Chapter 15
Pluralism and "Bad" Mathematical Theories: Challenging our Prejudices

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15.1 Introduction

In the philosophy of *logic*, we have fairly well worked out pluralist philosophies (Beall and Restall 2006); we also have good discussions and exchanges over variations of the position. In contrast, in the philosophy of *mathematics*, there is no well worked out position called "pluralism". When the attitude of pluralism is mentioned, it is listed as an attitude amongst others.[1] In this paper, pluralism is defended and developed as a philosophy of mathematics. A pluralist in the philosophy of mathematics is someone who places pluralism as the chief virtue in her philosophy of mathematics. She brings the attitude to bear on mathematical theories, including different foundations of mathematics, and on different philosophies of mathematics.

As a philosophy of mathematics, pluralism is founded on the conviction that we do not have the necessary evidence to think that mathematics is one unified body of truths, or is reducible to one or two theories (foundations). As pluralists, we might *hope* that mathematics will one day turn out to be so unified, but, and this is important, such hope is simply a subjective private feeling, and is not supported on present evidence. So, similarly, a pluralist might hope that there are several irreducible foundations in mathematics, and that there will never, nor can ever, even in principle, be a way of unifying these. Again this is a hope and a private conviction. The pluralist, as (public) philosopher is simply agnostic. Once this agnostic attitude is in place, then the pluralist is free to take an interest in mathematics as a series of theories, where each contains truths-relative-to-a-meta-theory. Or the pluralist might

[1] Examples of attitudes held by various philosophies of mathematics are: ontological parsimony, respect for the phenomenology of mathematics, respect for the views of mathematicians, simplicity etc. Some of the attitudes might be judged as a virtue or as a vice, depending on the philosophy.

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think of mathematics as a process, as opposed to concentrating on mathematics as a unified body of truths. The pluralist is also interested in “bad” mathematics, and how these parts inform the “good” parts. “Bad” mathematics include: intensional theories, intentional theories,[2] not yet completely formally represented theories, paraconsistent mathematics and trivial mathematics. Because of the last two, the pluralist philosopher underpins her philosophy with a paraconsistent logic.

In this paper I give two motivations for pluralism, and then discuss the position itself. I shall begin with three negative arguments why the present day foundationalist philosophies of mathematics are inadequate. The critique mainly comes from the naturalist insight that philosophers should respect mathematical compass and practice and reports by mathematicians concerning their philosophical views (Maddy 2007). In this way, naturalism is a motivation for pluralism. I then discuss the second motivation, which comes from structuralism: that there is no absolute notion of truth in mathematics. Rather, there is a perfectly robust notion of ‘truth in a structure’ (Shapiro 1997). As a philosophy, pluralism pushes both naturalist and structuralist insights beyond the present developments of these positions. This is what we do in the first two thirds of the paper.

The first of the three negative arguments for pluralism is that many foundationalist philosophies make a slide from a description of mathematics (in terms of a global foundation) to a norm for success for future mathematics, and this slide is illegitimate. The second argument is that the proposed foundations are artificial, so the foundations are a mis-description of mathematics in the first place. The mis-description is twofold, since, in addition to its being a mis-description of contemporary mathematics, whatever founding theory one has, ‘it’ will grow—by adding new axioms. Some new axioms are independent, and therefore there are no grounds for faith in there being one absolute true foundation for mathematics. The third negative argument is simply that many mathematicians today are pluralist, and that, following the naturalist, philosophers should respect and accommodate this interesting and philosophically challenging fact. Not only are many mathematicians pluralist, but they are highly aware of the context surrounding a theorem or proof, or the limitations of those theorems. In this respect, they are broadly structuralist. For this reason, pluralists are also motivated by structuralism. The structuralism I shall discuss is Shapiro’s structuralism (since it, too, is anti-foundationalist). But I want to push it further than Shapiro does, to include more ‘structures’ than what can be recognised as a ‘structure’ by model theory. These are the “bad” parts of mathematics. Once we take seriously the idea that we want to do philosophical work concerning such outlandish parts of mathematics, we need to articulate the pluralist position. This will be done in the last third of the paper. We need to fend from the suspicion that pluralism degenerates into history or sociology of mathematics, and that there is no room left for philosophy. We also need to fend from the related suspicion that pluralism is really a rampant relativism. To fend from this last suspicion, we deploy a paraconsistent logic. This allows the pluralist to cope

[2] We shan’t say much about these in this paper.

with situations when, (1) no decision can be made as to the truth of a well-formed formula in a theory (so we need a logic with truth-value gaps), and when, (2) a theory contains a well-formed formula which is both true and false (a logic with truth-value gluts).[3] By way of drawing some final conclusions about the limitations of pluralism we shall visit the semantics of the logic underlying the pluralist philosophy.

15.2 Setting up the Negative Arguments

15.2.1 Monism and Dualism, Revisionism and Foundationalism

Pluralism is anti-foundationalist. In this respect, pluralism takes issue with most of the standard traditional philosophies of mathematics. Let us study the enemy camp. In developing a foundationalist philosophy of mathematics, a philosopher seeks to give a unified philosophy of all of mathematics, where the unity is made plain by reduction to a foundation. These foundationalist philosophies of mathematics are monist and usually revisionist of mathematics. Alternatively, standard philosophies of mathematics split mathematics into two parts, the good part and the more suspect part; these are the dualists. Examples of monist philosophies are: platonism/realism, intuitionism and Whitehead and Russell’s logicism. Examples of dualist philosophies are: Fregean logicism, Hilbertian formalism and Cantorian realism. In the development of both monist and dualist philosophies,[4] it was presumed that to give a *philosophy of mathematics at all*, one had to give a foundation for mathematics. Only sporadically has this assumption been challenged, and the most recent challenge has come from structuralism. Interestingly, anti-foundationalism can be well motivated by some insights from Maddy’s naturalism, although, she does not consider herself to be pluralist in the sense developed here.

Since the terms “monism” and “dualism” are unusual in this context, let us briefly survey to what extent the monist and dualist foundationalist assumption has been made by more traditional philosophies of mathematics. Plato is an obvious example of a platonist monist, and so is Gödel.[5] Plato founded mathematics, especially geometry on the Forms. Gödel founds set theory on ZF set theory and the underlying truths to which it, and its extensions are responsible.

[3] Strictly speaking it is a relevant logic, which has truth-value gaps, and a paraconsistent logic has truth-value gluts. I need both, and the logic will also be more general than either a relevant or a paraconsistent logic. For lack of a better name, I still call it a paraconsistent logic, since one of the more interesting features is the presence of truth-value gluts.

[4] There were other philosophies of mathematics around too, such as conventionalism. However, these four have become the canon.

[5] I assume familiarity with the basic ideas of these positions, so I shall not discuss them further. For a good sketch of Gödel’s platonism see Maddy (1997, pp. 89–94).

A more nuanced example of a monist is Brouwer, who wants to found mathematics on a firm epistemological footing, where mathematics obeys our mathematical intuitions. Brouwer's philosophy does not neatly fit as a "monist" philosophy since he does not propose a formal foundational theory. Nevertheless, his attitude is still monist, since he founds mathematics in intuition. Brouwer resists giving a formal representation of intuition because he distrusts formal representations as flawed.

> Used as a means of communicating thought to others, language is bound to remain defective [*mutatis mutandis* for formal representations of mathematics], given the essential privacy of thought and the nature of the "sign," the arbitrary association of the thought with a sound or visual object. (van Stigt 1998, p. 7)

Instead, for Brouwer, mathematics is really carried out in the mind.

> Intuitionistic mathematics is a mental activity and for it every language, including the formalistic one, is only a tool for communication. It is in principle impossible to set up a system of formulas which would be equivalent to intuitionistic mathematics, for the possibilities of thought cannot be reduced to a finite number of rules set up in advance. (Mancosu 1998, p. 311; quoted from Heyting 1930)

For Brouwer, the founding intuition is not thought to be subjective and personal.[6] For this reason, he does not accept rival but equally legitimate intuitions. *Our participation in intuition* is personal and subjective but mathematical intuition *itself* is not, since Brouwer is drawing on a Kantian notion of intuition.[7] It is this "objective" intuition which founds mathematics for Brouwer, and since it is objective and unique, this makes Brouwer a monist.

An example of logicist monism can be found in *Principia Mathematica*, (Whitehead and Russell, 1997). They sought to give a foundation for most of mathematics in a logical type theory. The type theory was not proposed as "the new way of doing mathematics" or as "the new notation to adopt". Whitehead and Russell understood that it was impractical for mathematicians to carry out all of their work in type theory, they were not fanatics; but they thought it important to develop a logical foundation in which, *in principle*, mathematics could be carried out. This was enough to make the philosophical point that mathematics is really only logic.[8]

[6] 'Intuition' here is meant in the Kantian sense of a shared intuition in which we all participate. Degrees of participation might differ from one person to the next, but the mathematical intuition itself does not.

[7] Of course, philosophically, this is quite problematic, especially when Brouwer takes it upon himself to not recognise some mathematics (as accepted as such by others) as legitimate. He is not making the simple minded argument that "if I can't understand it, then it is not legitimate mathematics". However, he does end up restricting mathematics to what we would call today "effective" mathematics. This is difficult to defend in light of the problems with the Kantian or Brouwerian notion of intuition.

[8] Other examples of monists are: Curry, Dummett, Tennant and Hellman. Most of twentieth century philosophy of mathematics is monist, and this is partly because of an underlying idea that in order to be counted as a philosophy of mathematics at all, one has to give a unified philosophy of the whole of mathematics, and to do this, there has to be one founding discipline.

