By Fact 9.7, $\mathbf{K4_s5_c} = \mathbf{S5^M}$.

(iii) By (i), (ii) and Lemma 9.A.8(iii) we have $\mathbf{S5^M} \subsetneq \mathbf{KD4} = \mathbf{K45_c} \subsetneq \mathbf{KD45}$.

Since $\mathbf{S5^M} \subseteq \mathbf{KD45} \subseteq \mathbf{S5}$, so from Fact 9.3 and Corollary 9.1 we obtain:

Corollary 9.2. $\mathbf{KD45} \in \mathsf{NS5}_\diamond$ *and* $\mathbf{KD45}$ *defines* $\mathbf{D_2}$.

We can define a discussive consequence on the basis of any modal logic $\boldsymbol{L}$.

Definition 9.5 *For any* $\Pi \subseteq \mathrm{For^d}$ *and* $B \in \mathrm{For^d}$*:* $\Pi \vdash_{\mathbf{D}_{\boldsymbol{L}}} B$ *iff for some* $n \geqslant 0$ *and for some* $A_1, \ldots, A_n \in \Pi$ *we have* $\ulcorner \Diamond A_1^\bullet \to (\ldots \to \Diamond A_n^\bullet \to \Diamond B^\bullet)\ldots)\urcorner \in \boldsymbol{L}$. *In other words,*

$$\Pi \vdash_{\mathbf{D}_{\boldsymbol{L}}} B \qquad \textit{iff} \qquad \{\Diamond A^\bullet : A \in X\} \vdash_{\boldsymbol{L}} \Diamond B^\bullet,$$

where $\vdash_{\boldsymbol{L}}$ *is the pure modus-ponens-style inference relation based on* $\boldsymbol{L}$ *(see Definition 9.A.1 and Fact 9.A.1).*

If $\Pi = \{A_1, \ldots, A_n\}$*, then we will use notation:* $A_1, \ldots, A_n \vdash_{\mathbf{D}_{\boldsymbol{L}}} A$.

By ($\mathrm{R}^{\Diamond\Box}$) and (5!) we obtain

Lemma 9.2. *Let* $\boldsymbol{L}$ *be any normal logic such that* $\mathbf{KD45} \subseteq \boldsymbol{L}$. *Then for any* $A_1, \ldots, A_n, B \in \mathrm{For_m}$:

$$\ulcorner \Diamond(\Diamond A_1 \to (\ldots \to (\Diamond A_n \to B)\ldots))\urcorner \in \boldsymbol{L} \quad \textit{iff} \quad \ulcorner \Diamond A_1 \to (\ldots \to (\Diamond A_n \to \Diamond B)\ldots)\urcorner \in \boldsymbol{L}\,.$$

Corollary 9.3. *For any* $A_1, \ldots, A_n, B \in \mathrm{For^d}$:

$$\ulcorner A_1 \to^{\mathrm{d}} (\ldots \to^{\mathrm{d}} (A_n \to^{\mathrm{d}} B)\ldots)\urcorner \in \mathbf{D_2} \quad \textit{iff} \quad \ulcorner \Diamond(A_1 \to^{\mathrm{d}} (\ldots \to^{\mathrm{d}} (A_n \to^{\mathrm{d}} B)\ldots))^{\bullet}\urcorner \in \mathbf{KD45} \quad \textit{iff} \quad \ulcorner \Diamond A_1^\bullet \to (\ldots \to (\Diamond A_n^\bullet \to \Diamond B^\bullet)\ldots)\urcorner \in \mathbf{KD45}\,.$$

By definitions, Corollaries 9.2 and 9.3, and Fact 9.1 we obtain

Theorem 9.1. $\vdash_{\mathbf{D_2}} = \vdash_{\mathbf{D_{KD45}}}$.

Proof. For any $A_1, \ldots, A_n, B \in \mathrm{For}^{\mathrm{d}}$ we obtain

$$\begin{aligned} A_1, \ldots, A_n \vdash_{\mathbf{D_{KD45}}} B \quad &\text{iff} \quad \ulcorner \Diamond A_1^{\bullet} \to (\ldots \to (\Diamond A_n^{\bullet} \to \Diamond B^{\bullet}) \ldots)\urcorner \in \mathbf{KD45} \\ &\text{iff} \quad \ulcorner \Diamond (A_1 \to^{\mathrm{d}} (\ldots \to^{\mathrm{d}} (A_n \to^{\mathrm{d}} B) \ldots))^{\bullet}\urcorner \in \mathbf{KD45} \\ &\text{iff} \quad \ulcorner \Diamond (A_1 \to^{\mathrm{d}} (\ldots \to^{\mathrm{d}} (A_n \to^{\mathrm{d}} B) \ldots))^{\bullet}\urcorner \in \mathbf{S5} \\ &\text{iff} \quad A_1, \ldots, A_n \vdash_{\mathbf{D_2}} B \end{aligned}$$

Thus, for any $\Pi \subseteq \mathrm{For}^{\mathrm{d}}$ and $B \in \mathrm{For}^{\mathrm{d}}$: $\Pi \vdash_{\mathbf{D_2}} B$ iff $\Pi \vdash_{\mathbf{D_{KD45}}} B$.

In what follows, we prove that **KD45** is the smallest, while **S5** is the largest among normal logics which define the same consequence relation $\vdash_{\mathbf{D_2}}$. But neither $\mathbf{S5^M}$ nor **S4** is appropriate for this purpose.

Fact 9.11 $\vdash_{\mathbf{D_{S5^M}}} \subsetneq \vdash_{\mathbf{D_{S4}}} \subsetneq \vdash_{\mathbf{D_2}}$. □

The inclusions "$\subseteq$" are obvious. For "$\subsetneq$" we can use either the following examples or the next fact.

