Chapter 17
Notes on Inconsistent Set Theory

Zach Weber

17.1 Introduction

The standard axioms of naive set theory state existence and uniqueness conditions for sets (see Routley 1980; Priest et al. 1989; Brady 2006). The axioms are:

Axiom 17.1 (Abstraction) $x \in \{z : \Phi(z, u)\} \leftrightarrow \Phi(x, u)$.

Axiom 17.2 (Extensionality) $(\forall z)(z \in x \leftrightarrow z \in y) \leftrightarrow x = y$.

The purpose of this paper is to highlight and discuss two ideas that play in to the axiomatic development of a paraconsistent naive set theory, as detailed in Weber (2010b). We will focus on aspects of the theory that can be read right off the axioms, concerning intensional identity and unrestricted set existence. Both relate to inconsistency and are dealt with here as follows.

First, the extensionality axiom says that identity is governed by entailments. As we will define below, $\rightarrow$ is an intensional, relevant implication and so, as with an extensionality axiom formulated using a material conditional, this leads to some distinctive properties for identity. With these new properties in hand I extend some results of Arruda and Batens from da Costa's set theory (from da Costa 2000 in Batens et al. 2000).

Second, the set formation principle is fully unrestricted, so the set being defined may appear in its defining condition. We will explore how this makes modelling recursive phenomena particularly easy and natural, elaborating on ideas from Routley's set theory in Routley (1977).

To begin I lay out a relevant background logic, placing a strong emphasis on the restrictions such a logic must have in order to support an inconsistent set theory. The sections that follow proceed on the understanding that, while highly

Z. Weber (✉)
Department of Philosophy, University of Otago, PO Box 56, Dunedin 9054, New Zealand
e-mail: zach.weber@otago.ac.nz

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inconsistent, a good deal of control is being exerted on the theory through the weakened logic. The two features of a fully naive theory, identity and self-reference, dovetail throughout.[1]

17.2 Logic

The main purpose of this section is to summarize the known restrictions on a logic for naive set theory; see also Weber (2010a). The subsidiary purpose is to fix the logic used in this paper; the logic may be altered for different results, as long as all the restrictions are observed. Thus not much emphasis is placed on the particular choice here, except to provide exactness.

The language of first order set theory has primitives $\wedge, \neg, \rightarrow, \forall, =$ and $\in$, as well as a term-forming operator $\{\cdot : \cdot\}$; variables $x, y, z, \ldots$; names $a, b, c, \ldots$; and formulae $\Phi, \Psi, \Upsilon, \ldots$, built up by standard formation rules. The usual shorthand is used: $\Phi \vee \Psi$ for $\neg(\neg\Phi \wedge \neg\Psi)$; $\Phi \leftrightarrow \Psi$ for $(\Phi \rightarrow \Psi) \wedge (\Psi \rightarrow \Phi)$; $\exists$ is $\neg\forall\neg$. (Taking these as definitions means that e.g, $\Phi \vee \Psi \rightarrow \neg(\neg\Phi \wedge \neg\Psi)$ is no more than an instance of axiom I below.)

17.2.1 Axioms

All instances of the following schemata are theorems:

I $\Phi \rightarrow \Phi$

IIa $\Phi \wedge \Psi \rightarrow \Phi$

IIb $\Phi \wedge \Psi \rightarrow \Psi$

III $\Phi \wedge (\Psi \vee \Upsilon) \rightarrow (\Phi \wedge \Psi) \vee (\Phi \wedge \Upsilon)$ (*distribution*)

IV $(\Phi \rightarrow \Psi) \wedge (\Psi \rightarrow \Upsilon) \rightarrow (\Phi \rightarrow \Upsilon)$ *(conjunctive syllogism)*

V $(\Phi \rightarrow \Psi) \wedge (\Phi \rightarrow \Upsilon) \rightarrow (\Phi \rightarrow \Psi \wedge \Upsilon)$

VI $(\Phi \rightarrow \neg\Psi) \rightarrow (\Psi \rightarrow \neg\Phi)$ (*contraposition*)

VII $\neg\neg\Psi \rightarrow \Psi$ *(double negation elimination)*

VIII $\Phi \vee \neg\Phi$ *(excluded middle)*

[1]Following a distinction I first saw in Libert (2005), Axiom 17.1 is called *abstraction*, while the formulation in Theorem 17.3 below is called *comprehension*. There is a syntactic difference between abstraction and comprehension, and in weak paraconsistent logics the principles are not equally user-friendly, because the quantifier $\exists$ is sometimes tricky to eliminate. Nevertheless, both formulations capture a core intuition and in informal discussion the names are used interchangeably, without intending to mark an important difference.

