Chapter 4
Stability to Finite Disturbances: Energy Method and Landau's Equation

The main part of Chap. 2 and the whole of Chap. 3 were devoted to topics of linear stability theory dealing with the evolution of very small flow disturbances satisfying the linearized fluid dynamics equations. In Chap. 2 it was shown that the classical normal-mode method of the linear theory of hydrodynamic stability often leads to results which strongly disagree with experimental data. It was also indicated there that these disagreements are apparently due to nonlinear effects, which make linearization of the equations of motion physically unjustified. In Chap. 3 it was explained that the necessity for consideration of the full nonlinear dynamic equations often follows from the fact that many solutions of the initial-value problems for linearized fluid dynamics equations grow considerably at small and moderate values of the time t even in the cases when the normal-mode analysis shows that these solutions decay asymptotically (i.e. at $t \to \infty$).

The nonlinear theory of hydrodynamic stability has achieved a high level of development. Although the theory is still far from being completed, it has elucidated many formerly mysterious properties of fluid flows which are interesting for physicists and important for engineers. There is now an enormous literature on this subject and only a small part of it, dealing with relatively simple flows of incompressible fluids, will be considered in this book. In the present Chapter two topics from the nonlinear stability theory will be discussed: the energy method of stability analysis (short introductory consideration of this method was included in Sect. 3.4 above) and Landau's approach to the weakly nonlinear stability theory which described the initial period of the nonlinear development of flow disturbances.

4.1 The Energy Method of Stability Analysis and its Generalisations

4.1.1 Flows of Fluids of Constant Density

Remember first of all what was said about the energy method in Sect. 3.4 of Chap. 3. There, a flow of an incompressible constant-density fluid in a domain V was

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considered, where V is either bounded by solid walls or is unbounded in the directions of some coordinate axes x_j. It was assumed that the velocity and pressure fields of the flow are of the some of the form $\mathbf{U}(\mathbf{x}) + \mathbf{u}(\mathbf{x}, t)$ and $P(\mathbf{x}) + p(\mathbf{x}, t)$, where $\{\mathbf{U}(\mathbf{x}), P(\mathbf{x})\}$ are the velocity and pressure of some steady 'undisturbed flow'(which, in the case of unbounded flow, has the property that $\mathbf{U}$ and ∇P do not depend on those coordinates x_j that correspond to directions of flow unboundedness), while $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ are the velocity and pressure of some disturbance of arbitrary size (which in the case of unbounded V is periodic, with given periods $l_j = 2\pi/k_j$, with respect to the coordinates x_j). Hence $\{\mathbf{U}(\mathbf{x}), P(\mathbf{x})\}$ and $\{\mathbf{U}(\mathbf{x}) + \mathbf{u}(\mathbf{x}, t), P(\mathrm{x}) + p(\mathbf{x}, t)\}$ both satisfy the Navier–Stockes (for short, N-S) equations with "no-slip" boundary conditions at solid walls. Note also that the derivation of the energy-balance equation is unchanged if the undisturbed flow is unsteady and spatially periodic (with periods l_j) in directions in which V extends to infinity; moreover, the walls bounding the domain V can be moving, and V can depend on t.

The energy-balance equation for a flow disturbance follows easily from the equations of motion for $\mathbf{u} = (u_1, u_2, u_3)$, which are the differences between the N-S equations for $U_i + u_i$ and those for U_i alone, for $i = 1, 2, 3$:

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \tag{4.1a}$$

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{4.1b}$$

where, as usual, the summation is carried out over all three values of any indices which occur twice in single-term expressions ("repeated indices"). Let us now multiply Eq. (4.1a) by u_i, sum the equations obtained for $i = 1, 2, 3$, and then integrate the sum over the region V', where V' coincides with V if V is bounded, while if V is unbounded then V' includes only one period l_j in directions in which V extends to infinity. It is easy to see that the result of the integration can be written in the form

$$\frac{dE(t)}{dt} = -\int\limits_{V'} u_j u_i \frac{\partial U_j}{\partial x_i} d\mathbf{x} - \nu \int\limits_{V'} \sum_{j,i=1}^{3} \left(\frac{\partial u_j}{\partial x_i}\right)^2 d\mathbf{x} \tag{4.2}$$

where $d\boldsymbol{x}$ is an element of volume in the three-dimensional space of points $\boldsymbol{x}$ and

$$E(t) = \frac{1}{2} \int\limits_{V'} \sum_{j=1}^{3} u_j^2 d\mathbf{x} \tag{4.3}$$

is either the total kinetic energy of a disturbance (if V is bounded) or the energy density per wavelength (the unimportant factor ρ representing the constant density of the fluid is here omitted for simplicity). Equations (4.2) and (4.3) are just Eqs. (3.74) and (3.73) of Sect. 3.4 and the first of them is just the *Reynolds-Orr* (or R-O) *equation of the energy balance.* This was first derived more than a hundred years ago by Reynolds (1894) (who took $\mathbf{U}$ as the average flow velocity and $\mathbf{u}$ as the deviation of the velocity at a point from the average) and was later studied and used by Orr (1907) (whose interpretation of the velocities **U** and **u** was the same as that given above). It was noted in Sect. 3.4 that the single nonlinear term of Eq. (4.1a) for the velocity u_i—the last term of the left-hand side—makes no contribution to Eq. (4.2), since it produces a divergence term which drops out after the integration by virtue of boundary conditions. As a result, all the terms of the R-O equation turn out to be quadratic in the disturbance velocities u_i; therefore, the sign of the left-hand side of the R-O equation does not change when the velocity $\mathbf{u}(\mathbf{x}, t)$ is multiplied by some factor (i.e., this sign does not depend on the disturbance intensity). It was also noted in Sect. 3.4 that changing to dimensionless quantities transforms the energy-balance Eq. (4.2) into an equation of the same form but with dimensionless coordinates and velocities (measured in appropriate length and velocity units L and U) and with the dimensional factor ν replaced by the dimensionless combination $\nu/UL = 1/\mathrm{Re}$.

From the R-O Eq. (4.2), where all the velocities and coordinates are now assumed to be non-dimensionalized, it follows that if $UL/\nu = \mathrm{Re}$ takes a value which is greater than the value of the ratio

$$\frac{\left[\int\limits_{V'} \sum\limits_{j,i=1}^{3} \left(\frac{\partial u_j}{\partial x_i}\right)^2 d\mathbf{x}\right]}{\left[-\int\limits_{V'} u_j u_i \frac{\partial U_j}{\partial x_i} d\mathbf{x}\right]} = R[\mathbf{u}(\mathbf{x})] \tag{4.4}$$

for a given solenoidal (zero-divergence) vector field $\mathbf{u}(\mathbf{x}) = \{u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x})\}$ which satisfies all the necessary boundary and periodicity conditions, and if $\mathbf{u}(\mathbf{x}, \mathrm{t} = 0) \equiv \mathbf{u}(\mathbf{x})$, then $dE(t)/dt > 0$ at $t = 0$. On the other hand, if Re is smaller than (or equal to) the greatest value of $R[\mathbf{u}(\mathbf{x})]$ accessible for the class of solenoidal fields $\mathbf{u}(\mathbf{x})$ satisfying the necessary boundary and periodicity conditions, and if the undisturbed flow is a steady one, then for any shape and size of the initial field $\mathbf{u}(\mathbf{x}, \mathrm{t} = 0)$ of disturbance velocity for derivative dE/dt will be nonpositive at any $t \geq 0$. Therefore, we may conclude that *the minimal Reynolds number $Re_{cr\,min}$*, which first appears in the paper by Reynolds (1883), *coincides with the minimum value of* $R[\mathbf{u}(\mathbf{x})]$ *over all solenoidal vector fields* $\mathbf{u}(\mathbf{x})$ *representing possible initial values of the disturbance velocity.* Such a definition of $\mathrm{Re}_{\mathrm{cr\,min}}$ implies that at $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr\,min}}$ the undisturbed flow considered is *globally* (i.e., *unconditionally*) and *monotonically stable* (for more details about these concepts see, e.g., Joseph (1976); Manneville (1990); Dauchot and Manneville (1995), and Chap. 2 in Godreche and Manneville (1998)). Later it was also shown that if the flow region V is bounded in at least one spatial direction (and hence can be contained between some pair of parallel planes), then for any $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr\,min}}$ there exists a positive constant $\Lambda = \Lambda$ (Re) such that $E(t) \leq E(0)\exp(-\Lambda \mathrm{t})$ for any $t > 0$; therefore, in this case the disturbance energy falls off exponentially with time (see, e.g., Serrin (1959), and also the books by Joseph (1976), Sect. 4, Galdi and Rionero (1985), Chap. 1, Georgescu (1985), Sect. 1.1.5, and Straughan (1992), Chap. 3).

If Re^* is the smallest value of $R[\mathbf{u}(\mathbf{x})]$ corresponding to some subset of all admissible disturbance velocities $\mathbf{u}(\mathbf{x})$, then the inequality $\mathrm{Re}_{\mathrm{cr\,min}} < \mathrm{Re}^*$ is clearly valid, and hence Re^* is an estimate of $\mathrm{Re}_{\mathrm{cr\,min}}$ from above. Reynolds (1894) used his version of Eq. (4.2) for just such an estimate from above of $\mathrm{Re}_{\mathrm{cr\,min}}$ for plane Poiseuille flow. For this purpose he determined the minimum value of $R[\mathbf{u}(\mathbf{x})]$ for one special family of admissible two-dimensional vector fields $\mathbf{u}(\mathbf{x}) = \{u(x, z), 0, w(x, z)\}$ depending on two numerical parameters (not counting the amplitude whose value is unimportant) and thus proved that in this case $\mathrm{Re}_{\mathrm{cr\,min}} \leq 517$, where Re is based on the distance H between the walls and the mean velocity of the undisturbed flow $U_\mathrm{m} = 2U_{\max}/3$. Later Sharpe (1905) carried out a similar computation for a quite different two-parameter family of two-dimensional disturbances $\mathbf{u}(\mathbf{x})$, and in this way found a considerably lower estimate, $\mathrm{Re}_{\mathrm{cr\,min}} \leq 167$, of the minimum Reynolds number for plane Poiseuille flow. Sharpe also applied this method to estimation from above of $\mathrm{Re}_{\mathrm{cr\,min}}$ for the circular Poiseuille flow in a round tube; here the value of $\min_{\mathrm{u(x)}} R[\mathbf{u}(\mathbf{x})]$ for a particular two-parameter family of axisymmetric velocity disturbances gave $\mathrm{Re}_{\mathrm{cr\,min}} < 470$, where Re is based on the tube diameter D and the undisturbed mean velocity U_m. Then Lorentz (1907), computed the value of $\min_{\mathrm{u(x)}} R[\mathbf{u}(\mathbf{x})]$ for a class of 'elliptic whirls' disturbing a plane Couette flow and found that for this case $\mathrm{Re}_{\mathrm{cr\,min}} \leq 288$ where $\mathrm{Re} = HU/\nu$, H is the flow thickness and U is the velocity of the moving wall.

It was already clear to Orr (1907) that only very crude estimates of $\mathrm{Re}_{\mathrm{cr\,min}}$ can be found from investigations of special low-parametric subsets of disturbance velocities $\mathbf{u}(\mathbf{x})$. For this reason Orr did not consider any such subsets, but set up the variational problem of finding the solenoidal vector field $\mathbf{u}(\mathbf{x})$ which satisfies the required boundary conditions (and periodicity conditions, if V is unbounded), and minimizes the functional (4.4) where $\mathbf{U}(\mathbf{x})$ is a given undisturbed velocity field. Orr noted that he tried to solve this problem for three-dimensional vector fields u(x) but found it to be too difficult (remember that this was written in 1907). Therefore he considered only two-dimensional disturbances $\mathbf{u}(\mathbf{x}) = \{u(x, z), 0, w\,(x, z)\}$ (or, in the case of tube flow, $\{u_\mathrm{x}(x, r), 0, u_\mathrm{r}(x, r)\}$ assuming that such disturbances must be less stable than three-dimensional ones. For two-dimensional disturbances the solenoidal vector field u(x) may be represented in terms of the scalar stream function $\Psi(x, z)$ (or $\Psi(x, r)$) and substituted in this form into Eq. (4.4). In particular, for a plane-parallel undisturbed flow with velocity profile $U(z)$, the functional $R[\mathbf{u}(\mathbf{x})]$ in the case of a two-dimensional disturbance can be written as

$$R[\Psi(x,z)] = \frac{\iint (\Delta\Psi)^2 dx\,dz}{\iint \frac{\partial \Psi}{\partial x}\frac{\partial \Psi}{\partial z}\frac{dU}{dz} dx\,dz}; \tag{4.5}$$

and a similar equation may be obtained for tube flow. Therefore, determination of the value of $\mathrm{Re}_{\mathrm{cr\,min}}$ corresponding to two-dimensional disturbances can be reduced to the variational problem of finding the minimum value of functional (4.5) (or some similar functional for an axisymmetric primary flow) over the set of twice-differentiable functions $\Psi(x, z)$ (or $\Psi(x, r)$) satisfying the appropriate boundary conditions (in particular, conditions $\partial\Psi/\partial z = \partial\Psi/\partial x = 0$ at plane solid walls). This was just the variational problem Orr tried to solve for the cases of plane Couette flow and plane and circular Poiseuille flows. Unfortunately, his assumption that two-dimensional disturbances are the most unstable was later found to be incorrect, and also some of his numerical methods proved to be not sufficiently precise. However, this fact does not diminish Orr's main achievement, the first accurate formulation of the general variational problem of the energy method of stability theory.

Subsequent investigations of the stability of some simple parallel flows by the energy method were carried out during the 1910s and 1920s, in particular by Hamel (1911); Havelock (1921), and von Kármán (1924) (the first publication of the author's results of 1910, presented at the time only in a lecture). These workers also considered only two-dimensional disturbances, and used rather crude approximate solutions of Orr's variational problem. The papers by Tamaki and Harrison (1920) and Harrison (1921) were devoted to the study of the stability of circular Couette flow by the energy method, but the first of these papers was erroneous, while in the second the extremum was sought only among a rather special and narrow set of disturbances. However, for many years the inaccuracy of these calculations seemed to be an insufficient explanation of the fact that all estimates of the critical Reynolds numbers obtained by this method (often being 'estimates from above') turned out to be considerably lower than both the values of $\mathrm{Re}_{\mathrm{cr}}$ given by the normal-mode method of linear stability theory and the experimentally observed values of Re corresponding to transition of real flows to turbulence. This circumstance gave rise to extensive criticism of the energy method by a number of authors, proclaiming that, even in principle, this method can give only serious underestimates of $\mathrm{Re}_{\mathrm{cr}}$. The observed inadequacy of the method was usually explained by the fact that the minimization of the functional (4.4) (or (4.5)) was carried out over a set of disturbance velocities (or stream functions) satisfying only the required boundary and incompressibility conditions, while the equations of motion were not taken into account at all. Critical remarks of this kind can be found, e.g., in the books by Lin (1955), p. 59, Monin and Yaglom (1971), p. 152 (here a reference is given to the paper by Petrov (1938), where it was allegedly shown 'that the value of $\Psi(x, z)$ minimizing the functional (4.5) cannot generate a dynamically possible motion'), and Hinze (1975), p. 77, and also in papers by Serrin (1959), p. 4, and Joseph (1966), pp. 181–182 (these two papers will be considered below at greater length). However, this criticism is in fact unjustified; the energy method considers only the flow conditions at one instant of time, and at fixed time t the velocity $\mathbf{u}(\mathbf{x}, t)$ can take any value satisfying the above boundary and incompressibility conditions. (This fact was stressed by Lumley (1971) who also analyzed the arguments by Petrov (1938) to show their inconsistency). Since in the energy method $\min_{\mathrm{u(x)}} R[\mathrm{u(x)}]$ is taken over all possible instantaneous values of disturbance velocity, then—if the undisturbed flow is steady—at any Re below this minimum $dE(t)/dt$ will be negative at any non-negative value of t, i.e., the energy of the disturbance will decay monotonically with time for any intensity and shape of the initial disturbance.

Let us stress, however, that the validity of the inequality $\mathrm{Re} \leq \mathrm{Re}_{\mathrm{cr\ min}} = \min_{\mathrm{u(x)}} R[\mathbf{u}(\mathbf{x})]$, which guarantees the monotonic decrease of the disturbance energy with time, is only a *sufficient* (but not necessary) *condition for flow stability*. On the other hand, the validity of the opposite inequality $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr\ min}}$ is a *necessary* (but not sufficient) *condition for flow instability*. Remember also that in Chap. 2 it was noted

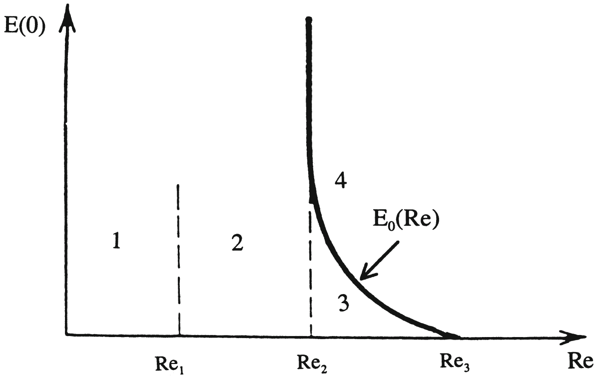

Fig. 4.1 Schematic representations of various stability regions of a given flow in the (E(0), Re)-plane. (After Joseph (1976)) $Re_1 = Re_{cr\,min}$; $Re_2 = Re_{0,cr}$; $Re_3 = Re_{cr}$; *1*-the region of global and monotic stability; *2*-the region of global nonmonotonic stability; *3*-the region of conditional stability; *4*-the region of instability

that the normal-mode method of the linear stability theory gives the value of Re_{cr} such that $Re > Re_{cr}$ is a *sufficient* (but not necessary) *condition for flow instability* (while the opposite condition $Re < Re_{cr}$ is a *necessary*, but not sufficient, *condition for flow stability*). Therefore, the value of $Re_{cr\,min}$ can be quite different from both the value or Re_{cr} of the linear stability theory and the value of Re characterizing real transition to turbulence. Thus it is only natural that, even when sufficiently precise computations of $\min_{u(x)} R[\mathbf{u}(\mathbf{x})] = Re_{cr\,min}$ are used, the energy method often leads to values of Re which are far below the Reynolds numbers observed at transition of flow to turbulence (and below the linear-theory values of Re_{cr} which are usually higher than Reynolds numbers at laminar-turbulent transition). Let us stress again in this respect that $Re_{cr\,min}$ determines only the threshold value of the Reynolds numbers corresponding to *global* (unconditional) *monotonic stability* of the flow considered, i.e., the validity of the condition $Re < Re_{cr\,min}$ is both necessary and sufficient for being sure that any initial disturbance will decay *monotonically* tending to zero as $t \to \infty$. However, certain range $Re_{cr\,min} < Re < Re_{0,\,cr}$ of Reynolds number exceeding $Re_{cr\,min}$ can exist, having the property that if Re belongs to it then any disturbance will necessarily decay to zero as $t \to \infty$ but the energy of some disturbance will transiently grow during some finite time intervals. This range corresponds to *global* (but *nonmonotonic*) *flow stability* and it is clear that transition to (undamped) turbulence cannot happen at $Re < Re_{0,\,cr}$. The range of Reynolds numbers corresponding to *conditionally stable flows* adjoins the globally-stable-flow range $0 \leq Re \leq Re_{0,\,cr}$; at values of Re from this range the disturbances satisfying some definite condition necessarily decay to zero while others can grow indefinitely. The most usual conditions guaranteeing the decay of disturbances have the form of energy limitations: the disturbance necessarily decays as $t \to \infty$ if its initial energy $E(0)$ does not exceed some threshold value $E_0(\mathrm{Re})$ depending on Re. The value of $E_0(\mathrm{Re})$ clearly must decrease monotonically with the increase of Re apparently tending to zero at $Re = Re_{cr}$ (where Re_{cr} is the critical value of the linear stability theory dealing with the infinitesimal disturbances) and to infinity at $Re = Re_{0,\,cr}$ (see schematic Fig. 4.1; additional information may be again found in Joseph (1976); Manneville (1990); Dauchot and Manneville (1995), and Godrèche and Manneville (1998)). As to the transition to turbulence, it occurs most often at some Reynolds number intermediate between $\mathrm{Re}_{0,\,\mathrm{cr}}$ and $\mathrm{Re}_{\mathrm{cr}}$.

Earlier in this section some very early papers on the energy method of the stability theory were mentioned. In the 1930s, 1940s, and early 1950s this theory did not attract much attention; note, however, two remarkable papers by Sorokin (1953, 1954) which will be discussed later in this section. Slightly later the important paper by Serrin (1959) appeared, stimulating a number of authors to resume stability investigations by the energy method. This resulted in a great number of new publications relating to many different problems on hydrodynamic stability.

Serrin began with an accurate derivation of the fundamental R–O Eq. (4.2) under rather general conditions (he considered a general unsteady flow in the presence of an external force in the region V bounded by walls which could be moving). Then he formulated the variational problem by Orr as a problem of finding the maximum of the functional $\prod[\mathbf{u}(\mathbf{x})] = \int\limits_{v'} u_j u_i(\partial U_j/\partial x_i)d\mathbf{x}$ under the following conditions: $D[\mathbf{u}(\mathbf{x})] = \int\limits_{v'} \sum\limits_{i,j=1}^{3} (\partial u_j/\partial x_i)^2\,\mathrm{d}\,\mathbf{x} = 1$ and div $\mathbf{u}(\mathbf{x}) = 0$, where $\mathbf{u}(\mathbf{x})$ satisfies the necessary boundary and periodicity conditions. Serrin wrote down the Euler-Lagrange (E-L) equations corresponding to this variational problem, which included Lagrange multipliers (since a conditional extremum was sought). He also showed that the equations obtained can be easily transformed into an eigenvalue problem for a system of partial differential equations similar to the N-S equations of fluid dynamics. (A slightly different derivation of these E-L equations, under slightly more general conditions, was given by Lumley (1971), while the corresponding eigenvalue problem was also considered by Galdi and Rionero (1985), Chap. 1, Geovgescu (1985), Sects. 1.1.2 and 1.3.1, and Straughan (1992), Chap. 3.) However, in the 1950s the determination of the exact solution of the eigenvalue problem seemed to be very difficult. Therefore Serrin concentrated his main efforts on the derivation of some approximate results, based on some relatively crude general inequalities.

In particular, Serrin showed that in the case of an arbitrary bounded region V with smooth enough boundary and a maximum diameter D the following inequality holds:

$$\int\limits_{v} \sum_{j,i=1}^{3} \left(\frac{\partial u_j}{\partial x_i}\right)^2 d\mathbf{x} \geq \frac{\mathrm{a}}{\mathrm{D}^2}\int\limits_{v} \sum_{\mathrm{i}=1}^{3} \mathrm{u}_{\mathrm{i}}^2 \mathrm{d}\mathbf{x}, \tag{4.6}$$

where $a = \frac{3+\sqrt{13}}{2}\pi^2 \approx 32.6$, for any solenoidal vector field $\mathbf{u(x)}$ in V vanishing on the boundary of V (this is a particular case of the known Poincaré inequality; see, e.g., Straughan (1992)). Using then the obvious relations

$$-\int\limits_{v} u_j u_{\mathrm{i}} \frac{\partial U_j}{\partial x_i} d\mathbf{x} = \int\limits_{v} \frac{\partial u_j}{\partial x_i} u_i U_J d\mathbf{x}, \frac{\nu}{2}\int\limits_{v} \sum_{i,j} \left(\frac{\partial u_i}{\partial x_i}\right)^2 d\mathbf{x} - \int\limits_{v} \frac{\partial u_j}{\partial x_i} u_i U_j dx$$
$$+ \frac{1}{2\nu}\int\limits_{v} \sum_{i=1}^{3} u_i^2 \sum_{j=1}^{3} U_J^2 d\mathbf{x} \geq 0,$$ Serrin obtained the inequality

$$-\int_V u_j u_i \frac{\partial U_j}{\partial x_i} d\mathbf{x} \le \frac{\nu}{2} \int_V \sum_{i,j} \left(\frac{\partial u_i}{\partial x_i}\right)^2 d\mathbf{x} + \frac{U_{\max}^2}{2\nu} \int_V \sum_{i=1}^{3} u^2 d\mathbf{x}, \tag{4.7}$$

where $U_{\max}$ is the maximum of the modulus of the undisturbed velocity $\mathbf{U(x)}$. Equations (4.2), (4.3), (4.6) and (4.7) easily imply that

$$E(t) \le E(0) \exp\left(\frac{U_{\max}^2}{\nu} - \frac{a\nu}{D^2}\right), \; i.e., \mathrm{Re}_{\mathrm{cr\,min}} = \left(\frac{U_{\max} D}{\nu}\right)_{\mathrm{cr\,min}} \ge \sqrt{a} \tag{4.8}$$

where $\sqrt{a} = [(3+\sqrt{13})/2]^{1/2}\pi \approx 5.71$. This 'estimate from below' of the critical Reynolds number may seem to be too low but we must remember that the value of $\mathrm{Re}_{\mathrm{cr\,min}}$ can be much smaller than that of $\mathrm{Re}_{\mathrm{cr}}$ found from transition experiments, and we should also take into account that the result (4.8) is based on rather crude inequalities and is very universal, being applicable to any bounded region of diameter D or less and to any flow in this region.

Serrin obtained similar estimates for flows in arbitrary straight channels of variable width not exceeding D (i.e., with width $H(y)$ which can depend on y and satisfies the condition $\max_y H(y) \le D$) and straight tubes of arbitrary cross section with diameter not exceeding D. Serrin proved that the inequalities (4.6) and (4.7) are valid for these cases too (with region V replaced by V'), except that the constant a in Eq. (4.6) is equal to π^2 in the case of a straight channel and to $2\pi^2$ in the case of a straight tube. Therefore, the new universal stability estimates have now the forms: $\mathrm{Re}_{\mathrm{cr\,min}} = (U_{\max} D/\nu)_{\mathrm{cr\,min}} \ge \pi \approx 3.14$ for channels of maximum width D and $\mathrm{Re}_{\mathrm{cr\,min}} = (U_{\max} D/\nu)_{\mathrm{cr\,min}} \ge \sqrt{2}\pi \approx 4.43$ for tubes of maximum diameter D.

Finally Serrin applied analogous arguments to a circular Couette flow between concentric cylinders of radii R_1 and R_2 rotating with angular velocities Ω_1 and Ω_2 (where index 1 relates to the inner cylinder and $\Omega_1 > 0$). Here (see e.g. Eq. (2.10), Sect. 2.6) the undisturbed velocity is given by

$$\begin{gathered}\mathbf{U(x)} = \mathbf{U}(r,\varphi,z) = \{U_r, U_\varphi, U_z\} = \left\{0, Ar + \frac{B}{r}, 0\right\}, \\ A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \quad B = \frac{R_1^2 R_2^2 (\Omega_2 - \Omega_1)}{R_2^2 - R_1^2}.\end{gathered} \tag{4.9}$$

Using these equations it is easy to show that

$$-\int_{V'} u_j u_i \frac{\partial U_j}{\partial x_i} d\mathbf{x} \le |B| \int_{V'} \sum_{i=1}^{3} u_i^2 \frac{d\mathbf{x}}{r^2} \tag{4.10a}$$

$$\int_{V'} \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j}\right)^2 d\mathbf{x} \ge b \int_{V'} \sum_{i=1}^{3} u_i^2 \frac{d\mathbf{x}}{r^2}, \quad b = \left[\frac{\pi}{\log(R_2/R_1)}\right]^2 \tag{4.10b}$$

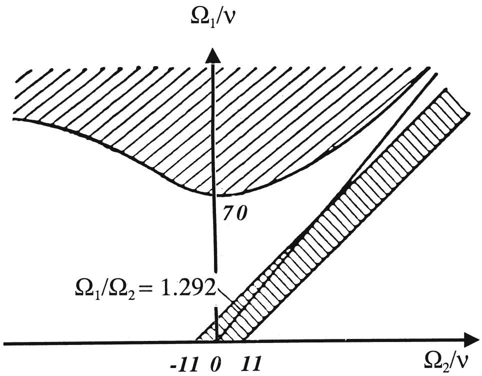

Fig. 4.2 Position of the region of instability to infinitesimal disturbances, and the region of stability to any finite disturbance, for Couette flow between rotating cylinders, studied by Taylor (1923). The *upper dashed* region corresponds to instability to infinitesimal disturbances, while flows corresponding to points of the shaded strip are definitely stable to any finite disturbance. (After Serrin (1959)) the continuous *straight line* in the figure is the boundary of the region of instability for the case of an inviscid fluid (Chap. 2)

where, as usual, V' is a part of the flow region having a width in the z direction equal to one disturbance wavelength (which can take an arbitrary value). Combining inequalities (4.10) with the R-O Eq. (4.2), Serrin obtained the relation

$$\frac{dE(t)}{dt} \leq (|B| - b\nu) \int\limits_{V'} \sum_{i=1}^{3} u_i^2 \frac{d\mathbf{x}}{r^2} \tag{4.11}$$

which, together with the expressions for the coefficients B and b, implies that circular Couette flow is stable to arbitrary disturbances if

$$\frac{|\Omega_2 - \Omega_1|}{\nu} \leq (R_2^2 - R_1^2)\left[\frac{\pi}{R_1 R_2 \log(R_2/R_1)}\right]^2 \tag{4.12}$$

(another proof of condition (4.12) was given by Joseph (1976), Sect. 37). The region (4.12) of the (Ω_1, Ω_2)-plane is usually only a small part of the region of stability to infinitesimal disturbances (see, for example, Fig. 4.2 for the case where $R_1 = 3.55$ cm, $R_2 = 4.03$ cm, so that $R_2/R_1 = 1.13$, which was studied experimentally by Taylor (1923); his results for this case were presented in Chap. 2, Fig. 2.10). The smallness of the region of universal stability in comparison to that of stability to infinitesimal disturbances may seem to be only natural, but let us also note that the result (4.12) is far from being exact, being based on the rather crude inequalities (4.10).

Shortly after the appearance of Serrin's paper of 1959, Velte (1962) improved the possible values of the coefficient a in the Poincaré inequalities (4.6) relating to the particular classes of fluid flows considered by Serrin. Namely, he showed that this coefficient is in fact not less than $6\pi^2$ in the case of flows in bounded regions of diameter D, not less than $3.74\pi^2$ for flows in straight channels of bounded width, and not less than $4.7\pi^2$ for flows in straight tubes of bounded diameter. These results imply the following sharpening of Serrin's estimates: $\mathrm{Re}_{\mathrm{cr\ min}} \geq \sqrt{6}\pi \approx 7.7$ for flows in bounded regions, $\mathrm{Re}_{\mathrm{cr\ min}} \geq \sqrt{3.74}\pi \approx 6.1$ for flows in straight channels, and $\mathrm{Re}_{\mathrm{cr\ min}} \geq \sqrt{4.7}\pi \approx 6.8$ for flows in straight tubes, where Reynolds number is based on the maximum flow velocity and the maximum diameter or width of the flow region.

The indicated estimates from below of $\mathrm{Re}_{\mathrm{cr\ min}}$ may be made more precise if their 'universality' is relaxed, i.e. if they are sought in more restricted sets of spatial regions V and/or velocity fields $\mathbf{U}(\mathbf{x})$. One of the first such attempts was made by Payne and Weinberger (1963) who considered the special case where V is a sphere of diameter D. These authors found that in this case the maximal possible value of the coefficient a in Eq. (4.6) is $4a_1^2$ where a_1 is the lowest positive root of the equation $\tan a_1 = a_1$. It follows from this that $a \approx 80$ and hence $\mathrm{Re}_{\mathrm{cr\ min}} \geq 8.94$ in the case considered. Since it is clear that the coefficient a cannot decrease when the region is shrinking, the last result gives the final improvement of Serrin's estimate of $\mathrm{Re}_{\mathrm{cr\ min}}$ for bounded regions of fixed maximal diameter, admitting no further corrections.

Later Sorger (1966a) (see also Joseph (1976), Sects. B7 and B8) independently considered the more general case of a region V bounded by two concentric spheres of radii R_1 and R_2, (where $0 \leq R_1 < R_2$, and $2R_2 = D$) and proved that here $\sqrt{a} = 4a_1^2$ where a_1 is the minimal zero of some combination of the Bessel functions of the first and second kinds, of order 3/2, taken at arguments a_1 and $\eta \mathrm{a}_1$ where $0 \leq \eta = R_1/R_2 < 1$ (for $\eta = 0$ this combination of Bessel functions becomes a function proportional to $\tan a_1 - a_1$, as it must do according to the result of Payne and Weinberger). Sorger (1966a) also found exact analytical solutions of the variational problem of determininig the largest possible value of a for two-dimensional flows in a planar region V, bounded by a circle of diameter D or by two concentric circles of radii R_1 and $R_2 = D/2 > R_1$; he used these solutions to determine relatively narrow ranges for the true values of a (and thus also for values of $\mathrm{Re}_{\mathrm{cr\ min}} = a^2$) in the cases of flows in a circular tube or in a circular channel between two concentric cylinders. Some energy-method estimates of $\mathrm{Re}_{\mathrm{cr\ min}}$ for flows in unbounded regions which cannot be confined between a pair of parallel planes were given by Galdi and Rionero (1985), Chaps. 2 and 3; see also Chap. 5 of Straughan's book (1992).

In the second half of the 1960s, numerical methods began to be widely applied to solution of the main variational problem of the energy theory of hydrodynamic stability, for a number of primary flows U(x) given in various spatial regions V (some of these methods were considered by Straughan (1992), pp. 217–224). This allowed the determination, with good accuracy, of values of the stability bounds $\mathrm{Re}_{\mathrm{cr\ min}} = \min_{\mathrm{u(x)}} R[\mathbf{u}(\mathbf{x})]$ (cf. Eq. (4.4)) for many important flows, both of homogeneous fluids of constant density ρ and of inhomogeneous fluids of variable density $\rho(\mathbf{x}, t)$ (dependent, for example, on the temperature $T(\mathbf{x}, t)$). The main results obtained in the late 1960s and early 1970s were summarized in the two-volume book by Joseph (1976). Let us recall in this respect that in Chap. 3, Sect. 3.4, it was mentioned that both Busse (1969) and Joseph and Carmi (1969) solved numerically the general (three-dimensional) variational problem of the energy stability theory for plane Poiseuille flow and found that $\mathrm{Re}_{\mathrm{cr\ min}} = 49.6$, while Joseph and Carmi simultaneously found that $\mathrm{Re}_{\mathrm{cr\ min}} = 81.5$ for circular Poiseuille flow in a round tube, and Joseph (1966) calculated that $\mathrm{Re}_{\mathrm{cr\ min}} = 20.7$ for plane Couette flow in a layer bounded by two parallel walls. (For Poiseuille flows Re is formed with the maximal velocity U_{max} and the channel half-width H_1 or the tube radius R, while in the case of Couette flow the half-difference of wall velocities U_0 and half-distance between walls H_1 are used as velocity and length scale). The paper by Joseph and Carmi also contains the energy-method determination of the value of $\mathrm{Re}_{\mathrm{cr\ min}}$ for Poiseuille flow (produced by a constant axial pressure gradient) in the annuli between two concentric round cylinders of different radii, while Joseph (1966) considered in addition the case of stratified Couette flow between parallel walls where the temperature of the lower wall is higher than that of the upper one (his main result for this case will be presented later). Note in conclusion that all the above-mentioned papers include the determination of the 'most dangerous' disturbances which correspond to the maximum value of $R[\mathbf{u}(\mathbf{x})]$ (i.e., are the most unstable). The results presented here, and also many results of the energy method of stability theory for more complicated flows (e.g., the pressure-gradient flows in annuli between concentric cylinders which are either sliding with respect to each other or rotating, or flows between rotating concentric spheres) can be found in the book by Joseph (1976). However, we wil not linger here to consider these more complicated flows. Instead, we will return to the applications of the energy method to the classical stability problem of Couette flow between concentric rotating cylinders.

Above, we mentioned the early, rather inaccurate, papers of Tamaki and Harrison (1920) and Harrison (1921) devoted to this problem, and also described the derivation by Serrin (1959) of the important universal stability condition (4.12). Note now that in the same paper Serrin supplemented the exact inequality (4.12) by some stronger but not fully rigorous conclusions. Namely, he assumed without proof that Orr's variational problem in the case of a Couette flow between rotating cylinders has an axially symmetric solution of the form $\mathbf{u}(\mathbf{x}) = \hat{\mathrm{u}}(r)e^{ikz}$ where cylindrical coordinates r, ϕ, z are now used and the wave number k takes arbitrary real values. Then the system of Euler-Lagrange differential equations determining the solution of the energy-method variational problem relating to such disturbances can be reduced to an eigenvalue problem for a linear system of two ordinary differential equations, with unkown functions $\hat{u}_r(r)$, $\hat{u}_\phi(r)$. Serrin could not solve this problem in the general case but he showed that in the case of a 'small gap' between the cylinders, i.e. where $R_2 - R_1 \ll (R_2 + R_1)/2$, his system of differential equations can be approximated by a pair of simpler equations, leading to an eigenvalue problem whose solution is known from previous work on hydrodynamic stability. Then the smallest eigenvalue of the problem studied (which depends on k so that the minimum over all real values of k must be considered) will determine the new stability criterion (with respect to axisymmetric disturbances) valid in the small-gap case. According to Serrin it has the form

$$\frac{|\Omega_2 - \Omega_1|}{\nu} \leq \frac{2\sqrt{1708}}{\sqrt{R_1 R_2}(R_2 - R_1)}. \qquad (4.13)$$

This somewhat tentative criterion gives the 'stability region' in the form of a strip similar to that presented in Fig. 4.2 but having much greater width.

Later Sorger (1966b, 1967) proved, under rather wide conditions, the existence of an axially symmetric solution of the form $\mathbf{u}(\mathbf{x}) = \hat{\mathbf{u}}(r)e^{ikz}$ for Orr's variational problem for the case of circular Couette flow. He also developed a method to determine numerical values of the function $\hat{u}_r(r) = \hat{u}_r(r;k)$, and of the corresponding critical Reynolds number $\mathrm{Re}_{\mathrm{cr\ min}}\,(k)$, where $\mathrm{Re} = U(R_1)(R_2 - R_1)/\nu$, for various values of k and $\eta = R_1/R_2$. (Reynolds number $\mathrm{Re}_{\mathrm{cr\ min}}\,(k)$ determines the boundary of stability with respect to axisymmetric disturbances with wave number k).The results obtained were then compared with those obtained from the linear theory of hydrodynamic stability, and used to determine the dependence of the value of $\mathrm{Re}_{\mathrm{cr\ min}} = \min_{0\le k\le\infty}$ $\mathrm{Re}_{\mathrm{cr\ min}}\,(k)$ on the value of η.

More detailed study of the stability of circular Couette flow was carried out by Hung (1968) and Joseph (see Joseph and Hung (1971) and Joseph (1976), Chap. 5). In particular, Hung (1968) solved numerically the general (three-dimensional) Orr's variational problem relating to the circular Couette flow for a number of values of $\eta = R_1/R_2$, A and B (see Eq. (4.9)). The found solution determined the region of 'universal stability' of Couette flow to arbitrary disturbances. According to Hung's results (partially presented by Joseph (1976) in Sect. 37) the stability region in all cases studied was of the form $|\Omega_2 - \Omega_1|/\nu \le \tilde{R}_{\mathrm{cr}}(\eta)(R_2^2 - R_1^2)/(R_1R_2)^2$, where $\tilde{R}_{\mathrm{cr}}(\eta)$ is some universal function of η. We see that here again the stability region has the shape of a strip, similar to that presented in Fig. 4.2, whose width depends on values of R_1 and R_2. It was also found that the disturbances which first become unstable when $|\Omega_2 - \Omega_1|/\nu$ is increasing are axisymmetric in all the cases studied, and similar to the Taylor vortices described in Sect. 2.6. Remember that in this section it was also noted that according to experimental results over a wide range of flow conditions, when circular couette flow becomes unstable the appearing unstable disturbance mode is a set of axisymmetric Taylor vortices. These results stimulated Joseph and Hung to began a more complete energy-balance investigation of the stability of Couette flow to axisymmetric disturbances.[1]

Joseph and Hung integrated the equations of motion for the squares $u_r^2 = w^2, u_\phi^2 = v^2$ and $u_z^2 = u^2$ of the velocity components of an axisymmetric disturbance over the spatial region V' (whose span in the z-direction is equal to the wavelength of the disturbance), and considered equations for $\frac{1}{2}\frac{d}{dt}\langle w^2 + u^2\rangle = \frac{d}{dt}E^{(1)}(t)$ and $\frac{1}{2}\frac{d}{dt}\langle v^2\rangle = \frac{d}{dt}E^{(2)}(t)$ (where angle brackets denote the integrals over V') neither of which contains pressure terms (since $\partial p/\partial\phi = 0$ in the case of axisymmetric disturbances). Summing these two equations one will obtain the R-O energy Eq. (4.2) for $E(t) = E^{(1)}(t) + E^{(2)}(t)$, which is used in the energy method of stability theory. However, it is easy to see that the convergence to zero of $E(t) = \frac{1}{2}\left\langle\sum_{i=1}^{3} u_i^2\right\rangle$ is only

[1] Joseph and Hung's paper of 1971 in fact represented a continuation of the work, unknown to them, of Pritchard (1968) who studied the same problem by the same method but restricted himself to consideration of linearized dynamic equations. (Pritchard's paper will be described at greater length in Sect. 4.12).

one of many consequences of the decay to zero of disturbance velocity u(x*, t*) as $t \to \infty$. Since the half-sum of squared velocity components has an important physical meaning, the requirement that $E(t) \to 0$ is a very attractive stability condition. Nevertheless, instead of this we may in principle require that the spatial average of some other nondegenerate positive-definite quadratic form of velocity components converges to zero as $t \to \infty$. Such convergence also shows that the disturbance decays to zero (and quite often it implies also the convergence to zero of $E(t)$); hence the absence of an explicit physical meaning of the selected quadratic form cannot be considered as a radical defect of the new method. These arguments clearly allow one to suggest a great number of modifications of the classical energy method of stability analysis.

Joseph and Hung (1971) (see also Joseph (1976), Sect. 40) proposed to use the condition $d[E^{(1)}(t) + \lambda E^{(2)}(t)]/dt = dE_\lambda(t)/dt < 0$, where λ is some positive constant and the disturbance velocity field u(x) is axisymmetric, as a new condition of stability with respect to axisymmetric disturbances (replacing the more usual requirement of negativity of $dE(t)/dt$). The new condition implies that $E_\lambda(t) = E^{(1)}(t) + \lambda E^{(2)}(t)$ decays to zero monotonically as $t \to \infty$; since $E(t) \leq \min[1, \lambda]\ E_\lambda\ (t)$, the energy $E(t)$ also decays to zero in this case. However, if $\lambda \neq 1$, the set of primary Couette flows, and of initial disturbances $\mathbf{u}(\mathbf{x})$ for which $dE_\lambda\ (t)/dt < 0$, does not coincide with the similar set corresponding to the condition $dE(t)/dt \equiv dE_1(t)/dt < 0$. Therefore, the study of exact conditions guaranteeing that $dE_\lambda\ (t)/dt < 0$ for $\lambda \neq 1$ leads to the possibility of finding some new classes of stable disturbances of circular Couette flows.

The Reynolds–Orr energy-balance Eq. (4.2) implies that the given class $\breve{\mathbf{D}}$ of velocity disturbances $\mathbf{u}(\mathbf{x})$ is certainly stable (and what is more, its kinetic energy decays monotonically as $t \to \infty$), if $\mathrm{Re} < \min_{u(x)\subset\breve{\mathbf{D}}} R[\mathbf{u}(\mathbf{x})] = \mathrm{Re}_{\mathrm{cr\ min}}$, where $R[\mathbf{u}(\mathbf{x})]$ is given by Eq. (4.4) and all lengths and velocities are measured in the units L and U used in the definition of Re. (Below it will be assumed that class $\breve{\mathbf{D}}$ consists of all axisymmetric velocity fields, therefore the value $\mathrm{Re}_{\mathrm{cr\ min}}$ will refer to axisymmetric disturbances only). However the balance equation for the 'modified energy' $E_\lambda(t)$ with $\lambda \neq 1$ differs from Eq. (4.2); therefore, conditions guaranteeing that $dE_\lambda(t)/dt < 0$ must also differ from conditions guaranteeing the negativity of $dE(t)/dt$. The most important difference between the balance equations for $E_\lambda(t)$ and for $E(t)$ is due to the fact that the nonlinear terms of the N–S equations for u(*x, t*) do not contribute to the dynamic equation for $dE(t)/dt$ but do affect the value of $dE_\lambda(t)/dt$. It was noted above that the absence from Eq. (4.2) of the terms produced by the nonlinear terms in these N–S equations means that all terms on the right-hand side of this equation are of second order with respect to the velocity components u_i. Therefore, the ratio $R[\mathbf{u}(\mathbf{x})]$ of such terms does not depend on the intensity ('amplitude') of the disturbance $\mathbf{u}(\mathbf{x})$. However, if $\lambda \neq 1$, then the equation for $dE_\lambda(t)/dt$ contains a term of third order in the velocity components (equal to $(1-\lambda)\,(wv^2/r)$). This makes the sign of $dE_\lambda\ (t)/dt$ dependent not only on ν (i.e. on Reynolds number Re), the parameters R_1, R_2 and Ω_1, Ω_2 of the primary Couette flow, and the shape of the initial disturbance $\mathbf{u}(\mathbf{x})$, but also on some characteristic of the

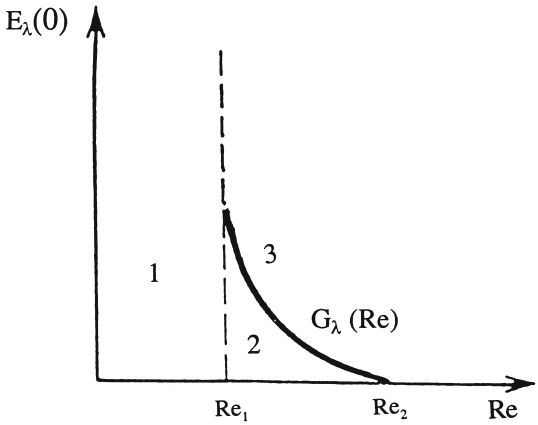

Fig. 4.3 Schematic representation of Joseph and Hung's (1971) results for stability of a flow between rotating cylinders to axisymmetric disturbances. $Re_1 = Re_{cr\ min}$; $Re_2 = Re_{\lambda,cr}$; *1*-the region of global stability and monotonic decay of $E(t)$; *2*-the region of conditional stability and monotonic decay of $E_\lambda\ (t)$; *3*-combined region of nonmonotonic stability and instability

intensity of $\mathbf{u}(\mathbf{x})$ (it is convenient to use the value of $E_\lambda(0)$ as such characteristic). As a result the main conclusion derived by Joseph and Hung from the study of conditions guaranteeing the negativity of $dE_\lambda(t)/dt$ has the form of a theorem about the *conditional stability* of axisymmetric disturbances in a circular Couette flow, determining a new stability region for such disturbances. The new stability region is a part of the (Re, E_λ)-plane which consists of such points that at the Reynolds number Re the 'generalized energy' $E_\lambda(t)$ of any axisymmetric disturbance with the initial 'energy' $E_\lambda\ (0) < E_\lambda$ decays monotonically to zero as $t \to \infty$ (and hence the energy $E(t)$ also decays to zero but its decay can be nonmonotonic). Note that for some nonnegative values of λ the new stability region can perfectly well include some points where $Re > Re_{cr\ min}$ and hence the energy $E(t)$ will not decay monotonically. In this case the new result represents an informative specification of the general statement about the possible existence of conditionally stable flows illustrated in Fig. 4.1 (see schematic Fig. 4.3 which represents graphically just this case of the Joseph and Hung theorem).

We will not give here the exact formulation of the theorem by Joseph and Hung but only its general character. The role of the functional (4.4) is now played by the functional $R_\lambda[\mathbf{u}(\mathbf{x})] = \langle D_\lambda[\mathbf{u}(\mathbf{x})\rangle / \langle P_\lambda[\mathbf{U}(\mathbf{x}), \mathbf{u}[\mathbf{x}]\rangle$, where $\langle D_\lambda[u(x)]\rangle$ is the sum of the viscous terms in the equation of motion for $E_\lambda\ (t) = \langle u^2 + \lambda\ v^2 + w^2\rangle /2$, divided by the kinematic viscosity ν (more exactly, by $(Re)^{-1}$ since the equation for E_λ (t) is now assumed to be non-dimensionalized) while $\langle P_\lambda\ [\mathbf{U}(\mathbf{x}),\ \mathbf{u}(\mathbf{x})]\ \rangle$ is the sum of production terms, linear in the undisturbed velocity gradient $dU(r)/dr$. The new functional $R_\lambda\ [\mathbf{u}(\mathbf{x})]$ is a natural replacement for the functional (4.4) when the 'modified kinetic energy' $E_\lambda\ (t)$ is considered instead of the energy $E(t)$. Let $Re_{\lambda,cr}$ denote $\min_{u(x)} R_\lambda\ [\mathbf{u}(\mathbf{x})]$, where the minimum is taken over the whole class of disturbance velocities considered (i.e., over the class of velocities of all axisymmetric disturbances). Then, if $Re < Re_{\lambda,cr}$, the sum of all right-hand-side terms of the equation for $dE_\lambda(t)/dt$ which are of second order in the components u_i will be negative. However, this does not mean that the derivative $dE_\lambda(t)/dt$ will necessarily be negative, since the term of the equation for $dE_\lambda(t)/dt$ which is of third order in the components u_i is not taken into account here. Remember that the size of this term, relative to the terms which are quadratic in u_i, depends on the intensity of the disturbance $\mathbf{u(x)}$ (which can be measured, e.g., by the modified kinetic energy $E_\lambda(0)$) and increases with increase of this intensity. Therefore, a condition guaranteeing the negativeness of $dE_\lambda(t)/dt$ must in some way restrict possible values of the initial disturbance intensity and thus diminish the possible influence of the third-order term.

These circumstances explain the following final form of the theorem found by Joseph and Hung (1971): *if* Re < $\mathrm{Re}_{\lambda,\mathrm{cr}}$ and $E_\lambda(0) < G\,(\mathrm{Re}_{\lambda,\mathrm{cr}} - \mathrm{Re}, \lambda, R_1, R_2, \Omega_1, \Omega_2)$, *where G is a definite function of given arguments proportional to* $(\mathrm{Re}_{\lambda,\mathrm{cr}} - \mathrm{Re})^2$, *then* $dE_\lambda\,(t)/dt < 0$ *for any nonnegative value of t, and* $E_\lambda\,(t)$ *decays to zero monotonically and not slower than expotentially* (hence $E(t) = \min[1, \lambda]\ E_\lambda\ (t)$ also decays to zero not slower than expotentially). This theorem clearly makes sense only if $\mathrm{Re}_{\lambda,\mathrm{cr}} > \mathrm{Re}_{\mathrm{cr\ min}} = \min_{\mathrm{u(x)}} R[\mathbf{u(x)}]$ and also $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr\ min}}$, since at $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr\ min}}$ the energy of any axisymmetric disturbance decays monotonically to zero. However, if $\mathrm{Re}_{\mathrm{cr\ min}} < \mathrm{Re} < \mathrm{Re}_{\lambda,\mathrm{cr}}$, then Joseph and Hung's theorem contains valuable information: it proves that here the 'generalized energy' $E_\lambda(t)$ of any axisymmetric disturbance, with an initial amplitude so small that $E_\lambda(0) < G = G_\lambda$ (Re) (for the sake of simplicity other arguments of the function G are here omitted) decays monotonically to zero in a circular Couette flow. This means that for Re within this interval, axisymmetric disturbances are *conditionally stable* (namely, stable under the condition that $E_\lambda(0) < G_\lambda(\mathrm{Re})$). Since the value of G is proportional to $(\mathrm{Re}_{\lambda,\mathrm{cr}} - \mathrm{Re})^2$, it vanishes at $\mathrm{Re} = \mathrm{Re}_{\lambda,\mathrm{cr}}$ and hence at this value of Re the theorem can be applied only to infinitesimal disturbances (see again the schematic Fig. 4.3 illustrating the Joseph–Hung theorem). In this figure $\mathrm{Re}_2 = \mathrm{Re}_{\lambda,\mathrm{cr}}$ represents the smallest Reynolds number at which there exists an axisymmetric disturbance having arbitrarily small value of E_λ (0) and such that its 'generalized energy' $E_\lambda(t)$ does not decay monotonically to zero as $t \to \infty$. The value $\mathrm{Re}_{\lambda,\mathrm{cr}}$, which clearly must be greater than $\mathrm{Re}_{\mathrm{cr}}$, depends on the choice of 'energy' $E_\lambda(t)$(and of the class of considered disturbances). This value differs from the Reynolds number $\mathrm{Re}_1 = \mathrm{Re}_{\mathrm{cr\ min}}$, determining the threshold below which the energy $E(t)$ of any axisymmetric disturbance decays monotonically, and can exceed this number. Figure 4.3 refers just to this case.

It was noted above that λ can be chosen as any positive number. Note now that the usefulness of the Joseph-Hung theorem increases as the number $\mathrm{Re}_{\lambda,\mathrm{cr}}$ and the function G_λ (Re) shown in Fig. 4.3 increase, leading to enlargement of the region of stable disturbances indicated in this figure. Joseph and Hung showed that when $\mathrm{Re} < \mathrm{Re}_{\lambda,\mathrm{cr}}$ is fixed, the value of $G_\lambda(\mathrm{Re})$ increases without limit as $R_1/R_2 \to 1$ or$\lambda \to 1$ (the last result agrees well with the known fact that no restriction of the disturbance amplitude is needed at $\lambda = 1$). Moreover, these authors also considered the problem of determination of the optimum value λ_0 of λ corresponding, at given values of R_1, R_2, Ω_1 and Ω_2, to the maximum possible value of $\mathrm{Re}_{\lambda,\mathrm{cr}}$. They proposed a relatively simple numerical method for computation of λ_0. Especially simple results were obtained for the case where $R_2^2\Omega_2 < R_1^2\Omega_1$ and $\Omega_2/\Omega_1 > 0$. In this case an analytic approximation of high precision was found for the optimal value λ_0. Using this approximation it was possible to compute quite accurately the values of $\mathrm{Re}_{\lambda_0,\mathrm{cr}}$ (i.e., of the maximum value of Re at which Joseph and Hung's theorem makes sense). It was found that here the values of $Re_{\lambda_0,\mathrm{cr}}$ practically coincide with the values of the critical Reynolds numbers $\mathrm{Re}_{\mathrm{cr}}$ given by the linear theory of hydrodynamic stability. This coincidence may be considered as being natural, since the critical values $\mathrm{Re}_{\mathrm{cr}}$ and $Re_{\lambda_0,\mathrm{cr}}$ both apply here only to infinitesimal axisymmetric disturbances $\mathbf{u}(\mathbf{x})$ (because for the class of Couette flows studied by Joseph and Hung the linear stability theory shows that $\mathrm{Re}_{\mathrm{cr}}$ is just the boundary of stability with respect to axisymmetric disturbances). Neverthelss, the coincidence is interesting, since it connects results obtained by two different approaches to the same problem. Note also that the approach by Joseph and Hung inspired many subsequent studies of various stability problems which will be considered at the end of Sect. 4.13.

An even more surprising coincidence relating to the same problem was found slightly earlier by Busse (1970). He considered the classical energy method of Reynolds and Orr, and compared stability results given by this method with those following from the linear theory of hydrodynamic stability. He analyzed the 'narrow gap' approximation, where $(R_2-R_1)/(R_2+R_1)\ll 1,$ assuming that the relative difference of angular velocities also asymptotically vanishes simultaneously, so that $(\Omega_2-\Omega_1)/(\Omega_2+\Omega_1)\ll 1.$ Busse found that then, if in addition $(\Omega_2-\Omega_1)/(\Omega_2+\Omega_1)=-4(R_2-R_1)/(R_2+R_1),$ the Reynolds-Orr energy method leads to an eigenvalue problem which coincides exactly with the eigenvalue problem (2.17–2.17′) arising in the linear theory of hydrodynamic stability for circular Couette flow. Therefore, in this case the stability boundaries (the critical Reynolds numbers, Re, or Taylor numbers, $\mathrm{Ta} = \Omega_1^2 R_1(R_2 - R_1)^3/\nu^2$ often used instead of Re) given by the linear stability theory for the case of infinitesimal disturbances and by the energy method for disturbances of arbitrary size are exactly the same (and hence $\mathrm{Re}_{\mathrm{cr}}=\mathrm{Re}_{\mathrm{cr\ min}}$, $\mathrm{Ta}_{\mathrm{cr}}=\mathrm{Ta}_{\mathrm{cr\ min}}$). A similar result was obtained by Busse for a plane Couette flow rotating around the y-axis with some definite angular velocity; in this case it was again found that $\mathrm{Re}_{\mathrm{cr}}=\mathrm{Re}_{\mathrm{cr\ min}}$. Some other examples (dating as far back as the 1950s) of flows where the critical values of the dimensionless flow parameter given by the linear stability theory and by the energy method coincide with each other will be considered in the following subsection; see also the paper by Wahl (1994) which contains further examples.

The discovery of flows where $\mathrm{Re}_{\mathrm{cr}}=\mathrm{Re}_{\mathrm{cr\ min}}$ evidently refutes the opinion, which was popular in the first half of the twentieth century, that the stability region given by the energy method must in principle be much smaller than the stability region determined by the linear theory of hydrodynamic stability. This discovery was then supplemented by the development by Joseph and Hung (1971) of the method which enlarged the region of validity of 'energy stability' results by introduction of the concept of 'conditional stability' and replacement of the energy density $E(t)$ by some other positive-definite functional of disturbance variables. This work led to a considerable revival of interest in the energy (and generalized-energy) methods of stability theory. Many of the papers devoted to this subject concerned motion of fluids with varying temperature (and hence also density) in a gravitational field producing a significant buoyancy effect. Therefore it will be reasonable to begin the next subsection by considering energy-method investigations of flow stability for a fluid with variable temperature.

4.1.2 Stability of Convective Motions and Related Stability Problems

In Subsect. 4.11 much attention was given to the classical Taylor problem of stability of Couette flow between coaxial rotating cylinders. However, there is another classical problem which also played a very important part in the early development of the linear theory of hydrodynamic stability. This is the famous Bénard–Rayleigh problem of stability of a stationary horizontal layer of fluid heated from below, which was considered in Sect. 2.7. Now we will turn to the applications of the energy method to this and to some other stability problems where buoyancy forces are of great importance.

In the case of motion of a fluid of variable temperature under gravity, the N–S dynamic equations must be replaced by some more general equations. Under rather general conditions, which in this book will be assumed to be always valid, we can neglect density variations except in the buoyancy term and, as in Sect. 2.7, use the Boussinesq equations. Let $\mathbf{U}(\mathbf{x})$ be the primary velocity field (it can also depend on time t but we will not consider this case) and $T(\mathbf{x})$ be the undisturbed temperature field. Then the nonlinear Boussinesq, continuity and heat conduction equations for the disturbances u_i and ϑ of the velocity and temperature will have the following form:

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_j}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + \delta_{i3} g \beta \vartheta, \quad i = 1, 2, 3, \tag{4.14a}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{4.14b}$$

$$\frac{\partial \vartheta}{\partial t} + u_i \frac{\partial T}{\partial x_i} + U_i \frac{\partial \vartheta}{\partial x_i} + u_i \frac{\partial \vartheta}{\partial x_i} = \chi \nabla^2 \vartheta, \tag{4.14c}$$

where p is the deviation of the pressure field from the undisturbed pressure P, g is the acceleration due to gravity, and β is the coefficient of thermal expansion of the fluid. The boundary conditions on stationary solid walls at constant temperature have a very simple form: $\mathbf{u}(\mathbf{x}, t) = \vartheta(\mathbf{x}, t) = 0$. More complicated boundary conditions must be used in the cases of moving walls, solid walls of non-constant temperature (i.e., those which have fixed finite thermal conductivity, or are characterized by fixed heat flux normal to the wall), and free surfaces of liquids; see e.g. the discussion of this question in Sect. 2.7 of this book, and in Sect. 55 of Joseph's book (1976) where some additional references relating to this subject can also be found. However, in the discussion below most attention will be given to the simplest case of zero boundary conditions for velocity and temperature disturbances at the walls.