Amongst the standard philosophies of mathematics, we also find dualists.[9] These include Frege, Hilbert and Cantor. Frege tried to prove that analysis, arithmetic and logic are analytic, whereas geometry is synthetic. We have two systems of mathematics working in parallel: the arithmetic/analysis system which is founded in logic and the geometrical theories[10] which are somewhat corrupted, since they require Kantian-type spatial intuition. Hilbert too was a dualist, but like Brouwer, his case is a little nuanced. Hilbert distinguished between real (finitistic) and ideal mathematics. Unlike Frege, Hilbert did not think that the two realms were necessarily irreconcilable. The hope behind his programme was to show that even ideal mathematics was real[11] (see Giaquinto 2002, pp. 142–151). So, arguably, while his programme was set up to ultimately vindicate monism, he recognised that mathematics in his day was (hopefully only temporarily) dualist. Cantor too divided mathematics into two realms: the mathematical realm of the finite and infinite, and the more metaphysical realm of "absolute infinite multiplicities" (see Giaquinto 2002, pp. 42–43). The realm of the finite and the infinite includes multiplicities on which we can perform mathematical operations. In contrast, "absolute infinite multiplicities" cannot be operated upon. They can just be thought up and wondered at. A close modern analogy to "absolute infinite multiplicities" is proper classes. We cannot operate on these, but we can think them up, and say some limited things about them.[12,13] From this brief survey, we can understand how the terms "monist" and "dualist" are being contrasted to pluralism and we can appreciate that many standard philosophies of mathematics are either monist or dualist.

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[9] I am not aware of other writers referring to Frege et. al. as dualists per se, but it is implicit in their writings. I hope that the terminology is helpful.

[10] Frege did not accept all of the geometrical systems as on a par. He thought of Euclidean geometry as "the true geometry".

[11] Note that there is some controversy over what Hilbert's programme really is: to reduce the whole realm of ideal mathematics to the formal finitistic part (in which case Gödel showed that this is impossible), or to investigate and maximally extend the scope of the finitistic realm.

[12] Examples of ways to 'think up' a proper class is to think of a totality or think of the complement class to a class which has been 'constructed' in appropriate (set theoretic) ways. More precise examples are: the proper class of all models which do not satisfy the Peano axioms or all of the ordinals.

[13] Kant is an interesting case. If logic is to be included in mathematics (a little anachronistically), then Kant too is a dualist because logic is analytic, whereas arithmetic and geometry are synthetic. He is not the same as the other dualists because he does not think that analytic truths are better. In fact, he needs synthetic mathematics in order to explain how it is possible for us to do metaphysics rigorously and to allow for the possibility of experience. So he does not show a preference for one of the mathematical foundations, in the way that Frege, Hilbert or Cantor did. Nevertheless, Kant does hold arguments for analytic truths and for synthetic truths to different standards.

15.2.2 The Foundationalist Argument

Let us now give the monist argument which is properly revisionist of mathematics. We shall then point out how this differs from the dualist case. The arguments will be important since we shall refer back to them when we run the negative arguments against foundationalism. To fix an example, consider ZFC (Zermelo-Fraenkel set theory with the axiom of choice) as a foundation for mathematics. Presenting a 'foundation' has two parts: a technical part and a philosophical part (which in turn has two parts). The technical result is achieved through a reduction of all (or most) existing successful mathematics to the foundation. We show, for most areas of successful mathematics, that they can be translated into the language of ZFC, and the theorems or results of the area of mathematics can be generated in ZFC too. In other words, we show that the language and proof apparatus of the reduced area of mathematics is strictly redundant with respect to ZFC. This is not enough to convince mathematicians to cease to work in the language of the reduced theories and to use the proof apparatus of the reducing area. For, the original language was designed to suit that area, and might be much more workable, less awkward, more suggestive etc. Nevertheless, the technical part is achieved since all the philosopher needs to know is that *in principle* it is possible (if a little awkward) to do all the work of the reduced discipline in the reducing discipline.

After we have the technical result, we make two philosophical moves. The first philosophical move is to state something to the effect that mathematics is 'essentially' the reducing discipline.[14] That is, we have captured almost all of mathematics in the reducing discipline, and therefore, all other languages, symbols, supposed ontology of reduced disciplines is strictly (philosophically/conceptually) redundant. Therefore, all we *strictly need* for most of mathematics is the apparatus of the reducing discipline. We have unified mathematics into one foundation. This is quite a philosophical coup!

The second philosophical move is to introduce a normative element; the essence of mathematics becomes a norm for *success* in mathematics. When we make this slide, we judge future "successful" mathematics against the backdrop of the essence capturing theory. If a proposed area of study does not fit into the founding discipline, then it is not "properly" or not "really" mathematics. At the very least it is not successful mathematics.

Dualists will run a similar argument, but it will include an added complication. The technical result will split mathematics into two, the best part and the suspect part. For an example, let us consider Fregean logicism. The two parts are the arithmetic part, and the geometrical part. Similarly, there will be two "essences" in mathematics, in the Fregean case, they will be analytic and synthetic, respectively. "Success" is relative to the different parts, there will be different norms, according

[14]Many philosophers are leery of using the term 'essence', so euphemisms are used instead. Feel free to replace 'essence' with your favourite substitute.

to which part of mathematics one is operating in. For example, a proof in the analytic-arithmetic part has to be able to be turned into a gapless proof. In contrast, a proof in geometry may invoke intuitive gaps (which draw on our spatio-temporal intuitions). The normativity in the dualist philosophy surfaces either when we favour one part of mathematics over another, or when we refuse to consider the purported mathematics which lie outside these two parts (for example, modal operators are seldom included in the language of founding mathematical theories, and modality is often not considered to be part of mathematics). Outlying parts of "mathematics" are either not recognised as mathematics or simply not discussed.

15.2.3 Naturalism

For my negative arguments against foundationalist philosophies of mathematics I shall capitalise on the naturalist insight that the philosopher is not there to set norms for success in mathematics on purely philosophical grounds. Thus, the naturalist rejects the normative move of both the monist and the dualist. Rather, the naturalist philosopher has two responsibilities towards the mathematician. One is to take seriously what the mathematician, himself, says about mathematics, on a philosophical level.[15] The other is to give a philosophy of mathematics which looks at all of mathematics, as it is practiced, and not just some philosophically convenient proper sub-part of mathematics. We now have enough information to run the negative arguments against formalism.

15.3 The Negative Arguments from Naturalism

15.3.1 Argument One: The Foundationalist's Slide from Description to Prescription is Illegitimate

The first argument broadly concerns the role of the philosopher vis-à-vis the mathematician. Implicitly, or explicitly, and to different degrees, foundationalist philosophies endorse the general idea that once the philosopher has developed a philosophy of mathematics, that philosophy should determine the limitations, and the future development, of mathematics. We saw this in the normative phase of the argument for the foundationalist philosophies.

[15] I take this insight from Maddy's development of naturalism. This departs from more Quinean naturalist philosophies who take their cues from scientists, and not mathematicians. I do not go as far as Maddy, in taking second place to mathematics. Instead, I follow Colyvan (2001) in thinking that philosophy of mathematics and mathematics sometimes influence each other. So the philosopher of mathematics is also allowed his input!

Illustrating the slide from description to setting the norm for success, Vopěnka complains that set theory is so powerful, that it comes to delimit our interests in mathematics, so while set theory was proposed as a reducing discipline; once this was done, set theory became a *norm for success* in mathematics: if a proposed mathematical topic is not interesting in set theory, then it should not be interesting for mathematicians.

> Set theory opened the way to the study of an immense number of various structures and to an unprecedented growth of knowledge about them. This caused a scattering of mathematics. [It is interesting that Vopěnka does not say "unifying"!] *Moreover, most results of this kind derive their sense only from the existence of the respective structure in Cantor set theory. Mathematics based on Cantor set theory changed to mathematics* [only being recognised in terms] *of Cantor set theory*. (Vopěnka 1979, p. 9)

In other words, Cantorian set theory became the standard by which proposed mathematics was judged to be "good" mathematics.[16] Today, ZFC has replaced Cantor's set theory as a point of reference. Under the ZFC norm for success, much of category theory is not mathematics, nor is the ramified type theory, nor, ironically, is all of Cantorian set theory. What is important here is that from Vopěnka's analysis it becomes plain that in keeping with a monist attitude we slide from description to norm for success, and that the norm precludes some potential further developments in mathematics, just because they are not recognised as *bona fide* mathematics. Unfortunately for the monist, history has not born out her slide from description to norm setting. Alternatives to ZFC have been developed. Some have been proved to be equi-consistent to ZFC, (and this is much weaker than using ZFC as the norm for success). Other areas of mathematics have not been shown to be equi-consistent, but might be so in the future. Other areas might never be, and might in principle never be able to be (such as, for example, a paraconsistent set theory, where a proposed proof of equi-*consistency* would be unrecognisable to the monist who wanted ZFC to set a norm for success in mathematics). Excluding mathematical developments, by means of norm setting, runs against the naturalist insight that philosophers should (if they are naturalists) want to observe not only what mathematicians say about their subject, but also observe their behaviour.

How do mathematicians think of set theory? It is true that a lot of present day mathematicians take it as a good verification of their work that it can be done in first-order set theory. But this does not mitigate against my point, since there is a difference between using first-order set theory as one, amongst other, means of verification, and counting set theory as the *only* means of verification. There is a further complication which should be addressed. When Vopěnka makes his anti-foundationalist complaint, the naturalist should observe that it was mathematicians,

[16]Vopěnka is highly revisionary of mathematics too. But he proposes a different founding theory. This does not interest me here. What is important is that we should realise that the platonist or realist proposal to found mathematics in set theory is, arguably, normative of mathematics.

not philosophers,[17] who set said norm. It seems then, that, as naturalists, respecting the practice of mathematics, we should give a philosophy that advocates ZFC or Cantorian set theory, as a foundation. According to the naturalist attitude, if mathematicians are setting norms in this way, then the philosophers should take the norm setting seriously. But, even as naturalists, we can be more careful. As we saw with the quotation from Vopěnka, not all mathematicians agree to follow the norm, trivially, since he is an example of a mathematician who does not. So then what are the *philosophers* to do about the rival norms internally set in mathematics? The pluralist observes that the Cantorian set theory norm was temporary. This is why Vopěnka's alarm is illegitimate. Contemporaneous with, and subsequent to, when Vopěnka's was writing the quoted passages, many developments in set theory have taken place. A number of higher-cardinal axioms have been proposed as extensions of ZFC set theory, and the prevailing attitude (I think) is that, in the light of the rival foundations, pluralism has succeeded set theoretic monism.

