Chapter 7
New Arguments for Adaptive Logics as Unifying Frame for the Defeasible Handling of Inconsistency

Diderik Batens

7.1 Introduction

A variety of formats is used to present defeasible logics. More often than not, the format is typical for the logic and derives from the accidental way in which the logic was discovered. Not only the object level description, but also the proof techniques needed for metatheorems vary with those formats. Unifying this domain seems highly useful if not necessary.

As soon as a standard format for adaptive logics was devised,[1] it seemed to offer an attractive means for unification. Today nearly all (first order) defeasible logics have been characterised by adaptive logics. Moreover, the unification is a strong one. If an adaptive logic is in standard format, the format itself defines the logic's proof theory and semantics. Moreover, most of the metatheory has been proved in terms of the standard logic alone. This includes soundness and completeness and a host of properties.

The standard format of adaptive logics may still prove not to be the right unifying frame. New defeasible logics may be discovered and may require that the format is modified or replaced. Or another format may turn out superior in the end. Nevertheless, especially in terms of the new arguments presented below, it is certainly worthwhile to continue the unification in terms of the standard format.

In the present paper, four new arguments are presented in favour of characterizing defeasible reasoning forms by adaptive logics in standard format. The arguments are diverse in nature, but all point in the same direction.

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[1]The first steps were taken in Batens (2001), but later the matter was refined. The best published formulation appears in Batens (2007). The most reliable reference on adaptive logics is Batens (201+), of which the central chapters are available on the web.

D. Batens (✉)
Centre for Logic and Philosophy of Science, Ghent University, Ghent, Belgium
e-mail: diderik.batens@ugent.be

K. Tanaka et al. (eds.), *Paraconsistency: Logic and Applications*, Logic, Epistemology, and the Unity of Science 26, DOI 10.1007/978-94-007-4438-7_7,

For the first two arguments, more technical papers are in preparation. This is why I shall consider them briefly, pointing out the results and commenting on their significance while referring, for technical matters, to the forthcoming papers. The third and fourth argument are presented a bit more at length.

7.2 Preliminaries

In order to make the paper minimally self contained, I shall first briefly summarise the standard format of adaptive logics. First, however, I need to introduce some logics.

Where **CL** is classical logic, let **CLuN** be the full positive fragment of **CL** together with the axiom $A \vee \neg A$.[2] **CLuN** is just like **CL** except that it allows for gluts with respect to negation (whence its name). So it is a paraconsistent logic and actually (with respect to **CL**) the most basic paraconsistent logic that is not also paracomplete. **CLuNs**, studied at length in Batens and De Clercq (2004), is the paraconsistent logic obtained by extending **CLuN** with double negation (in both directions) De Morgan axioms, axioms expressing the standard classical behaviour of negations of implications, negations of equivalences, and negations of the quantifiers, and Replacement of Identicals—its name refers to Schütte who first described its propositional fragment in Schütte (1960). **LP** is a fragment of **CLuNs**: all logical symbols have the same meaning as in **CLuNs** except for implication and equivalence, which are explicitly defined by $A \supset B =_{df} \neg A \vee B$ and $A \equiv B =_{df} (A \wedge B) \vee (\neg A \wedge \neg B)$ and hence are not detachable.

The sequel of this section may be skipped by people familiar with adaptive logics. An adaptive logic **AL** is defined by a triple:

1. A *lower limit logic* **LLL**: a reflexive, transitive, monotonic, and compact logic for which there is a positive test.
2. A *set of abnormalities* Ω: a set of **LLL**-contingent formulas, characterised by a (possibly restricted) logical form F which contains at least one logical symbol.
3. An *adaptive strategy*: Reliability, Minimal Abnormality, ...

The lower limit logic is the stable part of the adaptive logic; anything that follows from the premises by **LLL** will never be revoked. For technical reasons, all classical symbols are added to the lower limit logic, whence this extends **CL**. In the present context, this means that classical negation, $\check{\neg}$, is added next to the standard negation, $\neg$, which is paraconsistent. In standard applications, $\check{\neg}$ does not occur in the premises or in the conclusion. Its function is technical and metatheoretical. Abnormalities are supposed to be false "unless and until proven otherwise". Strategies are ways to handle derivable disjunctionsof abnormalities:

[2] Replacement of Identicals is not derivable in **CLuN** but can be added.

an adaptive strategy picks one specific way to interpret the premises as normally as possible. To keep the discussion with bounds, I shall only consider the Minimal Abnormality strategy—see below—in the present paper.

From now on, I shall take "adaptive logic" to mean adaptive logic in standard format. Inconsistency-adaptive logics are adaptive logics the lower limit of which has a paraconsistent standard negation.

Let us review some examples of inconsistency-adaptive logics. $\mathbf{CLuN}^m$ has **CLuN** as its lower limit logic, $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathscr{F}\}$, and Minimal Abnormality as its strategy—$\mathscr{F}$ is the set of open and closed formulas and $\exists(A \wedge \neg A)$ is the existential closure of $A \wedge \neg A$. $\mathbf{CLuNs}^m$ is similar, except that **CLuNs** is its lower limit and its set of abnormalities is $\Omega^a = \{\exists(A \wedge \neg A) \mid A \in \mathscr{F}^a\}$, in which $\mathscr{F}^a$ is the set of open and closed *primitive* formulas (those that contain no logical symbol except possibly for identity). $\mathbf{LP}^m$ is exactly like $\mathbf{CLuNs}^m$ except that **LP** is its lower limit.

If the lower limit logic is extended with an axiom by which all abnormalities entail triviality, one obtains the *upper limit logic* **ULL**. The upper limit logic of $\mathbf{CLuN}^m$, of $\mathbf{CLuNs}^m$, and of $\mathbf{LP}^m$ is **CL**. If a premise set Γ does not require that any abnormalities are true, the **AL**-consequences of Γ are identical to its **ULL**-consequences. In the opposite case, the **AL**-consequence set of Γ will in general be a superset of its **LLL**-consequences.

In the expression $Dab(\Delta)$, Δ is a finite subset of Ω and $Dab(\Delta)$ denotes the *classical* disjunction of the members of Δ. $Dab(\Delta)$ is called a *Dab-formula*. $Dab(\Delta)$ is a *minimal Dab-consequence of* Γ iff $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ whereas $\Gamma \nvdash_{\mathbf{LLL}} Dab(\Delta')$ for all $\Delta' \subset \Delta$. Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal *Dab*-consequences of Γ, $\Phi(\Gamma)$ comprises the minimal choice sets of $\{\Delta_1, \Delta_2, \ldots\}$. Where M is a **LLL**-model, $Ab(M)$ is the set of abnormalities verified by M.

Definition 7.1. A **LLL**-model M of Γ is *minimally abnormal* iff there is no **LLL**-model M' of Γ such that $Ab(M') \subset Ab(M)$.

Definition 7.2. $\Gamma \vDash_{\mathbf{AL}^m} A$ iff A is verified by all minimally abnormal models of Γ.

It was proved in Batens (2007) that a **LLL**-model M of Γ is *minimally abnormal* iff $Ab(M) \in \Phi(\Gamma)$.

Adaptive logics have also a dynamic proof theory, which is defined by rules of inference and by a marking definition. An annotated **AL**-proof consists of lines that have four elements: a line number, a formula, a justification and a condition. Where

$$A \qquad \Delta$$

abbreviates that A occurs in the proof as the formula of a line that has Δ as its condition, the (generic) inference rules are:

PREM If $A \in \Gamma$:

$$\frac{\ldots \quad \ldots}{A \quad \emptyset}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$:

$$\frac{\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \end{array}}{\begin{array}{ll} B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Theta)$

$$\frac{\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \end{array}}{\begin{array}{ll} B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}}$$

In RU, $\check{\vee}$ abbreviates classical disjunction. By applying the above rules, one moves from one stage of a proof to another. A *stage* is a list of lines—stage 0 of any proof is the empty list. Stage s' is an *extension* of s iff all lines that occur in s occur in the same order in s'. A dynamic proof is a chain of stages.