The Boussinesq Eq. (4.14a) differ from the N-S equations only by the additional term $g\beta\vartheta$ in the equation for $u_3 = w$. This term produces an extra term in the energy-balance Eq. (4.2) which now takes the form

$$\frac{dE(t)}{dt} = -\int\limits_{v'} u_j u_i \frac{\partial U_j}{\partial x_i} d\mathbf{x} + g\beta \int\limits_{v'} u_3\vartheta d\mathbf{x} - \nu \int\limits_{v'} \sum_{j,i=1}^{3} \left(\frac{\partial u_j}{\partial x_i}\right)^2 d\mathbf{x}$$

$$= -\left\langle u_j u_i \frac{\partial U_i}{\partial x_i}\right\rangle + g\beta\,\langle u_3\vartheta\rangle - \nu\left\langle \sum_{j,i=1}^{3} (\frac{\partial u_j}{\partial x_i})^2\right\rangle = \mathbf{P}_1 + \mathbf{P}_2 - \nu\mathbf{D} \quad (4.15)$$

which differs from the R-O Eq. (4.2) by the extra term $\mathbf{P}_2 = g\beta\,\langle u_3\vartheta\rangle$ on the right side. This term can be easily estimated by the following crude inequality

$$g\beta\langle u_3\vartheta\rangle \le g\beta\langle|u_3\vartheta|\rangle \le g\beta\langle u_3^2\rangle\langle\vartheta^2\rangle^{1/2} \le 2g\beta[E(t)E_T(t)]^{1/2} \quad (4.16)$$

where $E_T(t) = 0.5\langle\vartheta^2\rangle$ is an integral measure of the intensity of temperature disturbance (while $\langle\vartheta^2\rangle$ is often called the 'temperature variance'). Moreover, the heat-conduction Eq. (4.14c), together with the boundary conditions given above, leads to the following balance equation for the temperature-disturbance intensity $E_T(t)$

$$\frac{dE_T(t)}{dt} = -\left\langle \vartheta u_i \frac{\partial T}{\partial x_i}\right\rangle - \chi\left\langle \sum_{i=1}^{3}\left(\frac{\partial\vartheta}{\partial x_i}\right)^2\right\rangle = \mathbf{P}_T - \chi\mathbf{D}_T. \quad (4.17)$$

Here the first term on the right-hand side is clearly less than or equal to $2\gamma[E(t)E_T(t)]^{1/2}$ where $\gamma = \max_{\mathbf{x}\in v}|\nabla T(\mathbf{x})|$, while the following analog of the inequality (4.6) can be proved for the factor $\mathbf{D}_\mathrm{T}$ in the second term

$$\mathbf{D}_\mathrm{T} = \left\langle \sum_{i=1}^{3}\left(\frac{\partial\vartheta}{\partial x_i}\right)^2\right\rangle \ge \frac{a_T\pi^2}{D^2}\langle\vartheta^2\rangle = \frac{2a_T\pi^2}{D^2}E_T(t). \quad (4.18)$$

In Eq. (4.18) $a_T = 3$ if V is a bounded region of diameter D, and $a_T = 1$ for a horizontal layer of maximal thickness D. Combining the balance Eqs. (4.15) and (4.17) with the estimates of the terms of these two equations given above, we may obtain for the derivative $d[\sqrt{E(t)} + \lambda\sqrt{E_T(t)}]/dt$ (where the dimensional factor λ has a positive value) an inequality of the form

$$\frac{d[\sqrt{E(t)} + \lambda\sqrt{E_T(t)}]}{dt} \le \lambda_1\sqrt{E(t)} + \lambda_2\sqrt{E_T(t)} \quad (4.19)$$

where expressions for the coefficients λ_1 and λ_2 include the dimensional constants λ; $\gamma = \max|\nabla\mathrm{T}|$; and the coefficients entering Eqs. (4.6), and (4.15–4.18). If $\lambda = \sqrt{(a - \mathrm{Re}^2)g\beta\nu/2a_T\pi^2\gamma\chi} = \lambda_0\sqrt{g\beta\nu/\gamma\chi}$, where $Re = U_{max}D/\nu, \lambda_0^2 =$

$(a - \mathrm{Re}^2)/2a_T\pi^2$ is a dimensionless constant, and it is assumed that $a > \mathrm{Re}^2$, then the inequality (4.19) takes an especially useful form. In this case $\lambda_1 = -\xi[\sqrt{(a-\mathrm{Re}^2)a_T\pi/2} - \sqrt{Ra}] = -\xi(\lambda_0 a_T\pi^2 - \sqrt{Ra}), \lambda_2 = \lambda\lambda_1$, where $\xi = \lambda_0\nu/D^2$ if $\lambda_0(\mathrm{Pr})^{1/2} \le 1$ and $\xi = \chi/\lambda_0 D^2$ if $\lambda_0(\mathrm{Pr})^{1/2} > 1$, and where $\mathrm{Ra} = g\beta\gamma D^4/\nu\chi$ and $\mathrm{Pr} = \nu/\chi$ (cf. Joseph (1965)). It follows from this result that the convective motion will be universally (in other words, unconditionally or globally) stable to any disturbance of the velocity and/or temperature if

$$0 \le \mathrm{Ra} < \frac{a_T\pi^2(a - \mathrm{Re}^2)}{2} \tag{4.20}$$

since under this condition, for the value of λ indicated above, we have

$$\sqrt{E(t)} + \lambda\sqrt{E_T(t)} \le [\sqrt{E(0)} + \lambda\sqrt{E_T(0)}] \exp\left\{-\xi[\sqrt{a_T\pi^2(a-\mathrm{Re}^2)/2} - \sqrt{\mathrm{Ra}}]t\right\}. \tag{4.21}$$

The results (4.20–4.21) (obtained by Joseph(1965, 1966) in slightly different from) are similar to the Serrin-Velte-Sorger results of 1959–1967, derived for constant-density (non-convective) flows: they do not depend on any specific details of the flow geometry or on the distributions of the primary velocity and temperature fields. For the special case of a stationary horizontal fluid layer (for which $a_T = 1, a = 3.7\pi^{2,}$ and $\mathrm{Re} = 0$) we obtain the result: $\mathrm{Ra}_{\mathrm{cr\ min}} > 1.85\pi^4 \approx 180$. The last result can easily be improved; in fact the inequality (4.7) is clearly unsatisfactory in the case of stationary fluid where its left-hand side is equal to zero. If we simply omit the first term on the right-hand side of (4.15) and then repeat all the arguments, we obtain twice as good an estimate: $\mathrm{Ra}_{\mathrm{cr\ min}} > 360$ (which is still much smaller than the value $\mathrm{Ra}_{\mathrm{cr}} = 1{,}708$ given by linear theory). A similar improvement can also be made in the estimate (4.20) of the boundary of the universal stability region in the (Ra, Re)-plane if one uses a different estimate of the first term on the right-hand side of Eq. (4.15) (giving zero for fluid at rest) and another definition of Reynolds number (see Joseph (1965)). Note however that in the case of primarily stationary fluid all the results obtained in this way were much weaker than the older results of Sorokin (1953, 1954) and several other workers who studied conditions for the appearance of convection in fluids at rest.

Sorokin considered the stability problem for a stationary fluid in a given spatial region V (he assumed it to be bounded but his arguments can be applied to many unbounded regions too). Using Eqs. (4.14a–c) he proved that under very general conditions (which are satisfied in almost all situations of practical interest) $\mathrm{Ra}_{\mathrm{cr\ min}} = \mathrm{Ra}_{\mathrm{cr}}$ where $\mathrm{Ra}_{\mathrm{cr}}$ is the critical Rayleigh number determined by the linear theory of hydrodynamic stability, while $\mathrm{Ra}_{\mathrm{cr\ min}}$ is the stability boundary given by the energy method. (This means that for $\mathrm{Ra} < \mathrm{Ra}_{\mathrm{cr\ min}}$ both $E(t)$ and E_T(t) decay monotonically with time).

Morcover, Sorokin also proved that the principle of exchange of stabilities is valid here, i.e. that the eigenfrequency ω corresponding to the most unstable mode, if such a mode exists, is real (and that all other eigenfrequencies ω_j are also real here). (Remember, that for the Bénard problem, where V is an infinite horizontal layer, the principle of exchange of stabilities was first proved by Pellew and Southwell (1940); see Sect. 2.7). At first, these important papers by Sorokin did not attract much attention, and some of his results were later independently rediscovered by a number of authors (in particular, by Ukhovskii and Yudovich (1963); Howard (1963); Sani (1964), and Platzman (1965)). Then Joseph (1965, 1966) also independently derived Sorokin's results, and some of their generalizations, by a new method and under more general conditions than those used in the previous publications. His derivation was later described in the book by the same author (see Joseph (1976), Chap. VIII) which played a very important part in the revival of interest in energy methods. Therefore only Joseph's approach will be outlined below.

Seeking the stability boundary in the (Ra, Re)-plane, Joseph investigated conditions guaranteeing the decay with time of the quantity $E_\lambda(t) = E(t) + \lambda E_T(t)$ where, as above, λ is a dimensional factor having positive value. According to Eqs. (4.15) and (4.17), the right-hand side of the equation for $dE\lambda(t)/dt$ includes three "production terms", $\mathbf{P}_1$, $\mathbf{P}_2$ and $\lambda\mathbf{P}_{\mathrm{T}}$, and two "dissipation terms", $-\nu\,\mathbf{D}$ and $-\lambda\chi\mathbf{D}_{\mathrm{T}}$. The production terms can take positive values and they then describe the growth of the intensity of the velocity and temperature disturbances, caused by the interaction of flow disturbances with the primary flow. As to the dissipation terms, they are always negative and represent the decay of disturbance velocity and temperature fields caused by molecular viscosity and heat conductivity. Therefore, for decay of the 'modified energy' $E\lambda(t)$ of a disturbance with given velocity and temperature fields $\{\mathrm{u(x)}, \vartheta(\mathrm{x})\}$, the sum of the absolute values $\nu\,\mathbf{D}$ and $\lambda\chi\;\mathbf{D}_{\mathrm{T}}$ of the dissipation terms must be greater than $\mathbf{P}_1 + \mathbf{P}_2 + \lambda\mathbf{P}_{\mathrm{T}}$.

Joseph made the balance Eqs. (4.15), (4.17) and the equation for $dE\lambda(t)/dt$ dimensionless, replacing the dimensional independent and dependent variables $x_i, t, U_i, u_i, T, \vartheta, E, E_T$, and the coefficient λ by $x_i^+ = x_i/L, t^+ = t\nu/L^2, U_i^+ = U_i/U_0, u_i^+ = u_i L/\nu, T^+ = T/\Theta_0, \vartheta^+ = \vartheta(\chi g\beta L^3/\nu^3\Theta_0)^{1/2}, E^+ = E/\nu^2 L, E_T^+ = E_T\chi g\beta/\nu^3\Theta_0$, and $\lambda^+ = \lambda(\Theta_0/g\beta L)$ where L, U_0 and θ_0 are typical length, velocity and temperature scales of the primary flow (these scales must be chosen in a reasonable way for every specific problem). It is easy to verify that Eqs. (4.15) and (4.17) then take the following forms

$$\frac{dE^+(t^+)}{dt^+} = -\,\mathrm{Re}\left\langle u_j^+ u_i^+ \frac{\partial U_j^+}{\partial x_i^+}\right\rangle + \sqrt{\mathrm{Ra}}\langle u_3^+\vartheta^+\rangle - \left\langle \sum_{i,j=1}^{3}\left(\frac{\partial u_i^+}{\partial x_j^+}\right)^2\right\rangle, \qquad (4.15\mathrm{a})$$

and

$$\mathrm{Pr}\,\frac{dE_T^+(t^+)}{dt^+} = -\sqrt{\mathrm{Ra}}\left\langle \vartheta^+ u_i^+ \frac{\partial T^+}{\partial x_i^+}\right\rangle - \left\langle \sum_{i=1}^{3}\left(\frac{\partial \vartheta^+}{\partial x_i^+}\right)^2\right\rangle \qquad (4.17\mathrm{a})$$

where $\mathrm{Re} = U_0 L/\nu, \mathrm{Ra} = g\beta\Theta_0 L^3/\nu\chi, \mathrm{Pr} = \nu/\chi$, and angular brackets now denote integration, with respect to dimensionless coordinates x_i^+, over the region V'.

Also, changing to dimensionless variables makes the quantity $E_\lambda(t) = E(t) + \lambda E_T(t)$ proportional to $E^+(t^+) + \lambda^+ \mathrm{P}_T E_T^+(t^+)$. In the rest of this section we will use only dimensionless variables, and for simplification of notation we will omit the superscript 'plus' signs. Hence, the symbol $E_\lambda(t)$ will now denote the sum $E(t) + \lambda\,\mathrm{Pr}E_T(t)$, and according to Eqs. (4.15a)and (4.17a) the balance equation for this quantity has the form

$$\begin{aligned}\frac{\partial E_\lambda(t)}{\partial t} &= -\,\mathrm{Re}\left\langle u_i u_j \frac{\partial U_j}{\partial x_i}\right\rangle + \sqrt{\mathrm{Ra}}\left(\langle u_3 \vartheta\rangle - \lambda\left\langle \vartheta u_i \frac{\partial T}{\partial x_i}\right\rangle\right) \\ &\quad -\left\langle \sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2 + \lambda \sum_{i=1}^{3}\left(\frac{\partial \vartheta}{\partial x_i}\right)^2\right\rangle \\ &= \mathrm{Re}\,\mathbf{P}_1 + \sqrt{\mathrm{Ra}}(\mathbf{P}_2 + \lambda \mathbf{P}_T) - \mathbf{D} - \lambda \mathbf{D}_T. \end{aligned} \tag{4.22}$$

Equation (4.22) takes an especially simple form in the case of stationary fluid, where $\mathrm{Re} = 0$. Here the critical Rayleigh number of the energy theory, $\mathrm{Ra}_{\mathrm{cr\,min}}$, can be determined from the equation

$$\mathrm{Ra}_{\mathrm{cr\,min}} = \left[\max_\lambda \min_{\mathbf{u}(\mathbf{x}),\vartheta(\mathbf{x})} \frac{\mathbf{D} + \lambda \mathbf{D}_T}{\mathbf{P}_2 + \lambda \mathbf{P}_T}\right]^2 \tag{4.23}$$

where the minimum is taken over all solenoidal vector fields $\mathbf{u(x)}$ and scalar fields $\vartheta(\mathbf{x})$ satisfying the boundary conditions appropriate to the problem considered, and the maximum over all nonnegative values of λ (thus, the value of λ is varied in the search for the highest estimate of $\mathrm{Ra}_{\mathrm{cr\,min}}$). It can be shown that in the case where $\lambda = 1$ the Euler–Lagrange equations corresponding to the variational problem of finding the minimum in the right-hand side of (4.23) can be reduced to the same eigenvalue problem that appears in the linear stability theory applied to a given volume of stationary fluid with given temperature field $T(\mathbf{x})$. The critical Rayleigh number of linear stability theory, $\mathrm{Ra}_{\mathrm{cr}}$, is determined by the solution of this eigenvalue problem for the case of zero frequency (i.e. the eigenvalue $\omega = 0$) in exactly the same way that $\mathrm{Re}_{\mathrm{cr\ min}}$ is determined by the solution of the eigenvalue problem derived from the Euler-Lagrange equations.This means that $\mathrm{Re}_{\mathrm{cr\,min}} = \mathrm{Re}_{\mathrm{cr}}$ in this case, and that the optimal value of λ in Eq. (4.23) is $\lambda = 1$ (since the value of $\mathrm{Re}_{\mathrm{cr\,min}}$ clearly cannot be greater than $\mathrm{Re}_{\mathrm{cr}}$). In particular, for the Bénard-Rayleigh problem of stability of a horizontal layer of stationary fluid heated from below we find that $\mathrm{Re}_{\mathrm{cr\ min}} \approx 1{,}708$ in the case of two rigid walls at constant temperatures, while $\mathrm{Ra}_{\mathrm{cr\ min}} \approx 1{,}101$ for one rigid and one free boundary and $\mathrm{Ra}_{\mathrm{cr\ min}} \approx 657$ for the idealized case of two free boundaries, if the values $L = H$ (the distance between two walls) and $\Theta_0 = \Delta T$ (the difference between lower-wall and upper-wall temperatures) are used in the definition of the Rayleigh number (see Sect. 2.7).

In the more general case of a flow satisfying the Boussinesq equations and having given velocity and temperature fields $\mathbf{U(x)}$ and $T(\mathbf{x})$, the energy method can be used to find the stability boundary in the (Re, Ra)-plane, determining the region of (Re, Ra)-values that guarantes the decay of any initial disturbance regardless of its size. Such decay will clearly occur if, for at least one positive value of λ, the sum of dissipation terms $\mathbf{D}+\lambda\mathbf{D}_\mathrm{T}$ for any initial disturbance $\{\mathbf{u}(\mathbf{x}),\ \vartheta(\mathbf{x})\}$ is greater than the sum of production terms Re $\mathrm{P}_1+\sqrt{\mathrm{Ra}}(\mathbf{P}_1+\lambda\ \mathbf{P}_T)$. Hence the stability boundary in the (Re, Ra)-plane will now coincide with the boundary of the largest region in this plane in which, for some positive value of λ, the ineaquality $(\mathbf{D}+\lambda\mathbf{D}_\mathrm{T})\geq\ (\mathrm{Re}\mathbf{P}_1+\sqrt{\mathrm{Ra}}(\mathbf{P}_2+\lambda\mathbf{P}_T))$ is valid for any values of $\{\mathrm{u(x)},\ \vartheta(\mathrm{x})\}$ satisfying the boundary conditions of the problem considered.

Determination of the boundary curve in the (Re, Ra)-plane is a more difficult problem than in the case of fluid at rest, when only a boundary point on the Ra-axis must be found. Joseph (1966) proposed to assume at the beginning that that $\mathrm{Re}/\sqrt{\mathrm{Ra}}=\mu$ is fixed. Then at fixed values of λ and μ the boundary value of Ra (and hence also of $\mathrm{Re}=\mu\sqrt{\mathrm{Ra}}$) may be found from the equation

$$Ra(\lambda,\mu)=\left[\min_{\mathbf{u}(\mathbf{x}),\partial(\mathbf{x})}\frac{\mathbf{D}+\lambda\mathbf{D}_T}{\mu\mathbf{P}_1+\mathbf{P}_2+\lambda\mathbf{P}_T}\right]^{\mathbf{2}}. \tag{4.24}$$

It follows that, at a fixed value of μ, the optimal value of Ra (i.e., the value of $\mathrm{Ra}(\mu)_{\mathrm{cr\ min})}$ is equal to $\max_{\lambda\,>\,0}$ Ra(λ, μ). Then $(\mu\sqrt{\mathrm{Ra}_{\mathrm{cr\ min}}(\mu)},\ \mathrm{Ra}_{\mathrm{cr\ min}}\ (\mu))$ is a point of the boundary curve in the (Re, Ra)-plane and the set of all such points corresponding to nonnegative values of μ forms the whole of this curve.

As an example Joseph considered the case of a plane Couette flow heated from below, i.e., of a Couette flow in a layer between rigid planes at $z=0$ and $z=H$ having different temperatures T_0 and $T_1=T_0-\Theta_0$ where $\Theta_0>0$. The determination of the boundary curve in the (Re, Ra)-plane can be simplified here, since it can be proved that the optimal value of λ is 1 (hence $\max_{\lambda\,<\,0}\mathrm{Ra}(\lambda,\ \mu)=\mathrm{Ra}(1,\ \mu)$ at any μ). (This is connected with the fact that the temperature gradient ∇T is directed everywhere along the negative z-axis, i.e., has the same direction as the acceleration due to gravity; see Joseph (1976), Sect. 61, and Straughan (1992), pp. 60–61). Moreover, it can be shown that the most-unstable disturbance $\{\mathbf{u}(\mathbf{x}),\ \vartheta(\mathbf{x})\}$ minimizing the functional on the right-hand side of Eq. (4.24) has the form of streamwise rolls independent of the horizontal coordinate x. This allows the search for the minimum in the right side of (4.24) to be reduced to an eigenvalue problem for a system of ordinary differential equations. It turns out that this system may be transformed, by simple replacement of parameters, into a system equivalent to that appearing in the linear stability theory of a stationary layer of fluid heated from below. Using the known results of this theory, Joseph proved that any disturbance $\mathbf{u}(\mathbf{x})$, $\vartheta(\mathbf{x})$ will decay in a plane Couette flow heated from below if the values (Re, Ra) satisfy the inequality $4\mathrm{Re}^2+\mathrm{Ra}<1{,}708$, where Ra has the same meaning as above and $\mathrm{Re}=U_0H_1/\mathrm{v}$ where U_0 is the half-difference of the two wall velocities and $H_1=H/2$ is the half-width of the channel (this is the definition of Re for a plane Couette flow already used in Sects. 2.1, 3.3 and 3.4). This result shows again that for the case of fluid at rest (when $\mathrm{Re}=0$) the energy method gives the estimate $\mathrm{Ra}_{\mathrm{cr\ min}}=1{,}708$, the same as the critical value given by linear stability theory. At the same time, for the unstratified problem (when $\mathrm{Ra}=0$) the estimate found ($\mathrm{Re}_{\mathrm{cr}}\ \mathrm{min}=\sqrt{1{,}708}/2\approx 20.7$) is much smaller, not only smaller than the critical value $\mathrm{Re}_{\mathrm{cr}}=\infty$ given by the normal-mode method of linear stability theory but also smaller than the minimal values Re_1 of Re at which the instability of a plane Couette flow has been observed in the most accurate modern experiments and numerical simulations. (Recall that according to results presented in Sect. 2.1 of Chap. 2, Re_1 lies in the range from 320 to 370. Note also the conclusion by Hamilton et al. (1995) that turbulence cannot be sustained in a plane Couette flow at $Re \leq 300$, and the results of recent experiments by Bottin et al. (1998a, b) and Bottin and Chaté (1998), and numerical simulations by Barkley and Tuckerman (1998, 1999) according to which $Re_1 \approx 325$). It is, however, incorrect to say, as it often is, that it follows that the energy method is exact in the case of the pure convection problem but gives very poor results when applied to the non-convective Couette flow. In fact the results show only that for the Bénard-Rayleigh problem $Ra_{cr\,min} = Ra_{cr}$ (which is an exception), while for a plane Couette flow $Re_{cr\,min}$ is much smaller than $Re_{cr,}$ while the minimal value of Re at which instability is observed is here greater than $Re_{cr\,min}$ but smaller than Re_{cr} (this may be considered as being normal).

The methods for determination of the stability boundaries by the energy method and its modifications developed by Joseph (1965, 1966) can be applied to many other fluid-dynamic problems. A number of such problems was considered in Joseph's book (1976). Thus, for example, stability was studied for flows of a liquid with density depending on disturbed fields both of temperature, $T(\mathbf{x}) + \vartheta\,(\mathbf{x}, \mathrm{t})$, and of concentration of some admixture (e.g., salinity), $C(x) + c(x,\ t)$. The Boussinesq approximation was assumed to be valid here too but now it leads to an equation for $u_3 = w$ which includes a term proportional to c; therefore the diffusion equation must now be added to Eq. (4.14). In this case, Joseph replaced the function $E_\lambda(t) = E(t) + \lambda\, E_T(t)$ by the function $E_{\lambda_1,\lambda_2}(t) = E(t) + \lambda_1 E_T(t) + \lambda_2 E_C(t)$, where $E_C(t) = \langle c^2 \rangle$. It was shown that if the liquid is stationary, while the temperature gradient is directed downwards and the salinity gradient is directed upwards ('heating from below and salting from above') the critical parameters obtained from the linear and energy theories coincide, as in the case where only heating from below takes place. However, if both gradients ∇T and ∇C are directed downwards (the case of heating and salting from below) the two gradients produce opposite effects and here quite new solutions can appear. Other applications of energy methods considered in Joseph's book include, in particular, the cases of Boussinesq fluids with internal heat sources; convection in spherical layers, in porous media heated from below and in some non-Newtonian fluids; and stability of magneto-hydrodynamic flows. For more details relating to these and other applications of energy methods see, e.g., the papers by Joseph and Shir (1966); Joseph and Carmi (1966); Shir and Joseph (1968); Joseph (1970, 1988); Bhattacharyya and Jain (1971), and Ayyaswami (1971), and the numerous publications on this subject appearing in the 1980s and early 1990s. These newer publications include special monographs by Straughan (1982, 1992) and Galdi and Rionero (1985), a collection of papers edited by Galdi and Straughan (1988), an extensive survey paper by Galdi and Padula (1990) (these sources contain several hundred references), and a great number of research papers only a small part of which will be referred to below.

In the more recent literature on the energy method in hydrodynamic stability, most effort has been devoted to the extension of the classical Reynolds–Orr method of nonlinear stability analysis. Remember that in some of the above-mentioned stability investigations conditions were considered for the decay, not of $E(t)$ but of some other positive functions $E_\lambda(t)$ (or $E_{\lambda_1,\lambda_2}(t)$). Thus, the stability criteria were based, not on the kinetic energy of disturbance but on some other positive-definite quadratic forms of disturbance variables. It was therefore only natural that later some authors began the search for possible improvements of known results of the energy method by replacing the energy functional $E(t)$ by another integrated positive definite quadratic form. Some of these methods of stability analysis were called *weighted energy methods* while the name *generalized energy methods* was often applied to all such methods. However, even more often they are called *Lyapunov methods* since in fact they represent an application to fluid mechanics of the well-known direct (or second) Lyapunov method of stability analysis. (This method forms the most important part of the general theory of stability of motion developed by Lyapunov (1892) in his doctoral dissertation[2]). The direct Lyapunov method later gained wide popularity and was expounded in a great number of textbooks, special monographs, and collections of papers (see, e.g., Zubov (1957); LaSalle and Lefschetz (1961); Kazda (1962); Hahn (1963); Yoshizawa (1996), and Rouche et al. (1977)). In the first half of the twentieth century this method was mostly used to study the stability of dynamic systems having a finite number of degrees of freedom and described by ordinary differential equations; later, however, some of its applications to systems described by partial differential equations were also considered, e.g., by Zubov (1957); Movchan (1959); Knops and Wilkes (1966), and Lakshmikantham and Leela (1969). In the 1960s the first applications of the Lyapunov method to fluid mechanics appeared, quite independently of work based on the R-O Eq. (4.2). Later, Lyapunov's approach to stability of fluid motion underwent considerable development, and in fact formed a new branch of hydrodynamic stability theory having many points of contact, but nevertheless not merging, with work on generalizations of the classical energy method of Reynolds and Orr.

4.1.3 Applications of the Direct Lyapunov Method and Generalized Energy Functionals. Arnol'd's Variational Method

Lyapunov's stability was mentioned in Sects. 3.21 and 3.23, when the papers by Dikii (1960a, b) were considered. As was explained in Sect. 3.21 (see in particular footnote no. 1 there) Lyapunov's stability presupposed that some norm $\|\bullet\|$

[2] About 25 years later, in 1918, this brilliant Russian scientist, a member of the Russian Academy of Sciences, died at the age of 61 from hunger and lack of appropriate medical help in the city of Odessa enveloped in a civil war between bolsheviks and their opponents.

was introduced in the phase space H of the dynamical system considered, making **H** *a linear normed space*[3]. In problems on hydrodynamic stability*, Lyapunov's stability* of the 'primary flow' $\mathbf{U}_0(t)$, $0 \le t < \infty$ (where $\mathbf{U} = \mathbf{U}(\mathbf{x})$ is a collection of hydrodynamic fields uniquely determining the flow), means that for any $\varepsilon > 0$ there exists such a number $\delta(\varepsilon) > 0$ that the inequality $\|\mathbf{U}(0) - \mathbf{U}_0(0)\| < \delta(\varepsilon)$ implies that $\|\mathbf{U}(t) - \mathbf{U}_0(t)\| < \varepsilon$ for any nonnegative t. (Sometimes it is also additionally required that $\|\mathbf{U}(t) - \mathbf{U}_0(t)\| \to 0$ as $t \to \infty$, either for any $\mathbf{U}(0)$ or under the condition that $\|\mathbf{U}(0) - \mathbf{U}_0(0)\| < d$ for some given $d > 0$; if so then Lyapunov's stability is called *asymptotic*). The phase space H is here the functional space of all possible values of $\mathbf{U}(\mathrm{x})$ (in the cases where the velocity field uniquely determines the flow, **H** is the space of all solenoidal vector fields **u(x)** satisfying the appropriate boundary conditions). The norm in such a space is usually given by the square root of the integral, over the set of points **x**, of some non-degenerate positive-definite quadratic form of components $U_1(\mathbf{x})$, $U_2(\mathbf{x})$, $U_n(\mathbf{x})$ of the vector function $\mathrm{U}(\mathrm{x})$. Then $\|\mathrm{U}(\mathbf{x})\|^2$, the square of the norm of $\mathbf{U}(\mathbf{x})$, is a function of the functional argument $\mathbf{U}(\mathbf{x})$. Functions of functional arguments in mathematics are called *functionals;* therefore $\|\mathbf{U}(\mathbf{x})\|^2 = L[\mathbf{U}(\mathbf{x})]$ is a functional in the space H. The Lyapunov condition for stability (representing the main theorem of Lyapunov's second method) in application to stability of the primary flow $\mathbf{U}_0(\mathbf{x}, t)$ has the following form: If $\mathbf{U}(\mathbf{x}, t) = \mathbf{U}_0(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t)$ (so that $\mathbf{u}(\mathbf{x}, t)$ is a disturbance of the flow $\mathbf{U}_0(\mathbf{x}, t)$), then the flow $\mathbf{U}_0(\mathbf{x}, t)$ will be stable with respect to the norm $\| \bullet \|$ if $\mathrm{d}L[\boldsymbol{u}(\boldsymbol{x}, \mathrm{t})]/dt' < 0$ for any $\mathbf{u}(\mathbf{x}, t) \in H$ and any $t > 0$. The functional $\mathrm{L}[\mathrm{u}(\mathrm{x}, t)]$ satisfying the given conditions is called the *Lyapunov functional* (in the case of dynamic systems with a finite number of degrees of freedom the simpler name *Lyapunov function* is used). Some other formulations of conditions characterizing Lyapunov's functionals, and much additional information about the direct Lyapunov method of the study of stability, can be found in the literature on this subject mentioned above. Note only that since the definition of the norm $\| \bullet \|$, the existence of such a functional does not guarantee the stability of the given flow with respect to norms different form $\| \bullet \|$; in fact, a flow which is stable with respect to one norm can perfectly well be unstable with respect to some other norm.(Some examples of this phenomenon will be considered later in this subsection). Note also that, unfortunately, "there are no clear guidelines of how to choose Lyapunov's functionals; what is required is a little experience and a lot of luck" (this remark is due to Payne (1975); see also Rionero (1988)). However, Lyapunov's method of stability analysis has nevertheless proved to be very useful in many applications, and has been repeatedly applied to problems of hydrodynamic stability.

One of the first applications of Lyapunov's method to problems of hydrodynamic stability was due to Dikii (1960a, b), who did not indicate this explicitly but in fact investigated precisely the Lyapunov stability of the flows he considered. Since this author used only linearized dynamic equations, his results were given in Chap. 3 of

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[3] The definition of such spaces and description of their main properties can be found, for example, in the book by Kolmogorov and Fomin (1957).

this book, the present Chapter being nominally on nonlinear methods. Dikii studied the stability of two-dimensional disturbances of plane-parallel inviscid flows; therefore, here a scalar field of the stream function $\Psi(x, z, t) = \Psi(z, t)e^{ikx}$ (or of the vertical velocity $w = \partial\Psi/\partial z$) of a disturbance could be used as the field of functions $\mathbf{U}(\mathbf{x}, t) - \mathbf{U}_0(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, \mathrm{t})$. In Dikii's paper (1960b) the flow of homogeneous fluid between two solid walls was investigated and it was proved that under certain conditions the values of $|\Psi(x, z, t)|$ (where x and z are fixed, but t can take any nonnegative value) are bounded by a constant decreasing to zero when the initial values of the function Ψ, and of its spatial derivatives of the first and second orders, tend to zero. It is clear that this means that the flow is stable according to Lyapunov, with respect to a norm $\|\Psi(z,t)\|$ given by the square root of the integral with respect to z of a linear combination of $|\Psi|^2$, $|\Psi|^2$ and $|\Psi''|^2$ where primes denote d/dz (the appropriate norm is given by Eq. (2.75); see also Dikii (1976)). In the paper (1960a) the flow of an inhomogeneous fluid with the density profile $\rho(z) = \rho_0 \exp(-az)$, where $0 \le z < \infty$, was studied; here the Lyapunov stability was considered for a norm given by the square root of the integral with respect to z of a linear combination of $|\Psi|^2$ and $|\Psi'|^2$ only. The Lyapunov stability of the flows considered was proved by Dikii for the same conditions under which their asymptotic stability (i.e., asymptotic decay of the function $\Psi(x, z, t)$ as $t \to \infty$) was independently proved in the papers by Case (1960a, b) (see Chap. 3 for additional details).

Later Pritchard (1968) applied Lyapunov's method to a study of the two most famous problems of hydrodynamic stability—the Rayleigh-Bénard problem of convection in a layer of stationary fluid heated from below, and the Taylor-Couette problem of stability of flow between coaxial rotating cylinders. Like Dikii, he considered only linearized dynamic equations but took into account the effects of molecular viscosity and thermal diffusivity neglected by Dikii. In Sect. 2.7 it was shown that, in the case of the Rayleigh-Bénard problem, linearized equations for the disturbance $\mathbf{u}(\mathbf{x}, t)$, $\vartheta(\mathbf{x}, t)$ can easily be transformed into a system of two equations with unknowns $u_3 = w$ and ϑ; therefore here the space of pairs of scalar functions $\{\mathrm{w}(\mathbf{x}, t), \vartheta(\mathrm{x}, t)\}$, periodic with respect to coordinates $x_1 = x$ and $x_2 = \mathrm{y}$ and satisfying definite boundary conditions at $\mathrm{x}_3 = z = 0$ and $z = H$, can be taken as the space H. (The boundary conditions are naturally different for the cases of two rigid, two free, and one rigid and one free surfaces considered by Pritchard; see the discussion of this topic in Sect. 2.7). In the case of Taylor–Couette flow, only disturbances that were axisymmetric (independent of ϕ) and periodic in the z-direction were studied in Pritchard's paper. Therefore, here $\boldsymbol{H}$ was the space of functions $\{u(r, z, t), v(r, z, t), w(r, z, t)\}$ satisfying the axisymmetric continuity equation $r^{-1}\partial(ru)/\partial r + \partial w/\partial z = 0$, periodic with respect to z and vanishing on the walls at $r = R_1$ and $r = R_2$. The Lyapunov functional $L = \| \bullet \|^2$ in $\boldsymbol{H}$ in the case of the Rayleigh–Bénard problem was chosen to have the form $L[w, \vartheta] = \|(w, \vartheta)\|^2 = \int_{V'} [w^2 + k^{-2}(\partial w/\partial z)^2 + \lambda P\mathrm{r}\vartheta^2] d\mathbf{x}$, where $k = k_3$ is the wavenumber, $\mathrm{Pr} = \nu/\chi$ is the Prandtl number and λ is a positive constant whose value can be varied in search of the strongest stability criterion. In the case of the Talylor–Couette problem, Pritchard assumed that $L[u, v, w] = \|(u, v, w\|^2 = \pi \int_{V'} (u^2 + \lambda v^2 + w^2)\, r dr dz$, i.e. the norm $\|(u, v, w)\| =$

$[E\lambda(t)]^{1/2}$ was used which was independently applied, slightly later, to the nonlinear extension of the same problem by Joseph and Hung (1971). (Remember that these authors also considered only disturbances which were axisymmetric and periodic with respect to z). To find conditions guaranteeing the negativity of the derivative $dL(t)/dt$, where $L(t) = L[w(\mathbf{x}, t), \vartheta(\mathbf{x}, t))$ or alternatively $L(t) = L[u(r, z, t), v(r, z, t), w(r, z, t)]$, Pritchard derived a number of new integral inequalities. Using them he found that the inequality $dL/dt < 0$ is valid for values of the dimensionless primary-flow parameters Ra (the Rayleight number) or Ta (the Taylor number) smaller than some value of Ra_c or Ta_c, depending on λ and on the wavenumber $k = (k_1^2 + k_2^2)^{1/2}$or $k = k_3$. The maximum values, $\text{Ra}_\text{cr} = \max_{\lambda,k} \text{Ra}_\text{c}$ and $\text{Ta}_\text{cr} = \max_{\lambda,k} \text{Ta}_\text{c}$ are then just the critical values given by the version of the Lyapunov stability theory considered. Pritchard found that these critical values of Ra and Ta (and also the critical wave numbers k_{cr} (corresponding to them) agreed quite well with the critical values given by the normal-mode method of the linear stability theory. This clearly agrees with the earlier finding that the linear stability theory and the energy method lead to the same value of Ra_cr in the case of the Rayleigh–Bénard problem, and also agrees with subsequent results by Joseph and Hung (1971) relating to small disturbances in circular Couette flow.

Dikii's and Pritchard's applications of the Lyapunov method produced no appreciable repercussions. However the use of a related method by Arnol'd (alias Arnold) (1965a, 1966a, b, c) attracted much more attention which led to a definite revival of interest in the subject (see, e.g., the books by Arnol'd (1989a, Appendix 2); Marsden and Ratiu (1994); Marchioro and Pulvirenti (1994), and Arnol'd and Khesin (1998) and the references therein). Arnol'd considered two-dimensional disturbances in steady planar flows of inviscid ('ideal') fluid, but in contrast to Dikii and Pritchard he used in his studies the full nonlinear dynamic equations, not their linear approximation. Here we will pay most attention to the simplest case of two-dimensional disturbances having velocities $\mathbf{u}(\mathbf{x}, t) = \{u(x, z, t), w(x, z, t)\} = \{-\partial\Psi/\partial z, \partial\Psi/\partial x\}$ in a plane-parallel channel flow with velocity profile $U(z) = -d\Psi_0(z)/dz$, and only later will briefly describe the general results by Arnol'd relating to steady curvilinear plane fluid motions. Let us assume that all lengths are made dimensionless with a characteristic length L_0 and all velocities with a characteristic velocity U_0; then all quantities may be considered nondimensional (which means that we may take arbitrary functions of them, and add together any two quantities). The functions $\psi(x, z, t)$, $\psi_0(z)$ and $\psi(x, z, t) = \psi_0(z) + \psi(x, z, t)$ are non-dimensional stream functions of the disturbance, the undisturbed flow and the instantaneous disturbed flow, respectively, so $\Delta\psi = \partial w/\partial x - \partial u/\partial z$, $\Delta\psi_0 = d^2\psi_0/dz^2$ and $\Delta\psi$ are the corresponding vorticities. The nonlinear Euler equations of motion here reduce, as is well known, to a single equation for the conservation of vorticity $\Delta\psi$:

$$\frac{\partial}{\partial t}\Delta\psi - \frac{\partial\psi}{\partial z}\frac{\partial\Delta\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial z} = 0. \tag{4.25}$$

As usual, we will assume that disturbances are periodic in the coordinate x and that the period can take any value. Since the total energy of an inviscid flow is conserved in time and the vorticity $\Delta\Psi$ is also conserved, it is clear that both the integrals

$$E = \frac{1}{2}\int\int_{V'} (\nabla\psi)^2 dxdz \quad \text{and} \quad J_\Phi = \int\int_{V'} \Phi(\Delta\psi)dxdz$$

(where Φ is an arbitrary function of a single variable and V' is the rectangular region in the (x, z)-plane with width and length equal to the width of the channel and the length of disturbance period, respectively) are independent of time (i.e., are invariants of the disturbed motion). (Their invariance may easily be deduced from Eq. (4.25)). Following Arnol'd, let us consider the invariant functional $G = E + J_\phi$ of the stream function Ψ. It is easy to see that then the first variation of the functional G (i.e., the main part of the increment $\delta G = G[\psi_0 + \Psi] - G[\psi_0]$ for a small disturbance Ψ) may be represented in the form

$$\delta G[\psi]|_{\psi=\psi_0} = \int\int_{V'} [\Phi'(\Delta\psi_0) - \psi_0]\Delta\Psi dxdz \tag{4.26}$$

where Φ' is the derivative of the function Φ. Now let us assume that the velocity profile $U(z) = -d\psi_0(z)/dz$ has no inflection points. Then $d^2U(z)/dz^2 = -d^3\psi_0(z)/dz^3 \neq 0$ for all z, so that $\Delta\psi_0 = d^2\psi_0/dz^2$ is a monotonic function of z. This means that $\Delta\psi_0$ may be used as a new transverse coordinate instead of z. Hence, in particular, the stream function $\psi_0 = \psi_0(z)$ may also be considered as a function of $\Delta\psi_0$, i.e. it satisfies the equation

$$\psi_0 = \phi(\Delta\psi_0) \tag{4.27}$$

for some function ϕ. (Arnol'd showed that in fact Eq. (4.27) is also valid under a number of other conditions; in particular, under Fjørtoft's condition mentioned below). If now Φ is so chosen that $\Phi' = \phi$, then, according to (4.26) and (4.27), $\delta G[\Psi_0] = 0$, i.e., $\Psi = \Psi_0$ will be the stationary value of the functional $G[\psi]$. It is known that in the case of a function of a finite number of variables, the stationary points most often encountered are the points of its local maxima and minima. Now let $\Psi = \Psi(t)$ describe some dynamic system in a finite-dimensional space, with Ψ_0 a local extremal point of a time-invariant function $G[\Psi(t)]$ and Ψ_0 a disturbance of the initial value $\psi(t) = \psi_0$. The values of $G[\Psi_0 + \Psi(t)]$ corresponding to various disturbances Ψ_0 will clearly belong to the contour surfaces $G[\Psi_0 + \Psi(t)] = G[\Psi_0 + \Psi_0] =$ constant of the function $G(\Psi)$. At small values of the initial disturbance Ψ_0 the contour surfaces topologically have the appearance of the surfaces of small ellipsoids surrounding the extremal point Ψ_0. Therefore, if $\Psi_0 = \Psi\ (0)$ is small, then the values of $\Psi(t)$ will remain small at all values of t. This finite-dimensional analogy illustrates visually the main idea of the theory of Arnol'd. To make these arguments rigorous, we must now describe conditions guaranteeing that Ψ_0 is an external point of $G[\Psi]$, determine the strict sense of the statement that $\Psi = \Psi(x, z, t)$ is small and, finally, present a strict proof of the assertion for the case of an infinite-dimensional space of functions $\Psi(x, z, t)$.

The stationary point Ψ_0 of the functional $G[\Psi]$ will be a local extremum if, in some inertial system of coordinates, the second variation $\delta^2 G[\Psi_0]$ is either positive or negative definite, i.e. has the same sign for all disturbances $\Psi(x, z, t)$. It is easy to see that in the case considered the second variation of $G[\Psi_0]$ has the form

$$\delta^2 G[\psi]|_{\psi=\psi_0} = \int\int_{V'} \left[\left\{ \frac{U(z)}{U''(z)} \right\} (\Delta\Psi)^2 + (\nabla\Psi)^2 \right] dxdz \tag{4.28}$$

where $U''(z)$ denotes the second derivative of $U(z)$. It is clear that if $U''(z) \neq 0$ for all z (i.e., if Rayleigh's condition given in Sect. 2.82 is valid) then it is possible to choose an inertial coordinate system such that $U(z)/U''(z)$ will be positive everywhere, and hence Ψ_0 will correspond to a local minimum of the functional $G[\Psi]$. The same conclusion will also be true if there exists a constant K such that $[U(z) - K]/U''(z) \geq 0$ for all z, i.e., if the more general condition of Fjørtoft (given in the same section) is valid. The main stability theorem proved by Arnol'd states that the positive-definiteness of the quadratic form in the integrand on the right-hand side of Eq. (4.28) implies the Lyapunov stability of the flow with respect to the functional $L[\Psi(x, z, t)]$ on the right-hand side. (The proof of this statement can be found, e.g., in Monin and Yaglom (1971, 1971), pp. 158–160 of Vol. 1 and p. 853 of Vol. 2, while Arnol'd (1965a, 1966a, 1989a, App. 2) outlined the proof for a more general case of arbitrary steady planar motions). Note also that in the above-mentioned cases the ratio $U(z)/U''(z)$ (or, respectively, $[U(z) - K]/U''(z)$] is bounded from above and from below. Therefore in these cases the Lyapunov functional $L[\Psi(x, z, t)] = L(t)$ given by the right-hand side of Eq. (4.28) may be replaced by an equivalent but simpler function of the form

$$L(t) = ||\Psi||^2 = \int\int_{V'} [(\nabla\Psi)^2 + (\Delta\Psi)^2] dxdz = \int\int_{V'} [\mathbf{u}^2 + (\nabla \times \mathbf{u})^2] dxdz \tag{4.29}$$

representing the sum of integrated squares of velocity and of vorticity (i.e. kinetic energy and enstrophy). Arnol'd's stability theorem gives rigorous quantitative sense to the qualitative assertion in the paragraph preceding Eq. (4.28), and shows that the 'size' of the disturbances considered must be measured by the norm given by Eq. (4.29).

Let us now pass on to the general case of an arbitrary steady planar flow with the velocity field $\mathbf{U}(x, z) = \{U(x, z), W(x, z) = \{-\partial\psi_0(x, z)/\partial z, \partial\psi_0(x, z)/\partial x\}$, where $\mathbf{x} = \{x, z\} \in D$, D is an arbitrary (bounded or unbounded) two-dimensional domain with smooth impermeable boundaries (if they exist). Using the arguments similar to given above, Arnol'd (1966a, 1989a) (see also Marchioro and Pulvirenty (1994), Sect. 3.2, and Arnold and Khesin (1998), Sect. II.4) showed that in this case *if the condition* (4.27) *is valid and there exist two constant c and C such that*

$$0 < c \leq \frac{\nabla\psi_0}{\nabla\Delta\psi_0} \leq C < \infty, \tag{4.30}$$

then under sufficiently wide conditions the steady planar flow considered is stable in the Lyapunov sense with respect to the norm (4.29). (The inequalitites (4.30) make sense since for any steady flow in two dimensions the gradient vectors of the stream function and of its Laplacian are collinear; in particular, $\nabla\psi_0/\nabla\Delta\psi_0 = U(z)/U''(z)$ in the case of a plane-parallel flow with velocity profile U(z)). The statement printed in italics is the First Stability Theorem of Arnol'd. His second Stability Theorem is relating to the case where the ratio $\nabla\psi_0/\nabla\Delta\psi_0$ takes negative values. Here condition (4.30) must be replaced by the condition

$$0 < c \leq -\frac{\nabla\psi_0}{\nabla\Delta\psi_0} \leq C < \infty. \tag{4.30a}$$

The Second Theorem states that if inequalities (4.30a) are valid, then under all the other conditions guaranteeing the validity of the First Stability Theorem and one rather general additional condition the two-dimensional steady flow considered will be again stable in the Lyapunov sense with respect to a norm of the same type as the norm (4.29).

Marchioro and Pulvirenti (1994) noted that Arnol'd's condition (4.30) cannot be fulfilled in domains D without boundary. However, these authors also showed that the stability theorem is often valid for flows in such domains too, if the domain D and the primary flow in it possess some symmetry properties (see Sect. 3.3 in their book). Moreover, they showed that the inequality $c > 0$ can be replaced in the conditions of the First Stability theorem by the weaker inequality $c \geq 0$. Slight weakening of the conditions included by Arnol'd in the formulation of his Second Stability Theorem was indicated by Wolansky and Ghil (1996).

Arnol'd's results have a direct relation to the important question of the *admissibility of linearization* in the investigation of hydrodynamic stability. This question concerns the extent to which the stability (or instability) of solutions of linearized equations of fluid mechanics entails also the stability (or instability) of the corresponding solutions of the full nonlinear equations of motion. Before answering this question it is of course necessary to define exactly when the solution of the nonlinear system is called 'stable'. The most appropriate such definition is precisely that given by Lyapunov, who himself bore in mind this use of it (in application to motions described by systems of ordinary differential equations).

In the case of a finite-dimensional dynamic system described by the nonlinear vector equation $d\mathbf{x}(t)/dt = \mathbf{f}(\mathbf{x})$, the admissibility of linearization means that a one-to-one relationship exists between the stability in the sense of Lyapunov of a time-invariant solution $\mathbf{x_0}$ of this equation and its linear stability (the condition for the latter being that none of the eigenvalues of the equation linearized in the neighborhood of the point $\mathbf{x_0}$ shall have a positive imaginary part). For this case the existence of the one-to-one relation was proved under sufficiently general conditions of Lyapunov himself (indeed, the method of linearization is just Lyapunov's first method of stability analysis). However, in the case of dynamic systems in functional spaces described by nonlinear *partial* differential equations the situation is more complicated.

Let us remember that Luyapunov stability of fluid motion depends on the chosen norm in the functional space of fields of the flow quantities considered; therefore, there are in fact many different types of such stability. The proof of the admissibility of linearization must indicate which type of Lyapunov stability of solutions of the nonlinear system proves to be equivalent to the usual ('normal-mode') stability of solutions of the linearized equations, i.e., to the absence of normal-mode frequencies ω with positive imaginary parts. It easy to give simple arguments supporting the idea that in cases where the flow is unstable according to the linear theory (i.e., where there is an eigenvalue ω with $\Im m\omega = \omega^{(i)} > 0$), Lyapunov's stability conditions is usually also untrue. In fact, if the initial disturbance is chosen very small then it will evidently be well described by linearized equations. Hence in the cases considered a small initial disturbance may be chosen, such that for small t it grows proportional to $\exp[\omega^{(i)}t]$, where $\omega^{(\mathrm{i})} > 0$. Then, as the disturbance becomes relatively large, the linear approximation ceases to apply and the nonlinear terms changes the character of evolution of the disturbance (usually diminishing at first the rate of its growth and in many cases later even halting the growth entirely; see Sect. 4.21, below). If we now decrease the size of the initial disturbance (keeping its form), we merely achieve a longer time interval during which the linear theory is a suitable description of the flow, the subsequent fate of the disturbance being the same. Thus the maximal values achieved by the disturbance cannot be changed by diminishing its initial amplitude, and therefore it seems very likely that the flow considered must be unstable in the sense of Lyapunov. However, the rigorous proof of this assertion proved to be a far from easy matter.

It is quite plausible that under sufficiently broad conditions the reverse implication also holds—from the stability of a solution of linearized equations it follows that the corresponding solution of the complete non-linear system of equations is stable in the sense of Lyapunov. The assumption that linearization of equations of motion is possible for stability investigations has just this sense. In hydrodynamic stability theory this assumption is usually taken on trust (see, e.g., Lin (1955), Sect. 1.1, or Drazin and Reid (1981), Sect. 3; however, the book by Georgescu (1985) is an exception to this rule), but in most cases it is not at all easy to prove this rigorously. (Moreover, such a proof must clarify what disturbance norm provides Lyapunov stability of a flow in the case where all normal modes of linearized equations are decaying—this rather subtle question is also usually ignored in texts on hydrodynamic stability). The work of Arnol'd discussed above gives just such a proof for some particular cases. Remember, that Rayleigh's and Fjørtoft's conditions were introduced in Sect. 2.82 as sufficient conditions for the absence of unstable normal modes of the corresponding Rayleigh equation. Now we see that these conditions also guarantee the Lyapunov stability with respect to the norm (4.29) for two-dimensional solutions of the corresponding nonlinear equations. Arnol'd also showed that Fjørtoft's condition (which is weaker than Rayleigh's) can be replaced in his theorem on Lyapunov stability by some even weaker conditions which are valid, in particular, for velocity profiles which do not satisfy the Fjørtoft condition but, according to Tollmien (1935), nevertheless guarantee stability for solutions of linearized equations (again see Sect. 2.82). Thus, it was proved that, here again, linear stability implies Lyapunov instability for solutions of nonlinear equations.

Let us however emphasize that only two-dimensional disturbances of inviscid plane fluid flows were considered in the above-mentioned papers by Arnol'd. In fact, for three-dimensional disturbances of a flow (and three-dimensional flows), the reasoning presented above proved to be insufficient. Arnol'd (1965b) and Dikii (1965a) found only a few partial results relating to these cases, which do not resolve the question of interrelation between linear stability and nonlinear Lyapunov stability of three-dimensional disturbances. Abarbanel and Holm (1987) also tried to apply Arnol'd's method to nonlinear stability analysis of three-dimensional inviscid flows but they also found that the method does not work so successfully here as in the case of flows in two dimensions. Since the Squire theorem of the linear stability theory, given in Chap. 2, Sec. 2.8, cannot be generalized to the case of nonlinear stability theory (where only some much weaker statements are valid; cf. Sect. II.5.D in the book by Arnol'd and Khesin (1998)), the search for sufficiently general conditions of instability with respect to three-dimensional finite-amplitude disturbances presents a problem of considerable importance. Some arguments suggesting that the method developed by Arnol'd for investigation of stability of planar flows with respect to two-dimensional disturbances must be inadequate in the case of hydrodynamics in three dimensions were briefly noted by Arnol'd in the early paper (1966c); later this conclusion was explained more clearly by Arnol'd (1989a, App. 2); Rouchon (1991); Sadun and Vishik (1993) and in Sect. II.5.G of Arnol'd and Khesin's book (1998). Note however that, as early as the late 1960s and early 1970s, it was discovered that Arnol'd's variational approach (presented in the general form in his paper (1966b), which surprisingly linked up with some early ideas by Kelvin (1887)) can be successfully applied to studies of nonlinear Lyapunov stability for many types of disturbances encountered in a number of inviscid flows of practical interest. Such methods were first widely applied in geophysics; the works by Dikii (1965b, 1976); Blumen (1968, 1971); Dikii and Kurganskii (1971); Pierini and Salusti (1982); Benzi et al. (1982); Holm et al. (1983); Grinfeld (1984); Abarbanel et al. (1986), and Kurganskii (1993) are just typical examples. Somewhat later the same methods were used in many studies of stability magnetohydrodynamic flows and plasma oscillations. These new applications led, in particular, to the appearance of the excellent extensive survey by Holm et al. (1985) of the modern state of nonlinear stability investigations by methods developed by Arnol'd, which contains more than 150 references. For further examples of applications of this approach to the theory of hydrodynamic stability see, e.g., the books by Marsden (1992) and Marsden and Ratiu (1994), and papers by McIntyre and Shepherd (1987); Davidson (1998) and Vladimirov and Ilin (1998, 1999). Many other references to modern developments of the approach considered above can be found in Chap. II of the book by Arnol'd and Khesin (1998); here we will only mention the paper by Vladimirov (1990) where the direct Lyapunov method is applied to stability studies for some flows of viscous liquids affected by surface tension.

The question of the admissibility of linearization is also quite important in stability studies relating to steady flows of viscous fluids. In the case of viscous flows in smooth bounded domains one part of the linearization principle states that *if* all eigenfrequencies ω_j of the linearized dynamic equations corresponding to a given flow have negative imaginary parts, *then* the flow is also stable in the sense of Lyapunov (with respect to the norm (4.31), below). This was proved under sufficiently general conditions by Prodi (1962) (see also the detailed exposition of his proof by Georgescu (1985), Sect. 2.4.2, where a number of additional references relating to this topic can be found). The Lyapunov norm ||•|| used by these authors is given by the equation

$$||\mathbf{u}(\mathbf{x})||^2 = \int_v [\sum_{i=1}^{3} u_i^2(\mathbf{x}) + \sum_{i,j=1}^{3} \{\partial u_i(\mathbf{x})/\partial x_j\}^2] d\mathbf{x}. \quad (4.31)$$

The rigorous proof of the other part of the linearization principle (also for viscous flows in bounded domains) was briefly sketched in a note by Yudovich (1965) and was later given in detail in his special monograph (see Yudovich (1984)). A more elementary proof of admissibility of linearization for viscous flows in bounded domains, under slightly less general conditions, was given by Sattinger (1970). A quite different approach to the linearization principle was developed within the framework of the modern bifurcation theory (this theory will be briefly discussed in Sect. 4.22 and will be also mentioned in some subsequent parts of this book). Bifurcation theory allowed one to obtain some rather general conditions under which the solutions of the linearized equations certainly approximate faithfully the phase-space dynamics of a flow disposed in the vicinity of the steady primary flow. (The phase space has here the same meaning as in Sect. 2.3). These conditions are given by the so-called Hartman-Grobman theorem (see, e.g., Sect. 1.3 in Guckenheimer and Holmes (1993)), but they are based on the use of some new concepts which cannot be considered here.

Yudovich's monograph (1984) also contains a discussion of many other aspects of the general linearization problem, requiring the introduction of a number of different Lyapunov norms in functional spaces and the use of quite sophisticated mathematical techniques. Yudovich showed, in particular, that different norms are often needed for different purposes, and the answer to the question whether a flow is stable or unstable in Lyapunov's sense depends on the selection of the norm which is most appropriate for the given purpose. To illustrate the possibility of paradoxical disturbance behavior, Yudovich considered the simple case of a two-dimensional disturbances of an inviscid plane Couette flow. Here the velocity and vorticity of the disturbance remain bounded, but the vorticity derivatives grow unboundedly with time. Therefore in this case the vorticity $\Delta\Psi$ at large times t is reminiscent of a continuous but nowhere-differentiable Weierstrass function, and the flow is clearly unstable with respect to any norm which includes the square of the vorticity derivative. In the case of three-dimensional disturbances in the same flow, the velocity vector remains bounded but the vorticity vector grows unboundedly (see also Sect. 3.21, where related results were obtained for some other steady plane-parallel inviscid flows); hence the flow considered is unstable with respect to any norm including the square of the vorticity vector. However, there is no space for us to discuss the results in Yudovich's monograph in more detail. Let us only remember, in connection with the last remarks, the results by Arnol'd (1972) presented in Sect. 3.21 (and expounded in more detail in Sect. II.5 of Arnol'd and Khesin's book (1998)), which show that in three-dimensional inviscid flows disturbances can sometimes have extremely paradoxical asymptotic behavior, making the flow unstable with respect to rather simple norms.

Let us now return to the remark made at the end of Sect. 4.12, that Joseph's book (1976) prompted the appearance of a number of works investigating the possibility of improving the known energy-theory stability results by replacing the traditional energy functional $E(t)$ by some 'generalized energy' (i.e., by some new Lyapunov functional $L(t)$). We postponed the discussion of this remark until now, since the method used in the majority of these investigations is not in fact the traditional Lyapunov method considered above in this subsection. To explain this it is necessary to refer to particular examples. One of the first problems investigated in the above-mentioned way was that of convection in a horizontal fluid layer of thickness H, heated from below and rotating around a vertical axis with angular velocity Ω. Since the Coriolis force is orthogonal to the velocity and hence does no work, rotation does not change the energy-balance Eq. (4.2). Therefore the energy-method stability results are identical in the cases of rotating and non-rotating convection. However, the computations of the corresponding normal modes by linearized dynamic equations, carried out long ago by Chandrasekhar (1953, 1961), and the subsequent experiments by Rossby (1969) and some other workers (see, e.g., the survey by Bubnov and Golitsyn (1995)) both showed that the critical Rayleigh number $\mathrm{Ra}_{\mathrm{cr}}$, increases considerably with rotation rate (measured, e.g., by the so-called Taylor number $\mathrm{Ta} = \Omega^2 H^4/\nu^2$) and also depends on the Prandtl number Pr. Thus, while in the case of stationary layers of fluid the linear normal-mode theory and the energy method give the same value of $\mathrm{Ra}_{\mathrm{cr}}$, in the case of rotating layers, the values of $\mathrm{Ra}_{\mathrm{cr}}$ given by the energy method prove to be considerably smaller than those predicted by the linear stability theory or observed in experiments. Consequently, Joseph (1966) noted that the stabilizing influence of rotation on the emergence of convection in a fluid cannot be explained by the energy method of stability theory.

Later, however, some authors tried to replace Joseph's 'energy functional' $E_\lambda[\mathbf{u}(\mathbf{x},t), \vartheta(\mathbf{x},t)] = E(t) + \lambda \,\mathrm{Pr}\, E_T(t) = 0.5[\langle \mathbf{u}^2\rangle + \lambda \,\mathrm{Pr}\, \langle\vartheta\rangle^2]$ by another Lyapunov functional ('generalized energy') $L[\mathbf{u}(\mathbf{x},t), \vartheta(\mathbf{x},t)]$ in the hope of obtaining a larger value of the energy stability boundary $\mathrm{Ra}_{\mathrm{cr}} = \mathrm{Ra}_{\mathrm{cr}}$ (Ta) for rotating flows. In one of the first such attempts Galdi and Straughan (1985a) (see also the subsequent works by Mulone and Rionero (1989); Galdi and Padula (1990), and Straughan (1992), Sect. 6.1) tried to use a Lyapunov functional of the following form: $L[\mathbf{u}(\mathbf{x},t), \vartheta(\mathbf{x},t)] = \langle \mathbf{u}^2\rangle + \lambda_1 \,\mathrm{Pr}\, \langle\vartheta\rangle^2 + \lambda_2 \left\langle (\zeta + \lambda_3 \,\mathrm{Pr}\, \partial\vartheta/\partial z)^2\right\rangle + \lambda_4 \left\langle (\nabla\mathbf{u})^2 + \lambda_5 \,\mathrm{Pr}\, (\nabla\vartheta)^2\right\rangle$, where $\zeta = \partial v/\partial x - \partial u/\partial y$ is the vertical vorticity and λ_i, $i = 1, \ldots, 5$, are adjustable constants. Of course, the equation for $dL[\mathbf{u}(\mathbf{x}, t), \vartheta(\mathbf{x}, t)]/dt = dL(t)/dt$ will then also include terms which are cubic in the disturbance fields u_i, $i = 1, 2, 3$, and ϑ. To deal with the resulting nonlinear problem, all the above-mentioned authors used the approach by Joseph and Hung (1971), i.e., they neglected the cubic terms at first and only later calculated corrections to their results due to the nonlinearity of the system studied. Thus, the stability results obtained in the first stage of these investigations were only conditional, i.e., guaranteeing stability only for disturbances having very small norm $||(\mathbf{u}, \vartheta)|| = \{L[\mathbf{u}, \vartheta]\}^{1/2}$.

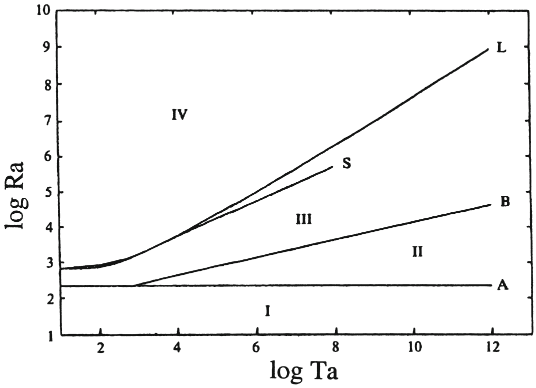

Fig. 4.4 Bounds in the (Ta, Ra)-palne of four stability regions for a rotating layer of fluid with $\mathrm{Pr} \geq 1$ heated from below and bounded by two free surfaces. (After Malkus and Worthing (1993)) L: Chandrashekhar's neutral curve of the linear stability theory; S-the boundary of the maximal region of conditional stability (i.e., of the limit as $\alpha \to 0$ of the regions of stability with respect to disturbances with the nondimensionalized 'initial amplitude' A_0 satisfying the inequality $A_0 < \alpha$); B: the boundary of the region of stability with respect to disturbances with $\mathrm{A}_0 < 10^{-6}$; A: the energy-theory stability boundary of the region of global monotonic stability

However, for disturbances with such a small norm that the cubic terms of the equation $dL/dt = 0$ can be neglected, it was found that the values of coefficients λ_I can be chosen in such a way that the stability region in the (Ra, Ta)-plane (i.e., the region where $dL/(t)/dt < 0$), turns out to be very close to the region determined by the linear theory of hydrodynamic stability (see, for example, Fig. 4.4 below in this section). This result is clearly analogous to the previously-mentioned results of Joseph and Hung (1971) relating to the Taylor–Couette stability problem. Similar results for the case of fluid layers heated from below (and also for some such layers of constant temperature) which are rotating with horizontal angular velocity $\Omega = \{\Omega_x, \Omega_y, 0\}$ were obtained by Wahl (1994), who used the ordinary energy norm but a special representation of divergence-free velocity field $\mathbf{u}(\mathbf{x}, t)$. The same representation of $\mathbf{u}(\mathbf{x}, t)$ was also used by Wahl (1994) and Kagel and Wahl (1994) in studies of Lyapunov stability (with respect to some particular Lyapunov functionals $L[\mathbf{u}(\mathbf{x}, t), \vartheta(\mathbf{x}, t)]$ including derivatives of fields $\mathbf{u}$ and ϑ) of arbitrary steady solutions of Boussinesq equations describing possible stationary disturbances in a horizontal fluid layer heated from below (see also the related paper by Schmit and Wahl (1993) where Lyapunov functionals of this type were used in detailed study of the onset of convection in a stationary layer of fluid heated from below).