IXa $(\Phi \to \Psi) \to [(\Psi \to \Upsilon) \to (\Phi \to \Upsilon)]$
IXb $(\Phi \to \Psi) \to [(\Upsilon \to \Phi) \to (\Upsilon \to \Psi)]$ *(hypothetical syllogisms)*
X $(\forall x)\Phi \to \Phi(a/x)$
XI $(\forall x)(\Phi \to \Psi) \to (\Phi \to (\forall x)\Psi)$
XII $(\forall x)(\Phi \vee \Psi) \to \Phi \vee (\forall x)\Psi$

Axioms XI and XII have the caveat that x does not appear free in Φ. The hypothetical syllogism pair *IXa* and *IXb* are called *suffixing* and *prefixing*, respectively.

17.2.2 Rules

The following rules are valid:

I $\Phi, \Psi \vdash \Phi \wedge \Psi$ (*adjunction*)
II $\Phi, \Phi \to \Psi \vdash \Psi$ *(modus ponens)*
III $\Phi, \neg\Psi \vdash \neg(\Phi \to \Psi)$
IV $\Phi \vdash (\forall x)\Phi$
V $x = y \vdash \Phi(x) \to \Phi(y)$ (*substitution*)

Brady proves that set theory in this logic has a model and is non-trivial (Brady 1989 and 2006, p. 242). If rule *III*, called *counterexample*, is brought up to arrow strength, the resulting logic is *DLQ* from Routley and Meyer (1976); with hypothetical syllogism, Axioms *IXa*, *b*, the logic is called *TLQ*. Non-triviality of naive set theory in these stronger logics is an open problem.

The fact that Brady's universal logic DJQ is not strong enough for some of these results is important. The key non-DJQ principles, excluded middle and counterexample, restores a connection between the intensional $\to$ and the extensional connectives, via the derived rule

$$\Phi \to \Psi \vdash \neg\Phi \vee \Psi$$

More to the point, the axiom does a lot of work. The preponderance of the results discussed below cannot be recovered (as given) using only DJQ. For more on the considerations going in to the choice of this particular logic, see Weber (2010a).

17.2.3 Restrictions

For a logic of naive set theory, *DLQ* is quite strong. For instance, it has a robust negation. But it is very spare, and for good reason. The first phase of paraconsistent set theoretical research has shown that there are several key restrictions to respect, on pain of triviality, which we recite here for ease of reference.

An inference is invalid if it does not preserve truth, and in the context of inconsistent set abstraction one must take extra care. *Disjunctive syllogism*,

$$\Phi, \neg\Phi \vee \Psi \vdash \Psi,$$

is invalid in this context, due to C.I. Lewis' famous argument in (Lewis and Langford 1959, p. 250). Also invalid is *contraction*,

$$\Phi \rightarrow (\Phi \rightarrow \Psi) \vdash \Phi \rightarrow \Psi$$

as shown by Curry (1942). Closely related is *axiom modus ponens* (or pseudo-modus ponens or mp-contraction),

$$\Phi \wedge (\Phi \rightarrow \Psi) \rightarrow \Psi$$

as found in Meyer et al. (1978) and Restall (1994). There is also a trouble with *permutation*,

$$\Phi \rightarrow (\Psi \rightarrow \Upsilon) \vdash \Psi \rightarrow (\Phi \rightarrow \Upsilon)$$

due to the argument in Slaney (1989). Slaney's argument shows that excluding the middle and permutation are not jointly tenable. The cause is, again, a close relative of Curry's paradox.

Since the logic is relevant it does not include weakening, $\Phi \vdash \Psi \rightarrow \Phi$. With weakening, we would have to drop contraposition. Else, we could argue from Λ to $\neg\Psi \rightarrow \Lambda$, then to $\neg\Lambda \rightarrow \Psi$. But if also $\neg\Lambda$, i.e. Λ is a true contradiction, then Ψ follows by *modus ponens*, where Ψ is arbitrary. The improper inference here is just $\Phi \vdash \neg\Phi \rightarrow \Psi$, a form of explosion.