That A is derivable on the condition Δ from the premise set Γ may be interpreted as follows: it follows from Γ that A or one of the members of Δ is true. As the members of Δ, which are abnormalities, are supposed to be false, A is considered as derived, unless and until the supposition cannot be upheld. The precise meaning of this depends on the strategy, which determines the marking definition (see below) and hence determines which lines are marked at a stage. If a line is marked at a stage, its formula is considered as not derived at that stage.

$Dab(\Delta)$ is a *minimal Dab-formula* at stage s of an **AL**-proof iff, at stage s, $Dab(\Delta)$ is derived on the condition $\emptyset$ and there is no $\Delta' \subset \Delta$ for which $Dab(\Delta')$ is derived on the condition $\emptyset$. Where $Dab(\Delta_1), \ldots, Dab(\Delta_n)$ are the minimal *Dab*-formulas at stage s of a proof from Γ, $\Phi_s(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, \ldots, \Delta_n\}$.

Definition 7.3. Marking for Minimal Abnormality: Line l is marked at stage s iff, where A is derived on the condition Δ at line l, (1) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, or (2) for some $\varphi \in \Phi_s(\Gamma)$, there is no line on which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.

This reads more easily: where A is derived on the condition Δ at line l, line l is *unmarked* at stage s iff (1) there is a $\varphi \in \Phi_s(\Gamma)$ for which $\varphi \cap \Delta = \emptyset$ and (2) for every $\varphi \in \Phi_s(\Gamma)$, there is a line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.

Definition 7.4. A is *finally derived* from Γ at line l of a stage s iff (1) A is the second element of line l, (2) line l is not marked at stage s, and (3) every extension of the stage in which line l is marked may be further extended in such a way that line l is unmarked.

Definition 7.5. $\Gamma \vdash_{\mathbf{AL}} A$ (A is *finally* **AL**-*derivable* from Γ) iff A is finally derived at a line of a proof from Γ.

As announced, most of the metatheory is provable in terms of the standard format, including that $\Gamma \vdash_{\mathbf{AL}} A$ iff $\Gamma \vDash_{\mathbf{AL}} A$.

7.3 Equivalent Premise Sets

This section reports on joint work with Peter Verdée and Christian Straßer (see Batens et al. 2009b). It is often important to determine whether two premise sets, Γ and Γ', are equivalent with respect to a logic $\mathbf{L}$, i.e. $Cn_{\mathbf{L}}(\Gamma) = Cn_{\mathbf{L}}(\Gamma')$. Thus, two theories may be 'identical' or not and two people may or may not share the same view on some topic. Determining whether two premise sets are identical by computing the sets $Cn_{\mathbf{L}}(\Gamma)$ and $Cn_{\mathbf{L}}(\Gamma')$ is obviously an impossible task. Fortunately certain criteria may be applied if the underlying logic is a Tarski logic (a reflexive, transitive, monotonic consequence relation), which is the common type of logics.[3]

Let $\mathbf{L}'$ be *weaker than* $\mathbf{L}$ iff $Cn_{\mathbf{L}'}(\Gamma) \subset Cn_{\mathbf{L}}(\Gamma)$ for some Γ and $Cn_{\mathbf{L}'}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma)$ for all Γ. The three most straightforward criteria are C1–C3 below. C1 is a direct criterion; the other criteria refer to a different logic. C2 and C3 are especially handy if $\mathbf{L}$ is a complicated logic.

C1 If $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$ and $\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma')$, then Γ and Γ' are $\mathbf{L}$-equivalent.
C2 If $\mathbf{L}'$ is a Tarski logic weaker than $\mathbf{L}$, and Γ and Γ' are $\mathbf{L}'$-equivalent, then Γ and Γ' are $\mathbf{L}$-equivalent.
C3 If every $Cn_{\mathbf{L}}(\Delta)$ is closed under a Tarski logic $\mathbf{L}'$ (viz. $Cn_{\mathbf{L}'}(Cn_{\mathbf{L}}(\Delta)) = Cn_{\mathbf{L}}(\Delta)$ for all Δ), and Γ and Γ' are $\mathbf{L}'$-equivalent, then Γ and Γ' are $\mathbf{L}$-equivalent.

For most defeasible logics, as formulated in the literature, one or more of the criteria break down. Easy examples are the Strong (or inevitable) consequence relation ($\Gamma \vdash_{Strong} A$ iff $\Gamma' \vdash_{\mathbf{CL}} A$ for every maximal consistent subset of Γ' of Γ) and the Weak consequence relation ($\Gamma \vdash_{Weak} A$ iff $\Gamma' \vdash_{\mathbf{CL}} A$ for some maximal consistent subset of Γ' of Γ)—see Rescher and Manor (1970) and Benferhat et al. (1997). Note that C1 does not hold for the Weak consequence relation and that C3 fails for the Strong consequence relation. The way in which some defeasible logics are presented causes the situation even to be worse. Thus criteria C1–3 require heavy reformulation before they even make a chance to be applicable to the many kinds of default logics or to the very transparent pivotal-assumption consequence relations defined in Makinson (2005).

The situation is completely different for adaptive logics: criteria C1–C3 provably hold for all of them. The proofs (in Batens et al. 2009b) rely on the fact that all adaptive logics have the following properties: reflexivity, fixed point ($Cn_{\mathbf{AL}}(Cn_{\mathbf{AL}}(\Gamma)) = Cn_{\mathbf{AL}}(\Gamma)$), cumulative monotonicity (if $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$, then

[3]Tarski logics that are compact and semi-recursive may be characterised as logics that have static proofs, whereas defeasible logics have dynamic proofs. A first version of the theoretical analysis of such notions is presented in Batens (2009a).

$Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma \cup \Gamma'))$, and cumulative transitivity (if $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ then $Cn_{\mathbf{AL}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{AL}}(\Gamma)$)—note that these properties are provable from the standard format. So, for adaptive logics, we have handy criteria for determining the equivalence of premise sets (and the identity of theories) and these criteria are the same as for Tarski logics.

Some will wonder how this is possible, given the claim that all defeasible first-order logics can be characterised by an adaptive logic. The reason is that the characterization often proceeds under a translation. An example might clarify this. Let the premises be formulated with classical negation, $\check{\neg}$. Let $\Gamma^{\neg\check{\neg}} = \{\neg\check{\neg}A \mid A \in \Gamma\}$ and let $\mathscr{W}^{\not\neg}$ be the set of closed formulas that do not contain $\neg$ (but may contain $\check{\neg}$). It was proved in Batens (2000) that $Cn_{Strong}(\Gamma) = Cn_{\mathbf{CLuN}^m}(\Gamma^{\neg\check{\neg}}) \cap \mathscr{W}^{\not\neg}$. So while C3 does not hold for the Strong consequence relation, C3 applies once the two premise sets are so translated and the 'logic' *Strong* is replaced by $\mathbf{CLuN}^m$.

There is a further result on extending premise sets. For every Tarski logic **L**, $\Gamma \cup \Delta$ and $\Gamma' \cup \Delta$ are **L**-equivalent if Γ and Γ' are. This does not hold for defeasible logics, not even for adaptive ones. However, for adaptive logics there is (apart from a specific criterion) a very close approximation: If **L** is a Tarski logic weaker than **AL** and Γ and Γ' are **L**-equivalent, then $\Gamma \cup \Delta$ and $\Gamma' \cup \Delta$ are **AL**-equivalent for all Δ.

Two other important results are proven in Batens et al. (2009b). Where **AL** is an adaptive logic and **LLL** is its lower limit logic: (1) every monotonic logic **L** that is weaker than **AL** is weaker than **LLL** or identical to it and (2) if $Cn_{\mathbf{AL}}(\Gamma)$ is closed under a monotonic logic **L**, then **L** is weaker than **LLL** or identical to it. This means that the lower limit logic provides very sharp versions of C2 and C3 and of the criterion mentioned in the previous paragraph.