At this point, the monist might well dig in her philosophical heels. For, the monist starts with the presumption that the reducing discipline is the correct foundation. A philosophical consequence of this use of "correctness" is that the reducing discipline is taken as an a priori norm for success for the whole of mathematics.[18] This runs directly counter to observation of practice in mathematics, and thus to the naturalist. Here the naturalist and the monist part company.

The dualist does not fare much better. For, he proposes a foundation for some part of mathematics, and this part will suffer from the same criticisms. The part of mathematics for which we provide a proper foundation: second-order logic for the Fregean logicist, finististic (real) mathematics for the Hilbertian, all of mathematics save "absolutely infinite magnitudes" for the Cantorian; is good mathematics, the rest is suspect. With Hilbert, we then engage in a project of trying to widen, or determine the scope of and the limitations of the "good" part of mathematics, to minimise the "suspect" part. The naturalist observer of mathematics will disagree with this nuanced normative attitude. He will observe that mathematicians work in both "good" and "suspect" areas of mathematics, and do not always agree that the "suspect" part of mathematics really is suspect. Take, for example, all of the work on higher cardinal axioms. A Hilbertian would find less value in this work than in the "proper" engagement in the Hilbertian programme of reducing the existing bad part of mathematics to the good part, since, for Hilbert, mathematicians

[17] The distinction is, of course, somewhat artificial, and if we do not accept it, then we re-phrase the structure of the foundationalist philosophy appropriately. Many mathematicians are also philosophers, and the same person can play both roles. I follow Colyvan (2001) in not recognizing a clear distinction between philosophy and mathematics, either in terms of persons or in terms of roles. Despite my agreement with Colyvan, it will be useful for the arguments here to adopt this artificial distinction.

[18] An example would be any attempts at intensional logics not counting as part of mathematics, just because "mathematics" i.e., set theory, is extensional, and cannot recognize intensional differences.

should not be *extending* the suspect part![19] Sporting my naturalist hat, I am not sure that the mathematician working on the higher-cardinals would agree! The very notions of "good" and "suspect" mathematics are not happily applied to the practice of mathematicians. So, here, the naturalist parts company with the dualist. In rejecting the normativity of monism and dualism. In this sense, pluralism is anti-foundationalist.[20]

15.3.2 Second Negative Argument Against the Foundationalist: The Argument from Mis-description

As was mentioned in the argument of the foundationalist, the foundationalist begins with the technical result that most of mathematics can be reduced to the founding discipline. This is a twofold mis-description. First, the reduction is sometimes artificial, and therefore not successful. The second mis-description concerns future growth of the foundation. Whatever the founding theory is, 'it' grows. So there is no fixed foundation.[21]

Vopěnka signals the reduction mis-description: "Some (mathematical) disciplines pursued in pre-set-mathematics [mathematics before the development of Cantorian set theory] had to be considerably violated in order to include them in set theory" (Vopěnka 1979, p. 9). Vopěnka cites the calculus as an example of such a violation. There are many other examples. Try proving that $7 + 92 = 99$ in Frege's logic (without using the inconsistency generated from Basic Law V) or in type theory. It is possible to "do calculus" in set theory, but it is so awkward that no one does it. Why? Because the proofs are too long or not explanatory, so we lose sight of what we are trying to do, and much of the proof is very mechanical, and should be skipped, since going through all of the mechanical steps is not informative, and certainly not "doing mathematics". In this way, the reductions do not give the "essence" of what mathematics is about, how it is practiced, what is interesting about it. Nevertheless, the reducing discipline does give some philosophical insights. For example, we might learn, with Frege, that arithmetic is really analytic, *pace* Kant.

[19] 'Bad', of course, is an over-simplification, especially in light of Hilbert's famously stating that he was not willing to be expelled from the paradise Cantor had introduced. Nevertheless, there is a tension in Hilbert's attitudes towards the finitistic and the ideal.

[20] More precisely, it is anti-foundationalist at the level of discourse where the foundational philosophies of mathematics do their work.

[21] Brouwer agrees with this, so in this respect, he too, parts company with the monist. The issue about where Brouwer fits in my account is quite subtle. Where Brouwer and I part company is in his emphasis on intuition. I think that mathematical intuition is interesting, but I disagree with Brouwer that "mathematics (all and only) takes place in the mind". I save this issue for another paper.

Apart from the artificiality of the reduction, there is the second problem of instability of the foundation. Even lovely, all-encompassing, mathematical theories grow. New axioms and techniques are suggested and tried. If we endorse the naturalist attitude, then we can observe (rather than resist) co-variance between "founding theories" and "essences": as the founding theory changes, so the essence changes.[22] For the pluralist, this observation makes a mockery of foundationalism as essentialism. This is not a logically necessary argument, since we could insist on fixing the foundation by reference to one formal theory, and resist new extensions. Rather, the argument is inductive: based on the history of mathematics. Foundational theories spawn new developments or additions to the mother theory, and foster the development of rival theories. No sooner had Whitehead and Russell introduced their simple type theory, than they developed the ramified type theory. Other type theories have sprung up since, some more successful (studied more), than others. After Cantor developed his naïve set theory, rivals were forthcoming: Zermelo-Fraenkel set theory, Gödel-Bernays set theory. Moreover, additions were made to these, with the axiom of choice, the development of class theory, higher cardinal axioms were added etc. Category theory too has seen development.

Looking more closely, we *extend* the foundational theory with new axioms which make a new theory (assuming we individuate theories by the language, plus axioms, plus inference rules). For example, we can extend ZF set theory with the axiom of choice, which gives us ZFC. Moreover, as is well documented, there is considerable dispute over the *admissibility* of new axioms which extend ZF set theory. Admissibility (classically) requires at least consistency with the original theory, but some proposed axioms are independent of the original theory; and therefore we can add the independent axiom, or an axiom which is inconsistent with it. The problem now is to arbitrate between the alternative proposed extensions, since some pairs of new axioms will lead to contradictions. To arbitrate, we have to modify our original notion of the *essence* of mathematics—since it no longer rests in the founding theory. It is strictly broader since we think it can accommodate extensions to the founding theory. Moreover, we have to do this in such a way as to accommodate only one subset of the proposed additional axioms. Witness the debates about $V = L$. If we choose $V = L$, then we preclude a number of other axioms. If we choose $V \neq L$ as a new axiom, then we preclude other proposed axioms. Accommodation, in the face of choices which preclude other additions, is no easy task, since the founding theory itself cannot arbitrate. Here, "choice" relies on some underlying sense of "the" theory—not individuated by a language, set of

[22]There is an important distinction I am glossing over, but it will be addressed in the next section. The distinction is between the presentation of the formal theory, and whatever it is that the formal theory is trying to capture. Here, I am assuming that the essentialist believes that he has in hand a formal theory which captures the essence of mathematics. The technical result is completed. If we draw apart the formal theory and what it is the theory is supposed to capture (which is intentionally different from an intended interpretation), then we might say that the formal theory imperfectly captures the essence, so the formal theory is allowed to 'grow' as and when we discover new aspects to the essence.

axioms and rules of inference—but by some vague intuitions which, one hopes, will become explicit through discovery and formal representation. But these vague intuitions are not good philosophical justifications for foundationalism, since these sorts of intuition vary from one mathematician to the next. We might dress up the intuitions by introducing considerations of fruitfulness, simplicity, elegance etc. But these considerations alone will not do, since, remember, we are providing a foundation, not for generating "lots" of mathematics,[23] or aesthetically pleasing mathematics, but for correct mathematics. If the foundationalist does go ahead, and opts for one extension over another, to fix the "essence" of mathematics, then he shows a weakness in the original monist argument for his first chosen foundation. For, arguing for one extension over another, is a covert admission that he did not have the full essence properly captured in the first place.

15.3.3 The Third Negative Argument Against the Foundationalist: De dicto and de re many Mathematicians Are Pluralist

De dicto many mathematicians are anti-foundationalist. Or, more mildly, they view foundations with suspicion.

> Many working mathematicians (though by no means all) are suspicious of logicians' [and philosophers'] apparent attempt to take over their subject by stressing its foundations. ... [Moreover,] I have been persuaded by Edwin Coleman that foundationalism in mathematics should be regarded with considerable suspicion; or at least that proper 'foundations', ... would be much more complex and semiotical than twentieth century mathematical logic has attempted. In which case it would be arguable whether 'foundations' is an appropriate term. (Mortensen 1995, p. 4)

In conversation, Ali Enayat, Joe Mourad, Jennifer Chubb, István Németi, Hajnal Andréka, Russell Miller and many other working mathematicians have all declared themselves to be pluralist, in some sense of "pluralist". I think that pluralism "is in the air", but it has not been worked out as a whole philosophical position, only as part of other positions.