Example 9.1. (i) $(p \vee \neg p) \wedge^{\mathrm{d}} p \vdash_{\mathbf{D_{S4}}} p$, while $(p \vee \neg p) \wedge^{\mathrm{d}} p \nvdash_{\mathbf{D_{S5^M}}} p$. Indeed, $(p \vee \neg p) \wedge^{\mathrm{d}} p \vdash_{\mathbf{D_{S5^M}}} p$ iff '$\Diamond((p \vee \neg p) \wedge \Diamond p) \to \Diamond p$' belongs to $\mathbf{S5^M}$ iff $(4^{\Diamond}) \in \mathbf{S5^M}$ iff (4) $\in \mathbf{S5^M}$. But (4) $\notin \mathbf{S5^M}$, by Lemma 9.1(ii).

(ii) $p, q \vdash_{\mathbf{D_2}} p \wedge^{\mathrm{d}} q$, while $p, q \nvdash_{\mathbf{D_{S4}}} p \wedge^{\mathrm{d}} q$.

(iii) $(p \vee \neg p) \wedge^{\mathrm{d}} p, q \vdash_{\mathbf{D_2}} p \wedge^{\mathrm{d}} q$, while $(p \vee \neg p) \wedge^{\mathrm{d}} p, q \nvdash_{\mathbf{D_{S4}}} p \wedge^{\mathrm{d}} q$. □

Fact 9.12

(i) *Let* $\boldsymbol{L}$ *be any regular logic such that* $\vdash_{\mathbf{D_2}} \subseteq \vdash_{\mathbf{D}_{\boldsymbol{L}}}$. *Then* $\boldsymbol{L}$ *contains* (D), (4), *and* $\ulcorner\Box\top \to (5)\urcorner$, *so* **CD45(1)** $\subseteq \boldsymbol{L}$.[10]

(ii) *Let* $\boldsymbol{L}$ *be any normal logic such that* $\vdash_{\mathbf{D_2}} \subseteq \vdash_{\mathbf{D}_{\boldsymbol{L}}}$. *Then* $\boldsymbol{L}$ *contains* (D), (4), *and* (5), *so* **KD45** $\subseteq \boldsymbol{L}$.

Proof. (i) For (D): Since $\emptyset \vdash_{\mathbf{D_2}} p \vee \neg p$, so—by the assumption—also $\emptyset \vdash_{\mathbf{D}_{\boldsymbol{L}}} p \vee \neg p$. Hence '$\Diamond(p \vee \neg p)$' $\in \boldsymbol{L}$, by the definition of $\vdash_{\mathbf{D}_{\boldsymbol{L}}}$. By Lemmas 9.A.5 and 9.A.7 we have that (D) $\in \boldsymbol{L}$.

For $\ulcorner\Box\top \to (5)\urcorner$: Since $p \to^{\mathrm{d}} \neg(p \vee \neg p), p \vdash_{\mathbf{D_2}} \neg(p \vee \neg p)$, so—by the assumption—also $p \to^{\mathrm{d}} \neg(p \vee \neg p), p \vdash_{\mathbf{D}_{\boldsymbol{L}}} \neg(p \vee \neg p)$. Therefore, by the definition of $\vdash_{\mathbf{D}_{\boldsymbol{L}}}$, we get that '$\Diamond[\Diamond p \to \neg(p \vee \neg p)] \to [\Diamond p \to \Diamond\neg(p \vee \neg p)]$' belongs to $\boldsymbol{L}$. Thus, by **PL**, '$\neg\Diamond(\Diamond p \to \neg\top) \vee (\Diamond p \to \Diamond\neg\top)$' belongs to $\boldsymbol{L}$. Thus, by ($\mathrm{R}^{\Diamond\Box}$) and **PL**, also '$\neg(\Box\Diamond p \to \Diamond\neg\top) \vee (\Diamond p \to \Diamond\neg\top)$', '$(\Box\Diamond p \wedge \neg\Diamond\neg\top) \vee \neg\Diamond p \vee \Diamond\neg\top$', '$(\Box\Diamond p \vee \neg\Diamond p \vee$

[10] The name '**CD45(1)**' is used in the sense of Segerberg (1971), vol. II. Notice that **CD45** = **KD45**.

$\Diamond\neg\top) \wedge (\neg\Diamond\neg\top \vee \neg\Diamond p \vee \Diamond\neg\top)$', and '$\Box\Diamond p \vee \neg\Diamond p \vee \Diamond\neg\top$' belong to $\boldsymbol{L}$. Thus, '$\Diamond p \wedge \Box\top \to \Box\Diamond p$' and $\ulcorner\Box\top \to (5^{\Diamond})\urcorner$ belong to $\boldsymbol{L}$. Hence, by the standard duality result, $\ulcorner\Box\top \to (5)\urcorner \in \boldsymbol{L}$ as well.

For (4): Since $p \wedge^{\mathrm{d}} q \vdash_{\mathbf{D_2}} q$, so '$\Diamond(p \wedge \Diamond q) \to \Diamond q$' and '$\Diamond(\top \wedge \Diamond q) \to \Diamond q$' belong to $\boldsymbol{L}$. However '$\Diamond\Diamond q \to \Diamond(\top \wedge \Diamond q)$' is a thesis of all regular logics. Thus, by transitivity, we obtain that $(4^{\Diamond}) \in \boldsymbol{L}$; so also $(4) \in \boldsymbol{L}$.

(ii) Since $\boldsymbol{L}$ is normal, so $\boldsymbol{L}$ is regular and $\ulcorner\Box\top\urcorner \in \boldsymbol{L}$.

Let $\mathbf{Cn}_{\Diamond}\mathbf{S5}$ be the set of modal logics which satisfy the following condition: for any logic $\boldsymbol{L}$

$$\boldsymbol{L} \in \mathbf{Cn}_{\Diamond}\mathbf{S5} \overset{\mathrm{df}}{\iff} \text{for any } \Pi \subseteq \mathrm{For_m} \text{ and } B \in \mathrm{For_m},$$

$$\Diamond\,\Pi \vdash_{\boldsymbol{L}} \Diamond B \text{ iff } \Diamond\,\Pi \vdash_{\mathbf{S5}} \Diamond B\,.$$

Let $\mathbf{NCn}_{\Diamond}\mathbf{S5}$ be the set of all normal logics from $\mathbf{Cn}_{\Diamond}\mathbf{S5}$. By definitions, Lemma 9.2, and Corollary 9.2 we obtain

Fact 9.13 **KD45** $\in \mathbf{NCn}_{\Diamond}\mathbf{S5}$.