17.2.4 A Case Study

Here is an example of how the weakened logic must be attended to. Consider the two way inference

$$\Phi \wedge \Psi \rightarrow \Upsilon \dashv\vdash \Phi \rightarrow (\Psi \rightarrow \Upsilon). \tag{17.1}$$

In classical logic (and set theory) this is obvious—because, materially, it just says

$$\neg(\Phi \wedge \Psi) \vee \Upsilon \dashv\vdash \neg\Phi \vee (\neg\Psi \vee \Upsilon). \tag{17.2}$$

The two-way derivation (17.2) is valid here, but in an intensional logic, the two sentences in (17.1) certainly do not say the same thing. They must, on pain of triviality, not be inter-derivable. Suppose the inference (1) from left to right. Now, we have as an axiom $\Phi \wedge \Psi \rightarrow \Phi$. So we would infer $\Phi \rightarrow (\Psi \rightarrow \Phi)$ as a valid scheme, which is weakening and so trivializing in this logic. From right to left on (1), because $(\Phi \to \Psi) \to (\Phi \to \Psi)$ is an instance of an axiom, $\Phi \wedge (\Phi \to \Psi) \to \Psi$ would be a valid scheme, which is mp-contraction. Given the relevance logic we are using, both directions of (17.1) must fail.

With this in mind, though, let us look at an example with a subset relation $\subseteq$.

Definition 17.1. $x \subseteq y := (\forall z)(z \in x \to z \in y)$. Then $x \subset y := x \subseteq y \wedge (\exists z)(z \in y \wedge z \notin x)$.

Then consider two ways of understanding transitivity,

$$y \subseteq z \to (x \subseteq y \to x \subseteq z),$$

$$x \subseteq y \wedge y \subseteq z \to x \subseteq z.$$

If subset is understood with arrows, as it is in Definition 17.1, then the first is an instance of hypothetical syllogism (Axioms IX) and the second of conjunctive syllogism (Axiom IV). But we can see that these are almost certainly independent of one another, based on the problems just discussed. So it is required in proofs that we be very clear about which forms we are using. More generally, in any formulation of definitions, for subset, ordinal number, or function, a great deal of thought is required. For example, f could be called a function when $\langle x, y\rangle \in f \wedge \langle x, z\rangle \in f \to y = z$, or when $\langle x, y\rangle \in f \to (\langle x, z\rangle \in f \to y = z)$, but with different results.

Although we do not need the following here, it is worth flagging a useful notion: a *relevant singleton* is written $\{x\}_y := \{z : z = x \wedge z \in y\}$. This is for relevance purposes, to fix $\{x\}_y \subseteq y$ iff $x \in y$.

17.3 Basics

Existential generalization (the contrapositive of axiom X) on the abstraction Axiom 17.1 immediately yields the principle:

Theorem 17.3 (Comprehension). $(\exists y)(\forall x)(x \in y \leftrightarrow \Phi(x, u))$.

Under abstraction, the substitution rule is $x = y \vdash (\forall z)(x \in z \to y \in z)$.

Proposition 17.1. $y = \{z : \Phi(z)\} \leftrightarrow (\forall x)(x \in y \leftrightarrow \Phi(x))$.

Proof. By extensionality, $y = \{z : \Phi(z)\} \leftrightarrow (\forall x)(x \in y \leftrightarrow x \in \{z : \Phi(z)\})$. By abstraction, $(\forall x)(x \in \{z : \Phi(z)\} \leftrightarrow \Phi(x))$.Then by conjunctive syllogism, $(\forall x)(x \in y \leftrightarrow \Phi(x))$. For the converse, we again invoke the abstraction scheme, where $(\forall x)(\Phi(x) \leftrightarrow x \in \{z : \Phi(z)\})$, so by conjunctive syllogism $(\forall x)(x \in y \leftrightarrow x \in \{z : \Phi(z)\})$. And this with the extensionality axiom completes the proof.

Abstraction and extensionality can then be reconnected, as in Frege's axiom:

Theorem 17.4 (Basic Law V). $\{x : \Phi\} = \{x : \Psi\} \leftrightarrow (\forall x)(\Phi \leftrightarrow \Psi)$.