Another stability problem which has often been studied by the method of Lyapunov is the magnetic Bénard problem of convection in a horizontal layer of a fluid conductor in the presence of a homogeneous vertical magnetic field; see, e.g., Galdi (1985); Rionero (1988); Rionero and Mulone (1988), and Galdi and Padula (1990). Here the main results found were similar to those obtained in the cases of the Taylor-Couette and rotational Bénard problems, but it was also shown that in this case the

Lyapunov functional can be chosen so that, for some range of flow parameters, the linear and Lyapunov nonlinear stability bounds coincide with each other. Note that in earlier studies of stability of magnetohydrodynamic flows by the energy method, carried out by Rionero (1967, 1968); Carmi and Lalas (1970); Bhattacharyya and Jain (1971), and Joseph (1976, Addendum to Chap. IX), a linear combination of integrated kinetic and magnetic energies was used as Lyapunov functional L, such that all cubic terms cancelled in the equation for dL/dt. However, this condition was not fulfilled in the cases of the more complicated Lyapunov functionals L used in publications appearing in 1980s and 1990s. Therefore in this later work the stability boundaries obtained were valid only under the condition that disturbances were small enough. This relates to the general conditions guaranteeing the coincidence of the critical parameters given by the linear and Lyapunov nonlinear stability theories, whose discussion plays a very important part in the work of Galdi and Straughan (1985b); Galdi and Padula (1990), and Straughan (1992). In fact, as a rule these conditions use only the linear parts of the differential equations determining the time evolution of flow disturbances, and hence presuppose the smallness of the latter—unless the cubic terms cancel in the equation for dL/dt.

The above-mentioned stability results, derived by the Lyapunov direct method employing Lyapunov's functionals L where dL/dt contains cubic terms, concern conditional stability only, and this clearly diminishes the practical usefulness of these results. This was specially emphasized in the review by Malkus and Worthing (1993) of the book by Straughan (1992). The reviewers considered the popular example of convection in a rotating horizontal layer of fluid. They illustrated the importance of amplitude restriction of results on conditional stability by supplementing curves L and S, shown in Fig. 6.2 of the book by Straughan (1992) (and relating to the case of a rotating layer of fluid with $\mathrm{Pr} \geq 1$ bounded by two free surfaces), by two additional curves A and B (see Fig. 4.4). The straight line A represents Joseph's (1966) energy-theory stability boundary $\mathrm{Ra}_{\mathrm{cr}} \approx 657$, which is independent of Ta and Pr. Hence points of region I in Fig. 4.4 correspond to flows stable with respect to disturbances of any size. The curve L is the linear stability curve computed by Chandrasekhar (1961) (and hence the region IV corresponds to instability with respect to arbitrarily small disturbances and the region below curve L—to stability with respect to infinitesimal disturbances). S is the boundary of the maximal region of conditional stability (corresponding to condition $L(0) = 0$) calculated by Galdi and Straughan (1985a) starting from the form of the functional $L[\mathbf{u}(\mathbf{x}, t), \vartheta(\mathbf{x}, t)]$ given above, with the optimal values of coefficients λ_i. Therefore, flows corresponding to points between curves L and S are linearly stable (i.e., exponentially-growing infinitesimal wave-like disturbances do not exist in these flows) but nevertheless $L[\mathbf{u}(\mathbf{x}, t), \vartheta(\mathbf{x}, t)] = L(t)$ can grow here with time for disturbances with an arbitrarily small value of $L(0)$. On the other hand, in the case of flows represented by points below curve S, such growth is impossible if $L(0)$ is small enough. However, the meaning of the words "small enough" was not explained in the book by Straughan. Trying to do this, Malkus and Worthing used an equation in the paper by Galdi and Straughan (1985a) which determines the maximal value $G(\mathrm{Ra, Ta, Pr})$ of the dimensionless initial 'energy' $L(0)$ of disturbances, which certainly do not destabilize a flow with given values of Ra, Ta, and Pr. This equation allowed them to compute the boundary B of the region of stability with respect to disturbances having initial 'amplitude' $[L(0)]^{1/2}$ less than10^{-6} times the appropriately defined 'unit amplitude' (so that the region III corresponds to flows unstable to at least one disturbance with initial dimensionless amplitude equal to 10^{-6} but stable to all smaller disturbances). We see that this region is rather large; therefore Malkus and Worthing were in doubt whether the curve S can be considered as a real boundary for the 'region of nonlinear stability', giving their opinion that, in almost all practical situations, even the curve B (bounding the region of stability to disturbances with dimensionless amplitudes not exceeding 10^{-6}) will be not useful as such a boundary.

Above, some applications of the "generalized energy method" determining the "conditional-energy bounds" were listed, and at the end we considered the review by Malkus and Worthing (1993) which sharply criticizes the usefulness of some of the results obtained by this method. (Note that this review also contains formulations of several interesting unsolved problems which are worth investigating by traditional and generalized energy methods). Let us now stress that the energy (and more general Lyapunov's methods) have already yielded some important new results concerning stability of fluid flows. The classical Reynolds–Orr energy-balance Eq. (4.2) and its generalization to the case of convective flows led to the discovery, for many cases, of exact or almost exact minimal-critical values of dimensionless global characteristics of laminar flows (e.g., of $\mathrm{Re}_{\mathrm{cr\ min}}$ or $\mathrm{Ra}_{\mathrm{cr\ min}}$) determining the boundary of the region of 'absolute' (i.e., 'unconditional' or 'global') stability of a flow to disturbances of any size. Such bounds, which have already been mentioned in Sect. 2.1, clearly have considerable theoretical and practical value. Energy methods also showed that there exist two quite different types of fluids flows. The first type consists of flows where the region of the normal-mode stability with respect to infinitesimal disturbances coincides with the region of energy stability with respect to disturbances of arbitrary size, while for flows of the second type the latter of these two regions covers only a small part of the first region. It is clear that the nonlinear development of disturbances and transition to turbulence must have quite different forms in flows of these two types. Moreover, Lyapunov's generalized energy method led to the discovery of a great number of explicit conditions for both nonlinear and linear flow stability, often concerning flows of great practical importance; see in this connection the survey by Holm et al. (1985) mentioned above, and the papers and books by Arnol'd (1965a, b; 1966a, b, c; 1989a); McIntyre and Shepherd (1987); Marsden and Ratiu (1994); Marchioro and Pulvirenti (1994); Arnol'd and Khesin (1998), and Davidson (1998). As to results relating to conditional Lyapunov stability, they imply physically-observable stability diagrams of the type shown in Fig. 4.3, where for given 'energy' E_λ the exact shape of the curve in the diagram can be determined from the equations of generalized energy theory. The possible extension of the region of conditional stability by means of replacement of the Reynolds–Orr energy functional $E(t)$ by some Lyapunov functional $L(t)$ also clearly leads to extension of the range of Re (or Ra, Ta, etc.) numbers covered by such a diagram. In particular cases where the Lyapunov method yields the same critical numbers that follow from the linear normal-mode theory, the diagram in Fig. 4.3 covers the whole range between the region of unconditional (global) stability with respect to arbitrarily large disturbances and the region of absolute instability with respect to arbitrarily small (infinitesimal) disturbances.

An important feature of the energy methods is their ability to determine the most-unstable types of disturbance, which capture the energy of the primary flow most efficiently and hence grow faster than all the others. In this connection Lumley (1971) conjectured that some modifications of the classical energy method might also be useful in investigations of developed turbulence. As an example, he tried to apply such a method to the study of the near-wall region of a turbulent boundary layer. Within this region he replaced the constant molecular viscosity ν by an empirical function $\nu_m(z)$ describing, with reasonable accuracy, the combined influence of the molecular viscosity and small-scale turbulent fluctuations on the mean flow and the accompanying large-scale structures. Then he appropriately modified the R–O energy-balance Eq. (4.2) and with its help determined the most unstable longitudinal (i.e., x-independent) disturbances. It was found that these disturbances agreed satisfactorily with the longitudinal structures actually observed in the near-wall regions of turbulent flows along flat plates. Later Poje and Lumley (1995) further developed the same idea, suggesting the use of the energy-balance method to identify the large-scale organized ('coherent') structures which, according to data accumulated during the second half of the twentieth century, exist everywhere in turbulent flows and play a rather important role in them. However, we cannot linger here on this subject which clearly lies outside the content of the present chapter.

4.2 Landau's Equation, its Generalizations and Consequences

4.2.1 The Landau Equation for the Amplitude of a Disturbance

The energy method of stability analysis deals with general (quite arbitrary) flow disturbances; the highly-developed linear theory of hydrodynamic stability is not used at all here. This theory suggests that in the case when the initial disturbance is rather weak its most-unstable normal-mode component (or the least stable, if unstable normal modes do not exist) will play the main part in the primary disturbance development. Therefore the study of the development of a normal-mode disturbance is important for understanding the behavior of disturbed flows, and such a study must take into account the influence of the nonlinear terms of the equations of motion, which clearly affect the disturbance evolution if the disturbance is not very small. The results obtained will be of interest both in the case where $Re < Re_{cr}$, where Re_{cr} is the critical Reynolds number[4] defined from the linear stability theory (in this case an investigation of the nonlinear normal-mode development can yield the

[4] For simplicity, we shall speak only of Reynolds number, although in some cases the initiation of instability will be determined by transition through a critical value of some other dimensionless control parameter of the same type.

critical Reynolds number for finite disturbances of fixed amplitude) and in the case where $\mathrm{Re} > \mathrm{Re}_{cr}$ (in this case the nonlinear results describe further evolution of weak disturbances, which increase exponentially according to linear theory).

The great importance of nonlinear effects in the development of flow disturbances was already fully appreciated by Reynolds in 1883, and some attempts to incorporate these effects into theoretical analysis were also made very early (in particular, by Noether (1921) and Heisenberg (1924)). However, the first really significant step towards the creation of the nonlinear theory of hydrodynamic stability was taken in a short note by Landau (1944) whose contents was described also in the books by Landau and Lifshitz (1944), Sect. 24; (1958), Sect. 27; and (1987), Sect. 26 (in the last of these, the presentation was partially changed to reflect more recent developments of the theory which will be considered later in this book). Landau's arguments were quite general and did not use any specific form of the equations of motion.

Landau considered simply the development of a normal-mode disturbance in a steady laminar flow. He was especially interested in the evolution of an unstable (exponentially-growing) wave-like mode of very small initial amplitude (which may be considered as being infinitesimal) at a slightly supercritical value of Re (i.e., only a little larger than Re_{cr}). However his reasoning can be equally well applied to slowly-decaying infinitesimal normal-mode disturbances at slightly subcritical $\mathrm{Re} < \mathrm{Re}_{cr}$; hence we will consider both these cases here. To Landau, it was only important that the velocity field of the mode considered could be represented in the form

$$\mathbf{u}(\mathbf{x},\, t) = A(t)\mathbf{f}(\boldsymbol{x}), \tag{4.32}$$

where $\mathbf{f}(\mathbf{x})$ is the eigenfunction of the corresponding eigenvalue problem while $A(t)$ is the complex disturbance amplitude, which can be represented in a form $A(t) = e^{-i\omega t} = e^{\gamma t - i\omega_1 t}$ for values of t at which the linear stability theory is valid. Here $\omega_1 = \Re\mathrm{e}\,\omega$ and $\gamma = \Im m\omega$ so that $\gamma > 0$ for growing waves, $\gamma < 0$ for decaying waves and $\gamma \to 0$ as $\mathrm{Re} \to \mathrm{Re}_{cr}$ (and therefore $|\gamma| \ll |\omega_1|$ for sufficiently small $|\mathrm{Re} - \mathrm{Re}_{cr}|$ if $\omega_1 \neq 0$). The form of $A(t)$ given above makes it clear that the real disturbance amplitude $|A(t)|$ satisfies the equation

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2. \tag{4.33}$$

However, Eq. (4.33) is correct only within the framework of linear stability theory. If $\mathrm{Re} > \mathrm{Re}_{cr}$ and $A(t)$ increases, there will inevitably come a point at which this theory is no longer valid and must be replaced by a more complete one, which takes into account those terms in the equations of motion that are nonlinear in the disturbances. Then the right side of Eq. (4.33) must be considered as the first term of the expansion of $d|A|^2/dt$ in a series of powers of A and A^* (where as usual the asterisk denotes the complex conjugate). In the case where $\mathrm{Re} < \mathrm{Re}_{cr}$, $A(t)$ is a decreasing function and here Eq. (4.33) is true for all t, but only in cases where the initial amplitude $A(0)$ is small enough. If, however, $A(0)$ is not sufficiently small, then at small values of t this equation represents only the first term of the expansion in powers of A and A^*.

If $|A(t)|$ is small, but not small enough for all the higher-order terms of the above-mentioned expansion to be neglected, then it is necessary to take into account the terms of the next order of the series, i.e. the third-order terms. However, it must also be remembered that the motion (4.32) is accompanied by periodic oscillations in the expression for $A(t)$, rapid in comparison with the characteristic time $1/|\gamma|$ of an appreciable change in the value of $|A(t)|$), and described by the factor $e^{-i\omega_1 t}$, where $|\omega_1| \gg |\gamma|$. These periodic oscillations do not interest us; hence to exclude them, it is convenient to average the expression $d|A|^2/dt$ over a period of time that is large in comparison with $2\pi/|\omega_1|$ (but small in comparison with $1/|\gamma|$). Since third-order terms in A and A^* will inevitably contain a periodic factor, they will all disappear during the averaging.[5] In the case of the fourth-order terms, there will remain, after averaging, only one term, which is proportional to $A^2 A^{*2} = |A|^4$. Thus, retaining terms of no higher than fourth order, we will have an equation of the form

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2 - \delta|A|^4. \tag{4.34}$$

Since the period of averaging is much less than $1/|\gamma|$ the terms $|A|^2$ and $|A|^4$ will be practically unchanged by averaging, so that Eq. (4.34) may be considered as an exact equation for the amplitude of the averaged disturbance. (In the case where $\omega_1 = 0$ the third-order terms also often disappear because of the symmetry properties of the problems considered, and hence Eq. (4.34) is valid here too; certain examples of this kind will be considered below). Equation (4.34) is called the *Landau equation,* and its coefficient δ, which can be either positive or negative (and can also be zero, but only in exceptional cases), is the *Landau constant*. Positive values of δ show that nonlinear effects stabilize the disturbance considered, decreasing the growth of its amplitude, while negativity of δ means that nonlinear effects destabilize the disturbance.

Equation (4.34) can be also rewritten as the following linear equation in $|A|^{-2}$

$$\frac{d|A|^{-2}}{dt} + 2\gamma|A|^{-2} = \delta, \tag{4.35}$$

whose general solution is easily seen to be

$$|A(t)|^2 = \frac{A_0^2 e^{2\gamma t}}{\left(1 - \frac{\delta}{2\gamma}A_0^2\right) + \frac{\delta}{2\gamma}A_0^2 e^{2\gamma t}} \tag{4.36}$$

Where $A_0 = |A(0)|$ is the initial amplitude of the disturbance. From Eq. (4.36) it follows that if $\delta > 0$, if the initial disturbance is sufficiently small, and if $\gamma > 0$ (i.e. $\mathrm{Re} > \mathrm{Re}_{cr}$ and the evolution of an unstable mode is studied), the amplitude $A(t)$ will

[5] To be more exact, we must say that third-order terms do not fully disappear after averaging but generate some terms of the fourth order which can be included in the fourth-order terms of the expansion considered.

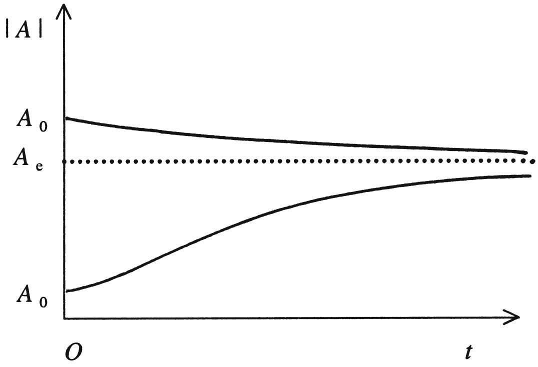

Fig. 4.5 The dependence of the disturbance amplitude $|A(t)|$ on time t in the case where $\delta > 0$ and $\mathrm{Re} > \mathrm{Re}_{cr}$ (and hence $\gamma > 0$) for disturbances with the initial amplitude $A_0 < Ae = (2\gamma/\delta)^{1/2}$ and $A_0 > A_e$ (but $A_0 - A_e$ small) according to Landau's Eq. (4.34)

first increase exponentially (in accordance with the linear theory), but then the rate of the increase slows, and as $t \to \infty$ the amplitude will tend to a finite 'equilibrium value' $A(\infty) = A_e = (2\gamma/\delta)^{1/2}$ independent of $A(0)$ (see the lower part of Fig. 4.5). Note now that γ is a function of the Reynolds number which becomes zero at $\mathrm{Re} = \mathrm{Re}_{cr}$ and may be expanded as a series in power of $\mathrm{Re} - \mathrm{Re}_{cr}$ (the latter fact may be deduced from the small-disturbance theory) while $\delta \neq 0$ for $\mathrm{Re} = \mathrm{Re}_{cr}$. Thus $\gamma \approx b\,(\mathrm{Re} - \mathrm{Re}_{cr})$ at small enough values of $|\mathrm{Re} - \mathrm{Re}_{cr}|$ where b is a positive constant. Consequently, $A(\infty) = |A|_{max} \propto (\mathrm{Re} - \mathrm{Re}_{cr})^{1/2}$ for $\delta > 0$ and small positive values of $\mathrm{Re} - \mathrm{Re}_{cr}$ (see Fig. 4.7a below). Hence $A(t)$ remains small at all values of t if $\mathrm{Re} - \mathrm{Re}_{cr}$ is small enough (therefore, even the inclusion in Eq. (4.34) of higher-order terms, for example one proportional to $|A|^6$, will not qualitatively change the behavior of the function $A(t)$). In the case where $A_0 > A_e = (2\gamma/\delta)^{1/2}$ but is nevertheless small (this is possible when $\mathrm{Re} - \mathrm{Re}_{cr}$ is small) Eqs. (4.34) and (4.36) can again be used as a reasonable first approximation; the corresponding behavior of $A(t)$ is shown in the upper part of Fig. 4.5. We see that here, at Re slightly exceeding Re_{cr}, any disturbance containing the unstable component transfomrs the primary laminar flow into a new laminar flow which is practically independent of the initial conditions. (In fact this new flow can turn out to be unstable to some disturbances neglected in the fluid-dynamic derivation of Landau's equation considered below. However, here we will not linger on this topic). If, however, $\delta > 0$ but $\mathrm{Re} < \mathrm{Re}_{cr}$ and hence $\gamma < 0$, then Eq. (4.36) shows that the disturbance decays monotonically and in accord with the linear theory (*i.e.*, $A(t) \propto e^{\gamma t}$ as $t \to \infty$). Here evidently neither the last term on the right side of Eq. (4.33), nor the terms of higher order omitted from this equation, significantly affect the disturbance evolution.

Let us now consider briefly the case where $\delta < 0$. If in this case $\gamma < 0$ (i.e., $\mathrm{Re} < \mathrm{Re}_{cr}$), then for $A_0 < (2\gamma/\delta)^{1/2}$ the solution $|A(t)|$ decays monotonically to zero (see the lower part of Fig. 4.6); hence in this case too the inclusion of the higher-order terms of the amplitude equation will not change the behavior of $A(t)$ qualitatively. If $\delta < 0$, $\gamma < 0$, but $A_0 = (2\gamma/\delta)^{1/2}$, then $A(t) = A_0$ at any $t > 0$; however; for $A_0 > (2\gamma/\delta)^{1/2}$ the function $A(t)$ grows with t (see again Fig. 4.6) and here the inclusion of higher-order terms in Eq. (4.34) becomes necessary at moderate

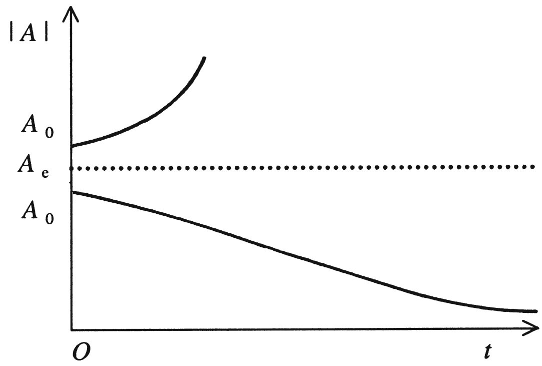

Fig. 4.6 The dependence of the amplitude $A(t)$ on t in the case where $\delta < 0$ and $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$ (i.e., $\gamma < 0$) for disturbances with the initial amplitude $A_0 < A_e = (2\gamma/\delta)^{1/2}$ and $A_0 > A_e$ according to Landau's equation

positive values of t. The possible influence of such terms will be illustrated later by a simple example; for now, we merely note that, according to the above argument, if $\delta < 0$ and $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$ then very small disturbances decay, but some disturbances which are not small enough grow with time; this is the *subcritical instability* of finite-amplitude disturbances. If now $\delta < 0$ but $\gamma > 0$ (i.e., $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$) then, for any $A_0 > 0$, solution (4.36) quickly becomes infinite; hence in this case the behavior of the amplitude $A(t)$ as $t \to \infty$ cannot be determined from Eq. (4.34) for any initial value A_0. To obtain a sensible result we must take into account the next term of expansion in the power of A and A^* and to assume it to be negative. Let the next term be $-\beta|A|^6$ where $\beta > 0$. Then, neglecting all terms of higher than the sixth order we obtain

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2 + |\delta||A|^4 - \beta|A|^6, \tag{4.37}$$

and hence

$$|A|^2_{\max} = \frac{|\delta|}{2\beta} \pm \left[\frac{|\delta|^2}{4\beta^2} + \frac{2\gamma}{\beta}\right]^{1/2} \tag{4.37a}$$

where $|A|^2_{\max}$ is the value of $|A|^2$ at which $d|A|^2/dt = 0$ and $\gamma \approx b(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$. The relation (4.37a) is shown in Fig. 4.7b, while Fig. 4.7a. corresponds to the case where $\delta > 0$. (The dotted lines in this figure correspond to amplitudes of unstable waves). In Fig. 4.7b two values $|A_1|^2_{\max} \approx \frac{|\delta|}{\beta} + \frac{2b}{|\delta|}(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ and $|A_2|^2_{\max} \approx \frac{2b}{|\delta|}(\mathrm{Re}_{\mathrm{cr}} - \mathrm{Re})$, given by Eq. (4.37a), are shown for the case where $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$. (If $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$, then only the first of these is meaningful, while the second becomes negative and must therefore be replaced by the value $|A_2|^2_{\max} = 0$ which also corresponds to a vanishing right-hand side of Eq. (4.37)). Since $d|A|^2/dt < 0$ for $|A| > |A_1|_{\max}$ and $|A| < |A_2|_{\max}$, while $d|A|^2/dt > 0$ if $|A_2|_{\max} < |A| < |A_1|_{\max}$, we see that for $\delta < 0$, $\beta > 0$ the primary flow is unconditionally stable only for $\mathrm{Re} < \mathrm{Re'}_{\mathrm{cr}}$ (where $\mathrm{Re}'_{\mathrm{cr}} \approx \mathrm{Re} - |\delta|^2/8b\beta$ is the value of Re at which two roots (4.37a) coincide). For $\mathrm{Re}'_{\mathrm{cr}}$ x $< \mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$ this flow is 'conditionally stable', i.e., stable with respect to small disturbances with

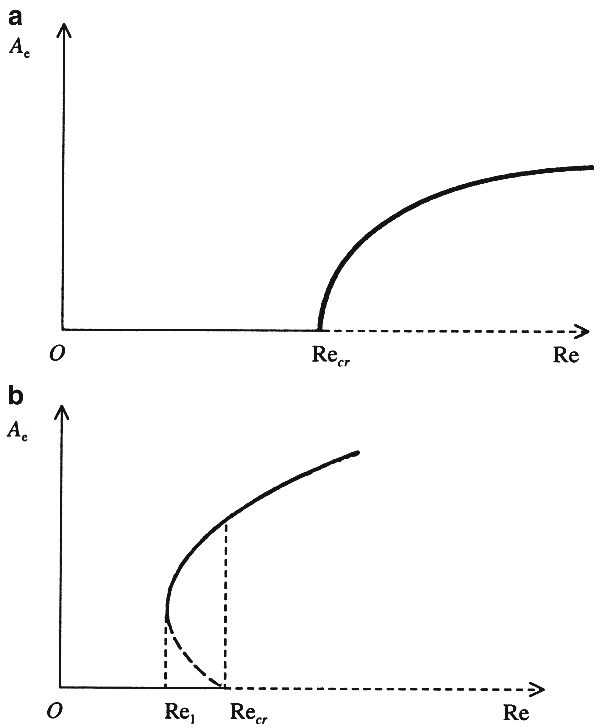

Fig. 4.7 The dependence of the equilibrium amplitude $|A(t)| = A_e$, satisfying the equation $d|A(t)|/dt = 0$, on the Reynolds number Re in the cases where either $\delta > 0$ (**a**), or $\delta < 0$ but the amplitude equation has the form (4.37) with $\beta > 0$ (**b**) Re_{cr} the critical Reynolds number; $Re_1 = Re_{cr}$ the threshold of subcritical instability. The *solid* and *dotted lines* represent amplitudes of stable and unstable equilibrium disturbances, respectively

$A_0 < |A_2|_{max,}$ but if $A_0 \geq |A_2|_{max}$ then the disturbance amplitude grows rapidly to the 'equilibrium value' $|A_1|_{max}$ (this conclusion makes more precise the above statements about the possibility of subcritical finite-amplitude instability when $\delta < 0$). For $Re > Re_{cr}$ the primary flow is unstable to disturbances of any amplitude and the normal-mode disturbance grows to the value corresponding to the point on the solid line (of course, this is correct only if $|Re - Re_{cr}|$ is small enough to justify the expansion in powers of A and A^* up to the approximation (4.37)).

The above results describe only a part of the contents of Landau's paper (1944). Landau, assuming that $\delta > 0$, considered the development of flow structures with further increase of Re beyond Re_{cr}. It was natural to assume that at some higher value of the Reynolds number, $Re_{2,cr} > Re_{cr}$, the oscillatory stable flow (with frequency ω_1) arising from the primary steady flow at $Re = Re_{cr}$ may itself become unstable to small disturbances, transforming it to a new stable oscillatory motion which includes oscillations of two frequencies ω_1 and ω_2 and therefore has two degrees of freedom. (Steady laminar motion is fully determined by the general flow conditions and hence has no degrees of freedom; in the case of oscillatory motion with fixed frequency ω_1 the phase θ_1 can take any value and hence this motion has one degree of freedom; while quasi-periodic oscillations with two periods $2\pi/\omega_1$ and $2\pi/\omega_2$ possess two degrees of freedom). This new motion in its turn becomes unstable at $Re = Re_{3,cr} > Re_{2,cr}$ generating a motion with three degrees of freedom, and so on. Several short series of such successive transformations of a steady flow into an oscillatory one, and then into more complicated oscillations, were in fact observed after 1944 in some particular flows when the corresponding value of Re (or of another appropriate dimensionless control parameter) was increased step by step; some of these series will be mentioned later in this chapter. However Landau also assumed that, as Re increases, the intervals between consecutive critical Reynolds numbers $Re_{n, cr}$ and $Re_{n+1, cr}$ will become smaller and smaller, so that at large, but not excessively large, values of $Re - Re_{cr}$ the number of degrees of freedom of the resulting motion will reach a very high value. According to Landau, the complicated and disordered motion appearing in this way just represents the fully developed turbulent flow. This Landau's (or, as it is also often called reflecting the contribution of Hopf (1948), Landau-Hopf's) scenario of transition to turbulence seemed at first to be physically quite convincing, and during many years it was considered by the majority of experts as being correct in its main features even though it was often stressed that its validity was not proved rigorously and that it cannot be universal; see, e.g., Monin and Yaglom (1971), p. 165, or Drazin and Reid (1981), p. 370. However later it was found that Landau's theory of transition to turbulence is far less satisfactory than was thought earlier and must be radically revised; this conclusion was based on some amazing new developments which will be described later in this book. These new results concern Landau's ideas about the development of irregular fluctuations at $Re \gg Re_{cr}$, but they do not diminish the importance of his equation for the description and explanation of the initial stage of evolution of small disturbances at values of Re close to Re_{cr}.

The coefficient γ of Landau's Eq. (4.34) is equal to the imaginary part of the eigenvalue $\omega = \omega_1 + i\gamma$ corresponding to the normal-mode disturbance considered (originally Landau assumed that this disturbance was the one with the greatest imaginary part of ω). So, to determine this coefficient one need merely solve the eigenvalue problem of linear stability theory (in the case of a plane-parallel flow this is the famous Orr-Sommerfeld eigenvalue problem). Solutions of this eigenvalue problem may nowadays be calculated rather easily. However Landau's derivation of Eq. (4.33) gave no instructions about possible methods for determination of the numerical value of δ. It was clear from the outset that here the full nonlinear equations of motion must be used, but at first it was not known how to do this. Three-dimensionality of the Navier-Stokes equations complicates the problem considerably; therefore in the book by Eckhaus (1965) (which was the first one on nonlinear stability theory) much attention was given to simplified model problems in one-dimensional space (a related model was considered also in Sect. 50 of Drazin and Reid's book (1981)) and then only two-dimensional disturbances of two-dimensional flows were studied. The first, still imperfect, attempts to estimate the numerical value of Landau's constant δ for some particular flows with the help of the equations of motion were made by Meksyn and Stuart (1951) and Stuart (1958). In the first of these papers much attention was given to nonlinear effects leading to distortion of the primary velocity profile by disturbances in a plane Poiseuille flow, while in the second paper an approximate estimate of the value of δ for two-dimensional plane waves in a plane Poiseuille flow, and axisymmetric wave-like disturbances in a circular Couette flow, was based on the assumption that the disturbance's shape is preserved during its evolution. In both papers it was assumed that the Reynolds number Re has a slightly supercritical value and that the disturbances studied are unstable according to linear stability theory. (Note also that $\omega_1 = \Re\mathrm{e}\ \omega$ differs from zero in the case of a plane Poiseuille flow but is equal to zero in a circular Couette flow). Meksyn and Stuart (1951) used the full Navier–Stokes system to compute the velocity distortion, and came to the conclusion that δ can have either sign. However, Stuart (1958) found that, according to the assumptions he made, the single Reynolds-Orr energy Eq. (4.2), which is only a particular consequence of the N-S system, implies Landau's Eq. (4.34) for disturbance amplitude $A = A\ (t)$ with a definite value of δ which is always positive.

Since some of the conclusions obtained by Meksyn and Stuart (1951) and by Stuart (1958) contradicted each other, Stuart (1960) (see also his survey papers (1962a, 1971)) and Watson (1960a) developed more precise methods to compute Landau's constant for small two-dimensional normal-mode disturbances in a plane Poiseuille flow. Stuart took into account that in the nonlinear development of a two-dimensional disturbance with given wave number k (i.e., having at $t = 0$ an initial velocity field of the form $\mathbf{u}(\mathbf{x}, 0) = \{u(z), 0, w(z)\}\ e^{ikx}$), higher harmonics (proportional to e^{inkx}, $n = 2$, 3,...) will also be generated. Therefore, he represented the disturbance velocity field $\mathbf{u}(\mathbf{x}, t)$ for $t > 0$ in the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_0(z, t) + \mathbf{u}_1(z, t)e^{ikx} + \mathbf{u}_2(z, t)e^{i2kx} + \dots . \tag{4.38}$$

Here $\mathbf{u}_\mathrm{n}(z, t)e^{inkx}$, $n = 0, 1, 2, \dots$, are two-dimensional solenoidal vectors (in general complex; remember that the true velocity is equal to the real part of the given expression) depending on t, and the term $\mathbf{u}_0(z, t)$ describes the distortion of the laminar Poiseuille-flow velocity profile by the disturbance. Further, it was assumed that as $t \to 0$, only the term on the right-hand side of Eq. (4.38) which is proportional to e^{ikx} is conserved, while for very small $t > 0$, this term becomes the solution $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_1(z)e^{i(-\omega t + kx)} = \mathbf{u}_1(z)e^{ik(x - ct)}$ of the Orr-Sommerfeld equation describing a growing or damped wave-like disturbance. Then, for slightly greater, but nevertheless small, positive t the first harmonic will be leading term on the right-hand side and $\mathbf{u}_1(z, t)$ may be written as

$$\mathbf{u}_1(z, t) = A(t)\mathbf{u}_1(z) + \text{higher-order terms.} \tag{4.39}$$

Stuart (1960) substituted Eqs. (4.38) and (4.39) into the nonlinear Navier–Stokes equations (which he replaced by the equivalent non-linear equation for the stream function $\Psi(x, z, t)$] corresponding to the velocity field $\mathbf{U} + \mathbf{u}(\mathbf{x}, t)$, where $\mathbf{U} = \{U(z), 0, 0\}$ is the Poiseuille-flow velocity (instead of using only Eq. (4.2) as in his 1958 paper). Assuming now that $|\gamma| = |\Im m\omega|$ is a small quantity (i.e., considering a disturbance with small amplification or damping corresponding to a point in the (k, Re)-plane close to the neutral-stability curve) and using expansion in powers of this quantity, he obtained for the complex amplitude $A(t)$ an approximate Landau-type equation of the form

$$\frac{dA}{dt} = -i\omega A - \tfrac{1}{2}l|A|^2 A \tag{4.40}$$

where ω is the same complex frequency as above and $l = \delta + i\delta'$ is another complex coefficient. (Later Fujimura (1989); Dušek et al. (1994), and Park (1994) reconsidered the derivation of Eq. (4.40) from the Navier–Stokes equations and indicated several sets of assumptions implying its validity, while Zhou (1991) indicated that in some cases the computation of the second term on the right-hand side of (4.39) is necessary for obtaining the satisfactory agreement with the experimental data). Equation (4.40) is usually called either the *complex Landau equation* or the *Stuat-Landau equation* (see e.g., Kuramoto (1984)) and l is the *complex Landau constant*. Representing the complex amplitude $A(t)$ as $|A(t)|\, e^{i\phi(t)}$, it is easy to show that the real part of Eq. (4.40) is equivalent to Landau's Eq. (4.34) for $|A|^2 = AA^*$, where $\gamma = \Im m\omega$ and $\delta = \Re el$. On the other hand, the imaginary part of Eq. (4.40) can be written as the following equation for the phase $\phi(t)$, supplementing Landau's equation:

$$\frac{d\phi}{dt} = -\omega_1 - \tfrac{1}{2}\delta'|A|^2 \tag{4.34a}$$

where $\omega_1 = \Re e\omega$, $\delta'' = \Im ml$.

According to Stuart's results, $\delta = \delta_1 + \delta_2 + \delta_3$ where the three terms correspond to three different physical processes affecting the nonlinear development of a wave-like disturbance. He also noted that only the term δ_1 (which is always positive) was taken into account in his paper of 1958 (hence the conclusion of this paper that δ was positive was an inevitable consequence of the assumptions made); and only terms δ_1 and δ_3 were considered (and imprecisely estimated) by Meksyn and Stuart (1951). For all three terms Stuart obtained explicit expressions, which were however rather cumbersome and contained the eigenvalues and eigenfunctions of the corresponding Orr-Sommerfeld equation (and also of the adjoint equation) in a complicated manner. These expressions clearly depend on k and Re; however, the numerical calculation of them (and of their sum δ) seemed to a very difficult problem in the early 1960s.

In the paper by Watson (1960a) accompanying that by Stuart a more complete Fourier representation of the disturbance velocity was used and the technique, traditional for the disturbance theory, of expansion into powers of the amplitude (instead of the powers of $\gamma = \Im m\omega$ considered by Stuart) was applied to the fluid-dynamic equations describing disturbance development. (However, expansion in powers of $|\gamma|$ was also used here and hence $|\gamma|$ was assumed to be small in Watson's derivations too). As a result, Watson obtained a new and more rigorous reformulation of Stuart's theory, leading to the generalized Landau equation of the form

$$\frac{d|A|^2}{dt} = |A|^2 \sum_{m=0}^{\infty} a_m |A|^{2m} \tag{4.41}$$

for the squared amplitude $|A|^2$. Here evidently $a_0 = 2\gamma, a_1 = -\delta$, while expressions for the coefficients a_m with $m > 1$ were found to be much more complex than for the Landau coefficient δ. Another rigorous analytical method allowing the investigation of streamwise periodic solutions of nonlinear equation for two-dimensional disturbances in a plane Poiseuille flow, which supplemented the expansion in powers of the small quantity $|\gamma| = |\Im m\omega|$ by expansion of all relevant functions of z in terms of the eigenfunctions of the linear O-S equation, was developed by Eckhaus (1965); it also led to confirmation of Stuart's (1960) results. Note, however, that the assumptions about the smallness of $|\gamma|$ used by Stuart, Watson, and Eckhaus made their theories inapplicable in principle to plane Couette and circular Poiseuille flows (for example), where unstable normal modes do not exist and therefore $|\gamma|$ cannot be very small. Therefore Ellingsen et al. (1970) and Itoh (1977a, b), who wanted to apply Stuart-Watson's theory to just these two exceptional flows, were forced to modify this theory to a form where only the smallness of the amplitude A was assumed. It was found in these papers that in fact the smallness of the disturbance amplitude is sufficient for the possibility of rigorous derivation of the Landau equation from the Navier-Stokes equations. More detailed analysis of assumptions utilized in the rigorous derivations of Eq. (4.41) was undertaken in particular by Herbert (1983b) and Fujimura (1989, 1991, 1997) whose papers will be discussed later in this subsection.

Stuart (1960) and Watson (1960a) investigated only the temporal nonlinear development of a two-dimensional wave disturbance in a steady plane Poiseuille flow. Two-dimensionality of the waves significantly simplified the theory, and could be justified to a certain degree by the results of Watson (1960b) and Michael (1961) mentioned in Chap. 2, they showed that, in the framework of the linear stability theory, there always exists for any steady plane-parallel flow a range of supercritical values of Re, $\mathrm{Re}_{cr} < \mathrm{Re} < \mathrm{Re}_1$, within which the most rapidly growing normal-mode disturbance is necessarily two-dimensional. However, Benney and Lin (1960) (see also Benney (1961, 1964)) indicated that when the nonlinear development is studied, interactions between two- and three-dimensional waves must be also of great importance. In this context Stuart (1962b) (see also his surveys (1962a, 1971)) generalized his and Watson's weakly-nonlinear disturbance theory of 1960 to the case of the evolution in plane Poiseuille flow of a disturbance which is composed of a two-dimensional and a three-dimensional plane wave with the same streamwise number k_1. Assuming that both disturbances are slowly growing or decaying, it is permissible, for relatively small values of t, to represent the velocity field of the disturbance considered in the form

$$\mathbf{u}(\mathbf{x},t) = A_1(t)\mathbf{u}_1(z)e^{ik_1x} + A_2(t)\mathbf{u}_2(z)e^{i(k_1x+k_2y)} + \text{higher-order terms} \quad (4.42)$$

including two time-dependent amplitudes $A_1(t)$ and $A_2(t)$. Then, using the expansion technique given in Stuart's and Watson's papers of 1960, Stuart obtained, for both amplitudes A_1 and A_2, two generalized Landau-type equations differing from (4.41) by the presence of their right-hand sides of the sums of composite terms $a_{m,n}|A_1|^{2m}|A_2|^{2n}$. In the lower non-linear approximation the "amplitude equations" for real amplitudes $A_1(t)$ and $A_2(t)$ (obtained when the complex exponential functions in Eq. (4.42) are replaced by real trigonometric functions) had the following form:

$$\frac{dA_1}{dt} = \gamma_1 A_1 - (\delta_1 A_1^2 + \beta_1 A_2^2)A_1,$$
$$\frac{dA_2}{dt} = \gamma_2 A_2 - (\beta_2 A_1^2 + \delta_2 A_2^2)A_2, \tag{4.43}$$

which, for $A_2 = 0$ or $A_1 = 0$, clearly yield an equation which is equivalent to Landau's Eq. (4.34) for the amplitude of a single wave disturbance. The system (4.43), under the condition that $\gamma_1/\delta_1, \gamma_2/\delta_2$ and the ratios of bilinear combinations of the coefficients entering Eq. (4.44) below are positive, evidently has the following four steady solutions:

$$\begin{aligned}
&\text{(I)} \quad A_1 = A_2 = 0,\\
&\text{(II)} \quad A_1 = 0,\ A_2 = (\gamma_2/\delta_2\,)^{1/2},\\
&\text{(III)} \quad A_1 = (\gamma_1/\delta_1\,)^{1/2},\ A_2 = 0,\\
&\text{(IV)} \quad A_1 = (\gamma_1\delta_2 - \gamma_2\beta_1)^{1/2}\,(\delta_1\delta_2 - \beta_1\beta_2)^{1/2},\\
&\qquad\quad A_2 = (\gamma_2\delta_1 - \gamma_1\beta_2)^{1/2}\,(\delta_1\delta_2 - \beta_1\beta_2)^{1/2}.
\end{aligned} \tag{4.44}$$

The stability of these solutions, which may be verified by known methods of stability theory of nonlinear differential equations (or nonlinear oscillations), is of considerable interest, and it was only natural that Stuart considered this question, paying special attention to cases where solution (IV), which represents an equilibrium state consisting of a combination of two- and three-dimensional wave oscillations, is stable. Stuart's two-mode weakly-nonlinear theory of 1962 was developed further by Itoh (1980) who supplemented it by some numerical examples illustrated by graphs.

In all the above-mentioned papers devoted to rigorous derivation of amplitude equations of the Landau type, only the nonlinear temporal development of wave-like disturbances with fixed wave numbers was considered. However, it was explained in Chap. 2 of this book that, in the case of steady flows with significant streamwise velocity $U(z)$ (e.g. boundary layers along flat plates or plane Poiseuille flows), the model of a streamwise developing disturbance of fixed real angular frequency ω corresponds better to observations in real experiments on flow instability, and therefore seems to be more appropriate. Taking this into account Watson (1962) modified the theory developed in his paper (1960a) assuming that a two-dimensional wave-like disturbance in a plane Poiseuille flow has fixed real frequency ω but complex streamwise wave number $k = k_1 + ik_2$, determined from the Orr-Sommerfeld eigenvalue problem with fixed real ω and unknown complex eigenvalue k. Then, according to the weakly nonlinear stability theory, the leading term of the evolving disturbance will have the form $\mathbf{u}(x,t) = A(x)\mathbf{u}(z)e^{i\omega t}$, where $\mathbf{u}(z)$ is the eigenfunction of the spatial O-S eigenvalue problem and $A(x) = e^{ikx}$ for very small values of x. Then, representing the velocity field $\mathbf{u}(\mathbf{x},t) = \mathbf{u}(x,z,t)$ (or the streamfunction field $\Psi(x,z,t)$) for $x > 0$ as a Fourier series in powers of $e^{i\omega t}$ (instead of the spatial Fourier series (4.38)) and applying appropriately-modified arguments from his paper

(1960a), Watson obtained for the spatially evolving amplitude $A(x)$ an equation of the form

$$\frac{d|A|^2}{dx} = |A|^2 \sum_{m=0}^{\infty} b_m |A|^{2m} \tag{4.41a}$$

which is completely similar to Eq. (4.41) (and turns into the spatial version of Landau's Eq. (4.34) when only the first two terms on the right-hand side are retained). It is clear that the value of $b_0 = -2k_2 = -2\Im mk$ can be now calculated by numerical solution of the spatial O-S eigenvalue problem (which is somewhat more complicated than the corresponding temporal problem but nevertheless accessible to computation; see Sect. 2.92). However, the expression found by Watson for the coefficients b_m with $m > 0$ turned out to be much more complex than the—also rather complicated—expressions for the corresponding coefficients a_m; therefore in the early 1960s their evaluation seemed to be impossible. But somewhat later Itoh (1974a, b) showed that by that time the values of the 'spatial Landau constant' $\delta_s = -b_1$ might already have been calculated with satisfactory accuracy for some important plane-parallel flows (see Figs. 4.11 and 4.17 below).

Note that Stuart (1960; a, 1962a,b) and Watson (1960a, 1962) used the fluid dynamics equations only for rigorous derivation of amplitude equations, and did not try to determine numerical values of the coefficients of the latter. Simultaneously, Stuart stressed that the early estimates of the value of δ by Meksyn and Stuart (1951) and Stuart (1958) are not trustworthy. Therefore it was natural to think that Stuart's and Watson's papers would stimulate other authors to find, at last, some accurate estimates of Landau's constant and of other coefficients of amplitude equations. And in fact papers devoted to such estimation began to appear soon after those mentioned above. We will now pass on to results of this subsequent work.

4.2.2 Evaluation of Coefficients of Amplitude Equations and Equilibrium Disturbances for Plane Poiseuille Flows

One of the first attempts to find a more or less reliable value for the Landau constant δ was made by Davey (1962) for the case of the growth of axisymmetric Taylor vortices in a Couette flow between rotating cylinders. Davey reformulated for this case all the arguments of Stuart (1960) and found that Stuart's equation $\delta = \delta_1 + \delta_2 + \delta_3$, where the three terms δ_i have the same physical meaning as in the case of plane Poiseuille flow, also appears here. He also found that in this case the expressions for these terms are again rather complicated but are nevertheless accessible to numerical computation. So, he calculated the value of δ for three particular combinations of the ratios $\mu = \Omega_2/\Omega_1$ and $\eta = R_1/R_2$. The values found turned out to be positive in all cases considered, for all vertical wave numbers k and Reynolds numbers Re, and these values agreed satisfactorily with the then-available experimental data. However, we will not linger here on these results of Davey, since nonlinear stability of circular

Couette flow will be considered separately later in this book. So now we will turn again to cases of plane-parallel (or nearly plane-parallel) primary flows.

Let us revert first of all to the model case of *plane Poiseuille flow.* Recall that Stuart's and Watson's nonlinear-stability papers of 1960 were both devoted to just this case, which was also considered rather early by Meksyn and Stuart (1951), then by Stuart (1958), and later by Eckhaus (1965). This was only natural, since plane Poiseuille flow is a classical example of steady, strictly plane-parallel, laminar flow having a very simple velocity profile, and had been extensively investigated within the framework of the linear theory of hydrodynamic stability. Thus, it was not surprising that relatively accurate estimates of the values of the Landau constant for disturbances in a plane Poiseuille flow were among the first applications of the Stuart-Watson theory to appear.

The above-mentioned estimates were calculated independently and almost simultaneously by Reynolds and Potter (1967) and Pekeris and Shkoller (1967). Reynolds and Potter used some extension and modification of the Stuart-Watson approach where determination of the equilibrium disturbances, introduced in application to another problem by Malkus and Veronis (1958), played a very important part, while Pekeris and Shkoller based their computations on the Eckhaus eigenfunction-expansion method. In both papers the computations were carried out for two-dimensional normal-mode disturbances corresponding to the unstable (or, if $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$, to the least stable) solution of the Orr-Sommerfeld eigenvalue problem under the condition that $|\gamma| = |\Im m\omega|$ is sufficiently small. However, Reynolds and Potter also included in their paper some remarks relating to three-dimensional disturbances, and presented some numerical results for the more general case of plane Couette–Poiseuille flows (these results will be discussed in Sect. 4.23). For plane Poiseuille flow Reynolds and Potter calculated values of δ at five different points of the neutral stability curve in the (k, Re)-plane (including the critical point (k_{cr}, $\mathrm{Re}_{\mathrm{cr}}$)), and at two points in the neighborhood of the neutral curve, while Pekeris and Shkoller evaluated the coefficient $\delta = \delta(k, \mathrm{Re})$ for an extensive region of the (k, Re)-plane (using equations which are in fact reasonable only in the vicinity of the neutral curve). The results of these two papers do not coincide numerically (one reason being that they used different normalizations and somewhat different definitions of the amplitude $|A|$, besides which some of the assumptions and approximations taken for granted in the two papers were different), but both results have the same general behavior and imply close agreement for ratios of the values $\delta = \delta$ (k, Re) at different points of the (k, Re)-plane.

In Fig. 4.8, results by Pekeris and Shkoller (agreeing, in general, with Reynolds and Potter's conclusions) are presented, including the neutral curve but without numerical values for γ and δ. (As to the values of Re and k, it is here assumed, as usually, that $\mathrm{Re} = U_{\max} H_1/\nu$ and k is made dimensionless by multiplication by H_1). These results show, in particular, that at the critical point (the point of the neutral curve farthest to the left), and at all points of the upper branch of the neutral curve, δ is negative. Some unstable two-dimensional disturbances of finite amplitude with wave number k must correspond to values of (k, Re) at points lying close to the neutral curve in the region where $\delta < 0$ and $\gamma < 0$; this means that at these values of (k, Re),

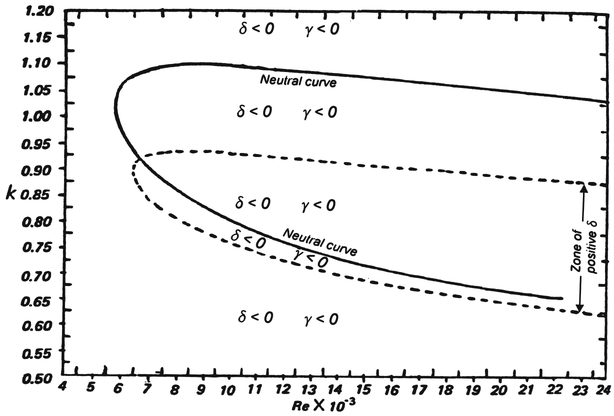

Fig. 4.8 The regions of positive and negative values of the coefficients γ and δ in the (k, Re)-plane for the case of plane Poiseuille flow. (After Pekeris and Shkoller (1967))

subcritical finite-amplitude instabilities exist in plane Poiseuille flow. Therefore, in the region where $\delta < 0$, the neutral curve (which bounds the set of points (k, Re) corresponding to unstable two-dimensional disturbances) shifts, in the case of finite disturbances, from the neutral-stability curve of linear stability theory (which relates to infinitesimal disturbances) and takes the shape shown in the schematic Fig. 4.9. On the other hand, for points (k, Re) in the region where $\delta < 0$, $\gamma > 0$ the negativity of δ means that supercritical finite-amplitude equilibrium states are rather unlikely to be observed here. Figure 4.8 shows also that $\delta > 0$ on the main part of the lower branch of the neutral curve. At the points (k, Re) close to this part the subcritical finite-amplitude instability does not exist for disturbances with $\gamma < 0$; however, if

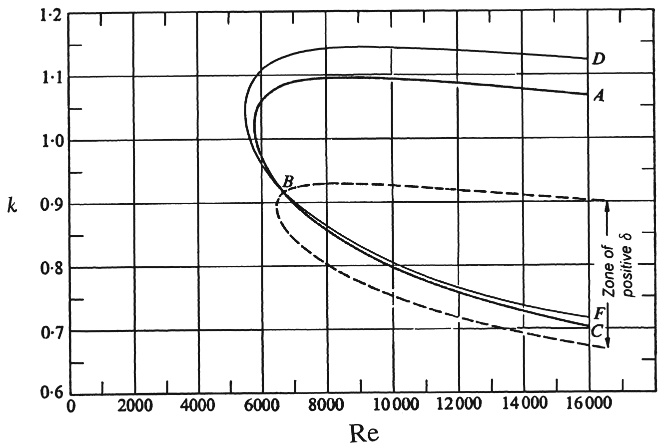

Fig. 4.9 Schematic form of the neutral-stability *curve DBF* for wave disturbances of plane Poiseuille flow having a fixed finite amplitude A. (After Pekeris and Shkoller (1969b)) the *curve ABC* is the neutral curve for infinitesimal disturbances where $\gamma = 0$, and the dotted *curve* represents points where $\delta = 0$

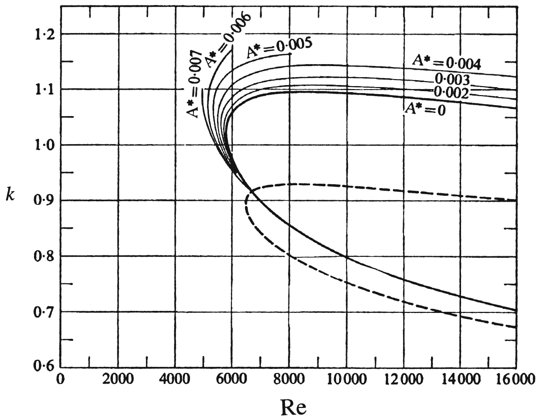

Fig. 4.10 Deviations of the neutral *curves* for wave disturbances of plane Poiseuille flow having finite amplitudes (characterized by the value of some dimensionless 'amplitude parameter' A^*) from the neutral curve for infinitesimal disturbances (corresponding to $A^* = 0$) in the region where $\delta < 0$, computed by Pekeris and Shkoller (1969b). The *dotted curve* have the same meaning as in Figs. 4.8 and 4.9

$\gamma = 0$ and $\delta > 0$ for a small but not infinitesimal disturbance, then this disturbance will decay according to Eq. (4.34). Thus, the neutral curve for finite disturbances corresponding to points where $\delta > 0$ must shift into the supercritical region where $\gamma > 0$, and hence finite-amplitude equilibrium states must exist.[6]

Reynolds and Potter's and Pekeris and Shkoller's papers stimulated the appearance of many subsequent papers on the nonlinear evolution of wave disturbances in a plane Poiseuille flow. These later papers, only some of which will be referred to below, include various amendments, modifications and revisions of results presented in the publications of 1967. In particular, Pekeris and Shkoller (1969a,b; 1971) computed some approximate solutions of the nonlinear initial-value problem for the least-stable Tollmien–Schlichting (T–S) wave with given wave number k, i.e., for the two-dimensional disturbance having the initial stream function of the form $\Psi(x,z,0) = Af_1(z)e^{ikx}$ where $f_1(z)$ is the normalized first (least stable) O-S eigenfunction of the plane Poiseuille flow and A is a disturbance amplitude which is finite (but small enough, since an expansion in powers of amplitude was used here). Using the computed results Pekeris and Shkoller tried to estimate quantitatively the shifts of the neutral curves for finite-amplitude disturbances of the form given above, for various values of A (see Fig. 4.10, taken from their paper (1969b)), and to determine the value of the finite-amplitude critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}(A)$ (which corresponds to the point which is farthest to the left on the neutral curve for disturbances of amplitude A). The same problem was studied by Georg and Hellums (1972) and Georg et al. (1974) who considered another initial form of disturbance (i.e. they did not use the traditional approach of considering the least-stable T–S wave) and another method of numerical solution of the nonlinear initial-value problem (which used neither the Eckhaus expansion into O-S eigenfunctions nor the expansion in powers of the amplitude, and hence was applicable to disturbances of any initial size).

[6] According to Eq. (4.34) and Fig. 4.8, the lower branch of the neutral curve in the case of finite disturbances must shift upward (to points where $\gamma \approx \delta|A|^2/2$). This very small shift is exaggerated in Fig. 4.9 to simplify its representation in the figure but later it will be neglected.

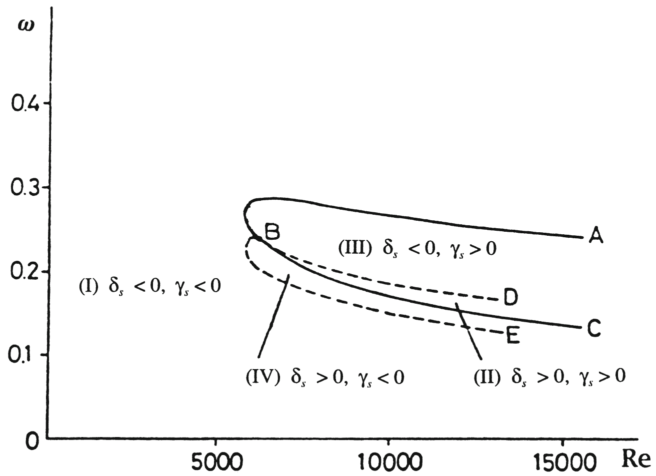

Fig. 4.11 The regions of positive and negative values of the coefficients γ_s and δ_s in the (ω,Re)-plane for the case of plane Poiseuille flow. (After Itoh (1974a))
ABC: the *curve* $\gamma_s(\omega$, Re) $= 0$ (the spatial neutral-stability *curve* of the linear stability theory bounding the region where $\gamma_s > 0$); *DBE*: the *curve* $\delta_s(\omega$, Re) $= 0$ bounding the region where $\delta_s > 0$

Georg et al. (whose estimates of the values of critical numbers $\mathrm{Re}_{cr}(A)$ were later found by Orszag and Kells (1980) to be too high because of the use of a non-optimal initial form of the disturbance) compared their results with those of several previous papers (including those by Reynolds and Potter and Pekeris and Shkoller). They found that the quantitative results of different authors sometimes do not agree adequately well, but all of them demonstrate the same general tendency. One more method for approximate determination of the neutral curve for two-dimensional finite-amplitude wave disturbances in a plane Poiseuille flow was proposed by Struminskii and Skobelev (1980), who used for this purpose the generalized Landau equation of the form (4.37). Later Luo (1994) reexamined the previously used methods of determination of complex coefficients ω and l in the Stuart-Landau Eq. (4.40). He suggested some improvements and showed that in the case of plane Poiseuille flow they lead to values of coefficients which agree well with those given by numerical simulation of disturbance evolution in this flow.

Itoh (1974a) studied the development of a spatially-evolving two-dimensional disturbance of frequency ω in a plane Poiseuille flow, using the theory by Watson (1962) modified by accounting more accurately for distortion of the mean flow by the disturbance. Using the modified version of Watson's theory, he computed approximate shapes of the curves $\gamma_s(\omega, \mathrm{Re}) = 0$ and $\delta_s(\omega, \mathrm{Re}) = 0$ (where $\gamma_s = b_0/2$ and $\delta_s = -b_1$ are coefficients of the 'spatial Landau equation', and ω is non-dimensionalized by multiplication by $H_1/U_{\max}$) on the (ω, Re)-plane. These curves are shown in Fig. 4.11; they determine location of the regions of positive and negative values of γ_s and δ_s in the (ω, Re)-plane and proved to be qualitatively similar to Pekeris and Shkoller's curves in Fig. 4.8 which correspond to temporally-evolving disturbances in the same flow.

Related computations were performed by Herbert (1976, 1977, 1978) (see also his review (1983a)) who used a quite different method. This author followed the approach initiated by Zahn et al. (1974) (and outlined in rudimentary form as far back as Noether (1921) and Heisenberg (1924)), and studied approximate numerical solutions of the nonlinear initial-value problem for stable Tollmien-Schlichting waves (i.e., those which are exponentially damped according to the linear stability theory), which at small values of t are represented by the O-S eigensolutions. He paid most attention to equilibrium solutions (i.e., to wave disturbances satisfying the condition that $d|A|^2/dt = 0$) at various values of k and Re. To find the value of the stream function $\Psi(x, z, t)$ corresponding to an evolving T–S wave, both Zahn et al. and Herbert represented Ψ by a strongly truncated Fourier series of the form (4.38), and then solved numerically a system of coupled nonlinear equations for the corresponding Fourier coefficients, simplifying this system greatly for the case of equilibrium solutions.

Herbert found numerous equilibrium two-dimensional disturbances in a plane Poiseuille flow which are periodic in the streamwise direction and have finite amplitudes. His results agree well with results of preceding numerical studies by Zahn et al. (1974), and of subsequent more accurate computations by Orszag and Kells (1980); Orszag and Patera (1980, 1981); Milinazzo and Saffman (1985); Ehrenstein and Koch (1991); Balakumar (1997); Hewitt and Hall (1998), and some others (see also the survey by Bayly et al. (1988)). Measuring the size of a two-dimensional wave disturbance by the ratio E of its kinetic energy (per unit length of the channel) to the energy of primary Poiseuille flow (E is clearly a single-valued function of A and is proportional to $|A|^2$ with good accuracy), Herbert determined the shape of the *neutral surface* (corresponding to the set of all two-dimensional equilibrium waves) in the three-dimensional (E, k, Re)-space; this surface is shown schematically in Fig. 4.12. (See also Ehrenstein and Koch (1991) and Sect. 2.8.3 in Godrèche and Manneville (1998) where a slightly different presentation of this surface is given. Two intersections of this surface with the plane Re = const. will be shown in Sect. 4.2.3 in Fig. 4.14a, b where, however, U_{max} is replaced by $U_{ave} = 2U_{max}/3$ in the definition of Re; some of its other intersections with planes Re = const. and k = const. can be found in Sect. 2.8.3 of Godrèche and Manneville (1998) and in the paper by Hewitt and Hall (1998)). The intersection of the neutral surface with the plane $E = 0$ clearly coincides with the Poiseuille-flow neutral curve of linear stability theory (shown, in particular, in Figs. 2.22 and 4.8), while the intersections of this surface with the planes E = const. (where also A = const.) coincide with the neutral-stability curves for finite-amplitude disturbances with given value of E (or A; cf. Figs. 4.9 and 4.10). The projection of the whole neutral surface in (E, k, Re)-space on the (k, Re)-plane is also indicated in Fig. 4.12; this projection determines the region of the (k, Re)-plane corresponding to unstable two-dimensional waves of any amplitude. This region is clearly much larger than the region of unstable infinitesimal waves, which is bounded by the neutral curve of linear theory. The projection of the leftmost point of the neutral surface in (E, k, Re)-space on the (k, Re)-plane determines the lowest Reynolds number Re^*_{cr} at which there exist also undamped two-dimensional waves of any amplitude, and the critical wave number k^*_{cr} corresponding to the undamped wave at Re $= \mathrm{Re}^*_{cr}$. Similarly,

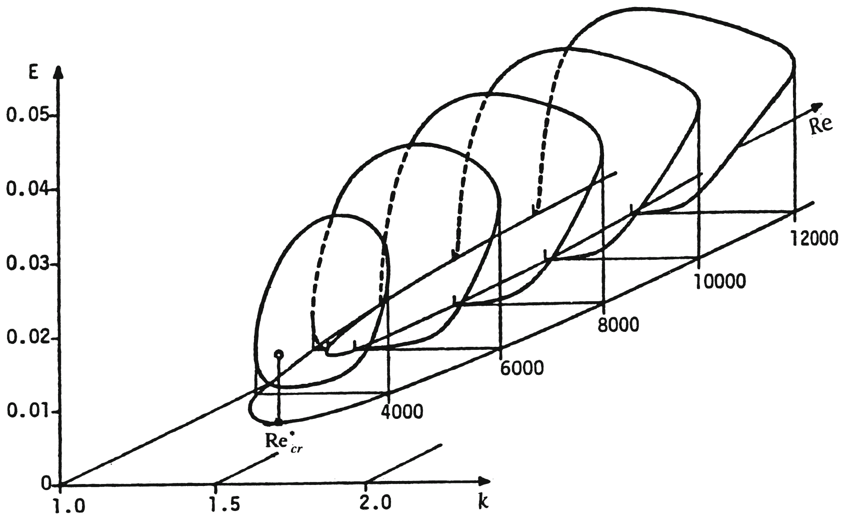

Fig. 4.12 Schematic shape of the nonlinear neutral surface in the three-dimensional (E, k, Re)-space corresponding to the set of all two-dimensional equilibrium waves in plane Poiseuille flow. (After Herbert (1977, 1978, 1983a))

the leftmost points of the neutral curves for waves with fixed energy E (and amplitude A) determine the critical Reynolds numbers $\mathrm{Re}_{cr}(E)$ (or $\mathrm{Re}_{cr}(A)$) for waves of fixed energy (and amplitude) and their wave numbers $k_{cr}(E)$ (or $k_{cr}(A)$). According to Herbert's approximate computations, $\mathrm{Re}^*_{cr} \approx 2935$ (as usual, channel half-thickness and Poiseuille-flow maximum velocity are used here as length and velocity scales) and to this corresponds the critical wave number $k^*_{cr} \approx 1.32$. Later Herbert's results were confirmed also by Orszag and Kells (1980); Ehrenstein and Koch (1991), and Balakumar (1997).