Lemma 9.3. (5_c) *and* (5) *belong to all logics from* $\mathbf{NCn}_{\Diamond}\mathbf{S5}$. *Thus, every logic from* $\mathbf{NCn}_{\Diamond}\mathbf{S5}$ *includes* **KD45**.

Proof. Firstly, '$\Diamond(\Diamond p \to p)$' and '$(\Diamond p \wedge \Diamond\neg\Diamond p) \to \Diamond\neg\top$' are theses of **S5**; so they are also theses of all logics from $\mathbf{NCn}_{\Diamond}\mathbf{S5}$. Secondly, these formulae are equivalent, respectively, to $(5_c^{\Diamond})$ and $(5^{\Diamond})$, on the basis of any normal modal logic. Thus, $(5_c^{\Diamond})$ and $(5^{\Diamond})$ belong to all logics from $\mathbf{NCn}_{\Diamond}\mathbf{S5}$. So every logic from $\mathbf{NCn}_{\Diamond}\mathbf{S5}$ includes $\mathbf{K55_c}$ (= **KD45**).

By Fact 9.13 and Lemma 9.3 we obtain:

Theorem 9.2. **KD45** *is the smallest element in* $\mathbf{NCn}_{\Diamond}\mathbf{S5}$.

Below we introduce a transformation from $\mathrm{For_m}$ to $\mathrm{For^d}$. It allows us to prove that if any normal logic defines the $\mathbf{D_2}$-consequence, it has to be located between **KD45** and **S5**.

Definition 9.6 *Let* $-^{\circ}$ *be the function from* $\mathrm{For_m}$ *into* $\mathrm{For^d}$ *such that:*

1. $(a)^{\circ} = a$, *for any propositional letter* a,
2. *And for any* $A, B \in \mathrm{For_m}$:

   (a) $(\neg A)^{\circ} = \ulcorner\neg A^{\circ}\urcorner$,
   (b) $(A \vee B)^{\circ} = \ulcorner A^{\circ} \vee B^{\circ}\urcorner$,
   (c) $(A \wedge B)^{\circ} = \ulcorner\neg(\neg A^{\circ} \vee \neg B^{\circ})\urcorner$,
   (d) $(A \to B)^{\circ} = \ulcorner\neg A^{\circ} \vee B^{\circ}\urcorner$,
   (e) $(A \leftrightarrow B)^{\circ} = \ulcorner\neg(\neg(\neg A^{\circ} \vee B^{\circ}) \vee \neg(\neg B^{\circ} \vee A^{\circ}))\urcorner$,
   (f) $(\Diamond A)^{\circ} = \ulcorner(p \vee \neg p) \wedge^{\mathrm{d}} A^{\circ}\urcorner$,
   (g) $(\Box A)^{\circ} = \ulcorner\neg A^{\circ} \to^{\mathrm{d}} \neg(p \vee \neg p)\urcorner$.

Lemma 9.4. *For any* $A \in \mathrm{For_m}$: $\ulcorner A \leftrightarrow A^{\circ\bullet}\urcorner$ *is a thesis of all classical logics.*

Lemma 9.5. *For any classical modal logic* $\boldsymbol{L}$

$$\vdash_{\mathbf{D}_{L}} = \vdash_{\mathbf{D_2}} \quad \textit{iff} \quad \boldsymbol{L} \in \mathsf{Cn}_{\diamond}\mathsf{S5}\,.$$

Proof. "⇒" $\diamond A_1, \ldots, \diamond A_n \vdash_{\boldsymbol{L}} \diamond B$ iff $\ulcorner \diamond A_1 \to (\ldots \to (\diamond A_n \to \diamond B)\ldots)\urcorner \in \boldsymbol{L}$ iff, by Lemma 9.4, **PL**, and (REP), $\ulcorner \diamond A_1^{\circ\bullet} \to (\ldots \to (\diamond A_n^{\circ\bullet} \to \diamond B^{\circ\bullet})\ldots)\urcorner \in \boldsymbol{L}$ iff $A_1^{\circ}, \ldots, A_n^{\circ} \vdash_{\mathbf{D}_L} B^{\circ}$ iff $A_1^{\circ}, \ldots, A_n^{\circ} \vdash_{\mathbf{D_2}} B^{\circ}$ iff $\ulcorner \diamond A_1^{\circ\bullet} \to (\ldots \to (\diamond A_n^{\circ\bullet} \to \diamond B^{\circ\bullet})\ldots)\urcorner \in \mathbf{S5}$ iff, by Lemma 9.4, **PL**, and (REP), $\diamond A_1, \ldots, \diamond A_n \vdash_{\mathbf{S5}} \diamond B$.

"⇐" Obvious.

Finally, we get the following

Theorem 9.3. *For any normal modal logic* $\boldsymbol{L}$:

$$\vdash_{\mathbf{D}_{L}} = \vdash_{\mathbf{D_2}} \quad \textit{iff} \quad \mathbf{KD45} \subseteq \boldsymbol{L} \subseteq \mathbf{S5}\,.$$

Proof. "⇒" For $\mathbf{KD45} \subseteq \boldsymbol{L}$ see Fact 9.12(ii).

For any $A \in \mathrm{For_m}$ we have: $\emptyset \vdash_{\mathbf{D_2}} A^{\circ}$ iff $\emptyset \vdash_{\mathbf{D}_L} A^{\circ}$. So by Definitions 9.1 and 9.5 we have: $\ulcorner \diamond A^{\circ\bullet}\urcorner \in \mathbf{S5}$ iff $\ulcorner \diamond A^{\circ\bullet}\urcorner \in \boldsymbol{L}$. Thus, by Lemma 9.4, **PL**, and (REP), we obtain that: $\ulcorner \diamond A\urcorner \in \mathbf{S5}$ iff $\ulcorner \diamond A\urcorner \in \boldsymbol{L}$. Thus, $\boldsymbol{L} \in \mathsf{NS5}_{\diamond}$. Therefore $\boldsymbol{L} \subseteq \mathbf{S5}$, by Facts 9.3 and 9.10.