As Routley (1977) points out, Zermelo's axioms now follow instantly as theorems—unsurprisingly, since Zermelo explicitly picked out instances of comprehension. For example, *Aussonderung* is just a weaker comprehension scheme, $(\exists y)(\forall x)(x \in y \leftrightarrow x \in a \wedge \Phi(x))$, while union, intersection and pairing are all as usual; e.g. the last is obtained by abstraction on the condition $x = a \vee x = b$. Given a working theory of functions, Fraenkel's replacement axiom scheme is easily obtainable, too.[2] Of some more interest is a proof of the axiom of infinity, which is an artefact of full comprehension, Proposition 17.3 below.[3]

A universe and an empty set both exist. The *universe* is

$$V = \{x : (\exists y)(x \in y)\},$$

and as one would expect, both $(\forall x)(x \in V)$ and $(\forall x)(x \subseteq V)$ hold. The *empty set* is the complement of V,

$$\emptyset = \{x : (\forall y)(x \in y)\},$$

and both $(\forall x)(x \notin \emptyset)$ and $(\forall x)(\emptyset \subseteq x)$ hold, too. See Dunn (1988) (in Austin 1988) for study of the uniqueness of these sets. For now the main fact to know about the empty set is that it is explosive. For example, to show that the empty set is empty, we argue by cases. Either $x \notin \emptyset$ or $x \in \emptyset$. If the former, stop. So suppose that $x \in \emptyset$. Then $(\forall y)(x \in y)$; then $x \in \{z : z \notin \emptyset\}$ and therefore $x \notin \emptyset$. More generally, $(\forall y)(x \in y) \rightarrow x \in \{z : \Psi\}$ for any Ψ at all. So $x \in \emptyset \rightarrow \Psi$. This property of $\emptyset$ is very useful; see also Slaney (1989).

17.4 Identity

The properties of $\rightarrow$ make identity an equivalence relation,

$$x = x,$$

$$x = y \rightarrow y = x,$$

$$x = y \wedge y = z \rightarrow x = z.$$

With hypothetical syllogism, additionally, $x = y \rightarrow (y = z \rightarrow x = z)$.

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[2]The first step in securing a set theoretic account of functions is defined ordered pairs and show them to behave according to the law $\langle a, b\rangle = \langle c, d\rangle \dashv\vdash a = c, b = d$. We will be assuming throughout that some approximation of standard mathematical functions is available.

[3]Without full comprehension, one can prove that the set of all sets is *Dedekind infinite* by producing an injection into itself, say by a map $x \mapsto \{x\}$, but, again, functions and cardinality arguments are mostly beyond our scope here.

By the counterexample axiom, $\rightarrow$ retains a connection to material implication, namely that if all Φs are Ψs, then everything is either not Φ or else Ψ. Contrapositively, if some Φs are not Ψs, then not all Φs are Ψs. This leads to the following surprising-and-intuitive result:

Proposition 17.2. *Sets that differ with respect to membership are not identical. In particular,* $(\exists x)(x \in a \wedge x \notin a) \vdash a \neq a$.

Proof. This is by rule III and the axiom of extensionality.

When a set a is such that its membership is inconsistent, some $b \in a$ and $b \notin a$, then a is *inconsistent*. And $(\exists x)(x \neq x)$, since by comprehension we have (at least) Russell's set,

$$R = \{x : x \notin x\}.$$

Excluding the middle, $R \in R \wedge R \notin R$. Since R differs from itself with respect to membership,

$$R \neq R.$$

Let us briefly expand on this theme, by seeing what happens when not only $=$ but parthood is tied to entailment,[4] as given by Definition 17.1.

For any a, we use the name $\mathscr{P}(a)$ for $\{x : x \subseteq a\}$.

The eccentricities of R enrich a result of Arruda and Batens (1982) from the set theory of da Costa (2000). Define by finite recursion (Theorem 17.7 below), $\mathscr{P}^0 = \mathscr{P}$ and $\mathscr{P}^{n+1} = \mathscr{P}\mathscr{P}^n$. Then

Theorem 17.5. $(\forall n)[\mathscr{P}^{n+1}(R) \subset \mathscr{P}^n(R)]$.