7.4 Reducing Tinkering

Both the structure of the $\mathbf{C}_n$ logics and certain statements of da Costa's seem to suggest that a certain stratagem should be applied to theories that turn out inconsistent. Whether da Costa had this application in mind or not, the stratagem is clearly interesting and suggested by the $\mathbf{C}_n$ logics. It is worthwhile to develop inconsistency-adaptive logics that have the $\mathbf{C}_n$ systems as their lower limit because these enable one to accomplish, in more comfortable circumstances, the task served by the stratagem. The results presented in this section are studied at length in Batens (2009). So I shall be brief here.

7.4.1 *The $\mathbf{C}_n$ Logics and the Stratagem*

The $\mathbf{C}_n$-logics form a hierarchy. A simple way to describe it—not da Costa's original one—goes as follows. Let $\mathbf{C}_{\overline{\omega}}$ be full positive (predicative) **CL** together with the axioms $A \vee \neg A$ and $\neg\neg A \supset A$ and the rule "if $A \equiv^c B$, then $\vdash A \equiv B$", in which $A \equiv^c B$ iff A and B are congruent in the sense of (Kleene 1952, p. 153) or one is obtained from the other by deleting vacuous quantifiers.[4]

Let A^1 abbreviate $\neg(A \wedge \neg A)$,[5] let A^2 abbreviate $\neg(A^1 \wedge \neg A^1)$, etc., and let $A^{(n)}$ abbreviate $A^1 \wedge A^2 \wedge \ldots \wedge A^n$. The logic $\mathbf{C}_n$ ($n \in \{1, 2, \ldots\}$) is obtained by extending $\mathbf{C}_{\varpi}$ with the following axioms

$$B^{(n)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$$

$$(A^{(n)} \wedge B^{(n)}) \supset (A \dagger B)^{(n)} \qquad \text{where } \dagger \in \{\vee, \wedge, \supset\}$$

$$\mathsf{Q}x(A(x))^{(n)} \supset (\mathsf{Q}x\, A(x))^{(n)} \qquad \text{where } \mathsf{Q} \in \{\forall, \exists\}$$

A formula of the form $A^{(n)}$ is a *consistency statement* in $\mathbf{C}_n$. It expresses that A behaves consistently—see for example da Costa (1974)—in that $A, \neg A, A^{(n)} \vdash_{\mathbf{C}_n} B$. Incidentally, $\neg^{(n)} A =_{df} \neg A \wedge A^{(n)}$ defines classical negation in $\mathbf{C}_n$.

The $\mathbf{C}_n$ logics form a hierarchy in that $\Gamma \vdash_{\mathbf{C}_n} A$ if $\Gamma \vdash_{\mathbf{C}_m} A$ for some $m > n$. $\mathbf{C}_{\varpi}$ forms a limit of this hierarchy. As it will be useful to have classical negation available even in $\mathbf{C}_{\varpi}$, let us extend the language with the symbol $\check{\neg}$ and give it the meaning of classical negation (by introducing the usual axioms)—the standard negation, $\neg$, is still paraconsistent. Note the difference between $\neg^{(n)}$ and $\check{\neg}$. The first is definable within the standard language and behaves like classical negation in all $\mathbf{C}_m$ with $m \leq n$, but is not definable in $\mathbf{C}_{\varpi}$. The second symbol does not belong to the standard language, and hence does not occur in the premises, but is added to the language for technical reasons.[6]

Two features of the $\mathbf{C}_n$ logics may cause some wonder. First, what is the use of having classical negation, viz. the symbol $\neg^{(n)}$, definable within paraconsistent logics? Next, what is the use of the hierarchy of $\mathbf{C}_n$ logics? The following paragraphs answer these questions, possibly with hindsight.

The paraconsistent $\mathbf{C}_n$ were introduced to replace **CL** in inconsistent contexts. Let $T_0 = \langle \Gamma_0, \mathbf{CL} \rangle$ turn out to be inconsistent. Replacing T_0 by $T_1 = \langle \Gamma_0, \mathbf{C}_1 \rangle$ saves the theory from triviality—I suppose that Γ_0 does not contain any formulas of the form $\neg(A \wedge \neg A)$ because these are **CL**-tautologies. At the same time, however, T_1 is much poorer than is desirable. Suppose that $A \vee B$ and $\neg A$ are $\mathbf{C}_1$-derivable from Γ_0 and that A is not. As Γ_0 was intended to be consistent, one would expect B to be derivable as well. But $A \vee B, \neg A \nvdash_{\mathbf{C}_1} B$. So, if A is not $\mathbf{C}_1$-derivable from Γ_0,

[4]All $\mathbf{C}_n$ logics defined below in the text are identical to da Costa's, except that he introduces $\mathbf{C}_{\omega}$ as the limit. $\mathbf{C}_{\omega}$ is like $\mathbf{C}_{\varpi}$ except that the former has positive intuitionistic logic where the latter has positive classical logic. An interesting study of limits of the hierarchy is presented in Carnielli and Marcos (1999). The logic $\mathbf{C}_{\varpi}$ is there called $\mathbf{C}_{min}$.

[5]While $\neg A \wedge A$ and $A \wedge \neg A$ are $\mathbf{C}_{\varpi}$-equivalent, $\neg(\neg A \wedge A)$ and $\neg(A \wedge \neg A)$ are not. Which of both is taken to express the consistency of A is a conventional matter.

[6]The approach is related to, but different from, the one followed in Carnielli et al. (2007), where a consistency operator, $\circ A$, belongs to the standard language and is implicitly defined by, for example, $\circ A \supset ((A \wedge \neg A) \supset B)$.

one might extend Γ_0 with the consistency statement $A^{(1)}$. This delivers the desired result because $A \vee B, \neg A, A^{(1)} \vdash_{\mathbf{C}_1} B$. Exactly the same situation arises if $\neg B \supset A$ and $\neg A$ are $\mathbf{C}_1$-derivable from Γ_0. So the addition of consistency statements to an inconsistent theory has dramatic effects. Within the paraconsistent context, it drastically enriches the theory. Moreover, the so enriched theory approaches the original theory, T_0, as it was originally *intended*.

Adding consistency statements involves a danger. Let $T_1' = \langle \Gamma_1, \mathbf{C} \rangle$ in which Γ_1 is obtained by adding a set of consistency statements of the form $A^{(1)}$ to Γ_0. T_1' may very well be trivial. When this is the case, one may retract some of the added consistency statements. There is, however, another possibility.

The transition from T_0 to T_1 involves the replacement of **CL**, which da Costa also calls $\mathbf{C}_0$, by $\mathbf{C}_1$ in order to avoid triviality. If T_1' turns out trivial, one may replace $\mathbf{C}_1$ by $\mathbf{C}_2$—let the result be T_2. In this way, triviality is avoided again; statements of the form $A^{(1)}$ are not consistency statements in the context of $\mathbf{C}_2$. Moreover, relying on the insights from the failed previous attempt, one may enrich Γ_1 with consistency statements of the form $A^{(2)}$, which have the desired effect in the context of $\mathbf{C}_2$. This process may be repeated. If $T_n' = \langle \Gamma_n, \mathbf{C}_n \rangle$, Γ_n comprising no statements $A^{(m)}$ for which $m > n$,[7] and is trivial, replacing $\mathbf{C}_n$ by $\mathbf{C}_{n+1}$ restores non-triviality because no $A^{(m)}$ occurring in T_n' is a consistency statement with respect to $\mathbf{C}_{n+1}$.

The stratagem demands the presence of classical negation and the $\mathbf{C}_n$ hierarchy and so motivates them. Certain phrases used by da Costa also suggest the stratagem. Thus he states that $\mathbf{C}_n$ logics isolate inconsistencies and he distinguishes between 'good' and 'bad' theorems of $\mathbf{C}_n$-theories, the bad ones being those whose negation is also a theorem. In order to isolate the bad theorems and to take advantage of the good ones, one needs to add consistency statements to the theory.

