Chapter 3
More About Linear Stability Theory: Studies of the Initial-Value Problem

3.1 Beginning of the Story: The Works of Kelvin and Orr

The normal-mode method of the linear stability theory, which was considered in Chap. 2, deals only with special "wave-like" infinitesimal disturbances of a given laminar flow. This method equates the strict instability of a steady flow to the existence of at least one wave-like disturbance (proportional to $e^{-i\omega t}$ and, in the case of homogeneity in the streamwise direction Ox, also to e^{ikx} which grows exponentially as $t \to \infty$ or, in the spatial formulation, as $x \to \infty$), and states that ordinary instability means that there exists a wave-like disturbance which is not damped at infinity. (The adjectives "strict" and "ordinary" will be omitted below in all cases where the difference between two types of instability is unimportant or it is clear from context which instability is considered.) However, is this definition of instability always appropriate? Is it not more reasonable to call a flow unstable, if there exists at least one small disturbance *of any form* which grows without bound after a long-enough time? Moreover, in practice even a bounded but large-enough initial growth of a small disturbance can violate the applicability of the linear stability theory, and make the flow unstable whatever be the asymptotic behavior of this disturbance according to linear theory. In Sect. 2.5 we have already noted in this respect that practical usefulness of the method of normal modes must not be exaggerated. In this chapter this topic will be considered at greater length.

A study of the time evolution of an arbitrary infinitesimal disturbance requires consideration of the solution of the general initial-value problem for linear equations (2.7a,b), obtained by linearization of the Navier-Stokes equations with respect to disturbances $u_i', i = 1, 2, 3$, and p'. The first attempt to construct the general solution of such an initial-value problem, for the simplest case of a plane Couette flow with linear velocity profile $U(r) = bz, 0 \le z \le H$, was made quite early by Kelvin (1887a) (who was still called William Thomson at this time, but was created Baron Kelvin of Largs in 1892). He found a family of exact solutions of Eq. (2.7) for this flow, which depended on three wave numbers k_1, k_2, k_3 (where $k_3 = n\pi/H$ with an integer value of n) and two amplitude coefficients W_0 and V_0 (the second of them for the spanwise velocity component $u_2' = v$). Kelvin's solution for the vertical velocity $u_3' = w$ has the form

A. M. Yaglom†, U. Frisch (ed.), *Hydrodynamic Instability and Transition to Turbulence,* Fluid Mechanics and Its Applications 100,
DOI 10.1007/978-94-007-4237-6_3, © Springer Science+Business Media Dordrecht 2012

$$w(\boldsymbol{x},t) = \frac{W(t)}{k^2 + (k_3 - k_1 bt)^2} \exp i[k_1 x + k_2 y + (k_3 - k_1 bt)z] + w^{(0)}(\boldsymbol{x},t) \quad (3.1)$$

where $W(t) = W_0 \exp\left\{-\nu t\left[K^2 - k_1 k_3 bt + (k_1{}^{2/3})b^2t^2\right]\right\}, k^2 = k_1^2 + k_2^2, K^2 = k_1^2 + k_2^2 + k_3^2$ and $w^{(0)}$ $(\boldsymbol{x}, t)$ is the vertical velocity component corresponding to the solution of Eqs. (2.7) which satisfies the initial condition $w^{(0)}$ $(\boldsymbol{x}, 0) = 0$ for any $\mathbf{x}$ and the boundary conditions guaranteeing that $w(\boldsymbol{x},t) = \partial w(\boldsymbol{x},t)/\partial z = 0$ for any x, y and t, if $z = 0$ or $z = H$. (Similar solutions found by Kelvin for disturbances of the other two velocity components and the pressure may be omitted here.) Since $w(\boldsymbol{x},0) = (W_0/K^2)\exp[i(k_1x + k_2y + k_3z)]$, the Fourier analysis allows one to represent any initial value of the vertical velocity disturbance in the form of an integral of the function $w(\boldsymbol{x}, 0)$ over all real values of k_1 and k_2 and a sum over all integer values of n. Noting now that the solution (3.1) decreases exponentially as $t \to \infty$ ($W(t)$ is an exponentially decreasing function and it seems natural to suppose that because of this $w^{(0)}$ $(\boldsymbol{x}, t)$ must also fall off exponentially with time), Kelvin came to the conclusion that any infinitesimal disturbance of a plane Couette flow must tend asymptotically to zero as t increases to ∞, i.e. that this flow is stable with respect to all such disturbances.

Kelvin's conclusion was disputed by Rayleigh (1892) and Orr (1907). Rayleigh's criticism was mainly directed at the arguments presented in Kelvin's paper (1887b), where Eq. (2.41) (now usually called the Orr-Sommerfeld equation) was first derived, but was used erroneously for proving the stability of plane Poiseuille and Couette flows since only real, and not complex, values of the frequency ω were considered by the author. Latter Orr noted (and Rayleigh agreed) that a similar objection can be applied to Kelvin's arguments in the paper (1887a), since here the function $w^{(0)}$ $(\boldsymbol{x}, t)$ was represented as an integral of a harmonic fluctuation $f(\boldsymbol{x},\omega)e^{i\omega t}$ over all real values of ω, while in fact the possible complex values of ω should also be taken into account. Thus, Kelvin considered only some special solutions of the initial value problem, and therefore their asymptotic decay did not prove the stability of Couette flow. (A more modern modification of the same criticism was given by Marcus and Press (1977), who showed that Kelvin's reasoning can be used to prove the linear stability of a flow with the linear velocity profile $\boldsymbol{U}(\boldsymbol{x}) = \{bz,\ 0,\ 0\}$ in an unbounded space $-\infty < z < \infty$, but not in a layer of finite thickness between two solid walls.) So, it became clear long ago that Kelvin's proofs (1887a, b) of the stability of plane Couette and Poiseuille flows to infinitesimal disturbances contained incorrigible flaws. Therefore, for a long time very little attention was paid to these papers in the literature on fluid mechanics.

However, Kelvin's papers of 1887 contained some valuable arguments as well. It has already been noted that in the paper (1887b) the very important equation (2.41) was derived. in the paper (1887a) an exact solution of the linearized fluid dynamics equations was found, which decays algebraically (as t^{-2}) as $t \to \infty$, if $\nu = 0$; it was quite different from exponentially decaying (or growing) normal-mode solutions, which Rayleigh began to study (for inviscid flows) a little earlier and which for almost a century completely ousted from the theory of hydrodynamic stability the study of solutions with algebraic asymptotic behavior. Moreover, it was mentioned in passing in the same paper that "solution $w^{(0)}(\boldsymbol{x}, t)$ rises gradually from zero at $t=0$ and later comes asymptotically to zero again as t increases to infinity." Discussing this paper, Orr (1907) remarked that for some values of viscosity ν and wave numbers k_i, $i=1, 2, 3$, the whole Kelvin solution (3.1) also at first rises rapidly with time from its initial value at $t=0$, and only later begins to decrease, tending to zero—this important remark by Orr will be considered in detail below.

Durnig the 20th century interest in exact solutions of dynamic equations was generally growing, and this led to revitalization, in the second half of this century, of attention to "Kelvin's modes of disturbance" (3.1). These modes were then re-examined by a number of scientists, some of whom (e.g., Moffatt (1967); Rosen (1971); Marcus and Press (1977)) apparently did not know Kelvin's old results, and rediscovered them (for more details see the interesting review by Craik and Criminale (1986)). It was, in particular, pointed out in this review that a single Kelvin mode is, in fact, an exact solution not only of the linearized equations (2.7) where $\boldsymbol{U}=\{bz, 0, 0\}$, but also of the full Navier-Stokes equations for the disturbed velocity field $\boldsymbol{u}(\boldsymbol{x}, t)=\boldsymbol{U}+\boldsymbol{u}\ (\boldsymbol{x}, t)$ with $\mathbf{U}$ as above. This interesting fact (which is, however, not especially important for stability studies since superpositions of exact solutions of nonlinear equations usually do not satisfy them, while stability theory has to do with superpositions of modes) was unknown to Kelvin and Orr. It was mentioned in passing by Moffatt (1967) and, according to Craik and Criminale, was independently discovered after 1965 (i.e., about 80 years after the appearence of Kelvin's paper) by a number of people (including both the authors) and was apparently first discussed in a publication on hydrodynamic stability only by Tung (1983). The above-cited review also contains a number of references to papers where Kelvin's solution was generalized to flows with a linear velocity profile incorporating either Coriolis force (Tung (1983) is just one of them), or density stratification, or both these effects; some of the results relating to stratified flows will be presented later in this chapter. Moreover, Craik and Criminale also found exact "Kelvin-like" solutions of the Navier-Stokes equations for the velocity field $\boldsymbol{u}=\boldsymbol{U}+\boldsymbol{u}'$ (without any restriction on the sizes of summands $\boldsymbol{U}$ and $\boldsymbol{u}'$) where $U_i(\boldsymbol{x}, t) = b_{ij}(t)x_j + b_i(t), (i, j = 1, 2, 3,$ and summation over the repeated index j is, as usual, assumed), so that the "basic velocity field" $\boldsymbol{U}\ (\boldsymbol{x}, t)$ is here, in general, neither steady nor parallel.

Let us now return to Orr's paper (1907). Here it is remarked (on pp. 74–75) that a superposition of an infinite number of functions $\exp(i\omega_n t)$, where $\Im\mathrm{m}\,\omega_n \geq 0$ for all n, "may at some time have a value which is exceedingly great compared with its initial value, and may even become infinite"; this is in fact a strong criticism of the method of normal modes (which, at the same time, underwent significant development in Orr's paper). Orr pointed out that a stability investigation requires the study of the time evolution of a general solution of the disturbance initial-value problem, but he limited himself to consideration only of some special exact solutions of it. In Part I of his paper the stability of inviscid (ideal) fluid flows was investigated, so the main references here were to Rayleigh's papers (1880, 1887, 1892, 1895) devoted to ideal-fluid stability studies. For an inviscid plane Couette flow with velocity profile $U(z) = bz, 0 \leq z \leq H$, Orr found an exact solution for the vertical velocity of the disturbance $w\ (\boldsymbol{x}, t)$ which, after some simple transformations, can be represented in the form

$$w(\boldsymbol{x},t) = \frac{W_0 \exp\left[i(k_1x + k_2y)\right]}{k^2 + (k_3 - k_1bt)^2}\left[e^{i(k_3-k_1bt)z} - e^{i(k_3-k_1bt)H}\frac{\sinh kz}{\sinh kH} - \frac{\sinh k(H-z)}{\sinh kH}\right] \tag{3.2}$$

where, as in (3.1), $k = (k_1 + k_2)^{1/2}$. Solution (3.2) can be considered as the limiting form of Kelvin's solution (3.1) as $\nu \to 0$. The last two terms in the brackets here correspond to the term $w^{(0)}(\boldsymbol{x}, t)$ in (3.1); they provide the fulfilment of the boundary condition that $w = 0$ at $z = 0$ and $z = H$, but do not vanish at $t = 0$ [therefore here $w(\boldsymbol{x}, 0) \neq (W_0/K^2)\exp\{i\ (k_1x + k_2y + k_3z)\}$]. However, replacing these two terms by slightly more complicated combination of trigonometric and hyperbolic functions, it is not difficult to obtain an exact solution with just such an initial value of the vertical velocity; see Orr (1907), pp. 26–27, or Drazin and Howard (1966), p. 28. Expressions for the other velocity components corresponding to solution (3.2) or Orr's related solution are not so simple; they were given by Orr only for the case of a two-dimensional disturbance where $v = 0$ and $k_2 = 0$.

Orr pointed out that the solutions obtained imply the existence of small disturbances of inviscid plane Couette flow, which can grow indefinitely before they begin to decay with time. In fact, let us assume that $k_1 > 0$, $k_3 > 0$ and $b > 0$ and exclude from consideration the close vicinity of solid walls, where $z \leq \varepsilon H$ or $z > H(1 - \varepsilon)$ for some very small number ε. Then it is possible to choose kH large enough (i.e., the horizontal wavelengths small enough compared to the flow thickness H) to make negligibly small the contributions of the second and third terms in the brackets on the right side of Eq. (3.2). In such a case $|w(\boldsymbol{x}, t)|$ will grow with time from its initial value $|w(\boldsymbol{x}, 0)| = |w|_o$ to a maximum $|w|_{\max}$ at time $t_{\text{opt}} \approx k_3/k_1\ b$ with $|w|_{\max}/|w|_0 \approx (k^2 + k_3^2)/k^2 = 1 + k_3^2/(k_1^2 + k_2^2)$. This shows that $|w|$ can reach an arbitrarily large value if k_3 is chosen to be large enough (and t_{opt} then also becomes very large). According to Eq. (3.2), $|w(\boldsymbol{x}, t)|$ diminishes with time without limit (asymptotically as $(t - t_{\text{opt}})^{-2}$) after the critical time t_{opt} ; however, if the disturbance grows greatly at smaller values of t, then the validity of the above equation (which follows from the linear stability theory) at $t > t_{\text{opt}}$ becomes quite questionable. This is the reason why Orr said that, according to his results, plane Couette flow of inviscid fluid is *practically unstable,* and this can explain the flow instabilities observed in other, but similar, types of flows.

In the case of a two-dimensional disturbance with $k_2 = 0$, similar results were obtained by Orr for the temporal evolution of the corresponding streamwise velocity component and kinetic energy density per unit mass, T^*, of the disturbance. It was found that for some values of k_1 and k_3 these quantities also increase greatly when time increases from $t = 0$ up to a certain critical time t_{cr} and only after this time do they decrease, tending to zero; see also an account of these results by Orr given by Farrell (1982). Plane-parallel inviscid flow with an arbitrary continuous velocity profile $U(z)$, $0 \leq z \leq H$, was also considered by Orr; however, no strict proofs were obtained for this case and only some qualitative reasons were presented, which gave the impression that as a rule the situation here does not differ very much from that of a plane Couette flow.