Proof. Arruda has found that

$$\ldots \mathscr{P}\mathscr{P}\mathscr{P}(R) \subseteq \mathscr{P}\mathscr{P}(R) \subseteq \mathscr{P}(R) \subseteq R.$$

To see that $\mathscr{P}(R) \subseteq R$, suppose $x \notin R$. Then $x \in x$. So $x \in x \wedge x \notin R$, meaning that $\exists y(y \in x \wedge y \notin R)$, so $x \nsubseteq R$. By contraposition, then, $x \subseteq R \rightarrow x \in R$, ergo $\mathscr{P}(R) \subseteq R$. Now suppose $x \in \mathscr{P}\mathscr{P}(R)$. Then $x \subseteq \mathscr{P}(R)$, so $x \subseteq R$ by transitivity. Therefore $x \in \mathscr{P}(R)$, and thus $\mathscr{P}\mathscr{P}(R) \subseteq \mathscr{P}(R)$.

To strengthen Arruda's finding, we employ contraposition at each arrow. For $\mathscr{P}(R) \subset R$, recall that $R \in R$ and $R \notin R$; this implies $R \nsubseteq R$. And $R \nsubseteq R \wedge R \in R$ gives $\mathscr{P}(R) \subset R$. For $\mathscr{P}\mathscr{P}(R) \subset \mathscr{P}R$, notice that $R \subseteq R$, but $R \in R \wedge R \nsubseteq R$; so by generalizing, $(\exists y)(y \subseteq R \wedge y \nsubseteq \mathscr{P}(R))$, as required. Again the argument may be continued:

$$\ldots \mathscr{P}\mathscr{P}\mathscr{P}(R) \subset \mathscr{P}\mathscr{P}(R) \subset \mathscr{P}(R) \subset R.$$

In general, then, this argument can be carried out for $\mathscr{P}^{n+1}(R)$ and $\mathscr{P}^n(R)$, which gives the full result by $\forall$-introduction.

[4]There is a debate about the right definition of subset—see (Mares, 2004, p. 198), and Beall et al. (2006), for instance using a more restricted implication.

That R 'implodes' in this way can be read as simple structure. With ordinal indices, one could go on to define by recursion (Theorem 17.7 below)

$$R_0 = R,$$
$$R_{\alpha+1} = \mathscr{P}(R_\alpha),$$
$$R_\lambda = \bigcup_{\kappa \in \lambda} R_\kappa,$$

where λ is a limit ordinal.

17.5 Full Comprehension

Since naive set theory formalizes the idea that all predicates determine sets, in the comprehension principle the occurrence of the set being defined in the defining predicate Φ is not ruled out. Following Routley, this is completely unrestricted or full comprehension. Priest and Routley write that

> The naive notion of set is that of the extension of an arbitrary predicate... This is as tight an account as can be expected from any fundamental notion. It was thought to be problematical only because it was assumed (under the ideology of consistency) that 'arbitrary' could not mean arbitrary. However, it does. (Priest et al. 1989, p. 499)

Set theory with a fully unrestricted comprehension principle is covered by the non-triviality proof in e.g. Brady (1989); Brady notes that Chang in 1965 had already noticed that a set theory with unrestricted comprehension can be consistent. In this section we look at some of the work a full comprehension principle can do—from supplying the concept of recursion to justifying a global choice principle.

When unrestricted, the abstraction axiom generates 'circular' or self-referring cases. These are neither necessarily inconsistent nor unique, e.g. cases like

$$x \in J \leftrightarrow x = J,$$
$$x \in K \leftrightarrow x = K,$$

mean that $J = \{J\}$ and $K = \{K\}$. But there is no way to say, absent further postulation, whether or not $J = K$. Compare this to other non-well-founded set theories, like Aczel's (discussed in Exercise 2.4 of Barwise and Moss (1996)). Aczel adds an axiom asserting, in effect, that $J = K$ in cases like these (since his anti-foundation axiom implies that all systems of equations have unique solutions). Here we allow the indeterminacy, in exchange for axiomatic simplicity.