Note that the 'nonlinear critical Reynolds number' Re^*_{cr} is considerably smaller than the ordinary (linear) critical Reynolds number $\mathrm{Re}_{cr} \approx 5772$ which relates to infinitesimal wave disturbances. However Re^*_{cr} is much greater not only than the value $\mathrm{Re}_{cr\,min} \approx 50$ which is given by the energy method and applies to disturbances of any shape and size, but also much greater than the value $\mathrm{Re}_1 \approx 1{,}000$ which, according to data by many authors (e.g., by Davies and White (1928); Patel and Head (1969); Kao and Park (1970); Nishioka and Asai (1985), and Alavyoon et al. (1986); see also Sect. 2.1) is typical for transition to turbulence in laboratory experiments on plane Poiseuille flow. During the 1980s and early 1990s a number of authors (in particular, Orszag and Kells (1980); Orszag and Patera (1982, 1983); Saffman (1983); Herbert (1983c, 1984, 1986); Soibelman and Meiron (1991); see also the surveys by Bayly et al. (1988) and Herbert (1988)) suggested the idea that the difference between values of Re^*_{cr} and of Re_1 can be explained by secondary instability of stable two-dimensional waves to small three-dimensional disturbances at values of Re smaller than Re^*_{cr}. To verify this idea these (and some other) authors performed a number of numerical simulations (i.e., solutions of the corresponding nonlinear initial-value problems for the N-S equations) of development, in a plane Poiseuille flow, of stable two-dimensional waves of finite amplitude in the presence of small three-dimensional disturbances with the same streamwise wave number. The results obtained showed that three-dimensional disturbances often destabilize two-dimensional waves, and cause rapid growth of two-mode disturbances at Reynolds numbers of the order of 700–1,000, much smaller than $\mathrm{Re}^*_{\mathrm{cr}}$ (and close to Re_1). (The simulations by Pugh and Saffman (1988) and the subsequent study by Barkley (1990) showed that the instability of two-dimensional equilibrium waves with respect to superimposed three-dimensional disturbances has a more complex character than was assumed earlier. Moreover, there were several attempts to explain the secondary instability of two-dimensional waves by triad interactions of such a wave with two three-dimensional ones, and these attempts also led to critical Reynolds number close to Re_1; see, e.g., Goldshtik et al. (1983, 1985); Craik (1985); Ehrenstein and Koch (1991), and Ehrenstein (1994). And still later Reddy et al. (1998) considered some quite different scenarios of the primary and secondary instabilities of a plane Poiseuille flow where two-dimensional waves play no part at all. However, we will not consider all these works in this section). The secondary instability of two-dimensional periodic waves usually generates, not a new equilibrium cellular state but a very complicated three-dimensional structure reminiscent developed turbulence; see, e.g., Saffman (1983); Rozhdestvensky and Simakin (1984); Bayly et al. (1988), and Jiménez (1987, 1990). In some of this work several successive transitions of Poiseuille flow to more and more complex behavior were also simulated numerically (more details of this will be given later).

Numerically-simulated equilibrium and developing wave disturbances in a Poiseuille flow may in principle be used to get some information about the values of Landau's constant and other coefficients of the amplitude equations for one-mode or composite two-mode waves. The estimates of δ implied by the results of Herbert's and Orszag and Kell's numerical simulations of two-dimensional equilibrium waves in a plane Poiseuille flow proved not to contradict values found earlier by Reynolds and Potter (1967); Pekeris and Shkoller (1967), and other authors by the quite different methods initiated by Stuart, Watson, and Eckhaus.

Quite another approach was applied to study of development of two-dimensional finite-amplitude waves in a plane Poiseuille flow by Andreichikov and Yudovich (1972) and Chen and Joseph (1973). This approach was based on the general theory of bifurcations, which is a special part of nonlinear science closely connected with stability problems. The word *bifurcation* means here the appearance of a supplementary solution of a given nonlinear 'dynamic equation' (or system of equations), describing the evolution of a definite object, when some dynamic parameters vary. The 'dynamic equation' (or equations) may be here algebraic, ordinary differential, partial differential or any other type. *Bifurcation theory* deals with the most typical features of the nonlinear evolution, namely, with the frequent occurrence of qualitative changes of the object's behavior corresponding to small variation of some dynamic parameters. Drazin and Reid (1981), p. 403, reasonably noted that this theory arose from particular early work by Poincaré and Lyapunov on figures of equilibrium of rotating self-gravitating masses of fluid, but the sphere of its applications has broadened enormously. Therefore it is natural that in recent years this theory attracted much attention, and gave rise to extensive and quite diverse literature. As typical examples we may mention here the books and survey papers by Sattinger (1973); Marsden (1978); Mittleman and Weber (1980); Seydel (1988); Arnol'd (1989b); Iooss and Joseph (1990); Baker (1991); Hale (1991); Iooss and Adelmeyer (1992); Guckenheimer and Holmes (1993); Arnol'd et al. (1994), and Field (1996).

An example of an instability-generated bifurcation in a fluid flow is given in Fig. 4.7a, where the dependence on Reynolds number Re of the equilibrium amplitude $A_e = |A|_{max}$ of a normal-mode disturbance in a primary steady flow is presented for the case where $\delta > 0$. Here $A_e = 0$ for $Re < Re_{cr}$; however, if $Re > Re_{cr}$ (but $Re - Re_{cr}$ is small), then $\gamma \propto Re - Re_{cr} > 0$, and the amplitude of a small disturbance tends to the equilibrium value $A_e = (2\gamma/\delta)^{1/2} \propto (Re - Re_{cr})^{1/2}$ (see the upper part of Fig. 4.5). Thus, the flow consisting of the primary flow and a superimposed two-dimensional periodic wave of amplitude A_e bifurcates at $Re = Re_{cr}$ from the pure primary flow. Figure 4.7b corresponds to the case where $\delta < 0$ and shows another type of bifurcation: here, according to the figure the 'secondary solution' which includes a finite-amplitude wave appears at $Re = Re'_{cr} < Re_{cr}$ but transition from the primary steady solution to this new solution can be caused only by a wave disturbance with amplitude exceeding $|A_2|_{max}$. Let us now consider the complex Landau amplitude $A(t)$ which satisfies Eq. (4.40) and describes the time-dependence of the leading term of the disturbance velocity $\mathbf{u}(\mathbf{x}, t)$. As we know, here $A(t) = |A(t)|\, e^{i(-\omega t+\theta)}$, where $|A(t)|^2$ satisfies the Landau Eq. (4.34) and the constant θ depends on the initial disturbance $\mathbf{u}(\mathbf{x}, 0)$. According to Fig. 4.7a, if $\delta > 0$, then for $Re < Re_{cr}$ the complex amplitude $A(t)$ for any initial value $A(0)$ tends to zero (i.e., to the origin of the complex-variable plane) as $t \to \infty$. In other words, for $Re < Re_{cr}$ all trajectories $A = A(t)$ in the complex-variable plane corresponding to various solutions of the complex Landau Eq. (4.40) are attracted to a focus at the origin. If, however, $Re > Re_{cr}$, then $|A(t)| \to A_e \propto (Re - Re_{cr})^{1/2}$ as $t \to \infty$ and hence the trajectory $A(t) = |A(t)|\, e^{i(-\omega t+\theta)}$ is here attracted to the circle of radius A_e in the complex-variable plane which makes up the *limit cycle* of the two-dimensional dynamical system corresponding to dynamic Eq. (4.40) (i.e., to a system of two Eqs. (4.34) and (4.34a) for real and imaginary parts of $A(t)$). This is just a specific case of the so-called *Hopf bifurcation*[7], where a periodic solution bifurcates from a steady

[7] This term reflects the contribution by Hopf (1942) to this subject. However sometimes its use meets objections since such bifurcations were in fact explicitly studied by A. A. Andronov (partially in collaboration with A. A. Vitt) in the early 1930s and were described at length in the book by Andronov and Khaikin (1937). It was also sometimes noted that the so-called 'Hopf bifurcation' first appeared in fact in the works of Poincaré; therefore, Marsden and McCracken (1976) wrote in the preface to their book that apparently the term 'Poincaré-Andronov-Hopf bifurcation' would be the most just. However, the short term 'Hopf bifurcation' is now universally accepted; so it will be used in this book too.

Note in conclusion that the classical book by Andronov and Khaikin was in fact written by three authors. Only in the late 1950's it was permitted to S. E. Khaikin, the only one author who was then one when the latter becomes unstable. Hopf bifurcations form the most elementary class of bifurcations, which are encountered very often in various applied fields (see, e.g., the books by Marsden and McCracken (1976); Hassard (1981), and Moiola and Chen (1996) specially devoted to such bifurcations); some more complicated bifurcations will also be discussed later in this book.

Above, for the sake of simplicity, we discussed only bifurcations of solutions of Landau's amplitude equations (which are ordinary and not partial differential equations). In fact only ordinary differential equations were considered in the early works on bifurcations by Poincaré, Andronov, Vitt, and Hopf. The general theory of periodic flow bifurcations from a steady solution of Navier–Stokes equations was developed independently by Yudovich (1971, 1972); Iooss (1972) and Joseph and Sattinger (1972) (see also Chaps. 9 and 9A in the book by Marsden and McCracken (1976), and references to early examples of such fluid-dynamic bifurcations in the book by Drazin and Reid (1981), p. 407). The papers mentioned contain, in particular, definite conditions under which such bifurcation necessarily occur. Then Andreichikov and Yudovich (1972) and Chen and Joseph (1973) showed that the results of the above-mentioned papers lead to definite assertions about the uniqueness, stability and properties of the two-dimensional periodic solutions which bifurcate from the steady Poiseuille flow at points of the corresponding neutral-stability curve. These assertions proved to be in good qualitative (and in satisfactory quantitative) agreement with the conclusions about disturbance development obtained earlier by other authors who used quite different, and often less rigorous, arguments based on the Stuart–Watson theory and its modifications.

Let us briefly discuss now results of some further work concerning the nonlinear evolution of normal-mode wave disturbances in plane Poiseuille flow. Recall that approximate estimates of the numerical values for the Landau constant for two-dimensional wave disturbances spatially evolving in a plane Poiseuille flow were first given by Itoh (1974a). Early comparisons of the available theoretical estimates with the experimental data by Nishioka et al. (1975), referring to development of waves generated by a vibrating ribbon in a laboratory channel flow, seemed to support both the results by Itoh (1974a) and the conclusions of Herbert (1977). However, subsequent more careful analysis detected some appreciable discrepancies between theory and experimental data, apparently connected with three-dimensional effects affecting measurements by Nishioka et al. and with some inaccuracies of Itoh's calculations; see, e.g., Zhou (1982); Herbert (1980, 1983a), and Sen and Venkateswarlu (1983). Another method for calculation of Landau's constant was proposed by Itoh (1977a); as was indicated by Davey (1978) and Herbert (1983b), this method differs from that of Reynolds and Potter (1967) only by rearrangement of the terms in some

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alive, to publish the revised edition of the book as a book by Andronov et al. (1959) with a strange remark in the Preface (which was repeated in the English translation of 1966 too) that 'the name of one of the authors was by an unfortunate mistake not noticed on the title page of the first edition'. The 'unfortunate mistake' was due to the fact that A. A. Vitt, a young talented scientist, was arrested in 1937 by Stalin's notorious secret police (which chose its victims for reasons incomprehensible to any normal mind) and died in prison the next year.

infinite series, and is appropriate only for the case where $A = A_e$ is the equilibrium amplitude of a disturbance.

The accuracy of estimates of the value of Landau's constant is clearly affected by the absence of a unique, universally-accepted definition of the disturbance amplitude A and by possible influence of further terms of Eqs. (4.41) and (4.41a) which were neglected in most of the early papers. (The first attempts to estimate, for a plane Poiseuille flow, the values of two coefficients a_m, $m = 1$ and 2, of Eq. (4.41) (in other words, of the coefficients δ and β of Eq. (4.37)) and of the corresponding complex coefficients λ_m, $m = 1$ and 2, of Eq. (4.41b) presented below were due to Gertsenshtein and Shtemler (1997) and Shtemler (1978). These authors applied the modified method of Reynolds and Potter (1967) to compute the values of the coefficients a_1, a_2, and λ_1, λ_2 for several points (k, Re) of the plane-Poiseuille-flow neutral curve and then, assuming that $A = A_e$, studied the influence of the terms with $m = 2$ on the values of the equilibrium amplitude A_e and the shape and stability of the equilibrium waves). Later it was stressed by Herbert (1980, 1983b) that many theories leading to determination of the higher-order terms do not exclude equally-justified alternative methods of computation, leading to changes in the values of these terms. In the paper of 1980 Herbert developed a consistent method of perturbation expansion for solution of the Navier-Stokes equations which included a unique definition of the real amplitude $A(t)$ and led to Eq. (4.41) with unique values of the Landau constants a_m of all orders. Then he showed how the values of these constants can be determined, and he calculated, for plane Poiseuille flow, the values of the first seven constants a_m at the critical point (k_{cr}, Re_{cr}) of the (k, Re)-plane and at one subcritical point with $\mathrm{Re} < \mathrm{Re}_{cr}$. The results obtained showed that the coefficients a_m increase rapidly with m. Therefore Eq. (4.41) is in fact useful only in the case of a very small amplitude A. In Herbert's paper (1983b) a survey and also a comparison of various expansion methods based on different assumptions was presented, and the ranges of applicability and shortcomings of these methods were discussed. In particular he showed that the method of Watson (1960a) is exact only at points of the neutral curve where $a_0 = 2\gamma = 0$, while if $\gamma \neq 0$, then Watson's value of δ differs from the value given by the more rigorous method of Herbert (1980) (see Fujimura (1987) for a more detailed analysis of this matter). Later Crouch and Herbert (1993) proposed a new general method for determination of the complex Landau constants λ_{m} of all orders $m \geq 0$ entering the equation for the complex disturbance amplitude A

$$\frac{dA}{dt} = A\sum_{m=0}^{\infty}\lambda_m|A|^{2m} \tag{4.41b}$$

which is a simple generalization of both the Stuart-Landau Eq. (4.40) and the Watson-Landau Eq. (4.41) (where $a_m = 2\Re e\lambda_m$). The same problem was also considered by Sen and Venkateswarlu (1983) and Fujimura (1989, 1991, 1997) whose papers will be discussed below.

Zhou (1982) developed an improved version of the classical Stuart-Watson method of 1960, assuming that both the amplitude $A(t)$ and the angular frequency $\omega_1(t)$ of the unstable wave disturbance vary with time. He expanded the derivatives *dA/dt* and $d\omega_1/dt$ in powers of a suitable small parameter ε (which represents the order of magnitude of nonlinear corrections), and numerically computed the solutions of the resulting system of coupled differential equations for the terms of ε-expansions up to fourth order. In this way Zhou obtained a much more detailed representation of the nonlinear development of subcritical (unstable) wave disturbances, for values of Re from 1,000 up to 5,500 and several values of *k*. It was found that the accuracy of the method decreases with increasing $\mathrm{Re} - \mathrm{Re}_{cr}$, but the experimental observations by Nishioka et al. (1986) concerning the terminal equilibrium states of disturbances at relatively small values of $\mathrm{Re}–\mathrm{Re}_{cr}$ are represented more satisfactorily by the new results than by the results presented in the preceding papers.

Weinstein (1981) applied Watson's (1960a) method to calculate values of the Landau constants $a_m = a_m$ (*k*, Re) up to $m = 3$ for the Poiseuille-flow wave disturbances corresponding to small values of both $|\mathrm{Re} - \mathrm{Re}_{cr}|$ and $|k - k_{cr}|$. His main purpose was to compare results following from his version of Watsons method with those given by quite another method, the so-called *method of multiple scales* first applied to some turbulent-flow calculations by Stewartson and Stuart (1971) (for other applications of the method see, e.g., Cole (1968); Kevorkian and Cole (1981); Nayfeh (1981), or Godrèche and Manneville (1998)). This method uses two different time scales (the 'slow' and 'rapid' ones) which allow the slow evolutionary processes to be isolated from the rapid high-frequency oscillations. (In Landau's original derivation of Eq. (4.34) averaging over a time period intermediate between 'slow' and 'rapid' time scales was used to the same end). Weinstein found that in the cases he considered both methods lead to exactly the same results; however, no numerical data were presented in this paper.

New calculations of the higher-order Landau coefficients for nonlinear wave disturbances in a plane Poiseuille flow, corresponding to both subcritical and supercritical regions of the (*k*, Re)-plane, were carried out by Sen and Venkateswarlu (1983) by both the Reynolds and Potter (1967) and the Watson (1960a) methods. It was found that in the supercritical region the results of both methods are relatively close (in the subcritical region the majority of the computations performed was based on the use of the R-P method). The authors supplemented the results of Pekeris and Shkoller shown in Fig. 4.8, by new lines separating the regions of positive and negative values for Landau's constants a_2 and a_3 in the (*k*, Re)-plane (see Fig. 4.13, based on their results). They also investigated the region of convergence of the Landau–Watson series (4.41) (it was found that the radius of convergence is rather short here, which agrees with the conclusions of Herbert (1980)) and indicated the summation methods appropriate for computations in the cases of slow convergence (or slow divergence) of this series. The equilibrium amplitudes and equilibrium velocity distributions were also determined for the subcritical region, and the values of a great number of complex Landau coefficients $\lambda_m = \lambda_m(k, \mathrm{Re})$ were presented for some particular cases. Some comparisons of the results obtained with experimental data by Nishioka et al. (1975) were discussed in the paper and were found to be encouraging. Note however that Reynolds and Potter's method and the original Watson methods, considered by Sen and Venkateswarlu, are not of high precision, and many

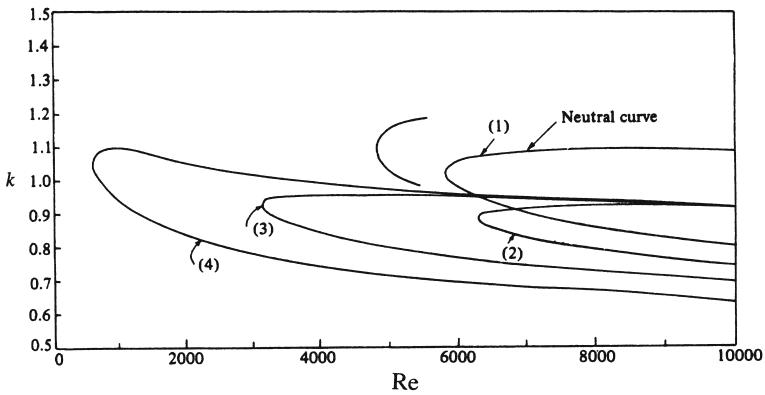

Fig. 4.13 Stability diagram showing the regions of positive and negative values of coefficients γ, δ, and the next two Landau's constants a_2 and a_3 in the (k, Re)-plane for the case of plane Poiseuille flow. (After Sen and Venkateswarlu (1983)) (*1*) *curve* where $\gamma = 0$ (neutral *curve* of the linear stability theory, inside it $\gamma > 0$); (*2*) *curve* where $\delta = 0$ (inside it $\delta > 0$); (*3*) *curve* where $a_2 = 0$ (inside it $a_2 > 0$); (*4*) *curve* where $a_3 = 0$ (inside it $a_3 < 0$)

researchers even supposed that they are inapplicable at points (k, Re) which are far from the neutral curve.

Fujimura (1989) compared two different methods of derivation of the general Landau Eq. (4.41b) for the complex disturbance amplitude $A(t)$ from the Navier-Stokes equations—his own modification of the amplitude-expansion method of Watson and the above-mentioned method of multiple scales (which can be applied to derivation of Eq. (4.41b) if a whole hierarchy of longer and longer time scales is introduced). He began by stressing that the results obtained by both methods depend essentially on the strict definition of the amplitude $A(t)$. Then he showed that if this definition is based on a special normalization condition for the fundamental mode, then in the case of slight supercriticality the method of multiple scales gives results equivalent to those which follow from the modified amplitude-expansion procedure (but not from its original form proposed by Watson). Some results of computations by both methods of the values of the first four complex Landau constants for slightly supercritical wave disturbances in a plane Poiseuille flow are also presented in this paper.

Later Fujimura (1991, 1997) studied one more method of derivation of the complex Landau Eq. (4.41b) from the equations of fluid motion. Note that this equation represents a crucial reduction of the infinite-dimensional dynamical system of flow disturbances evolving in time, to a one-dimensional system fully determined by its amplitude $A(t)$. On the other hand, the modern development of the dynamical system theory led to the appearance of a promising new method of the dimension reduction (i.e., reduction of the numbers of degrees of freedom), called the *method of center manifold* (see, e.g., the books by Carr (1981); Wiggins (1990); Manneville (1990), and Guckenheimer and Holmes (1993)). The method is based on the concept of a *center manifold*—a part S of the phase space R of all possible states of the consid-

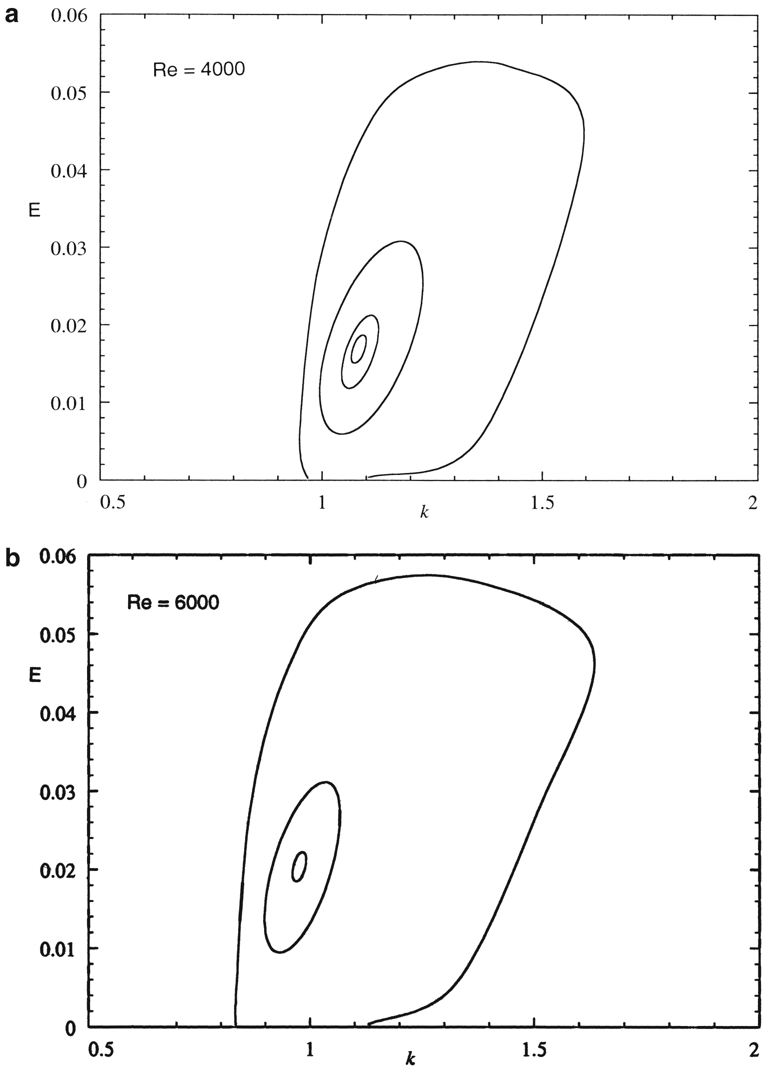

Fig. 4.14 Intersections of the nonlinear neutral surface in the three-dimensional (*E, k*, Re)-space with the plane Re = 4,000 for Couette-Poiseuille flows with Â = 0, 0.12, 0.144, and 0.147 (**a**), and with the plane Â = 6,000 for C-P flows with Â = 0, 0.2, and 0.218 (**b**). (After Balakumar (1997)). Here $\hat{A} = \hat{A}_B = U_W/[\frac{4}{3}U_{max} + U_w]$, $Re = U_{ave}H_1/\nu = [\frac{4}{3}U_{max} + U_w]H_1/2\nu$, and the increase of $\Re$ corresponds to s hrinking of the closed *curves* in the figure

ered system having some special properties. These properties imply, in particular, that any phase trajectory (the curve in R describing time evolution of the system), whose point at time t_0 belongs to S will remain in S also at any $t > t_0$, while a trajectory which is outside of S at time t_0 enters S, under rather general conditions, at some subsequent time moment. (For more information about specialized subspaces of a phase space and their properties see, e.g., Kelley (1967) In the case of a fluid flow the phase space R consists of all admissible values of the main fluid dynamical fields; see Sect. 2.3). The *center manifold reduction* consists in projection of the full phase space R, together with trajectories of a dynamical system lying in R, into some center manifold S of smaller dimension than that of the space R. In the extreme case when the dimension is reduced to one, the state of the nonlinear system becomes fully determined by the amplitude A, and hence the evolution of the system is described by the function $A(t)$ which under a wide range of conditions satisfies an equation of the Landau type. Note however that the center manifold (and thus also the center manifold reduction) may not be unique (see, e.g., Guckenheimer and Holmes (1993)). This nonuniqueness is analogous to the nonuniqueness of Landau's constants because of their dependence on the selected definition of the disturbance amplitude.

Some examples of derivations of Landau's amplitude equations for nonlinear systems with infinite dimensions by the method of center manifold may be found in Carr (1981) and Carr and Muncaster (1983); a number of applications of this method to fluid mechanical problems were considered by Guckenheimer and Knobloch (1983); Iooss (1987); Laure and Demay (1988); Renardy (1989); Manneville (1990); Cheng and Chang (1990, 1992, 1995); Chen et al. (1991), and Chossat and Iooss (1994), among many others. Fujimura at first considered (in the paper of 1991) the most common scheme of the center manifold reduction, applicable to infinite-dimensional systems arising from the partial differential equations (exemplified by the Navier–Stokes system). He applied the method to the classical example of the disturbance development in plane Poiseuille flow, which was also investigated by the center manifold method, in passing, by Renardy (1989) (whose paper was mainly devoted to more general problems). Renardy evaluated the Landau constant δ for a plane Poiseuille flow by this method, and compared her results with those found by Pekeris and Shkoller (1967) and Reynolds and Potter (1967). However, her comparisons had a serious deficiency, indicated by Fujimura (1991) who also showed that her value of δ was identical with that implied by the original Watson's method and hence, according to Fujimara's (1989) conclusion, was different from the value given by the method of multiple scales. Moreover, he also noted that, when applied to derivation of higher-order Landau equations, Renardy's reduction scheme leads to values of the higher Landau constants differing even from those given by Watson's original method. Therefore, Fujimura (1991) carried out a new careful evaluation of the complex Landau constants λ_{m}, with $m = 0$, 1, 2 and 3, for a plane Poiseuille flow by the methods of center manifold and of multiple scales, compared the results obtained by these two methods, and explained how the disturbance amplitude must be defined to make the results of two methods equivalent to each other.

In the paper of 1997 Fujimura applied, to the derivation of Landau's Eq. (4.41b), another center manifold reduction scheme (called by him "the reduction scheme of the second category"), which starts with an infinite, or finite, system of ordinary differential equations (in the cases where original equations are partial-differential, this system can be derived by means of a Galerkin projection or/and a normal-mode expansion). Such reduction scheme was used, in particular, in the above-mentioned papers by Guckenheimer and Knobloch, Cheng and Chang, and Chen et al. The main objecjtive of Fujimura (1997) was to prove the equivalence of Landau's equations, as given by this reduction scheme, to those derived by the method of multiple scales. To reduce the Navier–Stokes equations to a system of ordimary differential equations, a double expansion of flow fields in Fourier series and in eigenfunctions of the linear stability theory was used. Then the first and second Landau constants λ_2 and λ_3 were evaluated by the second-category method of center manifold, for plane Poiseuille flow and for two other simple fluid dynamical problems. Comparison of the values obtained with those given by the method of multiple scales showed that in all three cases the values of λ_2 and λ_3, computed by this version of the method of center manifold, approach their values given by the method of multiple scales as the truncation level of the eigenfunction expansion increases. Hence the three papers by Fujimura (1989, 1991, 1997), taken together, show that Landau's Eq. (4.41b) given by two versions of the center manifold reduction scheme, the method of multiple scales, and the modified Watson amplitude-expansion method are equivalent to each other if the disturbance amplitude is defined in a consistent way.

Stewartson and Stuart (1971) considered the propagation, in plane Poiseuille flow, of a group of two-dimensional waves undergoing both spatial and temporal development. In this case the disturbance amplitude A depends on both the time and the streamwise coordinate, i.e., $A = A\ (t, x)$. Therefore for small positive values of $\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}$, weakly nonlinear theory now leads to a nonlinear parabolic partial differential equation for $A\ (t, x)$, differing from the complex Landau Eq. (4.40) by an additional term proportional to $\partial^2 A/\partial \xi^2$, where $\xi = x - c_g t$, c_g being the streamwise group velocity. (This equation is now usually called *the Ginzburg–Landau equation* since it appeared in a quite different connection in the paper by Ginzburg and Landau (1950) on the theory of superconductivity. We will meet some other equations of the Ginzburg–Landau type in Sect. 4.24, parts (b) and (d)). To derive this equation, Stewartson and Stuart used the above-mentioned *multiple scale analysis*. Results similar to those by Stewartson and Stuart were found independently by DiPrima et al. (1971) while Hocking and Stewartson (1972) studied some exact solutions of the Ginzburg–Landau equations. Weinstein (1981) extended Stewartson and Stuart's theory, supplementing their amplitude equation by two more terms of higher order in A (and in addition showed that this equation may also be obtained by Watson's (1960a) method). A theory of Stewartson and Stuart's type, referring to groups of three-dimensional waves in a plane Poiseuille flow, was developed by Davey et al. (1974) but in this case it leads to a more complicated pair of coupled partial differential equations for the disturbance amplitude and for some characteristic of the pressure-gradient.

4.2.3 *Amplitude Equations and Equilibrium Disturbances in Other Parallel and Nearly Parallel Wall Flows*

4.2.3.1 Plane Couette-Poiseuille Flows

Passing to other parallel and nearly parallel fluid flows we will begin with the case of strictly-parallel *plane Couette-Poiseuille flows.* The combined *Couette-Poiseuille* (briefly C-P) *flows* are simpler in some respects than pure Couette flows, since unstable infinitesimal wave disturbances and a finite neutral-stability curve exist sin such combined flows, if only in cases where the relative strength of the Couette component is not too high, but they never exist in pure plane-Couette flows (see Sect. 2.91). Therefore methods developed by Stuart, Watson, Eckhaus, Reyunolds and Potter, and Pekeris and Shkoller, which are applicable only at (k, Re)-points close to the neutral curve, can be applied at least to some C-P flows, but are always inapplicable in the case of a Couette flow. Reynolds and Potter (1967), who were the first to investigate weakly nonlinear stability of C-P flows, considered only those relative strengths of the Couette component for which unstable infinitesimal disturbances exist. In these cases, the neutral curve in the (k, Re)-plane can be determined, and on this curve the critical point (k_{cr}, $\mathrm{Re}_{\mathrm{cr}}$) can be found. Reynolds and Potter carried out nonlinear stability analysis only for neutrally-stable wave disturbances with $\gamma = 0$, corresponding to critical points at various values of the relative strength of the Couette component. In this analysis they used the same method they applied to disturbances in plane Poiseuille flow. According to the results obtained, δ is negative at the critical point (and hence finite-amplitude instabilities exist at subcritical values of (k, Re) close to the critical point (k_{cr}, $\mathrm{Re}_{\mathrm{cr}}$)) in all C–P flows where there are unstable infinitesimal disturbances (and hence $\mathrm{Re}_{\mathrm{cr}}$ is finite though it can be arbitrarily large). This result makes it probable that some finite-amplitude disturbances are unstable in C-P flows, even in cases where all infinitesimal disturbances are stable (and hence $\mathrm{Re}_{\mathrm{cr}} = \infty$).

Shtemler (1978) supplemented Reynolds and Potter's computations by the estimation of the next-order coefficients a_2 and λ_2 of the generalized real and complex Landau Eqs. (4.41) and (4.41b) at the leftmost points (k_{cr}, $\mathrm{Re}_{\mathrm{cr}}$) of the neutral curves of C–P flows for a number of values of the relative strength of the Couette components corresponding to flows having finite values of $\mathrm{Re}_{\mathrm{cr}}$. A more detailed investigation of both linear and weakly-nonlinear stability of C–P flows was carried out by Cowley and Smith (1985) and Balakumar (1997). Studying the linear stability Cowley and Smith discovered that in a C-P flow the stability diagram of the linear theory can have a more complex form than was supposed by Potter (1966), Hains (1967), and Reynolds and Potter (1967). In these early papers it was assumed that if the streamwise wave number k is given, then at any values of Re and of the relative strength $\hat{A}$ of the Couette component either there exists one unstable two-dimensional normal mode or there are no such modes at all. Therefore, the above-mentioned authors thought that if the neutral curve in the (k, Re)-plane, which corresponds to the set of all neutrally-stable waves, exists in a C-P flow (and for this the inequality $\hat{A} < \hat{A}_{\mathrm{cr}}$ must be valid where $\hat{A}_{\mathrm{cr}}$ is some critical value of the relative strength $\hat{A}$), then this

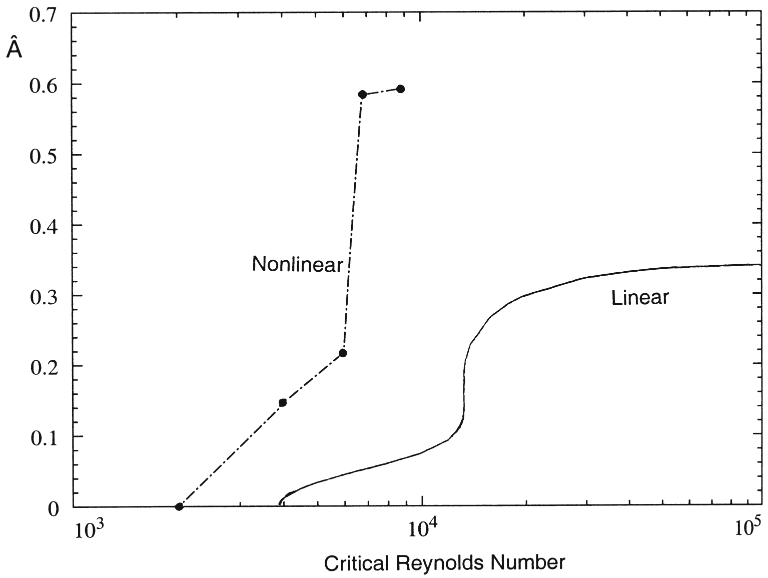

Fig. 4.15 Comparison of the nonlinear critical Reynolds numbers Re^*_{cr} (Â), where Â = $\hat{A}_B$, for C-P flows with various values of Â with the ordinary (*linear*) critical Reynolds numbers Re_{cr} (Â) for the same C–P flows. (After Balakumar (1997))

curve will have qualitatively the same form as the neutral-stability curve of a plane Poiseuille flow. However Cowley and Smith found that in a C–P flow with a relative strength of the Couette component that is sufficiently small (appreciably smaller than $\hat{A}_{cr}$), but non-vanishing, several neutral-stability curves (two or even three, if $\hat{A}$ takes very small values), corresponding to several unstable two-dimensional normal modes can exists simultaneously. This means that in addition to the critical value $\hat{A}_{cr}$ there also exist in C–P flows the critical values $\hat{A}_{2,cr} < \hat{A}_{cr}$ and $\hat{A}_{3,cr} < \hat{A}_{2,cr}$ corresponding to the appearance of additional unstable modes (growing more slowly than the most unstable mode appearing at $\hat{A} = \hat{A}_{cr}$); these new critical numbers clearly signify qualitative changes in the shape of the stability diagram. Balakumar (1997), in his study of the linear stability of C–P flows, considered only the most unstable modes; for them he computed, very accurately, first the value of $\hat{A}_{cr}$ then the neutral curves in the (k, Re) and (c, Re) planes (where $c = \omega/k$ is the phase velocity of a neutral wave) at a number of values of $\hat{A}$ in the range $0 \leq \hat{A} < \hat{A}_{cr}$, and finally the shape of the functions $\mathrm{Re}_{cr}(\hat{A})$ and $k_{cr}(\hat{A})$ (the first of these functions is shown in Fig. 4.15).

Note also that the relative strength $\hat{A}$ of the Couette component was defined differently by different authors, who also often used different forms of the C-P velocity profile $U(z)$ and different length and velocity scales L_0 and U_0. So, Potter, Hains, and Balakumar defined $U(z)$ as the sum of a parabolic profile $U_p(z)$ of a Poiseuille flow with maximal velocity $U_P(H/2) = U_{max}$ and a linear Couette's profile $U_C(z)$ growing from the value $U_C(0) = 0$ up to the value $U_C(H) = U_w$, while both Reynolds and

Potter, and Cowley and Smith assumed that $U_C(0) = -U_W/2$ and $U_C(H) = U_W/2$. Potter (1966) and Hains (1967) used the channel thickness H as the scale L_0, while in the papers by Reynolds and Potter (1967), Cowley and Smith (1985), and Balakumar (1997) L_0 was taken as $H_1 \equiv H/2$. Moreover, Hains assumed that $U_0 = U(H/2) = U_P(H/2) + U_C(H/2)$, Potter that $U_0 = U_P(H/2) = U_{\max}$, and Reynolds and Potter, Cowley and Smith, and Balakumar that $U_0 = U_{ave}$ (where U_{ave}, the averaged C-P velocity $U(z)$, clearly depends on the selected Couette-component profile $U_{\text{C}}(z)$). Thus, Hains (1967) measured the relative strength of the Couette component by the value of $\hat{A}_{\text{H}} = U_{\text{W}}/U(H/2) = U_{\text{W}}/[U_{\max} + \frac{1}{2}U_{\text{W}}]$ (this measure was used also in Sect. 2.91 where it was denoted as A); while Potter (1966) assumed that $\hat{A} = \hat{A}_{\text{P}} = U_{\text{W}}/U_{\max}$; and Reynolds and Potter (1967); Cowley and Smith (1985), and Balakumar (1997) defined $\hat{\text{A}}$ as $U_{\text{W}}/2U_{\text{ave}}$. Complying with this definition and with the accepted form of the profile $U_{\text{C}}(z)$, Reynolds and Potter, and Cowley and Smith used the measure $\hat{A} = \hat{A}_{\text{RP}} = \hat{A}_{\text{CS}} = 3U_{\text{W}}/4U_{\max}$, and Balakumar the measure $\hat{\text{A}} = \hat{\text{A}}_{\text{B}} = U_{\text{W}}/[\frac{4}{3}U_{\max} + U_{\text{W}}]$. It is easy to see that the measures $\hat{\text{A}}_{\text{H}}, \hat{\text{A}}_{\text{P}}, \hat{\text{A}}_{\text{RP}} = \hat{\text{A}}_{\text{CS}}$ and $\hat{\text{A}}_{\text{B}}$ of the relative Couette-component strength are in fact simple one-valued functions of each other so that the value of any of them determines the values of all the others. Moreover, the seemingly different critical values found by the above-mentioned authors, namely $\hat{A}_{\text{cr}} \approx 0.55$ (Hains; see also Sect. 2.91 of this book), $\hat{\text{A}}_{\text{cr}} \approx 0.7$ (Potter), $\hat{\text{A}}_{\text{cr}} \approx 0.528$ (Reynolds and Potter, and Cowley and Smith), and $\hat{A}_{\text{cr}} \approx 0.3455$ (Balakumar) only indicate that $\hat{\text{A}}_{\text{H,cr}} \approx 0.55, \hat{\text{A}}_{\text{P,cr}} \approx 0.7, \hat{\text{A}}_{\text{RP,cr}} = \hat{\text{A}}_{\text{CS,cr}} \approx 0.528$, and $\hat{\text{A}}_{\text{B,cr}} \approx 0.3455$; one may verify easily that these values agree rather satisfactorily with each other (only Hains' estimate is overstated by about 7 %).

As to the weakly nonlinear stability of the C-P flows, Cowley and Smith showed, in particular, that at all values of $\hat{\text{A}}$ which are close enough, above or below, to the critical value $\hat{\text{A}}_{\text{cr}}$, $\delta(k, \text{Re})$ is negative for the least-stable two-dimensional wave disturbances corresponding to some parts of the stable region of the (k, Re)-plane. Therefore, equilibrium wave disturbances of small but finite amplitudes can exist in a C-P flow with any such value of $\hat{\text{A}}$. (These results by Cowley and Smith also agree with conclusions by Milinazzo and Saffman (1985) who independently found that a family of two-dimensional equilibrium waves of finite amplitude exists in the C–P flows). In the case of a subcritical C-P flow, where $\hat{\text{A}} < \hat{\text{A}}_{\text{cr}}$, unstable disturbances correspond to periodic solutions of the N–S equations bifurcating from the steady C–P solutions at points of the neutral curve. However in the case of a supercritical flow with $\hat{\text{A}} > \hat{\text{A}}_{\text{cr}}$ the neutral curve does not exist at all. Therefore it is clear that the usual form of bifurcation theory, which requires the existence of a point of loss of stability at which the bifurcation begins, cannot be applied here. (A similar conclusion was also reached simultaneously by Milinazzo and Saffman). In this connection Cowley and Smith recalled rather exotic *bifurcations from infinity* of solutions of nonlinear equations which were considered by Rosenblat and Davis (1979) in their search of a possible origin of finite-amplitude equilibrium flow disturbances, observed in flows where stable infinitesimal disturbances do not exist. This recollection proved to be quite appropriate: Cowley and Smith (1985) succeeded in showing that just such a 'bifurcation from infinity' occurs in the supercritical C–P flows with $\hat{A} > \hat{A}_{cr}$ (where the 'critical value' may be considered as the infinite one, since one may assume that $(\hat{A} - \hat{A}_{cr})^{-1}$ and not $\hat{A}$ is the true stability parameter). Their results stimulated subsequent studies by Cherhabili and Ehrenstein (1995, 1997) and Nagata (1997) of some other types of bifurcations from infinity relating to finite-amplitude equilibrium states in C–P flows. Results of the last-named authors and also of the paper by Rosenblat and Davis (1979), where 'bifurcations from infinity' first appeared, will be discussed later in this section.

Balakumar (1997) did not use bifurcation theory at all in his studies of the nonlinear stability of C-P flows. He concentrated his attention on computations of finite-amplitude equilibrium two-dimensional waves at different values of $\hat{A} = U_W/2U_{ave} = U_W/[\frac{4}{3}U_{max} + U_W]$. His computations were based on application to C–P flows of the method outlined in the early papers by Noether (1921) and Heisenberg (1924), and then used by Zahn et al. (1974) and Herbert (1976, 1977, 1978) in their investigations of nonlinear stability of plane Poiseuille flows (for more details see Sect. 4.22 above). The main objective of Balakumar was to determine the evolution with $\hat{A}$ of the 'nonlinear neutral surface' in the three-dimensional $(E, k,$ Re)-space consisting of points corresponding to two-dimensional equilibrium waves (here E and k have the same meaning as in Fig. 4.12 in Sect. 4.22, and $\text{Re} = U_{ave}H_1/\nu$). Some of his results are presented in Figs. 4.14 and 4.15. Figure 4.14 shows the intersections of the neutral surfaces in $(E, k,$ Re)-spaces corresponding to C–P flows with several values of $\hat{A}$ with the planes Re = 4,000 and Re = 6,000. (For $\hat{A} = 0$ these intersections clearly coincide with those shown in Fig. 4.12, but the values of Re in Fig. 4.14 are equal to 2/3 of the values $\text{Re} = U_{max}H_1/\nu$ used in Fig. 4.12). All the intersections shown (whose boundaries represent the nonlinear 'neutral curves in the (E, k)-plane') have similar shapes but they gradually shrink in size with increasing $\hat{A}$ and, as Balakumar's extensive computations showed, completely disappear at $\hat{A} \approx 0.1472$ when Re = 4,000 and at $\hat{A} \approx 0.2182$ when Re = 6,000. However similar computations for Re = 7,000 showed that at this high Reynolds number the 'neutral curves in the (E, k)-plane' have the shape similar to that in Figs. 4.12 and 4.14 only for $\hat{A} < 0.2$, and when $\hat{A}$ increases further their shapes change very rapidly and begin to include a second loop at low values of E and k.

Figure 4.15 shows the dependence on $\hat{A} = U_W/2U_{ave}$ of the 'nonlinear critical Reynolds number' $\text{Re}^*_{cr} = \text{Re}^*_{cr}(\hat{A})$ the lowest Reynolds number at which unstable two-dimensional waves of finite amplitude exist in the C–P flow with relative strength $\hat{A}$ of the Couette component. For comparison the same figure includes also the computed values of a function $\text{Re}_{cr} = \text{Re}_{cr}(\hat{A})$ where Re_{cr} is the ordinary (linear) Reynolds number indicating the lowest value of Re at which there exist infinitesimal unstable two-dimensional waves (the computations of Re_{cr} are simpler than those of Re^*_{cr} and allow more precise results to be obtained). We see that $\text{Re}^*_{cr}(\hat{A})$ is always much smaller that $\text{Re}_{cr}(\hat{A})$, as it must be. The value of the linear critical Reynolds number $\text{Re}_{cr}(\hat{A})$ increases significantly as $\hat{A}$ grows from zero (where $\text{Re}_{cr} \approx 2 \times 5772/3 = 3848$) up to a value of about 0.1, then it remains almost constant until $\hat{A} \approx 0.3$, and later increases sharply to infinity as $\hat{A}$ approaches the value of $\hat{A}_{cr} \approx 0.3455$. The value of the nonlinear critical Reynolds number $Re^*_{cr}(\hat{A})$ increases as $\hat{A}$ increases from zero (where $Re^*_{cr} \approx 2 \times 2935/3 \approx 1957$) up to value of about 0.2, then remains approximately constant until $\hat{A} \approx 0.58$ and after this again begins to increase with the relative strength $\hat{A}$ of the Couette component. As to the higher values of $\hat{A}$, Balakumar found no steady two-dimensional waves in any C-P flow with $\hat{A} \geq 0.59$. Then he remembered that earlier several authors (in particular, Orszag and Kells (1980) and Milinazzo and Saffman (1985)) were unsuccessful in their attempts to simulate two-dimensional finite-amplitude equilibrium waves in a plane Couette flow, where $\hat{A} = \infty$, and they concluded that apparently such waves cannot exist in this flow. Therefore he assumed that there exists the nonlinear critical relative strength $\hat{A}$ (close to 0.59) above which equilibrium two-dimensional wave cannot exist in a C-P flow (and hence $Re^*_{cr}(\hat{A}) = \infty$). However, the real situation is not so simple, since Cherhabili and Ehrenstein (1995, 1997) (whose work was apparently unknown to Balakumar) found that even in a pure plane Couette flow (i.e. at $\hat{A} = \infty$) two-dimensional finite-amplitude equilibrium states exist if Re exceeds the critical value close to 1500, but these states are not of the form of traveling nonlinear two-dimensional waves, as considered by Balakumar, but are stationary, spatially localized (solitary-like) waves (more details of this will be presented below). Therefore the question of the possible two-dimensional equilibrium states in C-P flows with relatively high strength of the Couette component requires further investigation.

4.2.3.2 Plane Couette and Circular Poiseuille Flows

Now we will turn to the cases of plane Couette and circular Poiseuille flows. It is known that these flows are stable at any Re with respect to infinitesimal disturbances, i.e. are similar in this respect to C-P flows with $\hat{A} > \hat{A}_{cr}$. Rosenblat and Davis (1979) noted that in plane Couette and circular Poiseuille flows there exist sets of infinitesimal disturbances whose decay rates tend to zero as $Re \to \infty$. Therefore, they suggested that perhaps the value $Re = \infty$ may be regarded here as a bifurcation point in the following sense: a branch of finite-amplitude solutions of the complete nonlinear disturbance equations which, according to experimental data definitely exists in these cases, may have the property that for $Re \to \infty$ these solutions tend to coalesce with the primary ('basic') laminar solution of dynamic equations. Rosenblat and Davis proposed to say in such cases that the corresponding finite-amplitude solutions *bifurcate from infinity.* Then they showed that at least for some model nonlinear differential equations containing a real parameter μ, having the property that bifurcation of a steady solution cannot occur at any finite value of μ, such 'bifurcation from infinity' (i.e., at $\mu = \infty$) can really occur.

Let us begin with the case of *plane Couette flow* (briefly PCF). Since all infinitesimal wave disturbances are stable here (i.e., decay as $t \to \infty$), those methods for rigorous derivation of Landau's equations and evaluation of their coefficients which use the assumption that the disturbance studied corresponds to a point of the (k, Re)-plane lying near the neutral-stability curve cannot be applied here. On the other hand, studies of the nonlinear stability of PCF are of particular interest, since only nonlinear theory can explain the striking contradiction between the prediction of the linear stability theory about the stability of PCF and the experimental data which definitely show that, when Re is increasing gradually, PCF becomes unstable at Reynolds numbers Re (based upon half the channel depth H_1 and half-difference of the wall velocities U_1) in the range from 320 to 370, while with further increase of Re it rapidly becomes turbulent (see Sect. 2.1, pp. 16–17, and the discussion of stability of Couette flow heated from below in Sect. 4.12). These arguments stimulated a number of attempts to develop such nonlinear stability theory which can be applied to PCFs.

Relatively crude attempts of this type undertaken by Kuwabara (1967) and Lessen and Chiefetz (1975) will be briefly described later. However, we will first consider the papers by Ellingsen et al. (1970) and Coffee (1977), and, related to their results, the remarks about stability of PCF by Davey and Nguyen (1971); Itoh (1977a, b) and Davey (1978), who paid most attention to the nonlinear stability of circular Poiseuille flow in tubes. All these authors tried to apply, to computation of the development of two-dimensional normal-mode finite-amplitude disturbances in the PCF, some modifications of the Reynolds-Potter method which take into account that in the case considered $|\gamma|$ cannot be assumed to be small. Remember that in this method the determination of threshold ('equilibrium') amplitudes $A_e = A_e(k, \mathrm{Re})$ of the wave disturbances, having the property that $dA_e/dt = 0$, plays the main part.

Ellingsen et al. (1970) showed that in the case of a PCF with a high enough value of Re, a slightly modified Reynolds–Potter method yields Landau's Eq. (4.34) for the amplitude of the least damped two-dimensional wave with given wave number k and also yields an equation for the coefficient δ allowing its numerical computation. The computations showed that δ is negative over a large region of the (k, Re)-plane. Therefore, subcritical instability is possible here and hence the PCF is unstable with respect to finite-amplitude disturbances. Later Itoh (1977a) showed that the results of Ellingsen et al. may also be obtained by another more rigorous method without some of the simplifying assumptions of the latter authors.

Davey and Nguyen (1971) considered slightly different modifications of Reynolds and Potter's method. They applied this modification mainly to the study of nonlinear stability of a tube Poiseuille flow at high values of Re, but also presented some results of calculations for PCF, giving the dependence of the threshold disturbance energy E_e (corresponding to amplitude A_e) on k and Re. According to their results $E_e(\mathrm{Re})^{3/2}$ practically depends only on $k/(\mathrm{Re})^{1/2}$ in PCF with $\mathrm{Re} \geq 500$, and has a minimum value at $k/(\mathrm{Re})^{1/2} \approx 0.13$ (we recall that k is made dimensionless by multiplication by H_1). This means that in PCF two-dimensional waves with $k \approx 0.13(\mathrm{Re})^{1/2}$ are the most unstable. Later Davey (1978) found that the results of his paper with Nguyen relating to disturbances in PCF are very close to those which follow from the application to the same problem of another method of the same type proposed by Itoh (1977b). More detailed calculations of the values of $A_e(k, \mathrm{Re})$ and $E_e(k, \mathrm{Re})$ for numerous values of the arguments (k, Re), also based on a version of the Reynolds–Potter method, were made by Coffee (1977), whose results agree satisfactorily with earlier estimates by Ellingsen et al. and Davey and Nguyen. Since $\gamma < 0$ for all

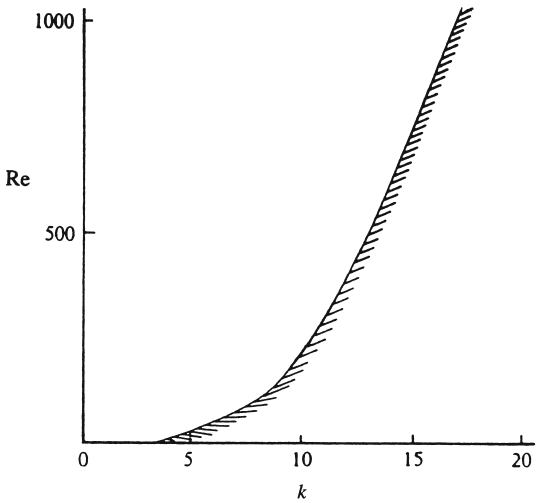

Fig. 4.16 Approximate location of the curve in the (k, Re)-plane separating the regions of positive and negative values of δ for plane Couette flow. (After Coffee (1977)) according to Coffee's calculations, $\delta < 0$ to the left of the given curve but $\delta > 0$ to the right of it

wave disturbances in a PCF, finite values of E_e and A_e show that $\delta < 0$, while for $\delta > 0$ the approximate theory based on Landau's equation leads to the conclusion that $E_e = \infty$, so that the flow is stable to disturbances of any size. Coffee's calculation implies that the region of the (k, Re)-plane where $E_e = \infty$ (and hence $\delta > 0$) is given approximately by the inequality $\mathrm{Re} \leq 1.7k^2$; his graph of the curve dividing the region where $\delta < 0$ from that where $\delta > 0$ is shown in Fig. 4.16 (here this curve replaces the dotted curve in Fig. 4.8 showing the points where $\delta = 0$ in the case of plane Poiseuille flow). Negativity of both γ and δ at a point (k, Re) means that a two-dimensional equilibrium wave with the wave number k can exist at this value of Re. Recall, however, that Orszag and Kells (1980) and Milinazzo and Saffman (1985) were unsuccessful in their attempts to stimulate two-dimensional equilibrium waves in PCF at any Re while Cherhabili and Ehrenstein (1995) found that some, quite specific, waves nevertheless exist in PCF, but only for Re close to 1,500 or even higher (this circumstance was mentioned in part (a) of this section and will be considered at greater length slightly later). Moreover, in Sect. 4.11 it was indicated that it follows from the Reynolds-Orr energy-balance Eq. (4.2) that disturbances of any shape and size must decay monotonically in a PCF if $\mathrm{Re} \leq 20.7$. These facts show that the early modifications of the Reynolds-Potter method discussed above, which had the object of making it applicable to linearly stable flows without a neutral curve, are apparently inaccurate and deserve no credit.

Let us say now a few words about the papers by Kuwabara (1967) and Lessen and Cheifetz (1975). Kuwabara's theory was based on crude assumptions, introduced by Meksyn and Stuart (1951), which do not require the smallness of the damping rate $|\gamma|$. Moreover, he also used some supplementary hypotheses which seemed dubious to some later authors (see, e.g., Lessen and Cheifetz (1975)). Kuwabara found that his assumptions imply the existence of some equilibrium two-dimensional finite-amplitude disturbances (and hence the positiveness of δ) in PCF if Re is high enough.

According to his calculations, $\mathrm{Re}^*_{\mathrm{cr}} \approx 45,000$ in the case of PCF. Such a high value of $\mathrm{Re}^*_{\mathrm{cr}}$ clearly disagrees with the experimental data and makes one suspect that the assumptions made are invalid.

A quite different 'quasilinear' theory, also strongly influenced by Meksyn and Stuart's arguments, was proposed by Lessen and Chiefetz (1975). They took into account only the distortion of the mean motion by a disturbance. This distortion affects the solutions of the Orr-Sommerfeld equation which determines the shapes of infinitesimal normal-mode disturbances. A rather crude finite-difference integration in time of the coupled equations for the distorted mean flow, and the least-stable disturbance corresponding to it, suggested a slow convergence of the disturbed Couette flow to some stable state.

Above, we mentioned some papers where the determination of the amplitudes for possible equilibrium two-dimensional finite waves in a plane Couette flow (PCF) played an important part. However, attempts to simulate such two-dimensional equilibrium waves numerically were unsuccessful for a long time. In this connection Orszag and Kells (1980); Patera and Orszag (1981a); Orszag and Patera (1981); Milinazzo and Saffman (1985), and Balakumar (1997) especially stressed that two-dimensional finite-amplitude equilibrium waves can be easily simulated in plane Poiseuille flow and combined Couette–Poiseuille (C–P) flows with not-too-high relative strength $\hat{\mathrm{A}}$ of the Couette component, but apparently such waves do not exist in plane Couette flow. (As was said above, Balakumar even tried to determine the upper bound of $\hat{\mathrm{A}}$-values at which such equilibrium waves exist in a C-P flow; see Fig. 4.15 and explanations relating to it in the text at the end of Sect. 4.2.3.1).

Recall now that Andreichikov and Yudovich (1972) and Chen and Joseph (1973) showed that finite-amplitude periodic waves in a plane Poiseuille flow bifurcate from the steady laminar solutions of the Navier-Stokes equations at the points of the neutral-stability curve, and Cowley and Smith (1985) found that bifurcations of the same type occur in C-P flows with $\hat{\mathrm{A}} < \hat{\mathrm{A}}_{\mathrm{cr}}$. Since such a curve does not exist in a PCF, bifurcations of this type are impossible here, and this fact was sometimes used to explain the non-existence of finite-amplitude wave solutions of the equations of motion in the case of pure Couette primary flow. However, when discussing the problem of equilibrium waves in combined Couette-Poiseuille flows we mentioned that in the 'supercritical' cases, where $\hat{A} > \hat{A}_{\mathrm{cr}}$ so that a neutral curve does not exist, such waves can be produced by a 'bifurcation from infinity'. Hence it is natural to think that such bifurcations can also lead to appearance of finite-amplitude equilibrium wave solutions in the case of primary plane Couette flow.

Apparently the first attempt to find some finite-amplitude solutions of the equations for disturbances in PCF which correspond to a 'bifurcation from infinity' was due to Nagata (1990). He applied such a bifurcation to find three-dimensional finite-amplitude standing waves in PCF. In order to find some finite-amplitude disturbance in PCF corresponding to 'bifurcation from infinity', one must first of all determine a family of auxiliary flows which i) depend on some parameter Λ and tend to PCF as $\Lambda \to \Lambda_0$, and ii) have the property that a neutral-stability curve corresponds to an auxiliary flow with a certain value of the parameter Λ, and that at a point on the neutral curve some finite disturbance bifurcates from the solution of the N–S equations describing this auxiliary flow. Then it is often possible to extend the 'composite solution' thus obtained (which includes the auxiliary flow and the finite disturbance superimposed on it) varying the value of Λ. Assuming now that $\Lambda \to \Lambda_0$ one will obtain the required finite-amplitude disturbance in PCF.

Nagata considered the family of primary flows between infinite concentric co-rotating cylinders (i.e., having angular velocities, Ω_1 and Ω_2 of the same sign). Here the steady solution (describing 'circular Couette flow') and solutions corresponding to flows appearing after the most common first supercritical bifurcation ('Taylor vortex flows') are well known, and the three-dimensional steady solutions which bifurcate from the Taylor vortices as Re increases further have also been investigated (in particular, by Nagata (1986, 1988)). Assuming that the dimensionless 'Coriolis parameter' $\mathrm{Co} = (\Omega_1 + \Omega_2)(R_2 - R_1)^2/\nu$ tends to zero (i.e., $d = R_2 - R_1 \to 0$), Nagata (1990) found numerically a branch of three-dimensional finite-amplitude steady solutions ('standing waves') in the limiting plane Couette flow which, according to his computations, appear abruptly at a Reynolds number $\mathrm{Re} = U_1 H_1/\nu$ around 125.

Nagata's paper led to a strong revival of interest in finding new finite-amplitude equilibrium disturbances in PCF arising from bifurcations from infinity. Nagata used the family of ciruclar Couette flows as auxiliary flows satisfying the above-mentioned conditions (i) and (ii), but shortly afterwards Clever and Busse (1992) considered, instead of this, the family of plane Couette flows between lower and upper walls at different temperatures. They began by considering the well-studied longitudinal convective rolls in a layer of motionless fluid heated from below, then passed to the wavy rolls that bifurcate from two-dimensional rolls when the latter become unstable, and finally replaced the motionless fluid layer by a layer having a linear velocity profile (the stability of convection rolls in such a flow was studied earlier by Clever et al. (1977)). Assuming now that $\mathrm{Ra} \to 0$ (where Ra is the Rayleigh number) Clever and Busse determined a family of finite-amplitude three-dimensional standing waves (of the same type as those found by Nagata) relating to the limiting (non-buoyant) case of PCF. At the same time Clever and Busse (see also Busse and Clever (1996a, b)) studied many interesting three-dimensional disturbances in a wide class of unstably-stratified Couette flows which are of great interest in geophysical fluid dynamics. And later Nagata (1996) considered disturbances in PCF in a conducting fluid, in the presence of a transverse magnetic field (which destabilizes the fluid motion and at large enough intensity makes the flow linearly unstable, i.e., a definite neutral-stability curve appears here, with a finite value of $\mathrm{Re}_{\mathrm{cr}}$; see Kakutani (1964)). Using such hydromagnetic auxiliary flows and then letting the intensity of the magnetic field tend to zero, Nagata again found three-dimensional standing waves of finite amplitude in PCF, as first found by him in 1990, and even succeeded in considerably improving the accuracy of computation of their characteristics.