Special attention was given by Orr in Part I to the important case of Poiseuille flow in a circular tube. This flow has the parabolic velocity profile $U(r) = A(R^2 - r^2)$, $0 \le r \le R$, and only the most simple axisymmetric two-dimensional velocity disturbances of the form $\boldsymbol{u}'(x,t) = \{u(r,x,t),\ 0, w(r,x,t)\}$, where $u = u'_x$ and $w = u'_r$ are streamwise and radial velocity components, were considered in his paper. Analyzing the time evolution of such disturbances in an inviscid fluid, Orr found that if $w(r,x,0) = U_0 \exp\left[i\left\{k_1 x + (k_2 r)^2\right\}\right]$ (the initial value of the component u can be easily determined in this case from the continuity equation), then the values of the disturbance amplitude and of its kinetic energy both increase with time at first (and the values of k_1 and k_2 may be chosen to give whatever growth is wanted) and only later begin to decay, tending to zero as $t \to \infty$. Orr supposed that the existence of such strongly growing disturbances can explain instability of a tube flow as studied by Reynolds (1883). (Note however that Orr's results do not agree well with the results of recent more accurate computations which will be considered below in Sect. 3.34. These new results suggest that only nonaxisymmetric disturbances can undergo substantial transient growth in a tube flow.)

In Part II of his paper Orr turned to the stability problem for viscous flows and therefore most attention was paid to Kelvin's papers (1887a, b). Orr presented detailed analysis of errors made by Kelvin in his reasoning and then considered the special exact solution (3.1) of linearized dynamic equations for disturbed plane Couette flow of viscous fluid. He explained that the supplementary solution $w^{(0)}\ (\boldsymbol{x}, t)$ of these equations, which provided the fulfilment of the boundary conditions at the walls, can be made as small as is wanted everywhere except in the close vicinity of the walls, if the wave numbers $k_i, i = 1, 2, 3$, are chosen to be very large compared with $1/H$ (i.e., wavelengths in all directions are much smaller that the thickness of the flow). Hence, if $k_i H \gg 1$ for all three values of i and the point $\boldsymbol{x}$ is not too close to a wall, then the time evolution of the first term on the right side of (3.1) will play the main part. The numerator of this term decreases exponentially with time [at first as $\exp(-\nu K^2 t)$ and finally as $\exp\{-\frac{1}{3}\nu(k_1 b)^2 t^3\}$], but the denominator also decreases with time until $t = t_{cr} = k_3/k_1 b$, and therefore it is clear that, if the viscosity ν is sufficiently small, the disturbance $w(\boldsymbol{x}, t)$ will at first grow with time in spite of the exponential decrease of $W(t)$. It is clear that if ν is so small that the decrease of $W(t)$ between $t = 0$ and $t = t_{cr}$ is negligible, the growth of $|w(\boldsymbol{x}, t)|$ may be made as large as desired by appropriate choice of wave numbers k_i. According to Orr, the existence of disturbances having such properties show that the plane Couette flow of a viscous fluid is *practically unstable* for sufficiently small viscosity ν (i.e. for sufficiently high values of the Reynolds number $\mathrm{Re} = H^2\, b/\nu$).

Orr also made some approximate calculations for the case of two-dimensional disturbances, where $k_2 = 0$ and $v(x,t) = 0$. He determined the size of a disturbance by specifying its kinetic energy density T^*, allowed moderate values for $k_1 H$ and $k_3 H$, and used two modifications of Kelvin's solution (3.1) which satisfied two different boundary conditions at the walls (both simplifying the standard conditions

Table 3.1 Characteristics of plane-wave disturbances optimally growing in plane couette flow at various values of Re. (After Butler and Farrell (1992))

Re	t	k_1	k_2	$E_{max}/E(0)$
4000	467	0.0088	1.60	18956
2000	234	0.0175	1.60	4739
1000	117	0.035	1.60	1184.6
500	59	0.067	1.60	296.0
250	30.2	0.12	1.61	73.9
125	16.1	0.144	1.63	18.55
62.5	8.2	0.0024	1.65	4.87
31.25	3.21	0	1.62	1.50

of vanishing velocity there). It was found that for moderate values of $k_1 H$ and $k_3 H$ the maximal growth of kinetic energy T^* not only depends on $\mathrm{Re} = H^2 b/\nu$, but is also very sensitive to the form of boundary conditions. According to these calculations, at Re $\approx$ 1900, the maximal value of $T^*(t)/T^*(0)$ can be close to 10,000, at least for one form of the boundary conditions used. This is, of course, only a crude estimate (since it was obtained for incorrect boundary conditions) but it strengthens Orr's conclusion about the practical instability of the flow considered, in spite of the asymptotic approach of $T^*(t)/T^*(0)$ to zero as $t \to \infty$. (The crude estimate by Orr of the maximum possible growth of the disturbance kinetic energy in Couette flow may be compared with the results of the first computation of this maximum by Butler and Farrell (1992), presented in Sect. 3.33, Table 3.1; note however that $\mathrm{Re} = H^2 b/4\nu$ in this table.)

It is worth noting that for Orr himself the proof of practical *instability* to infinitesimal disturbances of a plane Couette flow and of some other simple flows of an inviscid or slightly viscous fluid was apparently the main aim of his investigation. (This explains why Orr did not study the general initial-value problem for an arbitrary disturbance; for his purposes it was enough to consider only special exact solutions of disturbance equations). Curiously enough, although his paper of 1907 became a standard reference in all the literature on hydrodynamic stability, it was usually referred only in relation to the so-called Orr-Sommerfeld equation, which was, in particular, widely used (at first unsuccessfully and then successfully) to prove the *stability* (in the sense accepted in the normal-mode method) of Couette flow with respect to infinitesimal disturbances. At the same time, all other Orr's results (except, perhaps, those on the "energy method" of nonlinear hydrodynamic stability theory, which will be considered later in this book) were almost never mentioned in the literature for many decades (Willke's paper (1972) was apparently one of the earliest exceptions to this). It was only recently that Orr's concept of practical instability and his results related to it achieved wide popularity, began to be cited frequently, and became cornerstones of modern, quite sensational, developments in the linear theory of hydrodynamic stability—described, in particular, by Trefethen et al. (1993) and Grossmann (1995, 1996). Let us now consider these developments.

3.2 Studies of the Inviscid Initial-Value Problem for Disturbances in Plane-Parallel Flows

3.2.1 Discussion of General Results and Associated Examples

Many years passed by after Kelvin's unsuccessful attempt (1887a) to find the general solution of the initial-value problem for an infinitesimal disturbance in a particular steady laminar fluid flow before the next such attempt was made. One of the first new publication on the initial-value approach to hydrodynamic-stability theory was the interesting paper by Eliassen et al. (1953) who studied evolution of two-dimensional disturbances in a plane-parallel flow of an inviscid stratified fluid with the velocity profile $U(z) = bz$ and density profile $\rho(z) = \rho_0 \exp(-az), 0 \leq z \leq H$. However, these authors themselves commented that their mathematical derivations were not rigorous (see also critical remarks about this work by Dikii (1960a) and Hartman (1975)). Later a more rigorous approach to the same problem was made by Case (1960b); Dikii (1960a) (in both these papers it was assumed that $H = \infty$,) and some other authors. These works will be considered at greater length in Sect. 3.23. For now, we will discuss the more simple case of a plane-parallel flow of inviscid homogeneous (i.e., constant-density) fluid, whose stability was also studied by the initial-value-problem method by Case (1960a) and Dikii (1960b).

Let us assume at first, as Case and Dikii did, that the disturbance is two-dimensional, i.e., $\boldsymbol{u}'(\boldsymbol{x}, t) = \{u(x, z, t), 0, w(x, z, t)\}$. Then, substituting this disturbance and the mean velocity $\boldsymbol{U} = \{U(z), 0, 0\}$ into the linearized dynamical equations (2.7) with $v = 0$, and then eliminating the unknowns u and p' from the system obtained, we come to the following equation (often referred to as the *Rayleigh equation in space and time*) for the unknown function $w(x, z, t)$ satisfying the boundary conditions $w(x, z, t) = 0$ at $z = 0$ and $z = H$:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) w - U''(z)\frac{\partial w}{\partial x} = 0. \tag{3.3}$$

Here $U'' = d^2U/dz^2$; as in Chap. 2, we will denote by primes both differentiations on z and fluctuations of fluid-dynamic quantities, hoping that this will not cause confusion. Equation (3.3) has the same form as Eq. (2.53) for the stream function $\psi(x, z, t)$ of a two-dimensional disturbance, and it differs from the more general Eq. (2.38) only by the absence of terms containing $\partial/\partial y$ and v.

The method of normal modes consists of finding the "wave-like" solutions of Eq. (3.3) (proportional to $e^{i(kx-\omega t)} = e^{ik(x-ct)}$). The set of "eigenvalues" ω (or $c = \omega/k$), for which wave-like solutions exist, form the discrete frequency (or phase-velocity) spectrum of flow disturbances (depending, generally speaking, on k). It is however well known that the set of all wave-like solutions of Eq. (3.3) is not complete (i.e., their linear combinations do not exhaust all the possible disturbances) since a continuous spectrum also exists here (see Sect. 2.82). It has been already mentioned in this book (cf. also Drazin and Reid (1981), Sect. 21) that usually only a finite number of wave-like solutions of Eq. (3.3) exists for a given flow at each value of the wavenumber k. In the simplest case of a plane Couette flow, where $U''(z) \equiv 0$, it is very easy to show that wave-like solutions do not exist at all; here, therefore, the discrete spectrum is empty at any k. It follows from the results of Faddeev (1972) and Dikii (1976) that such solutions also cannot exist in the case of any velocity profile $U(z)$ having no inflection points, i.e., such that $U''(z)$ does not vanish within the flow (the absence here of complex eigenvalues c was proved as far back as 1880 by Rayleigh). All this made clear the inadequacy of the method of normal modes for the linear theory of hydrodynamic stability (at least, for inviscid fluids) and was an important stimulant for renewal of studies based on the consideration of the general initial-value problem.

The complicated form of the normal-mode spectrum of Eq. (3.3) suggests that double Fourier transforms with respect to x and t are not convenient for the study of the corresponding initial-value problem. Case (1960a, b) and Dikii (1960a, b) both found that combined Fourier-Laplace transforms (which were earlier applied to the solution of some initial-value problems arising in the linear theory of hydrodynamic stability by Eliassen et al. (1953) and Miles (1958)) are much more suitable for this purpose. Let us take a Fourier transform with respect to x and a Laplace transform with respect to t of Eq. (3.3). Then the unknown function $w(x, z, t)$ is replaced by the Fourier-Laplace integral

$$\hat{w}(k, p; z) = \int_0^\infty e^{-pt} dt \int_{-\infty}^{\infty} e^{-ikx} w(x, z, t) dx. \tag{3.4}$$

(Here the Fourier integral indicates that Fourier components with given wave number k are, considered, i.e., it is assumed that $w(x, z, t) \infty e^{ikx\cdot}$ However, the assumption about the proportionality of w to $e^{i\omega t}$, which is a cornerstone of the normal-mode method, is not used here.) Applying the Fourier-Laplace transform to all terms of Eq. (3.3) we obtain

$$\left[\{p + ikU(z)\}\left(\frac{\partial^2}{\partial z^2} - k^2\right) - ikU''(z)\right]\hat{w}(k, p; z) = \left(\frac{\partial^2}{\partial z^2} - k^2\right)\hat{w}(k; z, 0) \tag{3.5}$$

where $\hat{w}(k; z, t)$ is the Fourier transform with respect to x of the function $w(x, z, t)$. Replacement of the variable p by $c = ip/k$ transforms (3.5) to the form

$$ik\left[\{U(z) - c\}\left(\frac{\partial^2}{\partial z^2} - k^2\right) - U''(z)\right]\hat{w}(k, -ikc; z) = w_0(z) \tag{3.5$'$}$$

where w_0 (k, z) coincides with the right-hand side of Eq. (3.5). So, instead of the homogeneous partial differential equation (3.3) we now have to treat the inhomogeneous ordinary differential equation (3.5) or (3.5$'$) with its right-hand side determined by the initial value $w(x, z, 0)$. When the solution $\hat{w}(k, p; z)$ is found, the vertical velocity $w(x, z, t)$ can be determined by the inversion formula for a Fourier-Laplace integral (3.4):

$$w(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\hat{w}(k,p;z)dp \tag{3.6}$$

where $t > 0, -\infty < x < \infty, 0 < z < H$, and γ is chosen so that the integration contour in the complex p-plane is to the right of all singularities of the integrand.

The solution of the inhomogeneous equation (3.5′) can be written as

$$\hat{w}(k,-ikc;z) = \int_0^H G(z,z';k,c)w_0(z')dz' \tag{3.7}$$

where $G(z,z';k,c)$ is the appropriate Green's function (the solution of the same equation with the Dirac delta function $\delta(z-z')$ on the right). The Green's function G can be given as

$$G(z,z',k,c) = \frac{w_2(z)w_1(z')}{ik[U(z')-c]W(c)} \quad \text{for} \quad z' < z, \tag{3.8}$$

$$= \frac{w_1(z)w_2(z')}{ik[U(z')-c]W(c)} \quad \text{for} \quad z' > z,$$

where $w_1(z)$ and $w_2(z)$ are solutions to the homogeneous part of (3.5) satisfying conditions $w_1(0) = w_2(H) = 0$ and $w_1'(H) = w_1'(0) = 1$, while $W(c) = w_1w_2' - w_2w_1' = w_1(H) = -w_2(0)$ is the Wronskian of these two solutions (all primes denote here differentiation on z). For the special case of a plane Couette flow, where $U''(z) \equiv 0$, it is easy to find the explicit expression of the function G in terms of hyperbolic functions (see, e.g., Case (1960a); Drazin and Howard (1966); Dikii (1976); Drazin and Reid (1981); or Henningson et al. (1994); and also Criminale et al. (1991) where three different representations of this function are given). Equations (3.6–3.8) determine the general solution of the initial-value problem for the vertical velocity w, and the same equations with functions w's replaced by ψ's give the solution of the initial-value problem for the stream function $\psi(x,z,t)$ (which at any non-zero value of wavenumber k also satisfies the zero boundary conditions). In the case of plane Couette flow, the solution corresponding to the initial value of w or ψ represented by a single Fourier component naturally coincides with the solution found by Orr (1907) which falls off as t^{-2} when $t \to \infty$. For the much more general case of arbitrary, but sufficiently smooth, initial conditions, Case (1960a) found that in plane Couette flow $|w(x,z,t)|$ and $|\psi(x,z,t)|$ at any point (x, z) usually decrease as t^{-1} when $t \to \infty$. However, exact determination of the exponent in the decay law is a tricky problem and Case's results do not agree with the earlier deduction by Eliassen et al. (1953). who found that for rather general initial conditions $|w(x,z,t)| = |\partial\psi/\partial x|$ decays in Couette flow as t^{-2} when $t \to \infty$, and it is only $|u(x,z,t)| = |\partial\psi/\partial z|$ that decays as t^{-1}. The estimate by Eliassen et al. of the decay of $|w(x,z,t)| = |\partial\psi/\partial x|$ in Couette flow was later confirmed by Engevik (1966) and Brown and Stewartson (1980).