7.4.2 The Adaptive Logics

I shall proceed in two steps. First we need adaptive logics that interpret the premise set as consistently as possible with respect to a $\mathbf{C}_n$-logic. Let us call these $\mathbf{C}_n{}^m$ logics. These inconsistency-adaptive logics enrich a premise set with the consistency statements that are justifiable by logical means. The $\mathbf{C}_n{}^m$-logics should have been devised a long time ago, were it only because of the historical significance of the $\mathbf{C}_n$ logics. There was, however, a difficulty. $\mathbf{C}_n$ logics validate relations between contradictions and whenever this is the case there is a possibility that a flip-flop logic results. Flip-flop logics are adaptive logics, but are uninteresting for most application contexts. They behave like the upper limit logic whenever the premise set is normal, which is all right, and behave like the lower limit logic whenever

[7] Just as A^1 is a **CL**-theorem, viz. a $\mathbf{C}_0$-theorem, A^m is a $\mathbf{C}_n$-theorem whenever $m > n$. So one may suppose that no formula of the form A^m or $A^{(m)}$ is $\mathbf{C}_{n+1}$-derivable from the non-logical axioms of a theory that has $\mathbf{C}_n$ as underlying logic.

the premise set is abnormal (requires at least one abnormality to be true), which is not all right. Fortunately, a criterion for flip-flop behaviour, in terms of a specific indeterministic semantics, was developed for the application of the criterion to the $\mathbf{C}_n$ logics. In view of this result, the following logics are not flip-flops. For each n, $\mathbf{C}_n{}^m$ is defined as the triple consisting of (1) $\mathbf{C}_n$, (2) $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathscr{F}\}$, and (2) Minimal Abnormality—the result generalises to Reliability, which I do not consider for lack of space.

These logics assign as consequences of a premise set Γ all formulas true in the minimally abnormal $\mathbf{C}_n$-models of Γ—this obviously includes all $\mathbf{C}_n$-consequences of Γ.

Applying the adaptive logics has certain advantages over following the stratagem. First of all, the logic itself adds consistency statements that can be added on logical grounds; no tinkering is involved. Next, for some (actually most) premise sets, the consequence set will comprise an *infinite* number of consistency statements as well as all their consequences. Note that this effect cannot be obtained by tinkering. Moreover, it is possible that a *Dab*-formula is derivable, say $(p \wedge \neg p) \vee (q \wedge \neg q)$, of which no disjunct is derivable. In this case, there is no logical justification for either of the two disjuncts. So the logic will not chose between $\neg(p \wedge \neg p)$ and $\neg(q \wedge \neg q)$, but will have the disjunction of the consistency statements, $\neg(p \wedge \neg p) \vee \neg(q \wedge \neg q)$, as a consequence together with all that follows from it.

An interesting fact concerns the choice of a $\mathbf{C}_n{}^m$ logic that is suitable for a set of premises. It turns out that $\mathbf{C}_{\overline{\omega}}{}^m$ is the suitable choice for *all* premise sets. To be more precise, it holds for every $\mathbf{C}_n{}^m$ that $Cn_{\mathbf{C}_n{}^m}(\Gamma)$ is either trivial or identical to $Cn_{\mathbf{C}_{\overline{\omega}}{}^m}\Gamma$.

Now we come to the second step. Following the stratagem has also an advantage over applying the adaptive logic. Consider again a case where $(p \wedge \neg p) \vee (q \wedge \neg q)$ is derivable but none of both disjuncts is. A person following the stratagem is able to chose at this point, for example to consider $p \wedge \neg p$ as false, and hence $q \wedge \neg q$ as true.

It is possible to introduce such 'new premises' within an adaptive framework and it is actually possible to do this in a more elegant way than the stratagem permits. First of all, the minimal *Dab*-formulas that are derived evoke the question which of the disjuncts is true; so they indicate the points at which choices may be made. Next, there cannot be logical reasons for the choices. So the person applying the adaptive logics has to justify the new premises on the basis of *extra-logical* grounds. Moreover, the addition of new premises should proceed in a *defeasible* way in order to avoid possible triviality. Finally, each such new premise is better introduced in a prioritised way. Indeed, the justification of some consistency statements will be stronger than that of others. Given all this, the matter may be handled by a well known combined adaptive logic, which should only be adjusted to the circumstances in that the lower limit of the combining adaptive logics should be $\mathbf{C}_{\overline{\omega}}$. The combined logic *guides* the addition of prioritised consistency statements. To the $\mathbf{C}_{\overline{\omega}}{}^m$-consequences the combined logic first adds as many as possible of the consistency statements with the highest priority; to the result of this it adds as many as possible of the consistency statements with the next highest priority; and so on.

For the details of the combined logic, I refer to Batens (2009). It is interesting, however, to note that, while the hierarchy of $\mathbf{C}_n$ logics proves useless on the present approach, the priorities are expressed by formulas that largely follow da Costa's hierarchy of consistency statements. Thus $\neg\exists(A \wedge \neg A)$ is the least prioritised consistency statement concerning A, $\neg\exists(A \wedge \neg A) \wedge \neg(\exists(A \wedge \neg A) \wedge \neg\exists(A \wedge \neg A))$ is the next stronger consistency statement concerning A, and so on.

7.4.3 Two Comments

The enrichment that will be described in the next section may be introduced within the context of the $\mathbf{C}_n{}^m$ logics. This departs rather heavily from the stratagem, but is clearly meaningful in the present context.

The second comment concerns decidability. Not taking anything back of what I said about the advantages of the adaptive approach over the stratagem, let me try to avoid a misunderstanding. The adaptive approach clearly cannot make the situation more decidable than it is. For example, if the premise set is (finite and) propositional, the adaptive consequence set is decidable. In this case, an able logician may manage to obtain the right result in terms of the stratagem. Where the premise set is predicative, the stratagem may lead one to the wrong conclusions because one may never find out that an added consistency statement causes triviality. By following the adaptive approach, a similar situation may arise: one takes a conclusion as finally derived while it is not, because one does not manage to derive the required *Dab*-formulas. If matters are undecidable, no approach can repair this—see Horsten and Welch (2007) for a challenge and Batens et al. (2009a) for an answer.

The advantages of the adaptive approach are mainly threefold. First, it *defines* the consequence set in a correct way, even if this set is not recursive or not even semi-recursive. Next, there are proof procedures (see Batens 2005 and Verdée 201+) that, for some Γ and A, lead after finitely many steps to the conclusion that A is or is not a final consequence of Γ. It the answer is decidable, the proof procedure will provide it, and if it provides an answer, the answer is correct. Finally, the adaptive approach rigorously distinguishes between consistency statements that can be added on logical grounds and those that require an extra-logical justification. It guides the addition of the latter by delineating the choices to be made and it handles the added statements according to their priority.

7.5 Variations

The first inconsistency-adaptive logic, dating from around 1980, had the aim to offer a maximally consistent interpretation of premise sets, or theories, that were intended as consistent but had turned out to be inconsistent. So when it was recently found possible to realise the aim in a more efficient way, this came as a shock.

Two other problems are solved at once. Inconsistency-adaptive logics are instruments: formal characterizations of defeasible reasoning forms. We want to have a manifold of them around to suit specific application purposes. While there is a lot of variation with respect to the lower limit logic and the strategy, every lower limit logic seems to determine a unique set of abnormalities[8]—I disregard flip-flop logics (see previous section). In this paper, the limitation is overcome.

The second problem concerns the comparison between different lower limit logics. Stronger paraconsistent logics have in general larger consequence sets than weaker ones, but also spread inconsistencies. While the former property makes more formulas derivable on the empty condition, the latter restricts the number of formulas that are finally derivable but have a non-empty condition. In general, varying the lower limit logic often leads to incomparable adaptive consequence sets. The result presented in this paper changes the picture drastically. By varying the set of abnormalities, adaptive logics with a very weak lower limit logic may be given a very rich consequence set. I shall present comparative results below.

7.5.1 Characterization of the Abnormalities

The idea behind the enriched set of abnormalities is surprisingly simple. When certain complex $\mathbf{CLuN}^m$-abnormalities are derivable, these may have different causes. Thus if $(p \vee q) \wedge \neg(p \vee q)$ is **CLuN**-derivable from the premises, this may be because p is so derivable, or q is, or $p \vee q$ is whereas neither p nor q is. These three cases can be distinguished.