Moreover, many mathematicians are not only *de dicto* pluralist, many are *de re* pluralist. That is, their behaviour at conferences and in their written work, displays an open-mindedness and acceptance of alternative foundational theories. More than this, in their proofs and methodology, mathematicians will often avail themselves of whatever hypotheses are useful and can support the desired result. According

[23]We should be careful about the accolade 'fruitful'. It pre-supposes quantifying over mathematical results. For, adding any axiom will add an effectively enumerable number of new theorems, so axioms are equally fruitful. Alternatively, we might count only "important" new results, but how these are determined/chosen is again a problem; at least at any given time, since we might later discover that a theorem or result is important only many years later.

to Thurston, for mathematicians, the “*reliability* (of proof) does not need to come from mathematicians *formally* checking formal arguments (so working within one foundation): it comes from mathematicians *thinking carefully and critically* about mathematical ideas.” These ideas are not restricted to the ideas found in one foundation. The choice of which method or result to use in a proof is pragmatic, and there is a sense in which said method or result is considered to be trustworthy because it is “quite good at producing reliable theorems that can be solidly backed up.” (Thurston 1994, p. 171). Real “mathematical” proofs are non-deductive derivations of plausible hypotheses from problems, in some sense of “plausible”, where a problem is an open question; a hypothesis is any means that can be used to solve a problem (Cellucci 2008, p. 2). And, more important, a hypothesis is said to be plausible if and only if it is compatible with existing data—which includes any mathematical results and notions available at the time of inquiry (Goethe and Friend 2010).

This runs directly against the picture drawn by the monist philosophies of mathematics, but maybe the dualists are more accommodating. We might think that, as good dualists, mathematicians avail themselves of the better part of mathematics, when they can, and use the more suspect part with an uneasy conscience, such as when a constructive mathematician knows very well that there is a constructively unacceptable proof for a result, but, nevertheless, believes that the result is true, and works on giving a constructively acceptable proof of the same result. Of course this happens, and there are *bona fide* dualists amongst mathematicians. But the monist or the dualist stories are not the only stories to be told, and many mathematicians completely disregard the advice of the dualists. There is no “bad conscience”. In other words, for many mathematicians, the purported distinction between good and suspect mathematics completely dissolves. In the light of the *de dicto* and *de re* observations, the naturalist aspect of pluralism makes the pluralist anti-foundationalist.

We have some *prima facie* evidence for pluralism from the claims and behaviour of mathematicians. However, this is simply an observation about the state of play in mathematics today. As philosophers we have to decide whether or not to take the observations seriously, or to think of them as a temporary glitch. We might excuse the observations on the grounds that the working mathematician is simply “not a very good philosopher of mathematics and has not thought through the implications of her pluralism,”[24] or is engaged in “cognitive dissonance” or treats mathematical theories as tools and therefore her pluralism is due to a lack of philosophical thought about the matter. This might well be true in some cases. But, as philosophers, we should say more.

[24]“. . . what the mathematician says [about the philosophy of mathematics] is no more reliable as a guide to the interpretation of their work than what artists say about their work, or musicians [about theirs].” (Potter 2004, p. 4). Even if we do not quite have such a strong point of view, it remains that mathematicians express very different philosophical attitudes. At the risk of being repetitive, my personal observation is that most mathematicians are pluralists.

Pluralism motivated by naturalism does not prevent a philosopher or mathematician from working within the strictures of a philosophy, but we want to distinguish between being wedded to a theory for technical reasons, historical reasons or reasons of personal taste, on the one hand, and being wedded to a philosophical or mathematical theory for foundationalist reasons. For the pluralist, the normative force of foundationalist philosophies is confined to a class of theories. So, it is qualified, and not absolute. Speaking pluralistically: the normativity of a philosophical position stays internal to that philosophy, and is limited to the scope of the foundation. This should not upset traditional philosophers of mathematics too much: they have, after all, a whole class of theories at their disposal. Moreover this might be a proper class. So they have quite a large play-ground. But, at the end of the day, the pluralist asks them to admit to the parameters of the play-ground. In other words, according to the pluralist, what the foundationalist may not do is claim to give a philosophy for "all" of mathematics. There is perfectly legitimate and interesting mathematics outside the foundation.[25]

For the pluralist, there is no absolute mathematical truth, only truth within a theory. There is not one essence of mathematics characterised or represented, by a particular mathematical founding theory. Pluralism in mathematics is a challenge for philosophers, since it is foreign to the more traditional foundational philosophies of mathematics. However, it is not an insurmountable challenge. Shapiro, for one, considers himself to be a pluralist, and supplies a philosophy "without foundationalism".

15.4 The Second Motivation for Pluralism

15.4.1 Shapiro's Structuralism

Shapiro's structuralism is interesting to us because it is neither monist, nor dualist, it self-avowedly anti-foundationalist and pluralist, and advocates a "mathematics-first" attitude which is close to the naturalist insight we adopted in the previous section to make our negative arguments in favour of pluralism.[26] I think that his is the closest position extant to the pluralism supported here. Shapiro's structuralism is a philosophy of mathematics where the notion of "truth" is always qualified by that of "in-a-structure". He uses the highly expressive language of second-order logic to capture important mathematical concepts, such as "is Dedekind

[25]This is a comment about the state of play today. It might turn out one day that we have a unified foundation, which encompasses all of mathematics.

[26]The title of one of Shapiro's books is: *Foundations Without Foundationalism*. There is a sort of foundation, based on second-order logic and model theory. I am calling this an "organisational perspective" to distinguish it from a unifying revisionist foundation.

infinite"[27] and model theory to pick out structures (which, for Shapiro, are what mathematicians are interested in). For Shapiro, model theory is not a foundation, but an organisational perspective allowing for the clear individuation of mathematical theories, and for the comparison of various theories/structures from the point of view of chosen further meta-structures.[28] There is no ultimate structure, on pain of paradox. There is no absolute perspective. No structure is ultimately favoured over others (since model theory does not have one global structure). Model theory is not an axiomatised theory, and therefore has the flexibility to grow, without jeopardising stability. What is admitted as a structure will, undoubtedly, change over time, since model theory is a developing theory.

There are two types of individuation of theory taking place side-by side. We can either individuate theories in terms of the language of the theory, the proof theory and axioms, which is, roughly, how the model theorist thinks of a theory. A structure is just a set of objects together with some structure imposing relations which bear on the objects. Equally, we can individuate theories, in terms of an underlying idea which is not necessarily known to be fully captured by the formal representation of the theory. For example, if we think of model theory, then the formal representation is not yet fully achieved. In Shapiro's structuralism: model theory itself should be individuated in the latter way, since it is a growing theory; whereas particular structures should be individuated in the former way.[29] This is a very pluralist way of speaking. For the pluralist, the model theorist is able to "see" a lot of mathematics, make sense of it, organise it within the strictures of his model theory and make contributions and offer insights. He individuates structures up to isomorphism and recognises all concepts expressible in a second-order language.

The pluralist will now detect a limitation. Shapiro's structuralist can see quite a lot of mathematics, but not all of mathematics. As a result, Shapiro's pluralism is

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[27]The definition of Dedekind infinite is that: a set is Dedekind infinite iff it has a proper sub-set with which it can be placed into one-to-one correspondence. The natural numbers are Dedekind infinite, as are the integers, the rationals, the reals and so on. In contrast, finite sets have no proper sub-set which can be placed into one-to-one correspondence with them. To capture the notion of Dedekind infinite, we need the expressive power of second-order logic. See Shapiro (1991, p. 100). The formula for set X being infinite is: $INF(X) : \exists f[\forall x \forall y(fx = fy \rightarrow x = y) \& \forall x(Xx \rightarrow Xfx) \& \exists y(Xy \& \forall x(Xx \rightarrow fx \neq y))]$. This is read: There is a function which is such that if (two) of its values are identical, then the (two) arguments are equal. Moreover, the function operates on a proper subset of the set X.

[28]As previously mentioned, the title of Shapiro's first book on structuralism is: *Foundations Without Foundationalism The Case for Second-Order Logic*. Note the "Without Foundationalism". Foundationalism is identified with the monist or the dualist. Shapiro is anti-foundationalist in the sense that all mathematical theories which he recognizes are on a par. Insofar as he has a foundation, Shapiro's "foundation" is model theory. Model theory allows him to individuate mathematical theories (as structures). The model theory does not favour one structure as against another.

[29]This could be turned into a criticism of Shapiro's structuralism. It is inspired by Potter and Sullivan (1997, pp. 135–152). The criticism, adapted from the Potter and Sullivan paper is that Shapiro makes different ontological and metaphysical claims concerning individual models, on the one hand, and model theory itself, on the other. So there is a double standard.

restricted to what is recognized through the lens of model theory and what can be expressed in second-order logic, and this lens then sets a norm for what is to count as successful mathematics. In the spirit of friendly banter, we might say that Shapiro, too, is guilty of some of the sins of foundationalism.

15.4.2 Moving Beyond Shapiro's Structuralism

Where Shapiro and I part company is over the very important issues of what is to count as success in mathematics and what is of interest to the philosopher. The pluralist takes her cues from observed mathematical practice. So she can recognise parts of mathematics, which some mathematicians count as successful, but which are not recognised by model theorists (Friend 2006, 73). These include: intensional logics,[30] mathematical theories which are still in a stage of development and paraconsistent mathematics. Unsuccessful mathematics are trivial mathematics. Shapiro and I agree that these are unsuccessful. The difference between us, on this issue, is that the pluralist thinks that trivial theories are interesting and useful in mathematics. I shall call all of the above "bad" mathematics, since they are all implicitly rejected by Shapiro and each is rejected by most other philosophies of mathematics. The philosophical move of rejecting "bad" mathematics offends against the naturalist insight, runs the risk of instability or of begging the question. The pluralist philosophy developed here is more stable than Shapiro's pluralism and the more traditional philosophies as well[31] since the pluralist has the flexibility to adapt to changes over time in what counts as successful mathematics. Odd theories suddenly find an application; some obscure result proves useful to a more central mathematical concern. Theories which were viewed as highly suspect come to be accepted in more main-stream mathematics, such as the study of non-standard arithmetic. More important, there are revolutions in mathematics, such as the discovery on non-Euclidean geometries, or of the incompleteness results. These changes radically alter our conception of mathematics.