Dikii (1960b) used his solution of the initial-value problem for small disturbances in a plane Couette flow for the proof of its stability of another type with respect to such disturbances. Namely, he showed that for some wide enough class of smooth initial values the quantities $|w(x,z,t)|$ and $|\psi(x,z,t)|$ at any values of x and z are functions of t which are bounded by some constants, decreasing to zero when the initial values of w and ψ and of their spatial derivatives of the first two orders tend to zero. The different formulations of results by Dikii and the authors mentioned above were due to the fact that Eliassen et al., Case, Engevik, and Brown and Stewartson were looking for conditions of "asymptotic stability," i.e., of dying-out at infinity, of any small enough disturbance, while Dikii studied conditions for Lyapunov's stability, which means that any disturbance remains bounded at any t by a constant, which can be made arbitrarily small by sufficiently strong diminution of the initial disturbance.[1] Neither of these definitions of stability conflicts with Orr's "practical instability" mentioned in Sect. 3.1–in the case of "Lyapunov stability," this is because k is now assumed to be fixed, so an increase of k_3, as in Orr's arguments, increases the derivatives of the initial values.

In the case of an arbitrary smooth velocity profile $U(z)$ no explicit formula for the Green's function G can be found. Therefore, we must now investigate the asymptotic behavior of the second integral on the right side of Eq. (3.6), where $\hat{w}(k,p;z)$ is given by Eq. (3.7). For this aim it is convenient to deform the contour of integration on p to the left and thus to transform it into a new contour which is confined to the left half-plane of the complex-variable plane except for some loops surrounding the singularities of the function $G(z,z',k,c) = G(z,z',k,ip/k)$. It was shown by Dikii (1960b, 1976) and Case (1960a) (see also Drazin and Howard (1966), p. 31) that the only substantial singularities of this functions are poles at zeros of $W(c)$, i.e., at such values of c that the corresponding homogeneous version of Eq.(3.5′) has a solution $w(z)$ satisfying the conditions $w(0) = w(H) = 0$. For these and only these c's, wave-like solutions of Eq. (3.3), proportional to $e^{ik(x-ct)}$, exist and hence these c's form the discrete phase-velocity spectrum of the stability problem considered. The poles at zeros of $W(c)$ (under very broad conditions there are no more than a finite number of them at any k) make wave-like contributions to the vertical velocity w (or stream function ψ) with given longitudinal wavenumber k, and these contributions are proportional to e^{-ikct} (or have the form of these exponential functions multiplied by powers of t in the case of multiple eigenvalues c). The remaining part of the integral corresponds to a continuous spectrum of phase velocities (this spectrum is responsible for the singularities of G, which are due to the vanishing of $U(z') - c$]; asymptotic behavior of this part can be investigated as in the case of Couette flow, and for smooth enough velocity profiles $U(z)$ the results are the same as for this special case (see again the above-mentioned publications by Case and Dikii). We see that, in spite of the fact that in an inviscid plane-parallel flow there usually exist only a few possible wave-like disturbances, any sufficiently smooth, two-dimensional

[1] According to Lyapunov, a trajectory $U_0(t)$, $0 \le t < \infty$, of a dynamic system in a phase space with a norm $||U||$ is stable, if for any $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that for any initial value $U(0)$ satisfying the inequality $||U(0)-U_0(0)|| < \delta(\varepsilon)$ the inequality $||U(t)-U_0(t)|| < \varepsilon$ is valid for any t. For more details about such stability and discussion of its application to fluid mechanics, see Sect. 4.1 in Chap. 4 of this series. Lyapunov's stability clearly depends on the selection of the norm $||U||$ which in studies by Dikii included the absolute values of the function and its two derivatives on z.

initial disturbance (it was found later that both stated conditions are essential) can grow in such a flow, without bound as $t \to \infty$, only at the expense of unstable wave-like disturbances, i.e., only in cases where there exist complex or multiple real eigenvalues of the corresponding Rayleigh's equation.

At first sight, the results of Dikii and Case, related to the general velocity profile, give the complete solution of the initial-value problem for small enough disturbances of steady inviscid flows; and, apart from this, they rehabilitate the normal-mode approach, showing that instability of these flows can be produced only by unstable normal modes. However, this first impression is incorrect. To say nothing of the fact that both these authors were dealing only with the simplest plane-parallel flows with smooth velocity profiles $U(z)$, we must stress again that here only smooth two-dimensional disturbances were studied (in spite of the fact that there is no analog of Squire's theorem valid for initial-value problems) and the possible strong transient growth of initially small disturbances (studied long ago by Orr for both two-dimensional and three-dimensional disturbances) was not even mentioned. Thus, important restrictions of the problem were accepted in the above-mentioned papers, and many questions related to the initial-value-problem approach were left there unsolved.

Interesting results about the asymptotic behavior of three-dimensional infinitesimal velocity disturbances of inviscid steady flows were obtained by Arnold (1972). He showed that in the case of some such flows the growth of three-dimensional unstable disturbances differs considerably from the case of the more ordinary flows and disturbances usually considered in the linear theory of hydrodynamic stability. In some exceptional flows studied by this author, an infinite number of very different types of unstable disturbances can exist and the absolute value of the disturbance vorticity $\zeta = \{\zeta_1, \zeta_2, \zeta_3\}$ can grow exponentially with time, regardless of the character, and location in the complex-variable plane, of the spectrum of exponents ω corresponding to "normal modes," i.e., to velocity disturbances proportional to $e^{i\omega t}$. All these exceptional flows are strictly three-dimensional and fairly complicated; therefore, the will not be considered in this book. However, it was remarked by Bogdat'eva and Dikii (1973) (see also Dikii (1976), Sect. 9) that Arnold's arguments show also that in the case of three-dimensional disturbances of a steady inviscid flow the length $|\zeta|$ of the disturbance vorticity vector can grow without bound with time, even in simple plane-parallel flows with velocity $U = \{U(z), 0, 0\}$ having no complex eigenvalues ω of the corresponding Rayleigh's equation (and thus being stable according to normal-mode formulation of linear stability theory). The growth of vorticity is linear in time in these simple flows and it does not indicate that the flow is unstable in the ordinary sense since all velocity components (and also the vertical component of vorticity) are here bounded for all values of t and only the horizontal vorticity components are rising without bound.

Bogdat'eva and Dikii based their modification of Arnold's arguments on the study of evolution of three-dimensional disturbances in a plane-parallel flow with given velocity profile $U(z)$. Here the equation for the vertical velocity $w(\boldsymbol{x}, t) = w(x, y, z, t)$

has the form

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2 w-U''(z)\frac{\partial w}{\partial x}=0 \tag{3.9}$$

(cf. again Chap. 2, Eq. (2.38)), i.e., it differs from Eq. (3.3) only by replacement of the two-dimensional Laplacian by the three-dimensional Laplacian $\nabla^2=\Delta$. Note now that in the case of two-dimensional disturbances, where $\boldsymbol{u}'(\boldsymbol{x},t)=\{u(x,z,t),0,w(x,z,t)\}$, it is enough to have only an equation for w, since here, when w is known, $u(x,z,t)$ can be easily determined from the continuity equation $\partial u/\partial x+\partial w/\partial z=0$ (and the pressure disturbance, if needed, can be determined form Eq. (2.37)). However, for general three-dimensional disturbances the values of $w(x,y,z,t)$ do not determine the velocity field $\boldsymbol{u}'(\boldsymbol{x},t)=\{u,v,w\}$; therefore, in this case at least one more equation is needed. (As the third and fourth equations needed for determination of all the velocity-component and pressure disturbances, the continuity Eq. (2.36) and (2.37) can then be used).

The most convenient equation to supplement Eq. (3.9) is the equation for the vertical vorticity component $\zeta_3=\partial v/\partial x-\partial u/\partial y$. It follows easily from Eq. (2.35) that this equation has the form

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\zeta_3-U'\frac{\partial w}{\partial y}=0. \tag{3.10}$$

When w is determined from Eq. (3.9), (3.10) allows ζ_3 to be determined, and when w and ζ_3 are known, the continuity Eq. (2.36) allows the horizontal velocities u and v to be found (cf. Eq.(3.15) below). Note also, that Eqs. (3.9) and (3.10) imply the equation

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left[U''\frac{\partial\zeta_3}{\partial x}-U'\frac{\partial\nabla^2 w}{\partial y}\right]=0, \tag{3.11}$$

which shows that the combination in the brackets can depend only on $x-Ut$, y, and z. let us now apply the Fourier transformation with respect to horizontal coordinates x and y and consider, for the sake of simplicity, only one Fourier component. This means that the velocity and vorticity components of the disturbance considered are assumed to be proportional to $\exp[i(k_1x+k_2y)]$ with amplitudes depending on z and t (but, contrary to the normal-mode method, the form of dependence on t is now not restricted). Let $k_1\neq 0$ (the case where $k_1=0$ will be considered later); then it follows easily from Eq. (3.11) that

$$\zeta_3=\frac{k_2U'}{k_1U''}(w''-k^2w)+e^{ik_1Ut}\left[\zeta_3-\frac{k_2U'}{k_1U''}\left(w''-k^2w\right)\right]_{t=0} \tag{3.12}$$

where again $k^2=k_1^2+k_2^2$. (If the initial-value problem has already been solved for the vertical velocity w, Eq. (3.12) gives the explicit solution of the same problem for the vertical vorticity ζ_3). Note now that if $U''(z)\neq 0$ for all values of z (i.e., if Rayleigh's condition is valid) and also in more general cases where Fjørtoft's condition is valid (and hence there exists such constant velocity K that $[U(z)-K]/U''(z)$ is a continuous nonnegative function of z) the functions $|w|, |w'|$ and $|w''|$) are bounded by some constants c_0, c_1, and c_2 which do not depend on t. (This statement follows, in particular, from Dikii's conservation law

$$\frac{d}{dt}\int_0^H [|w'|^2 + k^2|w|^2 + \frac{U-K}{U''}|w'' - k^2w^2|^2]dz = 0 \tag{3.13}$$

which can be proved in exactly the same way as its special form (2.57) was proved for the case of two-dimensional disturbances, if Eq. (2.53) is replaced by Eq. (3.9).[2]) Hence it follows from Eq. (3.12) that for flows where $U''(z) \neq 0$ everywhere, $|\zeta_3|$ is also bounded by some constant. Equation (3.12) also shows that $|\zeta_3|$ does not tend to zero as $t \to \infty$ even if the vertical velocity does, since the second term on the right-hand side of this equation represents a harmonic oscillation with fixed amplitude. However, this amplitude decreases to zero when the initial values of ζ_3, w, and w'' tend to zero; therefore the behavior of the vertical vorticity is not in conflict with Liapunov's stability (with appropriate definition of the norm) of the flow with respect to infinitesimal disturbances.

If $U''(z)$ vanishes at some point (or points), then Eq. (3.12) implies that $|\zeta_3 U''|$ is bounded by some constant. Therefore, in this case $|\zeta_3|$ can possibly grow with time without bound, at inflection points of the velocity profile.

For a given Fourier component of the disturbance the definition of the vorticity component z_3 and the equation of continuity take the forms

$$\zeta_3 = ik_1 v - ik_2 u, w' = -ik_1 u - ik_2 v, \tag{3.14}$$

where $w' = \partial w/\partial z$, and hence

$$u = i\left(k_1 w' + k_2\zeta_3\right)/k^2, v = i\left(k_2 w' - k_1\zeta_3\right)/k^2. \tag{3.15}$$

It follows from this that in Rayleigh's case (when $U''(z) \neq 0$ everywhere $|u|$ and $|v|$ do not tend to zero as $t \to \infty$ but are bounded by some constants (which become zero when the initial disturbance and its first and second derivatives on z tend to zero).

However, the horizontal vorticity components ζ_1 and ζ_2 in this case can increase infinitely when $t \to \infty$. To illustrate this Bogdat'eva and Dikii considered the simplest solution of Eqs. (3.9) and (3.10) where $w = 0$ (i.e., $\boldsymbol{u}'(\boldsymbol{x}, t) = \{u(x, y, z, t), v(x, y, z, t), 0\}$). Then Eq. (3.12) shows that here the Fourier components of the vertical vorticity have the form $\zeta_3 = A(z)\exp[i\{k_1(x - U(z)t) + k_2 y\}]$ where $A(z)\exp[i(k_1 x + k_2 y)] = \zeta_3(x, y, z, 0)$. According to Eq. (3.15), components u and v are proportional to ζ_3 when $w = 0$; hence these three functions of x, y, z, and t all have the same form. Therefore, here $-\partial v/\partial z = \zeta_1$ and $\partial u/\partial z = \zeta_2$ include terms of the

[2] It is true that Eq. (3.13) implies only that if the initial values of $|w|$, $|w'|$ and $|w''|$ are small enough, then their root-mean-square values will be bounded by some small constants at any value of t. However, using results of the initial-value-problem investigations, it is possible to prove that in fact the values of these functions of t and z will be uniformly bounded by some small constants for all $t > 0$ and $0 < z < H$; see, e.g., Dikii (1976).

form $B(z)k_1tU'(z)\exp[i\{k_1(x-U(z)t)+k_2y\}]$ representing harmonic oscillations with amplitudes growing linearly with time. If $w(x,y,z,0)\neq 0$, then the arguments become somewhat more complicated but the situation here, as a rule, is the same as for disturbances with vanishing vertical velocity. In fact, according to Eq. (3.12), in this case $\zeta_3(x,y,z,t)$ also includes the summand of the same form as above (with $A(z)$ equal to the initial Fourier amplitude of the combination in the brackets on the right-hand side of Eq. (3.12)), and, according to Eq. (3.15), u and v include the summands of this form too. However, ζ_1 and ζ_2 by definition include the derivatives $-\partial v/\partial z$ and $\partial u/\partial z$, respectively, and this implies that these vorticity components include harmonic oscillations with amplitudes proportional to time t. Thus we see that the horizontal velocity components u and v of a three-dimensional small disturbance of a plane-parallel steady flow with velocity profile $U(z)$ without inflection points are bounded at any t by small constants (but do not go asymptotically to zero as $t\to\infty$) while horizontal vorticity components ζ_2 and ζ_3 can here increase indefinitely with time. Whether under such conditions a flow must be called stable or unstable depends on the precise definition of the term "stability" employed (in particular, Arnold (1972) regarded the unbounded growth of vorticity as an indication of flow instability).