To guarantee, meanwhile, that such instances are valid abstractions—to ensure that every predicate, even groundless ones, determines a set—we have abstraction instances of the form

$$x \in \{z : \Phi(z, u)\} \leftrightarrow \Phi\left[z/x,\ u/\{z : \Phi(z, u)\}\right]$$

where the right-hand-side indicates a simultaneous substitution in Φ of z by x, and u by the term $\{z : \Phi(z, u)\}$. (At first, in Brady and Routley (1989, p. 419), a new quantifier, formation rule, and reflection axiom were added to handle circular predicates; but by Brady (2006, p. 177), the idea is streamlined as above.) Axiom 17.1 in this way includes cases

$$x \in \{z : \Phi(z, u)\} \leftrightarrow \Phi(x, \{z : \Phi(z, u)\}).$$

To start, the axiom makes for some very direct expressions of natural phenomena. For example, (consistent) infinite descents have extreme expressions, like

$$x \in \Delta \leftrightarrow (\exists y)(y \in x \wedge y \in \Delta).$$

The simplest members of Δ could be a pair a, b such that $a \in b$ and $b \in a$. That there are such sets at our disposal might have application to models of inconsistent arithmetic (see Priest 2000), where circular periods occur in the successor relation, if the ordering $<$ on natural numbers is reduced to $\in$.

To take a simpler, and inconsistent, example, Routley identifies the limiting case of diagonal sets,

$$x \in \mathscr{Z} \leftrightarrow x \notin \mathscr{Z}$$

which is a kind of 'ultimate Russell set'. Non-self-identity $\mathscr{Z} \neq \mathscr{Z}$ is by Proposition 17.2, but actually something much stronger follows. By excluded middle, either $x \in \mathscr{Z}$ or not, for every x, from which it follows that $(\forall x)(x \in \mathscr{Z})$ and $(\forall x)(x \notin \mathscr{Z})$.

While this in some sense does make $\mathscr{Z}$ both universal and empty, we do not have $\mathscr{Z} = V$ or $\mathscr{Z} = \emptyset$, since identity is controlled by relevance. Because of relevance, $(\exists y)(x \in y)$ does not entail $x \in \mathscr{Z}$, so $V \subseteq \mathscr{Z}$ does not obtain and a fortiori neither does $\mathscr{Z} = V$. This is actually good news; the alternative is triviality (see Weber 2010a).

The universe is not the only set to have a highly inconsistent '$\mathscr{Z}$'-part. Any non-empty set a, for example, will have a subset $\mathscr{Z}(a) = \{x : x \in a \wedge x \notin \mathscr{Z}(a)\}$. Now, just as with unrestricted $\mathscr{Z}$, we have $(\forall x)(x \notin \mathscr{Z}(a))$. For $x \in a$, though, this is just the property needed to show $x \in \mathscr{Z}(a)$. So every member of a both is and is not a member of $\mathscr{Z}(a)$. This subset of a acts as a reflection of a over which inconsistency can be 'dialled up' as high as we like. Some points to note about this $\mathscr{Z}(a)$ phenomenon:

- Full comprehension is not required to give this result. Instead of $\mathscr{Z}$, just take $\{x : x \in a \wedge R \in R\}$, for R the Russell set. The same arguments go through. This is the inconsistent aspect of the dopplegänger phenomenon (see Weber 2010a).
- While $\mathscr{Z}(a)$ is inconsistent for non-empty a, this does not prove that a is inconsistent. By the $\rightarrow$-logic of parthood, a set can have inconsistent parts and yet be perfectly consistent as a whole. The universe V is only the biggest example.

- There are consequences here for cardinality. For example, one can provide a proof of Cantor's theorem, of the form $|a| < |\mathscr{P}a|$, essentially by appealing to $\mathscr{Z}(a) \in \mathscr{P}a$. In a sense, this is good news, as it confirms an important theorem. On the other hand, consider singletons:

$$\{a\} = \{x : x = a\}$$
$$\mathscr{Z}(\{a\}) = \{x : x = a \wedge x \notin \mathscr{Z}(\{a\})\}$$

It is simple to check that $\mathscr{Z}(\{a\}) \subseteq \{a\}$. In fact, though, by the argument for Theorem 17.5, it is almost as straightforward that $\mathscr{Z}(\{a\}) \subset \{a\}$, a *proper* subset. Now, if a set X is *Dedekind infinite* when there is an injection from X to a proper subset of X, then we just proved that $\{a\}$ is Dedekind infinite for any set a. This strongly suggests that a finer grained notion of cardinality is required than in the classical definitions of infinity.