[30]Model theory is extensionalist, and only individuates structures and objects in those structures "up to isomorphism", only recognizing certain properties (predicates, relations, functions) as "counting" for mathematics. But we find, in mathematical practice, that considerations, not recognized by model theory, are also pertinent to mathematics.

[31]One might think that I am being somewhat unfair, and ignoring a lot of philosophical activity. For example one might point out that Russell was much aggrieved by the paradoxes, and theorised a lot about them. And Russell's investigation into the paradoxes shaped his philosophy and formal system. Moreover, some very important philosophical work has been done in looking very closely at Frege's trivial theory—such as the work of Dummett, Wright and Heck. I appropriate such activity, and call it pluralist. What is anti-pluralist is any accompanying norm setting revisionism. So, we should be carful about our interpretation of the intension behind the excellent work cited above; we might say that these philosophers engage in pluralist work despite themselves.

To remedy the instability, we can be more circumspect by qualifying our measure of success by means of a temporal index, then any "rejection" is made relative, stable and harmless. For example, we might say that we are giving a philosophy of early nineteenth century mathematics. Provided we are historically accurate, the pluralist will not object. But the pluralist is more ambitious than this.

The rejection of bad mathematical theories might also beg the question, as when the reductionist foundationalist philosopher re-trenches and says that whatever fails to conform to her conception of what counts as successful mathematics is, by definition, unsuccessful. That is, she sets an a priori norm for success in mathematics. But the force of such an argument is limited, for it begs the question against the naturalist perspective. My diagnosis is that there is an inevitable tension between the naturalist attitude and the desire to give a traditional philosophical account of successful mathematics.

15.4.3 Prejudices: Optimal Versus Maximal Pluralism

To distinguish "success", from the "rest" of mathematics, while remaining pluralist, let us distinguish between an *optimal* pluralist philosophy and a *maximal* pluralist philosophy. The optimal pluralist gives norms for philosophically well motivated theories, i.e., for "successful" mathematics. Shapiro is an example of an optimal pluralist. There might be several competing norms. They might include: consistency, constructive considerations, definitions of validity, search for a robust ontology etc. In contrast, the maximal pluralist is maximally descriptivist: tries to philosophically account for the whole corpus of mathematical activity. The maximal pluralist is loath to set a norm for success in mathematics, and he will *accommodate, account for, or study* "bad" theories (without slipping into triviality; see the next section). Under the maximalist attitude, the pluralist can, of course, *observe* norms, but does not arbitrate between competing norms (unlike the optimal pluralist). We had a taste of this earlier. For this reason, the pluralist has to entertain, what were traditionally thought of as "bad" mathematical theories.

Since, in this paper I am advocating maximal pluralism, let us give the motivations for considering "bad" mathematics at all. Beginning with paraconsistent mathematics: there is now a corpus of literature on paraconsistent logics and paraconsistent mathematics. These are taken seriously by some mathematicians.[32] A philosophy of mathematics which does not treat of these is incomplete, and

[32]There is plenty of sociological evidence for this. Witness publications by "major" publishers, both as books and in journals; numbers, sizes, and sections newly contained in conferences. One telling example is the history of the world congress on paraconsistent logic.

violates the naturalist attitude.[33] The second sort of important (and potentially successful) mathematics, for the pluralist, is nascent theories. These are ignored by other philosophies, so count as "bad" mathematics for everyone else. These are theories which are still in progress. All theories go though a stage of "construction", "becoming" or (more platonistically) "coming to be known" or "coming to be formally represented". Depending on how we individuate theories in mathematics, we might even say, with the Gödelian optimist,[34] that set theory is nascent! More specifically: if we do not individuate theories in the standard way in terms of a language, a set of axioms and rules of inference, but rather in terms of "some theory to be discovered" or as a "construction of the mind", or as having an "informative semantics",[35] then many mathematical theories are nascent. A philosophy of mathematics which did not accommodate nascent theories would be found lacking by the pluralist.

Trivial theories are the most controversial of the "bad" theories. A trivial mathematical theory is one where every well formed formula in the language of the theory is true. They are distinguished from each other by their language.[36] For a trivial mathematical theory two factors have to be in place. The underlying logic of the theory has to be classical (has to allow *ex falso quodlibet* inferences) and there has to be a contradiction derivable from the axioms using the rules of inference of the theory. Historically, there are three (to my knowledge) mathematical theories which had a profound impact on mathematics or logic, and were found to be trivial. These are: Cantor's naïve set theory, Frege's formal theory of logic and the first version of Church's formal theory of mathematical logic. All three had profound repercussions on subsequent mathematics. None led to the collapse of "all

[33] Shapiro's pluralist structuralism cannot recognize paraconsistent logics and mathematics, since they cannot have a structure, since the logic Shapiro uses is classical second-order logic, and only consistent theories have a model—in a classical theory.

[34] The Gödelian optimist thinks that in the end, given an open problem, we shall discover a technique to make an absolute decision about that problem. Tennant has several good discussions about the Gödelian optimist in Tennant (1997).

[35] For the distinction between an "informative" and a merely "technical" semantics see Priest (2006, p. 181). A semantics is uninformative if it is developed simply for technical reasons, to prove consistency (in a classical setting). In contrast, a semantics can be informative in two ways. Either it is informative in the sense of being the intended interpretation. That is, the semantics is developed with complete reference to some logical or mathematical meaning. In these cases the syntax is developed after, or conceptually comes after, and is developed soundly—to be in harmony with the semantics. The more subtle case of an informative semantics is found when we developed a semantics for technical reasons, so we know that the syntax is consistent. But then we find an application, or an interpretation, that suits the syntax. This semantics is informative *post facto*.

[36] A trivial theory is got by espousing a classical logic (i.e., which allows *ex falso quodlibet* inferences) that contains a contradiction. We then have the result that every sentence written in the language is provable. If the languages of (what we suppose are) two trivial theories are the same, then the theories are the same. However, if (what we suppose to be) two trivial theories have different languages, then they can be distinguished from each other. Some sentences will be true in one, but not recognizable in the other. I thank Priest for pressing me on this point at the Logica conference 2005.

of mathematics". None led even to the collapse of "that part of mathematics infected by the theory".[37] The important proofs contained in the above trivial theories do not proceed as *ex falso quodlibet* inferences, which is one of the reasons *why* the theories are considered to be important despite their being trivial. The good trivial theories are studied and trawled for good ideas and insights. Contrast these trivial theories to Prior's "Tonk" theory (Prior 1960, pp. 8–39). This is not an interesting theory because we see immediately that it is inconsistent. Not all inconsistent theories are uninteresting as witnessed by the work of Dummett, Wright and Heck on Frege's trivial theory. In general, after spotting an inconsistency, mathematicians try to fix the theory with *minimal* changes. To ignore the mathematical and philosophical influence of such theories, again, would be to provide a philosophy of mathematics which is lacking in scope.

Having considered various "bad" theories, and finding that they should not be dismissed as not part of the mathematical corpus by the naturalist, it follows that we should try to adopt a maximal pluralist attitude, and not only an optimal pluralist attitude. The virtues of maximal pluralism are greater inclusion and stability. We should turn to the criticisms of maximal pluralism before giving a more detailed account of the maximal pluralist view.[38]

15.4.4 Critique of Maximal Pluralism from Trivialism

The criticism runs: if the maximal pluralist is so loath to set norms or arbitrate between existing norms, then everything goes, and the position is actually trivialist.[39] For any theory, we can find a meta-theory or an attitude that endorses the theory, so there is no real philosophical judgment, there are just relative judgments or descriptions. No one wants a trivial philosophy. We might end up with a trivial philosophy because we took seriously some trivial mathematical theories. The language of these theories is a proper sub-part of the philosophy, so the triviality spreads through the philosophy. Oddly, for the trivialist, but it comes as a relief to

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[37]For example, we did not stop doing arithmetic when Russell discovered paradox in Frege's reduction of arithmetic to logic. This is also evidence against trivialism.

[38]I should like to thank Norma B. Goethe and Göran Sundholm for sustaining some of these criticisms against me in conversation. Note that they were much more delicate and kind in their tone than what is reported in the imagined quotation!

[39]Trivialism is the position that every grammatical, categorically correct, sentence is true. A sentence is categorically correct if it makes no "category mistakes": where we confuse what type of object we are talking about. For example, it makes no sense to talk of water dreaming, angry chairs, kilograms travelling etc., unless, of course, we are in a fantastical/super-natural setting or using a metaphor. Trivialism is the dual of scepticism: where every grammatical, categorically correct sentence is subject to doubt. However, unlike the sceptic, the trivialist position does not "implode" since its own very trivialism is true. It is an entirely robust and stable position. However, it is highly uninteresting to maintain it.

the maximal pluralist that this has not in fact occurred.[40] In practice we observe a clean break between trivial mathematical theories and philosophical positions which entertain them. The infection does not spread to the philosophy.

A trivial philosophy of mathematics holds that every well-formed mathematical formula, in any language of mathematics is true (and its negation is true), and any philosophical sentence about mathematics is also true.[41] Anything goes, and all judgments are as good or as correct as the next.[42] The notion of judgment is so degenerate that we might say it is absent, since it is not discriminating. Trivialism is pretty hopeless *as a philosophy*, although it is very easy to defend/maintain verbally! The main criticism against it which is pertinent to this project is well expounded in Priest (2006, pp. 68–69), and it is an argument from meaning. It is not clear that a trivialist can mean anything by his utterance or written statement, since there is no recognisable *judgment* attending sentences. They are all true, so there is no distinction between true and false, since they are also all false. So there can be no meaningful intentionality,[43] since there is no distinction between a belief, a known fact, a subject of fear, desire or what have you. Since there is neither judgment nor intentionality attending the use of language, the philosophy of mathematics being presented is degenerate. According to someone who is not a trivialist, the trivialist theory renders[44] mathematics and philosophy meaningless. Provided that we hold that some wffs are false (and not true), we do not have a trivial theory. An example of a wff which the maximal pluralist holds false (and not true) is: $\vdash_{PA} 2+9=34$.[45] We read this: "in Peano Arithmetic, two plus nine equals thirty-four". This is enough to distinguish the maximal pluralist from the trivialist about arithmetic, the whole of mathematics or philosophy. Note that we have not *defeated* the trivialist. Rather, we have simply shown that maximal pluralism is distinct from trivialism which is

[40] I know of no discussion of trivialism which has degenerated into trivialism, except in moments of jest.