The existence of the strong dependence of the evolution of a disturbance on smoothness of its initial value was demonstrated by Willke (1972) on some rather peculiar examples relating to two-dimensional disturbances in an inviscid plane Couette flow. However, he began with consideration of the old Orr's solution (3.2), where $k_2=0$, of the linearized inviscid Navier-Stokes equations for the velocity field $\boldsymbol{u}=\boldsymbol{U}+\boldsymbol{u'}$, where $\boldsymbol{U}=\{bz,0,0\}$, $\boldsymbol{u'}=\{u(x.z.t),0,w(x,z,t)\}$, $0<z<H$. He neglected in this solution the terms involving hyperbolic functions, which are of importance only in close proximity to the solid walls at $z=0$ and $z=H$, and instead of the vertical velocity $w(x,z,t)$ he used the stream function $\psi(x,z,t)$ which satisfies the same equation as w. It has in fact already been noticed in Sect. 3.1 that, if k_1,k_3 and b are positive, then solution (3.2) for ψ implies that $|\psi(x,t)|=|\psi|$ increases from the initial value $|\psi|_0$ at $t=0$ to the value $|\psi|\max\approx[1+(k_3/k_1)^2]|\psi|_0$ at $t=t_{\max}\approx k_3/k_1b$, and then decreases, tending to zero as $t\to\infty$. According to Eq. (3.2) $|\psi|$ falls off like $(t-t_{\max})^{-2}$ on either side of the time $t_{\max}$. Willke noted in this respect that the above conclusions (with appropriate change of the values for $|\psi|_{\max}$ and $t_{\max}$) are valid not only for wave-like disturbances, where $\psi(x,z,0),w(x,z,0)$ and also the initial vorticity $\zeta(x,z,0)=-(\partial^2/\partial x^2+\partial^2/\partial z^2)\psi(x,z,0)=\partial u(x,z,0)/\partial z-\partial w(x,z,0)/\partial x$ are proportional to $\exp[i(k_1x+k_3z)]$, but also for disturbances with much more general initial values of the form $\zeta(x,z,0)=\exp(ik_1x)f(z)$, where $k_1\neq 0$ and the function $f(z)$ is twice continuously differentiable. (To obtain this last result, which according to Willke, was already known to Orr, it is only necessary to expand the function $f(z),0\leq z\leq H$, into Fourier's series and apply Orr's solution to all Fourier components. Note that proportionality to t^{-2} of the asymptotic decay rate of $|w|$ was later proved by Henningson et al. (1994) also for smooth three-dimensional disturbances in Couette flow with $k_1\neq 0$; See below about this matter.)

Then Willke considered more complicated cases where the initial disturbance is very irregular and does not satisfy the smoothness requirements used in previous investigations of Couette-flow stability. His investigation of these cases employed a nonstandard mathematical technique and some subtle analytical results; the conclusion obtained will be described only briefly below.

Willke assumed that the initial disturbance was specified not by a smooth ordinary function but by a "generalized function" (or, what is the same, a "distribution") which can have any degree of irregularity (see literature on such functions listed in Sect. 2.82, p. 84). In such a case it is natural to look for solutions of the corresponding dynamic equations which are also represented by generalized functions, i.e., to use the generalized-function (or else distribution-theoretic) approach to these differential equations (see the book by Gel'fand and Shilov (1958) devoted to discussion of this approach). This allowed Willke to analyse rather easily the laws of growth and decay for arbitrarily irregular solutions and to find estimates for the dependence of the highest possible growth rate on a numerical characteristic of the degree of irregularity of the generalized function describing the initial disturbance. To show that his estimates are strict, Willke considered a special sequence of complicated solutions represented by lacunary series of solutions of the form (3.2) (i.e., by infinite sums of functions of this form with $k_2 = 0$, fixed value of k_1, rapidly increasing values of k_3 and decreasing amplitudes W_0's; these sums do not converge at fixed points (x, z, t) but converge in some special sense to a definite generalized function). The first term of the above-mentioned sequence of solutions is an ordinary (but nondifferentiable) function; all the further terms are generalized functions related to first- or higher-order derivatives of continuous nondifferentiable functions. (These generalized functions can be accepted as flow variables in the same way as more common examples of such functions which include Dirac's δ-functions and their derivatives.) With the aid of some analytical results Willke showed that his "generalized solutions" can match all the growth-rate bounds found by him for irregular two-dimensional disturbances of a Couette flow. It turned out that these solutions can grow (for any length of time) like any positive integer power of t, and then decay arbitrarily slowly (like an arbitrarily small negative fractional power of t). The transient growth proportional to a high positive power of t can be reached only for very irregular disturbances represented by complicated generalized functions, but arbitrarily slow ultimate decay is possible for disturbances with continuous, but not differentiable, initial vorticity $\zeta(x, 0)$.

Willke's paper (1972) was devoted to investigation of rather exotic two-dimensional disturbances of an inviscid plane Couette flow, which are interesting only theoretically but not in practice (at least until now). However, in the 1980s and 1990s many more realistic examples of disturbance developments in plane-parallel inviscid flows were also studied, and some of them were again related to plane Couette flow. As typical examples we can mention the papers by Shepherd (1985) and Farrell (1987) where development of simple two-dimensional disturbances to an inviscid plane Couette flow in unbounded space $-\infty < x, y, z < \infty$ was considered as a proper model of some important meteorological phenomena. The unboundedness of space makes unnecessary the terms of Eq. (3.2) containing hyperbolic functions, which were added to satisfy the boundary conditions at the walls; hence here Orr's solution corresponding to two-dimensional disturbances with $k_2 = 0$ takes the form: $w(x,t) \propto \psi(\boldsymbol{x},t) \propto [k_1{}^2 + (k_3 - k_1bt)^2]^{-1} \exp[i(k_1x + (k_3 - k_1bt)z)]$, while $\zeta(\boldsymbol{x},t) \propto \exp[i(k_1x + (k_3 - k_1bt)z)]$. Using these equations Shepherd studied the evolution of a standing wave composed of a pair of two-dimensional Orr's waves with the same amplitude W_0 and wave vectors $(k_1, 0, k_3)$ and $(k_1, 0, -k_3)$. If $k_1 > 0, k_3 > 0$ and $b > 0$, then the first of these two waves will gain energy from the mean motion until time k_3/k_1b and will lose it after this time, while the second wave will lose energy at any time t. As to the total energy density of a standing wave, it will decrease monotonically if $k_1^2 > 3k_3^2$, i.e., $\theta = \arctan\ (k_3/k_1) < \pi/6$, and will at first increase and then decrease if $\theta > \pi/6$. For the isotropic collection of standing waves with homogeneous circular distribution of angles θ the total energy was found to remain constant in time, while for some other simple distributions of wave-vector directions moderate transient growth of energy was discovered. Farrell (1987) tried to estimate the value of the inverse-shear time scale b^{-1} appropriate for modeling the mid-latitude free-atmosphere processes and found that it is typically of the order of 10 h. Therefore, he concluded that the asymptotic laws of wave development in a steady Couette flow at $t \gg b^{-1}$ are usually irrelevant for modeling real atmospheric processes since for such times this model is unsuitable, but the transient growth of waves can serve as a reasonable model of the initial stage of the development of a disturbance at the expense of the energy of mean atmospheric motion. The accumulated wave energy can then be transferred to some quasi-stationary large- or medium-scale atmospheric structures (e.g., in cyclogenesis) or be spent to generate modal disturbances whose subsequent development must be studied within the framework of the normal-mode theory. In this respect Farrell studied the energetics of the solitary-wave and wave-packet developments, and considered the temporal evolution of the Couette flow disturbances for a number of specific initial values of the corresponding stream function (such as the "checkerboard initial value" $\psi(x,z,0) = A\cos(k_1x)\cos(k_2z)$; Shepherd's isotropic wave packet where $\psi(x,z,0) = AJ_0(kr)$, J_0 is the Bessel function and $r = (x^2 + z^2)^{1/2}$; a Gaussian isotropic wave packet where $\psi(x,z,0) = A\exp[-(kr)^2)]$; and an anisotropic localized disturbance where $\psi(x,z,0) = A\exp[-(k_0r)^2]\cos(k_1x + k_2z)$.) Most attention was paid to the last of these examples, where a considerable transient growth of disturbance energy (depending on and increasing with $s = k_2/k_1$) was found and where the time evolution of the disturbance shape agreed qualitatively with data of some meteorological observations.

General three-dimensional disturbances in bounded inviscid plane Couette flow between walls at $z = 0$ and $z = H$ can be analyzed by the method applied by Case (1960a) and Dikii (1960b) to the study of two-dimensional disturbances. Let us replace the one-dimensional Fourier transform on the left-hand side of Eq. (3.4) by the two-dimensional Fourier transform of the function $w(x,y,z,t)$ with respect to horizontal coordinates x and y (or, what is the same, assume that $w(x,y,z,t) = \tilde{w}(k_1,k_2,z,t)\exp\{i(k_1x + k_2y)\}$ and take the Laplace transform of $\tilde{w}(k_1,k_2,z,t)$ with respect to t). Using Eq. (3.9) instead of (3.3), it is easy to show that the resulting Fourier-Laplace transform of $w(x,y,z,t)$ (or the Laplace transform of $\tilde{w}((k_1,k_2,z,t))$, which we will denote by $\hat{w}(k_1, k_2, p; z)$, satisfies an equation very like Eq. (3.5). The only differences are that the two factors ik entering Eq. (3.5) must now be replaced by ik_1; $\tilde{w}(k; z, 0)$ must be replaced by $\tilde{w}(k_1, k_2, z, 0)$, the Fourier transform of $w(x, y, z, 0)$ with respect to x and y (or the value of the coefficient $\tilde{w}(k_1, k_2, z, t)$ at $t = 0$) while k^2 must be interpreted as $k_1^2 + k_2^2$. Denoting the Laplace-transform variable p by $-ik_1c$ we arrive at an equation of the form (3.5′) for $\hat{w}(k_1, k_2, -ik_1c; z)$ with ik replaced by ik_1 and $w_0(z) = (\partial^2/\partial z^2 - k^2)\tilde{w}(k_1, k_2, z, 0)$. Solution of this inhomogeneous linear equation can again be represented in the form (3.7), with Green's function G given by (3.8) with k replaced by k_1 (recall that in the case of Couette flow an explicit expression of the function G can be easily obtained). When $\hat{w}(k_1, k_2, p; t)$ is known, the vertical velocity w can be determined by the inversion formula for either a composite triple Fourier-Laplace integral generalizing (3.6) or, if w is assumed to be proportional to $\exp[i(k_1x + k_2y)]$, a one-dimensional Laplace integral.

The general expression for $w(x, y, z, t)$ obtained is close to that found by Case (1960a) for two-dimensional disturbances in Couette flow. Henningson et al. (1994) showed that according to this expression $\tilde{w}(k_1, k_2, z, t)$ decays as t^{-2} as $t \to \infty$ if $k_1 \neq 0$ and the above function $w_0(z)$ is smooth enough. Transient growth of the vertical velocity at small and moderate values of k_1t must also occur here for reasons explained by Orr as for back as 1907. However, the temporal behavior of the horizontal velocity components is different and the simplest way to show this is based on the study of the vertical vorticity ζ_3.

The simple Eq. (3.12) cannot be used in the case of Couette flow, where $U''(z) = 0$ at all values of z. However, when one Fourier component of the disturbance is studied and hence derivatives $\partial/\partial x$ and $\partial/\partial y$ can be replaced by factors ik_1 and ik_2, Eq. (3.10) for ζ_3 can be easily integrated to yield the result

$$\zeta_3(z, t) = \zeta_3(z, 0)e^{-ik_1U(z)t} + ik_2U'(z)e^{-ik_1U(z)t}\int_0^t w(z, t')e^{ik_1U(z)t'}dt' \qquad (3.16)$$

(the dependence of the vorticity ζ_3 and velocity w on horizontal coordinates, given by the factor $\exp[i(k_1x + k_2y)]$, is not indicated here). The first term represents the advection of the initial vertical vorticity by the flow velocity $U(z)$, while the second term represents the integrated effect of the vertical velocity, the so-called *lift-up* effect (see Landahl (1975)). Note that, according to Eqs. (3.15), the horizontal velocity components u and v include terms proportional to ζ_3; so this effect consists of the generation of horizontal velocity perturbations by lifting-up of the fluid elements in the presence of the mean shear. The lift-up effect increases with increasing U', as it must, and with decreasing spanwise wavelength $2\pi/k_2$ (for a physical explanation of this last dependence see, e.g., Henningson (1988) or Henningson et al. (1994)). In cases where vertical velocity disturbances decay fast enough as $t \to \infty$, the integral in (3.16) converges to a finite limit and the second term on the right side describes a *permanent scar* in the disturbance, convecting downstream with the local mean velocity, discovered by Landahl (1975) (see also Bogdat'eva and Dikii (1973) and Gustavsson (1978)). In Rayleigh's plane-parallel flows, where $U''(z) \neq 0$ for all z, the value of the scare is given by the last term in brackets on the right side of (3.12); for a plane Couette flow it was shown by Henningson et al. (1994) that $\zeta_3(z,t) \approx \zeta_3(z,0)\exp[-ik_1bzt] + ik_2w(z,0)\,bt$ at small values of t and $\zeta_3(z,t) \approx \left[\zeta_3(z,0) - i\pi\frac{k_2}{k_1}\frac{\sin hkz \sin hk(H-z)}{k \sin hkH}w_0(z)\right]e^{-ik_1bzt}$ at large values of t, where b and $w_0(z)$ have the same meaning as above. We see that $\zeta_3(t)$, aside from the convected initial value, contains a term which grows linearly for short times and for large times represents a permanent scar convected downstream, depending on the initial value of the vertical velocity w and on the wave numbers k_1, k_2, and $k = (k_1^2 + k_2^2)^{1/2}$.