Consider the premise set $\Gamma_1 = \{\neg(p \vee q), q, p \vee r\}$ and let the underlying logic be $\mathbf{CLuN}^m$. Note that $\neg p$ is derivable on the condition $\{(p \vee q) \wedge \neg(p \vee q)\}$ and hence r is derivable on the condition $\{(p \vee q) \wedge \neg(p \vee q), p \wedge \neg p\}$. By the presence of q and $\neg(p \vee q)$, however, $(p \vee q) \wedge \neg(p \vee q)$ is derivable from Γ_1 on the empty condition and so cannot be taken to be false. So neither $\neg p$ nor r are $\mathbf{CLuN}^m$-derivable from Γ_1. At first sight, this seems justified. Note, however, that the derivability of $(p \vee q) \wedge \neg(p \vee q)$ is caused by the presence of q, not by the presence of p.

It is possible to turn this idea in a technically feasible definition? It is. In the presence of $\neg(p \vee q)$, each of $p \vee q$, p, and q may cause the abnormality. The disjunction is derivable from either disjunct. Moreover, any **CLuN**-model verifying $p \vee q$ verifies p or q, but not necessarily both. This suggests that we consider $(p \vee q) \wedge \neg(p \vee q)$, $p \wedge \neg(p \vee q)$, and $q \wedge \neg(p \vee q)$ as separate abnormalities. The gain is clear: as $\neg(p \vee q) \vdash_{\mathbf{CLuN}} \neg p \vee (p \wedge \neg(p \vee q))$, p is derivable from $\neg(p \vee q)$ on the condition $\{p \wedge \neg(p \vee q)\}$ if the member of this singleton counts as an abnormality. Moreover, while $q \wedge \neg(p \vee q)$ is unconditionally derivable from Γ_1, $p \wedge \neg(p \vee q)$ is provably not a disjunct of any minimal *Dab*-consequence of Γ_1. Of course, this is merely an example; the matter requires elaboration.

[8] This is typical for inconsistency-adaptive logics, not for other adaptive logics.

Primitive formulas and their negations will be called *atoms*. Formulas that are not atoms are classified as $\mathfrak{a}$-formulas or $\mathfrak{b}$-formulas, varying on a theme from Smullyan (1968). To each of them, two other formulas are assigned according to the following table.

$\mathfrak{a}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{b}$	$\mathfrak{b}_1$	$\mathfrak{b}_2$
$A \wedge B$	A	B	$A \vee B$	A	B
$A \equiv B$	$A \supset B$	$B \supset A$	$A \supset B$	$\check{\neg} A$	B
			$\neg A$	$\check{\neg} A$	$\check{\neg} A$
$\check{\neg}(A \vee B)$	$\check{\neg} A$	$\check{\neg} B$	$\check{\neg}(A \wedge B)$	$\check{\neg} A$	$\check{\neg} B$
$\check{\neg}(A \supset B)$	A	$\check{\neg} B$	$\check{\neg}(A \equiv B)$	$\check{\neg}(A \supset B)$	$\check{\neg}(B \supset A)$

Next, a set sp(A) of *specifying parts* is assigned to every open or closed formula A as follows:

1. Where A is a conjunction of (one or more) atoms, possibly preceded by a sequence of quantifiers, $\text{sp}(A) = \{A\}$.
2. $\text{sp}(\mathfrak{a}) = \{\mathfrak{a}\} \cup \{\text{sp}(A \wedge B) \mid A \in \text{sp}(\mathfrak{a}_1); B \in \text{sp}(\mathfrak{a}_2)\}$.
3. $\text{sp}(\mathfrak{b}) = \{\mathfrak{b}\} \cup \text{sp}(\mathfrak{b}_1) \cup \text{sp}(\mathfrak{b}_2)$.
4. $\text{sp}(\forall \alpha A) = \{\text{sp}(\forall \alpha B) \mid B \in \text{sp}(A)\}$.
5. $\text{sp}(\exists \alpha A) = \{\text{sp}(\exists \alpha B) \mid B \in \text{sp}(A)\}$.

The adaptive logic $\mathbf{CLuN}_1^m$ is defined by the following triple: (1) lower limit: **CLuN**, (2) set of abnormalities: $\Omega^s = \{\exists(B \wedge \neg A) \mid A \in \mathscr{F}; B \in \text{sp}(A)\}$, and (3) strategy: Minimal Abnormality.

The mechanism is one of *refinement*. Even if $(p \vee q) \wedge \neg(p \vee q)$ is true in some models of a premise set, either $p \wedge \neg(p \vee q)$ or $q \wedge \neg(p \vee q)$ may be false in some of those models and this enables us to rule out some further models as more abnormal than required by the premises.

We have seen that the logic $\mathbf{CLuN}_1^m$ is richer than $\mathbf{CLuN}^m$ with respect to Γ_1. However, the enrichment is not restricted to similar cases. Let me mention two further examples. Consider first $\Gamma_2 = \{p \vee q, \neg(p \vee q), p \vee r, q \vee s\}$. In view of the explicit contradiction between the first two premises, one might expect to obtain no gain in this case. Yet, there is one. It is easily seen that r is derivable from Γ_3 on the condition $\{p \wedge \neg(p \vee q)\}$ and that s is derivable on the condition $\{q \wedge \neg(p \vee q)\}$. So $r \vee s$ is derivable on both conditions. Moreover, the only minimal *Dab*-consequences of Γ_3 are $(p \vee q) \wedge \neg(p \vee q)$ and $(p \wedge \neg(p \vee q)) \vee (q \wedge \neg(p \vee q))$. It follows that $r \vee s$, which is not a $\mathbf{CLuN}^m$-consequence of Γ_3, is a $\mathbf{CLuN}_1^m$-consequence of this premise set.

Another enrichment is illustrated by $\Gamma_3 = \{\neg\neg(p \wedge q), \neg p, \neg q \vee r\}$. Neither q nor r is a $\mathbf{CLuN}^m$-consequence of Γ_2, but both are $\mathbf{CLuN}_1^m$-consequences of it.

7.5.2 A Combined Inconsistency-Adaptive Logic

For all that was said, one might have the impression that $\mathbf{CLuN}_1^m$ offers a net gain over $\mathbf{CLuN}^m$, but this is false. In order to obtain a net gain, we need a combined adaptive logic. To see this, consider $\Gamma_4 = \{\neg(\neg s \vee (\neg p \wedge \neg r)), \neg(\neg p \vee \neg q), \neg(s \vee p)\}$.

The only members of minimal *Dab*-consequence of Γ_4 (with respect to both Ω and Ω^s) are provably 1–9 below. All are members of Ω^s and only 1–3 are members of Ω.

1	$(\neg s \vee (\neg p \wedge \neg r)) \wedge \neg(\neg s \vee (\neg p \wedge \neg r))$
2	$(\neg p \vee \neg q) \wedge \neg(\neg p \vee \neg q)$
3	$(s \vee p) \wedge \neg(s \vee p)$
4	$\neg s \wedge \neg(\neg s \vee (\neg p \wedge \neg r))$
5	$(\neg p \wedge \neg r) \wedge \neg(\neg s \vee (\neg p \wedge \neg r))$
6	$\neg p \wedge \neg(\neg p \vee \neg q)$
7	$\neg q \wedge \neg(\neg p \vee \neg q)$
8	$s \wedge \neg(s \vee p)$
9	$p \wedge \neg(s \vee p)$

It is also provable that we may restrict our attention, in this specific propositional case, to models of Γ_4 that verify the premises together with some of the relevant propositional letters and the classical negation of the others. A survey is displayed in Table 7.1. Unmentioned letters may receive an arbitrary value, provided they are not inconsistent. The numbers in the table refer to the abnormalities listed before. The first row of stars depicts the (kinds of) models that are minimally abnormal with respect to $\mathbf{CLuN}^m$; the second row of stars those that are *moreover* minimally abnormal with respect to $\mathbf{CLuN}_1^m$. The two-step selection is required because the second, fourth, sixth, eight, and ninth models are minimally abnormal with respect to Ω^s-abnormalities, but none of them is minimally abnormal with respect to Ω-abnormalities. The so combined selection delivers the consequences q, $p \vee r$, $s \vee r$, ... on top of those delivered by $\mathbf{CLuN}^m$.