"⇐" By Corollary 9.2 and Fact 9.3, $\boldsymbol{L} \in \mathsf{NS5}_{\diamond}$. Thus, $\boldsymbol{L} \in \mathsf{NCn}_{\diamond}\mathsf{S5}$, by Lemma 9.2. Hence $\vdash_{\mathbf{D_2}} = \vdash_{\mathbf{D}_L}$, by Lemma 9.5.

9.5 The Smallest Regular Modal Logic Defining D_2-Consequence

We will show that consequence relation $\vdash_{\mathbf{D_2}}$ is also closely connected with the regular logic **CD45(1)**.

Definition 9.7 *Let* **CD45(1)** *be the smallest regular logic which contains* (D), (4), *and* (5(1)), *i.e.* $\ulcorner \Box\top \to (5)\urcorner$.

Remark 9.2. In the notation of Segerberg a regular logic $\mathbf{CN^1D(1)4(1)5(1)}$ corresponds, by the definition, to the normal logic **KD45**. Yet in **C2** the formulae (D), (4) and (5_c) are respectively equivalent to (D(1)), (4(1)) and (5_c(1)), i.e., $\ulcorner \Box\top \to (\Box p \to \diamond p)\urcorner$, $\ulcorner \Box\top \to (\Box p \to \Box\Box p)\urcorner$ and $\ulcorner \Box\top \to (\Box p \to \diamond\Box p)\urcorner$ (see Segerberg 1971, p. 208). Moreover, the formula (N^1), i.e. $\ulcorner \Box\top \to \Box\Box\top\urcorner$ (see Segerberg 1971, p. 198), is an instance of (4). Thus, $\mathbf{CD45(1)} = \mathbf{CN^1D(1)4(1)5(1)}$. Hence, by Lemma 9.A.9, i.e. Corollary 2.4 from Segerberg (1971), vol. II, we obtain:

$$\mathbf{CD45(1)} = \mathbf{CF^1} \cap \mathbf{KD45}\,,$$

$$\mathbf{CN^1 5_c 5(1)} = \mathbf{CF^1} \cap \mathbf{K55_c}\,,$$

where $\mathbf{CF^1}$ is the *falsum* logic. □

By the above remark and the equality $\mathbf{KD45} = \mathbf{K55_c}$ we obtain[11]:

Fact 9.14 $\mathbf{CD45(1)} = \mathbf{CN^15_c5(1)}$.

Fact 9.15 *The logic* **CD45(1)** *is not normal. In other words,* **CD45(1)** *has no thesis of the form* $\ulcorner\Box B\urcorner$.

Proof. It is enough to use a model from Fact 3.1 of Nasieniewski and Pietruszczak (2008): Let v be a valuation from $\mathrm{For_m}$ into $\{0,1\}$ which preserves classical truth conditions for classical connectives and let $v(\Box A) = 0$ and $v(\Diamond A) = 1$, for any $A \in \mathrm{For_m}$. Notice that for any thesis of **CD45(1)** we have $v(A) = 1$, while, for example, $v(\Box\top) = 0$.

Fact 9.16 $\mathbf{rS5^M} \subsetneq \mathbf{CD4} \subsetneq \mathbf{CD45(1)} \subsetneq \mathbf{KD45} \subsetneq \mathbf{S5}$.

Proof. Notice that, by Lemma 9.9, $\mathbf{rS5^M} = \mathbf{C4_s5_c}$. Moreover, $(5^{\Diamond}_{c})$, $(4_s) \in \mathbf{CD4} = \mathbf{C45_c}$, respectively by Lemmas 9.A.8(ii) and 9.1(i). Thus, $\mathbf{rS5^M} \subseteq \mathbf{CD4}$. This inclusion is proper, since $\mathbf{rS5^M} \subsetneq \mathbf{S5^M} \subsetneq \mathbf{KD4}$ and $(4) \notin \mathbf{S5^M}$ (see Lemma 9.1).

Besides, we have $\mathbf{CD4} \subseteq \mathbf{KD4}$. But $(5) \notin \mathbf{KD4}$, so also $(5(1)) \notin \mathbf{KD4}$, since in all normal logics we have the thesis '$(5) \leftrightarrow (5(1))$'. Hence $(5(1)) \notin \mathbf{CD4}$. Moreover, $\mathbf{CD45(1)} \subseteq \mathbf{KD45}$. This inclusion is proper by Fact 9.15.

Lemma 9.6. *The formulae* (†) *and for any* $n \geqslant 2$

$$\Diamond p_1 \to (\Diamond p_2 \to \dots (\Diamond p_n \to (\Diamond(p_1 \wedge (\Diamond p_2 \wedge \dots (\Diamond p_{n-1} \wedge \Diamond p_n)\dots)))))$$

and for any $n \geqslant 1$

$$\Diamond(\Diamond p_1 \to (\Diamond p_2 \to \dots (\Diamond p_n \to q)\dots)) \to \\ \to (\Diamond p_1 \to (\Diamond p_2 \to \dots (\Diamond p_n \to \Diamond q)\dots))$$

are theses of $\mathbf{CN^15(1)} \subseteq \mathbf{CD45(1)}$.

Proof. By Lemma 9.A.8(vi), $(\dagger) \in \mathbf{K5}$. Obviously $(\dagger) \in \mathbf{CF^1}$. So, we use Lemma 9.A.9. The proof in the case of remaining formulae is analogous. It is by induction on n.

Let $\mathbf{RCn_{\Diamond}S5}$ be the set of all regular logics from $\mathbf{Cn_{\Diamond}S5}$. We have:

Lemma 9.7. $\mathbf{CD45(1)} \in \mathbf{RCn_{\Diamond}S5}$.

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[11]We have also a proof of the following fact without the use of Lemma 9.A.9. Firstly, by Lemma 9.A.8(ii), $(5_c) \in \mathbf{CD4}$; so $\mathbf{CN^15_c5(1)} \subseteq \mathbf{CD45(1)}$.