Cherhabili and Ehrenstein (1995) tried to apply the same method to find two-dimensional equilibrium states in PCF. They began by considering the family of two-dimensional equilibrium traveling waves of finite amplitude in plane Poiseuille flow found by Herbert (1977, 1978). Then, adding a Couette component to the primary Poiseuille flow, they numerically extended the Poiseuille-flow wave solutions to the combined Couette-Poiseuille (C-P) flow and then also to the limiting case of pure Couette flow. The limiting Couette-flow solutions unexpectedly proved to have the form of spatially-localized two-dimensional standing waves which can exist at Reynolds numbers (defined in terms of the channel half-thickness and half-difference of wall velocities) exceeding the 'critical value' of about 1,500. The authors suggested that previous attempts to compute finite-amplitude two-dimensional waves in PCFs failed because everybody looked for the usual traveling waves whereas only standing two-dimensional waves exist in PCF. Later Cherhabili and Ehrenstein (1997) investigated stability of these two-dimensional equilibrium states with respect to secondary two-dimensional and three-dimensional disturbances. The authors found that the three-dimensional disturbances are the most destabilizing ones; they give rise to some specific three-dimensional stationary equilibrium states (spanwise-periodic but streamwise-localized, and thus differing from the three-dimensional states found by Nagata and by Clever and Busse), bifurcating at points of the neutral-stability surface corresponding to equilibrium two-dimensional waves of finite amplitude. These new equilibrium states were found at values of Re close to 1,000.

In 1997 Nagat noted that none of the available experimental data relating to disturbances in PCF confirmed the existence of time-independent two- and three-dimensional waves of finite amplitude corresponding to the solutions found numerically by Cherhabili and Ehrenstein (1995, 1997) and by Nagata himself (see his papers (1990, 1996)). Therefore he returned to computations of various finite-amplitude solutions of equations for disturbances in C–P flows, and of their limits when the relative intensity of the Poiseuille component Q (which can be, e.g., set equal to the ratio $U_{\max}/U(H)$ of the maximal velocity of the Poiseuille component to the velocity of the upper wall) tends to zero. This time the main attention was given to traveling-wave solutions (well known in plane Poiseuille flows). Nagata (1997) showed that in C-P flows at not too high values of Q there exist two different branches of finite-amplitude three-dimensional traveling-wave solutions. Only the first of them was considered in Nagata's paper (1990); as $Q \to 0$ these solutions tend to time-independent ('standing') three-dimensional waves discovered and studied in his papers (1990, 1996) (and also found in PCF by Clever and Busse (1992)). However, there is also a second branch of three-dimensional traveling-wave disturbances in C–P flows, which was unknown earlier. It was found now that this second branch may also be located over a wide range of values for Q, and as $Q \to 0$ it turns into two branches of finite three-dimensional shape-preserving traveling waves. These waves represent a new class of finite equilibrium wave disturbances which can appear in PCF if its Reynolds number $U_1 H_1/\nu$ exceeds 150.

Clever and Busse (1997) (see also Busse and Clever (1996a) and the more general earlier discussion of this matter by Busse (1991)) stressed that the steady three-dimensional equilibrium disturbances found in PCF by Nagata (1990, 1996) and by themselves (1992), which are also present in circular or stratified Couette flows, correspond to tertiary solutions of the equations of motion, arising from the solution describing a steady laminar flow after two subsequent bifurcations. The two-dimensional steady waves found by Cherhabili and Ehrenstein (1995) correspond to secondary solution, but the three-dimensional streamwise localized equilibrium states discovered by Cherhabili and Ehrenstein in 1997 (and also the three-dimensional traveling waves found in 1997 by Nagata) again represent tertiary solutions. Clever and Busse noted the large difference between the 'critical Reynolds number' Re $\approx$ 125 corresponding to Nagata's three-dimensional steady equilibrium states (and also the 'critical value' Re $\approx$ 150 found by Nagata's (1997) for three-dimensional finite-amplitude traveling-wave solutions), and the values of Re in the range from 1,000 to 1,500 which determine the thresholds above which Cherhabili and Ehrenstein (1995, 1997) found the two- and three-dimensional streamwise-localized steady equilibrium solutions of equations of motion. Nagata's 'critical values' are appreciably smaller than the results of experiments and numerical simulations for the lowest Reynolds numbers Re_{cr} at which some disturbances are not decaying in a PCF but produce persistent turbulent spots there, and also much smaller than the smallest values of Re at which the turbulence can be sustained in a PCF (those values do not differ much from Re_{cr}). At the same time the 'critical Reynolds numbers' found by Cherhabili and Ehrenstein are much greater than all observed values of Re_{cr}.

These facts forced Clever and Busse (1997) to consider anew the data relating to the tertiary steady three-dimensional states (having the form of wavy rolls similar to those often observed in the case of convection) found by Nagata (1990, 1996) and by themselves in 1992. They recalled that instability of these states had already been proved by Clever and Busse (1992); Nagata (1993) and Busse and Clever (1996a), and noted that because of this it was important to study the quaternary solutions bifurcating from the tertiary ones. Then they found that some interesting quaternary solutions bifurcate from the tertiary ones at Reynolds numbers not too much exceeding the 'critical Reynolds numbers' at which steady tertiary solutions start to exist. These quaternary solutions have the form of oscillatory way rolls, basically differing from tertiary steady waves only by the time variation of their amplitudes. The comparison of the solutions found with available experimental and numerically simulated data relating to instabilities in PCFs is not an easy matter, but the authors noted that some features of the quaternary solutions are similar to those of the longitudinal vortices found in Couette-flow experiments by Dauchot and Daviaud (1995) and Dauchot et al. (1996) and in Couette-flow simulations by Bech et al. (1995) and Hamilton et al. (1995). (Results of more detailed experimental investigations of the instabilities in PCFs by Bottin et al. (1997, 1998a, b) and Bottin and Chaté (1998), and numerical simulations by Barkley and Tuckerman (1998, 1999) appeared only later. These papers showed very convincingly the leading role of streamwise vortices in transition of PCFs to turbulence, and Bottin et al. (1998a), noting some qualitative differences between the structures detected by the indicated authors and the equilibrium solutions of Navier–Stokes equations found by Nagata, Busse and Clever, and Cherhabili and Ehrenstein, nevertheless related these two types of vortical formations with each other). The possible relation of sequences of three bifurcations, each of which decreases flow symmetry and makes the flow structure more complicated, to final transition to turbulence was also discussed in the papers by Clever and Busse (1993) and Busse and Clever (1996a); moreover, then Busse and Clever (1996b, 1998) considered also some tertiary and quaternary equilibrium states in plane Couette flows between differently heated horizontal walls. However, at present there are not enough data to make the situation clear. Note also that the numerical methods used for the study of solutions produced by several subsequent bifurcations are very complex and their complexity increases greatly with any loss of symmetry properties; therefore, the accuracy of the current computations of higher-order states may not be very good.

Now we will pass to the case of *circular Poiseuille flow* (CPF) in tubes. It has been already noted in Sect. 2.94 that, in many respects relating to stability, this flow is similar to plane Couette flow but is much more complicated. Its greater complexity is reflected, in particular, in the fact that the strict proof of the linear stability of plane Couette flow was found as long ago as the 1970s, while for CPF such a proof is unknown up to now although there is no doubt that this flow is linearly stable. Greater complexity also explains why studies of the nonlinear stability of CPF are appreciably less numerous than those relating to plane Couette flow and mostly deal only with axisymmetric disturbances; moreover, the same complexity has led to some contradictions between the results of different authors.

One of the first attempts to investigate the nonlinear stability of the CPF to axisymmetric wave disturbances, and to estimate the value of the corresponding Landau constant δ, was made by Davey and Nguyen (1971). They applied Reynolds and Potter's (1967) method to this problem and found that δ takes negative values for a wide range of wave numbers k and Reynolds numbers Re. This means that nonlinearity destabilizes the flow. Hence the tube flow must be unstable to finite axisymmetric disturbances, and evaluation of δ allows the determination of the equilibrium amplitudes $A_e = A_e$ (k, Re) and of the neutral-stability surface in the three-dimensional (A, k, Re)-space. Itoh (1977b), who also considered only axisymmetric disturbances, developed another method of stability analysis. His theory showed that the spatial Landau constant δ_s is positive for all values of Re and of frequency ω considered by him. (Itoh studied spatial, not temporal, development of disturbances; therefore in his work the frequency ω replaced the wave number k, and δ_s replaced δ). Thus, according to Itoh's theory, nonlinear effects stabilize CPF and therefore finite-amplitude instabilities and equilibrium disturbances cannot exist in this flow (at least to the approximation that neglects higher powers of amplitude A). The evident contradiction between Davey and Nguyen's and Itoh's conclusions clearly cannot be due only to the difference between temporal and spatial stability analysis, and in fact Itoh easily showed that his results directly contradict those of Davey and Nguyen.

In this connection Davey (1978) reconsidered the derivations of the equations for the Landau constant proposed in his 1971 paper with Nguyen and in Itoh's paper (1977b). He found that slightly different approximations were used in these papers and this led to some difference in the final equations; however, according to Davey, a special investigation was needed to determine which approximations are more accurate. He noted also that in applications to plane Couette flow the two theories imply almost identical results, and the difference is also relatively small for some particular axisymmetric disturbances in CPF which were not considered by Itoh; but in applications to disturbances in CPF which were actually studied in both papers, the results of the two theories prove to be contradictory. To clarify the situation, Patera and Orszag (1981b) applied direct numerical simulation to development of axisymmetric disturbances in CPF, i.e. they solved the corresponding nonlinear (N-S) initial-value problems numerically. They paid particular attention to those disturbances which were found to be undamped either by Davey and Nguyen (1971) or by Davey (1978) (who mentioned some axisymmetric disturbances which tended to equilibrium states according to both the theory proposed by Davey and Nguyen and that of Itoh). Numerical simulation showed, however, that in fact all these disturbances (and also all the other axisymmetric disturbances considered by Patera and Orszag) are damped. Therefore, Patera and Orszsag concluded that apparently all axisymmetric disturbances decay in CPF and that the methods used by Davey and Nguyen (1971) and Itoh (1977b) are probably both inapplicable to CPF. (Remember however that the remark by Orszag and Patera (1980) about the nonexistence of two-dimensional equilibrium waves in plane Couette flow was dismissed by Cherhabili and Ehrenstein (1995)).

Another method for the study of nonlinear stability of CPF, applicable to small but finite, and in general non-axisymmetric, disturbances in high-Reynolds-number tube flow was proposed by Smith and Bodonyi (1982). Their theory further develops the approach initiated independently by Benney and Bergeron (1969) and Davis (1969), applied to two-dimensional disturbances in plane-parallel flows and then used in a large number of subsequent papers (see, e.g., discussion of this topic in the book by Drazin and Reid (1981), Sect. 52.5, and more recent survey papers by Maslowe (1986) and Churilov and Shukhman (1995)). Benney and Bergeron, and Davis noted that if $\mathrm{Re} \gg 1$ and $\mathrm{A} \ll 1$ (where A is the dimensionless amplitude of the disturbance), then the linear stability theory (i.e., the linear Orr-Sommerfeld equation) is applicable only when $\lambda = A(\mathrm{Re})^{2/3} \ll 1$. However, if $\lambda \gg 1$ or $\lambda \approx 1$, then some specific nonlinear effects play an important part in the vicinity of the 'critical layer' where the phase velocity c of a normal-mode disturbance coincides with the undisturbed flow velocity $U(z)$. Smith and Bodonyi considered the time evolution of a normal-mode disturbance with velocity of the form $\mathbf{u}(x,t) = A \exp[i\{k(x-ct)+n\phi\}]\mathrm{F}(r)$, where all independent and dependent variables are non-dimensionalized by using the maximal Poiseuille-flow velocity U_0 and the tube radius R as units of velocity and length, $\mathbf{F}(r)$ is an $O(1)$ vector function (having all components of the order of one) and A is a small amplitude factor which determines the order of magnitude of the true amplitude (whose definition is not unique, though this topic was not considered in the paper). For the sake of simplicity it was also assumed here that $A = \mathrm{Re}^{-2/3}$, although it was noted that the majority of the conclusions obtained is also valid in the case where $1 \gg A \gg \mathrm{Re}^{-2/3}$. The authors looked for equilibrium (neutrally-stable) solutions and hence the dimensionless phase velocity c (which varies with k, n, and Re) was assumed to be real; moreover, they also accepted that $0 < c < 1$. Careful analysis of the dynamic equations for the disturbance velocities showed that here (exactly as in the problems studied by Benney and Bergeron (1969) and Davis (1969)), the nonlinear terms prove to be quite important in the thin 'nonlinear critical layer' (whose thickness is determined just by this condition) where $U(z) \approx c$. It was also found that neutrally-stable disturbances of the form considered here exist in CPF for $0.284 < c < 1$ and $n = 1$ (and the shapes of these disturbances were also determined by Smith and Bodonyi); at the same time, arguments were presented suggesting that no neutral solutions of this form exist for other values of c and n. The existence of neutrally stable disturbances implies that the Landau constant δ is negative, and hence unstable disturbances of finite amplitude can exist here. Thus, Smith and Bodonyi proved that at large values of Re the CPF is unstable to some small non-axisymmetric disturbances of finite amplitude.

Slightly later the nonlinear stability of the CPF was investigated by Sen et al. (1985), who used in their work the same version of the equilibrium-amplitude method of Reynolds and Potter (1967) that was applied by Sen and Venkateswarlu (1983) to the problem of the stability of plane Poiseuille flow. Sen et al. disagreed with the popular opinion that Reynolds and Potter's method has an acceptable precision only at points (k, Re) near the neutral curve (this opinion prompted Itoh (1977b) to announce that the indicated method is inapplicable to CPF). Therefore they tried to use it to study the stability of tube flow to both axisymmetric (with the azimuthal wave number $n=0$) and non-axisymmetric (with $n=1$) least-stable central normal-modes of disturbance (i.e., the modes with the disturbance energy concentrated mainly near the tube axis; it was for this mode with $n=0$ that the results by Davey and Nguyen (1971) and by Itoh (1977) proved to be contradictory). As in all versions of Reynolds and Potter's method, it was assumed beforehand that there exists the equilibrium state of the normal mode considered, with the time-independent finite amplitude A_e (i.e., the existence of undamped finite-amplitude disturbances was postulated). Then the disturbance stream function $\Psi(x, r, t)$ if $(n=0)$ or velocity and pressure $\mathbf{u}(x, r, \phi, t)$ and $p(x, r, \phi, t)$ if $(n=1)$ where expanded in powers of $e^{i[n\phi+k(x-ct)]}$ (where n and k are the azimuthal and streamwise wave numbers and c is the phase speed of the normal wave given by the linear stability theory) and the terms of the series obtained were represented as the appropriate powers of amplitude multiplied by the normalized disturbance functions. When such forms of the flow fields were substituted into the equations of motion and the boundary conditions, the solvability conditions for the equations for different terms of power series allowed successive determination of the values of the complex Landau constants $\lambda_m(k, \mathrm{Re}), m = 1, 2, 3, ...$, and then to evaluate the equilibrium amplitude A_e from the real part (4.41) of Eq. (4.41b).

According to the numerical results of Sen et al. there is, for axisymmetric or non-axisymmetric disturbances and at any Re, a definite range of wave numbers k for which a finite equilibrium amplitude A_e exists, showing that there are some undamped finite-amplitude disturbances. As Re$\to\infty$, $A_e \to 0$ as $\mathrm{Re}^{-4/3}$, and hence the velocities of the equilibrium disturbances tend to zero as $\mathrm{Re}^{-2/3}$. Some examples of the radial velocity distributions for equilibrium disturbances, of the dependences of amplitudes of velocity components on Re, and of the numerical values of about ten Landau constants λ_m for some specific values of k and Re, and for $n=0$ and 1, are also presented in the paper. The authors stressed that their analysis had a number of limitations (relating, e.g., to the ranges of k and n studied, and to the choice of normal modes), and was based on very complicated calculations which used a number of approximations; therefore, a check of these results by other methods, and their further extension, would definitely be worthwhile. However, apparently no attempts to carry out such a check were undertaken up to now.

4.2.3.3 Nearly Plane-parallel Boundary-layer Flows

In the constant-pressure *boundary-layer flow* (BLF) on a flat plate a neutral-stability curve, and unstable modes of infinitesimal disturbances, certainly exist; in this respect the stability properties of the flow are simpler to investigate than those of the linearly-stable plane Couette and circular Poiseuille flows. However in some other respects BLF is more complex than the two flows mentioned above. First of all, BLF is not strictly parallel; here the primary steady flow has non-zero vertical velocity $W(x, z)$ and the conventional thickness of BLF, $d(x)$,[8] is not constant but increases in the streamwise direction Ox. Moreover, BLF extends to infinity in the vertical direction Oz; therefore the Orr-Sommerfeld eigenvalue problem (corresponding to a simplified plane-parallel flow model) now has a spectrum which includes both discrete and continuous components. Infinite vertical extent of the flow also complicates the upper boundary conditions for the BLF disturbances. It was explained in Chaps. 2 and 3 that in linear-stability studies of boundary layers the parallel-flow approximation is usually used, i.e., the real BLF with the thickness $d(x)$ is usually replaced by a definite plane-parallel model (usually by the so-called *parallel Blasius model* of a flow in the half-space $0 \le z < \infty$ with the velocity field of the form $\{U(z), 0, 0\}$, where $U(z)$ is the standard Blasius profile corresponding to some fixed boundary-layer thickness $d(x_0)$ which does not depend on the x coordinate). Such a parallel-flow approximation is often used in studies of the nonlinear stability of BLF too, but here it has a much more narrow domain of applicability (therefore Stuart in his highly authoritative review (1971) of nonlinear stability theory expressed doubt about the validity of the approximation in this case). All these reasons complicate the determination of the Landau equations for disturbance amplitudes in BLF.

Apparently the first investigation of the Landau equation for boundary-layer flow was carried out by Itoh (1974b) (some of his results were previously announced by Tani (1973)). Exactly as in his Poiseuille-flow paper (1974a), Itoh studied the spatial development of disturbances, using a modification of Watson's (1962) theory combined with an extension of the Stuart–Watson approach of 1960. He took it for granted that at large values of x and Re the streamwise variation of the flow conditions is of minor importance, and hence the streamwise growth of the boundary-layer thickness $d(x)$ (and of $\mathrm{Re}(x) = U_0 d(x)/\nu$ where $U_0 = U(\infty)$ is the free-stream velocity outside the boundary layer) may be taken into account rather crudely. Thus, he introduced the contracted streamwise coordinate $\xi = (x - x_0)/\varepsilon$, where x_0 corresponds to a point far from the leading edge of a plate at $x = 0$ and $\varepsilon = d(x_0)/x_0$ is

[8] Since the letter δ is now used to denote the Landau constant, the boundary-layer thickness will be denoted in this (and only in this) subsection as $d = d(x)$. Similarly, the displacement thickness of the BLF, which is the most widely used vertical length scale of this flow, will be denoted here as $d^* = d^*(x)$.

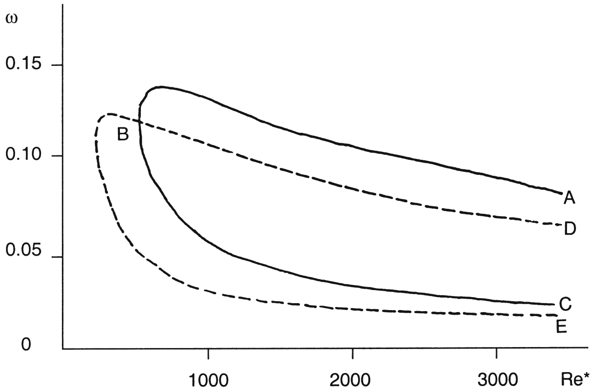

Fig. 4.17 The regions of positive and negative values of the coefficients γ_s and δ_s in the (ω, Re*)-plane for the constant-pressure boundary-layer flow. (After Itoh (1974b)) *ABC*: the *curve* $\gamma_s(\omega, \mathrm{Re}^*) = 0$ (the spatial neutral-stability *curve* of the linear stability theory, bounding the region where $\gamma_s > 0$); *DBE*: the *curve* $\delta_s(\omega, \mathrm{Re}^*) = 0$ bounding the region where $\delta_s > 0$

a small parameter. Then he treated the flow in the neighborhood of the point x_0 as homogeneous with respect to the coordinate ζ, and neglected the terms in the equations of motion which are of order ε^2 or higher. The assumption used (which is close to the plane-parallel approximation) allowed Itoh to evaluate both coefficients of the spatial Landau equation (corresponding to a two-dimensional wave-like disturbance proportional to $e^{i(k\xi-\omega t)}$ where ω is real but k is complex) by a method similar to that applied in his paper (1974a) to strictly plane-parallel Poiseuille flow. He thus determined the spatial neutral-stability curve $\gamma_s(\omega, \mathrm{Re}^*) = 0$ in the (ω, Re*)-plane (where $\mathrm{Re}^* = d^*\ U_0/\nu;\ U_0$ and d^*—the displacement thickness of the BLF—will now be used as velocity and length units in all considerations of the results relating to this flow making in all physical quantities dimensionless). Then the curve $\delta_s(\omega, \mathrm{Re}^*) = 0$ was also computed (recall that $\gamma_s = b_0/2$) and $\delta_s = -b_1$ are coefficients of the 'spatial Landau equation'). The curves obtained are shown in Fig. 4.17; they are of the same general shape as the curves for plane Poiseuille flow in Fig. 4.11, and again show that the spatial Landau constant δ_s is negative along the upper branch of the neutral curve but positive on the main part of the lower branch. Finally Itoh tried to compare his theoretical results with the experimental data of Klebanoff et al. (1962), relating to disturbances generated by a vibrating ribbon located not far from the leading edge of a plate in a wind tunnel. However he found that his theory could explain the behavior of real periodic disturbances only in the case of disturbances with rather small initial amplitude.

Independetly of Itoh, Herbert (1975) also studied Landau's equations for the BLF. However, he considered not spatial but temporal development of two-dimensional disturbances of given wave number k, i.e. he tried to evaluate coefficients of the temporal Landau Eqs. (4.34) and (4.40) and the corresponding equilibrium amplitudes $A_e = (2\gamma/\delta)^{1/2}$. The results obtained were then used to determine the curves $\gamma\ (k, \mathrm{Re}^*) = 0$ and $\delta\ (k, \mathrm{Re}^*) = 0$ in the (k, Re*)-plane. Herbert's computations were based on an approximation of the same type as that introduced by Itoh (1974b) and naturally led to quite similar results. Similar approximation was used also by Gertsenshtein and Shtemler (1997), who applied it to computation of the real coefficients

a_1 and a_2 and complex ceofficients λ_1 and λ_2 of Eqs. (4.41) and (4.41b) at the points of the BLF neutral curve in (k, Re)-plane.

The results of Itoh (1974b) provoked Smith's (1979b) distrust, since in his paper (1979a) Smith found that nonparallelism of BLF appreciably affects the disturbance development. Therefore Smith (1979b) proposed a quite different asymptotic theory of the nonlinear evolution of two-dimensional disturbances in BLF. He considered the case where Re is very high and the disturbance amplitude A is sufficiently small and, using the results of his paper (1979a), derived new values of the coefficients of the spatial Landau equation for the disturbance amplitude. This derivation will not be considered here at length; note only that Smith's computations, relating to a nonparallel model of BLF, confirmed Itoh's (1974b) conclusion that the spatial Landau constant δ_s, corresponding to two-dimensional disturbances, takes positive values near the main part of the lower branch of the BLF neutral curve. At the same time Gajjar and Smith (1985), who used similar methods which also took into account the flow nonparallelism, found that the influence of nonparallelism does not change the conclusion, obtained for the parallel model of BLF, according to which δ_s is negative near the upper branch of the BLF neutral curve. Let us remind the reader in this respect that in Chap. 2 it was noted that direct numerical simulations by Fasel and Konzelmann (1990) and Bertolotti et al. (1992) of the disturbance development in BLF, and also the careful measurements of this development by Klingmann et al. (1993), led to the conclusion that the actual effect of nonparallelism of the BLF is apparently considerably smaller than was suggested in many previous theoretical papers on this subject (which often contradicted each other). The comparison of the results of Itoh (1974b); Herbert (1975); Smith (1979b), and Gajjar and Smith (1985) with each other shows that this conclusion is at least qualitatively (when only the signs of quantities are taken into account) applicable to values of Landau's constants of the BLF too. The same conclusion also follows from the results of a study by Itoh (1984) of the values of Landau's constants in the BLF, supplementing his earlier investigation (1974b).

Trying to improve the simplified treatment of flow non-parallelism used in his paper (1974b); Itoh (1984) referred to his paper (1977a) where a more accurate approach to derivation of Landau's equations for two-dimensional normal-mode disturbances was suggested. He stressed that this approach is applicable only to subcritical (i.e., linearly stable) disturbances, and therefore proposed a new modification of the Stuart–Watson method, which leads to results similar to those found in his paper (1977a); this modification made the results applicable to the cases where supercritical (linearly unstable or neutral) disturbances are studied. Simultaneously, he also developed a more accurate method for taking the slight flow nonparallelism into account. Using these modifications he re-evaluated the neutral-stability curve in the (k, Re*)-plane for two-dimensional temporally-evolving infinitesimal wave disturbances and computed a new the location of the maximum-growth-rate line of the supercritical region in this plane, and also the values of $\mathrm{Re}^*_{\mathrm{cr}}$, k_{cr} and ω_{cr} (he found that $\mathrm{Re}^*_{\mathrm{cr}} \approx 519$, $k_{\mathrm{cr}} \approx 0.30$, and $\omega_{\mathrm{cr}} \approx 0.12$). Then he computed the values of the complex Landau coefficient $l = \delta + i\delta'$ of Eq. (4.40) at the points of the neutral-stability curve and of the maximum-growth-rate line of the supercritical region (consisting of the points (k, Re*) where $\gamma(k, \mathrm{Re}^*) = \max_{k'} \gamma(k', \mathrm{Re}^*)$ and is positive). Computations of the values of l at points on the maximum-growth-rate line were carried out by two different methods, the first of which used a version of the parallel-flow approximation while the second took the flow nonparallelism into account more accurately. The results given by both methods showed that the real Landau constant δ takes positive values on the main part of the line considered, and that corrections due to the more accurate accouinting for nonparallelism are inessential at points far from the neutral-stability curve, but become significant at points near the 'critical point' ($k_{\mathrm{cr}}, \mathrm{Re}^*_{\mathrm{cr}}$) where this line intersects the neutral-stability curve. Therefore the computations of the values of l on the neutral curve were now performed only by the second ('non-parallel') method. The new computations led to negative values of δ at all points of the upper branch of the neutral curve, and to positive values of δ at almost all points of the lower branch (except only the 'critical point' and its small surroundings, where δ takes slightly negative values). These results agree with Itoh's previous results shown in Fig. 4.17, and with the above-mentioned results of Herbert (1975); Smith (1979b), and Gajjar and Smith (1985), showing also that the temporal and spatial Landau constants $\gamma(k, \mathrm{Re}^*)$ and $\gamma_s(k, \mathrm{Re}^*)$ apparently usually have the same signs.

Numerical values of l and $\delta = \Re e\, l$ clearly depend on the definition of the complex amplitude A. In the earlier discussions, the approach developed in the papers by Stuart and Watson of 1960 was always used, and therefore it was assumed that $A(t)$ represents the numerical factor entering the leading term of the Fourier expansion of the initially-infinitesimal normal-mode disturbance satisfying the Orr-Sommerfeld equation (see, e.g., Eqs. (4.38) and (4.39)). However Itoh (1984) used in the beginning of his paper another particular definition of the disturbance amplitude, based on the distribution of the vertical velocity $w(x, z, t)$. This definition is mathematically convenient but it is difficult to measure the corresponding amplitude A in laboratory experiments and thus to compare the proposed theory with experimental data. Therefore Itoh later repeated the computation, now using as A some typical value of the streamwise disturbance velocity u at the height $z = d^*/2$. The new values of l were approximately four times greater than the old ones, but the form of their dependence on Re* proved to be practically the same. Itoh also computed the values of the Landau constant l for three-dimensional disturbances of a special type, namely, for some special wave packets composed of three-dimensional plane waves. In this case the values of real and imaginary parts of l proved to be much smaller than the values corresponding to two-dimensional waves.

Recall that when two coefficients of Landau's equation (either temporal, or spatial) are of the same sign, they determine the value of the amplitude A_e of the equilibrium disturbance (subcritical if $\gamma < 0$, $\delta < 0$, and supercritical if $\gamma > 0$, $\delta > 0$). In the theories where some higher-order real Landau constants a_m, $m \geq 2$, are also taken into account, the equilibrium amplitude A_{e} can be determined as the smallest positive root of the appropriately-truncated Eq. (4.41), if such root exists. On the other hand according to Reynolds and Potter (1967), the existence of an equilibrium disturbance can considerably simplify the derivation of the corresponding Landau's equation from the equations of motion. Sen and Vashist (1989) applied the method of Reynolds and Potter to derivation of the higher-order complex Landau equations for two-dimensional normal-mode wave disturbances in the plane-parallel model of the Blasius boundary layer. This derivation was carried out quite similarly to those of Sen and Venkateswarlu (1983) and Sen et al. (1985) for two-dimensional wave disturbances in plane and circular Poiseuille flows. Sen and Vashist again considered the unstable (or the least stable) two-dimensional wave corresponding to given values of k and $\mathrm{Re} = U_0 d/\nu$ (or $\mathrm{Re}^* = U_0 d^*/\nu$—they used both definitions of the Reynolds number) and computed the values of the complex coefficients λ_m, $m = 1, 2, \ldots, 8$, for a number of values of Re and k. Then they determined the nonlinear neutral curve in the (k, Re*)-plane corresponding to their nonlinear model of the eighth order. It was shown that the nonlinear effects decrease the value of Re_{cr} and increase the values of k_{cr} approaching the non-linear neutral curve to the experimental data then available. However, at that time the authors had no accurate enough experimental data for quatitative comparison with their theory, and they did not try to estimate the influence of the non-parallelism of BLF, which they neglected in the computations.

Note now that in the case of a strictly parallel flow, equilibrium disturbances can also be computed by direct numerical simulation (DNS), i.e., by numerical solution of the corresponding N-S equations (see, e.g., Herbert's work (1976, 1978, 1983a) relating to plane Poiseuille flow). However, in the case of BLF an additional difficulty arises from the fact that the Blasius boundaryu layer is not an exact solution of the N-S equations with standard boundary conditions. Moreover, the boundary-layer thickness $d = d(x)$ depends on x and hence the disturbance cannot be assumed to be proportional to e^{ikx}, with $k = \mathrm{const}$. Therefore, to compute the equilibrium two-dimensional wave-like disturbances in the BLF, Milinazzo and Saffman (1985) supplemented the N–S equations by a fictitious counter-streamwise 'force' which suppresses the boundary-layer growth and makes the two-dimensional flow with velocity $\{U(z)\ 0,\ 0\}$, where $U(z)$ is a standard Blasius profile, an exact solution of the equations of motion considered. (The authors noted that the inclusion of such a force is an 'old well-known idea' which apparently was due originally to L. Prandtl). Later Fischer (1995) used the same modification of the equations of motion for careful evaluation of the Landau constants $\delta(k, \mathrm{Re})$ and equilibrium amplitudes A_e corresponding to a plane-paralle model of the Blasius BLF. On the other hand Lifshits and Shtern (1986); Lifshits et al. (1989), and Koch (1992) also used the plane-paralle approximation in their calculations of the BLF equilibrium solutions, but modified, not the equations of motion but the boundary conditions. Note also that local parallelism of the flat-plate boundary layer and streamwise periodicity of the disturbances were simply assumed to be valid in the important studies of BLF nonlinear stability of Laurien and Kleiser (1989) and Zang and Hussaini (1990).

Milinazzo and Saffman (1985) and Lifshits et al. (1989) considered some particular examples of equilibrium two-dimensional disturbances in BLF (Lifshits et al. also presented some examples of special periodic-halving bifurcations). Lifshits and Shtern (1986) and Koch (1992) tried to determine the neutral surface in the thre-dimensional (E, k, Re)-space similar to that computed by Herbert (1978, 1983a) for plane Poiseuille flow (see Fig. 4.12 above). However, in their computations of equilibrium solutions Lifshits and Shtern (1986) used only the terms of orders zero and one in the Fourier expansion of the disturbance stream function $\Psi(x, z, t)$ similar to (4.38), while Koch discovered in 1992 that in the case of BLF such severe truncation of Fourier series leads to results which can even be qualitatively incorrect. Therefore Koch also showed results of neutral-surface computations where the second harmonic was included in truncated Fourier series for $\Psi(x, z, t)$, and additionally presented graphs of several cross-sections of this surface by the planes $\mathrm{Re} = \text{const.}$ and $k = \text{const.}$ computed with the help of Fourier series truncated after the nth harmonic, where n varied from 1 to 6. The results obtained gave much information about the complicated shape of the neutral surface in the case of BLF and also allowed an estimate of what truncation is sufficient for obtaining the necessary degree of precision. Then Koch passed to the important problem of secondary instability of two-dimensional equilibrium disturbances to small three-dimensional disturbances. His results relating to this topic, and also the results of simultaneously-published papers by Stewart and Smith (1992) and Smith and Bowles (1992), provide a very valuable supplement to the survey of the same subject by Herbert (1988) and shed additional light on the process of boundary-layer transition.

4.2.4 Amplitude Equations for Disturbances in Free Flows in an Unbounded Space

4.2.4.1 Plane Mixing Layers and Jets

Now we will pass to consideration of parallel (or nearly parallel) free flows in the unbounded space $-\infty < z < \infty$ and begin with the case of a strictly plane-parallel *plane mixing layer* between two parallel flows in contiguous half-spaces $-\infty < z < 0$ and $0 < z < \infty$, having constant but different velocities $\{-U_0, 0, 0\}$ and $\{U_0, 0, 0\}$ where U_0 is positive. In Sect. 2.93 it was mentioned that a very convenient and widely-used analytic approximation to the mixing-layer profile is the hyperbolic-tangent profile: $U(z) = U_0 \tanh(z/H)$ where H characterizes the mixing-layer thickness. Therefore we will also use this approximation.

In Sect. 2.93 it was explained that $\mathrm{Re}_{\mathrm{cr}} = 0$ for the hyperbolic-tangent mixing layer, i.e. this flow is linearly unstable at any value of $\mathrm{Re} = U_0H/\nu$. The corresponding neutral-stability curve in the (k, Re)-plane was shown in Fig. 2.35; it suggests that in an inviscid fluid, where $\mathrm{Re} = \infty$, this flow must be linearly unstable with respect to two-dimensional wave-like disturbances if $kH < 1$. This is in fact so, as was proved long ago by Tatsumi et al. (1964) (see also Sect. 31.10 in the book by Drazin and Reid (1981)). Assuming that the influence of viscosity must be insignificant at large values of Re, Schade (1964) tried to calculate the value of the Landau constant δ for the neutral two-dimensional disturbances with $kH = 1$ in inviscid flow with a hyperbolic-tangent velocity profile. He based his calculation on the method of Stuart (1960) but supplemented it by some simplifying assumptions (in particular, he neglected the mean-flow-distortion effect on δ). To overcome the difficulty arising from the singularity of the inviscid Rayleigh Eq. (2.48) (see Sect. 2.82) at the 'critical level' where $U(z) = c$, Schade introduced viscosity in some of his equations (but, as we will see below, this was insufficient for obtaining the correct results). His calculation led to the conclusion that $\delta > 0$ (equal to $32/3\pi$, if U_0 and H are taken as the velocity and length units) at $kH = 1$, and hence small unstable disturbances in the mixing layer with wave numbers slightly smaller than $(H)^{-1}$ must tend to a finite equilibrium state as $t \to \infty$. This conclusion also agreed with the results of Stuart's (1967) study of equilibrium finite-amplitude disturbances in various inviscid laminar mixing layers (including the hyperbolic-tangent one). Later Maslower (1977a), who used a method quite similar to that of Schade (1962), computed, for a hyperbolic-tangent mixing layer with finite value of Re, the values of δ corresponding to two-dimensional disturbances with wave numbers k which are equal to or slightly smaller than the wave number k_0 of the neutrally-stable disturbance. His results for neutral disturbances with $k = k_0$ agreed with Schade's result for the case where $\mathrm{Re} = \infty$, and showed that the value of δ is positive at any Re and decreases with decreasing Re. Simultaneously Maslowe also noted at the end of his paper, that the effect of the mean-flow distortion, which was neglected in his and Schade's studies, apparently also affects the value of δ but he did not elaborate on this remark.

Maslowe (1977a) apparently did not know at the time about the paper by Gotoh (1968) who also calculated values of the Landau constant δ and of the equilibrium amplitude $A_e = (2\gamma/\delta)^{1/2}$ for small finite disturbances in viscous mixing layers, with very large but finite values of Re and with values of k near the neutral-stability curve. Gotoh gave special consideration to the contribution of the 'nonlinear critical layer' (which also included the effect of the mean-flow distortion) and found that in the case of a hyperbolic-tangent mixing layer, δ is positive at all the values of Re and k he considered, and is given by equations

$$\delta = 16.35(\mathrm{Re})^{1/3}\left[1 + 0.25\left(\frac{\gamma}{k}\right)\mathrm{Re}\right], \quad \text{if}\,\frac{\gamma}{k} < (\mathrm{Re})^{-1/3}, \tag{4.45}$$

and

$$\delta = \frac{0.5k^4}{\gamma^3}, \quad \text{if}\,\frac{\gamma}{k} > (\mathrm{Re})^{-1/3}, \tag{4.45a}$$

where δ, γ and k are non-dimensionalized in the usual way. (Note also that γ is a smooth function of k, vanishing at the wave number k_0 of the neutral disturbance; therefore, under the natural assumption that this function is differentiable with nonzero first derivative, $\gamma \propto (k_0 - k)$ near the neutral curve). These results clearly disagree with Schade's conclusion relating to $\mathrm{Re} = \infty$. However, Michalke (1972), who supported Gotoh's criticism of the results of Schade (1964), asserted at the same time that Gotoh's results are also erroneous, since the value of δ must be independent of the value of γ. As will be indicated below, this conclusion by Michalke was later found to be groundless; nonetheless, it was possibly one of the reasons why Gotoh's paper was totally forgotten for a number of years, while the incorrect result by Schade was repeatedly reproduced, as a particular case of some more general results, in papers by Stuart (1967); Benney and Maslowe (1975); Maslower (1977a,b), and Huerre (1977) (Fig. 4.18).

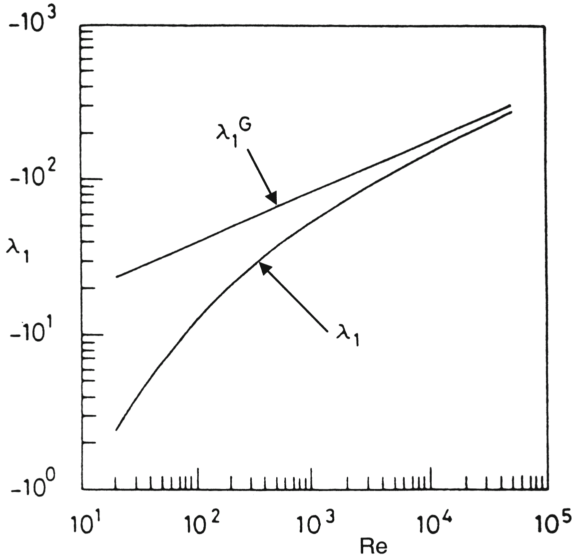

Fig. 4.18 Comparison of the values of $\lambda_1(\text{Re}) = -\delta(\text{Re})/2$ for the neutral wave disturbances in a hyperbolic-tangent mixing layer, computed by Fujimura, with Gotoh's asymptotic equation $\lambda_1 = \lambda_1^{\text{G}} = -8.177\text{Re}^{1/3}$ for $\text{Re} \gg 1$ (After Fujimura (1988))

Huerre (1980) tried to develop a new theory of the nonlinear stability of small free-shear-layer disturbances, based on the approach applied by Benney and Bergeron (1969); Stewartson and Stuart (1971) and Benney and Maslower (1975) to studies of the space-time development of wave packets with amplitude $A = A\ (t, x)$ in high-Reynolds-number parallel shear flows and leading to a nonlinear parabolic partial differential equations of the Ginzburg–Landau type for the function $A\ (t, x)$ (see the last paragraph of Sect. 4.22 and the related paper by Huerre and Scott (1980)). In his paper of 1980 Huerre came to the incorrect conclusion that in the hyperbolic-tangent mixing layer Landau's constant $\delta = \delta\ (k, \text{Re})$ is negative for wave numbers k near the neutral curve, and hence no equilibrium finite-amplitude states can exist here for waves which are slightly unstable according to linear theory. However later he found an error in his paper of 1980, whose correction (presented in Huerre (1987)) led him to the conclusion that δ is positive at all large enough values of Re and small values of $k_0 - k$, and is proportional to $\text{Re}^{1/3}$ (with the same coefficient of 16.35 which was earlier found by Gotoh) in cases when Re is sufficiently large and $k = k_0$ (i.e. $\gamma = 0$). The inaccuracy of Huerre's paper (1980) was discovered independently by Churilov and Shukhman (1987) who also solved the same problem more accurately and obtained, under the condition that $\text{Re} \gg 1$ and $(\text{Re})^{-1} \ll \gamma \ll (\text{Re})^{-1/3}$, the same Eq. (4.45) for δ which was found by Gotoh (1968). (Apparently neither Churilov and Shukhman, nor Huerre, knew in 1987 about Gotoh's paper of 1968). Finally Fujimura (1988) applied the general amplitude expansion proposed by Herbert (1983b) to the computation of Landau's constant δ for small, slightly-unstable, disturbances in a hyperbolic-tangent mixing layer. He found that, for large enough values of Re, numerical values of δ obtained in this way agree well with Gotoh's asymptotic Eqs. (4.45) and (4.45a), and computed also the Landau constants a_2 and a_3 of the next two

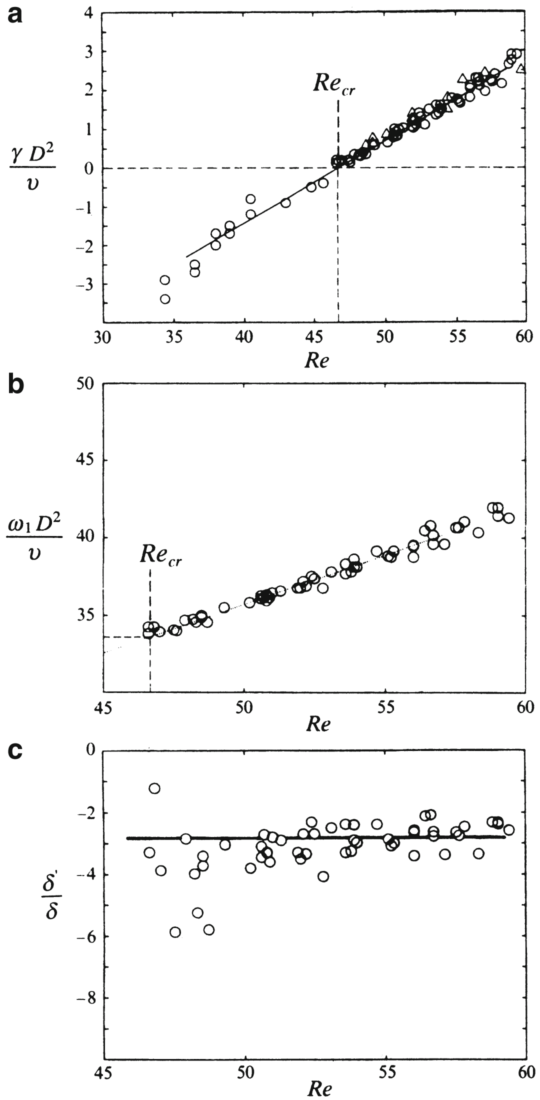

Fig. 4.19 The dependence on $\mathrm{Re} = U_0 D/\nu$ of the values of $\gamma D^2/\nu$ (**a**), $\omega_1 D^2/\nu$ (**b**), and δ'/δ (**c**) in the circular-cylinder wake according to measurements of the development of controlled wake oscillations at the point $\mathbf{x} = \{x, y, z\} = \{8D, 0, 4D\}$ behind the cylinder. (After Schumm et al. (1994)). *Symbols* O and Δ correspond to two different methods of wake-oscillation control

orders, showing in particular that a_2 is negative (i.e., stabilizes the flow) in a wide ragne of Re and k values. An example of Fujimura's results is given in Fig. 4.19 where computed values of $\lambda_1 = -\delta/2$ for neutral disturbances with $k = k_0$ (i.e., with $\gamma = 0$) and Reynolds numbers in the range $20 \le \mathrm{Re} \le 50{,}000$ are compared with Gotoh's asymptotic equation $\lambda_1^G = -\delta_G/2 = -8.177(\mathrm{Re})^{1/3}$, valid at $\mathrm{Re} \gg 1$.[9] Note also

[9] Fujimura found numerically, and also proved analytically, that for neutral two-dimensional disturbances for the difference $\delta - \delta_G$ in fact tends to a constant $\Delta\delta$ as $\mathrm{Re} \to \infty$. According to both his computations and his analytical results $\Delta\delta$ is close to 57.

that still later, in their extensive survey of work on the influence of the critical layer on nonlinear development of small disturbances in weakly supercritical shear flows, Churilov and Shukhman (1995) also indicated that the old paper by Gotoh (1968) gave correct results. Simultaneously they showed that the critical-layer contribution alone often leads to very high values of the Landau constant δ, and sometimes produces amplitude equations of a form quite different from that proposed by Landau in 1994.

Comparision between experimental data and the above theoretical estimates for the values of Landau's constant in a mixing layer (or similar estimates relating to other parallel flows in unbounded space) is rather difficult. In fact the the theory considered above deals mainly with slowly-growing wave disturbances in such flows corresponding to (k, Re)-points near the neutral curve, while in real life the most important role in instability phenomena is played here by modes which are maximally (or almost maximally) amplified, and hence far from neutral. The rapidly-growing most-unstable waves later generate subharmonic waves with half the frequency of the dominant mode, and the interaction of the dominant mode with subharmonic ones and with the mean flow cannot be described by the Landau-type theory (see, e.g., the old survey of appropriate experimental data by Miksad (1972) and the more recent paper by Monkewitz (1988a) containing many additional references). However, this does not mean thtat Landau's theory is useless for quantitative description of instability phenomena in free shear flows; see in this respect the discussion of wake-flow instabilities below.

Plane jets in an unbounded space represent type of plane-parallel flows having some similarities with the parallel mixing layers. It was mentioned in Sect. 2.93 that the most widely-used model of the corresponding velocity profile $U(z)$ is the so-called *Bickley jet profile* $U(z) = U_0 \operatorname{sech}^2 (z/H)$, where $-\infty < z < \infty$ and H characterizes the jet thickness (see Eq. (2.87), unlike the hyperbolic-tangent approximation for the mixing layer this profile is an exact analytical solution of the boundary-layer equations). The problem of nonlinear evolution of normal-mode disturbances in the Bickley jet has attracted less attention than the same problem for the hyperbolic-tangent mixing layer and we will not discuss it in detail. Note only that Gotoh (1968), in parallel with his work on development of disturbances in the mixing layer, considered the same problem for the case of Bickley's jet and found that here again equations of the form (4.45) and (4.45a) are valid. However, now the numerical coefficients 16.35 and 0.25 in Eq. (4.45) must be replaced by coefficients 2.19 and 1.5., while the new value of the coefficient in Eq. (4.45a) was not indicated by Gotoh. His results relating to the Bickley jet, unlike his results for the hyperbolic-tangent mixing layer, have not yet been confirmed by other authors but one may conjecture that they are valid too. Some remark about the instabilities of round jets will be made at the very end of this section.

4.2.4.2 Wake Flows: the Case of a Circular-cylinder Wake

Let us now consider nearly plane-parallel *wake flows* with velocity profiles of the type shown in Fig. 2.31c. In Chap. 2 it was indicated that the 'Gaussian' velocity profile (2.89) describes, accurately enough, the velocity distribution in the laminar wake behind a thin flat plate parallel to the free stream. Some remarks about the nonlinear instability of wakes behind flat plates will be made in the part (c) of this section, but most attention will be paid here to the most important and most widely studied *plane wakes behind long cylindrical bluff bodies* of constant cross-section, in uniform flows with free-stream velocity U_0. constant and normal to the body length. As in Sect. 2.93 it will be assumed below that the axis Oy is parallel to the cylinder axis (and defines the 'spanwise' direction), while the axis Ox is directed along the direction of the oncoming uniform flow and the midpoint of the body is chosen as the origin of coordinates.

Let us begin with the case of *circular-cylinder wakes,* while the wakes behind some other spanwise homogeneous bodies will be briefly considered later. It is well known that when Reynolds number $\mathrm{Re} = U_0D/\nu$ is gradually increased the flow around a circular cylinder of diameter D undergoes a whole series of remarkable transformations produced by a number of instability phenomena (see, e.g., Sects. 3.3 and 17.8 in the textbook by Tritton (1988), the survey by Coutanceau and Defaye (1991), the nice old survey paper by Morkovin (1964), and-for more details-the recent book by Zdravkovich (1997), vol. 1 of which (vol. 2 has not appeared at the time of writing) is about 700 s long and contains a huge bibliography which, however, does not intersect too much with that at the end of this chapter). In the present section devoted to Landau's equation, the first two transformations of the cylinder wake are the most interesting. The first of them takes place at $\mathrm{Re} \approx 4$ (this $\mathrm{Re}_{cr} = \mathrm{Re}_{0,\,cr}$ corresponds to the origin of linear instability of a laminar wake) and leads to a steady wake flow of a new type characterized by the appearance of the recirculation zone just behind the cylinder, the size of which slowly increased with Re and which consists of two symmetrical stationary vortices attached to the rear of the cylinder (for more data about this flow see, e.g., Coutanceau and Bouard (1977) or Zdravkovich (1997)). The second transformation leads to the formation at some $\mathrm{Re} = \mathrm{Re}_{1,\,cr}$ above 40 of the *von Kármán* (or, as it is also sometimes called, the *Bénard-von Kármán*) *vortex street*[10], consisting of a double row of opposing vortices, convected downstream and producing wake oscillations (see, e.g., the excellent Photos 94–98 in the album by Van Dyke (1982)). The appearance of the vortex street is due to the 'shedding' of vortices periodically torn away from the back of a cylinder with a frequency f coinciding with the frequency of the wake oscillations. The next transition to a three-

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[10] These names mark the contributions by Kármán (1911) (and Kármán and Rubach (1912)) and by Bénard (1908) to the investigation of this phenomenon. Note, however, that in fact the formation and subsequent 'shedding' of vortices behind bluff bodies was observed and repeatedly sketched by Leonardo da Vinci about the year 1,500 (one of his brilliant drawings opens Zdravkovich's book of 1997) and has been studied at least from the days of Strouhal (1878) who, in particular, first measured the frequency f of arising wake oscillations.

Experimental data show that critical Reynolds number $\mathrm{Re}_{1,cr}$ depends on the cylinder aspect ratio L/D (where L is the length of the cylinder) and boundary conditions at the cylinder ends; usually this number takes values between 40 and 50. It was however noted that under some special conditions a short vortex street (which is not stable and is wholly located in a region near the cylinder) can be excited at smaller values of Re between 22 and 40 (see, e.g., Plaschko et al. (1993)).

dimensional flow regime occurs usually at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}} \approx 170$–$190$; it will be briefly considered at the end of the present part b of this subsection.

The Reynolds number $\mathrm{Re}_{1,\mathrm{cr}}$ (below it will often be simply denoted as $\mathrm{Re}_{\mathrm{cr}}$) is the threshold value for the appearance of instability of the steady wake flow arising at $\mathrm{Re} = \mathrm{Re}_{0,\mathrm{cr}}$, which leads to its transition to a new oscillating regime. Such a transition clearly represents a Hopf bifurcation. The corresponding value of $\mathrm{Re}_{\mathrm{cr}}$ was theoretically evaluated by a number of researchers-in particular, by Zebib (1987); Jackson (1987); Morzyński and Thiele (1991, 1992, 1993), and Noack and Eckelmann (1992, 1994a) whose results do not differ too muich from each other, from the available experimental data, or from estimates of this number given by numerical simulations. The methods used by by these authors were different from those described in Sects. 2.8 and 2.9, since here *non-parallel stability analysis* was used (i.e. the flow around the cylinder was not assumed to be plane-parallel). However, as a rule this flow was assumed to be two-dimensional (independent of the spanwise y coordinate) and was given as the steady solution of the two-dimensional Navier-Stokes equations satisfying the appropriate boundary conditions. The use of the two-dimensionality assumption clearly means that here only the centrial part of the wake behind a long cylinder with large enough value of L/D (where L is the cylinder length) is considered. As to the temporal development of the wake oscillations occurring at $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$, it was successfully described by Landau's equations in a number of papers which will be considered below. Note that in contrast to the above discussion of the cases of plane mixing layers and jets, these papers concentrated, not on the mathematical evaluation of the Landau coefficients for some given primary velocity profiles $U(z)$, but on the investigations of the disturbance development in real wake flows. Therefore below Landau's equations will not be applied to the idealized neutral or nearly-neutral normal modes, corresponding to points of the (k, Re)-plane neighboring the neutral curve, but to the most-unstable disturbances, which suppress all the others and play the dominant part in the observed disturbance development. This implies, in particular, that the coefficients of these equations will now depend on Re but not on k, since the value of Re uniquely determines the wave number of the most unstable wave disturbance.

Apparently Mathis (1983) and Mathis et al. (1984) where among the first experimenters to show that the 'shedding of vortices' and formation of the vortex street in a flow around a long circular cylinders represents a Hopf bifurcation which can be described by Landau's equation. Therefore the complex Landau Eq. (4.40). (which, as mentioned above, is also often called the Stuart–Landau equation), having the complex coefficients ω and l, was introduced here for the complex amplitude, $A(t) = |A(t)|\, e^{i\phi(t)}$, of the 'vertical' (i.e. 'transverse' or z-wise) velocity $w\,(t)$ of wake oscillations at a fixed point inside the wake (namely, at the point with coordinates ($5D$, 0, 0)). The complex equation for $A(t)$ was then replaced by two real equations for the functions $|A(t)|$ and $\phi(t)$ (both of which have been already given in Sect. 4.21):

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2 - \delta|A|^4, \tag{4.34}$$

$$\frac{d\phi}{dt} = -\omega_1 - \tfrac{1}{2}\delta'|A|^2 \tag{4.34a}$$

where $\omega_1 + i\gamma = \omega$, $\delta + i\delta' = l$.

Laser-Doppler-anemometer measurements by Mathis, and Mathis et al., of the velocity w (t) in the wakes of a number of cylinders plaed in a wind-tunnel were made at various values of Re and confirmed that $\gamma \approx b(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ at small values of Re–$\mathrm{Re}_{\mathrm{cr}}$ where $\mathrm{Re}_{\mathrm{cr}} \approx 47$ and $b = \mathrm{const.} \approx \nu/5D^2$ if the aspect ratio L/D is large enough. (At small values of L/D, $\mathrm{Re}_{\mathrm{cr}}$ takes greater values—this observation by Mathis et al. agreed with results of some preceding experiments and later it was confirmed, in particular, by Lee and Budwig (1991) and Norberg (1994)). At the same time the coefficients δ, δ' and ω_1, in contrast to γ, do not vanish at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$, and their values at small values of Re – $\mathrm{Re}_{\mathrm{cr}}$ may be approximated by two-term relations:

$$\begin{aligned} &\delta \approx \delta_0 + \delta_1(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}), \quad \delta' \approx \delta'_0 + \delta'_1(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}), \\ &\omega_1 \approx \omega_{10} + \omega_{11}(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}), \end{aligned} \tag{4.46}$$

where δ_0, δ'_0 and ω_{10} are the values of these coefficients at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$, and δ_1, $\delta\,'_1$ and ω_{11} are their derivatives with respect to Re at this point. As has been already repeatedly noted above, it follows from Eq. (4.34) that if $\delta > 0$, then a Hopf bifurcation of the disturbed flow occurs at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$ and, at slightly supercritical conditions (i.e., when $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ but Re – $\mathrm{Re}_{\mathrm{cr}}$ is small), a small initial disturbance tends to a equilibrium state with the amplitude $A_e = (2\gamma/\delta)^{1/2} \approx (2b/\delta_0)^{1/2}$ $(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})^{1/2}$. The existence of equilibrium amplitude A_e in supercritical wake flows was confirmed by the experimental data of Mathis et al. (and of many other authors); thus, the data definitely show that $\delta > 0$ in the case of the most unstable disturbance in the wake behind a circular cylinder. The data show also that the relation $A_e \propto (\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})^{1/2}$, which corresponds to the first term of the Taylor-series expansion of $(2\gamma/\delta^{1/2})$ in powers of Re – $\mathrm{Re}_{\mathrm{cr}}$, is valid even when Re – $\mathrm{Re}_{\mathrm{cr}}$ is not too small. Hence the derivative δ_1 is rather small in absolute value and may usually be neglected. (The same conclusion follows from the validity of the relation (4.47), below, over a wide range of Reynolds numbers). Moreover, measurements of the values of δ/δ' at various Reynolds numbers, which will be described below (see, in particular, Fig. 4.19c) show that this quantity also is independent of Re over a considerably range of supercritical Reynolds numbers. Hence the derivative δ'_1 may also be neglected, and both coefficients δ and δ' may be considered as being independent of Re.

Note that $(1/2\pi)(d\phi/dt) = f$ is just the local frequency of oscillations of the wake amplitude $A(t)$, while peak-to-peak value $2|A|$ of these oscillations is equal to the double equilibrium $2A_e$. Hence Eqs. (4.34) and (4.34a) (the first of which determines the value of A_e) together with the equation $\gamma = b(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ imply the equation

$$\mathrm{Ro} = a\mathrm{Re} - a_1, \tag{4.47}$$

Where $\mathrm{Ro} = fD^2/\nu$ is the so-called *Roshko number*, $a = -[(b(\delta'/\delta) - \omega_{11}]D^2/2\pi\nu$, and $a_1 = \{\omega_{10} - [b(\delta'/\delta) - \omega_{11}]\mathrm{Re}_{\mathrm{cr}}\}D^2/2\pi\nu$. The dimensionless quantity

fD^2/ν was introduced by Roshko (1953, 1954), who also showed that over a wide range of Reynolds numbers its dependence of $\mathrm{Re} = U_0D/\nu$ is given by an equation of the form (4.47) with constant coefficients a and a_1. Therefore Eq. (4.47) is often called the *Roshko equations* though in fact the same equation, written in the form

$$St = a - a_1/\,\mathrm{Re}, \tag{4.47a}$$

where $\mathrm{St} = fD/U_0 = \mathrm{Ro}/\mathrm{Re}$ is the so-called *Strouhal numbers* was employed by Rayleigh (1915) (see also Rott (1992) and Williamson (1995, 1996a)). Thus, the empirical 'Ro-Re' and 'St-Re' relations (4.47–4.47a) are fully compatible with Landau's equation.

The experimental data by Mathins et al. (1984), for cylinders with not too small values of the aspect ratios L/D, agreed with Roshko's eqution (4.47) with constant coefficients a and a_1 only in a limited range of Reynolds numbers from $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}} \approx 47$ to $\mathrm{Re} \approx 90$. When the value of Re was increased further, the character of the wake oscillations changed discontinuously and then the values of a and a_1 also changed. Mathis et al. noted that abrupt changes of the regime of wake oscillations found by them agree with earlier results of Tritton (1959, 1971) and Gaster (1971). Later the nature of these changes, their dependence on the value of L/D and on the end conditions at $y = \pm L/2$, and possible methods for getting rid of the changes were discussed by a number of authors; see, eg., Slaouti and Gerrard (1981); Lee and Budwig (1991); Szepessy (1993), and the subsequent discussion of this topic after Eq. (4.49) where additional references will be given.

More detailed experimental studies of disturbance behavior in wakes behind circular cylinders were later carried out both by the group with which Mathis collaborated (see Provansal et al. (1987); Provansal (1988)) and by some other researchers (see, e.g., Strykovski (1986), whose dissertation covered much the same ground as that of Mathis (1983); Sreenivasan et al. (1987); Strykovski and Sreenivasan (1990); Schumm (1991); Schumm et al. (1994); Park (1994), and the survey by Monkewitz (1996)). These authors also based their studies on the Landau model and performed a number of careful measurements which allowed them to determine, at some points of the cylinder wake, the values of all coefficients of Landau's Eqs. (4.34) and (4.34a) at various values of Re. These determinations used methods of wake control allowing the wake oscillations (always existing if $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$) to be switched off (completely or partially) and then switched on again very rapidly. Observing, at different values of Re, the rate of growth with time of the amplitude $|A(t)|$ of the disturbance velocity from the initial small value to the final equilibrium value A_e, one may find the coefficients γ and δ of Eq. (4.34) and their dependence on Re. In this way it was found that δ is usually independent of Re, while γ satisfies the relation $\gamma = b(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ where the values of $\mathrm{Re}_{\mathrm{cr}}$ and b can also be determined from experimental data. Moreover, measurements of the frequency f of equilibrium wake oscillations at various Reynolds number determined the dependence of $\mathrm{Ro} = fD^2/\nu$ on Re, verified the Roshko Eq. (4.47) and gave the values of coefficients a and a_1. Using these values, and also the values of δ, b and $\mathrm{Re}_{\mathrm{cr}}$ given by the results of amplitude measurements, one may also determine the values of $b\delta'/\delta - \omega_{11}$ and ω_{10}. On the other hand, one may observe, at various values of Re, the increase with time of the frequency f of wake oscillations from the moment of their switching on (when $|A| = 0$ and hence $f = -\omega_1/2\pi$) to the final equilibrium conditions (when $|A| = A_e$). Such observations make it possible to determine the dependence of ω_1 on Re (and the values of coefficients ω_{10} and ω_{11}) and to chek the value of Re_{cr} already found. When this is done, the coefficients a and a_1 may be computed a new, to compare their new values with those implied by the experimental verification of the Rayleigh-Roshko laws (4.47) and (4.47a).

The method of control used by Strykovski, and Sreenivasan et al. (and also by Mathis, and Mathis et al). consisted of the quick reduction, perhaps to zero, of the free-stream velocity, with a subsequent quick return to its initial value U_0 (corresponding to given $\mathrm{Re} > \mathrm{Re}_{cr}$). Schumm, Schumm et al., and Park, also employed several other control methods such as bleeding of fluid from the rear part of the cylinder, wake heating, or forced vertical vibrations of a cylinder with a small amplitude $a_0 \ll D$. (All these operations at supercritical $\mathrm{Re} > \mathrm{Re}_{cr}$ strongly suppress vortex shedding; see, e.g., Monkewitz's surveys (1993, 1996) and the papers on wake control by Roussopoulos (1993); Schumm et al. (1994); Park et al. (1993, 1994); Park (1994); Roussopoulos and Monkewitz (1996); Gunzburger and Lee (1996), and Gillies (1998) containing many additional references). However, the above-mentioned control methods are applicable only at supercritical Reynolds numbers and can provide no information about the values of coefficients of Eqs. (4.34–4.34a) at $\mathrm{Re} < \mathrm{Re}_{cr}$. To obtain such information Sreenivasan et al., Schumm et al., and Park used some methods of 'subcritical wake control', i.e. of artificial forcing of the vortex shedding and wake oscillations of the appropriate frequency at subcritical conditions characterized by the given value of Re which is smaller than Re_{cr}. Applying this forcing, and then switching it off rapidly and observing the subsequent damping of oscillations, one may obtain data relating to values of the Landau coefficients at subcrtical Reynolds numbers.