[41] We might come to this position by supposing, say, that ZF contains a contradiction. More precisely, we need a theory which is considered to be foundational to mathematics, we need for it to be a classical theory: allowing *ex falso quodlibet* inferences, and we need to be able to derive a contradiction from the axioms using the rules of inference.

[42] For a good discussion of trivialism see Priest (2006, pp. 56–71).

[43] There might, of course be reported or avowed intentionality, such as when the trivialist reports: "I believe that snow is white". He will equally assent to: "I believe that snow is any colour but white."

[44] The trivialist will "hold", in the sense of assert, any position. This is not the point. Trivialism arises from the idea that mathematics is classical and there is a contradiction in mathematics, and therefore (under our old classical reasoning) all of mathematics is true, we then get to the meaninglessness of any particular mathematical statement, and wallow in our degenerate theory. There is a sequence to the reasoning which gets us to the degenerate position. Once there, reasoning, as such, is impossible.

[45] The trivialist will, of course, agree that "$\vdash_{PA} 2+9=34$ is false", since the trivialist will agree to everything. The maximal pluralist will disagree that " "$\vdash_{PA} 2+9=34$" is true". (I think that) this is all we need is to distinguish the positions.

enough to fend from the criticism that maximal pluralism is a trivial philosophy. The differences will be fleshed out when we look more closely at the paraconsistent logic underpinning pluralism.

15.4.5 Critique from “Disdain for Sociology”

A less technical critique comes from a “disdain for sociology”. An imagined interlocutor might object: “Michèle, if you give up on giving a philosophical account of successful mathematics, then you let in all sorts of abominations: trivial theories, crankish scribbles, numerology... Moreover with your moral-high-ground pluralism you are loath to judge, rate and order rubbish-posing-as-mathematics as quite inferior to very good and fruitful mathematics. What sort of a philosophy are you hoping to give here? It might be stable, but it will also be empty/uninteresting. Have you lost all philosophical ambition? Have you turned Wittgensteinian (later, and only under some interpretations)? Are you not left with only doing sociology, history or historiography of mathematics since your naturalist attitude only allows description?” There are a number of complaints included in the imagined quotation. The interlocutor accuses the maximal pluralist of philosophical, or logical, degeneration in the sense that whatever philosophy there was initially threatens to “degenerate” to the rank of sociology. The pluralist has a response. There is, in fact, lots of philosophical work to be done under a pluralist banner. The best way to see this is to look at the philosophical position of pluralism, and pay special attention to the sort of judgments which pluralists make. Even at the level of description, there is good description, and mis-description. Aside from the descriptions, there are *bona fide* value judgments, and these are modeled by the semantics of the logic.

15.5 Maximal Pluralism

15.5.1 The View

There are three levels of philosophical activity and three corresponding levels of mathematical activity. The first level of both mathematics and philosophy concerns particular results in mathematics. Examples on the mathematical side are particular theorems, lemmas, definitions and proofs. On the philosophical side we have discussions concerning particular results. For example, we might discuss: theorems, definitions or the completeness of a theory, a compactness result or a proof in a theory. These might include discussions about limitative results, since these results are given within a particular mathematical theory. Thus, we might include the proof of equi-consistency of two theories at this first level. At the second level, we have full mathematical theories, or theories which are being developed. Examples of fully developed theories are: Euclidean geometry, first-order arithmetic, modal logic S4, Zermelo-Fraenkel set theory and Topos theory. The larger of these mathematical theories are theories *within which* we make mathematical comparisons between other (smaller)[46] theories. For example we might show the reduction of one theory to another, we might give an equi-consistency proof between two smaller theories, we might show embeddings, and so on. The larger whole theories are often thought of, by philosophers, as foundational and are often accompanied by a philosophy. For this reason, on the philosophical side, at this level, we have the more traditional philosophies of mathematics, such as: set theoretic realism, Maddy's set theoretic naturalism, the constructive philosophies, logicism and so on. In fact, just about every philosophical position in the philosophy of mathematics is found at this level. *At the third level, we have pluralism as a philosophy which is pluralist towards the activity which takes place at the first and second level.*

The logic accompanying pluralism is paraconsistent.[47] The paraconsistent logic provides the "space of reason"[48] within which mathematical activity can take place. The view is that the forces of mathematical history: particular inspirations, insights, the publication of articles, the disseminating of mathematical information, conspire to trace complex paths which merge and split within the paraconsistent space of reason. As a philosophy, pluralism favours the pluralist attitude over other philosophical virtues. The pluralist attitude combines anti-foundationalism while maintaining an interest in foundations—as good mathematical theories in their own right, and as accompanied by philosophies of mathematics which affect the development of mathematics. A pluralist who took an interest in "foundations" would have plenty to say about axioms which are independent of a foundational mathematical theory, such as the higher-cardinal axioms. The pluralist observes the bifurcations of set theory with the addition of different sets of axioms. The pluralist will not feel any need to favour one extension over another. Note that this demurring is not due to "lack of knowledge", but, rather, to an acceptance that in the present state of play in mathematics, there simply is no definitive mathematical way to arbitrate between theories. There is no unique absolute perspective.[49] Since some pairs of mathematical theory contradict each other, pluralism requires a paraconsistent logic at the third level.

[46]The terms "smaller" and "larger" refer to the expressive power of a theory. Roughly, the more theories can be reduced to, or embedded in a theory, the more expressive power the theory has.

[47]There are actually different versions of pluralism, varying with choice of underlying logic, but to simplify, here, I give only one logic, which in this case is paraconsistent.

[48]This is not a term I like, but it is useful in this context.

[49]It might be instructive to compare this attitude to Gödelian optimism, which is the thought that in the end, given an open problem, we shall discover a technique to make an absolute decision about that problem. Tennant has several good discussions about the Gödelian optimist in Tennant (1997). In contrast, here we have the agnostic, who demurs. This character is either a pessimist (the demurring is then based on an inductive argument, and the pessimism might be reversed in a particular instance), or the character is a principled agnostic. It is the principled agnostic position which is explored in this paper.

The logic helps us to individuate and reason over theories, foundations and philosophies of mathematics. The virtue of paraconsistent logic is that in deploying it, we can cope with contradictions within and between theories, and this is very important. Byers remarks: “No description of mathematics would be complete without a discussion of its *subtle* relationship to the contradictory (my emphasis)” (Byers 2007, p. 81). Our only hope of engaging in a subtle discussion is through the use of a paraconsistent logic, since the more traditional philosophies are anything but subtle in this respect! The same author remarks later:

> Moreover, paradox has great value. Thus paradox should be seen as a generating force within the domain of mathematical practice. ... Where do that power and dynamism come from? Well, they come from ambiguity, contradiction and paradox. These things are therefore of great value. They need to be unravelled, explored, developed, and not excised. (Byers 2007, p. 112)[50]

Ambiguity, paradox and contradiction need to be unravelled if one wants to give an account of the practice and development of mathematics. This is partly a psychological task, but it is also philosophical, since it raises epistemological questions largely ignored by traditional philosophies of mathematics. For, if Byers is correct, then it is through awareness of, and confrontation with: ambiguity, paradox and contradiction that we develop mathematics. They are epistemological tools, not strict limitations or parameters on reasoning or on the corpus of mathematics.

As a step towards developing this epistemological sophistication, the logic the maximal pluralist uses is a little different from regular paraconsistent logics in that one type of variable ranges over whole mathematical theories (sets of wffs)[51] and another type of variable will range over classes of mathematical theory.[52] For, it is with these units that we meet conflict and contradiction. With careful use of the Routley/Priest Characterisation Principle,[53] which we shall explore in the next section, the maximal pluralist can compare “bad” theories to each other: cordoning off the trivial theories, and comparing mutually contradictory theories. We can even compare trivial theories to each other. To distinguish between different trivial theories, we look to the differences in characterisation between, say, Frege’s theory and Cantorian set theory. The difference lies in the vocabulary and languages of the theories. In both cases we have, what are sometimes called “consistent contradictions”.[54] In contrast, if we compare classical Euclidean geometry to

[50]Note that Byers makes no mention of paraconsistent or relevant logics. I therefore point out that he, himself, is not advocating a paraconsistent point of view or anything of the sort. Nevertheless, the quotations, and in many other places in the book, I found support for the position advocated in this paper. I do not know what Byers’ reaction would be to the mention of paraconsistent logics.

[51]It is not very different, for, we could imagine a very long conjunction of wffs, each conjunct of which is put in normal form and arranged in some ordering.

[52]To preserve the pluralism, we allow all symbols of mathematics to be included in the language. The language is growing, not fixed.

[53]The principle is: *A theory is just whatever it is characterized to be*.

[54]The notion of “consistent contradiction” was introduced to me by Marcelo Coniglio in the presentation of Coniglio and Carnielli (2008). “Consistent contradictions” are explosive. Anything

projective geometry, we find that, together they give us a “normal contradiction”; one we can resolve by keeping the theories separate. This sort of observation, made by a pluralist, is enough to parry the accusation, made by the imaginary interlocutor who accused the pluralist of precluding judgment of theories (letting in abominations). There is plenty of properly philosophical work to be done for the pluralist. However, we should be aware that the use of a paraconsist logic brings with it its own philosophical stamp concerning the “truth” of a mathematical theory.