Let us call the combined adaptive logic $\mathbf{CLuN}_c^m$ and let $Cn_{\mathbf{CLuN}_c^m}(\Gamma) = Cn_{\mathbf{CLuN}_1^m}(Cn_{\mathbf{CLuN}^m}(\Gamma))$, which offers the right selection of models. Proof theoretically such logics seem to be disastrous: it seems that one needs to compute $Cn_{\mathbf{CLuN}^m}(\Gamma)$ before one can even start to apply $\mathbf{CLuN}_1^m$. But this is not so. As was spelled out already in Batens (2001), the dynamic proof theory of thus combined adaptive logics is hardly more complex than that of the combining logics.

7.5.3 Some Comparisons

As promised, I shall now show that the combined logic $\mathbf{CLuN}_c^m$ does not only better than $\mathbf{CLuN}^m$, but does also very well in comparison to inconsistency-adaptive

Table 7.1 **CLuN**-models of Γ_4

p	p	p	p	p	p	p	p	$\check{\neg}p$	$\check{\neg}p$	$\check{\neg}p$	$\check{\neg}p$	$\check{\neg}p$	$\check{\neg}p$	$\check{\neg}p$	$\check{\neg}p$
q	q	q	q	$\check{\neg}q$	$\check{\neg}q$	$\check{\neg}q$	$\check{\neg}q$	q	q	q	q	$\check{\neg}q$	$\check{\neg}q$	$\check{\neg}q$	$\check{\neg}q$
r	r	$\check{\neg}r$	$\check{\neg}r$	r	r	$\check{\neg}r$	$\check{\neg}r$	r	r	$\check{\neg}r$	$\check{\neg}r$	r	r	$\check{\neg}r$	$\check{\neg}r$
s	$\check{\neg}s$	s	$\check{\neg}s$	s	$\check{\neg}s$	s	$\check{\neg}s$	s	$\check{\neg}s$	s	$\check{\neg}s$	s	$\check{\neg}s$	s	$\check{\neg}s$
	1		1		1		1		1	1	1		1	1	1
				2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3		3		3		3	
*		*							*		*		*		*
	4		4		4		4		4		4		4		4
										5	5			5	5
								6	6	6	6	6	6	6	6
				7	7	7	7					7	7	7	7
8		8		8		8		8		8		8		8	
9	9	9	9	9	9	9	9								
*		*							*						

logics that have a richer lower limit. Below, I consider five premise sets to compare $\mathbf{CLuN}^m_c$ with the corresponding adaptive logics that have as their lower limit logic respectively the maximal paraconsistent logic **CLuNs** and **LP**. I list the results for the latter logics together where they are identical with respect to the formulas that are listed—they differ from each other with respect to formulas that contain implications or equivalences.

$$\Gamma_5 = \{\neg(p \vee q), q \vee r, p, \neg p \vee s\}$$

$\mathbf{CLuN}^m$	$\mathbf{CLuN}^m_c$	$\mathbf{CLuNs}^m/\mathbf{LP}^m$
p	p	p
		$\neg p$
	$\neg q$	$\neg q$
$q \vee r$	r	r
s	s	

$$\Gamma_6 = \{p \vee q, \neg(p \vee q), p \vee r, q \vee s\}$$

$\mathbf{CLuN}^m$	$\mathbf{CLuN}^m_c$	$\mathbf{CLuNs}^m/\mathbf{LP}^m$
		$\neg p$
		$\neg q$
$p \vee q$	$p \vee q$	$p \vee q$
$p \vee r$	$p \vee r$	$p \vee r$
$q \vee s$	$q \vee s$	$q \vee s$
	$\neg p \vee \neg q$	
	$r \vee s$	

$\Gamma_7 = \{p, \neg p \vee q, \neg(p \vee r), \neg\neg p \supset s\}$

$\mathbf{CLuN}^m$	$\mathbf{CLuN}_c^m$	$\mathbf{CLuNs}^m$	$\mathbf{LP}^m$
p	p	p	p
		$\neg p$	$\neg p$
$\neg\neg p$	$\neg\neg p$	$\neg\neg p$	$\neg\neg p$
	$\neg r$	$\neg r$	$\neg r$
q	q		
s	s	s	

$\Gamma_8 = \{p, \neg p \vee q, \neg(p \vee r), \neg\neg p \supset s, \neg q \vee t, r \vee u\}$

$\mathbf{CLuN}^m$	$\mathbf{CLuN}_c^m$	$\mathbf{CLuNs}^m$	$\mathbf{LP}^m$
p	p	p	p
		$\neg p$	$\neg p$
$\neg\neg p$	$\neg\neg p$	$\neg\neg p$	$\neg\neg p$
	$\neg r$	$\neg r$	$\neg r$
q	q		
s	s	s	
t	t		
	u	u	u

It is interesting to study the difference between the consequence sets. In all cases, (1) the $\mathbf{CLuNs}^m$-consequences or $\mathbf{LP}^m$-consequences that are not $\mathbf{CLuN}_c^m$-consequences cause additional inconsistency and (2) some $\mathbf{CLuN}_c^m$-consequences are neither $\mathbf{CLuNs}^m$-consequences nor $\mathbf{LP}^m$-consequences and they do not cause additional inconsistency. I am not claiming, however, that $\mathbf{CLuN}_c^m$ is better than the other adaptive logics. An instrument should be used where it is suitable. The only point I wanted to make is that $\mathbf{CLuN}_c^m$ maximally isolates inconsistencies, just as much as $\mathbf{CLuN}^m$, but nevertheless offers an extremely rich consequence set.

7.6 Parsimonious Axiomatisations

7.6.1 The Problem

Let $\mathscr{L}_A$ be the language of arithmetic (with one constant, 0, and three functions, $'$, $+$, and $\times$). In several places, for example Priest (1994, 1997, 2000, 2006), Graham Priest considers inconsistent models of arithmetic (see also Paris and Pathamanathan 2006; Paris and Sirokofskich 2008). In these models, the logical symbols are interpreted in terms of Priest's **LP**—implication and equivalence are defined and non-detachable. I shall only consider the so-called collapsed models.

$\mathcal{M}^n_p$ denotes the model with the following successor graph:

$$\begin{array}{ccccc} 0 \rightarrow 1 \rightarrow \ldots \rightarrow & n & \rightarrow n+1 \\ & \uparrow & \downarrow \\ & n+p-1 \leftarrow & \ldots \end{array}$$

In order to simplify the subsequent argument, let us concentrate on models $\mathcal{M}^n_1$, which have the following successor graph

$$\begin{array}{c} 0 \rightarrow 1 \rightarrow \ldots \rightarrow n \\ \hphantom{0 \rightarrow 1 \rightarrow \ldots \rightarrow} \circlearrowright \end{array}$$

Let us more particularly concentrate on $\mathcal{M}^2_1$. In order to avoid confusion between numbers and numerals, let the domain of the model be $\{\mathfrak{f}, \mathfrak{m}, \mathfrak{a}\}$ and let the interpretation of the successor function be characterised by the following graph:

$$\begin{array}{c} \mathfrak{f} \rightarrow \mathfrak{m} \rightarrow \mathfrak{a} \\ \hphantom{\mathfrak{f} \rightarrow \mathfrak{m} \rightarrow} \circlearrowright \end{array}$$

with $v(0) = \mathfrak{f}$, viz. the constant 0 is taken to name $\mathfrak{f}$. So $0'$ names $\mathfrak{m}$, and $0''$, $0'''$, etc. all name $\mathfrak{a}$.

Every $\mathcal{M}^n_1$ can be seen as modelling a specific inconsistent arithmetic $\mathbf{A}^n_1 = \{A \mid \mathcal{M}^n_1 \Vdash A\}$ (the formulas of $\mathcal{L}_A$ that are verified by $\mathcal{M}^n_1$). As every $\mathcal{M}^n_1$ is a finite model, $\mathbf{A}^n_1$ can be finitely axiomatised with **LP** as the underlying logic. This means that there is a recursive, and actually finite, set of formulas Γ such that $\mathbf{A}^n_1 = \{A \mid \Gamma \vdash_{\mathbf{LP}} A\}$.