On this note, we derive a classical axiom of infinity.[5]

Proposition 17.3 (Infinity). *There is a non-empty set i isomorphic to an ω-sequence of Zermelo ordinals,*

$$i = \{i, \{i\}, \{\{i\}\}, \{\{\{i\}\}\}, \ldots\}$$

Proof. Consider $i = \{x : x = i\}$. Since $i \in i$, the set is not empty. Since $i = \{i\}$, by substitution, $\{i\} \in i$.

For the development of Peano arithmetic, we could then define the natural numbers as

$$\omega = \{x : [x = \emptyset \vee (\exists y)(\{y\} = x)] \wedge x \subseteq \omega\}$$

using full comprehension to ensure that numbers are preceded only by other numbers.

We turn then to the theory of ordinal numbers, which includes the natural numbers. In standard set theory, ordinals are understood as the set of all preceding ordinals, ordered by membership. This is plainly recursive and can be captured in a definition: An ordinal is a transitive, well-ordered set of ordinals.

Let $Wo(x)$ mean that x is well-ordered—that there is a linear $\in$-order on x where also every non-empty subset of x has a least member. Let $Conn(x)$ mean that for every $y \in On$, either $x \subseteq y$ or else $y \subseteq x$. The formalism of this is not a concern now; the rendering of (Brady, 2006, p. 310), could do. The matter at hand is full comprehension.

[5] Compare this to Petersen's characterization of the natural numbers, (Petersen, 2000, p. 386).

Proposition 17.4. *There is a set On such that*

$$
\begin{aligned}
x \in On \leftrightarrow \quad & x \subseteq On \\
& \wedge\ Conn(x) \\
& \wedge\ y \in x \rightarrow y \subseteq x \\
& \wedge\ Wo(x).
\end{aligned}
$$

For short, $On = \{x : x \text{ is an ordinal}\}$.

Notice immediately that On is transitive. Since $\alpha \in On \rightarrow \alpha \subseteq On$, the set of all ordinals satisfies one of the key conditions for being an ordinal. With this definition, one can work from $\emptyset \in On$ up to Burali-Forti's paradox that $On \in On$.

Theorem 17.6. *[Burali-Forti 1897]* $On \in On$.

Proof. On is a transitive, well-ordered set of connected ordinals (see Weber 2010b). Checking the definition of 'ordinal' gives the result.

Because $\in$ is irreflexive on ordinals, and because of what we know about identity from the last section (Proposition 17.2), we have some contradictions:

Corollary 17.1. *$On \notin On$, and then $On \neq On$.*

Full comprehension is well suited to modelling recursive processes, as we have been seeing. We would like a transfinite recursion theorem. Barwise and Moss (1996) use the nice example of a function $g : \omega \longrightarrow \omega \times \omega$ defined as $g(n) = \langle n, g(n+1)\rangle$, which delivers a sequence $g(0) = \langle 0, \langle 1, \langle 2, \langle \ldots \rangle\rangle\rangle\rangle$, and with full comprehension, it is easy enough to prove that something like g exists, namely

$$\langle x, y\rangle \in g \leftrightarrow x \in \omega \wedge y = \langle x, g(x+1)\rangle.$$

There is no guarantee that this g is a function, though. Instead, a general form of recursion on the ordinals (and ipso facto the natural numbers) is captured in the next proof. Let $f|x$ be the restriction of f to x, defined as $\{\langle u, v\rangle \in f : u \in x\}$.

Theorem 17.7 (Transfinite Recursion). *Let h be a function from V to V. There is a function f from On to V such that*

$$f(\alpha) = h(f|\alpha).$$

Proof. The set $\langle x, y\rangle \in f \leftrightarrow y = h(f|x)$ exists, and is a function because h is.

Full comprehension, then, is very powerful. It is time to consider one of its most arresting, and earliest, applications, to the axiom of choice.

Routley (1977) produced an argument for the axiom of global choice from full comprehension. He did this by defining a function to be either univocal or empty, since classically an empty set is a function by dint of material implication. The instance of comprehension

$$x \in f \leftrightarrow \exists u \exists v(u \in X \wedge x = \langle u, v\rangle \wedge v \in u) \wedge f \textit{ is a function}.$$

then allows the following proof: Either f is empty or not. Either way, f is a function, because if it is non empty then f is a function by the definition of f, while if it is empty then f is a function by definition of function. So there is a choice function on any X—including the universe, V. This is the axiom of *global choice*.