Three-dimensional disturbances of an inviscid plane Couette flow between solid walls were also studied by Criminale and Drazin (1990) and Criminale et al. (1991). Their method for solution of the general initial-value problem was based on the transition to the "convected coordinate system" ($\xi = x - U(z)t, y, z$) used much earlier by Kelvin (1887a) and Orr (1907) (see also Craik and Criminale (1986)). Criminale and Drazin considered two particular solutions of the initial-value problem, while Criminal et al. found the explicit non-Fourier-transformed form of the general solution and then considered at length the case where $w(x,y,z,0) \propto \exp[i(k_1x + k_2y)]$. Most attention was paid here to the particular case where $(\partial^2/\partial z^2 - k^2)w(x,y,z,0) = \exp[i(k_1x + k_2y)]W_0(z)$ and $W_0(z) = W_0(z; z_0, L)$ has the form of a rectangular pulse of unit area with the center at $z = z_0$ and the thickness $2L$ (hence $W_0(z) \to \delta(z - z_0)$ as $L \to 0$). Since equations obtained for the velocity components (u,v,w) proved to be complicated, an integrated positive measure of disturbance size (the "energy," whilch for purely two-dimensional disturbances coincided with the ordinary kinetic energy density) was introduced. Evaluation of this measure showed that solution of the initial-value problem considered usually decays monotonically with time (or, as an exception, preserves their size); hence, the phenomenon of the transient disturbance growth is here mostly lacking. The rate of decay is practically independent of the position of the pulse $W_0(z)$, but depends strongly on its relative thickness $\mu = L/H$, dimensionless wave number $\kappa = kH$, and wave-vector orientation $\theta = \arctan(k_2/k_1)$, growing with increasing μ and κ and with a decrease of θ from $\pi/2$ to zero. In particular, if $\theta = 0$ (i.e., the disturbance is two-dimensional) and either $\mu = 0$ or $\kappa \ll 1$, then the "energy" of the disturbance remains constant with time; the same is true for cases where $\mu \neq 0$ and κ is not small but $\theta = \pi/2$. Moreover, if $\mu = 0$ (i.e., $W_0(z) = \delta(z - z_0)$) and $\theta \neq 0$ (i.e., the disturbance is really three-dimensional), then the horizontal velocities velocities u and v at $z = z_0$ grow linearly with time and at $|z - z_0|/H \ll 1$ their growth is practically linear up to very large values of bt (i.e., here there is a considerable transient algebraic growth of the disturbance). This shows again that in cases of singular initial conditions the behavior of disturbances can differ considerably from that for smooth initial values (see also the paper by Willke (1972) discussed above).

Criminale and Drazin (1990) considered, along with the case of a plane Couette flow, development of disturbances in two other steady inviscid plane-parallel flows with piecewise linear velocity profiles: in two-layered unbounded flow where $U(z) = b_1z$ for $z > 0, U(z) = b_2z$ for $z < 0$, and $b_2 \neq b_1$; and in a piecewise-linear model of a boundary layer where $0 \leq z < \infty$, and $U(z) = bz$ for $0 \leq z < H$, while

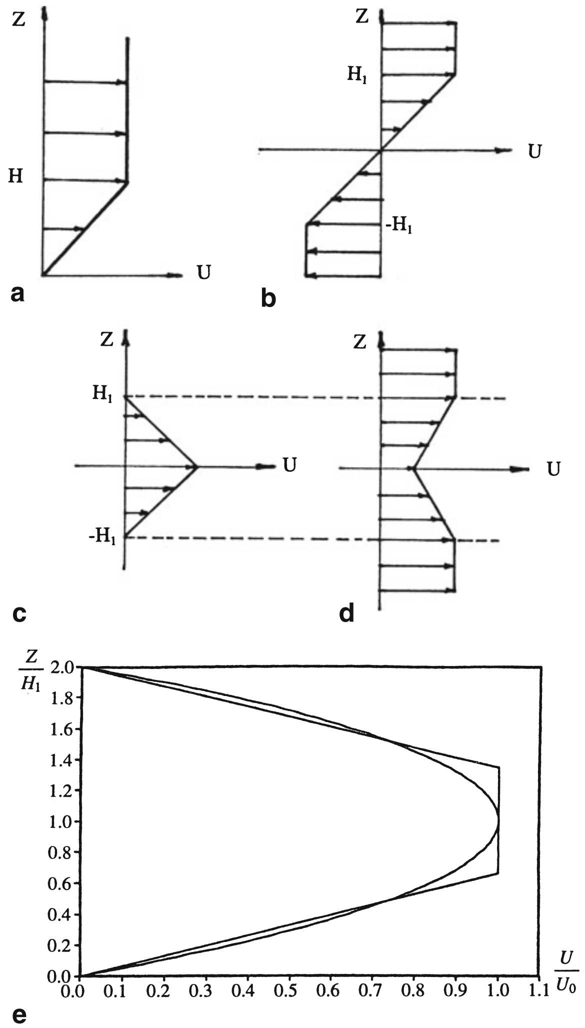


Fig. 3.1 Piecewise-linear models of velocity profiles for some plane-parallel fluid flows: **a** model of a boundary-layer profile used by Gustavsson (1978) and Criminale and Drazin (1990); **b** model of a mixing-layer profile used by Bun and Criminale (1994) and Criminale et al. (1995); **c** and **d** models of plane-jet and plane-wake profiles by Criminale et al. (1995); **e** model of a plane Poiseuille-flow profile by Henningson (1988) compared with the exact parabolic profile

$U(z) = bH = U_0$ for $z > H$ (see Fig. 3.1a). For disturbances in these flows the following schematic initial conditions at $t = 0$ were used; (*a*) unit point pulse of velocity, (*b*) unit point pulse of vorticity, (*c*) monochromatic three-dimensional plane wave of velocity, and (*d*) a similar wave of vorticity. The possibility of stimulation of nonlinear effects by transient algebraic growth of an initially small disturbance was discussed by the authors, and such growth was illustrated by results related to the case of the initial condition (*c*), first considered, for disturbances of a Couette flow, by Orr (1907).

Later Bun and Criminale (1994) and Criminale et al. (1995) applied the initial-value-problem approach to detailed study of the evolution of three-dimensional disturbances in schematic piecewiselinear models of an inviscid plane mixing layer (with profile $U(z)$ shown in Fig. 3.1b (and also in Fig. 2.31e in Chap. 2)) and (in the second of those papers) also of a plane jet (Fig. 3.1c) and a plane wake (Fig. 3.1d). It was mentioned in Sect. 2.93 that as long ago as 1894 Rayleigh proved that unstable normal modes (growing exponentially as $t \to \infty$) exist in a plane mixing layer with the velocity profile given in Fig. 3.1b. It is easy to show that the same statement is also true for inviscid piecewise-linear free shear flows in an unbounded space with the velocity profiles shown in Figs. 3.1c and 3.1d. (Recall that in Sect. 2.93 models of viscous plane jets and wakes with analytic velocity profiles, differentiable everywhere, were considered and the results showed that such flows definitely have unstable normal modes of disturbance in the inviscid case too; see also the remarks following Eq. (2.87), which contains a number of references related to this topic. In the case of piecewise-linear jet and wake models the corresponding proofs are even simpler since here the exact analytic solutions for equations of the linear stability theory may be used instead of the approximate numerical solutions used in the cases of analytic velocity profiles.) It might be concluded from this that consideration of the general initial-value problem for piecewise-linear plane free flows in an unbounded space is superfluous, since the classical normal-mode theory has already proved that these flows are unstable with respect to small disturbances, which can grow here as exp ($\omega^{(i)}\, t$) as $t \to \infty$, where $\omega^{(i)}$ is the greatest imaginary part of the discrete eigenvalues of Rayleligh's equation (2.48) with $c = \omega/k$. However, results by Bun and Criminale (1994) and by Crilminale et al. (1995) show that this conclusion is incorrect in many cases. According to these results the behavior of three-dimensional disturbances in these flows is dominated by the exponential growth of unstable normal modes only for very large times, while for earlier times the transient algebraic growth, which is in fact due to the continuous spectrum of Rayleigh's equation, plays the main part. This transient growth can lead to a quite substantial rise of the velocity disturbances before the exponentially-growing normal modes become dominant. Just this rise apparently produces the early nonlinear transformation of the whole flow structure which has often been observed experimentally. Similar results were deduced by Criminale et al. from the numerical solution of the appropriate initial-value problem for the case of inviscid jets and wakes with differentiable analytic velocity profiles.

Let us now mention two other investigations of the initial-value problem for small disturbances in plane-parallel steady inviscid flows with piecewise linear velocity profiles. In the PhD thesis by Gustavsson (1978) a general solution of the problem was given for the same piecewise linear model of a boundary-layer flow that was later considered by Criminale and Drazin (and is sketched in Fig. 3.1a). Gustavsson paid special attention to the case of a localized three-dimensional initial disturbance. Then Henningson (1988) applied Gustavsson's method to study the evolution of disturbances in the piecewise linear model of a plane Poiseuille flow shown in Fig. 3.1e. Note also the paper by Breuer and Haritonidis (1990) who numerically solved the initial-value problem for a localized disturbance in a plane-parallel boundary-layer flow with the Blasius velocity profile. (As mentioned above, in the case of curved velocity profiles the initial-value problem for small velocity disturbances can only be solved numerically.) In these investigations (and also in the survey by Henningson and Alfredsson (1996)) it was stressed that the general solution of the initial–value problem includes terms of two different types, which are directly separated in analytical solutions and can also be detected in numerical results.

The first of these types is represented by terms of the Fourier–transformed solutions for velocity components which contain the factor $e^{-ik_1U(z)t}$ (see e.g. Eqs. (3.1), (3.2), (3.11) and (3.16) above, which include a number of such terms). Here the inversion of the Fourier transform leads to functions of *y, z* and $\xi = x - U(z)t$; this indicates that the corresponding disturbances are convected streamwise with the local flow velocity $U(z)$. Such disturbances were, in fact, first discovered by Kelvin (1887a) and Orr (1907) and were also at length studied by Criminale and his collaborators in the papers indicated above; they often undergo considerable transient growth followed by decline. These disturbances were called *convective* by Gustavsson (1978); in the case of inviscid flow they correspond to a continuous spectrum of Rayleigh's equation (which, for bounded flows, fills the interval $U_{\min} \le c \le U_{\max}$ of the real axis; see Sect. 2.82, p. 85). Besides convective terms, Fourier transforms of solutions of the initial-value problem also include "terms of the second type," proportional to $e^{-ik_1c(k)t}$, where $k = (k_1^2 + k_2^2)^{1/2}$ and $c(k)$ is a special function that appears in the course of the solution. In some cases there are several functions $c(k)$, appearing in different terms of the second type; these functions can be either real or complex, and they can also be determined by analysis of the corresponding Rayleigh's equation (which, according to Chap. 2, has the same form (2.48) for two- and three-dimensional disturbances, and includes only k but not k_1 and k_2). Inversion of the Fourier transform translate these terms into functions of $x\text{-}c(k)t$, which describe three-dimensional waves with the wave vector $k = (k_1, k_2)$ and streamwise phase velocity $c(k)$ (or $\Re e\ c(k)$ if $c(k)$ is complex), and are related to normal modes studied in Chap. 2.[3] Since the frequency $\omega = k_1c(k)$ (or frequencies $\omega_i = k_1c_i(k)$, if there are several functions $c(k)$) of waves corresponding to second-type terms) depend on k, these waves are *dispersive*; therefore Gustavsson called the collection of these waves the *dispersive* disturbances.

The normal-mode approach to stability theory paid most attention to individual normal modes with given values of k and c (or ω). However, in studies of the initial-values problems, all plane waves making non-zero contribution to the Fourier expansion of the initial disturbance must be simultaneously taken into account. In the important case of an initially-localized disturbance, the Fourier expansion includes a vast collection of waves with different wave vectors $k = (k_1, k_2)$. Hence here the laws of wave-packet evolution must be applied.

The convective disturbances are convected streamwise with velocity $U(z)$; hence their evolution is relatively simple. However, the evolution of dispersive disturbances with angular velocities $\omega = k_1c(k)$ is more complicated. According to kinematic wave theory (see, e.g., Landau and Lifshitz (1958, 1987), Sects. 66, 67, or Whitham

[3] They are not identical to normal modes since the functions $c(k)$ do not coincide with the discrete eigenvalues of Rayleigh's eigenvalue problem, which do not exist in many important cases. In fact, functions $c(k)$ correspond to limits, as $\mathrm{Re} \to \infty$, of discrete eigenvalues of the Orr-Sommerfeld eigenvalue problem for given profile $U(z)$; their determination from the Rayleigh equation requires careful examination of the analytic continuation fo this equation into the complex-variable plane.

(1974)), if there is a wave packet which is concentrated in a bounded spatial region and is composed of waves with various wave numbers (k_1, k_2), then the group of waves with the "central wave" of the form $A(z)\exp\{i[k_1(x-ct)+k_2y]\}, c = c(k) = c\left(\sqrt{k_1^2+K_2^2}\right)$, is moving, not simply streamwise with the phase velocity $c\,(k)$ but with the two-component horizontal group velocity $\boldsymbol{C} = \{C_x(k_1, k_2), C_y(k_1, k_2)\}$ where

$$C_x = \frac{\partial \omega}{\partial k_1} = c + \frac{{k_1}^2}{k}\frac{dc}{dk}, C_y = \frac{\partial \omega}{\partial k_2} = \frac{k_1 k_2}{k}\frac{dc}{dk}. \tag{3.17}$$

If the initial disturbance is concentrated in a close vicinity of the point $(0, 0, z_o)$, then at time $t > 0$ its dispersive waves with wave numbers (k_1, k_2), where $k_1^2 + k_2^2 = k^2$ is fixed, will form a packet whose horizontal projection will be concentrated near the point (x, y) where

$$\frac{x}{t} = c + \frac{k_1^2}{k}\frac{dc}{dk}, \quad \frac{y}{t} = \frac{k_1 k_2}{k}\frac{dc}{dk}. \tag{3.17$'$}$$

Gustavsson (1978) noted that Eqs. (3.17′) imply the following simple result

$$\left(\frac{x}{t} - c - \frac{k}{2}\frac{dc}{dk}\right)^2 + \left(\frac{y}{t}\right)^2 = \left(\frac{k}{2}\frac{dc}{dk}\right)^2. \tag{3.18}$$

It follows from this that in the case of disturbance initially located near the point $(0, 0, z_0)$, the dispersive wave components with wave numbers (k_1, k_2), where ${k_1}^2 + {k_2}^2 = k^2 =$ const., spread horizontally over a circle whose center at time t is at the point $(ct + ktdc/dk,\ 0)$, with radius $(kt/2)dc/dk$. The location of these circles corresponding to different values of k is shown, for piecewise-linear models of boundary-layer and plane Poiseuille flows, in figures presented by Gustavsson (1978); (see also Henningson (1988); Henningson et al. (1994); and Henningson and Alfredsson (1996)). (In Poiseuille flow there are two different functions $c_1(k)$ and $c_2(k)$ corresponding to disturbances symmetric and antisymmetric with respect to the channel midplane $z = H/2$, However, the waves with phase velocity $c_1(k)$ are characterized by much greater spreading than waves with velocity $c_2(k)$, which moreover take quite different values in the cases of piecewise-linear and of real, parabolic, Poiseuille profiles.) Note also that, according to the above-mentioned papers, the spreading of localized disturbances by wave dispersion (i.e., the dispersive effect) forms only a small part of the total disturbance spreading, which is mainly due to Landahl's lift-up effect mentioned above.