There is, however, an oddity. Not only $\mathcal{M}^2_1$, but also $\mathcal{M}^1_1$ as well as the trivial model $\mathcal{M}^0_1$ are models of $\mathbf{A}^2_1$. This is related to the fact that $\mathbf{A}^0_1 \supseteq \mathbf{A}^1_1 \supseteq \mathbf{A}^2_1 \supseteq \ldots$. It is also related to the fact that $\mathcal{M}^2_1$ is a model of classical arithmetic,[9] which, provided it is consistent, is a limit of this sequence of sets. The sentences of $\mathcal{L}_A$ that are true in the standard model of arithmetic are also true in the finite and inconsistent model $\mathcal{M}^2_1$. In the same way the sentences of $\mathcal{L}_A$ that are true in $\mathcal{M}^2_1$ are also true in $\mathcal{M}^1_1$ and in $\mathcal{M}^0_1$.

It follows from Gödel's first theorem that no consistent axiomatisation of first-order sentences true in the standard model of arithmetic identifies the standard model. Every such axiomatisation also has non-standard models, the domain of which comprises objects not named by any numeral. So no (first-order) axiomatisation identifies the standard model. The situation is similar for every $\mathbf{A}^n_1$, except that the domains of the non-intended models comprise not more but less objects than the domain of the intended model—the larger n, the greater the number of non-intended models. In many other respects, the situation is dissimilar from the situation of classical arithmetic, but in this sense it is similar.

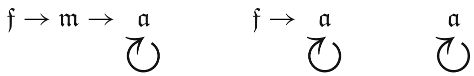
[9]By "classical arithmetic" I obviously mean the set of formulas true in the standard model and not the theorems of some axiom system.

The failure to identify a single model, say $\mathscr{M}_1^2$, is obviously contingent on the object language and on the underlying logic. Let us first have a look at variant logics.

7.6.2 A **LPm**-*Axiomatisation*

One might hope to identify $\mathscr{M}_1^2$ by presenting an axiomatisation that has **LPm** as its underlying logic rather than **LP** (see Priest 1991, 2006). Indeed, **LPm** selects the 'minimal abnormal' **LP**-models of a premise set—see below for the quotation marks. In $\mathscr{M}_1^2$, the denotation of $0''$ and of all higher numerals are inconsistent with respect to identity (that is $0'' = 0'' \wedge \neg\, 0'' = 0''$ is a theorem), but the denotations of $0'$ and of 0 are consistent with respect to identity. In $\mathscr{M}_1^1$, the denotation of $0'$ is also inconsistent with respect to identity, and in $\mathscr{M}_1^0$ the denotation of every numeral is inconsistent with respect to identity—$\mathscr{M}_1^0$ is a trivial model.

Unfortunately, **LPm** does not provide a solution. The cause lies with the way in which minimal abnormal models are defined in **LPm**. Here are, again, the successor graphs of $\mathscr{M}_1^2$, $\mathscr{M}_1^1$, and $\mathscr{M}_1^0$:



The 'abnormal part' of a model is represented in **LPm** by the atomic inconsistent 'facts' that hold in the model. In other words, for every n-ary predicate R, the n-tuples that belong to both the extension of R, $v^+(R)$, and to the anti-extension of R, $v^-(R)$ (see Priest 2006 for details). The only predicate that matters in the present context is identity and all three models have the same abnormal part, viz. $v^+(=) \cap v^-(=) = \{\langle \mathfrak{a}, \mathfrak{a} \rangle\}$. So all three models are **LPm**-models of $\mathbf{A}_1^2$. This means that no **LPm**-axiomatisation identifies $\mathscr{M}_1^2$ and that the difficulty remains.

Incidentally, we obviously need $v(0) = \mathfrak{a}$ instead of $v(0) = \mathfrak{f}$ in the displayed model $\mathscr{M}_1^0$. Some isomorphic models have $\mathfrak{f}$ as the only element of the domain, and these have exactly the same abnormal part as some models isomorphic with $\mathscr{M}_1^1$ and $\mathscr{M}_1^2$.

7.6.3 **LP**^{m}-*Axiomatisation*

Unlike **LPm**, **LP**^{m} is an adaptive logic in standard format; it was described earlier. The difference with **LPm** is that abnormalities are not 'inconsistent' n-tuples of members of the domain, but *formulas*, viz. existentially closed contradictions. The abnormal part of a **LP**-model M, $Ab(M)$, is the set of abnormalities verified by M. So $Ab(\mathscr{M}_1^2)$ comprises all formulas of the form $0^i = 0^i \wedge \neg\, 0^i = 0^i$ in which i is a sequence of two or more names of the successor function, as well as the

LP-consequences of these, for example $\exists x(x = x \wedge \neg x = x)$. The set $Ab(\mathscr{M}_1^1)$ *moreover* comprises $0' = 0' \wedge \neg 0' = 0'$ and $Ab(\mathscr{M}_1^0)$ even comprises $0 = 0 \wedge \neg 0 = 0$. So, of the three considered models, only $\mathscr{M}_1^2$ is a minimally abnormal model of $\mathbf{A}_1^2$. An $\mathbf{LP}^m$-axiomatisation of $\mathbf{A}_1^2$ is obtained, for example by adding the axiom $0''' = 0''$ to the Peano Axioms. Let this set of axioms be called $\mathbf{PA}_1^2$—there are obviously simpler, viz. finite, sets that do exactly the same job. The axiom system $\langle \mathbf{PA}_1^2, \mathbf{LP}^m \rangle$ (the axioms $\mathbf{PA}_1^2$ closed under $\mathbf{LP}^m$) identifies $\mathbf{A}_1^2$.

It is important to realise that the effect results from changing the underlying logic. If this is $\mathbf{LP}^m$, the models of $\mathbf{A}_1^2$ have to be $\mathbf{LP}^m$-models, and the only such model is $\mathscr{M}_1^2$.

Some may wonder whether an axiomatisation with $\mathbf{LP}^m$ as underlying logic is really an axiomatisation. Indeed, a $\mathbf{LP}^m$-proof of A from the premise set Γ requires a list of formulas together with a reasoning in the metalanguage establishing that A is finally derived in the list of formulas (see for example Batens et al. 2009a for details). So this kind of proofs, which are called dynamic, do not form a positive test for (final) derivability. In the present context, however, this complication does not arise. Given the model $\mathscr{M}_1^2$, which is finite, and the language, there are prospective proofs, see for example Batens (2005), that form a decision method for derivability. In other words, $Cn_{\mathbf{LP}^m}(\mathbf{PA}_1^2)$ is a decidable set and the couple $\langle \mathbf{PA}_1^2, \mathbf{LP}^m \rangle$ is a legitimate axiomatisation of $\mathbf{A}_1^2$. For those who are still mistrusting, let $\langle \Delta, \mathbf{LP} \rangle$ be an axiomatisation of $\mathbf{A}_1^2$—so $Cn_{\mathbf{LP}}(\Delta) = \mathbf{A}_1^2$. Next, consider the axiomatisation $\langle \Delta, \mathbf{LP}^m \rangle$ and note that $Cn_{\mathbf{LP}^m}(\Delta) = \mathbf{A}_1^2$.[10] As every $\mathbf{A}_1^2$-theorem is **LP**-derivable from Δ, it is unconditionally $\mathbf{LP}^m$-derivable from Δ. So in view of this metatheoretic fact, there is a positive test for $\mathbf{A}_1^2$-theoremhood.

7.6.4 A Richer Language

Other axiomatisations are possible, even with a Tarski logic as the underlying logic, but they all have the disadvantage that they require replacing $\mathscr{L}_A$ by a richer language.

The first alternative is that one adds classical (or Boolean) negation, $\check{\neg}$. A suitable axiomatisation is obtained by extending $\mathbf{PA}_1^2$ with, for example, $\check{\neg} 0' = 0''$. In the presence of classical negation, $\check{\neg} 0 = 0'$ is derivable from this and, in general, $\check{\neg} A$ is derivable whenever A is "false only" in $\mathscr{M}_1^2$. Apart from requiring an extension of the language, this approach has the further disadvantage that it is opposed to Priest's philosophical views—he has argued against the meaningfulness of classical negation, a point which I shall not discuss here.