Secondly, $5^{\Diamond}(1)$ belongs to $\mathbf{C5_c5(1)}$, so by (US) we have: '$\Box\top \to (\Diamond\Box p \to \Box\Diamond\Box p)$'. Moreover, by $5(1)$, (RM), (K) and **PL**, we obtain: '$\Box\Box\top \to (\Box\Diamond\Box p \to \Box\Box p)$'. So, by **PL**, we receive: '$(\Box\Box\top \wedge \Box\top) \to (\Diamond\Box p \to \Box\Box p)$'. Hence, by (5_c) and **PL**, we get '$(\Box\Box\top \wedge \Box\top) \to (\Box p \to \Box\Box p)$'. Hence, by (N^1), **PL** and (RM), we have that $(4) \in \mathbf{CN^15_c5(1)}$. Thus, $\mathbf{CD45(1)} \subseteq \mathbf{CN^15_c5(1)}$, since by Lemma 9.A.8(i), $(\mathrm{D}) \in \mathbf{C5_c}$.

Proof. For any $A_1, \ldots, A_n, B \in \mathrm{For_m}$ by Lemma 9.2 and Fact 9.16, and Fact 9.8(iii): $\ulcorner \Diamond A_1 \to (\ldots \to (\Diamond A_n \to \Diamond B)\ldots)\urcorner \in \mathbf{S5}$ iff $\ulcorner \Diamond(\Diamond A_1 \to (\ldots \to (\Diamond A_n \to B)\ldots))\urcorner \in \mathbf{S5}$ iff $\ulcorner \Diamond(\Diamond A_1 \to (\ldots \to (\Diamond A_n \to B)\ldots))\urcorner \in \mathbf{CD45(1)}$.

By Lemma 9.6, it follows from the last statement that $\ulcorner \Diamond A_1 \to (\ldots \to (\Diamond A_n \to \Diamond B)\ldots)\urcorner \in \mathbf{CD45(1)}$. The reverse implication is obvious.

By the above lemma we have directly:

Corollary 9.4. $\vdash_{\mathbf{D_2}} = \vdash_{\mathbf{D_{CD45(1)}}}$.

By Fact 9.12(i) and Definition 9.7 we obtain:

Lemma 9.8. *For any regular logic* $\boldsymbol{L}$ *such that* $\vdash_{\mathbf{D_2}} = \vdash_{\mathbf{D}_{\boldsymbol{L}}}$ *it is the case that* $\mathbf{CD45(1)} \subseteq \boldsymbol{L}$.

By Lemmas 9.7, 9.5, and 9.8 we conclude that

Corollary 9.5. $\mathbf{CD45(1)}$ *is the smallest element in* $\mathsf{RCn}_\Diamond\mathsf{S5}$.

We have of course also a regular version of Theorem 9.3:

Lemma 9.9. $\mathbf{S5}$ *is the biggest element in* $\mathsf{RCn}_\Diamond\mathsf{S5}$.

Proof. Let us assume that $\boldsymbol{L} \in \mathsf{RCn}_\Diamond\mathsf{S5}$ and $A \in \boldsymbol{L}$. By the classical logic we have $(p \vee \neg p) \to A \in \boldsymbol{L}$ and by monotonicity $\Diamond\Box(p \vee \neg p) \to \Diamond\Box A \in \boldsymbol{L}$ i.e, $\Diamond\Box(p \vee \neg p) \vdash_{\boldsymbol{L}} \Diamond\Box A$. Thus, by the assumption $\Diamond\Box(p \vee \neg p) \to \Diamond\Box A \in \mathbf{S5}$ and by (MP) we obtain that $\Diamond\Box A \in \mathbf{S5}$, so using the standard reduction of modalities we obtain that $A \in \mathbf{S5}$.

We have a lemma that is analogous to Lemma 9.8:

Lemma 9.10. *For any regular logic* $\boldsymbol{L}$ *such that* $\vdash_{\mathbf{D_2}} = \vdash_{\mathbf{D}_{\boldsymbol{L}}}$ *it is the case that* $\boldsymbol{L} \subseteq \mathbf{S5}$.

Proof. Assume that $A \in \boldsymbol{L}$. By Lemma 9.4 we have also $A^{\circ\bullet} \in \boldsymbol{L}$.

Since $\Diamond(\Diamond\neg(p \vee \neg p) \to \neg(p \vee \neg p)) \in \mathbf{S5}$ thus, $\neg(p \vee \neg p) \to^{\mathrm{d}} \neg(p \vee \neg p) \in \mathbf{D_2}$ and by the assumption also $\neg(p \vee \neg p) \to^{\mathrm{d}} \neg(p \vee \neg p) \in \mathbf{D}_{\boldsymbol{L}}$. By the definition of $\mathbf{D}_{\boldsymbol{L}}$ it means that $\Diamond(\Diamond\neg(p \vee \neg p) \to \neg(p \vee \neg p)) \in \boldsymbol{L}$. But for every regular modal logic the last statement is equivalent to: $\Diamond\Box(p \vee \neg p) \in \boldsymbol{L}$. It follows from Lemma 9.A.6 that $\Diamond\Box A^{\circ\bullet} \in \boldsymbol{L}$. But again for every regular modal logic this condition is equivalent to $\Diamond(\Diamond\neg A^{\circ\bullet} \to \neg(p \vee \neg p)) \in \boldsymbol{L}$, which means that $\neg A^{\circ} \to^{\mathrm{d}} \neg(p \vee \neg p) \in \mathbf{D}_{\boldsymbol{L}}$, so $\neg A^{\circ} \to^{\mathrm{d}} \neg(p \vee \neg p) \in \mathbf{D_2}$. Therefore, $\Diamond(\neg A^{\circ} \to^{\mathrm{d}} \neg(p \vee \neg p))^{\bullet} \in \mathbf{S5}$, equivalently $\Diamond\Box A^{\circ\bullet} \in \mathbf{S5}$. From this follows that $A^{\circ\bullet} \in \mathbf{S5}$ while by Lemma 9.4 we conclude that $A \in \mathbf{S5}$.