Henningson (1988) and Breuer and Haritonidis (1990), who considered quite different flows, both made careful calculations for the case where the initial disturbance had the form of two pairs of counter-rotating eddies, schematically shown in Fig. 3.2. Here the initial streamwise velocity disturbance u is equal to zero and therefore $v = -\partial\psi/\partial z, w = \partial\psi/\partial y$ where the stream function $\psi(x, y, z)$ is very close to zero everywhere outside a small spatial region surrounding the "central point" with coordinates $(0, 0, z_0)$. [This form of the initial disturbance was chosen be-

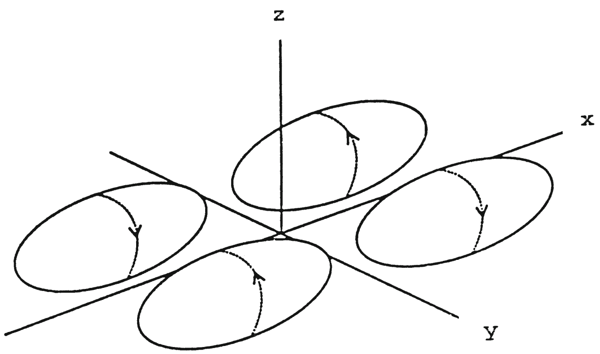


Fig. 3.2 Schematic shape of the initial velocity field for a localized disturbance considered by Russell and Landahl (1984), Henningson (1988), and Breuer and Haritonidis (1990)

cause it was used for similar purposes by Russell and Landahl (1984).) In Figs. 3.3a, b taken from Breuer and Haritonidis (1990) (and reprinted also by Henningson et al. (1994)), contours of vertical and horizontal disturbance velocities w and u are plotted in the (x, z) plane for $y = 0$ and for several values of t. The distribution of the vertical velocity does not change qualitatively with t, but typical values of w decrease, and the entire structure moves downstream with a velocity approximating the typical group velocity of waves in the boundary layer. Simultaneously, the velocity distribution expands in the streamwise direction, which also agrees well with theoretical predictions for dispersive disturbances. Contours of $w = \text{const.}$ in the (x, y) plane, also presented by Breuer and Haritonidis (1990) for one value of z, show quite definitely the development of the wave-packet-like character of the vertical velocity distribution with increasing time t. The only feature in Fig. 3.3a resembling convective disturbances is the patch of low-speed fluid moving streamwise at large heights (the edge of the boundary layer is located near $z = 3\delta^*$) ahead of the main disturbance, with a speed approaching the free-stream velocity U_0.

The distributions of the streamwise disturbance velocity u shown in Fig. 3.3b contrast strongly with the distributions of w. Since initially $u = 0$, transient growth of streamwise velocity clearly must take place. Computational results in Fig. 3.3b demonstrate that the growth of $|u|$ is dominated by the lift-up effect. This effect at first produces a region of negative values of u which travels at the local undisturbed velocity and is immediately followed by a high-speed region of fluid. The mean velocity gradient existing in the lower part of the boundary layer generates the tilting of the shear layer between low-speed and high-speed regions and the stretching of the structure in the x direction; as a result, an inclined shear layer is formed whose intensity and streamwise length increase with time. Thus, the streamwise velocity disturbances are mainly of a convective nature.

Constant-velocity contours in the (x, y) plane were also presented by Henningson (1988) (see also Henningson et al. (1994)) for the piecewise-linear model of plane Poiseuille flow, showing normal and streamwise disturbance velocities at a fixed value of z and several values of t. These contours show the same typical features as the later results of Breuer and Haritonidis. Here again vertical velocity disturbances w are mostly dispersive, and their contours show that a wave-packet with wave-crests

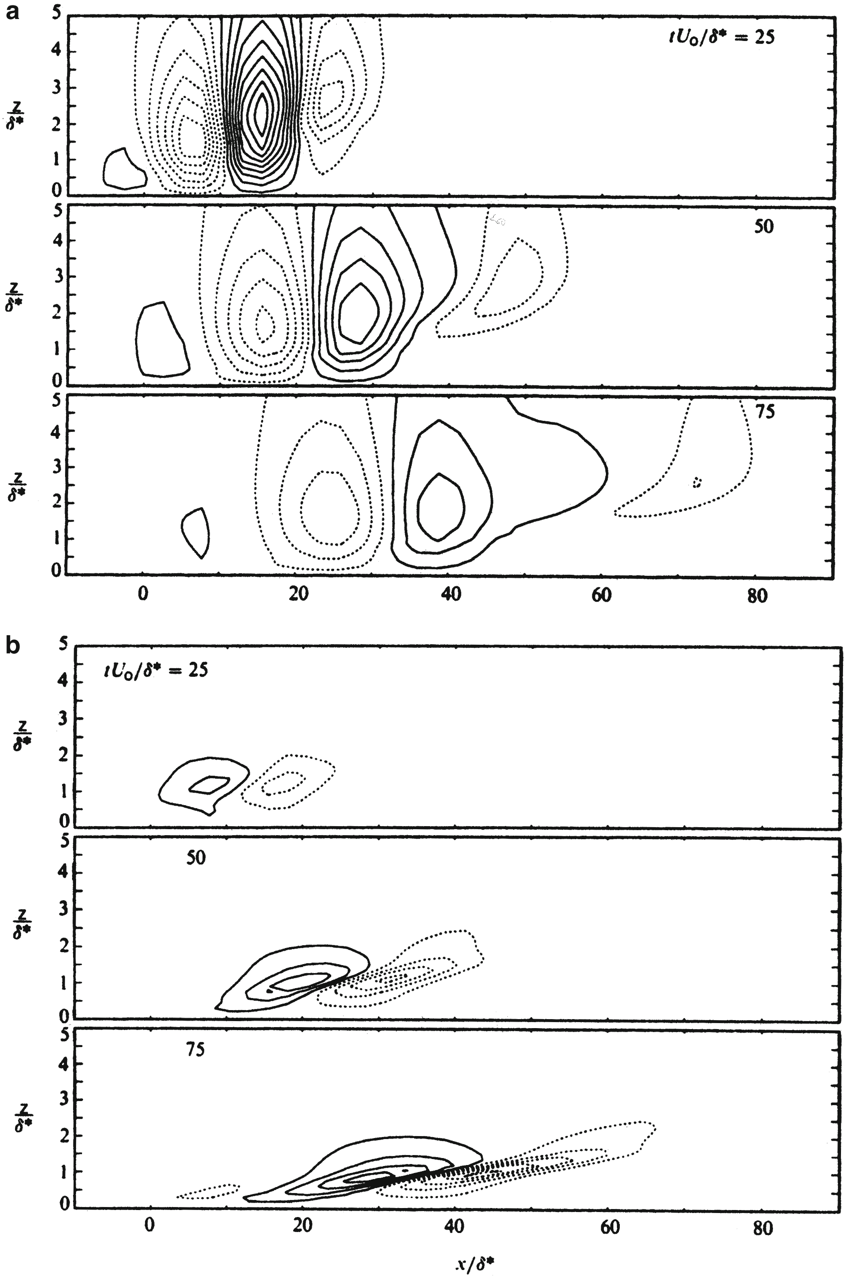


Fig. 3.3 Computations by Breuer and Haritonidis (1990) of contours in the (x, z)-plane of the vertical velocity $w(x, y, z, t)$ (**a**), and the streamwise disturbance velocity $u(x, y, z, t)$ (**b**) in an inviscid boundary-layer flow at $y = 0$ and several values of t. Velocities, lengths, and times are scaled with the free-stream velocity U_0, displacement thickness δ^*, and ratio δ^*/U_0, respectively. *Solid* and *dotted lines* represent positive and negative velocity values; contour spacing is $0.2w_0$ for w-contours, and to $2w_0$ for u-contours, where w_0 is the maximum value of $w(x, y, z, 0)$

swept back at 45° emerges rather quickly from the initial disturbance. In this case the amplitude of w disturbances also decreases with time, while the disturbance as a whole spreads in the horizontal plane. In contrast to this, the streamwise component u grows quickly, and a moderate values of t its typical value exceeds that of w more than tenfold and is dominated, not by the wavepacket, but by an intense shear layer. Comparison of these results with those of Breuer and Haritonidis gives the impression that the main features of the disturbance development are not too sensitive to the details of the undisturbed velocity profile.

Breuer and Haritonidis (1990) also performed an experimental investigation of disturbance evolution in a laboratory boundary layer on a flat plate. The initial disturbance was created by the impulsive motion, first up and then down, of a small flush-mounted membrane at the wall and thus consisted of two small-amplitude three-dimensional disturbances of opposite signs. The observed disturbance evolution during small enough initial time intervals was found to be in good qualiltative agreement with the results of inviscid calculations, showing the rapid formation of an intense inclined shear layer and a strong increase of streamwise disturbance velocity. Further downstream, viscous effects were detected; at not too small initial amplitude of disturbance, the nonlinearity clearly manifests itself.

3.2.2 Further Examples of Unstable Disturbances in Inviscid Plane-Parallel Flows

It was shown above that transient growth of initially small disturbances often takes place in plane-parallel flows of inviscid fluid, and can lead to immense increases of disturbance size and energy. It was also proved that, under rather general conditions, the horizontal components u and v of the disturbance velocity do not decay with time. Now we will consider several specific examples of unstable disturbances, some of which, apparently, can be of importance in many flows encountered in nature and engineering.

It was mentioned, in particular, by Willke (1972) and Criminale et al. (1991) that in a plane-parallel inviscid flow the vertical velocity w of a small-enough disturbance, which is independent of the streamwise coordinate x (i.e., such that $k_1 = 0$ in its Fourier representation) does not damp with time. It was, however, simply shown by Ellingsen and Palm (1975) that if in the case of such disturbance the velocity shear $U'(z)$ of the primary flow and the initial vertical disturbance velocity w do not vanish, then the streamwise velocity u of the disturbance increases indefinitely with time so that the flow is clearly unstable with respect to this disturbance. In fact, in this case Eqs. (2.35) and (2.37), with $n = 0$, take the forms

$$\frac{\partial u}{\partial t} + U'w = 0, \quad \frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{\partial w}{\partial t} = -\frac{1}{p}\frac{\partial p}{\partial z}, \tag{3.19}$$

$$\nabla^2 p = 0, \tag{3.20}$$

where $\nabla^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$ and the disturbances of flow variables are denoted by small letters without primes. Since $p(y, z, t)$ denotes a pressure disturbance vanishing at infinity, it follows from (3.20) that $p = 0$. Equation (3.20) shows that all nonconstant Fourier components of p vanish and thus $p = 0$ is the only physically acceptable solution; recall also that to prove instability it is enough to show that there exists a growing solution.) This implies that w (and also v) do not depend on time; therefore, according to the first Eq. (3.19)

$$u(y, z, t) = u(y, z, 0) - U'(z)w(y, z, 0)t, \tag{3.21}$$

and hence, if $U'(z) \neq 0$ and $w(y, z, 0) \neq 0$, then $|u|$ increases linearly with time. This result of Ellingsen and Palm was apparently one of the first examples of disturbances growing algebraically (and not exponentially) without bound as $t \to \infty$.

Of course, disturbances independent of x, which preserve their intensity from $-\infty$ to ∞ in the streamwise direction, are not those that are really encountered in turbulent flows. It was, however, shown by Landahl (1980) that similar arguments can also be applied to the much more important class of three-dimensional localized initially-infinitesimal disturbances, vanishing at time $t = 0$ outside some bounded region surrounding the coordinate origin. Let us now present his arguments.

In the general case where independence of x is not assumed, the partial derivative $\partial/\partial t$ must be replaced in Eqs. (3.19) by $D/Dt = \partial/\partial t + U(z)\partial/\partial x$, while zero right-hand sides of the first Eq. (3.19) and Eq. (3.20) must be replaced by $-\rho^{-1}\partial p/\partial x$ and $-2\rho U'(z)\partial w/\partial x = -\partial(2\rho U'w)/\partial x$, respectively (see Eqs. (2.35) and (2.37) in Chap. 2). Let us now integrate the corrected form of the first Eq. (3.19) over the whole x-axis. Then for the localized disturbance we obtain

$$\frac{\partial \overline{u}}{\partial t} = -\overline{w}U'(z), \quad \text{where} \quad \overline{u} = \int_{-\infty}^{\infty} u dx, \quad \overline{w} = \int_{-\infty}^{\infty} w dx, \tag{3.22}$$

and it is assumed that both the integrals in the given definitions of $\overline{u}$ *and* $\overline{v}$ exist. Integration over x of the corrected third of the Eqs. (3.19) yields

$$\frac{\partial \overline{w}}{\partial t} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z}, \quad \text{where} \quad \overline{p} = \int_{-\infty}^{\infty} p dx. \tag{3.23}$$

However, it follows from the solution of the corrected Eq. (3.20) (i.e., from Eq. (2.37)) that p may be represented as a derivative with respect to x of a finite integral, which vanishes for large values of x. Hence $\overline{p} = 0$, and from Eq. (3.23) it follows that $\overline{w}$ is independent of time. Integration of Eq. (3.22) then gives

$$\overline{u}(y, z, t) = \overline{u}(y, z, 0) - U'(z)\overline{w}(y, z, 0)t. \tag{3.24}$$

This result has the same form as Eq. (3.21), but now velocities of a localized flow disturbance integrated over the x-axis replace the x-independent point values. It is clear that the new equation has an immeasurably wider domain of application than Eq. (3.21) due to Ellingsen and Palm.