15.5.2 The Semantics of the Logic Underlying Pluralism: Judgment Values

The semantics of the logic concerns the first and second level of analysis.[55] We use the semantics to make sense of the following judgments, where we have a notion of truth indexed to a philosophy X. The units ‘y’ being judged are whole mathematical theories (sets of wffs) or individual wffs. Philosophies and mathematical theories are individuated by the *characterisation principle*. The original characterisation principle, as developed by Routley is:

An object is characterised by its properties.

Which object it is depends therefore only on its properties, so an object just is (is individuated by) its characterisation (see Priest 2003, p. 4). We use a more general version of the principle: namely, that

A mathematical theory just is its wffs.

Note that the set of wffs might not be closed (under certain operations) since we might not have decided which are the admissible operations for generating new wffs. The truth of a mathematical theory is indexed to a philosophy at the second level. A philosophy, X, is not as easy to characterise, since developing a philosophy is not as rigorous and formal a task as developing a mathematical theory. So, we shall have to be more circumspect. A philosophy of mathematics (at the second level) just is the set of sentences used to characterise the philosophy. The easiest way to do this is to refer to a definition, if there is one, or a chapter in a book. Thus we have the philosophy of whatnot as characterised by whoever in chapter whatever of some book. More broadly, how we determine philosophies might be no

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can be derived logically from them. These are contrasted to “normal contradictions” from which not everything follows, only a very few things follow. With normal contradictions, we have a very controlled explosion. The use of the word “normal” refers to the fact that we encounter what, at first appear to be, contradictions quite frequently in “real life”, but we deal with these quite well.

[55] I prefer the term “judgment values” to “semantics”, since “semantics” comes with too many connotations about giving truth-values, interpretations and domains of interpretation. “Judgment values” are part of the semantics, in a broad sense of “semantics”.

easy matter, whence the subtleties, intricacies and delights philosophy. Therefore, the characterisation is not always fixed, but we can usually fix it temporarily for the sake of coming up with some judgments.

The value judgments are as follows. In all of the clauses, fallibility and revisability are understood. That is, words like "success", "recognise", "determine", "wrong" make the judgments revisable.

1. "y is a true mathematical theory given philosophy X" gets the value judgment (T), iff X recognises y as a successful mathematical theory.

   An example of such a judgment is Euclidean geometry, given Zermelo-Fraenkel set theoretic realism is true. Projective geometry is also true if we are Zermelo-Fraenkel set theoretic realists. Intuitionist ordinal arithmetic is also true if we are Zermelo-Fraenkel set theoretic realists, as is any theory which can be reduced to Zermelo-Fraenkel set theory.
2. "We do not yet know if y is true given philosophy X", or "the philosophy X is neutral with respect to the mathematical theory y" gets the value judgment (U) (for "unknown" or truth-value gap), iff philosophy X is not able to determine whether or not mathematics y is true.[56]

   Examples show up if we choose, say, a Gödelian optimist philosophy and consider some of the set theories made by taking Zermelo-Fraenkel set theory as a base and adding some of the higher cardinal axioms. Another example of 2 shows up if we ask whether particular modal logics are true given Hellman's structuralism, as presented in Hellman (1989) since he does not come clean on either his metaphysical views concerning modality, nor on the formal theory which best represents mathematical possibility.
3. "y is false, given philosophy X" gets the value judgment (F), X recognises y as false, incorrect, ill conceived or wrong in some sense.

   Examples of number 3 are: if we start with Martin Löf's constructive type theory, then any mathematical theory which ineluctably contains the full classical law of excluded middle will be false. The law of excluded middle is eliminable in theories which allow only a finite domain, or which can be interpreted by the ordinals. Otherwise the law of excluded middle is ineluctable, and the theory is false or meaningless or misguided, when indexed to Martin-Löf's constructivism. Other examples can be found if we consider "bad" theories. Shapiro's structuralism considers paraconsistent mathematical theories to be false, since they are not recognised by model theory (since, in classical model theory, models demonstrate consistency).

[56] There is an ambiguity between our *not knowing* that philosophy X endorses y, and in principle, philosophy X is neutral with respect to y. This ambiguity runs through all of the judgments. I leave it in place with the counsel to make it clear when deploying judgments whether one means them in the epistemic or the ontological sense. Of course in a constructive vein, the distinction does not arise.

4. "y has contradictions, but these are true (and false), given philosophy X" gets the value judgment ($-\!\!\!\!\!T$) (Truth value glut, T favoured), iff Philosophy X recognises, entertains and tolerates the mathematical theory y which contains contradictions.

   Examples of 4 are: relevant or paraconsistent theories with their accompanying philosophies.
5. "y has contradictions but these are false (and true), given philosophy X" gets the value judgment ($-\!\!\!\!\!F$) (truth-value glut, F favoured) iff the philosopher works with a trivial theory, despite its being trivial.

   Examples of 5 are trivial theories. These are true and false: studied and trawled, but badly flawed, since they enjoy "consistent contradictions". They are not rejected altogether. So, for example, Dummett's work on Frege's inconsistent formal system gives us plenty of examples of sentences which would be judged as $-\!\!\!\!\!F$ because they are embedded in a trivial formal theory. The theory gets this judgment because there are quite acceptable things Frege writes within his own formal theory.[57]

The notions of truth and falsity of a whole mathematical theory is not classically bivalent. Nor should it be, for the pluralist. The five sorts of judgment are a bit unusual and call into question some of our philosophical prejudices. In particular, we are replacing the more traditional "truth-value" notion with the more nuanced notion of value judgment. The motivation for this is the simple thought that, for the pluralist, it does not make sense to say of a mathematical theory that it is true as such. It is more interesting, and informative, to say that it is consistent, or, it is a theory to which most of mathematics can be reduced, or, that it can be used to analyse another theory, and then reveal interesting problems. The choice of value judgments forces us to be explicit about our perspective, philosophy X. This mechanism in the judgment turns what used to be a normative claim into a descriptive claim. The force of this turn is to be more subtle, more precise, and then allows us to move on to other judgments. Note also that we have two sorts of judgment value glut: the one where contradiction is handled and constrained already in the theory, and the one where we have a trivial theory. As far as I know, this is original to the semantics for this pluralist theory, and is not a distinction found in other paraconsistent logics. The draw to be more nuanced in our value judgments is echoed in Byers.

> Mathematics is so commonly identified with its formal structure that it seems peculiar to assert that an idea [in mathematics] is neither true nor false. What I [William Byers] mean by this is similar to what David Bohm means when he says "theories are insights which are neither true nor false, but, rather, clear in certain domains, and unclear when extended

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[57]The definition of the logical connectives and operators has not yet been set. There are several possibilities. Consider, for example conjunction. We can define this as true when: both conjuncts are true, when both conjuncts are favoured (truth by itself is truth favoured) or when neither conjunct is false. Negation also merits careful consideration in the face of value gaps and gluts. In face of such choices, we can make particular choices, so conjunction is one thing, or we could even introduce several conjunctions, several negations, several conditionals—defined in terms of the other connectives and so on. Presumably, the syntax would then be designed to make, at least a sound system.

> beyond those domains” (Bohm 1980, p. 4). Classifying ideas as true or false is just not the best way of thinking about them. Ideas may be fecund; they may be deep; they may be subtle; they may be trivial. These are the kinds of attributes we should ascribe to ideas. Prematurely characterising an idea as true or false rigidifies the mathematical environment. Even a “false” idea can be valuable. For example, Goro Shimura once said of his late colleague Yutaka Taniyama, “He was gifted with the special capability of making many mistakes, mostly in the right direction. I envied him for this and tried in vain to imitate him, but found it quite difficult to make good mistakes” (Singh 1997, p. 174). A mistake is “good” precisely because it carries within it a legitimate mathematical idea. (Byers 2007, pp. 256–257)

It is too easy for philosophers of mathematics to restrict their task to giving an account of the “realm of mathematical truths”. This conception of the task of the philosophy of mathematics is a relic of Platonism which is rejected by the pluralist.

The pluralist is not alone in his rejection. There have been notable exceptions to this conception of the philosophy of mathematics: Brouwer was interested in mathematics as living in the mind. For Brouwer, there is a supervenience relationship between the psychology of learning about mathematics and the content of mathematics. In some ways this was explored further by Husserl. Husserl was interested in the phenomenology of mathematics, in our interaction with whatever it is that mathematicians treat as objects of study, in studying the phenomenology of these objects, since they are presented to us as objective and rigid, see Tieszen (2005). Husserl was not interested in the traditional metaphysical question concerning the ontology of mathematics: whether the objects of mathematics are independent of us. Instead, Husserl “bracketed” the traditional/metaphysical question of ontology and focused on what it is for us to encounter or experience such an object. These philosophies of mathematics break with our inherited Platonistic tradition, and, those of us who were raised in that tradition typically have difficulty recognising the importance of their place in the philosophy of mathematics, since we cannot easily classify them as “realist or anti-realist” etc. So, we cannot easily compare them to the more standard philosophies of mathematics. But this is not enough to claim that pluralism is correct. We have yet to answer the traditional philosopher’s concerns.