Sreenivasan et al. (1987) measured (by both hot-wire and laser-Doppler anemometers) wake velocity fluctuations behind the central parts of three cylinders with aspect ratios $L/D = 60$, 27 and 14 at several values of x/D and z/D and values of Re in the range $35 < \mathrm{Re} < 100$. They found (as Mathis et al. did earlier) that the characteristics of wake oscillations vary (though not too much) with the cylinder aspect ratio, and most attention was paid to the case where $L/D = 60$, in the hope that the results would also be representative of greater values of L/D. It is natural to think that the complex constant $\omega = \omega_1 + i\gamma$ is simply the most unstable eigenvalue (i.e., that having the greatest imaginary part) of the Orr-Sommerfeld equation corresponding to the plane-parallel model of the wake velocity profile. If so, then this constant is a *global stability characteristic* which does not depend on the point in the wake at which observation is carried out (see the closing paragraph in Sect. 2.93, and the supplementary discussion of this topic at the beginning of the subsequent small-type text). However, the constants δ and δ' are apparently position-dependent and depend also on the choice of the measured flow characteristic and the definition of the amplitude A. (However Sreenivasan et al. found that the in the range $3 < x/D < 7$ the spatial variations of these constants are small and may be neglected). As to the ratio δ'/δ, it affects the values of coefficients a and a_1 of the Roshko equation and hence must be independent of both the point of observation and the value of the Reynolds number. According to measurements by Sreenivasan et al., $\mathrm{Re}_{\mathrm{cr}} \approx 46$ in the wake behind a circular cylinder with $L/D > 60$ and

$$\gamma D^2/\nu \approx 0.20(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}), \quad \omega_1 D^2/\nu \approx -34.3 - 0.7(R - \mathrm{Re}_{\mathrm{cr}}), \tag{4.48a}$$

$$\delta D^2/\nu \approx 134, \quad \delta' D^2/\nu = -404 \tag{4.48b}$$

(so that $\mathrm{Ro}_{\mathrm{cr}} = (-\omega_1 D^2/2\pi\nu)_{\mathrm{cr}} \approx 5.45, \delta'/\delta \approx -2.90$). Note that Sreenivasan et al., who did not know about the work of Mathis (1983) and Mathis et al. (1984), found exactly the same dependence of $\gamma \mathrm{D}^2/\nu$ on $\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}$ as the latter authors and nearly the same value of $\mathrm{Re}_{\mathrm{cr}}$. Results of more numerous and careful measurements by Schumm (1991) and Schumm et al. (1994), who investigated wakes behind several circular cylinders with $L/D \geq 50$ and applied several different methods of wake control, prove to be very close to that found by Sreenivasan et al.: according to Schumm et al.

$$\mathrm{Re}_{\mathrm{cr}} = 46.7 \pm 0.3, \quad \gamma D^2/\nu = [0.21 \pm 0.005](\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}), \tag{4.49a}$$

$$\frac{\omega_1 D^2}{\nu} = -[33.6 \pm 0.3] - [0.64 \pm 0.02)(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}), \quad \frac{\delta'}{\delta} = -[2.90 \pm 0.45] \tag{4.49b}$$

(see Fig. 4.19 where results of their measurements of coefficients of Eqs. (4.34) and (4.34a) at different values of $\mathrm{Re} = U_0 D/\nu$ are shown). Close results were obtained also by some other researchers; for example, Albarède and Monkewitz (1992) came to the conclusion that $\delta'/\delta = -3 \pm 0.6$, while numerical simulations of wake flows by Dusék et al. (1994) led to the estimate $\delta'/\delta \approx -2.7$, and according to laboratory measurements by Albarède and Provansal (1995) $\delta'/\delta = -2.6 \pm 0.7$.

Sreenivasan et al. (1987) noted that the values of coefficients a and a_1, implied by their estimate of $\mathrm{Re}_{\mathrm{cr}}$ and of the Landau coefficients (4.48), do not differ too much from empirical values of a and a_1 recommended by Roshko (1954), while Monkewitz (1996) more methodically compared values of a and a_1 given by estimates (4.49) with values which agree best with empirical St-Re relations. Such a comparison is not an easy matter, since the Strouhal number in a cylinder wake depends on a number of factors. As was shown by Gerrard (1978) and Williamson (1989, 1995, 1996a), the empirical forms on the St-Re relation for the wakes behind circular cylinders, collected over a period of more than one hundred years (beginning with the frequency data of Strouhal (1878)), are very scattered. This scatter evidently cannot be explained by errors of measurements since both Re and St numbers can be easily measured with a high accuracy. (Oscillations of two-dimensional wakes have the unique frequency $f = f(\mathrm{Re})$ coinciding with the frequency of vortex shedding; in the case of three-dimensional wakes, several discrete oscillation frequencies and even the continuous frequency spectrum often exists, but at small and moderate values of Re here too the unique dominant frequency f can be measured accurately by means of numerical or instrumental spectral analysis). Therefore, the scatter must have another explanation. Recall now that, according to the above discussion of the experimental data by Mathis et al. (1984), the values of a and a_1, which correspond to these data depend on both the cylinder aspect ratio L/D and the range of Reynolds numbers considered, and this dependence was also found to be in agreement with results of some earlier observations of wakes behind circular cylinders. Let us add to this that more recent experimental and numerically-simulated data both show that the character of the vortex street behind a cylinder strongly depends on the boundary conditions at the cylinder's ends, and that usually the ordinary 'parallel shedding' is replaced at some $\mathrm{Re}_s > \mathrm{Re}_{cr}$ by 'oblique shedding' at some angle θ to the cylinder axis (see the papers mentioned at the end of the first new paragraph after Eq. (4.47a), and the papers by Williamson (1988a, 1989, 1995, 1996a); Norberg (1994); Persillon and Braza (1998) and references therein). The data show also that the frequency of wake oscillation f and the Roshko and Strouhal numbers Ro and St, which are proportional to it, in the case of oblique shedding depend on the 'shedding angle' θ. It is clear that such dependence must affect the Ro-Re and St-Re relations violating their universality. Moreover, the data presented in the above-mentioned papers (and in those by Williamson (1988b, 1996b, c); Coutanceau and Defaye (1991); Konig et al. (1990, 1992, 1993); Hammache and Gharib (1993); Brede et al. (1994); Zhang et al. (1995); Thompson et al. (1996); Henderson (1997), and Leweke and Williamson (1998), among many others) show that at some greater value of Re the primary mode of 'oblique shedding' is replaced by another three-dimensional mode, which in its turn can be replaced by a more complicated flow regime in the range of still greater Re numbers.

In the late 1980s and 1990s it was also proved that at moderate values of Re the oblique shedding is always due to 'end effects' caused by finite length L of a cylinder and that the shedding angle θ depends on the spanwise boundary condition at cylinder ends which play a very important part even at large aspect ratios $L/D > 100$. This dependence allows the value of θ to be changed by a proper modification of either the cylinder end conditions (dependent on the method of supporting the cylinder) or the flow near the cylinder ends. Therefore one may pass to the parallel regime of vortex shedding by appropriate change of flow configuration near the cylinder ends. In particular, it was found that the parallel regime may be caused by small increase of the undisturbed velocity $U_0 = U_0(y)$ near $y = \pm L/2$, or by suction of small amounts of fluid from just downstream of the ends of a cylinder, whereas without any manipulation affecting boundary conditions, parallel shedding at relatively large values of Re can be attained only at an aspect ratio wellover 1000 (see again the papers referred above and also those by Eisenlohr and Eckelmann (1989); Hammache and Gharib (1989, 1991); Albarède and Monkewitz (1992); Norberg (1994); Miller and Williamson (1994); Monkewitz (1996), and Monkewitz et al. (1996)).

Since the changes of boundary conditions may often be achieved in laboratory experiments by means of some simple mechanical devices, and can also be easily incorporated in numerical simulations, a number of high-quality frequency determinations was carried out during the last decade, in circular-cylinder wakes near mid-span, under conditions guaranteeing the regime of 'parallel vortex shedding'. The results obtained in numerous experiments were collected by Williamson (1988a,

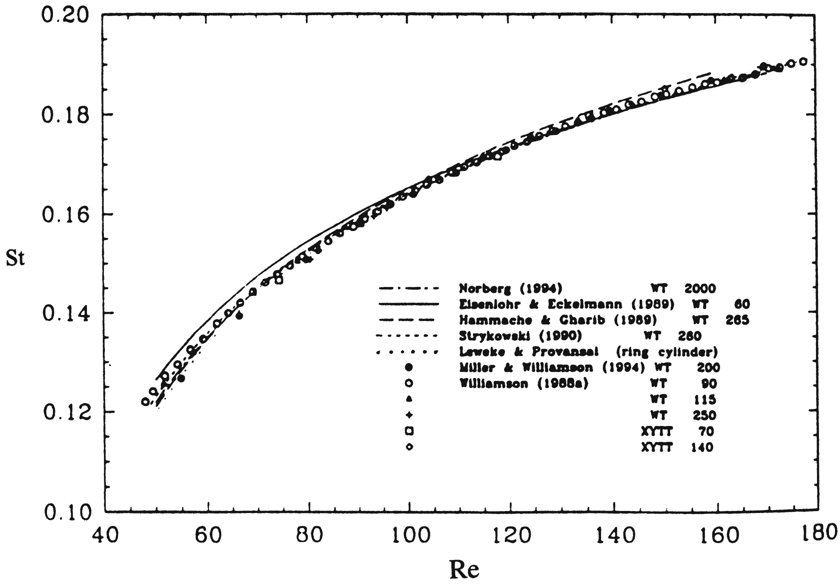


Fig. 4.20 'Universal St-Re relation' found by Williamson for the case of purely parallel vortex-shedding regime of the wake behind a circular cylinder at moderate Reynolds numbers, and its comparison with available experimental data. (After Williamson (1996a)). Various *symbols* represents experimental data by Williamson and his coworkers, various curves–the data of other authors; *WT*: wind-tunnel data; *XYTT*: results obtained in a special water tank facility. The numbers after the facilty marks indicated the cylinder aspect ratios *L*/*D*

1989, 1995, 1996a) in the form of 'universal St-Re relation' shown in Fig. 4.20. The data in this figure include the measurements made both in wind tunnels and water-tank facilities, by a number of different techniques, and covering the Re-range from $\mathrm{Re}_{cr} \approx 50$ and to $\mathrm{Re} \approx 180$. Very similar results were also found in numerical simulations of the cylinder wake by Karniadakis and Triantafyllou (1989); Thompson et al. (1996); Persillon and Braza (1998), and some others. Moreover, Prasad and Willaimson (1997) showed also that, by the appropriate adjustment of boundary conditions at cylinder ends, one can make vortex shedding parallel also in the case of wakes characterized by high Reynolds numbers much exceeding the values considered in Fig. 4.20. However, in this case the parallel-shedding flow regime quickly becomes three-dimensional and its St-Re relation is no longer universal (this matter will be discussed at greater length at the end of this part of Sect. 4.24).

According to Williamson the empirical St-Re relation shown in Fig. 4.20 may be best approximated by a three-term equation of the form $\mathrm{St} = a - a_1/\mathrm{Re} + a_2\mathrm{Re}$ (where $a = 0.1816$, $a_1 = 3.3265$, and $a_2 = 1.6 \times 10^{-4}$). However, Monkewitz (1996) found that two-term approximation (4.47a), with coefficients $a = 0.199$ and $a_1 = 3.94$, which corresponds to estimates (4.49), in indistinguishable at small and moderate values of $\mathrm{Re} - \mathrm{Re}_{cr}$ from approximation suggested by Williamson, and only at $\mathrm{Re} \approx 100$ this two-term equation leads to results which fall slightly below those giving by Williamson's approximation. Thus, we must conclude tha the Landau model gives quite a good description of the data relating to wake oscillations generated by 'parallel shedding' under the conditions of small and moderate (but not great) supercriticality, and that the empirical estimates (4.49) give, with good accuracy, the values of coefficients of the corresponding complex Landau equation. In fact, it is quite surprising that Landau's equation, with coefficients computed under that conditions that $\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}$ is small, leads to results which agree so well with experimental data for $\mathrm{Re}/\mathrm{Re}_{\mathrm{cr}}$ up to 1.5.

Note that time-amplified global oscillations of the entire near wake are intimately connected with local *absolute* (in contrary to *convective*, see the closing part of Sect. 2.93) instability of the wake flow. In fact, the wake flow is not strictly plane-parallel, and hence its local velocity profile, and the Orr-Sommerfeld eigenvalues depending on it, vary slowly with the streamwise coordinate x. Hence, the local values of all coefficients of the complex Landau (i.e., Stuart-Landau) Eq. (4.40) here on x (and this dependence becomes more significant with the increase of non-parallelism of the flow). This means, in particular, that the local oscillation frequency $f = -\omega_1/2\pi$ slowly changes with increase of distance from the cylinder. However the observation definitely show that the near wake, having a considerable streamwise extent, usually oscillates as a whole with constant frequency f, somehow selected from the collection of weakly varying local values. Such 'oscillation as a whole' characterizes the *global instability mode*, which occurs in the wake behind a solid body only in the cases where a considerable regions of the absolute flow instability exists near a body. Thus, one may say that the *Bénard-von Kármán vortex street is due to the absolute instability of the flow in the near wake.* Just this circumstance stimulated numerical investigations of wake regions of local absolute instability, typified by the papers of Koch (1985); Huerre and Monkewitz (1985); Monkewitz and Nguen (1987); Monkewitz (1988b, c); Yang and Zebib (1989); Hannemann and Oertel (1989), and Delbende and Chomaz (1998). The complex amplitude of global wake oscillations can depend on the spatial coordinates (on x and z in the case of a two-dimensional vortex street, and on three coordinates in more complicated cases) but its dependence on t in the case of a non-steady regime of global mode development will satisfy Landau's equation with the same coefficient $\omega = \omega_1 + i\gamma$ at all points $\mathbf{x}$. The data relating to the spatial distribution of the oscillation amplitudes $A(\mathbf{x}, t)$ will be considered at greater length below, for more details and additional references concerning the general properties of the global instability modes of nearly plane-parallel flows see, e.g., the papers by Triantafyllou et al. (1987); Karniadakis and Triantafyllou (1989); Huerre and Monkewitz (1990); Monkewitz (1990, 1996); Chomaz et al. (1991); Monkewitz et al. (1993); Le Dizès (1994), and Le Dizès et al. (1996).

As to the problem of 'oblique shedding', williamson (1988a, 1989, 1995, 1996a) showed that, in the cases where the 'shedding angle' θ is fixed, the 'universal St-Re relation' of Fig. 4.20, which corresponds to parallel shedding, is valid with good accuracy for 'modified Strouhal number' $\mathrm{St_m} = \mathrm{St}/\cos\theta$. This Williamson's 'cosine law' of oblique vortex shedding was confirmed in a number of experimental papers (see, e.g., König et al. (1993); Miller and Williamson (1994), and Monkewitz et al. (1996)) but its theoretical explanation requires the use of some special analytical techniques. Since it was shown that 'oblique shedding' is strongly affected by the 'spanwise boundary conditions' at $y = \pm L/2$, the 'cosine law' can be derived theoretically only from a model which takes into account the influence of the flow configuration near the cylinder ends on the oscillations of the middle part of the wake. This simplest way to achieve this is to introduce a y-dependent oscillation amplitude $A\ (y, t)$ and replace the complex Landau Eq. (4.40) by the more general *complex Ginzburg–Landau* (G–L) *equation* for this amplitude, having the form

$$\frac{\partial A}{\partial t} = -i\omega A + \mu \frac{\partial^2 A}{\partial y^2} - \frac{1}{2} l |A|^2 A \tag{4.50}$$

where ω, μ, and l are three complex coefficients and the second term on the right-hand side describes the spanwise diffusion of oscillations. (For more information about this equation see, e.g., the extensive survey by Cross and Hohenberg (1993) containing a comprehensive bibliography, Chap. 5 of the book of Bohr et al. (1998), the paper by van Saarloos (1995) and other papers in Cladis and Palffy-Muhoray (1995) where a number of modifications, generalizations, and various applications of Eq. (4.50) are collected. A typical example of the useful generalization of Eq. (4.50) is provided by the 'quintic G-L equation' containing an additional term proportional to $|A|^4 A$; this equation was used, in particular, by Shtemler (1978) and Bottin and Lega (1998), who applied it to stability studies relating to plane Poiseuille and Couette flows, and by Iwasaki and Toh (1992), who based on this equation their model description of turbulence structures at high Reynolds numbers). Equation. (4.50) and some other related nonlinear model equations were applied to description of the spanwise-varying cylinder wakes, in particular, by Albarède et al. (1990); Albarède (1991); Noack et al. (1991); Park and Redekopp (1992); Albarède and Monkewitz (1992); Triantafyllou (1992); Chiffaudel (1992); Albarède and Provansal (1995), and Monkewitz et al. (1996). Models by Albarède and Monkewitz, Triantafyllou, Monkewitz et al., and some others lead to results which explain the approximate validity of the 'cosine law'. However, this was not the primary purpose of introduction of these models.

The point is that according to available experimental data of a number of authors (e.g., of Williamson (1988a, 1989, 1992, 1995, 1996a, b); Ohle and Eckelmann (1992); König et al. (1992, 1993); Brede et al. (1994), and Miller and Williamson (1994)), wakes behnd circular cylinders at relatively low Reynolds numbers often have rather complicated spanwise structure. It was found, in particular, that at moderately subcritical values of Re spanwise cell structures frequently appear in such wakes, i.e., several spanwise regions with constant shedding frequency are formed which are separated by the so-called 'nodes' where the frequency changes discontinuously and vortex dislocation is observed. In the cases of 'perfectly symmetric' boundary conditions at the two ends of the cylinder and at large values of Re (and sometimes at relatively small Re but not too small values of x), symmetrical V-shaped (downstream-pointing) 'chevron' structures are often observed, i.e., the vortices on both sides of the cylidner midpiot have shedding angles of equal magnitude but opposite sign. The search for an explanation of these strange features of the observed wakes behind circular cylinders stimulated the introduction of the G–L model (4.50) and its investigation by Albarède et al. (1990); Albarède and Monkewitz (1992); Albarède and Provansal (1995), and Monkewitz et al. (1996) (see also Monkewitz's survey (1996)).

The G-L model can in principle describe the influence of the end conditions on the angle of obliquie shedding and explain the experimental result that the oblique shedding can be converted back into the parallel shedding by changing the flow configuration near the cylinder ends. However, to derive even qualitative conclusions from the G-L model, it is necessary first of all to determine the values of all the coefficients of Eq. (4.50). Since the complex coefficients ω and l have the same meaning here as in Eqs. (4.40) and (4.34–4.34a), it seems natural to make, as a first approximation, the assumption that these two coefficients of Eq. (4.50) do not depend on y, and have the same values as in the case of strictly parallel vortex shedding where oscillations are spanwise homogeneous. This simplifying assumption was accepted in the above-mentioned papers, where the empirical estimates of $\omega = \omega_1 + i\gamma$ and $l = \delta + i\delta$ quite close to the above estimates (4.48) and (4.49) were used. However the third coefficient $\mu = \mu_r + i\mu_1$ of Eq. (4.50) is a new one, and it can be determined only from data of measurements relating to the dependence of cylinder wakes on the spanwise end conditions.

Albarède and Monkewitz (1992) tried to use for this purpose the data for the dependence of Re_{cr} on the aspect ratio L/D of the cylinder generating the wake. If the oscillation amplitude A depends on y and satisfies Eq. (4.50), then the growth of A from the initial infinitesimal value will be described, not by the linearized Landau Eq. (4.32), but by the linearized G-L equation, which differs from Eq. (4.50) by the absence of the cubic term on the right-hand side. Also the measured rate of amplitude growth at $\mathrm{Re} > \mathrm{Re}_{cr}$ must evidently be equal, in this case, to the rate of growth of the most unstable spanwise-inhomogeneous mode. The normal modes are now given by the eigenfunctions of the linearized Eq. (4.50), which depend on the boundary conditions at $y = \pm L/2$. However, it seemed natural to assume that, at large values of L/D, the boundary conditions will not very essentially affect the rate of growth of normal modes. Therefore Albarède and Monkewitz used the simplest boundary conditions $A\ (y, t) = 0$ at $y = \pm L/2$, hoping that their use could hardly lead to very significant errors. The above arguments allow Re_{cr} to be determined approximately, as the smallest value of Re at which the imaginary part of at least one eigenvalue of the linearized G–L equation is not negative but equal to zero. Re_{cr} clearly depends on the aspect ratio L/D and of μ (recall that ω and l are assumed known); hence $\mathrm{Re}_{cr} = \mathrm{Re}_{cr}\ (L/D, \mu)$. Therefore, the measured values of Re_{cr} at various values of L/D may be used for estimation of the value of μ.

Albarède and Monkewitz at first attempted to use the results of the measurements by Mathis et al. of the values of Re_{cr} at a number of values of L/D but found that their data were insufficiently accurate eand complete. Therefore they carried out additional careful measurements of the values of Re_{cr} at various aspect ratios L/D and the results led them to the conculsion that $\mu_r/\nu = 32 \pm 6$. To find the imaginary part μ_i of the complex coefficient μ, two different methods were used by Albarède and Monkewtiz, both based on data for the angular frequency ω_1 of the most unstable mode at different values of Re and *L*/*D*. The two methods led to not-too-different results, and showed that apparently $(\mu_i/\mu_r) = -0.3 \pm 0.6$. Later Albarède and Provansal (1995) arranged a more careful determination of the values of the various coefficients of Eq. (4.50) (first of all of μ_r). They used somewhat modified boundary conditions, and carried out more complete and accurate measurements of the dependence of characteristics of steady cylinder wakes on Re and *L*/*D*. As a result they obtained the new estimate $\mu_r/v = 10 \pm 4$ for Re < 100, which differs considerably from the preceding estimate by Albarède and Monkewitz. (This great difference was apparently mainly due to the change of boundary conditions, which were found to be more important than it was assumed earlier). The value of μ_i was unimportant for the majority of applications considered by Albarède and Provansal; in rare cases where it was needed they used the estimate by Albarède and Monkewitz.

A quite different method of determining the values of μ_r and μ_i was used by Monkewitz et al. (1996). Here, special experiments were arranged in which nonsymmetric time-dependent boundary conditions were realized at the cylinder ends. The coefficients of the G-L model were then determined from both the steady shedding data (the only data used previously) and the data of measurements of the 'spanwise wave number shocks', i.e. abrupt increases in shedding angle across the span of a cylinder initiated by appropriate impulsive changes of ends conditions. The observed gradual reduction of the shedding angle θ along the Oy axis was then compared with predictions of the G-L model. Under the condition that the G-L model with coefficients independent of y is valid, this comparison allowed the values of μ_r/v and μ_i/μ_r to be determined with considerably greater accuracy than was achieved in the previous investigations. Monkewitz et al. published the results obtained for Re = 100, 120 and 140; the values of μ_i/μ_r proved to be practically independent of Re and close to -1, while all values of μ_r/v were found to be fairly close to 20, growing slightly with Re (from 18.7 at Re = 100 to 25.6 at Re = 140).

Albarède and Monkewitz (1992) found that their version of the G-L model describes, quite well, many phenomena observed in cylinder wakes in the laboratory. The model led to correct dependence of Re_{cr} on *L*/*D* and showed, in full agreement with the experimental data, that after the impulsive switching on of an external stream of constant velocity, vortex shedding always starts as the parallel mode while the regions of 'oblique shedding' develop from the cylinder ends and, in the case of symmetric end conditions, lead to steady-state 'chevron patterns'. The possibility of forcing the transition from the 'oblique' to 'parallel' vortex shedding by means of change of flow configuration at the cylinder ends can also be derived from the G-L model considered. Moreover, the plan views (in the (x, y)-plane) of cylinder wakes observed in flow visualizations agree well with results of model computations. Albarède and Provansal (1995) showed that their improvements of the previous version of the G-L model gives a theoretical explanation of a number of even more subtle features of wake development. In addition to this, Monkewitz et al. (1996) demonstrated that the samc G-L model satisfactorily describes many surprising non-steady wake phenomena which can be produced in laboratory experiments where non-symmetric, impulsive (i.e., time-dependent) spanwise boundary conditions are realized. Note however the remark by Leweke and Williamson (1998) indicating that the explanation of the loss of stability of a two-dimensional cylinder wake at supercritical values of Re proposed by Leweke and Provansal (1995), which was based on the G-L model, disagrees with some known properties of the observed cylinder-wake instability. On the other hand, while Landau's Eqs. (4.34) and (4.40) were derived from Navier–Stokes equations as long ago as the early 1960s by Stuart, Watson, and Eckhaus (and then more thoroughly by Fujimura (1989) and Duśek et al. (1994)), who used for this purpose definite asymptotic expansion procedures (see Sect. 4.21 above), apparently no rigorous derivation of this type has yet been given for the G-L Eq. (4.50) (the references of the G-L equations at the end of Sect. 4.22 concerned quite different flows and other equations of the Ginzburg-Landau type). Thus, the problem of the strict derivation of this equation and the accurate determination of conditions for its validity remains unsolved.

Equation (4.50) is the 'transverse' Ginzburg-Landau equation, taking into account the spanwise 'diffusion' of wake oscillations which often becomes apparent in laboratory experiments and numerical simulations. As to the spatial development of these oscillations, it was always neglected above, i.e., it was assumed that none of their characteristics depends on the streamwise coordinate x. This assumption was based mainly on the fact that, according to the available wake observations, the oscillation frequency f is practically the same within a large spatial region, as it must be in the case of a global instability mode. However visualisations of wake flows clearly show that some local characteristics of the oscillations vary considerably when coordinates of the observation point are changed. In particular, it will be explained below that the local oscillation amplitude at the point $(x, 0, 0)$ first gorws with the value of x but then reaches a maximum and begins to decrease when x increases further. Recall that when discussing the experiments by Sreenivasan et al. (1987) we noted (just above Eq. (4.48)) that the assumption about complete streamwise homogeneity of oscillations is just a convenient simplification, applicable only to regions of short streamwise extent.

To take into account the possible dependence of wake oscillations on the streamwise coordinate x one must use some new analytical models differing from the Landau and transverse Ginzburg-Landau models (4.40) and (4.50) by the presence of terms describing the streamwise variability of the flow characteristics. One of the simplest methods of accounting for the streamwise variability is to replace the Stuart–Landau Eq. (4.40) by the 'longitudinal' Ginzburd-Landau equation for the streamwise-dependent oscillation amplitude $A(x,\ t)$. The simplest version of this G-L equation includes, instead of the transverse-diffusion term of Eq. (4.50), a streamwise-diffusion term proportional to $\partial^2 A/\partial x^2$. Then the streamwise advection may be taken into account by the inclusion of the term $U\partial A/\partial x$ on the left-hand side of th G-L equation and/or by the replacement of the simple second derivative $\partial^2 A/\partial x^2$ on the right-hand side by $\partial^2 A/\partial \xi^2$, where $\xi = x - Ut$. As was indicated at the end of Sect. 4.22, the longitudinal G-L equation has definite theoretical grounds, and it has been repeatedly used in studies of the weakly nonlinear instability of plane-parallel and nearly plane-parallel flows. Some attempts to apply the longitudinal G-L equation to the study of plane wake flows were briefly considered by Park and Redekopp (1992) (in the initial part of their paper), Le Dizès et al. (1993, 1996) and

Hunt (1993). In additional, Park et al. (1993) used the longitudinal G-L equation for the quantitative analysis of control methods for a two-dimensional x-dependent global mode of circular-cylinder wake oscillations, and Xiao et al. (1998) briefly outlined a new application of the longitudinal G-L model to development of control methods regulating the value of the amplitude $A\ (x,\ t)$.

A more complete two-dimensional Ginzburg–Landau equation for anoscillation amplitude $A = A\ (x,\ y,\ t)$ dependent on two spatial coordinates was applied to wake flows by Park and Redekopp (1992) and Chiffaudel (1992), and Roussopoulos and Monkewitz (1996). Park and Redekopp considered the G–L equation of the form

$$\frac{\partial A}{\partial t} + U\frac{\partial A}{\partial x} = -i\omega A + \mu_1\frac{\partial^2 A}{\partial x^2} + \mu_2\frac{\partial^2 A}{\partial y^2} - \frac{1}{2}l|A|^2 A, \tag{4.51}$$

while Chiffaudel used a more complicated model equation which included also the third- and fourth-order derivatives of the amplitude. (The fourth-order G-L equation, containing amplitude derivatives up to the fourth order, was also studied by Raitt and Riecke (1995); however, this model will not be considered in the present book). Generally speaking all four coefficient ω, μ_1, μ_2 and l of Eq. (4.51) can be complex and dependent on the two coordinates x and y (and the real advection velocity U can also depend on x and y), but Park and Redekoppp restricted themselves to the model where $U =$ const., μ_1, μ_2 and l are complex constants while $\omega(x, y) = i\gamma(x, y)$ is purely imaginary (i.e., γ is real) and has the form $\omega(x, y) = i[c_0(y) - c_1(y)x]$ where c_0 and c_1 are real functions of one variable and $c_1(y) < 0$ at any y. Analyzing solutions of Eq. (4.51) in the region $0 \leq x < \infty, -L/2 \leq y \leq L/2$, under the boundary conditions $A(0, y, t) = 0, A(x, -L/2, t) = F_1(x, t), A(x, L/2, t) = F_2(y, t)$, and choosing reasonable values of constants μ_1, μ_2 and l and functions c_1, c_2, F_1 and F_2, the authors determined the $(x,\ y)$-region of the absolute instability of the wake flow considered, and showed that many observed features of the spatial and temporal development of circular-cylinder wake oscillations (e.g., the observed interrelation of parallel and oblique sheddings and formation of 'chevron patterns') can be explained if one assumes that oscillation amplitude satisfies Eq. (4.51). Roussopoulos and Monkewitz, who studied the feedback control of oblique vortex shedding for Reynolds numbers close to $\mathrm{Re}_{\mathrm{cr}}$ considered another model: they assumed that the oscillation amplitude $A_1\ (x,\ y,\ z,\ t)$ can be represented as a product $A\ (x,\ y,\ t)\ B(z)$ where $A\ (x,\ y,\ t)$ satisfies the G-L Eq. (4.51) in which $U = U\ (x)$ depends linearly on x, $\omega = \omega(x)$ is a complex function quadratic in x, and μ_1, μ_2 and l are complex constants. Then the authors used the results of the stability theory for circular-cylinder wakes and the data of wake oscillation measurements presented in Monkewitz's paper (1988b) to evaluate approximately all coefficients of Eq. (4.51). To apply the G-L amplitude equation to description of wake-oscilltion control methods, Roussopoulos and Monkewitz added to the right-hand side of Eq. (4.51) a function $F(x,\ y,\ t)$ representing the effect of the feedback control. Then solving numerically the obtained equation under the appropriate initial and boundary conditions and varying the values of $F(x,\ y,\ t)$ they could calculate the influence of various control actions on the wake oscillations and compare the calculation results with conclusion following from their laboratory measurements of control effects.

Another method of investigating the dependence of cylinder wake flows on streamwise coordinate x was used by Dušek et al. (1994) and Dušek (1996). Dušek et al. systematically studied the interrelation between the coupled nonlinear equations for the spatially-varying temporal Fourier components (corresponding to expansion of the disturbance velocity $u(\mathbf{x}, t)$ in powers of $e^{i\omega_1 t}$ where ω_1 is the oscillation frequency) and the local Landau equations for oscillation amplitudes A of the dominant harmonic at various points $\mathbf{x}$. They found, in particular, that for validity of the Landau equation the shape of the unstable mode must vary much more slowly than its amplitude. Then Dušek et al. considered the application of the results obtained to a cylinder wake flow, and compared the conclusions implied by direct numerical simulation of this flow, at Re slightly above the first Hopf bifurcation threshold $\mathrm{Re}_{\mathrm{cr}}$, with predictions based on approximate amplitude equations. Later Dušek (1996) used the results of the above-mentioend paper of 1994 to develop a numerical method for computing the spatially-varying temporal Fourier coefficients of velocity components in the cylinder wake. He evaluated the spatial structure of several terms of the Fourier series (the zeroth term describing the distortion of the primary steady flow by a disturbance, the first one which usually corresponds to the dominant harmonic, and a few subsequent terms describing higher harmonics) at two different supercritical values of Re, and showed that far downstream all harmonics behave like parallel traveling waves. Dušek also found that global characteristics of the dominant wave (its frequency, wavelength and phase velocity) agreed well with the experimental data of Williamson (1989). However, he did not try to compare the results of his computations with more complete experimental data for the spatial structure of the cylinder wake since very few such data were then available. Nevertheless, some experimental and numerically-simulated data on the spatial structure of two-dimensional wakes were obtained in the mid 1990s and these data, which will be considered below, agree in general with numerical results by Dušek et al. (1994) and Dušek (1996).

Let us begin with the paper by Goujon-Durand et al. (1994) who investigated the velocity oscillations at various spatial positions behind a spanwise homogeneous bluff body placed in a water tunnel. (In this paper a cylinder with the trapezoidal cross-section shown in Fig. 4.21a, and not a circular cylinder, was used for generation of the wake, but the general features of wake oscillations are similar in this case to those in a circular-cylinder wake). The authors measured the transverse flow velocity $w(\mathbf{x}, t)$ at a number of points $\mathbf{x}$ and numerous Reynolds number $\mathrm{Re} = U_0D/\nu$ (where D is the 'trapezoid thickness' indicated in Fig. 4.21a) ranging from $\mathrm{Re}_{\mathrm{cr}} \approx 58$ to $2\mathrm{Re}_{\mathrm{cr}}$. Instead of characterization the disturbance intensity by the value of the equilibrium amplitude A_e of velocity oscillations at a fixed spatial point $\mathbf{x}$, Goujon-Durand et al. measured the peak to peak amplitudes $A(x)$ at a number of points $(x,\ 0,\ 0)$ and then analyzed the values of the maximal amplitude $A_{\max} = \max_{x\,>\,0} A(x)$ and of the distance from the body, $x_{\max}$, at which the amplitude $A_{\max}$ was observed. They found that in the range of Reynolds numbers from $\mathrm{Re}_{\mathrm{cr}}$ to about $1.6\mathrm{Re}_{\mathrm{cr}}$, power laws of the form $\mathrm{A}_{\max} \propto (\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ and $x_{\max} \propto (\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})^{-1/2}$ are valid. In the same range of Reynolds numbers the local oscillation amplitude $A(x)$ satisfies the following similarity law: $A(x)/A_{\max} = \mathrm{F}(x/x_{\max})$ where $\mathrm{F}(\zeta)$ is an universal function which does not deped on Re.

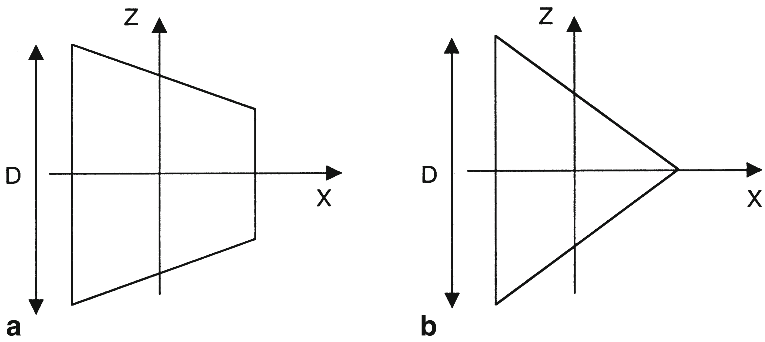


Fig. 4.21 **a** Trapezoidal cross-section of the cylinder used in experiments by Goujon-Durand et al. (1994) and Wesfreid et al. (1996). **b** Equilateral triangular cross-section of the cylinder used in the numerical simulations of a cylinder wake by Zielinska and Wesfreid (1995) and Wesfreid et al. (1996)

Simple similarity laws found by Goujon-Durand et al. for $A_{\max}$, $x_{\max}$ and $A(x)/A_{\max}$ had not yet received a theoretical explanation. Moreover, the relation $A_{\max} \propto (\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ seems strange, since it is known that the equilibrium amplitude A_e at a fixed point $\mathbf{x}$ is proportional to $(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})^{1/2}$ over a considerable range of positive values of $\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}$. Therefore, Zielinska and Wesfreid (1995) tried to verify these laws from the results of a numerical simulation of the purely two-dimensional wake behind a cylinder with a cross-section in the form of an equilateral triangle with the apex pointing upstream (see Fig. 4.21b). Their data were based on the analysis of numerical solutions of the two-dimensional Navier-Stokes equations describinb the flows in the (x, z)-plane around an impenetrable equilateral triangle; the solutions were computed for various values of $\mathrm{Re} = U_0 D/\nu$ (where D is the length of triangle sides, and U_0 is the velocity of the uniform flow upstream of the body). The solutions gave the values of vertical and horizontal velocity oscillations $w(x, z, t)$ and $u(x, z, t)$ and of the mean-flow distortion (i.e., of the zeroth harmonic $\Delta \mathrm{u}(x, z, t)$ of the streamwise disturbance velocity) at a number of the points (x, z) where x ran through a set of positive values, while z took two values, $z = 0$ and $z = 0.5\mathrm{D}$. (Note that the oscillations $w(x, z, t)$ and $u(x, z_1, t)$, where z can take arbitrary values but $z_1 \neq 0$, mainly represent the contributions of the dominant first harmonic with frequency ω_1, while the main contribution to the value of the streamwise velocity $u(x, 0, t)$ at the symmetry axis $z = 0$ is due to the second harmonic with doubled frequency $2\omega_1$; see, e.g., Stuart (1960); Hannemann and Oertel (1989); Dušek et al. (1994) and Dušek (1996)). Then the values of the peak-to-peak oscillation amplitudes A_{w}, A_{u} and $A_{\Delta u}$ of the two velocity components w and u, and of the mean flow distortion at the chosen points, were computed for a number of Re values. The results showed that the flow undergoes a Hopf bifurcation at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}} \approx 38$, which can be described by Landau's Eq. (4.40) with a coefficient ω which depends only on Re (and represents a linear function of $\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}$ at small values of $|\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}|$) and a coefficient l depending on $\mathbf{x}$. The maximum values of $A_{\mathrm{w,\,max}}$, $A_{\mathrm{u,\,max}}$ and $A_{\Delta\mathrm{u,\,max}}$ of the three amplitudes on the lines $z = 0$ and $z = 0.5D$ and streamwise coordinates $x_{\mathrm{w,\,max}}$ etc. of the points corresponding to these maximum amplitudes were also determined by Zielinska and Wesfreid.

Zielinska and Wesfreid then showed that the normalized streamwise and transverse velocity amplitudes $A_u(x)/A_{u,\max}$ and $A_w(x)/A_{w,\max}$ (where both the local and maximum amplitudes correspond to the wake oscillations at points with $z = 0$) are represented in the case considered by two different universal functions $F_u(x/x_{u,\max})$ and $F_w(x/x_{w,\max})$ of the normalized coordinate $x/x_{\max}$. These conclusions clearly agree with those of Goujon-Durand et al. (1994) for the transverse velocity oscillations in a slightly different but related wake flow. As to the distances $x_{\max}$ from the bluff body to the points where the oscillation amplitudes take maximal values, it was shown that the values of $x_{w,\max}$, corresponding to lines $z = 0$ and $z = 0.5D$, and of $x_{u,\max}$, corresponding to the line $z = 0.5D$, are proportional to $(\mathrm{Re} - \mathrm{Re}_{cr})^{-1/2}$ in the range of supercritical Reynolds numbers extending up to about $1.3\mathrm{Re}_{cr}$. This result agrees with the similar conclusion found by Goujon-Durand et al. by analysis of the experimental data. However, the values of $x_{\max}$ corresponding to oscillations of the streamwise velocity u, and of the mean-flow distortion $U_0 - u$ on the symmetry axis $z = 0$, which are unrelated to the dominant harmonic of the velocity field, depend on $\mathrm{Re} - \mathrm{Re}_{cr}$ in a more complicated manner which cannot be described by a simple power law. Moreover, according to numerical simulations of Zielinska and Wesfreid, the maximal oscillation amplitudes $A_{w,\max}$ at the axis $z = 0$ and $A_{u,\max}$ at the line $z = 0.5D$, which characterize the dominant first harmonic of the wake velocity, are both proportional to $(\mathrm{Re} - \mathrm{Re}_{cr})^{1/2}$ (and not to $(\mathrm{Re} - \mathrm{Re}_{cr})$, as Goujon-Duran et al. claimed) at $\mathrm{Re}_{cr} < \mathrm{Re} < 1.3\mathrm{Re}_{cr}$.

Since some of the results found by Goujon-Durand et al. (1994) and by Zielinska and Wesfreid (1995) contradicted to each other, it was decided to repeat the corresponding measurements and the analysis of the numerically-simulated data, to extend to span of the investigation and to improve its accuracy. Results of this new work were presented in the paper by Wesfreid et al. (1996). The new experiments used the same trapezoidal bluff body and water tunnel as before, but now a laser-Doppler anemometer was used to scan the values of the streamwise velocity $u(x, y, z, t)$ in the central part of the wake (near $y = 0$ where no variations of the oscillation frequency were fond) and the (x, z)-region extendind from $x = 0.7D$ to $x = 25D$ and from $z = 0$ to $z = 2.8D$. The time series of $u(\mathbf{x}, t)$ was fed to a spectrum analyzer to determine the frequency and amplitude of the dominant harmonic of velocity oscillations. The measurements covered the range of Reynolds numbers from $1.1\mathrm{Re}_{cr}$ to $1.6\mathrm{Re}_{cr}$, where this time it was found that $\mathrm{Re}_{cr} = 60.8$. The numerical simulation repeated the previous computations of two-dimensional wake oscillations behind a triangular cylinder with the cross-section shown in Fig. 4.21b. However, now the fluctuations $u(x, z, t)$ of the streamwise velocity was evaluated for the region $0.7D < x < 25D$, $0 < z < 2.75D$ of the (x, z)-plane, and the range of Reynolds numbers was from $\mathrm{Re} = 1.016\ \mathrm{Re}_{cr}$ to $\mathrm{Re} = 1.6\mathrm{Re}_{cr}$ (where $\mathrm{Re}_{cr} = 36.2$). The measured values of $u(x, y, z, t)$ and calculated values of $u(x, z, t)$ were both used to find the amplitude $A(x, z, \mathrm{Re})$ of the u-velocity oscillations at various points (x, z) and various values of Re. Then the maximal amplitude $A_{\max}(\mathrm{Re}) = \max_{x,z} A(x, z, \mathrm{Re})$ was determined for various values of Re and the Re-dependent point $(x_{\max}, z_{\max})$ was found where the amplitude $A_{\max}$

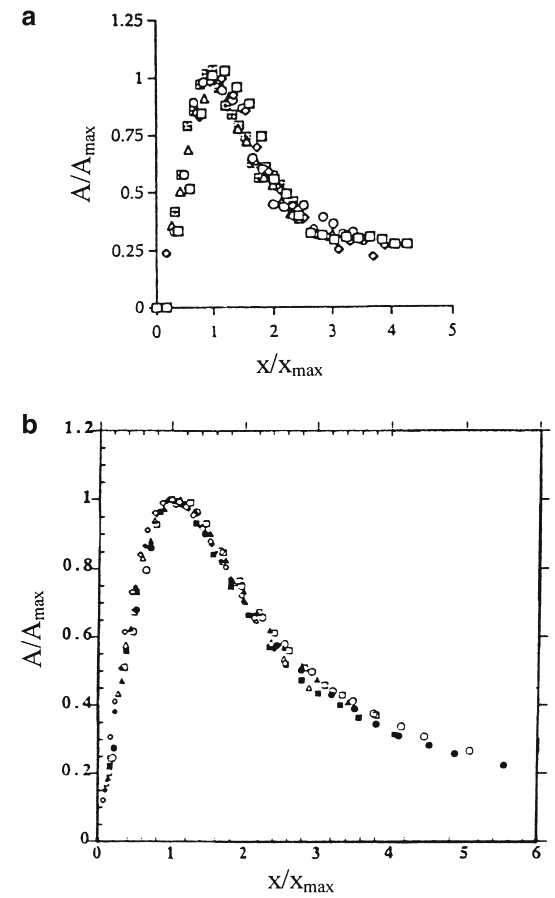


Fig. 4.22 Universal representation of the dependence of the normalized amplitude $A(x, z, \mathrm{Re})/A_{max}(z, \mathrm{Re})$ of cylinder-wake oscillations on the coordinate x. (After Wesfreid et al. (1996)) (**a**) Values of $A(x, z, \mathrm{Re})/A_{max}(z, \mathrm{Re})$ corresponding to velocities $u(x, y, z, t)$ measured in the wake behind a cylinder of trapezoidal cross-section at $y \approx 0$ and $z = z_{max} \approx 0.7\mathrm{D}$; (**b**) values of $A(x, z, \mathrm{Re})/A_{max}(z, \mathrm{Re})$ corresponding to numerically simulated velocities $u(x, z, t)$ in the two-dimensional wake behind a triangular cylinder at $z = 0.5\mathrm{D} < z_{max}$. The various *symbols* correspond to different values of Re in the ranges $1.21\mathrm{Re}_{cr} \leq \mathrm{Re} \leq 1.59\mathrm{Re}_{cr}$ (**a**) and $1.02\mathrm{Re}_{cr} \leq \mathrm{Re} \leq 1.31\mathrm{Re}_{cr}$ (**b**)

is reached. The experimental and numerical results had the same general character and both showed that, at given values of z and Re, the amplitude $A(x, z, \mathrm{Re})$ increases with x at small values of x, takes a maximal value $A_{max}(z, \mathrm{Re})$ at some point $x_{max}(z, \mathrm{Re})$ and then begins to decrease as x increases further. The values of $A_{max}(\mathrm{Re})$ and $A_{max}(z, \mathrm{Re})$ for $z > 0$ increase with Re in proportion to $(\mathrm{Re} - \mathrm{Re}_{cr})^{1/2}$ over a wide range of Re values (this conclusion agrees with results of Zielinska and Wesfreid (1995)), $x_{max}(\mathrm{Re}) \propto (\mathrm{Re} - \mathrm{Re}_{cr})^{-1/2}$ in the same range of Re values, but $z_{max}(\mathrm{Re})$ changes very little when Re is changing. Finally, according to both the experimental and the numerical data, the normalized amplitude values $A(x, z, \mathrm{Re})/A_{max}(z, \mathrm{Re})$ are represented rather accurately by universal functions of $x/x_{max}(z, \mathrm{Re})$, both for a fixed arbitrary value of z and $z = z_{max}$ where $A_{max}(z, \mathrm{Re}) = A_{max}(\mathrm{Re})$; see, e.g., Fig. 4.22. This result clearly extends the conclusions found earlier by Goujon-Durand et al. and Zielinska and Wesfreid.

Above we considered the wake flow behind a circular cylinder only in the restricted range of Reynolds numbers from $\mathrm{Re}_{cr} \approx 47$ up to about 100–170 or even less (see, e.g., Figs. 4.19–4.20). This was quite natural, since we were interested in the regime of wake oscillations which an be described by the simple Landau Eq. (4.40). Generated by a Hopf bifurcation at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$, the two-dimensional regime of parallel vortex shedding is often then transformed into a three-dimensional regime of oblique shedding by the influence of spanwise end conditions, but, as indicated above, one may prevent this transformation (and thus return to a two-dimensional wake regime) by some modification of the flow conditions at the cylinder ends. However, as Re increases, the wake flow inevitably acquires three-dimensional features. This circumstance was discovered rather early, in particular, by Roshko (1953, 1954); Hama (1956) and Bloor (1964) and was later studied and described (often together with descriptions of some subsequent wake bifurcations at still larger values of Re) in numerous sources dealing with either experimental or numerically-simulated data (see, e.g., Williamson (1988b, 1995, 1996a, b, c); König et al. (1990, 1993); Coutanceau and Defaye (1991); Karniadakis and Triantafyllou (1992); Tomboulides et al. (1992); Hammache and Gharib (1993); Roshko (1993); Norberg (1994); Mansy et al. (1994); Brede et al. (1994, 1996); Willams et al. (1995); Zhang et al. (1995); Mittal and Balachandar (1995a, b); Thompson et al. (1996); Wu et al. (1996a, 1966b); Zdravkovich (1997); Henderson (1997); Persillon and Braza (1998), and Leweke and Williamson (1998)). Results of different authors sometimes contradict each other in detail, but all show that at some $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}}$, in the range $150 < \mathrm{Re}_{2,\mathrm{cr}} < 200$, the regime of parallel vortex shedding becomes unstable with respect to some spanwise-periodic modes of disturbance, and transforms into a three-dimensional vortical regime. A number of the cited papers also include information about the appearance, at a Reynolds number of around 160 (clearly exceeding the threshold value for the primary instability of a two-dimensional wake), of the second three-dimensional unstable mode, which has smaller spanwise period and different symmetry properties. The existence of these two unstable modes was pointed out by Williamson (1988b, 1989) and was later confirmed in the experiments of Mansy et al. (1994); Williams et al. (1995); Brede et al. (1996); Wu et al. (1996a, b) and of some other researchers, and also in a number of direct numerical simulations of circular-cylinder wakes (e.g., those by Karniadakis and Triantafyllou (1992); Mittal and Balachandar (1995a, b); Zhang et al. (1995), and Thompson et al. (1996)). At present these modes are suaully referred to as modes A and B (see, e.g., the papers by Williamson (1996a, b, c); Henderson (1997) and Leweke and Williamson (1998) where the symmetry properties of these modes and the physical mechanisms of their instability are discussed in detail; in particular, Henderson also considered the Landau constants corresponding to development of the A and B modes). Both modes A and B oscillate with the same dominant frequency coinciding with the frequency of vortex shedding (i.e., they are periodic in time with period T equal to the shedding period). However, the simultaneous existence, at large enough values of Re, of two unstable modes makes transitions between them possible, producing discontinuities in the frequency of mode oscillations and the appearance of oscillations of double period $2T$ (or having even period mT of higher multiplicity); see, e.g., the general theory presented by Ioos and Joseph (1990) and the specific examples of period doubling of wake oscillations found by Tomboulides et al. (1992); Karniadakis and Triantafyllou (1992); Mittal and Balachandar (1995a) and Thompson et al. (1996).

The experimental and numerical-simulation data which illustrate wake transition to three-dimensional regimes of vortex shedding are characterized by considerable scatter in the observed values of the transition Reynolds number $Re_{2,cr}$. According to Roshko (1953, 1954) and Tritton (1959) $Re_{2,cr} = 150$ (though in the later survey by Roshko (1993) the higher estimate $Re_{2,cr} = 180$ was proposed), while Zhang et al. (1995) found that $Re_{2,cr} = 160$, Norberg (1994)–that $Re_{2,cr} = 165$, Williamson (1989)–that $Re_{2,cr} = 178$ (but in the survey of 1995 the latter author gave the much higher estimate $Re_{2cr} = 205$, and in the survey (1996a) he came to the conclusion that $Re_{2,cr} = 194$ is the best estimate). In parallel, Williamson (1996a, b) stated that the next transition, leading to the emergence of the mode B, takes place at $Re = Re_{3,cr}$ in the range between 230 and 260. The scatter of experimental values of $Re_{2,cr}$ and $Re_{3,cr}$ can be explained by the influence of free-stream turbulence, the difference btween between parallel and oblique vortex shedding, and/or the influence of the variability of end conditions (see Williamson, 1996a). Less scattered resutls are given by the careful theoretical investigations of the linear stability of parallel-shedding flows by Noack et al. (1993); Noack and Eckelmann (1994a, b); Barkley and Henderson (1996) and Henderson and Barkley (1996). These stability papers prove that at $Re = Re_{2,cr}$ lying between 170 and 190 the two-dimensional Kàrmàn street generated by paralle vortex shedding becomes unstable with respect to small three-dimensional disturbances with a spanwise wavelength equal to a few cylinder diameters. According to the most precise computations by Barkley and Henderson, $Re_{2,cr} = 188.5$ and the spanwise wavelength $\lambda_{y,\,cr} = 2\pi/k_{y,\,cr}$ of the mode A, the three-dimensional disturbance losing stability at this Re, is close to $4D$ (the authors suggested the even more precise estimate $\lambda_{y,\,cr} = 3.96D$). Barkley and Henderson computed neutral-stability curves in the (λ_y, Re)-plane, corresponding to neutrally-stable wave disturbances in a two-dimensional Kármán-street flow; these curves are shown in Fig. 4.23. The upper curve in this figure bounds the region of A-mode instability, while the lower curve represents the neutral-stability curve for the mode B of three-dimensional disturbances which becomes unstable at round $Re = Re_{3,cr} \approx 260$ and at this Re has the spanise wavelength $\lambda_{2,y,cr} \approx D$ (more precisely $Re_{3,cr} \approx 259$, $\lambda_{2,y,cr} \approx 0.82D$). The results of Barkley and Henderson for mode A agree, to high accuracy, with williamson's (1996b, c) laboratory measurements. As to the results of Barkley and Henderson for mode B, the validity of their comparison with experimental data may raise some doubts, since these results were obtained by application of the linear stability theory for two-dimensional wake flows to conditions in which the two-dimensional wake is always unstable and where nonlinear effects are inherent. Therefore the fact that the main features of the observed second instability mode do not deviate much from those given by the application of the linear stability theory to a two-dimensional primary flow may be considered as somewhat surprising. However, the agreement of the linear theory developed for the second unstable mode of three-dimensional disturbances in the two-dimensional wake with the experimental data for mode B was confirmed by many authors, and it

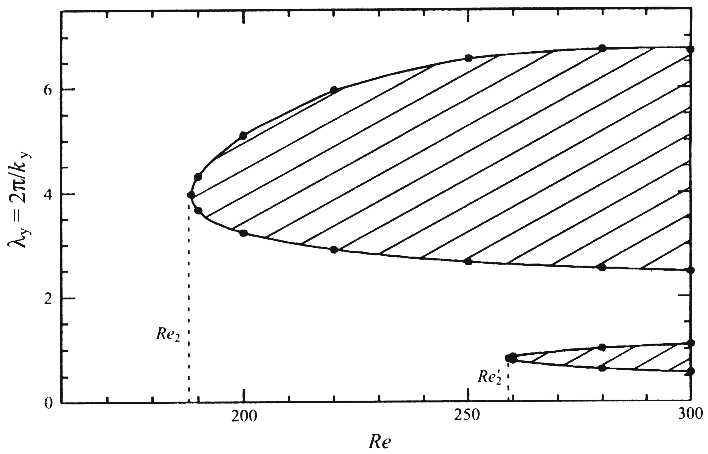


Fig. 4.23 The neutral-stability curves in the (λ_y, Re)-plane, where $\lambda_y = 2\pi/k_y$ is the spanwise wavelength (and U_0 and D are used as velocity and length units), which correspond to two types of neutral three-dimensional wave disturbances in the two-dimensional circular-cylinder wake. (After Barkley and Henderson (1996)). The upper shaded region corresponds to unstable A modes, while points of the lower shaded region correspond to unstable B modes; $\mathrm{Re}_2 \equiv \mathrm{Re}_{2,\mathrm{cr}} \approx 188.5$, $\mathrm{Re}_2' \equiv \mathrm{Re}_{3,\mathrm{cr}} \approx 260$

will be shown later that a similar situation occurs also in the study of wakes behind a square cylinder and a sphere.

The final transition to fully turbulent wake flow apparently takes place after several successive transformations, at higher and higher values of Re, into more and more asymmetric flow regimes. Breaking of symmetry properties leads not only to more complicated spatial patterns but also to increasingly complex dynamics, i.e., makes the flow more and more tangled (see, e.g., Crawford and Knobloch (1991); Dangelmayr and Knobloch (1991), and Hirschberg and Knobloch (1996)). Some of these further transformations possibly represent Hopf bifurcations which increase by one the number of degrees of freedom of the considered flow and may be described by modified Landau's equations.

Experimental data relating to circular-cylinder wake oscillations at very high Reynolds numbers showed quite early that here the standard definition of the Strouhal number does not allow a universal form of the St-Re relation to be obtained. Therefore Roshko (1961) concluded that at such values of Re the cylinder diameter D cannot be used as an appropriate length scale entering the definition of St; instead, he recommended using the wake thickness H as a length scale and changing the definition of the velocity scale (let us recall in this respect that just H was used as the length scale in the linear stability theory of wake flows considered Sect. 2.93). Later Bearman (1967) and Griffin (1981), trying to obtain the universal form of the St-Re relation, suggested some other choices of length and velocity scales to make the wake-oscillation frequency f dimensionless. Still later Adachi et al. (1966) measured, in a range $1.5 \times 10^4 < \mathrm{Re} < 10^7$ of Reynolds numbers $\mathrm{Re} = U_0 D/\nu$, the vortex-shedding frequencies f for eight rough circular cylinders of a fixed diameter D with surfaces covered by homogeneous roughnesses with the heights h of roughness elements satisfying the inequalitites $4.54 \times 10^{-6} < h/D < 2.5 \times 10^{-3}$. Then they calculated for all round frequencies f four different dimensionless combinations $\mathrm{St} = fL/V$ (differing by the used length and velocity scales L and V; the definitions of St proposed by Roshko, Bearman and Griffin were included in their list) and analyzed the dependence of the obtained values of St on h/D and Re. They found that at $h/D < 5 \times 10^{-4}$ the roughness of the cylinder does not affect the wake characteristics and that at such values of h/D the St-Re relation has the most universal form when Bearman's definition of St is used (such St preserves practically the same value in the whole studied range of Reynolds numbers).

4.2.4.3 Wakes Behind Non-Circular Cylinders and Rectangular Plates

Above, wakes behind circular cylinders were considered almost exclusively. The only exceptions were brief remarks about the two-dimensional wakes behind two particular non-circular cylinders: one which was studied in experiments by Goujon-Durand et al. (1994) and Wesfreid et al. (1996) and the other which was numerically simulated by Zielinska and Wesfreid (1995) and Wesfreid et al. (1996) (see Fig. 4.21a, b). In the remarks it was stated that these wakes are similar in many respects to the circular-cylinder wake and, exactly like the latter, undergo a Hopf bifurcation at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$ of the order of a few tens. Now these remarks will be supplemented by brief considerations of some other results relating to *wakes behind non-circular cylindrical bodies.*

Let us beign with the results of Jackson (1987), who calculated the points of onset of vortex shedding in flows past a whole collection of non-circular cylinders. He considered only purely two-dimensional wakes (i.e., the wake flows were assumed to be independent of spanwise coordinate y) and did not try to apply time-consuming direct numerical simulation to this problem. Instead, Jackson used a modification of the simple method of direct location of the Hopf-bifurcation points outlined by Griewank and Reddien (1983) (who in their turn relied on some ideas presented in the collection edited by Mittelmann and Weber (1980)). This method deals with dynamical systems described by systems of ordinary differential equations, and employs some general properties of bifurcating solutions at the Hopf-bifurcation points to compute the position of these points without solving the given equations and computing their eigenvalues.

In the case of flow around a cylindrical body the equations of motion depend on the parameter Re, and its threshold value above which the periodic solution exists is just the critical value $\mathrm{Re}_{\mathrm{cr}} = (U_0 D/\nu)_{\mathrm{cr}}$, symbolizing the emergence of a Hopf bifurcation. Jackson's method allows this value $\mathrm{Re}_{\mathrm{cr}}$ to be computed directly, together with the coordinate $i\omega_1$ of a point of the imaginary axis where, at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$, the spectrum of the Navier–Stokes eigenvalues crosses this axis indicating the appearance of flow instability. The value of ω_1 determined the shedding frequency $f_{\mathrm{cr}} = -\omega_1/2\pi$ and the Strouhal and Roshko numbers $\mathrm{St}_{\mathrm{cr}} = f_{\mathrm{cr}} D/U_0$ and $\mathrm{Ro}_{\mathrm{cr}} = f_{\mathrm{cr}} D^2/\nu = \mathrm{St}_{\mathrm{cr}} \mathrm{Re}_{\mathrm{cr}}$ at

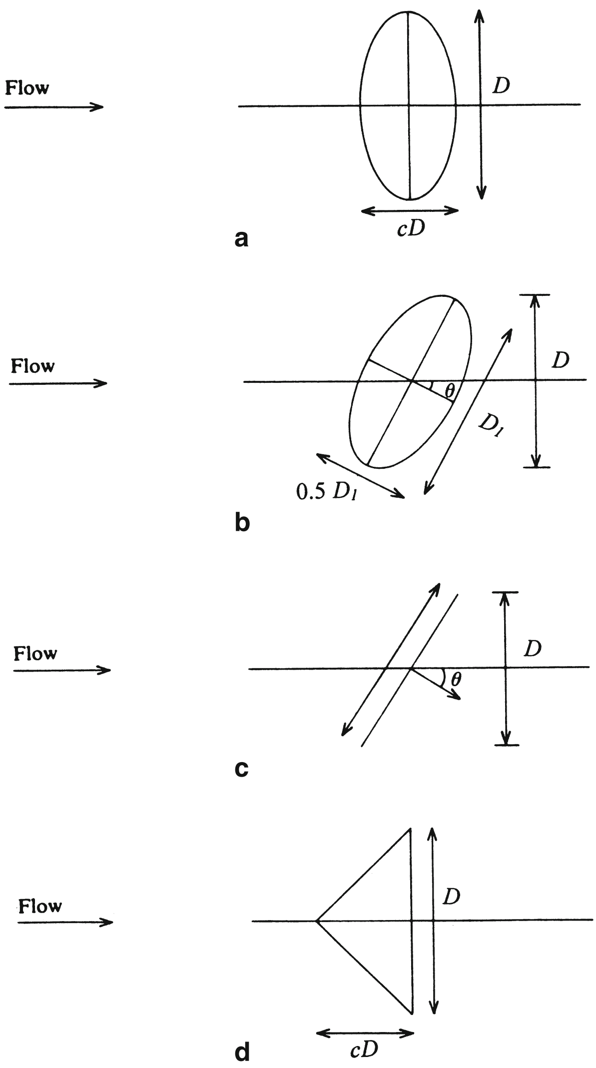


Fig. 4.24 Cross-sections of the non-linear cylinders for which Jackson (1987) determined the points of onset of the vortex shedding from the cylinder body. (**a**) ellipses oriented along the flow; (**b**) ellipses oriented at angles θ to the flow; (**c**) flat plates with normals at angles θ to the flow; (**d**) isosceles triangles with apexes directed upstream

$\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$. (Here U_0 is the constant velocity of the oncoming flow and D is the cross-stream 'thickness' of this body indicated in Fig. 4.24). Jackosn made calculations for cylindrical bodies with the following cross-sections: (a) ellipses with a principal axis of length D perpendicular to the flow and a principal axis of length cD along the flow direction, where c varies from 10^{-4} to 2; (b) ellipses with the major axis twice as long as the minor axis, oriented at various angles θ to the flow direction where $0° \leq \theta \leq 90°$; (c) straight segments of finite length at various orientations θ to the flow where $0° \leq \theta \leq 60°$ (here 'cylindrical bodies' turn into thin flat plates, at $\theta = 0$ such a plate does not differ in fact from ellipse (a) with $c = 10^{-4}$); (d) isosceles

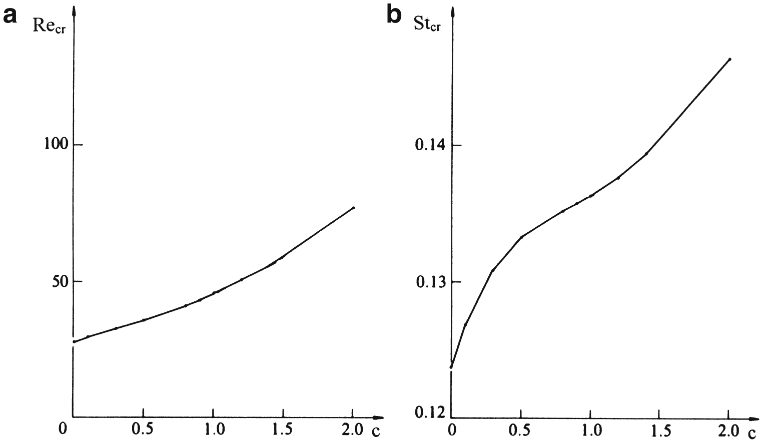


Fig. 4.25 Dependence of the critical Reynolds number Re_{cr} (**a**), and the critical Strouhal number St_{cr} (**b**) on the parameter c for wakes behind elliptic cylinders with cross-sections shown in Fig. 4.24(**a**). (After Jackson (1987))

triangles with base of lengh D perpendicular to the flow, the apex toward the flow and the height h of length cD where $0 \le c \le 2$ (see again Fig. 4.24). For all these bodies the values of Re_{cr} and St_{cr} were computed, and their dependence on the parameters c and θ was presented in the form of tables and graphs (as an example, Fig. 4.25 shows the graphs for elliptic cylinders (a)). It was noted that in the case of a circular cylinder (corresponding to the shape (a) with $c = 1$) the results agree well with those of the previous experiments and numerical simulations by various authors (cf. the similar remark on p. 110 where some references to earlier papers were given). The results relating to some other elliptic cylinders (shapes (a) with $c \neq 1$) were later verified by Morzyński and Thiele (1991, 1992) who used another numerical method and obtained the results close to those by Jackson. (Direct numerical simulation of flows past some elliptic cylinders were carried out, in particular, by Mittal (1994) and Mittal and Balachandar (1995b, 1996); here the values of St were determined for several supercritical values of Re sometimes also exceeding the threshold value $Re_{2,cr}$ for wake transition to three-dimensionality). For the case of the equilateral triangular cross-section Jackson found that $Re_{cr} \approx 35$; this estimate proved to be slightly lower than the estimate $Re_{cr} \approx 38$ found for this case by Zielinska and Wesfreid (1995) but it agreed somewhat better with the subsequent results by Wesfreid et al. (1996) according to which $Re_{cr} = 36.2$.