Equation (3.24) does not imply that the velocity u itself necessarily increases with time. In fact, the integrated velocity $\overline{u}$ could increase because the disturbance spreads streamwise without becoming more intense, and Landahl showed that this is exactly what occurs. He carefully inspected the general solution of the initial-value problem for a localized three-dimensional disturbance in a steady plane-parallel flow between solid walls, with velocity profile $U(z)$, and proved that under rather general conditions, guaranteeing the absence of exponentially-growing normal modes in the flow, the streamwise propagation speed of such disturbance lies between minimum and maximum values of $U(z)$, $U_{\min}$ and $U_{\max}$, and that asymptotically the front of the disturbance propagates just with the velocity $U_{\max}$ and its back with the velocity $U_{\min}$. Then, using Eq. (3.24) Landahl showed that as $t \to \infty$

$$E(y,z,t) \equiv \frac{1}{2}\int_{-\infty}^{\infty}(u^2+v^2+w^2)dx > \frac{[U'(z)]^2\overline{w}^2(y,z,0)t}{2(U_{\max}-U_{\min}+\Delta)} \tag{3.25}$$

for any $\Delta > 0$. Hence, if $U'(z) \neq 0$ and $\overline{w}(y,z,0) = \overline{w}_0 \neq 0$, then the total kinetic energy $E(t)$ grows at least as fast as linearly as $t \to \infty$, but this growth is naturally explained by linear growth of the streamwise extent of the disturbed region. Landahl's analysis of the initial-value-problem solution for a three-dimensional localized disturbance shows that here, under wide conditions, $|w|$ decays as t^{-1} at large times; since the size of the disturbed region is growing linearly in time, such decay is consistent with constancy of the integrated vertical velocity $\overline{w}(t)$. However the same analysis leads to the conclusion that, in the case of a localized three-dimensional disturbance, the value of $|u|$ remains bounded as $t \to \infty$ (see also Eqs. (3.15) and (3.12) above). Hence the linear growth of the integrated streamwise velocity, which follows from Eq. (3.24), must be explained by streamwise elongation of the disturbed region. Such elongation clearly transforms any group of small localized three-dimensional disturbances of a shear flow into a streaky structure–a collection of longitudinal narrow streaks of either high-speed or low-speed fluid. At present there are numerous data, both from the flow-visualization experiments and from direct numerical simulations, which show that longitudinal streaky structures arise very often in transitional and fully turbulent shear flows. Such structures are typical, in particular, for flows behind the "turbulent spots" that appear during the initial stage of transition to turbulence (see, e.g., Sect. 2.1, Chap. 2), and in the near-wall regions of turbulent flows bounded by solid walls. The widespread occurrence of streaky turbulent structures gives ground for the suggestion that Landahl's (1980) algebraic growth of the energy of streamwise velocity disturbances can be of fundamental importance in many flows where transition to turbulence and formation of complicated eddy structures occur.

Landahl's simplified stability analysis of 1980 predicted the asymptotic behavior at $t \to \infty$ of the streamwise and vertical velocity disturbances integrated over the x-axis, $\overline{u}(y,z,t)$ and $\overline{w}(y,z,t)$. However for better understanding of the nature of eddy structures generated by algebraic instability it was necessary to study, at greater length, the behavior of disturbances in fluid flows of various types. Having this in mind and taking into account the important role played by near-wall flows in all phenomena related to interactions of fluid flows with the contiguous solid bodies (in particular, in fluid friction and heat or mass exchange), Landahl (1990, 1993a, b, 1996, 1997) gave much attention to the evolution of a weak three-dimensional disturbance arising in the near-wall layer of a turbulent boundary layer.

Landahl (1993a, b) stressed that there are three different physical processes involving (and affecting) weak disturbances: *i*) disturbance interaction with the shear of the undisturbed flow; *ii*) viscous damping of velocity disturbances; and *iii*) their nonlinear interactions with themselves. The first two processes are described by the linearized dynamic equations, and only the study of the third process requires the use of the full nonlinear Navier-Stokes equations. These three processes are characterized by the following quite different time scales: *i*) shear-interaction time scale $t_s = [dU/dz|_{z=0}]^{-1}$; *ii*) viscous-interaction scale $t_v = [L^2\nu^{-1}(dU/dz)^{-2}]^{1/3}$; and *iii*) nonlinear-interaction scale $t_n = L/u^{(0)}$, where L is the typical streamwise length of the initial disturbance and $u^{(0)}$ is the scale of the streamwise disturbance velocity. Taking into account observational data on weak velocity disturbances in the near-wall regions of turbulent boundary layers on a flat plate, Landahl estimated that here typically $t_v/t_s \approx 20$, $t_n/t_s \approx 100$. Hence usually $t_s \ll t_v \ll t_n$ and therefore the viscous damping beging to play a role at relatively late times (namely, at $t = O(t_v)$); at earlier times the evolution of a disturbance can be accurately enough described by the inviscid linear stability theory. Moreover, the nonlinear terms of dynamic equations need not be taken into account until still later (at $t = O(t_n)$). In this section, devoted to the inviscid linear theory, we shall consider only the initial stage of the evolution of weak disturbances in a boundary layer flow.

Let us restrict ourselves to the plane-parallel model of a boundary-layer flow, which neglects the influence of the weak nonparallelism caused by the dependence of the boundary-layer thickness on x. Then the inviscid linearized dynamic equations will have the form of Eqs. (2.35–2.36) with $\nu = 0$, implying Eqs. (2.37) and (2.38), again with $\nu = 0$ (cf. also remarks preceding Eqs. (3.22) and (3.23)). Replacing the streamwise coordinate x by the convected coordinate $\xi = x - U(z)t$, and then integrating the first two of Eqs. (2.35) and Eq. (2.38) with $\nu = 0$ (i.e., the first two Eqs. (3.19) and (3.9) corrected as above) over "Lagrangian time," following a fluid element moving with the undisturbed velocity $U(z)$, Landahl obtained the equations:

$$u(\xi, y, z, t) = u_0(\xi, y, z) - U'(z)l - \Pi_x/\rho, \tag{3.26a}$$

$$v(\xi, y, z, t) = v_0(\xi, y, z) - \Pi_y/\rho, \tag{3.26b}$$

$$\nabla^2 w(\xi, y, z, t) = \nabla^2 w_0(\xi, y, z) + U'' l_x. \tag{3.26c}$$

In these equations the subscript zero denotes the initial values, subscripts x and y denote the partial derivatives, and the *liftup* of a fluid element l and the *pressure impulse* Π are determined as follows

$$l(x, y, z, t) = \int_0^t w(x - U(z)(t - t_1), y, z, t_1)dt_1, \tag{3.27}$$

$$\Pi(x, y, z, t) = \int_0^t p(x - U(z)(t - t_1), y, z, t_1)dt_1.$$

Substituting the definition (3.27) of the liftup l into Eq. (3.26c) we obtain an integro-differential equation for the vertical velocity w. On the other hand, substituting the explicit solution of the Poisson equation (3.26c) (see Eq. (3.30) below) in the first Eq. (3.27) we arrive at an intergro-differential equation for the liftup l. Differentiating Eq. (3.26a) on x and Eq. (3.26b) on y and taking into account that $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$ for an incompressible fluid, we find the following two-dimensional Poisson equation for the pressure impulse Π:

$$\nabla_h^2 \Pi = -w_z - u_{o\xi}(\xi, y, z) - v_{0y}(\xi, y, z) + U'(z)l_x, \quad \nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{3.28}$$

where again subscript zero denotes initial values while subscripts ξ, y and x denote differentiation. Hence Π can be expressed in terms of w, l, and the initial disturbance velocities. When Π and l are known, horizontal velocities u and v can be easily found from Eqs. (3.26a,b). (The value of l determines the most important “lift-up term” $U'(z)l$ of the streamwise-velocity equation (3.26a), which was in fact first introduced long ago by Prandtl (1925); see also Landahl (1985)). Equations (3.26–3.28) were applied by Landahl (1993a, 1996, 1997) to determination of the asymptotic (“long-time”) behavior of all the velocity and vorticity components of a localized disturbance with initial velocities u_0, v_0 and w_0 vanishing outside a bounded domain having the streamwise length scale L and the “center” at a point with $x = y = 0$.

Analysis of the behavior of l, w, u and v for $t \to \infty$ (i.e., for $T \to \infty$, where $T = tU'(z)$ is the dimensionless time) proves to be quite complicated but the explicit solution (3.30) of the Poisson equation (3.26c) nevertheless allows one to obtain some interesting results. Studying the vertical velocity w, Landahl found that

$$w(x, y, z, t) \approx f_1(\xi, y, z)T^{-1} \quad \text{for} \quad \mathrm{T} \gg 1, \tag{3.29a}$$

where f_1 is a function of three variables determined by the initial values of $\nabla^2 w$. This result clearly agrees with the conclusion about integrated velocity $\overline{w}$ given in Landahl (1980). The long-time behavior of the streamwise velocity u was found to be particularly complicated but it is also admissible to analysis. For fixed bounded values of $|\xi|/L$ Landahl showed that

$$u(\xi, y, z, t) \approx -U'(z)f_2(\xi, y, z)T\gamma \quad \text{for} \quad T \gg 1, \tag{3.29b}$$

where f_2 is another function of three variables given by some special integral transform of $\nabla^2 w_o(x, y, z)$ while $\gamma = zU''(z)/U'(z)$. As to the spanwise velocity

$v(x, y, z, t)$, its values at $T \gg 1$ may be determined with satisfactory accuracy from the approximate equality

$$\partial u/\partial x + \partial v/\partial y \approx 0. \tag{3.29c}$$

We see that the horizontal velocity components approach a "frozen shape" in a coordinate system moving with the local velocity $U(z)$, and simultaneously decay or grow (depending on the sign of γ) algebraically with time. For velocity profiles without inflection points $U''(z) < 0$ for all z. Hence in flows with such profiles the disturbances will decay algebraically at the rate $\gamma = \gamma\ (z)$ which takes very small values near the wall (and tends to zero when $z \to 0$).

Different results were found by Landahl for the case of a fixed streamwise location (i.e., for fixed x where $|x| = \mathrm{O}(L)$). Here

$$u(x, y, z, t) = f_3(y, z) \quad \ln T + f_4(y, z) \quad \text{for} \quad T \gg 1, \tag{3.29d}$$

where f_3 and f_4 are some integral transforms of the function $F(y, z) = \int_{-\infty}^{\infty} \nabla^2 w_0(x, y, z) dx$. Therefore, there will be a logarithmic growth with time of the streamwise velocity of a disturbance before the algebraic decay takes over. Such logarithmic growth was observed by Lundbladh (1993) in numerical solutions of the linearized Navier-Stokes equations describing the evolution of a weak localized disturbance in plane Couette and Poiseuille flows (which differ from the boundary-layer flow studied by Landahl but apparently must be subjected to the same asymptotic laws). Streamwise velocities of disturbances independent of x lead to the appearance of streaks where fluid is flowing with velocity unequal to $U(z)$. The streaks have cross-flow structure which is approximately independent of x, while their lengths grow with time in proportion to $tU(z)$.

The formation of streaks in the near-wall region of a turbulent boundary layer, first observed by Kline et al. (1967) and later confirmed and investigated by many authors, was the main subject of Landahl's paper (1990). Here, in particular, some results of the mostly inviscid numerical calculations were given for the case of the development of a localized disturbance in a boundary layer with velocity profile U(z) close to the mean-velocity profiles observed in turbulent boundary layers. The initial shape of the disturbance was a pair of counter-rotating streamwise rolls (similar to those shown in Fig. 3.2) either fully symmetric or slightly non-symmetric with respect to the plane $y = 0$. The results obtained (partially presented in Fig. 3.4) showed that the streaks are weakly represented in the case of a symmetrical initial structure, but even small initial asymmetry in the spanwise direction y makes them much longer and more persistent.

To find the unknown function $w(x,\ y,\ z,\ t)$ the Poisson equation (3.26c) may be handled by standard methods. According to known results for this equation in the half-space $-\infty < x, y < \infty, 0 \le z < \infty$, with zero boundary conditions at $z = 0$ and at infinity, the formal solution of Eq. (3.26c) may be written as

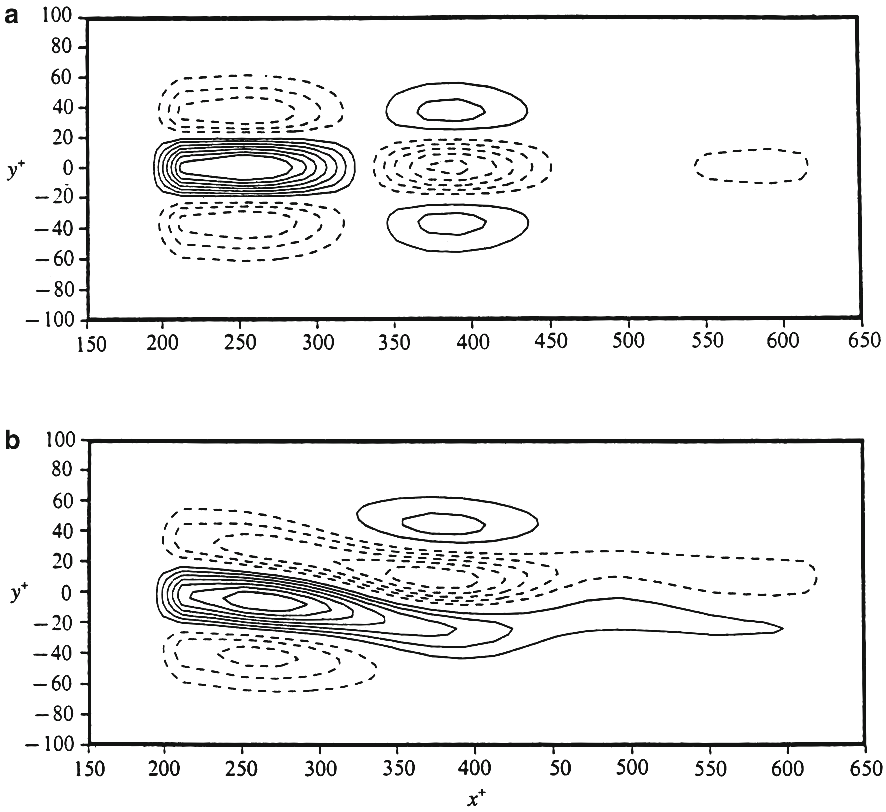


Fig. 3.4 Contours of constant streamwise velocity u in the plane $z^+ \equiv zu^*/\nu = 15$ of a boundary layer at $\mathrm{t}^+ \equiv t(u^*)^2/\nu = 40$ [where $u^* = (\nu dU/dz^|z = 0)^{1/2}$], for **a** symmetrical initial structure of the disturbance, and **b** slightly asymmetrical initial structure (after Landahl (1990)). *Solid* and *dotted lines* represent positive and negative values, respectively; contours start at $u = -W_0$, with spacing equal to $0.25w_0$ where w_0 is the velocity scale characterizing the intial distribution of vertical velocity

$$w = -\frac{1}{4\pi}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{0}^{\infty} dz' [\nabla^2 w_0(\xi', y', z') + U''(z')) l_x(x', y', z', t)] \left[\frac{1}{R} - \frac{1}{R_1}\right], \tag{3.30}$$

where $\xi' = x' - U(z')t$ and

$$R = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{\frac{1}{2}},$$
$$R_1 = [(x - x')^2 + (y - y')^2 + (z + z')^2]^{\frac{1}{2}}. \tag{3.30a}$$

In Eq. (3.30), it is convenient to replace integration over x' by integration over ξ'; then dx', x' and $x-x'$ must be replaced in this equation by $d\xi', \xi' + U(z')t$ and $\xi - \xi' + [U(z) - U(z')]t$. This explicit solution was the basis of the asymptotic analysis by Landahl whose main results were briefly outlined above.