So taking together Lemmas 9.8 and 9.10 we receive:

Corollary 9.6. *For any regular logic* $\boldsymbol{L}$ *such that* $\vdash_{\mathbf{D}_{\boldsymbol{L}}} = \vdash_{\mathbf{D_2}}$ *we have* $\mathbf{CD45(1)} \subseteq \boldsymbol{L} \subseteq \mathbf{S5}$.

Lemma 9.11. *For any regular logic* $\boldsymbol{L}$ *such that* **CD45(1)** $\subseteq \boldsymbol{L} \subseteq$ **S5** *we have* $\boldsymbol{L} \in \mathsf{RCn}_{\diamond}\mathbf{S5}$.

Proof. Assume that **CD45(1)** $\subseteq \boldsymbol{L} \subseteq$ **S5**. We have to prove that for any $A_1, \ldots, A_n$, $B \in \mathrm{For_m}$: $\ulcorner\Diamond(\Diamond A_1 \to (\ldots \to (\Diamond A_n \to B)\ldots))\urcorner \in \mathbf{S5}$ iff $\ulcorner\Diamond(\Diamond A_1 \to (\ldots \to (\Diamond A_n \to B)\ldots))\urcorner \in \boldsymbol{L}$. Left-to-right implication follows from Lemma 9.7. The reverse implication is obvious.

From this lemma and Lemma 9.5 we obtain

Theorem 9.4. *For any regular logic* $\boldsymbol{L}$ *such that* **CD45(1)** $\subseteq \boldsymbol{L} \subseteq$ **S5** *we have* $\vdash_{\mathbf{D}_{\boldsymbol{L}}} = \vdash_{\mathbf{D_2}}$.

Finally, directly from Corollary 9.6 and Lemma 9.11 we get the following

Theorem 9.5. *For any regular modal logic* $\boldsymbol{L}$

$$\vdash_{\mathbf{D}_{\boldsymbol{L}}} = \vdash_{\mathbf{D_2}} \quad \textit{iff} \quad \mathbf{CD45(1)} \subseteq \boldsymbol{L} \subseteq \mathbf{S5}\,.$$

Appendix: Some Facts from Modal Logic

As in Chellas (1980) modal formulae are formed in a relational way from propositional letters: 'p', 'q', 'p_0', 'p_1', 'p_2', …; truth-value operators: '$\neg$', '$\vee$', '$\wedge$', '$\to$', and '$\leftrightarrow$' (connectives of negation, disjunction, conjunction, material implication, and material equivalence, respectively); modal operators: the necessity sign '$\Box$' and the possibility sign '$\Diamond$'; and brackets. Let $\mathrm{For_m}$ be the set of modal formulae, and—as in Chellas (1980)—let **PL** be the set of modal formulae which are instances of classical tautologies. Let $\top :=$ '$p \to p$'.

As in Bull and Segerberg (1984) and Chellas and Segerberg (1996), a set $\boldsymbol{L}$ of modal formulae is a (*modal*) *logic* iff

- **PL** $\subseteq \boldsymbol{L}$,
- For any $C, A \in \mathrm{For_m}$: $\boldsymbol{L}$ contains the following formula

$$C[{}^{\neg\Box\neg A}/_{\Diamond A}] \leftrightarrow C\,, \qquad (\mathtt{rep}^{\Box})$$

where $C[{}^{A}/_{B}]$ is any formula that results from C by replacing one or more occurrences of A, in C, by B, i.e. using $(\mathtt{rep}^{\Box})$ we are *replacing* in C one or more occurrences of '$\neg\Box\neg$' by '$\Diamond$'.[12]

[12]In Bull and Segerberg (1984) and Chellas and Segerberg (1996) the symbol '$\Diamond$' is only an abbreviation of '$\neg\Box\neg$'. In the present paper '$\Diamond$' is a primary symbol, thus, we have to admit an axiom of the form $(\mathtt{rep}^{\Box})$. Theses of this form are equivalent to the usage of '$\Diamond$' as the abbreviation of '$\neg\Box\neg$'.

- $\boldsymbol{L}$ is closed under the following three rules: *modus ponens* for '→':

$$\text{if } A \text{ and } \ulcorner A \to B \urcorner \text{ are members of } \boldsymbol{L}, \text{ so is } B. \tag{MP}$$

uniform substitution:

$$\text{if } A \in \boldsymbol{L} \text{ then } \mathrm{s}\,A \in \boldsymbol{L}, \tag{US}$$

where $\mathrm{s}\,A$ is the result of uniform substitution of formulae for propositional letters in A.

Definition 9.A.1 *Let $\boldsymbol{L}$ be any modal logic. We define the consequence $\vdash_{\boldsymbol{L}}$ as follows. For any $\Pi \subseteq \mathrm{For_m}$ and $B \in \mathrm{For_m}$: $\Pi \vdash_{\boldsymbol{L}} B$ iff for some $n \geqslant 0$ and for some $A_1, \ldots, A_n \in \Pi$ we have $\ulcorner A_1 \to (\ldots \to (A_n \to B)\ldots)\urcorner \in \boldsymbol{L}$.*

Notice that $\Pi \vdash_{\boldsymbol{L}} B$ iff there is a derivation of B from $\boldsymbol{L} \cup \Pi$ with the help of *modus ponens* for '→' as the only rule of inference, i.e., $\vdash_{\boldsymbol{L}}$ is the pure *modus-ponens*-style inference relation based on $\boldsymbol{L}$.