Jackson determined values of Re_{cr} and St_{cr} making use of the nonlinear bifurcation theory, but these results relate to the linear stability theory and hence they might as well have been discussed in Chap. 2. (However, in Chap. 2, as a rule, only results obtained in the framework of the parallel-flow approximation were considered while Jackson's and Morzyński and Thiele's computations dealt with the two dimensional but non-parallel model). Similar stability computations were performed

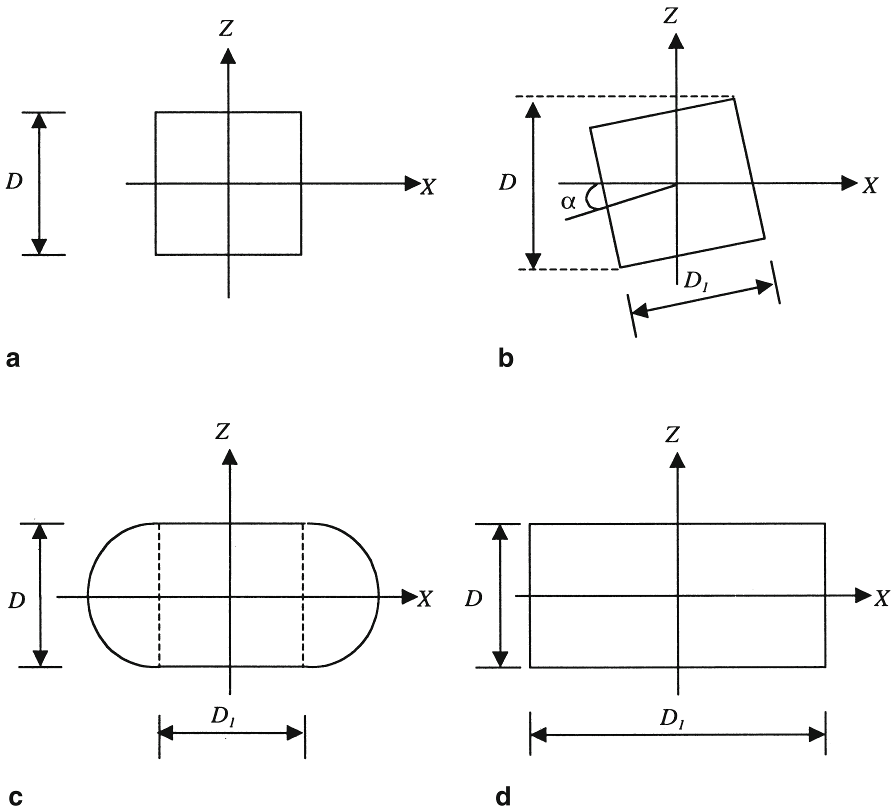


Fig. 4.26 Cross-sections of the square cylinder at zero incidence (**a**); of the square cylinder at nonzero incidence angle α (**b**); of the oblong cylinder (**c**); and of the rectangular cylinder (**d**)

by Kelkar and Patankar (1992) for the case of a flow around a *square cylinder* at zero incidence (i.e., with a plane face perpendicular to the stream; see Fig. 4.26a). These authors calculated the solutions of two-dimensional Navier–Stokes equations describing laminar steady flows around a square cylinder having constant velocity $\mathbf{U} = \{U_0, 0, 0\}$ far from this cylinder and corresponding to several moderate values of $\mathrm{Re} = U_0D/\nu$. Then the onset of unsteadiness (i.e., the emergence of a Hopf bifurcation) was determined by numerical solution of the linear stability problem for the computed laminar flows. Thus, the values of Re_{cr} and St_{cr}, corresponding to the beginning of vortex shedding, were found for the wake behind a square cylinder placed normal to an uniform flow (in particular, it was found that $\mathrm{Re}_{cr} = 53$). Values of $\mathrm{St} = \mathrm{St}(\mathrm{Re})$ computed by Kelkar and Patankar for values of Re close to Re_{cr} were compared with the results of Okajima's (1982) laboratory measurements of Strouhal numbers of the square-cylinder wake, and it was found that the numerical simulation leads to results which agree well with the experimental ones.

Table 4.1 Critical values Re_{cr} and Ro_{cr} corresponding to the start of vortex shedding from a square cylinder at incidence, versus incidence angle α. (After Sohankar et al. (1997, 1998))

α	0°	10°	20°	30°	45°
Re_{cr}	51.2	51.0	48.7	44.0	42.0
Ro_{cr}	5.9	6.2	6.1	5.4	5.2

Later Sohankar et al. (1997, 1998) and Sohankar (1998) carried out the numerical simulation of flows around square cylinders at variable incidence (with $0° \leq \alpha \leq 45°$ where α is the angle of incidence shown in Fig. 4.26b) at a number of values of Re and deduced the dependence on the angle α of $Re_{cr} = (U_0D/\nu)_{cr}$ and $Ro_{cr} = (fD^2/\nu)_{cr} = St_{cr}Re_{cr}$ (where $D = (\cos\alpha + \sin\alpha)\ D_1$ is the cross-stream 'thickness' of the cylinder and D_1 is the length of the square side; see again Fig. 4.26b). Found by them values of Re_{cr} and Ro_{cr} at $\alpha = 0°$ proved to be close enough to Kelkar and Patankar's results; they are presented in Table 4.1 together with the results for other values of α.

The method used by Sohankar et al. to obtain these results will be described at greaterlength a little later, but now we will return to some results of Schumm et al. (1994) which were omitted in discussion of this paper earlier in this section. The point is that the results relating to the vortex-shedding flow behind a circular cylinder (which were summarized in Eq. (4.49) and Fig. 4.19 above) were supplemented by Schumm et al. by results of similar experimental studies of wakes behind some non-circular cylinders. Namely, together with the case of a circular-cylinder wake, Schumm et al. investigated also the vortex shedding from an *oblong cylinder* without sharp corners, with the cross-section sketched in Fig. 4.26c (where $D = 0.69$ mm, $D_1 = 1.68$ mm), and two *rectangular cylinders* (thick plates parallel to the flow direction) with cross-sections of the shape shown in Fig. 4.26d. The wake behind a piezoceramic oblong cylinder with the same cross-section was first studied by Berger (1964, 1967) (see also Berger and Wille (1972)) who paid most attention to the influence of cylinder oscillations on the wake flow. Schumm et al. used the same cylinder and measured the oscillations of the transverse ('vertical') velocity $w(x, y, z, t)$ at the point $(x/D, y/D, z/D) = (10, 0, 1)$ of its wake at different values of $Re = U_0D/\nu$ and different stages of oscillation development. These measurements allowed the calculation, in exactly the same way as for a circular-cylinder wake, of the values of Re_{cr} and of all the coefficients in the corresponding Landau Eqs. (4.34) and (4.34a). It was found that here $Re_{cr} \approx 79.2$; $\gamma D^2/\nu \approx 0.116\ (Re - Re_{cr})$; $\omega_1 D^2/\nu \approx 58.1$; $\delta'/\delta \approx -1.85$. Moreover, it was also shown that here the growth rate γ and the oscillation frequency ω_1 have the same values at all points $(x/D, 0, z/D)$ with $z/D = 1$ and $10 \leq x/D \leq 40$. These facts confirm that at $Re = Re_{cr}$ a Hopf bifurcation occurs, leading to a global mode of oscillations satisfying the Landau equation (see the final paragraph of Sect. 2.93 and the beginning of the small-type text above).

Similar, but less complete, experimental results were obtained by Schumm et al. for the wake behind a *rectangular plate* with $D = 4$ mm, $D_1 = 60$ mm, and the spanwise length $L = 200$ mm. (The second rectangular plate used by the authors was noticeably larger, and its wake was studied only at greater values of Re which will not be considered here). It was shown that in the case of the first plate $\mathrm{Re}_{cr} = (U_0 D/\nu)_{cr} \approx 135$, and $\gamma D^2/\nu \approx 0.083$ ($\mathrm{Re} - \mathrm{Re}_{cr}$). The authors remarked that the coefficient 0.083 in the latter equation proved to be close to the value obtained by Hannenmann and Oertel (1989) for the numerically-simulated two-dimensional wake behind the rectangular plate (values of D and D_1 were especially chosen by Schumm et al. so that the ratio D_1/D had the same value 15 as in the numerical simulation). However, the value of Re_{cr} in the numerically-simulated wake was considerably smaller than the value found in the laboratory experiment. Schumm et al. assumed that this discrepancy can be due to deviation of the experimentally-produced wake from the idealized purely two-dimensional numerical model of Hannemann and Oertel.

For the case of *square cylinders at zero incidence* (where $D_1 = D_2 = D$) more detailed investigations of wakes at moderate Reynolds numbers were carried out by Sohankar et al. (1995, 1997, 1998, 1999) and Robichaux et al. (1999) (some parts of this work were also considered in detail in Robichaux's (1997) and Sohankar's (1998) theses). These authors based their work mainly on the analysis of DNS ('direct numerical simulation') data but Sohankar et al. also included in their papers experimental results of Norberg (partially presented in his paper of 1989), which were also compared with the data of Okajima (1982, 1995) and of a few other experimenters. It has been already stated above that Sohankar et al. (1997, 1998) used numerical simulations of flows around square cylinders at various angles of incidence α to determine the dependence of the critical values Re_{cr} and Ro_{cr} on the value of α (the results were presented in Table 4.1). Now a little more will be said about this work. Referring to Schumm et al. (1994) and Park (1994); Sohankar et al. stressed that the onset of flow oscillations, caused by a Hopf bifurcation, can be described by the Stuart-Landau Eq. (4.40) for the complex disturbance amplitude $A(t)$ (or, what is the same, by two real Eqs. (4.34) and (4.34a) for th real amplitude $|A|$ and phase ϕ). Following Park (1994), they chose the amplitude of the lift force on the cylinder to be the amplitude $A(t)$ (recall that the wake oscillations produced by vortex shedding are due to a global instability mode where the values of $\gamma(\mathrm{Re})$ and $\omega_1(\mathrm{Re})$ do not depend on the choice of amplitude A). Then they investigated the growth of $|A(t)|$ with t at various values of Re and α (in the ranges $45 < \mathrm{Re} < 200$, $0° \leq \alpha \leq 45°$) and determined the growth rate $\gamma(\mathrm{Re}, \alpha)$ (representable as $b(\alpha)[\mathrm{Re} - \mathrm{Re}_{cr}(\alpha)]$ at small and moderate values of $\mathrm{Re} - \mathrm{Re}_{cr}$) and the Landau constant $\delta(\mathrm{Re}, \alpha)$. Values of $\gamma(\mathrm{Re}, \alpha)$ were used to determine the function $\mathrm{Re}_{cr}(\alpha)$ while values $\mathrm{St}_{cr}(\alpha) = -\omega_{1,cr}(\alpha)D/2\pi U_0$ were found with the help of Eq. (4.34a).

Moreover, the wake-flow simulations and/or measurements at supercritical Reynolds numbers $\mathrm{Re} > \mathrm{Re}_{cr}$ allowed determination of the dependence of a number of a physical characteristics of vortex shedding (the Roshko and Strouhal numbers Ro and St are typical examples) on Re and α. In particular, it was shown by Sohankar et al. (1997, 1998) (who based their conclusion on the unpublished experimental data of Norberg, supplemented by some new DNS data) that at $\alpha = 0°$ (i.e., for square cylinders with one plane side facing the flow) the dependence of the Roshko number $\mathrm{Ro} = \mathrm{St} \times \mathrm{Re} = fD^2/\nu$ on $\mathrm{Re} = U_0 D/\nu$ at $\mathrm{Re}_{cr} < \mathrm{Re} < 200$ is described with reasonable accuracy by the Roshko law (4.47) (which agrees well with the Stuart–Landau

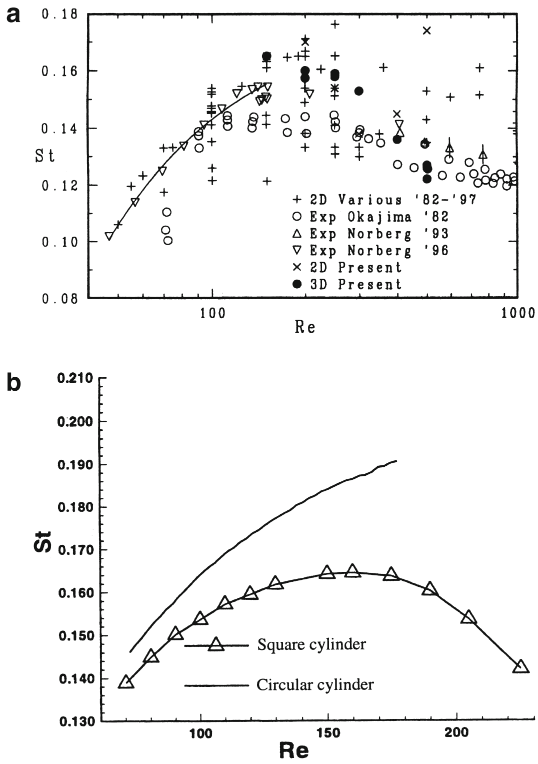


Fig. 4.27 The dependence of the Strouhal number $\mathrm{St} = fD/U_0$ on the Reynolds number $\mathrm{Re} = U_0D/\nu$ for wakes behind square cylinders at zero incidence according to various experimental and numerically-simulated data. (After Sohankar et al. (1997, 1999) and Robichaux et al. (1999)) (**a**) Summary graph by Sohankar et al. collecting various experimental (*Exp*) and numerically simulated (relating to a two-dimensional (*2D*) or a three-dimensional (*3D*) wake model) data. The solid line represents the empirical law (4.52): St = 0.18-3.7/Re. (**b**) Numerically simulated data (Robichaux et al. 1999) corresponding to a 3D wake model, and their comparison with the experimental data of Williamson (1996b) for circular-cylinder wake oscillations

Eq. (4.34) and (4.34a)). According to Sohankar et al. this law here as the form:

$$\mathrm{Ro} = 0.18\,\mathrm{Re} - 3.7 \tag{4.52}$$

(and hence $a = 0.18$, $a_1 = 3.7$ for wakes behind square cylinders at zero incidence). This conclusion was repeated for the indicated range of Reynolds numbers in the next paper by Sohankar et al. (1999); see Fig. 4.27a reproduced in the papers of 1997 and 1999. Then Robichaux et al. (1999) independently determined from their two-dimensional DNS data the St-Re relation for the square-cylinder wake in the range $70 < \mathrm{Re} < 230$; their result is presented in Fig. 4.27b together with the similar curve for the circular-cylinder wake (this curve is based on the results collected by Williamson (1995, 1996a, b, c) and it extends only slightly the 'universal St-Re relation' shown in Fig. 4.20). One can see that St-Re data in Fig. 4.27b relating to square-cylinder wakes are much less scattered than those in Fig. 4.27a, where data from a number of quite different sources (having different accuracy) sources are collected, but on the whole data of Robichaux et al. for $70 < \mathrm{Re} < 200$ do not disagree greatly with the results presented in Fig. 4.27a and with Eq. (4.52).

As to the comparison of St-Re relations for square—and circular-cylinder wakes, Robichaux et al. noted that the considerably smaller values of St, and their non-monotonic dependence on Re, in the case of a square cylinder may be explained by the fact that such a cylinder is a much bluffer body than the circular one. The blunt upstream face of the square cylinder, and its sharp edges, lead to flow separation and the formatin of recirculation regions on the top and bottom faces. These features lead to increase of the effective cross-stream thickness of the body. Therefore, the flow upstream of the cylinder actually sees a body with the increased 'effective thickness' $D^* > D$. Since $\mathrm{St} = fD/U_0$ contains the body thickness as a factor, the use of an underestimated value of the effective thickness leads to an underestimate of the prompted by physical arguments value of St and, since this underestimate increases with the growth of Re, it can lead to non-physical decrease of St as Re increases. Robichaux et al. introduced a plausible Re-dependent estimate of the 'effective thickness' D^* of the square cylinder and showed that replacement of D by D^* in the expression for St implies an St-Re relation for a square cylinder which does not differ much from the relation for a circular cylinder. It will be shown later that similar reasoning can be used to explain the form of the measured St-Re relation for wakes of flat plates parallel to the stream direction.

Let us stress, however, that the study of the St-Re relation for the square-cylinder wake in a limited range of moderate Reynolds numbers was not the main purpose of the papers by Sohankar et al. (1999) and Robichaux et al. (1999). Both groups of authors took into account the available results of investigations of circular-cylinder wakes, which showed that the simple two-dimensional wake transforms into a more complicated three-dimensional form at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}} \approx 190$, while at still greater Re the wake even contains two different three-dimensional modes, A and B, having specific symmetry properties (see the end of part (b) of this section). Therefore, they decided to check whether or not a similar transition to three-dimensionality takes place in the square-cylinder wake. With that end in view, Sohankar et al. collected and analyzed numerous results of measurements in air and water flows and of two- and three-dimensional (2D and 3D) direct numerical simulations of unsteady flows around a square cylinder at zaero incidence for a wide ragne of Reynolds numbers, $\mathrm{Re} = 150 - 1{,}000$ (see Fig. 4.27a). Note that the analyzed data included the experimental and 2D an 3D simulated results of the authors themselves at $\mathrm{Re} = 150 - 500$; this range also extends well above the circular-cylinder critical value of $\mathrm{Re}_{2,\mathrm{cr}}$. The data in Fig. 4.27a show that $\mathrm{Re} = 200$, which was the highest value of Re inspected in the papers of 1987 and 1988, is close to the upper bound of the Re-region where Eq. (4.52) is valid. At higher values of Re this equation is clearly incorrect, and there the results of 3D numerical simulations agree much better with the experimental data than the results of 2D simulations. (The incorrectness, at large values of Re, of the results of 2D simulations of flows around cylindrical bodies was also noted by Tamura et al. (1990)). The 3D numerical simulations performed by Sohankar et al. also showed that the two-dimensional square-cylinder-wake flow becomes unstable and undergoes transition to a three-dimensional form at some Re between 150 and 200. It was also shown that three-dimensional wake flow includes both the three-dimensional instability modes, A and B, which were observed in circular-cylinder wakes, and in a square-cylinder wake these modes have spatial structures similar to those of circular-cylinder modes A and B. There are, however, also some new features specific to square-cylinder wakes; e.g., at $\mathrm{Re} = 200 - 300$ in such wakes some low-frequency lift force pulsations were detected, which apparently do not exist in circular-cylinder wakes. At the same time the Strouhal numbers and mean drag values given by 3D numerical simulations were found to be in satisfactory agreement with experimental results (for St, the validity of this conclusion is seen in Fig. 4.27a).

Robichaux et al. (1999) performed only 2D numerical simulations of the square-cylinder wake and considered a restricted range of Reynolds numbers, $70 \leq \mathrm{Re} \leq 300$. However, they then applied to the simulated two-dimensional models of wake flows a three-dimensional linear theory of hydrodynamic stability of the same type as that used by Barkley ad Henderson (1996) on a 2D model of the circular-cylinder wake. That is, they investigated the stability of 2D wake flows to infinitesimal 3D disturbances depending periodically on the spanwise coordinate y. This investigation showed that at $\mathrm{Re} \equiv \mathrm{Re}_{2,\mathrm{cr}} \approx 160$ (more precisely, at some Re in the range 162 ± 12) the 2D square-cylinder wake becomes unstable to 3D disturbances with a spanwise wavelength (non-dimensionalized by the side length D) $\lambda_{\mathrm{y,\,cr}} \approx 5.22$. The corresponding three-dimensional unstable mode oscillates with a frequency equal to that of the vortex shedding and has a spatial structure similar to that of mode A of the circular-cylinder wake; therefore it was natural to call it mode A too. The second 3D unstable mode ('mode B'), with the same frequency as the first one and a spatial structure similar to that of mode B of the circular-cylinder wake, was also discovered in the square-cylinder wake by stability analysis of Robichaux et al.; it becomes unstable at a slightly greater Reynolds number $\mathrm{Re}_{3,\mathrm{cr}} \approx 190$ (more precisely, 190 ± 14) and has dimensionless spanwise wavelength $\lambda_{2,\mathrm{y,\,cr}} \approx 1.2$. Moreover, Robichaux et al. found that in the square-cylinder wake there also exists a third mode of unstable 3D disturbances (having specific spatial structure) which apparently does not exist in the wake of a circular cylinder; this mode (which was called 'the mode S' by the authors) becomes unstable at $\mathrm{Re} \equiv \mathrm{Re}_{4,\mathrm{cr}} \approx 200$ (more precisely, 200 ± 5), which differs very little from $\mathrm{Re}_{3,\mathrm{cr}}$, and has dimensionless wave length $\lambda_{3,\mathrm{y,\,cr}} = 2.8$ intermediate between $\lambda_{\mathrm{y,\,cr}}$ and $\lambda_{2,\mathrm{y,\,cr}}$. However, this new mode is subharmonic, with an oscillation period twice the shedding period of the primary two-dimensional state (and hence with half the shedding frequency). The discovery of modes A and B by the linear stability analysis of Robichaux et al. confirmed the corresponding results by Sohankar et al. found by a quite different method, specifically a fully-nonlinear three-dimensional DNS, while the discovery of the subharmonic mode S by Robichaux et al. had something in common with the discovery by Sohankar et al. of low-frequency oscillations of the DNS data. (The lack of complete coincidence of the results of two groups seems only natural since the methods used were too different; in particular, the 3D DNS results depend on the choice of the spanwise aspect ratio L/D, which took values of only 6 and 10 in the simulations of Sohankar et al., while the two-dimensional primary flow of the stability analysis correspond to $L/D = \infty$).

Let us now return to the *elongated rectangular cylinders* with $D_1/D = 15$ used in the wake studies of Hannemann and Oertel (1989) and Schumm et al. (1994).

These cylinders represent some examples of rectangular plates of finite thickness placed parallel to the flow direction. Other examples of such plates were considered by Nakayama et al. (1993) who performed 2D numerical simulations of the wakes behind plates, parallel to the flow, of thickness D with values of D_1/D varying from 3 to 10 for two Reynolds numbers $U_0D/\nu = 200$ and 400. For both values of Re it was found that the Strouhal number, $\mathrm{St} = fD/U_0$, varies when the value of D_1/D changes. More detailed numerical simulation of velocity oscillations in the wake behind a rectangular plate of finite thickness were performed by Hammond and Redekopp (1997) who used another idealized two-dimensional model of such a wake. Namely, these authors assumed that the plate, of thickness D, is semi-infinite (filling the volume $-\infty < x \leq 0$, $-\infty < y < \infty$, $-D/2 < z < D/2$) and that along opposite sides of this plate two independent plane-parallel streams are flowing in the Ox direction, with the same (nominally Blasius) velocity profile corresponding to given velocity U_0 outside the boundary layer. (The authors also investigated the case of an asymmetric wake where the limiting velocities U_1 and U_2 outside the upper and lower boundary layers differ from each other; however, we will not linger on the results of this case). Hammond and Redekopp studied the oscillations of the streamwise and transverse ('vertical') velocity components $u(x, z, t)$ and $w(x, z, t)$ at the point $(x/D, z/D) = (1, 0.5)$ and found that at not too large positive values of $\mathrm{Re} - \mathrm{Re}_{cr}$ (where again $\mathrm{Re} = U_0D/\nu$) the amplitudes of both these oscillations satisfy, with high accuracy, the same Landau Eqs. (4.34) and (4.34a) with the coefficients: $\gamma \mathrm{D}/U_0 \approx 0.0078(\mathrm{Re} - \mathrm{Re}_{cr})$ where $\mathrm{Re}_{cr} \approx 120$ (this value is greater than that found by Hannemann and Oertel and does not differ too much from experimental value of Schumm et al)., $\omega_1(\mathrm{Re}_{cr})D/U_0 \approx -0.61$, and $\delta'/\delta \approx -1.37$. It was also verified that the values of these coefficients were independent of position over a large region of the (x, z)-plane. Thus we see that this numerical simulation also confirms the fact that at $\mathrm{Re} = \mathrm{Re}_{cr}$ a Hopf bifurcation occurs in the flow behind a rectangular plate, and leads to the appearance of a global mode of oscillation with a complex amplitude $A(t)$ that satisfies the Landau Eq. (4.34–4.34a).

Plates of rectangular section, whose wakes were investigated by Hannemann and Oertel (1989); Nakayama et al. (1993); Schumm et al. (1994) and Hammond and Redekopp (1997), can be considered as models of an idealized infinitely thin flat plate parallel to the flow direction. It was indicated in Sect. 2.93 and recalled again on p. 108 of the present section that the laminar wake behind such a plate has the 'Gaussian' velocity profile of Eq. (2.89). The results of linear stability analysis were presented in Fig. 2.34, and from these the values of Re_{cr}, k_{cr} and $\omega_{1,cr}$ for the wake of a thin flat plate can be evaluated. However, selection of the most appropriate length and velocity scales is not a trivial matter in this case, since it is clear that the very small 'thickness' of the plate cannot be used now as a reasonable length scale. In Sect. 2.93 and Fig. 2.34 the half-width of the laminar wake was used as the length scale H (the increase of the width with x was neglected) and the difference between U_0 and the velocity at the laminar wake center-line was chosen as the velocity scale, but both these scales are irrelevant when wake behavior at supercritical Reynolds numbers $\mathrm{Re} > \mathrm{Re}_{cr}$ is considered. Therefore, when Eisenlohr and Eckelmann (1988) investigated, in a wind tunnel, the wakes behind eight different thin plates with blunt

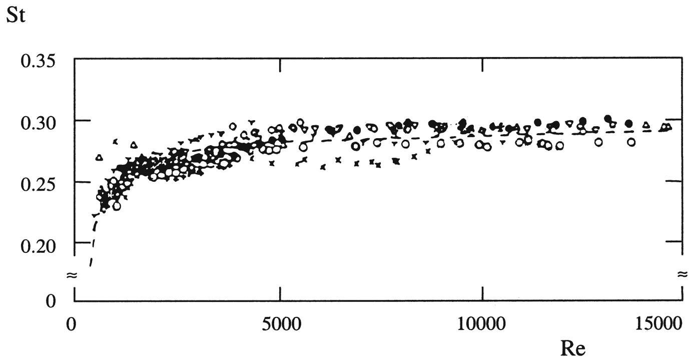


Fig. 4.28 The dependence of the Strouhal number $St = fD^+/U_0$ on the Reynolds number $Re = U_0D^+/\nu$, where $D^+ = D + 2\delta^*$, in the case of wake oscillations behind thin plates of thickness D. (After Eisenlohr and Eckelmann (1988)) the *dotted line* represents the empirical relation (4.52a): $St = 0.286 - 39.2/Re$. Different *symbols* (which are often superimposed on each other) correspond to different plates

trailing edges (and having a thickness D varying from 1 to 8 mm, with spanwise width L and streamwise length D_1 in the ranges from 280 to 500 mm and from 200 to 800 mm, respectively), they utilized quite different scales for reduction of wake characteristics to dimensionless form. Namely, they used the undisturbed velocity U_0 of the oncoming stream as the velocity scale while the sum $D^+ = D + 2\delta^*$, where δ^* is the displacement thickness of the upper or lower boundary layer near the trailing edge of the plate, was taken to be the length scale. (the length D^+, which was first introduced by Bauer (1961), evidently characterizes the real 'height' of a barrier restraining the flow. This length is similar in many respects to the length scale D^* used by Robichaux et al. (1999) for reduction of the great difference between the St-Re relations for circular-cylinder and square-cylinder wakes; see Fig. 4.27b above and explanations relating to it in the text). Eisenlohr and Eckelmann showed that this definition of the length scale leads to a universal value of the critical Reynolds number, $Re_{cr} = (U_0D^+/\nu)_{cr} \approx 140$, and to a universal form of the general *flat-plate Roshko law* (4.47):

$$Ro = 0.286\,Re - 39.2 \tag{4.52a}$$

(where $Ro = fD^{+2}/\nu$ and f is the frequency of wake oscillations) which was found to be valid with quite satisfactory accuracy for all the plates and all the considered (rather large) values of Re (see Fig. 4.28). Since the boundary-layer thickness δ^* grows with the stream length D_1 of the plate, one may try to use Eq. (4.52a) to explain of the dependence of the values of $St_D = fD/U_0$ at fixed $Re_D = U_0D/\nu$ on D_1/D found by Nakayama et al. (1993), but Nakayama et al. did not do this. However later Hammond and Redekopp (1997) recalculated some of their results, in which the plate thickness D had been used as the basic length scale, by including the displacement thickneses of the two boundary layers in the length scale. They found that, with this normalization, their numerically-simulated data led to dimensionless values of f which agreed quite satisfactorily (with a difference of about 4 %) with those observed by Eisenlohr and Eckelmann (1988) at the same values of Re (calculated using the new length scale). This agreement clearly provides additional validation of the results of both these papers.

As was explained above, the law (4.47) (whose particular cases include (4.52) and (4.52a)) is in fact a consequence of the Landauf-Stuart Eqs. (4.34–4.34a) and Eq. (4.46) Since in the case of a flat-plate boundary layer $\delta^* \propto (U_0)^{-1/2}$ (more precisely, $\delta^* \approx 1.73(\nu L_1/U_0)^{1/2}$; see, e.g., Eq. (1.56) in the book by Monin and Yaglom (1971)), Eq. (4.52a) explains the old observation by Taneda (1958), who discovered that in the wake behind a thin flat plate parallel to the flow the oscillation frequency f grows with the flow velocity U_0, not linearly (as in the case of a circular-cylinder wake where the linear relation between f and U_0, i.e. the constancy of St, was established, for a wide range of Reynolds number, by Strouhal (1878) and Rayleigh (1894) and at large values of Re follows from Eq. (4.47a)), but in proportion to $(U_0)^{3/2}$. In fact, when the thickness D of a plate is much smaller than the boundary-layer thickness δ^*, $D^+ = D + 2\delta^*$ is practically proportional to $(U_0)^{-1/2}$ while $\mathrm{Re} = U_0 D^+/\nu \propto (U_0)^{1/2}$. Then Eq. (4.52a) implies that $f \propto (U_0)^{3/2}$ at large values of U_0—this is just the result found by Taneda (1958) which was confirmed by the data by Eisenlohr and Eckelmann shown in Fig. 4.28. However, in flows around circular cylinders δ^* is much smaller than the cylinder diameter D; therefore here the boundary-layer thickness may be neglected and hence $f \propto U_0$ approximately.

Computations of Hannemann and Oertel, Nakayama et al., and Hammond and Redekopp, and also the measurements by Schumm et al. and Eisenlohr and Eckelmann, concern two-dimensional flat-plate wakes only. However Meiburg and Lasheras (1988) and Lasheras and Meiburg (1990) have demonstrated, both experimentally and by numerical simulations, that two different three-dimensional vorticity modes can be generated at moderate values of Re in the two-dimensional wake behind a thin flat plate, by introducing spanwise-varying disturbances in the flow near the trailing edge of the plate. The authors described the symmetry properties of these two modes, which later proved to be practically coincident with the symmetries of the modes A and B in circular-cylinder wakes, first discovered at approximately the same time (in particular, by Williamson (1988b)), but investigated in detail only later. According to Julien et al. (1997) both these modes can also occur in the undisturbed flat-plate wake (apparently at greater values of Re). Therefore, one may surmise that the evolution of the wake of a thin flat plate with increasing Re is similar to that of the wake behind a circular cylinder. Let us recall in this respect that the same similarity to circular-cylinder wakes was discovered by Sohankar et al. (1999) and Robichaux et al. (1999) for wakes behind square cylinders facing the flow.

A thin flat plate parallel to the flow direction corresponds to the special case of a rectangular cylinder with cross-section shown in Fig. 4.26d where $D \ll D_1$ (and hence it is possible to consider the limiting case where $D/D_1 \to 0$). Another interesting limiting case occurs when $D \gg D_1$; it corresponds to flows around long thin plates of finite width D placed in a uniform stream of velocity U_0 but this time normal to the stream direction. The two-dimensional vortex-shedding regime of the wake behind such a plate was briefly considered by Jackson (1987) (the case

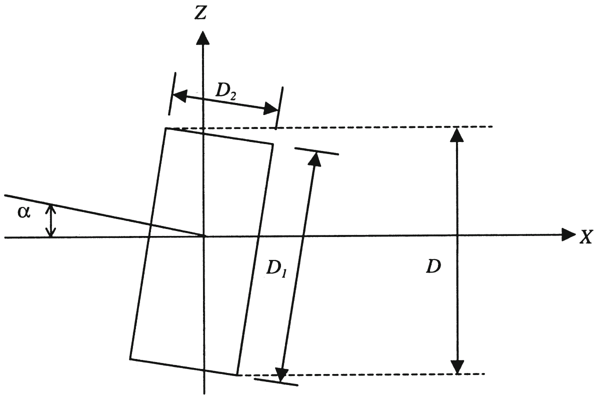


Fig. 4.29 General view of the cross-section of a rectangular cylinder at angle of incidence α

of the cross-section shown in Fig. 4.24c corresponding to $\theta = 0$); according to this computations the transition from a steady wake regime to an oscillating, vortex-shedding, regime occurs here at $\mathrm{Re} = \mathrm{Re}_{cr} \approx 27.77$ and the frequency of oscillations arising at this Re corresponds to a Strouhal number $\mathrm{St}_{cr} \approx 0.1237$. More detailed investigations of the normal-plate wake regime at higher values of Re were carried out by many researchers; here we will mention only Roshko's (1993) survey paper and the short announcement, and rather long subsequent paper, by Najjan and Balachandar (1996, 1998) devoted to discussion of the recent DNS results and also containing (in the paper of 1998) an extensive list of references relating to this subject.

The cited papers on square-cylinder and flat-plate wakes represent only a few examples of numerous studies of wakes behind square and non-square rectangular cylinders, placed along the spanwise axis Oy, in a uniform stream at different angles of attack α between 0 and 90° (see Fig. 4.29). Many characteristics of such wakes (in particular, frequencies of wake oscillations, fluctuating velocities at various points, and pressure, drag and lift forces) were measured by Okajima (1982); Okajima and Sugitani (1984); Knisely (1990), and Norberg (1993), among others, while papers by Davis and Moore (1982); Davis et al. (1984); Franke et al. (1990); Okajima (1990, 1995); Okajima et al. (1992); Li and Humphrey (1995); Sohankar et al. (1995, 1997, 1998, 1999), and some other authors concentrated mainly on analysis of numerical-simulation data but often included supplementary experimental results and cited many additional references. Below we will briefly consider only a small part of the material presented in the above list of papers, which is itself very far from being complete.

Franke et al. (1990) numerically simulated square-cylinder wakes at zero incidence and $40 < \mathrm{Re} < 300$, and compared the resulting St-Re relation with the experimental data of Okajima (1982) and the experimental and numerical data of Davis and Moore (1982) and Davis et al. (1984). They found relatively large discrepancies between the results, and came to the conclusion that there were apparently some significant uncertainties in both experiments and simulations. Knisely (1990) performed numerous measurements (both in a wind tunnel and a water channel) of characteristics of wakes behind square and non-square rectangular cylinders with side ratios D_2/D_1 ranging from 0.04 to 1 and with angles of attack α from 0 to 90° (the data for $\alpha = 0$ and 90° were naturally the most numerous) and supplemented his experimental results by an informative review of similar data from other researchers. In particular, Knisely presented many graphs showing the dependence of the Strouhal number $St = fD/U_0$ (where f is the frequency of wake oscillations, U_0 is the free-stream velocity, and $D = D_1 \cos\alpha + D_2 \sin\alpha$ is the apparent thickness of the rectangle seen from the front, as indicated in Fig. 4.29) on the angle of attack α, for wakes of cylinders with various D_2/D_1 (but Re was often not held constant in his experiments). Norberg (1993) measured, in a wind tunnel, the values of the Strouhal numbers St and pressure forces for wakes behind rectangular cylinders of high aspect ratio $L/D_1 > 50$ (where L is the spanwise length of a cylinder) having various side ratios D_2/D_1 (in the range from 1 to 5), and placed at various angles of attact α in streams corresponding to various Reynolds numbers $Re = U_0 D/\nu$. The values of St were first of all measured for the case where $\alpha = 0^o$ is fixed but the ratios D_2/D_1 and Reynolds numbers Re take various values. This allowed Norberg to determine the dependence of the number St on D_2/D_1 at different values of Re, and on Re (in the range $400 \leq Re \leq 3 \times 10^4$) at a number of values of D_2/D_1. Then the values of St were measured at various values of all three parameters Re, D_2/D_1 and α and the dependence of St on α was graphically presented at a number of values of Re and D_2/D_1. Li and Humphrey (1995) analyzed the numerically-simulated data on the St-Re relation for wakes behind square cylinders at various orientations and $100 < Re < 1{,}000$.

The examples of rectangular-cylinder-wake studies presented here should give a general idea of this extensive field of research, which is quite important in practice. The studies of the wakes behind non-circular and non-rectangular cylindrical bodies are much less numerous than those for the cases of circular and rectangular cylinders, and here only two typical examples of such studies will be mentioned. Eibeck (1990) compared, for $Re = U_0 D/\nu = 1.3 \times 10^5$, the data of circular-cylinder wake measurements with results of similar measurements behind a cylinder with the tapered cross-section having a circular (of diameter D) upstream part turning smoothly into a triangular downstream part with a sharp angle at the apex (so that the streamwise length D_1 of the considered cylindrical body was almost 2.5 times greater than its thickness D). He found that the vertical structures differed appreciably in two compared wakes. Breier and Gatzmanga (1995) measured, in a wide range of Reynolds numbers, the St-Re relations for wakes behind cylindrical bodies of rectangular, triangular, trapezoidal, and a more complicated combined cross-sections, trying to determine in which case St is practically independent on Re in the most wide range of Reynolds numbers. Their purpose was to find the cross-section guaranteeing that the wake-oscillation frequency is proportional to flow velocity U_0 in a wide ragne of velocities, and hence the velocity measurements can be replaced by more simple frequency measurements. (The utilization of wake-frequency measurements for determination of flow velocity was first suggested by Roshko (1953, 1954) and later was practiced on a large scale; see, e.g., the discussion of this subject by

Takamoto (1987)). Some recommendations relating to this matter are included in the Breier and Ganzmanga's paper."

4.2.4.4 Wakes Behind Tapered Cylinders and Circular Rings

Now we will turn to wakes behind bluff bodies nonhomogeneous in the 'spanwise' direction (in contract to 'spanwise homogeneous bodies' considered above). We will begin with the case of vortex shedding from *linearly tapered cylinders* of length L with diameters D_1 and $D_2 < D_1$ of two ends at the points with coordinates (0, 0, 0) and (0, L, 0) (if $D_2 = 0$, the cylinder clearly becomes a cone). As in the cases considered above, the axis Oy (directed along the cylinder or cone axis) is assumed to be orthogonal to the stream direction Ox, but now circular cross-sections of a cylinder have diameters diminishing linearly with y. The study of vortex shedding from tapered cylinders was initiated by two papers by Gaster (1969, 1971) who investigated the wakes behind such cylinders placed in a water tunnel at first (in the paper of 1969) for the cases of the taper ratios $R_T = L/(D_1 - D_2)$ equal to 36 and 18 and then (in 1971) for the case of a more mildly tapered cylinder with $R_T = 120$ (the wake behind a circular cylinder was also studied in the latter paper which has been already referred to above in this connection). Later, further measurements of vortex shedding from linearly tapered cylinders and cones, with different values of R_T (ranging from 13 to about 600) and $\phi = \tan^{-1} [(D_1 - D_2)/2L]$ were obtained, in particular, by Piccirillo (1990); Van Atta and Piccirillo (1990); Noack et al. (1991); Papangelou (1991, 1992), and Piccirillo and Van Atta (1993), while Jespersen and Levit (1991) carried out a numerical simulation of the flow past a tapered cylinder with $R_T = 100$.

In 1969 Gaster found that wake oscillations behind a tapered cylinder do not have one dominant frequency f but are characterized by a combination of two quite different main frequencies f_1 and $f_2 \ll f_1$ (the frequency f_2 modulates the high-frequency wake oscillations and depends only on $(U_0)^2/\nu$ but not on the body length scales). In the second paper (1971) his measurements at $R_T = 120$ showed that the wake oscillations have a definite cellular nature, i.e. are composed of spanwise cells with a constant dominant shedding frequency which changes from cell to cell. Later such cells were discovered in all the wakes of tapered cylinders and cones considered in the above-mentioned papers, whenever $\mathrm{Re}_{max} = U_0 D_1/\nu$ was not too large. It was found that the cells often have clear boundaries and quite definite dominant frequencies (see, e.g., a typical example shown in Fig. 4.30a). Recall that cellular structure was also found by many authors in the wakes of circular cylinders, but there the cells usually depended essentially on conditions at the cylinder ends, while in the cases of tapered cylinders no influence of the end conditions on the cell structure was found. (In this respect the cells behind tapered cylinders are similar to cells of circular-cylinder wakes in shear flows with undisturbed velocity $U_0 = U_0(z)$ having a constant velocity gradient dU_0/dz, studied, e.g., by Griffin (1985) and Woo et al. (1989)).

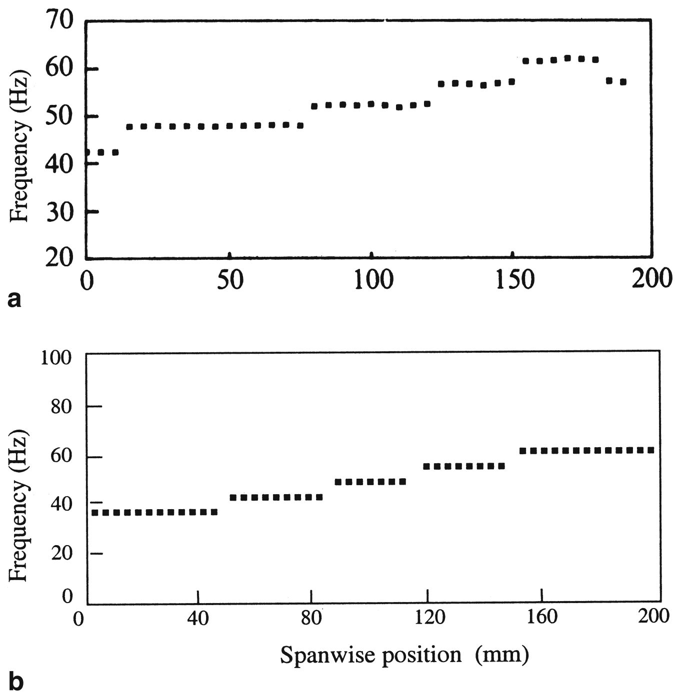


Fig. 4.30 (**a**) The dependence of the measured dominant frequency f of wake oscillations at points $\{x, 0, y\}$ (where $x \approx 15$ mm is fixed) behind the tapered cylinder on the spanwise coordinate y. (After Papangelou (1991, 1992)). The cylinder with the end diameters $D_1 = 2.57$ mm, $D_2 = 1.55$ mm and the length $L = 202$ mm was placed normal to the air flow of velocity U_0 such that $\mathrm{Re}_{max} = U_0 D_1/\nu = 123$, (**b**) Values of the frequencies $f(y)$ computed by the Ginzburg–Landau model Eq. (4.50) with the appropriately chosen coefficients. (After Papangelou (1991, 1992))

The dependence of the cell lengths and frequencies on the values of the taper ratio R_T and of the maximal and mean Reynolds numbers Re_{max} and $\mathrm{Re}_{mean} = U_0(D_1 + D_2)/2\nu$ was investigated carefully by Van Atta and Piccirillo (1990); Piccirillo and Van Atta (1993) and Papangelou (1991, 1992). It was found in particular that the difference in shedding frequencies between adjacent spanwise cells is a constat, coinciding with Gaster's modulation frequency f_2, and that the spanwise length of a cell divided by the cylinder diameter at the cell midpoint D_{cm} multiplied by R_T is also constant if $\mathrm{Re}_{cm} = U_0 D_{cm}/\nu > 100$. Piccirillo and Van Atta (1993) also found that the dependence of the cell Strouhal number $\mathrm{St}_c = f_c D_{cm}/U_0$, where f_c is the frequency of cell oscillations, on the cell Reynolds number $\mathrm{Re}_c = U_0 D_{cm}/\nu$ may be approximated with reasonable accuracy by the Rayleigh-Roshko law (4.47a) with constant coefficients $a \approx 0.195$ and $a_1 \approx 5.0$.

The Roshko law is a consequence of the Landau Eq. (4.34) and (4.34a) but now St_c and Re_c vary with the spanwise coordinate y. Therefore an analytic model describing wake oscillations behind tapered cylinders and cones must include the dependence on y in some way. The first, rather crude, model of this type was proposed by Gaster (1969) who described the wake oscillations $u(\mathbf{x}, t)$ by a system of coupled equations representing nonlinear van der Pol oscillators (with a coupling described by a spanwise-diffusion term proportional to $\partial^2 u/\partial y^2$) and corresponding to different spanwise coordinates y. This model was later refined by Noack et al. (1991), who applied their modification of Gaster's model to describe the cellular structure of wakes behind both untapered and tapered circular cylinders. However Papangelou (1992) found that the model of Noack et al. successfully describes only the appearance of spanwise cells, not their quantitative characteristics. Therefore he tried to utilize the Landau–Ginzburg model (4.50) for this purpose. The estimates (4.48) of complex coefficients $\omega = \omega_1 + i\gamma$ and $l = \delta + i\delta'$ given by Sreenivasan et al. (1987) were used in Papangelou's model, together with their estimate $\mathrm{Re}_{\mathrm{cr}} \approx 46$ of the critical Reynolds number (but now the values of D and $\mathrm{Re} = U_0 D/\nu$ were dependent on y) while the coefficient μ was assumed to be real and positive (contrary to the applications of Eq. (4.50) to modeling of wakes behind non-tapered cylinders described above, where μ was always assumed to be complex). Solutions of the corresponding Eq. (4.50) with various positive values of μ showed that this value may be chosen in such a way that the solution will satisfactorily describe many (though not all) quantitative features of the observed cell structure (see, e.g., Fig. 4.30b).

Tapered cylinders and cones with axes orthogonal to the stream direction represent only one special class of spanwise-inhomogeneous bluff bodies. Now we will turn to another class of such bodies, to whose wakes the G-L model (4.50) was also applied with definite success. Recall first of all that this model, supplemented by the appropriate boundary conditions at cylinder ends, allows a number of important characteristics of wakes behind circular cylinders to be calculated with satisfactory accuracy. However the experiments show that the flow regime of such a wake depends very substantially on the details of flow conditions near the cylinder ends, and this circumstance essentially complicates the determination of the boundary conditions which are 'appropriate' for a given experiment. Therefore as a rule, calculations based on the G-L model use some artificial boundary conditions selected by the requirement to produce results consistent with the available data. Because of this, Leweke et al. (1993a, b) and Leweke and Provansal (1994, 1995) applied the same model to the case where a cylinder of finite length was curved into a torus (a circular ring) so that no end conditions were needed.

Roshko (1953, 1954) was apparently one of the first researchers to study the *wake behind a circular* (*toroidal*) *ring* placed perpendicular to a uniform stream of velocity U_0. He showed that for $L/D \geq 10$ (where L is the ring outside diameter and D is its cross-section diameter) and for a not-too-small value of $\mathrm{Re} = U_0 D/\nu$, vortices are shed from a ring in almost the same way as from a straight cylinder, and form an annular vortex street. Later it was realized that frequency measurements in such wakes can be successfully used for the flow-velocity determinations (see the remark at the end of the previous part (c) relating to this matter) and this fact stimulated more detailed experimental investigations of wakes behind rings by Takamoto and Izumi (1981); Monson (1983); Takamoto (1987) and Bearman and Takamoto (1988) (in the two latter papers, wakes behind circular rings of trapezoidal cross-section were studied in detail, and wakes of some rings with rectangular and triangular cross-sections were also considered in passing), and finally by Leweke and his coworkers, whose studies of 1994 and 1995 of wakes behind toroidal rings contain the most interesting experimental data for the subject considered here. It was found in these works that the vertical structure behind a ring of circular shape can have a number of different forms: the wake can consist of an array of counter-rotating vortex rings parallel to the central plane of the toroidal solid ring, or of counter-rotating inclined vortex rings (i.e., shed at some angle θ with respect to the plane of symmetry of the torus), or of a pair of counter-rotating helical vortices (i.e., any inclined vortex after one 'round' connects to the next one) with discrete helix steps of $2\pi n/k$ (where n is an integer and k is a fixed streamwise wave number of wake oscillations), or of groups of interwoven helical vortices, and so on. Thus, a number of different normal modes can exist in the ring wakes. The number n (which can take either sign) also determines the dependence of the phase Φ of the wake velocity oscillations on the 'spanwise coordinate' $y = L\phi/2$ (where ϕ is the angular coordinate of the cylindrical coordinate system (x, r, ϕ) with the origin at the center of symmetry of the toroidal ring). Namely, as y increases from $y = 0$ (at an arbitrary point of the ring) to $y = \pi L = L_1$ (where L_1 is the length of the outer circle of the torus), the difference $\Phi(y) - \Phi(0)$ changes from zero to $2\pi n$. If $n = 0$, the vortex rings are parallel to the torus midplane and hence correspond to 'parallel shedding' with $\theta = 0$, while the wake structures with $n \neq 0$ are produced by 'oblique shedding', with shedding angle $\theta \neq 0$ depending in a definite way on n, D, L and k. Leweke and Provansal (1995) constructed graphs representing the St-Re relations (where again $\mathrm{St} = fD/U_0$, $\mathrm{Re} = U_0D/\nu$ and f is the frequency of wake oscillations) for various values of n and showed that Williamson's 'cosine law' of oblique shedding is valid here too, with high accuracy. (This means that if St is the Strouhal number corresponding to oblique shedding at angle θ, the $\mathrm{St_m} = \mathrm{St}/\cos\theta$ practically coincides with the value of St corresponding to parallel shedding, i.e. $n = 0$, at the same values of Re and L_1/D). Generally speaking, the values of St depend on three variables-the aspect ratio L_1/D, n and Re, if $n = 0$ the St-Re relation for the ring wake tends, as $L_1/D \to \infty$, to the straight-cylinder relation shown in Fig. 4.20 (see Fig. 4.31).

The experiments also show that in the wake of a ring, every mode of wake oscillations is characterized by its own critical Reynolds number $\mathrm{Re_{cr,\,n}}$, so that at $\mathrm{Re} < \mathrm{Re_{cr,\,n}}$ the nth-mode disturbances cannot exist at all (any such disturbance dies down to zero whatever the initial amplitude). The Reynolds numbers $\mathrm{Re_{cr,\,n}}$ are ring-wake equivalents of the critical Reynolds number $\mathrm{Re_{cr}} \approx 46$ characterizing the beginning of the vortex shedding in the circular-cylinder wake, but now the transition Reynolds number depends on the number n of the emerging vertical mode, and hence the whole family of integers n must be considered. Moreover, the nth mode is itself stable only for a definite Reynolds-number range $\mathrm{Re_{cr,n}} < \mathrm{Re} < \mathrm{Re^*_{cr,n}}$ while at $\mathrm{Re} > \mathrm{Re^*_{cr,n}}$ this mode becomes unstable to small disturbances and therefore transforms into a different, more complicated, vortical structure. (Reynolds number $\mathrm{Re^*_{cr,n}}$ is the nth-mode ring-wake equivalent of the Reynolds number $\mathrm{Re_{2,cr}} \approx 190$ characterizing the beginning of instability of the two-dimensional Bénard-Kármán vortex street produced by parallel vortex shedding; now it also depends on the mode

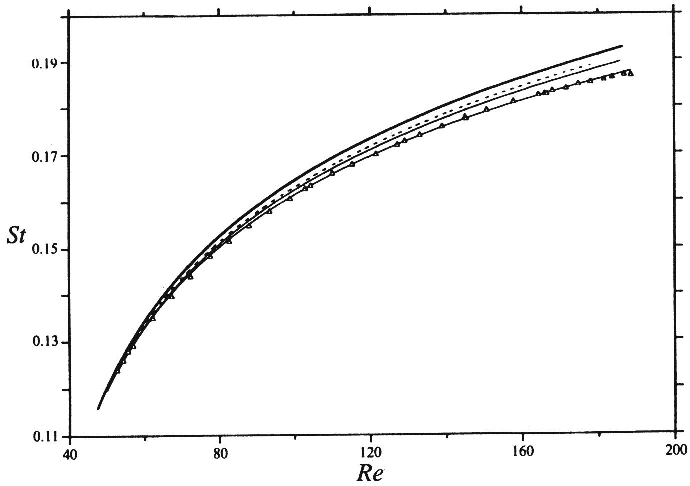
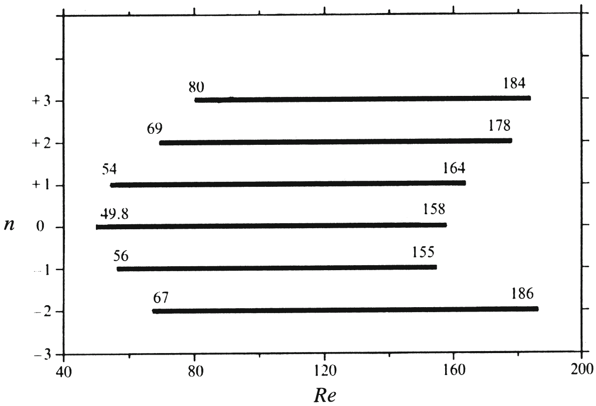


Fig. 4.31 Comparison of the St-Re curves for parallel vortex shedding (i.e., $n = 0$) from rings of different aspect ratios L_1/D with the curve for a straight cylinder: $L_1/D = 99.5$ (- - -), 59.0 (- - - - -), and 31.5 (- - Δ - -); straight cylinder (——) (After Leweke and Provansal (1995))

Fig. 4.32 Stability domains $\mathrm{Re}_{\mathrm{cr,n}} < \mathrm{Re} < Re^*_{\mathrm{cr,n}}$ of periodic-vortex-shedding modes with different values of n for the wake of a ring with aspect ratio $L_1/D = 59.0$. (After Leweke and Provansal (1995)). The numbers indicate the critical Reynolds numbers $\mathrm{Re}_{\mathrm{cr,n}}$ and $\mathrm{Re}^*_{cr,n}$



number n). The 'stability regions' $\mathrm{Re}_{\mathrm{cr,n}} < \mathrm{Re} < Re^*_{\mathrm{cr,n}}$ corresponding to various modes of ring-wake oscillations often overlap (see the typical Fig. 4.32 showing some experimental data of Leweke and Provansal (1995)). Therefore for many values of Re several normal modes are stable simultaneously. Apparently the initial conditions alone determine which mode will dominate the wake oscillations in such cases. It was also shown that the vortical structure of the wake depends significantly on the aspect ratio L_1/D. In particular, for aspect ratios smaller than about 20, the ring wake behaves similarly to the wake of a solid disk. On the other hand, for $20 < L_1/D < 100$ the ring curvature plays relatively minor role, and locally the wake has an appearance similar to that of the wake of a straight long cylinder. (However the minimal critical Reynolds number $\mathrm{Re}_{\mathrm{cr}} = \mathrm{Re}_{\mathrm{cr},0}$ depends here on the body-curvature parameter $K = D/L_1$ and increases nearly linearly with K; see again the paper by Leweke and Provansal (1995)). Referring to the similarity of the ring wake to that of straight cylinder, Leweke et al. (1993a, b) applied the G-L Eq. (4.50) to the wake of a ring, with the same numerical coefficients as were used successfully in the case of the wake of a straight circular cylinder. However, later Leweke and Provansal (1994, 1995) carried out a direct experimental determination of some coefficients of Eq. (4.50) for ring wakes.

Leweke and Provansal used the fact that the boundary conditions for the amplitude $A(y, t)$ of wake oscillations in the case of the wake of a ring have a very simple form: here evidently $0 \le y \le L_1$ and $A(0, t) = A(L_1, t)$ for any $t \ge 0$. It follows from this that the amplitude $A(y, t)$ can be represented as a sum of Fourier components of a form $A_{\mathrm{n}}(y, t) = B_{\mathrm{n}}\exp\{i[\Omega_{\mathrm{n}} t + Q_{\mathrm{n}} y]\}$, whre n takes integer values (only components with $|n| = 0$, 1, 2 and 3 were in fact detected in their experiments), $Q_{\mathrm{n}} = 2\pi n/L_1$, and the real amplitudes B_{n} and angular frequencies Ω_{n} can be determined from the G-L Eq. (4.50). In particular, the real and imaginary parts of Eq. (4.50) imply that the equilibrium values of amplitude B_{n} and angular frequency Ω_{n} (which do not depend on t) are given by the following expressions

$$B_n = \left[\frac{2(\gamma - \mu_r Q_n^2)}{\delta}\right]^{1/2}, \quad \Omega_n = -\left(\omega_1 + \gamma\frac{\delta'}{\delta}\right) - \left(\mu_i - \mu_r\frac{\delta'}{\delta}\right) Q_n^2 \qquad (4.53)$$

where, as usual, $\omega_1 + i\gamma = \omega$, $\delta + i\delta' = l$, and $\mu_r + i\mu_i = \mu$. (Equation (4.53) generalized the known equations determining the equilibrium amplitude A_{e} and Strouhal frequency $f = \Omega_0/2\pi$ which follow from Landau's Eqs. (4.34) and (4.34a) and correspond to parallel shedding where $\mu = \mu_r + i\mu_{\mathrm{I}} = 0$). Leweke and Provansal used for γ and δ'/δ the values $\gamma = 0.2(v/D^2)\,(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ and $\delta'/\delta = -3.0$ which were obtained earlier from data of circular-cylinder wake experiments; this means that the effect of the ring curvature was neglected (relying on measurements by the authors which show that this effect does not play an important part if D/L_1 is small enough; see, e.g., Fig. 4.31). However to find μ_{r} Leweke and Provansal used the equations

$$B_0^2 = \frac{0.4v}{\delta D^2}(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}), \quad \frac{B_n^2}{B_0^2} = 1 - \frac{4\pi^2\mu_r/v}{0.2(L_1/D)^2(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})}n^2 \qquad (4.54)$$

which follows from the first Eq. (4.53) and the expression for γ given above. Eq. (4.54) were verified by measurements of $(B_0)^2$ and $(B_n/B_0)^2$, where B_0 and B_{n} are the amplitudes of the zeroth and n th modes of the streamwise-velocity oscillations, in the wake behind a ring of outer diameter $L = 56.9$ mm and cross-sectional diameter $D = 3.03$ mm (so that the aspect raito $\pi L/D = L_1/D$ was 59.0). The oscillations of the streamwise velocity $u(\mathbf{x}, t)$ were measured at the point with coordinates (7D, 0, −2D) in a coordinate system with the origin at the ring center and the Ox axis pointing in the downstream direction. Measured values of the squared normalized amplitude $(B_0 D/v)^2$ of the zeroth oscillation mode at various values of $\mathrm{Re} = U_0 D/v$ are shown in Fig. 4.33; they confirm the proportionality of $(B_0)^2$ to $\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}$ over

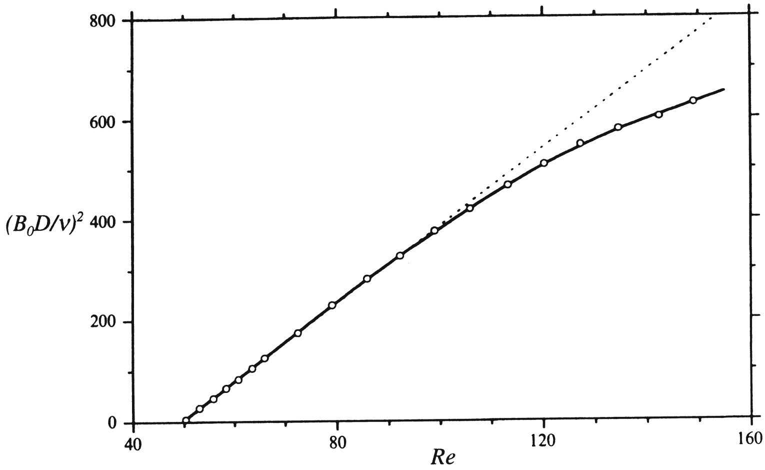


Fig. 4.33 Dependence of the normalized square of streamwise velocity fluctuations, $(B_0D/v)^2$, on $\mathrm{Re} = U_0D/v$ for the parallel shedding mode with $n = 0$ at the point $\{x, y, z\} = \{7D, 0, -2D\}$ behind a ring with aspect ratio $L_1/D = 59.0$. (After Leweke and Provansal (1995))

a considerable range of Re, and allow the critical Reynolds number Re_{cr} and the value of δ corresponding to streamwise-velocity oscillations at the measurement point to be estimated with good accuracy. The measured values of $(B_nB_0)^2$, where n took the values 1, 2, and 3, proved to be more scattered than the values of $(B_0)^2$, but on the whole they agreed with the second Eq. (4.54) and led to the conclusion that $\mu_r/v \approx 10$ over a range of not-too-high values of Re – Re_{cr}. Above $\mathrm{Re} = 100$, however, μ_r/v begins to increase with Re. Moreover, the second Eq. (4.53) allows $\mu_i - \mu_r(\delta'/\delta) = \mu_r[(\mu_i/\mu_r) - (\delta'/\delta)]$ to be determined from measurements of the difference of two angular frequencies $\Omega_n - \Omega_m$ (or of two ordinary frequencies $f_n - f_m = (\Omega_n - \Omega_m)/2\pi$) corresponding to two different oscillation modes of the ring wake. Leweke and Provansal (1994) measured the differences $f_n - f_m$ for a number of integer values of n and m and various values of Re, and deduced the dependence of $(\mu_r/v)[(\mu_i/\mu_r) - (\delta'/\delta)]$ on Re over a wide range of Reynolds numbers. The results were compared with estimates of $(\mu_r/v)[(\mu_i/\mu_r) - (\delta'/\delta)]$ from measurements of wake oscillations behind circular cylinders made by Williamson (1989) and Monkewitz, Williamson and Miller (whose results were known in 1994 but were published only in 1996). The comparison showed that the estimates derived from data on wake oscillations behind straight cylinders and behind rings agree rather satisfactorily with each other. Analyzing the data of both types Leweke and Provansal recommended in 1994, for a wide range of not too high supercritical values of Re, the estimate: $(\mu_i/\mu_r) - (\delta'/\delta) = 2.9 \pm 0.8$, but in 1995 they replaced this by two separate estimates: $\mu_i/\mu_r \approx -0.65, \delta'/\delta \approx 3.0$. Then they showed that the G–L Eq. (4.50) with the above values of coefficients describes, quite satisfactorily, the general development of wakes behind rings placed normal to the flow and also many observable features of such wakes. Let us recall, however, the remark by Leweke and

Williamson (1998), which has been already mentioned at the end of the discussion of the transverse Ginzburg-Landau Eq. (4.50). They commented that the application by Leweke and Provansal (1995) of the G-L model for the determination of the instability threshold for the wake flow implied a type of instability differing from that observed in laboratory experiments or numerical simulations of wake flows.