There is something unsatisfactory about the argument. Full comprehension is not even required here, since a 'function' like

$$\{\langle u, v\rangle : R \in R\},$$

with R the Russell set, supports the same reasoning.[6] (Since $R \notin R$, the set has no members, and so satisfies Routley's criteria to be a function.) But this does not appear to be a function in any mathematical sense, since every ordered pair whatsoever is a member.

Later Routley (Priest et al. 1989, p. 374) reprised the attempt with the comprehension instance

$$x \in f \leftrightarrow \exists u \exists v(u \in X \wedge x = \langle u, v\rangle \wedge v \in u)$$
$$\wedge \forall u \forall y \forall z(\langle u, y\rangle \in f \wedge \langle u, z\rangle \in f \rightarrow y = z),$$

again looking to say that f is a function on X. But the argument is really just a version of Curry's paradox, and is blocked by the failure of contraction, since to show that f is a function leads us to consider $\langle u, v\rangle \in f \rightarrow \forall z(\langle u, v\rangle \in f \wedge \langle u, z\rangle \in f \rightarrow v = z)$. If f is non-empty, then it is a function, but there is no telling whether or not f is empty; we first need to know whether or not it is a function. So this second formulation is a contraction away from choice, but also from proving anything at all.

In a sense, Routley is trying to use a paradox to make choice true. The first attempt uses a paradox of material implication (that when f is empty, $\langle x, y\rangle \in f$ materially implies that y is unique). That idea can be presented in terms of Russell's paradox, or it can be rephrased in terms of the implicational form of Russell's antinomy, Curry's paradox. But none of these are making meaningful use of full comprehension per se, and more seriously, none of these give us reason to think that the axiom of choice is true.

[6] Conrad Asmus pointed this out.

On the other hand, defining the ordinals self-referentially by full comprehension leads to Burali-Forti's paradox, and, as I now outline, this paradox not only delivers an equivalent theorem, but gives us a good mathematical reason to think that what we have proved is true. We derive Cantor's well-ordering principle, by giving an easy way for the universe to be injected into a particular subset of On. Suppose we say that a function $f : a \longrightarrow b$ is *injective*, or one-one, iff $(\forall x)(\forall y)\neg(x \neq y \wedge f(x) = f(y))$.

Theorem 17.8. *The universe can be well-ordered.*

Proof. An injection $f : V \longrightarrow On$ is required. Consider the constant function $f(x) = On$. The range of f is a segment of the ordinals. Because $On \neq On$, we have that $(\forall x)(\forall y)(x = y \vee On \neq On)$, so $(\forall x)(\forall y)(x = y \vee f(x) \neq f(y))$. Therefore f is an injection. Thus

$$\{x_{f(x)} : f(x) \in On\}$$

is a well-order on V.

17.6 Conclusion

Whether a useful choice principle really obtains, and so whether this line fares better than Routley's arguments, remains to be seen. Indeed, most of an elementarily paraconsistent set theory—elementary in the sense that no appeal is made to classical results—remains to be seen. From the point of view of inconsistent mathematics, I only hope to have suggested there is a great deal of the universe of sets still waiting to be explored. Drawing again on one venerable tradition in paraconsistent set theory, I join da Costa in his structural initiative:

> It would be as interesting to study the inconsistent systems as, for instance, the non-Euclidian geometries: we would obtain a better idea of the nature of certain paradoxes, could have a better insight on the connections amongst the various logical principles necessary to obtain determinate results, etc. (da Costa 1974, p. 498)

And drawing again on another tradition, Routley claimed more. In a programmatic polemic, Routley (1977) hypothesized that standard mathematics, beginning with set theory, can be recaptured using a suitable "ultramodal" logic. Later reprinted in his magnum opus, he writes

> There are whole mathematical cities that have been closed off and partially abandoned because of the outbreak of isolated contradictions. They have become like modern restorations of ancient cities, mostly just patched up ruins visited by tourists.
>
> In order to sustain the ultramodal challenge to classical logic it will have to be shown that even though leading features of classical logic and theories have been rejected, . . . by going ultramodal one does not lose great chunks of the modern mathematical megalopolis. . . . The strong ultramodal claim—not so far vindicated—is the expectedly brash one: we can do everything you can do, only better, and we can do more. (Routley 1980, p. 927)

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