Fact 9.A.1 Lemmon (1977). *$\Pi \vdash_{\boldsymbol{L}} B$ iff there exists a sequence $A_1, \ldots, A_n = B$ in which for any $i \leqslant n$, either $A_i \in \Pi$, or $A_i \in \boldsymbol{L}$, or there are $j, k < i$ such that $A_k = \ulcorner A_j \to A_i \urcorner$.*

All members of the set $\boldsymbol{L}$ are called *theses* of the logic $\boldsymbol{L}$. By ($\mathtt{rep}^{\Box}$), every modal logic has the following thesis:

$$\Diamond p \leftrightarrow \neg \Box \neg p. \tag{df $\Diamond$}$$

A modal logic $\boldsymbol{L}$ is *classical* (*congruent*) iff $\boldsymbol{L}$ is closed under the following rule for any $A, B \in \mathrm{For_m}$:

$$\text{if } \ulcorner A \leftrightarrow B \urcorner \in \boldsymbol{L} \text{ then } \ulcorner \Box A \leftrightarrow \Box B \urcorner \in \boldsymbol{L}. \tag{RE}$$

Every classical logic $\boldsymbol{L}$ is closed under the rule of replacement, i.e. for any $A, B, C \in \mathrm{For_m}$:

$$\text{if } \ulcorner A \leftrightarrow B \urcorner \in \boldsymbol{L} \text{ then } \ulcorner C \leftrightarrow C[{}^{A}/_{B}] \urcorner \in \boldsymbol{L}. \tag{REP}$$

It is known (cf. e.g. Chellas 1980) that while defining classical logics one uses ($\mathtt{df}\,\Diamond$) instead of ($\mathtt{rep}^{\Box}$), i.e. treats them (logics) as subsets of $\mathrm{For_m}$ which include **PL** and ($\mathtt{df}\,\Diamond$) and which are closed under rules (MP), (US) and (RE). We also have an analogous situation in the case of monotonic, regular, and normal modal logics defined further.

Every classical modal logic has the following thesis

$$\Box p \leftrightarrow \neg \Diamond \neg p \tag{df $\Box$}$$

Lemma 9.A.1 *A classical modal logic contains, respectively, the following formulae*

$$\Box(p \to q) \to (\Box p \to \Box q) \tag{K}$$

$$\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q) \tag{R}$$

$$\Box p \to p \tag{T}$$

$$\Box p \to \Box\Box p \tag{4}$$

$$\Diamond\Box p \to \Box p \tag{5}$$

$$\Box p \to \Diamond\Box p \tag{5_c}$$

$$\Box p \leftrightarrow \Diamond\Box p \tag{5!}$$

if and only if it contains, respectively, their dual versions

$$\Box(p \to q) \to (\Diamond p \to \Diamond q) \tag{$K^\Diamond$}$$

$$\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q) \tag{$R^\Diamond$}$$

$$p \to \Diamond p \tag{$T^\Diamond$}$$

$$\Diamond \Diamond p \to \Diamond p \tag{$4^\Diamond$}$$

$$\Diamond p \to \Box \Diamond p \tag{$5^\Diamond$}$$

$$\Box \Diamond p \to \Diamond p \tag{$5_c^\Diamond$}$$

$$\Diamond p \leftrightarrow \Box \Diamond p \tag{$5^\Diamond!$}$$

Lemma 9.A.2 *For any classical modal logic* $\boldsymbol{L}$ *the following conditions are equivalent*:

(a) *For any* $\tau \in \mathbf{PL}$, $\ulcorner\Box\tau\urcorner \in \boldsymbol{L}$ (*resp.* $\ulcorner\Diamond\tau\urcorner \in \boldsymbol{L}$, $\ulcorner\Diamond\Box\tau\urcorner \in \boldsymbol{L}$).
(b) $\ulcorner\Box\top\urcorner \in \boldsymbol{L}$ (*resp.* $\ulcorner\Diamond\top\urcorner \in \boldsymbol{L}$, $\ulcorner\Diamond\Box\top\urcorner \in \boldsymbol{L}$).

Lemma 9.A.3 *Let* $\boldsymbol{L}$ *be any classical modal logic such that*

(a) *either* $\ulcorner\Box\top\urcorner \in \boldsymbol{L}$,
(b) *or* (5), $\ulcorner\Diamond B\urcorner \in \boldsymbol{L}$, *for some* $B \in \mathrm{For_m}$.[13]

Then $\boldsymbol{L}$ *is closed under the rule of necessitation*:

$$\textit{if } A \in \boldsymbol{L} \textit{ then } \ulcorner\Box A\urcorner \in \boldsymbol{L}. \tag{RN}$$

[13] Notice that (b) implies (a).

Lemma 9.A.4 *Chellas (1980). Let* $\boldsymbol{L}$ *be any classical modal logic such that* (T), (5) $\in \boldsymbol{L}$. *Then* $\boldsymbol{L}$ *has as its theses* $\ulcorner\Box\top\urcorner$, $\ulcorner\Diamond\top\urcorner$, $\ulcorner\Diamond\Box\top\urcorner$, (4)*, and*

$$\Box p \to \Diamond p \tag{D}$$

and $\boldsymbol{L}$ *is closed under* (RN) *and the following rules*:

$$\textit{if } A \in \boldsymbol{L}, \textit{ then } \ulcorner\Diamond A\urcorner \in \boldsymbol{L}, \tag{RP}$$

$$\textit{if } A \in \boldsymbol{L}, \textit{ then } \ulcorner\Diamond\Box A\urcorner \in \boldsymbol{L}. \tag{RPN}$$

A modal logic $\boldsymbol{L}$ is *monotonic* iff $\boldsymbol{L}$ is closed under the monotonicity rule, i.e. for any $A, B \in \mathrm{For_m}$:

$$\text{if } \ulcorner A \to B\urcorner \in \boldsymbol{L}, \text{ then } \ulcorner\Box A \to \Box B\urcorner \in \boldsymbol{L}, \tag{RM}$$

Every monotonic logic $\boldsymbol{L}$ is classical and it is closed under the dual form of (RM), i.e. for any $A, B \in \mathrm{For_m}$:

$$\text{if } \ulcorner A \to B\urcorner \in \boldsymbol{L}, \text{ then } \ulcorner\Diamond A \to \Diamond B\urcorner \in \boldsymbol{L}. \tag{RM$^\Diamond$}$$

Lemma 9.A.5 *For any monotonic logic* $\boldsymbol{L}$ *the following conditions are equivalent*:

(a) *For any* $\tau \in \mathbf{PL}$, $\ulcorner\Box\tau\urcorner \in \boldsymbol{L}$ (resp. $\ulcorner\Diamond\tau\urcorner \in \boldsymbol{L}$, $\ulcorner\Diamond\Box\tau\urcorner \in \boldsymbol{L}$).
(b) $\ulcorner\Box\top\urcorner \in \boldsymbol{L}$ (*resp.* $\ulcorner\Diamond\top\urcorner \in \boldsymbol{L}$, $\ulcorner\Diamond\Box\top\urcorner \in \boldsymbol{L}$).
(c) For some $B \in \mathrm{For_m}$, $\ulcorner\Box B\urcorner \in \boldsymbol{L}$ (*resp.* $\ulcorner\Diamond B\urcorner \in \boldsymbol{L}$, $\ulcorner\Diamond\Box B\urcorner \in \boldsymbol{L}$).