4.2.4.5 Wakes Behind Spheres and Other Axisymmetric Bodies

Wakes behind circular rings placed normal to the flow represent a special example of wakes behind axisymmetric bluff bodies. However in the above discussion of ring wakes, we emphasized first of all their similarity to wakes behind straight circular cylinders, paying only secondary attention to their axial symmetry. Now we will consider some other axisymmetric wakes, concentrating mainly on the consequence of axisymmetry.

Axisymmetry wakes appear behind any body of revolution submerged in an uniform stream directed along the body axis. Vortex shedding from the downstream parts of such bodies, and global oscillations of the resulting wakes, have been observed by many researchers. It was found that these features are related to the existence in the wakes behind axisymmetric bodies, in the cases when the Reynolds number Re is not too small, of zones of absolute instability with respect to non-axisymmetry disturbances with azimuthal wave number $n=1$ (see, e.g., Monkewitz (1988c)). Since the sphere is a prototype axisymmetric body, the *wakes behind spheres* are clearly the most significant axisymmetric wakes. Flows past spheres can be easily produced in the laboratory and are encountered in some engineering devices and natural phenomena; therefore sphere wakes began to attract attention very early and were studied quite extensively. In Sect. 2.2 it was mentioned that the dependence of the drag of a sphere submerged in a fluid flow on the Reynolds number Re was studied long ago by Eiffel (1912) and Prandtl (1914) (in fact there were also many other early studies of sphere drag); all these studies inevitably included the consideration of sphere wakes. The formation of vortices behind a sphere and vortex shedding from spheres were described in the 1930s in particular by Winny (1932); Foch and Chartier (1935), and Möller (1938), while later the vortical structures and quantitative characteristics of sphere wakes were studied by Taneda (1956, 1978); Torobin and Gauvin (1959); Magarvey and Bishop (1961a, b); Magarvey and MacLatchy (1965); Goldburg and Florsheim (1966); Zikmundova (1970); List and Hand (1971); Calvert (1972); Masliyah (1972); Achenbach (1972, 1974); Nakamura (1976); Pao and Kao (1977); Perry and Lim (1978); Kim and Durbin (1988); Sakamoto and Haniu (1990, 1995); Berger et al. (1990); Bonneton and Chomaz (1992); Wu and Faeth (1993); Provansal (1996); Provansal and Ormières (1998); Ormières et al. (1998); Ormières and Provansal (1999), and many other experimenters. Nevertheless experimental data for sphere wakes continue to be scattered and sometimes contradictory. The scatter can be explained by a number of factors complicating the wake measurements, such as the influence of the sphere supports, the effect of free-stream turbulence, and the weakness and slowness of wake oscillations at values of Re near the instability threshold. The influence of support devices can be diminished or even annulled by the use of spheres towed through, or freely falling or rising in, quiescent fluid but here some other complications often emerge. However the general features of sphere wakes are now known rather well, and many of them are quite similar to those of wakes behind circular cylinders.

The available data show that in the case of uniform external stream the flow around a sphere at low Reynolds numbers is steady, axisymmetric, and attached to the whole sphere body. At some greater value of Re, flow separation occurs and an axisymmetric, toroidal recirculation eddy, which is attached to the segmental area on the downstream side of the sphere, appears. According to experiments by Taneda (1956), the separation is first observed at $\mathrm{Re} = \mathrm{Re}_{0,\mathrm{cr}} \approx 24$ (where $\mathrm{Re} = U_0 D/\nu$ is based on the sphere diameter and free stream velocity). This estimate agrees with the results of some relatively early theoretical investigations of flows around a sphere, using either analytical or numerical approximations of the corresponding solutions of the Navier-Stokes equations (see, e.g., the summary of a number of such studies by Pruppacher et al. (1970) which implies that $\mathrm{Re}_{0,\mathrm{cr}} \approx 20$). There were also some experimenters who obtained different estimates of $\mathrm{Re}_{0,\mathrm{cr}}$ (e.g., Nakamura (1976) found that $\mathrm{Re}_{0,\mathrm{cr}} \approx 10$, and this estimate was also given by Wu and Faert (1993) who, however, made no measurements at so small value of Re). On the other hand, numerical simulations of flow past a sphere by Shirayama (1992), and the subsequent more careful and explicit simulations by Tomboulides (1993) (see also Tomboulides et al. (1993)) and Johnson (1996) (see also Johnson and Patel (1999)), which will be discussed at greater length later, confirmed the old estimates of $\mathrm{Re}_{0,\mathrm{cr}}$ given by Taneda and Pruppacher et al. (all of them show that $\mathrm{Re}_{0,\mathrm{cr}} \approx 20$). As Re increases further, the flow remains axisymmetric and steady, but the downstream extent of the recirculating wake zone, and the separation angle which determine the sphere segment adjoining to this zoen, progressively increase. The increase with Re of the streamwise length of the recirculating zone and of the separation angle were measured in Taneda's and Nakamura's experiments and were also determined from the numerically-simulated data by Pruppacher et al. (1970); Fornberg (1988); Shirayama (1992); Tomboulides (1993); Magnaudet et al. (1995), and Johnson (1996) who found that the numerical results agree quite well with each other and with the experimental ones (with the sole exception of Taneda's values of the length of recirculating zone at large values of Re, which were obviously underestimated; see, e.g., Fig. 4.34 and the above-mentioned papers by Shirayama, Tomboulides et al., and Johnson and Patel). The same is also true for values of the drag coefficient of a sphere which were also computed by Shirayama, Tombulides, and Johnson in a range of Reynolds numbers not too far above the critical value $\mathrm{Re}_{0,\,\mathrm{cr}}$; here again the computed values agree excellently with values given by Ross and Willmarth (1971) who accurately measured the sphere drag and compared their results to those of numerous previous drag studies. However, at some $\mathrm{Re} = \mathrm{Re}_{1,\mathrm{cr}}$ in the range between 100 and 350, the steady axisymmetric wake flow becomes unstable, and this leads to an abrupt change of the wake structure. At this value of Re a new wake regime emerges which, according to the results of many recent studies, is non-axisymmetric and steady, while in some older work it was found to be non-axisymmetric and oscillating (more will be said about this

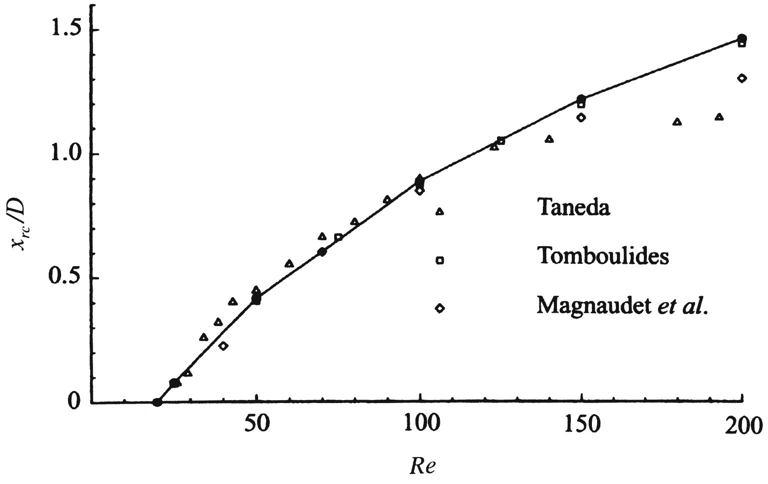


Fig. 4.34 Comparison of the dependence of the dimensionless streamwise length x_{rc}/D of the recirculation zone (measured on the wake axis) on Re given by Johnson and Patel's computations (the *solid line* with *black dots*) with results of Taneda's experiments and of earlier computations by Tomboulides, and Magnaudet et al. (After Johnson and Patel (1999))

below). The transition to an oscillating flow regime means that the periodic shedding of vortices begins at this Re, and signifies a Hopf bifurcation which may be described analytically by the complex Landau Eq. (4.40); while the replacement of one steady flow by another is a regular (non-Hopfian) bifurcation whose description does not require consideration of a complex amplitude equation.

In many cases, values of the critical Reynolds number $\mathrm{Re}_{1,\mathrm{cr}}$ (which for the sake of simplicity will often be denoted below by $\mathrm{Re}_{\mathrm{cr}}$) given by different experimenters disagree with each other. Recently Johnson and Patel (1999) stated that the observed onsets of the oscillatory shedding regime of a sphere wake covers the range $290 < \mathrm{Re}_{\mathrm{cr}} < 400$; however, if all the results indicated below were taken into account, then this range would be expanded to at least $130 \leq \mathrm{Re}_{\mathrm{cr}} < 400$. According to the experiments of Möller (1938), who towed a sphere through water, $170 < \mathrm{Re}_{\mathrm{cr}} < 200$. Later Taneda (1956) found that a weak oscillation with a long period appears in the sphere wake at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}} \approx 130$. A value of $\mathrm{Re}_{\mathrm{cr}}$ close to this was also found by Zikmundova (1970), who concluded from her observation of aluminum spheres dropped through the solutions of glycerol and water that $130 < \mathrm{Re}_{\mathrm{cr}} < 150$. Taneda's value of $\mathrm{Re}_{\mathrm{cr}}$ was accepted by some other authors (e.g., by Fornberg (1988)) but almost all recent data show that Taneda's and Zikmundova's estimates of this value were appreciably too low. (Note, however, that Provansal and his coworkers, whose work will be discussed at the end of this paragraph, found in the late 1990s that at $\mathrm{Re} \approx 150$ the sphere wake undergoes a bifurcation, but of a different type from that found by Taneda and Zikmundova). Magarvey and MacLatchy (1965), who made rather accurate observations of the wakes behind freely-falling solid spheres, found that the recirculation zone becomes unstable, and the wake begins to oscillate, only at $\mathrm{Re} \approx 300$. The start of wake oscillations at $\mathrm{Re} \approx 300$ (accompanied by an abrupt change of the vortical wake structures leading to the appearance of hairpin-shaped vortex loops) was later detected also by Levi (1980) and Sakamoto and Haniu (1990). Magarvey and Bishop (1961a, b) presented a number of photographs of wakes produced by liquid drops settling in a immiscible liquid; these photos show that the wake became non-axisymmetric at $\mathrm{Re} \approx 210$ but lost its steadiness only at $\mathrm{Re} \approx 270$. Goldburg and Florsheim (1966) also studied the wakes behind freely-falling solid spheres at moderate values of Re, and found that the dependence of the Strouhal numbers $\mathrm{St} = fD/U_0$ of wake oscillations on Reynolds number is described, with good accuracy, by the Rayleigh–Roshko Eq. (4.47a) with $a \approx 0.387$ and $a_1/a \approx 270$ over a considerable range of Re. As was shown above, Eq. (4.47a) follows from Landaus equation for the complex amplitude of wake oscillations. These values of coefficients a and a_1 show that the oscillatory wake regime was observed at $\mathrm{Re} > 270$, but Goldburg and Florsheim also noted that in their experiments the wake lost its axisymmetry at $\mathrm{Re} \approx 210$. Ross (1968) and Roos and Willmarth (1971) stated that their observations of spheres towed through water showed that $215 < \mathrm{Re}_{cr} < 290$. According to Nakamura's (1976) experiments with falling spheres, some change in the nature of the wake occurs at $\mathrm{Re} = 190$, but the change was not described in detail and therefore Kim and Pearlstein (1990) interpreted it as a transition to non-axisymmetric oscillating wake regime while Natarajan and Acrivos (1993) took it as the loss only of the axisymmetry, but not the steadiness, of the wake flow. Shirayama (1992) described some experiments according to which $\mathrm{Re}_{cr} \approx 250$, but he paid his main attention to flow simulation for $\mathrm{Re} = 500$. Then Wu and Faeth (1993) towed a polished plastic ball through a rectangular bath filled with quiescent water and glycerol mixture, visualized the flow near the towed sphere, and measured by laser velocimeter the mean streamwise velocities and root-mean-square velocity fluctuations at a number of points. Their measurements cover the range of Reynolds number $\mathrm{Re} = U_0D/\nu$ from 30 to 4,000, but for the topic discussed in this subsection the range $30 \leq \mathrm{Re} \leq 400$ represents the main interest. According to the results of these authors, the recirculation region on the downstream side of the sphere was steady and axisymmetric at $\mathrm{Re} < 200$, steady but non-axisymmetric at $200 < \mathrm{Re} < 280$, and unsteady with vortex shedding at $\mathrm{Re} > 280$. Still later the French researchers (Provansal (1996); Provansal and Ormières (1998); Ormières et al. (1998) and Ormières and Provansal (1999); see also beautiful photos presented by Leweke, et al. (1999)) used flow visualization in a water channel to observe the flow behind a fixed sphere (held by a thin upstream metallic pipe with three holes allowing to inject the dye into the water), and laser-Doppler and hotwire anemometers to measure velocities in a wind tunnel flow behind another sphere held inside the tunnel by four thin wires. According to their data, the sphere wake is steady and axisymmetric at $\mathrm{Re} < 150$, while at $\mathrm{Re} \approx 150$ its axisymmetry breaks and for $150 < \mathrm{Re} < 180$ the wake is non-axisymmetric but remains steady and has the vortical structure including a single linear vortical thread. At $\mathrm{Re} \approx 180$ this structure changes and becomes more complicated (begins to include a pair of vortical threads) but at $180 < \mathrm{Re} < 280$ the wake continues to be steady and non-axisymmetric. However, if Re grows further, then at $\mathrm{Re} = \mathrm{Re}_{cr} \approx 280$ the sphere wake begins to oscillate with a frequency f which

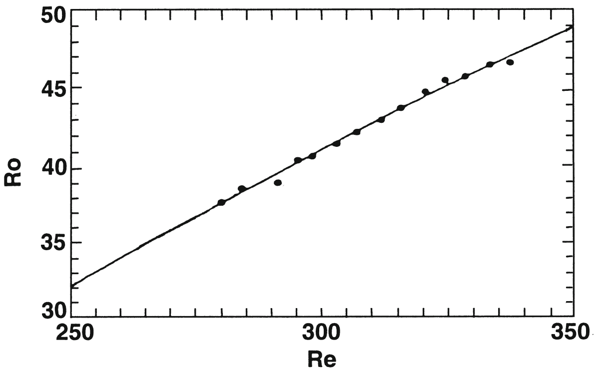


Fig. 4.35 Dependence of $\mathrm{Ro} = fD^2/\nu$ on $\mathrm{Re} = U_0D/\nu$ in the wake behind a sphere at supercritical Reynolds numbers $\mathrm{Re} \geq \mathrm{Re}_{\mathrm{cr}}$ according to measurements of Ormières et al. (1998)

does not depend on the point of observation, and corresponds to a Roshko number $\mathrm{Ro} = fD^2/\nu \approx 38$. Spectral analysis of streamwise velocity fluctuations was then used to measure the values of Ro in the Reynolds-number range $280 < \mathrm{Re} < 340$, corresponding to the periodic vortex-shedding regime. Data by Ormièdes et al. (see Fig. 4.35) show that in this range the Ro-Re relation can be approximated by the linear Eq. (4.47), while according to the 1998 and 1999 papers by Provansal and Ormières even higher precision can be reached if Eq. (4.47) will be replaced by the three-term equation $\mathrm{Ro} = a\mathrm{Re} - a_1 + a_2\mathrm{Re}^2$ where $a = 0.391$, $a_1 = -48.2$, and $a_2 = -3.6 \times 10^{-4}$ (recall that an equation of such form was earlier proposed by Williamson for the Ro-Re relation in the supercritical circular-cylinder wake; see the explanation relating to Fig. 4.20 in part (b) of this section).

The experimental results listed above (which clearly do not exhaust all the available results) must be supplemented by consideration of a few attempts to compute the value of $\mathrm{Re}_{\mathrm{cr}}$ by applying linear stability theory to the axisymmetric steady flow around a sphere. The first such attempt was due to Kawaguti (1955), but his results ($\mathrm{Re}_{1,\mathrm{cr}} = 51$, corresponding to instability of the steady sphere wake to unsteady axisymmetric disturbances) contradicts all other available results of stability computations (and also of experiments or simulations), and must therefore be disregarded. However the paper by Kim and Pearlstein (1990), whose results are apparently also incorrect, signified a more serious attack on the problem. Modifying Fornberg's (1988) approach, the authors computed a new the axisymmetric solution of the Navier–Stokes equations corresponding to the laminar flow past a sphere in a free stream with constant velocity $\mathrm{U}_0 = \{\mathbf{U}_0, 0, 0\}$. Then they investigated, in the framework fo the linear theory of hydrodynamic stability, the stability of this solution to infinitesimal disturbances proportional to $\exp[i(n\phi - \omega \mathrm{t})]$, where ϕ is the angular cylindrical coordinate, $n = 0, 1, 2..$, and possible values of ω are determined by the eigenvalue problem of linear stability theory. (Hence, both axisymmetric (azimuthal wave number $n = 0$) and non-axisymmetric ($n \neq 0$) disturbances were considered by the authors). The analysis showed that as $\mathrm{Re} = U_0D/\nu$ increases the disturbance which becomes unstable first of all has the azimuthal wave number $n = 1$. According to Kim and Pearlstein's computations, the instability of disturbances with $n = 1$ emerges at

$\mathrm{Re} = \mathrm{Re}_{cr} = 175.1$ and leads to a non-axisymmetric oscillating flow regime. As to disturbances with other values of n, the authors did not find instability for any of them in the whole investigated range of Reynolds numbers. Let us recall from Sect. 2.94 that the linear-stability-theory results of Batchelor and Gill (1962) and some other authors showed that the disturbances with $n = 1$ are the most unstable in a number of other axisymmetric jet and wake flows, and that Monkewitz's (1988c) results also indicated the paramount role of disturbances with $n = 1$ in formation of the global instability modes in axisymmetric spatially-developing flows.

Kim and Pearlstein compared their theoretical results with the results of previous experimental work and concluded that the agreement of their theory with the experimental data is more or less satisfactory. However later Natarajan and Acrivos (1993), who solved the same stability problem by a more advanced numerical method leading to different results, reconsidered Kim and Pearlstein's conclusion. The new authors applied to the computation of the solution of the equations of motion, describing the steady axisymmetric flow past a sphere, the numerical procedures developed for other purposes by Fornberg (1991) and Natarajan et al. (1993). This allowed them to describe the flow more explicitly than was possible earlier. Then Natarajan and Acrivos applied a new numerical method to solution of the linearized equations describing the evolution of small disturbances in the flow past a sphere. This method confirmed the result of Kim and Pearlstein, according to which the disturbances which become unstable at the smallest value of Re have the azimuthal wave number $n = 1$ (and instability to disturbances with $n \neq 1$ was again not found for any Re). However, the new computations showed that the unstable disturbance with $n = 1$ first appears at Reynolds number $\mathrm{Re}_{1,cr} \approx 210$, greater than was found by Kim and Pearlstein, and the disturbance differs qualitatively from the unstable disturbance of Kim and Pearlstein's theory. Namely, according to Natarajan and Acrivos the disturbance which becomes unstable at $\mathrm{Re} = \mathrm{Re}_{1,cr}$ is non-axisymmetric but also nonoscillatory, i.e., it corresponds to a purely imaginary eigenvalue $\omega = i\gamma$ with the imaginary part γ (determining the growth rate of the disturbance) proportional to $(\mathrm{Re} - \mathrm{Re}_{1,cr})$ which is negative for $\mathrm{Re} < \mathrm{Re}_{1,cr}$ but positive for $\mathrm{Re} > \mathrm{Re}_{1,cr}$ (see Fig. 4.37b below). Hence the critical Reynolds number $\mathrm{Re}_{1,cr}$ signifies a regular bifurcation (not of the Hopf type), the replacement of the axisymmetric steady flow by a new steady flow which includes a non-axisymmetric velocity mode with azimuthal wave number $n = 1$. This means that the transition of the axisymmetric wake regime to instability here proceeds through a steady state, corresponding to zero eigenvalue $\omega = 0$, i.e., it is of the same "exchange of stabilities" type which was encountered in this book when the instabilities of the Taylor-Couette flow between two rotating cylinders and of an immovable fluid layer heated from below were considered (see Sects. 2.6 and 2.7).

Natarajan and Acrivos computed, for the stability problem relating to small non-axisymmetric disturbances with $n = 1$, not only the eigenvalue $\omega = \omega_0$ with the greatest imaginary part γ_0 (reducing to zero at $\mathrm{Re} = \mathrm{Re}_{1,cr}$) but also a number of other complex eigenvalues $\omega_j = -\omega_{1,j} + i\gamma_j,\ j = 1,\ 2, \ldots$. Some of these eigenvalues are represented in Fig. 4.36 which shows that besides the eigenvalue ω_0 (which at $\mathrm{Re} = \mathrm{Re}_{1,cr}$ crosses the imaginary axis at the zero point), there is another eigenvalue (which will be temporarily denoted as ω_1) whose imaginary part

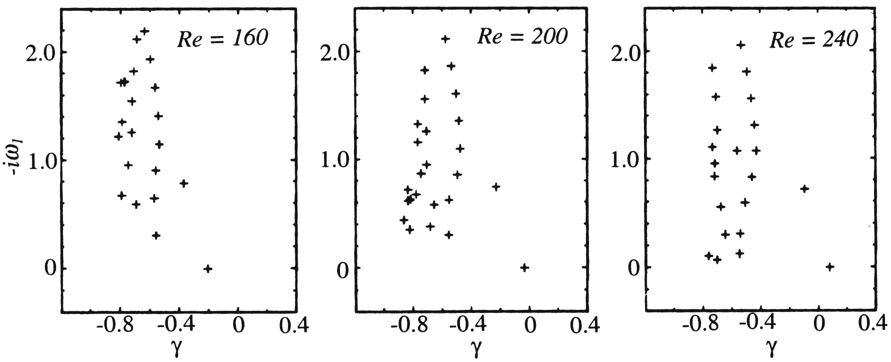


Fig. 4.36 Values of complex eigenvalues $i\omega = \gamma - i\omega_1$ corresponding to the eigenvalue problem of the linear stability theory for non-axisymmetric disturbances with $n = 1$ in the steady axisymmetric flow of velocity U_0 past a sphere of diameter D. The eigenvalues are made dimensionless by using U_0 and D as velocity and length units and are presented for three values of $\mathrm{Re} = U_0D/\nu$. (After Natarajan and Acrivos 1993)

also approaches zero (only slightly more slowly than that of ω_0) as Re increases. Accurate computations showed that the eigenvalue ω_1 crosses the imaginary axis at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}} \approx 277.5$. Figure 4.36 showed that ω_1 has nonzero real part; according to the computations, this eigenvalue crosses the imaginary axis at the point where $-\Re e\omega_1 = \omega_{1,1} \approx 0.710U_0/D$. Thus, Natarajan and Acrivos found that the axisymmetric steady sphere wake loses its axisymmetry (but not steadiness) and acquires the azimuthal wave number $n = 1$at $\mathrm{Re} \approx 210$, and at $\mathrm{Re} \approx 277.5$ the second unstable mode of disturbance, which is also non-axisymmetric with $n = 1$ but is unsteady, appears in the flow. This could mean that at $\mathrm{Re} > 277.5$ the flow preserves the azimuthal wave number $n = 1$ but begins to oscillate with the frequency $f = \omega_{1,1}/2\pi \approx 0.113U_0/D$. If so, then the wake transformation at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}}$ clearly represents a Hopf bifurcation produced by the emergence of periodic shedding of vortices from the sphere; the wake oscillations arising at this Re correspond to a Strouhal number $\mathrm{St}_{\mathrm{cr}} \approx 0.113$.

The cautious description (using the expression "could mean...") of the result relating to $\mathrm{Re}_{2,\mathrm{cr}}$ is due to the fact that the theory only shows that at $\mathrm{Re} > 277.5$ the axisymmetric flow past a sphere becomes unstable with respect to non-axisymmetric oscillatory disturbances. However, the theory also shows that at some lower value of Re axisymmetric flow becomes unstable to infinitesimal disturbances of another type. The situation here is quite similar to that in the case of the stability studies for the circular-cylinder wake performed in 1996 by Barkley and Henderson, and Henderson and Barkley. As was noted in part (b) of this section, the critical Reynolds number $\mathrm{Re}_{3,\mathrm{cr}} \approx 260$ (and the whole lower stability curve in Fig. 4.23) found by these authors was also obtained by application of the linear stability theory to an obviously-unstable primary flow. It was explained, however, that the resulting value of $\mathrm{Re}_{3,\mathrm{cr}}$ nevertheless agrees well with the experimental threshold for the appearance of the second unstable mode B. A similar situation apparently occurs in the case of the square-cylinder wake (see the discussion of the paper by Robichaux et al. (1999) in part (c) of this section). As will be indicated below, the value of $Re_{2,cr}$ determined by Natarajan and Acrivos by means of linear stability analysis also agrees well with the available experimental data.

Natarajan and Acrivos noted that the loss of axisymmetry of the sphere wake, at a smaller value of Re than that at which the wake becomes unsteady, was observed in experiments by Magarvey and Bishop (1961a, b) and Goldburg and Florsheim (1966). Moreover, the values of the two critical Reynolds numbers, indicating the thresholds of the two bifurcations, which were found by these authors, are quite close to the values of $Re_{1,cr}$ and $Re_{2,cr}$ given by Natarajan and Acrivos' stability calculations. The experimental results by Nakamura (1976) may also be considered as being in good agreement with the calculated results, if one assumes that the bifurcation observed by this author corresponds to the loss of wake axisymmetry but not steadiness. Thus, Natarajan and Acrivos concluded that the available experimental data are substantially more favorable to their results than to those of Kim and Pearlstein. Note however that, during the preparation of the paper of 1993, Natarajan and Acrivos did not know about the paper by Wu and Faeth (1993), which contains a very convincing experimental confirmation of their theoretical results[11]. In fact, the latter authors observed both bifurcations predicted by Natarajan and Acrivos and gave, quite independently, the estimates $Re_{1,cr} \approx 200$ and $Re_{2,cr} \approx 280$ for the two critical Reynolds numbers, which are very close to the values computed by Natarajan and Acrivos. The later experimental results of Provansal and coworkers (Provansal (1996); Provansal and Ormières (1998) Ormières et al. (1998) and Ormières and Provansal (1999)), and the results of the sphere-wake observations by Johnson and Patel (1999), also show that the sphere wake loses its axisymmetry at a smaller value of Re than its steadiness and begins to oscillate only at $Re \approx 280$. The results of Johnson and Patel in fact agree in many other details with Natarajan and Acrivos' theoretical predictions. These new discoveries increase considerably the cogency of the statement made by Natarajan and Acrivos, that the available experimental data agree much better with their results than with Kim and Pearlstein's stability computations.

Of course, Natarajan and Acrivos did not analyze all the available experimental data relating to sphere wakes which, as mentioned above, are rather scattered. Moreover, they also did not look for a possible error in Kim and Pearlstein's complicated and tedious computations, which could explain the difference between the conclusions of two papers devoted to the same problem. However, Natarajan and Acrivos indicated one more very important confirmation of their results: namely, they stressed that their results agree very well with the results of independent stability computations for three-dimensional flows past a sphere, by a completely different method, carried out by Tomboulides (1993) and Tomboulides et al. (1993) practically simultaneously with Natarajan and Acrivos' investigation.

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[11] Experimental results presented by Wu and Faeth (1993) are described at greater length in the thesis by Wu (1994). In addition Wu's thesis also contains descriptions of experimental studies of wakes behind spheres placed in a uniform stream where considerable velocity disturbances are presented; see in this respect also the papers by Wu and Faeth (1994, 1995).

Tomboulides, and Tomboulides et al., did not use the linear theory of hydrodynamic stability as in the papers discussed above but used a nonlinear approach based on direct numerical simulation (i.e., on numerical solution of the full nonlinear equations of fluid dynamics). First of all, they numerically solved the nonlinear equations describing the steady flow of a uniform stream with a constant velocity $\mathbf{U}_0 = \{U_0, 0, 0\}$ past a sphere of diameter D. The solutions found were axisymmetric, and the results for the Reynolds-number dependence of the streamwise length of the recirculation zone, the separation angle, and the sphere drag coefficient, discussed above, were computed from just these solutions. After this, at Reynolds numbers 200, 220, 250, 270, 285, and 300 a non-axisymmetric velocity disturbance with the azimuthal wave number $n = 1$ was added to initial conditions corresponding to the computed axisymmetric solution of equations of motion. The disturbance had the total energy equal to 10^{-8} of the energy of the axisymmetric flow and was randomly generated in such a way that its initial energy was distributed over all the eigenmodes with $n = 1$. Then the full equations of motion were solved numerically for the new initial conditions and the energy of all modes with $n = 1$ was traced in time.

It was found that the energy of the initial non-axisymmetric disturbance decayed in time, and the flow eventually returned to full axisymmetry, only at Re = 200. For all other inspected values of Re the energy of the disturbance grew and asymptoted to a finite constant value. This showed that $200 < \mathrm{Re}_{1,\mathrm{cr}} < 220$. The observed dependencies of the disturbance amplitudes $A(t)$ on time (exemplified in Fig. 4.37a) allowed easy detection of the regions of initial exponential decay or growth, yielding the initial growth rate (positive or negative) $\gamma = \Re\mathrm{m}\,\omega$ (where ω is the corresponding eigenvalue of the linear stability problem) of the least-stable mode with $n = 1$. Values of γ obtained in this way are shown in Fig. 4.37b, together with the same quantity computed by Natarajan and Acrivos from linear stability theory. One may see that the agreement between the results of linear and nonlinear computations is remarkable. Note also that the data shown in Fig. 4.37b agree well with the approximate equation $\gamma \approx b\,(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ which according to Landau's theory must be valid at small values of $|\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}|$.

As to the exact value of $\mathrm{Re}_{1,\mathrm{cr}}$, Natarajan and Acrivos found that $\mathrm{Re}_{1,\mathrm{cr}} = 210$ while a thorough investigation of this question by Tomboulides led to the conclusion that $\mathrm{Re}_{1,\mathrm{cr}} = 212$; the difference between these two estimates is clearly negligible. The above-mentioend experimental estimates by Magarvey and Bishop (1961a, b) ($\mathrm{Re}_{1,\mathrm{cr}} \approx 210$) and Wu and Faeth (1993) ($\mathrm{Re}_{1,\mathrm{cr}} \approx 200$) of the threshold Reynolds number signifying the transition to non-axisymmetric wake regime are also very close to the corresponding results of Tomboulides, and Tomboulides et al. The recent experiments of Provansal (1996); Provansal and Ormières (1998); Ormières et al. (1998), and Ormières and Provansal (1999), confirmed that the sphere wake loses its axial symmetry at a value of Re below the onset of wake oscillations, and implied an estimate of the oscillation threshold $\mathrm{Re}_{2,\mathrm{cr}}$ which is very close to that found by Natarajan and Acrivos and by Tomboulides. However this recent work led to results relating to the value of $\mathrm{Re}_{1,\mathrm{cr}}$ which deviate from the conclusions of Natarajan and Acrivos' linear and Tomboulides' nonlinear stability theory. In fact, according to Provansal and his coworkers there exist two different bifurcations, both leading

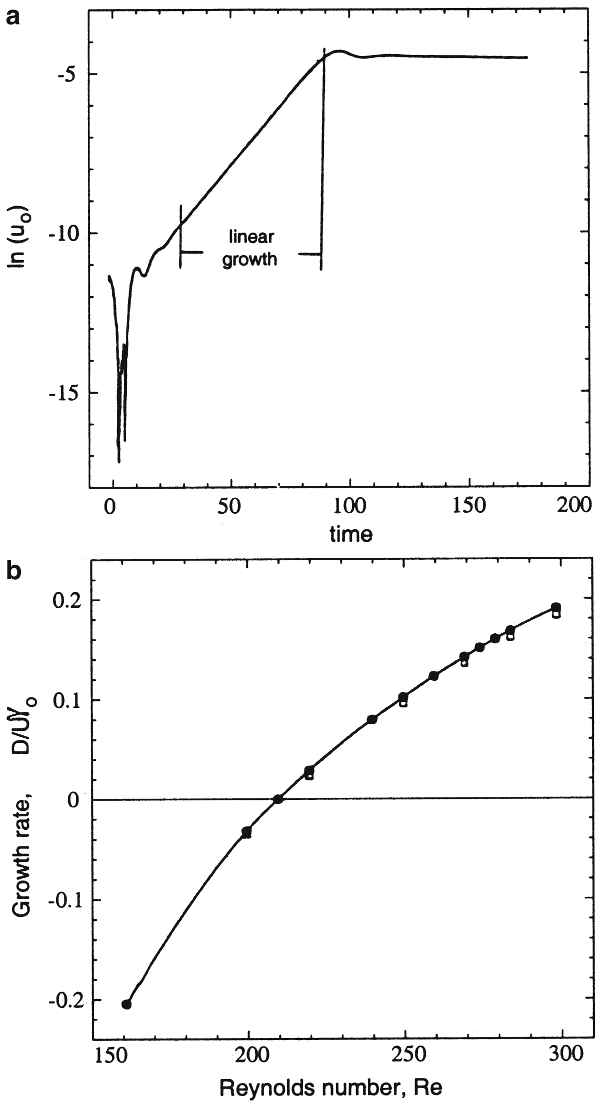

Fig. 4.37 **a** Dependence on time (measured in conventional units) of $ln(u_\phi)$ where u_ϕ is the non-dimensionalized azimuthal disturbance velocity at the point $\{x, r\} = \{D, 0.2D\}$ of the wake behind a sphere at Re = 250. (After Tomboulides 1993) **b** The dependence of the dimensionless growth rate $\gamma D/U_0$ of the least stable mode with $n = 1$ on Re in the wake behind a sphere. (After Tomboulides 1993 and Tomboulides et al. 1993). Circles: results following from the linear stability analysis of Natarajan and Acrivos, filled circles: values given by nonlinear direct numerical simulation of Tomboulides

to the emergence of some non-axisymmetric steady flow regimes, which occur at smaller Re than the value $Re_{1,cr} \approx 210$ at which the wake regime first becomes non-axisymmetric according to Natarajan and Acrivos' and Tomboulides' computations. However, only the second of the found by the French researchers non-axisymmetric regimes conforms to the non-axisymmetric steady wake regime predicted by the linear and nonlinear stability theories and the corresponding to it critical Reynolds number $Re \approx 180$ does not differ very much from the value $Re_{1,cr}$ implied by the above-mentioend stability studies. As to the another non-axisymmetric sphere-wake regime detected by the French experimenters at $150 < \mathrm{Re} < 180$, it apparently required further investigation. Without it this specific result (which may have been affected by the influence of the sphere supports) can hardly outweigh the available data supporting Natarajan and Acrivos' and Tomboulides' conclusions.

The linear-theory results of Natarajan and Acrivos determine only the character of the initial evolution of a very small disturbance in the sphere-wake flow. However, the results of Tomboulides' nonlinear analysis lead to asymptotic values $A\infty$ of the disturbances as $t \to \infty$ and hence to values of the Landau constants $\delta = 2\gamma/A_\infty^2$ (proved to be positive) at various Reynolds numbers and positions in the sphere wake. (Note that Landau's equation describing the emergence of sphere-wake oscillations was considered by Ormières et al. (1998) and Ormières and Provansal (1999) who, in particular, showed that the increase of the energy of the streamwise velocity fluctuations with the Reynolds number Re is linear at small supercritical values of Re, as it must be according to Landau's theory). It was also found by Tomboulides that the non-axisymmetric steady wake structure emerging at $\mathrm{Re} = \mathrm{Re}_{1,\mathrm{cr}}$ preserves planar symmetry with respect to some plane parallel to the flow direction. Such symmetry, which is weaker than the axial symmetry but not incompatible with azimuthal wave number $n = 1$, was also observed by Magarvey and Bishop (1961a) and Levi (1980), and was later found in the numerically-simulated supercritical sphere wakes computed by Shirayama (1992) and Johnson and Patel (1999). The time history of velocity disturbances described by numerical solution of the nonlinear equations of motion computed by Tomboulides show that the mode with $n = 1$ begins to oscillate at some value of Re in the interval $270 \leq \mathrm{Re} \leq 285$. Recall that according to the linear theory by Natarajan and Acrivos the transition to an oscillating wake regime takes place at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}} = 277.5$ while according to the experimental results by Magarvey and Bishop $\mathrm{Re}_{2,\mathrm{cr}} \approx 270$, and both Wu and Faeth, and Provansal and his coworkers (who used wake control to observe the time evolution of velocity disturbances at subcritical and supercritical values of Re close to the critical value $\mathrm{Re}_{2,\mathrm{cr}}$) found that $\mathrm{Re}_{2,\mathrm{cr}} \approx 280$. We see that here again the conclusions of nonlinear numerical stability analysis agree very well with results given by linear stability theory and by several trustworthy experimental investigations. Let us also re-emphasize that the computations by Natarajan and Acrivos, and by Tomboulides, and experimental investigation by Wu and Faeth were carried out practically simultaneously and independently from each other. Therefore the remarkable coincidence of the results obtained independently by three very different methods gives good reason to trust them. It is now possible to support this conculsion by reference to results of a subsequent successful numerical simulation of the flow past a sphere at a number of moderate Reynolds numbers, combined with visual observation of the wake regimes at different values of Re.

Let us mention at first a numerical simulation by Gebing (1994) of the flow of a compressible fluid past a sphere at Reynolds numbers from 20 to 1,000 and a Mach number of 0.4. This simulation also showed the existence of two subsequent transition of the same type as those found for incompressible flows—a loss of axial symmetry at $\mathrm{Re} \approx 300$ and the emergence of an oscillatorty wake regime at $\mathrm{Re} \geq 400$. However, up to now compressible flows have not been considered at all in this book; therefore, the main attention will be given below to the results of accurate numerical simulations of the incompressible flows past a sphere at $20 \leq \mathrm{Re} \leq 300$ performed by Johnson (1996). He employed a numerical method differing from that of Tomboulides, and presented the final version of his results in the paper by Johnson and Patel (1999), where numerically simulated data were accompanied by results of dye-injection observations of the wake flow behind a sphere towed through a water tank. Both the simulations and visual observations showed that, at Reynolds numbers from 20 to approximately 210, the wake flow is steady, axisymmetric, and does not undergo any substantial topological transformations. As has been noted above, the length of the recirculation zone, the separation angel, and the drag coefficient computed for Re-values in this range coincided very well with many previous experimental and numerical results. However, at a Reynolds number of 211 the calculated solution of the equations of motion becomes non-axisymmetric, but preserves planar symmetry with respect to some plane parallel to the flow direction, and remains steady.

Steady non-axisymmetric numerical solutions were found for all investigated values of $\mathrm{Re} \geq 211$ up to $\mathrm{Re} = 270$. However at $\mathrm{Re} = 280$, which was the next higher Reynolds number considered, the solution, obtained for the initial conditions leading at $\mathrm{Re} = 270$ to steady non-axisymmetric solution, was found to be oscillating with a fixed frequency f. Hence, Johnson's numerical simulations show that $\mathrm{Re}_{1,\mathrm{cr}} = 211$ and $270 < \mathrm{Re}_{2,\mathrm{cr}} \leq 280$. These results agree excellently with those found in numerical studies of Natarajan and Acrivos (1993); Tomboulides (1993), and Tomboulides et al. (1993), and in experiments of Magarvey and Bishop (1961a, b) and Wu and Faeth (1993) (and in the part relating to the onset of wake oscillations at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}}$ also with experimental results of Provansal and his group). Wishing to understand (and to explain) the physical mechanisms leading to the loss of wake axisymmetry at $\mathrm{Re} = 211$ and the transition to unsteady vortex-shedding regime at $\mathrm{Re} \approx 275$, Johnson and Patel analyzed very thoroughly all the numerical and visualization data relating to $\mathrm{Re} = 250$ and $\mathrm{Re} = 300$, and presented in their paper of 1999 an extensive collection of graphs, photos, and model pictures illustrating the properties of the wake regimes at these two Reynolds numbers.

The collected data gave reasons to associated the transition to a non-axisymmetric steady regime at $\mathrm{Re} = \mathrm{Re}_{1,\mathrm{cr}} = 211$ with an azimuthal instability of the low-pressure core of the toroidal vortex, emerging at $\mathrm{Re} = \mathrm{Re}_{0,\mathrm{cr}} \approx 20$ and then growing with Re, becoming more unstable with the decrease of the role of viscosity. Relying on this general idea, Johnson and Patel proposed a physical mechanism describing the transition process. This mechanism allowed them to interpret physically their visualization results for $\mathrm{Re} = 250$ and 300, and to explain the appearance at $\mathrm{Re} > \mathrm{Re}_{1,\mathrm{cr}}$ behind a sphere of two streamwise vortices extending downstream and forming two parallel vortical threads. (These vertical threads were first observed in the liquid-drop experiments by Magarvey and Bishop, whose results were later confirmed by visualization experiments of Levi (1980), Provansal and his coworkers (who found the two-thread regime for $180 < \mathrm{Re} < 280$), and Johnson and Patel, and by numerically-simulated data of Shirayama (1992), Tomboulides, and Johnson and Patel). The value of $\mathrm{St} = fD/U_0$ at $\mathrm{Re} = 300$ computed by Johnson was equal to 0.137, and coincided almost exactly with the result $\mathrm{St} = 0.136$ of Tomboulides' computations and with the value given by the experimental form of the Roshko law (4.47) given by Provansal and Ormiėdes in their papers of 1998 and 1999. (The experimental values of St for vortex shedding from a sphere found by Johnson and Patel were slightly higher than the corresponding numerical results but they agree well with experimental values of St found by Sakomoto and Haniu (1990, 1995) for nearby values of Re). The calculated drag coefficient at $\mathrm{Re} = 300$ was also close enough to previous experimental and numerical results.

A similar physical mechanism was also proposed for explanation of the transition to unsteadiness at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}} \approx 275$. This mechanism explains not only the observations of Achenbach (1974); Perry and Lim (1978), and Sakamoto and Haniu (1990, 1995) of periodic shedding, at $\mathrm{Re} > \mathrm{Re}_{2\mathrm{cr}}$, of hairpin vortices of consistent orientation, but also the shedding of previously-unrevealed oppositely-oriented hairpin vortices which were seen in the new visualizations of the sphere wake and, according to Johnson and Patel, may have a rather simple physical origin. However, space limitations forbid more detailed discussion of this subject.

The results of Natarajan and Acrivos, Tomboulides (and Tomboulides et al)., and Johnson (and Johnson and Patel) may be applied in principle to determination of the coefficients γ and δ of the real Landau Eq. (4.34), describing the bifurcation at $\mathrm{Re} = \mathrm{Re}_{1,\mathrm{cr}}$ of the steady axisymmetric wake flow observed at smaller values of Re (see, in particular, Fig. 4.37b where data relating to the coefficient $\gamma = \gamma(\mathrm{Re})$ are presented). The transition at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}}$ of a steady non-axisymmetric wake flow to a non-axisymmetric oscillating vortex-shedding regime represents a Hopf bifurcation and requires the use of a complex Landau Eq. (4.40) (or, what is the same, two real Eqs. (4.34) and (4.34a)) for its theoretical interpretation. Coefficients γ and δ in both cases can be estimated if some method of control of wake development is used, so that the time history of the real amplitude of some appropriately chosen characteristics of the wake flow can be observed from the initial instant of this development (cf. the discussion of Eqs. (4.48) and (4.49) in part (b) of this section). This procedure was applied to the study of the sphere wake at Re near $\mathrm{Re}_{2,\mathrm{cr}}$ by Ormières et al. (1998); Provansal and Ormières (1998), and Ormières and Provansal (1999) who, in particular, determined the values of $\gamma(\mathrm{Re})$ (it was found that $\gamma D^2/\nu \approx 0.9(\mathrm{Re} - \mathrm{Re}_{2,\mathrm{cr}})$ at small values of $\mathrm{Re} - \mathrm{Re}_{2,\mathrm{cr}}$) and the coefficients of the Ro-Re relation corresponding to the vortex-shedding regime of the sphere wake. As to the coefficients ω_1 and δ', which are needed for the description of a Hopf bifurcation at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}}$, the first coincides with the real part of the corresponding complex eigenvalue (denoted by ω_1 in the above discussion of the paper by Natarajan and Acrivos), while the second can be easily determined from the values of γ, δ, and the frequency f of the observed wake oscillations. The values of f were given for a number of values of Re by both Tomboulides (1993) and Johnson (1996), who used their own numerical simulations for this purpose, and by Provansal and his group who used spectral analysis of measured velocity fluctuations in the wake; the results of all studies were practically the same. Moreover, the French researchers also measured the dependence of the energy of streamwise-velocity oscillations E (more exactly, of the normalized energy E/E_{max}) on the streamwise coordinate x of the observation point in a sphere wake and of Re, and the dependence on Re of the coordinate x_{max} at which the amplitude of the velocity oscillations takes the greatest value. The results of these measurements were found to be similar in many respect to the results of Wesfreid et al. (1996) relating to spatial variations of velocity oscillations in the wake of a circular cylinder.

Many details of the flow past a sphere at larger values of Re can be found, in particular, in the papers by Achenbach (1974); Pao and Kao (1977); Perry and Lim (1978); Taneda (1978); Kim and Durbin (1978); Sakamoto and Haniu (1990, 1995); Shirayama (1992); Bonneton and Chomaz (1992); Wu and Faeth (1993); Tomboulides (1993), and Tomboulides et al. (1993). The values of the wake-oscillation frequency f and of the Strouhal number St at many values of Re were determined, in particular, by Achenbach (1974); Taneda (1978); Kim and Durbin (1988), and Sakomoto and Haniu (1990, 1995); the last-named of them includes a general sketch of the shape of the St-Re relation for a wide range of Re, both for a sphere in a constant-velocity stream and in streams with various constant transverse velocity gradients. Note, however, that at large enough values of Re wake oscillations often have the shape of supperpositions of several harmonics of different frequencies. (For example, Shirayama (1992) found that at $\mathrm{Re} = 500$ two frequencies, corresponding to different Strouhal numbers, are clearly seen in the spectrum of sphere-wake oscillations). With further growth of Re the number of different spectral components of wake oscillations increases and the transition to turbulence leads to the appearance in the wake of a continuous frequency spectrum. In the above-mentioned papers, many topological transformations of the vorticity field of sphere wakes are described; however, these high-Re wake transitions will not be considered in this chapter.

Let us now say a few words about the wakes behind some other axisymmetric bodies. We will begin with the *wakes behind flat circular disks* perpendicular to a uniform steady flow. Such wakes were studied in experiments by Schmiedel (1928) (who considered spheres and round disks freely falling in a liquid), Marshall and Stanton (1931); Fail et al. (1957) (here wakes behind circular plates were considered, together with those behind some other plates perpendicular to the flow), Carmody (1964); Willmarth et al. (1964); Calvert (1967a, b) (who also studied wakes behind cones with axes parallel to the stream direction and flat disks non-orthogonal to the stream); Roos (1968); Roos and Willmarth (1971); Fuchs et al. (1979); Takamoto (1987); Bearman and Takamoto (1988); Berger et al. (1990); Lee and Bearman (1992); Cannon et al. (1993); Provansal (1996) (who indicated that he had studied wakes behind discs and cones parallel to the stream, together with the sphere wakes discussed above, but mentioned only one specific result relating to cones), Miau et al. (1997), and some other researchers. However, the results of this work are much less definite than those relating to sphere wakes. The vortical structures in disk wakes were investigated at various Reynolds numbers and by various experimental methods, in particular, by Fuchs et al., Berger et al., Lee and Bearman, Cannon et al., and Miau et al., but the results obtained are still very scattered. Apparently the only attempt to calculate the stability characteristics of wakes behind circular disks was due to Natarajan and Acrivos (1993), using the same method as in their study of the stability of sphere wakes. They found that, as in the case of a sphere wake, a steady axisymmetric disk wake loses its stability first of all to a nonoscillatory non-axisymmetric disturbance with $n = 1$. According to their calculations, this loss occurs at $\mathrm{Re} = \mathrm{Re}_{1,\mathrm{cr}} = 116.5$ (where Re is based on the disk diameter and free-stream velocity). However, the non-axisymmetric steady flow past a circular disk emerging at this Re loses its stability at only slightly lager Reynolds number, $Re_{2,cr} = 125.6$, when a new oscillating non-axisymmetric wake regime (again with $n = 1$) appears with a frequency of oscillation corresponding to $St_{cr} \approx 0.125$. Natarajan and Acrivos noted also that the results of experiments by Willmarth et al. (1964), who oserved the behavior of freely falling circular disks, can be interpreted as a crude confirmation of their theoretical results.

A number of observations of wake regimes behind axisymmetric bodies differing from spheres and round disks can be also found in the literature, but only a few quantitative conclusions can be obtained from the results. It was noted above that Calvert (1967a) studied wakes behind cones with axes parallel to the flow direction, apexes directed upstream, and various apex angles. Such wakes were investigated in more detail by Goldburg and Florsheim (1966) who observed wakes behind freely falling cones (with apexes directed downwards) together with wakes behind falling cone-spheres (hemispheres attached to the base of cones). They showed that the Rayleigh-Roshko law (4.47a) with constant coefficients a and a_1 is valid for the oscillation frequencies of these wakes (in particular, in the case of a cone with 20° apex angle, $a \approx 0.454$ and $a_1/a \approx 160$; this means that periodic vortex shedding from such a cone is observed for Re > 160). Provansal (1996) indicated that, according to his experiments, $Re_{cr} \approx 185$ determined the threshold of a periodic vortex-shedding regime behind an upstream-pointing cone, but gave no further details. Zikmundova (1970) (whose results relating to the value of $Re_{1,cr}$ for the sphere wake gave rise to doubt) studied, together with the sphere wake, the wake behind a spheroid, and also Masliyah (1972) observed both sphere wakes and wakes behind several oblate spheroids. Hama and Peterson (1976); Hama et al. (1977), and Peterson and Hama (1978) studied the wakes behind slender bodies of revolution, and found that here instability appears at much greater Reynolds numbers (based on the body diameter) than in the cases of bluff bodies (such as disks, spheres and cones). The number of references to papers dealing with wakes behind various axisymmetric bodies can be easily increased, but we will not linger on this subject here.

4.2.4.6 Axisymmetric Jet Flows

At the beginning of this section a short remark was made relating to the Landau constant δ for the plane Bickley jet. Now, in conclusion of the present section we will mention several papers dealing with amplitude equations for unstable disturbances in axisymmetric jets issuing from a circular orifice into a space filled with a fluid at rest. Let us recall that at the end of Sect. 2.93 it was indicated that if the fluid in a jet does not differ from the fluid in the surrounding space, then only convectively (but not absolutely) unstable disturbances can exist in jet flow, while in the case of a jet which is heated (or for some other reason has appreciably smaller density than that of the surrounding fluid) absolute instability can take place. It has been mentioned several times in this section that the presence of regions of absolute instability is necessary for the excitation of the global mode of self-sustained oscillations in a nearly-parallel fluid flow. Therefore the development of disturbances in non-heated jets must inevitably differ from the same process observed in wakes or heated jets above the Hopf-bifurcation threshold.

Danaila et al. (1997) applied direct numerical simulation to investigate the spatial disturbance development in round (unheated) jets with some widely-used initial velocity profiles from the list given by Michalke (1984) (see also Sect. 2.9.4 in Chap. 2) and several initial Reynolds numbers U_0D/ν (where U_0 is the typical jet velocity at the orifice and D is the orifice diameter). It was shown that at relatively small, slightly-supercritical Reynolds numbers 'helical modes' with $n = \pm 1$ are most unstable (i.e. their amplitudes grow most quickly) while at highly-supercritical Reynolds numbers the axisymmetric mode with $n = 0$ becomes the most amplified. At some stage of the disturbance development in a slightly-supercritical round jet a Hopf-like bifurcation was detected which however led to a quasiperiodic (and not purely periodic) final state. In the subsequent paper by Danaila et al. (1998) the nonlinear disturbance development of a Hopf bifurcation leading to the production of oscillating helical modes with $n = \pm 1$ was studied by analysing the corresponding amplitude equations. Since here amplitudes of two modes, with $n = 1$ and $n = -1$, must be considered and the higher harmonics (whose frequencies are multiples of the dominant frequency) also play a definite role (cf. the discussion of papers by Dušek et al. (1994) and Dušek (1996) in part (b) of this section), these amplitude equations are more complicated than the simple Landau equation and may be considered as its generalizations (of the same type as Stuart's Eqs. (4.43) which were considered in Sect. 4.21).

Let us now pass to the case of heated round jets. At the end of Sect. 2.93, literature was cited, in which it was proved that absolute instability can emerge under certain conditions in heated jets, and some conditions making such emergence possible were indicated. (Sect. 2.93 was devoted to plane free flows in an unbounded space but in discussion of heated jets it was specially noted that the statements made are valid for both plane and round jets). Some examples of experimental confirmation of results relating to the absolute instability of heated jets with negligible buoyancy effects can be found in papers by Monkewitz and Sohn (1988) and Sreenivasan et al. (1989) referred to in Sect. 2.93; valuable supplementary data of the same type can be found in papers by Monkewitz et al. (1989, 1990). In the case of a heated jet the regime of jet oscillations depends on two dimensionless parameters: the Reynolds number $\mathrm{Re} = U_0D/\nu$ and the ratio $\rho_0/\rho_\infty = S$ of the density of fluid issuing from the orifice to the ambient density far from the jet. (Here we again assume that the influence of buoyancy can be neglected in comparision with the influence of the inertia of moving fluid. This assumption is usually true near the orifice; the case where buoyancy is essential was considered by Krizhevsky et al. (1996) but will be not treated here). The above-mentioned experiments show that over a wide range of Reynolds numbers strong global oscillations of the 'jet column' arise automatically, if the value of the parameter S lies below $S_{cr} \approx 0.62$. It follows from this that at such values of S a Hopf bifurcation takes place, which corresponds to Landau's equation with positive Landau constant $\delta > 0$. Just this situation will be considered below in line with the presentation given by Raghu and Monkewitz (1991).

Raghu and Monkewitz analyzed the experimental data for a jet of hot air issuing from a round nozzle of diameter $D = 15$ mm into unheated still air (this arrangement and experimental conditions were practically the same as used by Monkewitz et al. (1990)). Since wake oscillations were observed only at $S < S_{cr} \approx 0.62$, it was possible to suppress the oscillations by extending the length of the nozzle by another 15 mm and then cooling the nozzle extension to reduce the air temperature, in this way increasing the density ratio S above the critical value S_{cr}. Thus, jet control could be realized by regulating the jet temperature. By switching off the cooling system it was possible to return S quickly to its initial low value $S_0 < S_{cr}$ and hence to create conditions promoting the excitation of jet oscillations. After this the researchers could observe, at a selected point of the jet, the transient growth of the complex oscillations amplitude $A(t) = |A(t)|e^{i\phi(t)}$ from zero up to its equilibrium value corresponding to the selected position of the observation point and the value $S_0 < S_{cr}$ of the parameter S. This transient growth is determined by the complex Landau (otherwise, Stuart–Landau) Eqs. (4.34) and (4.34a); the observations described allow evaluation of all four real coefficients γ, ω_1, δ and δ'' of these equations as in the papers by Mathis et al. (1984); Provansal et al. (1987); Sreenivasan et al. (1987) and Schumm et al. (1994) on cylinder-wake oscillations, described in part (b) of this section.

Let us replace the complex amplitude $A(t)$ by the normalized amplitude $A(t)/A_e = R(t)e^{i\vartheta(t)}$, where $Ae = (2\gamma/\delta)^{1/2}$ is the real equilibrium amplitude and $R(t) = |A(t)|/A_e$ is the real normalized amplitude which can vary in the range $0 \leq R(t) \leq 1$ (it is assumed here that $S < S_{cr}$ and therefore $\gamma > 0$ and $\delta > 0$). Then Eqs. (4.34) and (4.34a) can be rewritten in the following form:

$$\frac{1}{R}\frac{dR}{dt} = \gamma(1 - R^2), \tag{4.55}$$

$$\frac{d\phi}{dt} = -\omega_1 - \frac{\delta'\gamma}{\delta}R^2. \tag{4.55a}$$

Switching off the cooling system at first and then switching it on again, one could measure, at a given point of observation, values of the real amplitude $|A(t)|$ (gradually growing from zero at $t = 0$ to its equilibrium value A_e at large t) together with the jet oscillation frequency $(1/2\pi)(d\phi/dt) = f(t)$. Values of $|A(t)|$ and A_e determine $R(t)$, and in Fig. 4.38 the values of $[dR/dt]/R$ and $d\vartheta/dt$ measured by Raghu and Monkewitz at the point with coordinates $(x, r) = (1.3D, 0.5D)$ (where x is the streamwise coordinate measured from the jet orifice and r is the radial cylindrical coordinate indicating the distance from the jet axis) are presented in their dependence on the value of R^2, varying from zero to one, for the case where $S = 0.546$. We see that the experimental data agree well with the linear dependence of both presented in Fig. 4.38 quantities on R^2, predicted by Eqs. (4.55) and (4.55a), and allow evaluation of all coefficients of these equations for the given observation point, value of S, and flow conditions. (Recall that in the case of the global mode of wake oscillations, the values of γ and ω_1 do not depend on the observation points and that the results of similar cylinder-wake observations presented in part (b) of this section showed that the ratio δ'/δ is also practically constant over a large spatial region). Resutls of Raghu

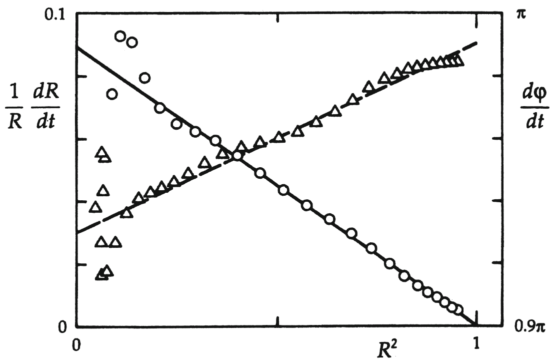

Fig. 4.38 Dependence of $[dR/dt]/R$ (o) and of $d\phi/dt$ (Δ) on R^2 at the point $\{x, r\} = \{1.3D, 5D\}$ of a heated circular jet with $S = 0.546 < S_{cr}$. (After Raghu and Monkewitz 1991)

and Monkewitz's measurements at different values of the parameter S showed that the global oscillations of the heated jet come to an end at $S = S_{cr} \approx 0.62$ (this value is slightly less than the estimate $S_{cr} \approx 0.63$ found by Monkewitz et al. in 1990). More precisely, Raghu and Monkewitz found that at their chosen point of observation the critical value S_{cr} and the coefficients of Landau's Eqs. (4.55) and (4.55a) for the heated jet take the following values:

$$S_{cr} = 0.62 \pm 0.01, \quad \gamma D/U_0 = [1.15 \pm 0.15](S_{cr} - S), \tag{4.56a}$$

$$\omega_1 D/U_0 = -[0.68 + 0.01] - [0.88 + 0.02](S_{cr} - S), \quad \delta'/\delta = -2.5 + 0.6. \tag{4.56b}$$

We see that the measurements of the transient growth of the jet oscillations confirm the emergence of a Hopf bifurcation at a critical density ratio $S = S_{cr}$, and yield rather accurate estimates of the values of S_{cr} and of coefficients of Landau's Eqs. (4.55–4.55a).

This example will conclude the present section of the book, devoted to various applications of the real and complex Landau equations (and in some cases also of more general Ginzburg–Landau equations) to description of the nonlinear instabilities of fluids flows.

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