To traditional philosophers of mathematics, discussion of “judgments” and “insights” can sound rather suspect. After all, “indexed truth” is not far removed from “subjective truth”: “true for me” but “false for you”, since this is a type of indexing—indexing to a person at a time. Prompted by such worries, our opponent interlocutor of the previous critique might step in and ask if we are not really interested in the psychology of mathematics more than the philosophy of mathematics. The answer is that we are interested in the psychology, but only insofar as it can inform the philosophy. For example, it would be quite significant if we were to discover that there is a neuro-psychological block to our conceiving a consistent contradiction.[58] Or, there might be a neuro-psychological basis for

[58] I’m afraid I only have an anecdote. A student of mine, Thom Genarro gave a talk on some recent findings in psychology which he reckoned had some impact on the philosophy of mathematics. One such finding was that at the very primitive level, our brains are so constructed as to preclude

our holding some mathematical theorems as ineluctable. For example, we might discover that when faced with said theorem, a very primitive part of the brain is active, as opposed to a more esoteric theorem, where several complex areas of the brain are activated, and therefore, tenuously, we might think that esoteric mathematics cannot be grasped by everyone, but that they are easier to accept in the sense of there being several alternative neurological pathways which serve the function of allowing us to grasp the concepts. For pluralists, this line of enquiry is legitimate, if highly tenuous. Pluralists are able to acknowledge the insightful work done by neuro-scientists, or before them, Brouwer and Husserl, because we are able to acknowledge that psychological findings might partly explain the paths our mathematical investigations have taken. If we replace the traditional absolute truth values with judgment values, then we have more subtle tools to use for our philosophical work of comparing and evaluating philosophical theories and their fit with formal mathematical theories.

There is other philosophical work to be done, which is quite untainted by psychology or neuro-science. Take a paraconsistent logic "off the shelf" and put the value-judgments to work. The pluralist blocks *ex falso quodlibet* inferences in order to entertain pair-wise inconsistent theories, and whole trivial theories. Our new judgment values can accommodate nascent theories, where the "U" judgment will be useful. Contradictory theories are ones where the $\not{T}$ and $\not{F}$ judgments become important. Add indexes to the turnstile symbol to indicate the context of derivation (so essentially these turnstiles will pick out a class of formal theories). We can now prove things about combining mathematical theories. When the pluralist puts the value-judgments to work, he can give a subtle and precise interpretation of what is going on in mathematics. Working on the logic which best accommodates maximal pluralism is a philosophical and logical task. Giving philosophies for mathematical theories or classes of theories is another philosophical task (at the second level). Bringing philosophical insights to bear on particular results in mathematics is a philosophical task of the first level. See Corfield (2003). Thus there is plenty of philosophical work to be done; and it is work that cannot be done purely by a sociologist, historian or psychologist. This parries, again, the critique from "disdain for sociology". But what about the third level: the one occupied by the pluralist? Is the pluralist a pluralist about pluralism, or is he a foundationalist?

15.5.3 Conclusion: The Third (Paraconsistent) Level

The third level is pluralist *about* the levels below, but what happens at the third level? If we wanted to be optimally pluralist, and, say, we were constructivist, we

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our conceiving a contradiction. Some of the audience, including myself shot our hands up at this point. Either paraconsistent logicians are some sort of ubermenschen since they have overcome this primitive block, or the experiments which indirectly "show" this pre-suppose that we cannot conceive of a contradiction. I'll let you decide which is the more likely disjunct.

would choose a constructive logic at this level. We would then insist on a criterion for success in mathematics, namely, that the theory be constructive. What falls under which value judgments would then change. In this sense, pluralism has a perspective which is informed by our logic. Within these constraints, we could then be pluralist about different constructive mathematical theories—all of those which fit within the constraints. Similarly, if we are ultimately convinced by classical logics and convinced that contradictions can only be supported in the short term, in the sense that we have to adapt when faced with contradictory information, then we would probably favour a classical adaptive logic.[59] Optimal pluralism is not pluralist towards itself, since an optimal pluralist cannot tolerate competing optimal pluralist philosophies.

In this paper, I have presented maximal pluralism, and maximal pluralism is pluralist towards itself. The maximal pluralist favours some form of paraconsistent logic. The paraconsistent logic brings with it its own to philosophical stamp. In this case, the stamp is (at least) dialetheist, since there are true contradictory theories (which are also false). Truth is favoured. But it is not only dialetheist, since there are also false contradictory theories (which are also true). These are the trivial theories. Falsity is favoured. Thus, the logic might be more than dialetheist. When making judgments about the first and second levels, we have to hold the logic and philosophy of the third level fixed. However, pluralism can also be pluralist concerning the third level by moving up to a fourth level. When we judge pluralism, as a philosophy, we are occupying a fourth level of analysis. This is the level we have occupied in this section. Occupying this fourth level, we can say that, for example, pluralism accommodates mathematics as it is practiced today. So, maximal pluralism is pluralist about pluralism at the third level. In this sense, we might say that pluralism enjoys “closure” of its theory, as Priest uses the term, see Priest (2002). But we transcend pluralism too. Pluralists are well aware that philosophy and mathematics are historically (conceptually) situated. That is, philosophy and mathematics are developing with reference to each other, what looks like a good philosophy of present day mathematics, might not look so good of future mathematics. More important dialetheically, as pluralists concerning the third level, we take seriously the possibility that there could be significant improvement to the logic or class theory, in which case we would revise the underlying logic.

To sum up: if we take a paraconsistent logic to underpin our pluralism, and we think that mathematics contains dialetheias. We think that pluralism commits us to recognising some contradictions as true and to being pluralists about our own pluralism. This is because we take seriously the idea that there might be rival paraconsistent foundations, each meriting its own take on the lower levels of analysis. After all, the term “paraconsistent” is adopted by quite different philosophical and logical traditions. Each has its own motivations, and will bring its own philosophical stamp to bear on its version of pluralism. Maximal pluralism is a thoroughgoing pluralism.

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[59]See Batens’ http://logica.rug.ac.be/adlog/al.html for an introduction to adaptive logics.

References

Batens, D. 1997. Inconsistencies and beyond. A logical-philosophical discussion. *Revue International de Philosophie* 200: 259–273.

Beall, J.C., and G. Restall. 2006. *Logical pluralism*. Oxford: Clarendon Press.

Bohm, D. 1980. *Wholeness and the implicate order*. London: Routledge.

Byers, W. 2007. *How mathematicians think: Using ambiguity, contradiction and paradox to create mathematics*. Princeton: Princeton University Press.

Cellucci, C. 2008. Why proof? What is a proof? In *Deduction, computation, experiment. Exploring the effectiveness of proof*, ed. G. Corsi and R. Lupacchini, 1–27. Berlin: Springer.

Colyvan, M. 2001. *The indispensability of mathematics*. Oxford: Oxford University Press.

Coniglio, M.E., and W. Carnielli. 2008. *On discourses addressed by infidel logicians*. Presented at Fourth World Congress of Paraconsistency in Melbourne, Australia.

Corfield, D. 2003. *Towards a philosophy of real mathematics*. Cambridge: Cambridge University Press.

Friend, M. 2006. *Meinongian Structuralism*. The Logica Yearbook 2005. Marta Bílkova and Ondřej Tomala (eds.). Filosofia, Prague, pp. 71–84.

Giaquinto, M. 2002. *The search for certainty: A philosophical account of the foundations of mathematics*. Oxford: Clarendon Press.

Goethe, N.B., and M. Friend. 2010. Confronting ideals of proofs with the ways of proving of the research mathematician: Philosophy, history and practice-based mathematics. *Studia Logica* 96: 273–288.

Hellman, G. 1989. *Mathematics without numbers*. Oxford: Oxford University Press.

Heyting, A. 1930. Die formalen Regeln der intuitionistischen Logik. *Sitzungsberichte der Preussischen Akademie der Wissenschaften* 1930: 42–56. Translated and quoted in Paulo Mancosu (1998); *From brouwer to Hilbert: The debate on the foundations of mathematics in the 1920s*. New York: Oxford University Press.

Körner, S. 1962. *The philosophy of mathematics: An introductory essay*. New York: Harper and Row.

Maddy, P. 1997. *Naturalism in mathematics*. Oxford: Clarendon Press.

Maddy, P. 2007. *Second philosophy*. Oxford: Oxford University Press.

Mancosu, P. 1998. *From Brouwer to Hilbert: The debate on the foundations of mathematics in the 1920s*. New York: Oxford University Press.

Mortensen, C. 1995. *Inconsistent mathematics and its applications*. Dordrecht: Kluwer.

Potter, M. 2004. *Set theory and its philosophy*. Oxford: Oxford University Press.

Potter, M., and P. Sullivan. 1997. Hale on caesar. *Philosophia Mathematica* 5: 135–152.

Priest, G. 2002. *Beyond the limits of thought*. Oxford: Clarendon Press.

Priest, G. 2003. Meinongianism and the philosophy of mathematics. *Philosophia Mathematica* vol. XI Series III(1): 3–15.

Priest, G. 2006. *Doubt truth to be a liar*. Oxford: Oxford University Press.

Prior, A. 1960. The roundabout inference ticket. *Analysis* 21: 8–39.

Shapiro, S. 1991. *Foundations without foundationalism: The case for second-order logic*, Oxford logic guides, vol. 17. Oxford: Clarendon Press.

Shapiro, S. 1997. *Philosophy of mathematics: Structure and ontology*. Oxford: Oxford University Press.

Singh, S. 1997. *Fermat's enigma: The epic quest to solve the world's greatest mathematical problem*. New York: Penguin Books.

Tennant, N. 1997. *The taming of the true*. Oxford: Clarendon Press.

Thurston, W.P. 1994. On proofs and progress in mathematics. *Bulletin of the American Mathematical Society* 30(2): 161–177.

Tieszen, R. 2005. *Phenomenology, logic and the philosophy of mathematics*. Cambridge: Cambridge University Press.

van Stigt, W.P. 1998. Brouwer's intuitionist programme. In *From Brouwer to Hilbert: The debate on the foundations of mathematics in the 1920s*, ed. Paulo Mancosu. New York: Oxford University Press.

Vopěnka, P. 1979. *Mathematics in the alternative set theory*. Leipzig: Teubner-texte zur Mathematik.

Whitehead, A.N., and B. Russell. 1997. *Principia mathematica to *56*. Cambridge, Cambridge University Press.