Lemma 9.A.6 *Let a monotonic logic* $\boldsymbol{L}$ *has a thesis of the form* $\ulcorner\Box B\urcorner$ (*resp.* $\ulcorner\Diamond B\urcorner$, $\ulcorner\Diamond\Box B\urcorner$). *Then* $\boldsymbol{L}$ *is closed under the rule* (RN) (*resp.* (RP), (RPN)).

A modal logic $\boldsymbol{L}$ is *regular* iff $\boldsymbol{L}$ is monotonic and (K) $\in \boldsymbol{L}$. A logic $\boldsymbol{L}$ is regular iff $\boldsymbol{L}$ is closed under the *regularity rule*, i.e. for any $A, B, C \in \mathrm{For_m}$:

$$\text{if } \ulcorner A \land B \to C\urcorner \in \boldsymbol{L} \text{ then } \ulcorner\Box A \land \Box B \to \Box C\urcorner \in \boldsymbol{L}. \tag{RR}$$

Every regular modal logic has the following theses: (K$^\Diamond$), (R), (R$^\Diamond$) and

$$\Diamond (p \to q) \leftrightarrow (\Box p \to \Diamond q) \tag{R$^{\Diamond\Box}$}$$

By (R$^{\Diamond\Box}$) we obtain.

Lemma 9.A.7 *For any regular logic* $\boldsymbol{L}$: $\ulcorner\Diamond\top\urcorner \in \boldsymbol{L}$ *iff* (D) $\in \boldsymbol{L}$.

A modal logic is *normal* iff it contains (K) and is closed under (RN) iff it is regular and contains $\ulcorner\Box\top\urcorner$.

Let **K** (resp. **C2**) be the smallest normal (resp. regular) modal logic. Using names of formulae from Lemma 9.A.1, to simplify naming normal (resp. regular) logics we write the *Lemmon code* $\mathbf{KX}_1 \ldots \mathbf{X}_n$ (resp. $\mathbf{CX}_1 \ldots \mathbf{X}_n$) to denote the smallest normal (resp. regular) logic containing formulae ($\mathtt{X}_1$), ..., ($\mathtt{X}_n$) (see Bull and Segerberg 1984; Chellas 1980; Lemmon 1977). We standardly put $\mathbf{T} := \mathbf{KT}$, $\mathbf{S4} := \mathbf{KT4}$ and $\mathbf{S5} := \mathbf{KT5}$. As it is known, $\mathbf{T} \subsetneq \mathbf{S4} \subsetneq \mathbf{S5}$, $\mathbf{KD45} \subsetneq \mathbf{S5}$, $\mathbf{KD45} \not\subseteq \mathbf{S4}$ and $\mathbf{T} \not\subseteq \mathbf{KD45}$.

Lemma 9.A.8

(i) ($\mathtt{D}$) $\in \mathbf{C5_c} \subseteq \mathbf{K5_c}$; ($\mathtt{D}$) $\in \mathbf{KT}$.
(ii) ($\mathtt{5_c}$) $\in \mathbf{CD4} \subseteq \mathbf{KD4}$.
(iii) $\mathbf{KD4} = \mathbf{K45_c}$ and $\mathbf{CD4} = \mathbf{C45_c}$.
(iv) ($\mathtt{4}$) $\in \mathbf{K5!}$.
(v) $\mathbf{KD45} = \mathbf{K5!} = \mathbf{K55_c}$.
(vi) *In* $\mathbf{K}$ *the formula* ($\mathtt{5}$) *is equivalent to the following formula*

$$(\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge \Diamond q) \tag{†}$$

Proof. (i) '$\Diamond(p \to \Box p)$' belongs to $\mathbf{C5_c}$, by ($\mathtt{R}^{\Diamond\Box}$). So, we use Lemma 9.A.7.
(ii) By ($\mathtt{4}$), (US), ($\mathtt{D}$) and **PL** we obtain that ($\mathtt{5_c}$) $\in \mathbf{CD4}$.
(iii) By (i) and (ii).
For (iv) see Exercise 4.46 in Chellas (1980).
(v) By (i), (ii) and (iv).
For (vi) see Exercise 4.37 in Chellas (1980).

Notice that from Lemmas 9.A.3, 9.A.4, and 9.A.7 we obtain:

Corollary 9.A.1 $\mathbf{CD5} = \mathbf{KD5}$, $\mathbf{CD45} = \mathbf{KD45}$ *and* $\mathbf{CT5} = \mathbf{KT5} := \mathbf{S5}$.

Thus, while defining strictly regular logics one uses some additional formulae. We adopt a convention from Segerberg (1971), p. 206. For the formula ($\mathtt{X}$) and any $i \geqslant 0$ we put ($\mathtt{X(i)}$) := $\ulcorner \Box^i \top \to (\mathtt{X}) \urcorner$.

Lemma 9.A.9 *Segerberg (1971), vol. II, Corollary 2.4. For any* $i > 0$:

$$\mathbf{CN^iX}_1\mathbf{(i)} \ldots \mathbf{X}_n\mathbf{(i)} = \mathbf{CF^i} \cap \mathbf{KX}_1 \ldots \mathbf{X}_n\,,$$

where

$$\Box^i \top \to \Box^{i+1} \top \tag{$\mathtt{N^i}$}$$

$$\Diamond^i \neg \top \tag{$\mathtt{F^i}$}$$

Of course, in any modal logic $\mathtt{N}^0$ is equivalent to $\ulcorner \Box \top \urcorner$; so $\mathbf{CN^0} = \mathbf{K}$.

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