Another alternative is to extend the language with a relevant implication, $\rightarrow$, as well as with bottom, $\bot$, and adding to $\mathbf{PA}_1^2$ axioms like $0' = 0'' \rightarrow \bot$, $0 = 0' \rightarrow \bot$,

[10]This further clarifies the claim made in the previous paragraph. Although $Cn_{\mathbf{LP}}(\Delta) = Cn_{\mathbf{LP}^m}(\Delta)$, $\langle \mathbf{PA}_1^2, \mathbf{LP}^m \rangle$ identifies $\mathscr{M}_1^2$ whereas $\langle \mathbf{PA}_1^2, \mathbf{LP} \rangle$ does not.

and so on. If the relevant implication is the one from Priest (2006, § 18.3), the "and so on" should not be underestimated; even $0' = 0 \rightarrow \bot$ is not a consequence of $0 = 0' \rightarrow \bot$. If the n in $\mathscr{M}_1^n$ is large, the number of required axioms will be impressive, but obviously finite.

This approach too seems to involve difficulties. If the relevant implication is not extremely poor, one will have as a theorem $\forall x \forall y(x = y \rightarrow f(x) = f(y))$ for every one argument function f. So, in particular, one will have $\forall x \forall y(x = y \rightarrow x' = y')$ as a theorem. But then $0' = 0' \rightarrow 0'' = 0''$ is a theorem. As $\neg 0'' = 0''$ is a theorem of $\mathbf{PA}_1^2$ and $\rightarrow$ is contraposable, $\neg 0' = 0'$ would be a theorem of $\mathbf{PA}_1^2$. But this is wrong: $\neg 0' = 0'$ is false in $\mathscr{M}_1^2$ and so should not be a theorem of $\mathbf{PA}_1^2$. Of course, the difficulty will not occur if the relevant implication is weaker, for example is the one from Priest (2006, § 18.3). One wonders, however, whether this implication will be sufficient to formalise the whole body of our knowledge, empirical and mathematical. Indeed, Priest is a monologist. So he opposes using different logics in different contexts.

The presence of an enthymematic implication does not repair the situation. Indeed, while one might prefer to replace the relevant implication in $\neg 1 = 1 \rightarrow \bot$ by an enthymematic one, there is no reason to perform the same replacement in $\forall x \forall y(x = y \rightarrow f(x) = f(y))$ in case this is a theorem. However, the presence of a non-contraposable relevant implication would remove this specific difficulty, might very well be justifiable,[11] and seems to provide a sufficiently strong statement $\forall x \forall y(x = y \rightarrow f(x) = f(y))$.

More serious difficulties are lurking around the bend. First, the relevant implication is *ad hoc* in the present context—it occurs nowhere else in the inconsistent arithmetic, just like the classical negation from two paragraphs ago. Next, I cannot see any sense in which $\neg 0' = 0'$ can be said to *relevantly imply* every statement of the language. Adding the implicative axioms comes to a technical trick. It does the job, but can only be justified by the argument that it provides a warrant that is as good as the one the classical logician invokes by recurring to classical implication (which connects classical inconsistency to triviality)—but see below.

Another difficulty is related to the fact that everything is true in the trivial model, in the present context $\mathscr{M}_1^0$. So, even if it can be avoided that $\mathscr{M}_1^1$ is a model of $\mathbf{A}_1^2$, this theory still has both $\mathscr{M}_1^2$ and $\mathscr{M}_1^0$ as models, and so does not identify $\mathscr{M}_1^2$ in a unique way—please compare with the $\mathbf{LP}^m$-axiomatisation which *does* rule out the trivial model $\mathscr{M}_1^0$.

Incidentally, the classical logician seems to do better in this respect *on her understanding*. She can claim that adding $\check{\neg} 0' = 0''$ identifies $\mathscr{M}_1^2$ in a unique way. On the classical logician's understanding, there is no trivial model because the truth values, say t and f, are distinct, v_M is a function, and $v_M(\check{\neg} A) = t$ iff $v_M(A) = f$. So there are no models in which $v_M(\check{\neg} A) = t = v_M(A)$. Of course Graham Priest has argued that the classicist's understanding makes no sense, a point not discussed here.

[11]The most obvious justification for contraposition is consistency. So I always wondered why so many relevant logicians want their implications to be contraposable.

7.6.5 Describing the Models

Until now, I phrased the difficulty as one of axiomatising the formulas true in the models $\mathscr{M}_p^n$, while I took those models at face value. However, a similar difficulty affects the *description* of the models. Whether one considers the description I gave above or the description in Priest (2006), the model $\mathscr{M}_1^1$ actually agrees with the description of the model $\mathscr{M}_1^2$. The domain counts three different objects, $\mathfrak{f}$, $\mathfrak{m}$, and $\mathfrak{a}$. Of these, $\mathfrak{m}$ and $\mathfrak{a}$ are not only different but also identical and the successor function holds between them. Note, incidentally, that "$\mathfrak{m}$" and "$\mathfrak{a}$" are not the elements of the domain, but the names of these elements; just as the drawing is not the successor graph, but a representation of it. That the *characters* "$\mathfrak{m}$" and "$\mathfrak{a}$" are not identical, but different, and different only for that matter, does not prevent them from naming the same entity.[12] By a similar reasoning, the model $\mathscr{M}_1^0$ agrees with the description of $\mathscr{M}_1^2$.

So the description of $\mathscr{M}_1^2$ does not identify this model as we understood it, *unless* we presuppose that the description is as consistent as possible, viz. is presented in terms of $\mathbf{LP}^m$. Unlike **LPm**, $\mathbf{LP}^m$ will select the right description and will select the right models of the description—these are not the models described *by* the description.

7.7 Concluding Comment

Rather than commenting on the promise made in the introduction, I shall comment on a consequence of the preceding section.

In Mortensen (2008), Chris Mortensen writes that, according to inconsistency-adaptive logics, "only consistent conclusions are deduced *pro tem*" and continues "In the opinion of this (opinionated) writer, consistentising strategies are useful for the context of discovery, but fail to do justice to *a priori* reasoning from inconsistent premises, where one should be acknowledging the full role of all the premises without dodging the inconsistencies in them." These claims are actually false,[13] but the reason to quote them lies elsewhere, viz. in the presupposed status of *a priori* reasoning. If there is any truth in the previous section, one needs "consistentising

[12]One shouldn't make too much of the "different only" phrase. In Priest's view it may be true together with "the characters are the same", for otherwise "This sentence is false and only false." would produce triviality.

[13]The first quoted claim is obviously false: *all* formulas derivable by the lower limit logic are adaptively derivable, whether consistent or inconsistent. However, some *further* consequences are adaptively derivable by taking as many *other* inconsistencies to be false as the premises permit. So inconsistency-adaptive logics do acknowledge the full role of all the premises and do not dodge any inconsistencies in them. They presuppose that inconsistencies are false *unless and until* proven otherwise, from the premises that is.

strategies" in order to describe the models $\mathscr{M}_p^n$ and this is apparently required before any *a priori* reasoning about them can even start. Inconsistency-adaptive logics were always presented as *instruments* (or methods), which may be more or less suited to a specific context, and not as candidates for "the true logic" or "the standard of deduction" or "the canon of *a priori* reasoning". Nevertheless, the situation depicted in this section seems to present a further argument, apart from many others, to mistrust a strict separation between sensible reasoning instruments and *a priori* reasoning. It also suggests that, while it is easy to explain the paraconsistent viewpoint by relying on classical results, such as the supposedly consistent standard model of arithmetic, it might be more difficult for the monologist dialetheist to offer her teachings from scratch. That Graham Priest has been persistently working in that direction, including the development of a dialetheisticly sound set theory, deserves the admiration and sympathy of every logician, even of those who (like me) consider the standard of reasoning as context dependent.

Acknowledgements Research for this paper was supported by subventions from Ghent University and from the Fund for Scientific Research—Flanders. I am indebted to Graham Priest for comments on a former draft of this paper.

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