3.2.3 *Initial-Value Problems for Disturbances in Inviscid Stratified Flows*

Applications of the method of normal modes to stability analysis of inviscid stratified plane-parallel flows, whose horizontal velocity $U(z)$ and density $\rho(z)$ depend on the vertical coordinate z, were considered in Chap. 2 Sect. 2.8.3. Recall now that at the beginning of Sect. 3.21 it was pointed out that the papers by Eliassen et al. (1953), Dikii (1960a) and Case (1960b), devoted to applications of the initial value method of stability analysis to some stratified plane-parallel flows, were among the earliest papers using such an approach to hydrodynamic stability theory. Let us additionally remark that in the papers by Miles (1958), Hartman (1975), and Brown and Stewartson (1980), which were also mentioned in Sect. 3.21 as examples of this approach, the primary flows considered were also stratified. We see therefore that publications dealing with applications of the initial-value method of stability analysis to stratified flows are definitely not rare. Hence it seems reasonable to devote some space to consideration of such publications.

As everywhere in Sect. 3.2 we shall assume that the primary flow is plane-parallel and inviscid. Let us begin with the paper by Dikii (1960a), which differs from the other above-mentioned papers by the definition of stability used. Namely, Dikii proved the Lyapunov stability of exponentially-stratified Couette flows, where $U(z) = bz,\ \rho(z) = \rho_0 \exp(-az)$, in a half-space $0 \le z < \infty$ (bounded by a flat solid wall at $z = 0$), while the other authors interpreted stability as the absence of disturbances growing unboundedly with time. It was also assumed by Dikii that $\mathrm{Ri} = ga/b^2 > 0$, i.e., that the density is decreasing with height and hence the density stratification is stable. (It seems to be obvious that in the case of unstable stratification, where $\rho(z)$ increases with z and $\mathrm{Ri} < 0$ everywhere, the flow will be unstable; a proof of this fact will be indicated below in this section.) We have already mentioned in Sect. 2.83 that this problem was first studied by Taylor (1931) by the method of normal modes; this author presented convincing arguments showing that the spectrum of eigenvalues c for the eigenvalue problem related to Eq. (2.66′) (at present usually called the Taylor-Goldstein, or T-G, equation) is here quite different in cases where $\mathrm{Ri} > 1/4$ and where $0 < \mathrm{Ri} < 1/4$. However, Taylor's results did not imply a clear answer to the question about the stability (or instability) of the flow to small disturbances. Therefore Dikii did not give much attention to the normal modes, but studied the solution of the initial-value problem for Eq. (2.63), which determines the vertical velocity w of a disturbance. The disturbance was assumed to be two-dimensional and hence w depended only on x, z and t. Note also that the Boussinesq approximation, which simplifies all the equations, was not used by

Dikii, but he noticed that the introduction of this approximation would not change his results.

To find the required solution, Dikii first replaced the streamwise coordinate x in Eq. (2.63) by the convected coordinate $\xi = x - bzt$, and then applied to the unknown function $w(\xi, z, t)$ the combined Fourier-Laplace transform (where Fourier and Laplace transforms were taken with respect to variables ξ and z, respectively). As a result, an ordinary differential equation of the second order was obtained for the Fourier-Laplace transform $\hat{w}(k, p; t)$ of $w(\xi, z, t)$, where this transform was considered as a function of t dependent on two parameters, the Fourier- and Laplace-transform variables k and p. The solution of this equation was then found in the form of a sum of two indefinite integrals of some combinations of hypergeometric functions with weight functions determined by complicated integral equations, which included initial values of w and $\partial w/\partial t$. Applying the inversion formula (similar to Eq. (3.6)) to the Fourier-Laplace transform $\hat{w}(k, p; t)$, Dikii found the general solution $w(\xi, z, t)$ of the initial value problem. A cumbersome investigation of the asymptotic behavior of this solution allowed him to prove that if initial values of w and $\partial w/\partial t$ are smooth enough, then for any $\mathrm{Ri} > 0$ the absolute value of the vertical velocity $|w(\xi, z, t)|$ remains bounded at any time $t > 0$ by a constant which can be made arbitrarily small by sufficient diminution of the absolute values of these initial values. This statement just proved the Lyapunov stability of the considered stratified flows with respect to small and smooth initial disturbances. In his paper Dikii paid most attention to a single Fourier component of the disturbance, i.e., to the case where $w(x, z, t) = e^{ikx}W(z, t)$. In this case asymptotic behaviors of $|w(x, z, t)|$ and $|w(\xi, z, t)|$, when $t \to \infty$ but the other two independent variables have fixed values, are the same; however, in some other cases, considered below, they can differ considerably.

The initial-value approach was also used for the study of time evolution (first of all as $t \to \infty$) of disturbances in exponentially stratified Couette flows (where the vertical extent $0 \le z < \infty$ was sometimes replaced by $0 \le z \le H < \infty$ or $-\infty < z < \infty$ with appropriate change of the boundary conditions) by Eliassen et al. (1953), Case (1960b), Kuo (1963), Hartman (1975), Chimonas (1979), Brown and Stewartson (1980), and Farrell and Ioannou (1993a) (this list surely is not complete). Since these papers are quite typical of applications of the initial-value approach to stability of stratified flows, we shall consider below, almost exclusively, flows with linear velocity and exponential density profiles. Note that in all the above-mentioned papers the Boussinesq approximation was used, and, as a rule, only two-dimensional disturbances were studied, with the aid of the combined Fourier-Laplace transform (3.4) with respect to variables (x,t), applied to the vertical velocity $w(x, z, t)$ (or, what is practically the same, to the stream function $\psi(x, z, t)$) (rarely-met deviations from this procedure will be noted below). However, the investigation of the asymptotic behavior of these transforms as $t \to \infty$ proved to be sophisticated and requiring great skill; therefore it is not surprising that some of the results obtained were inaccurate and differed from more precise results given in other publications.

In the early investigation by Eliassen et al., where the thickness H was assumed to be finite, arguments were presented which made very plausible the assumption (which later was proved to be correct) that for $\mathrm{Ri} > 1/4$ the T-G eigenvalue problem has an infinite number of real discrete eigenvalues (note that for the case where $0 \leq z < \infty$ the existence of such real eigenvalues was proved by Taylor (1931); however, if $-\infty < z < \infty$, then, as we shall see later, no discrete eigenvalues exist at any value of Ri). Eliassen et al. assumed also that the eigenfunctions corresponding to the set of all discrete eigenvalues form a complete system in a space of admissible initial values of w or ψ; however, this assumption was proved later to be incorrect. (If it were correct, any solution $w(x, z, t)$ or $\psi(x, z, t)$ would be representable in the form of a linear combination of neutral normal modes, i.e., would be a bounded undamped function of t; hence the flow would be stable. It was in fact later proved to be stable, but the proof turned out to be not so simple.) As for the case where $0 < \text{Ri} < 1/4$, Eliassen et al., relying on some nonstrict arguments, came to the conclusion that in this case the T-G eigenvalue problem has no discrete eigenvalues and that the flow is stable, since here, as $t \to \infty, |w| \propto t^{-1}$ and $|u| \propto t^{\mu-1/2}$, where $\mu = (1/4 - \text{Ri})^{1/2}$ (here and below μ always denotes the positive value of the square root of $1/4 - \text{Ri}$, or that having a positive imaginary part). For $0 > \text{Ri} > -3/4$ (i.e., $1/2 < \mu < 1$) Eliassen et al. found that the T-G eigenvalue problem also has no discrete eigenvalues, but the flow is unstable since here again $|u| \propto t^{\mu-1/2}$ as $t \to \infty$ (however $|w| \propto t^{\mu-3/2}$, i.e., it tends to zero). Moreover, it was also found in this paper that for $\text{Ri} < -3/4$ there exists at least one pair of complex eigenvalues (the number of such pairs increases with decrease of Ri) of the corresponding T-G eigenvalue problem; therefore, here the flow is unstable and some disturbances in it grow exponentially with time. This last conclusion and the majority of the results on discrete spectra and on flow stability or instability were later rigorously proved by other authors; however certain suggested asymptotic relations were found to be incorrect.

Case (1960b) followed Taylor and studied stability of exponentially-stratified Couette flows in the half-space $0 \leq z < \infty$. Based on the analytical results by Dyson (1960) (which were independently found also by Dikii (1960c)) he proved rigorously that for such flows at any $\text{Ri} > 0$ there are no complex eigenvalues satisfying the T-G eigenvalue problem, but $\text{Ri} > 1/4$ then at each wave number k there exists an infinite number of real eigenvalues, while for $0 < \text{Ri} < 1/4$ there are either one or zero real eigenvalues at any k (see also discussion of this topic in Sect. 2.83). However, it was also showed by Case that in the case considered the T-G equation at any $\text{Ri} > 0$ has a continuous spectrum which fills the half-lines $0 < c < \infty$ and $-\infty < c < 0$. According to Case's calculations (which were later found to be inaccurate), $|w|$ includes not only undamped oscillations, corresponding to discrete real eigenvalues, but also a contribution from the continuous spectrum which is represented by a function tending to zero as $t^{-1/2}$, if $\text{Ri} > 1/4$, and as $t^{\mu-1/2}$, if $0 < \text{Ri} < 1/4$; therefore Case concluded that the flow is stable at any $\text{Ri} > 0$.

Kuo (1963) considered the general solution of the initial-value problem for a three-dimensional vertical velocity disturbance $w(x, y, z, t)$ in an exponentially-stratified Couette flow, having either stable or unstable stratification and filling either a layer of finite thickness H or a half-space $0 \leq z < \infty$. The solution found by him used a preliminary transformation from $w(x, y, z, t)$ to a Fourier-Laplace transform $\hat{w}(k_1, k_2, p; z)$, combining a two-dimensional Fourier integral with respect to the horizontal coordinates and a Laplace integral with respect to time. Following Taylor (1931), Kuo showed that the study of stability for three-dimensional waves proportional to $\exp[i(k_1x+k_2y)]$ can be reduced to the corresponding two-dimensional problem with modified Richardson number $\mathrm{Ri}_1 = (1+k_2^2/k_1^2)\,\mathrm{Ri} > Ri$ (cf. also the corresponding discussion in Sect. 2.83). Therefore, it is enough to consider only two-dimensional disturbances below. Note also that most attention was given by Kuo to the investigation of normal-mode disturbances; since we are currently discussing the initial-value problem, the results of this investigation will be only briefly indicated here.

First, Kuo noticed that in the case of a stratified Couette flow, where $U''(z) \equiv 0$, the Boussinesq approximation implies the following simple form of the integral relation (2.69) found by Howard (1961) for $n = -1$:

$$\int_0^H \rho\{(U-c)^4(|\partial F_{-1}/\partial z|^2 + k^2|F_{-1}|^2) - (U-c)^2(2+\mathrm{Ri}\,)(U')^2|F_{-1}|^2\}dz = 0 \tag{3.31}$$

where c is an eigenvalue satisfying the T-G eigenvalue problem and, contrary to Eq. (2.69), all the variables are now assumed to be dimensional (here the height H can take both finite and infinite values). If $\mathrm{Ri} \le -2$ then for real c the integrand in the left-hand side of Eq. (3.31) is everywhere positive, and hence no real eigenvalues c can exist in this case. For exponentially stratified Couette flows in an infinite layer, where $H = \infty$, Kuo showed that complex eigenvalues c exist if and only if $\mathrm{Ri} < -2$ (i.e., for $\mathrm{Ri} < -2$ wave-like disturbances exponentially growing with time surely exist). In addition to this he also calculated the number of discrete eigenvalues c for any $\mathrm{Ri} < 1/4$ (for $\mathrm{Ri} > 1/4$ this number is infinite while for $\mathrm{Ri} < 1/4$ it is always finite) and repeated without criticism Case's conclusion that $|w| \propto t^{\mu-1/2}$ as $t \to \propto$, if $0 < \mathrm{Ri} < 1/4$, supplementing it with the statement that this conclusion holds also for $0 > \mathrm{Ri} > -2$ (i.e., for $1/2 < \mu < 3/2$ when $t^{\mu-1/2}$ grows unboundedly with t).

For a Couette flow of finite height H having unstable exponential stratification (so that $\rho^{-1}\partial\rho/\partial z = b > 0$) Kuo found that here complex eigenvalues c (and hence exponentially growing normal modes of disturbance) exist only for $\mathrm{Ri} < -3/4$, while for $1/4 > \mathrm{Ri} > -3/4$ no discrete eigenvalues exist at any wavenumber k. Therefore the time evolution of the disturbance velocity w in this case is determined by the contribution from the continuous spectrum of T-G eigenvalues. According to Kuo, this contribution leads to the same asymptotic law as for $H = \infty$, so that here again $|w| \infty t^{\mu-1/2}$ for large enough values of t. Moreover, Kuo also investigated the spectrum of discrete eigenvalues c (which depends on k) at various values of Ri for both $H = \infty$ and finite H. (Note that according to (3.31) Kuo's real eigenvalues c in cases of strong stability (large negative Ri) must be fictitious. In fact, the corresponding eigenfunctions have singularities and therefore do not represent true solutions; cf. discussion by Eliassen et al. (1953)). Finally, Kuo investigated the shapes of the most unstable disturbances in unstably-stratified Couette flows and found that they can explain the results of some meteorological observations and laboratory experiments. However, we have no space to discuss this matter in more details.

Later Chimonas (1979) analyzed anew the asymptotic behavior of disturbances to exponentially stratified Couette flow, for $-\infty < z < \infty$ and Ri < 1/4. (The unboundedness of the low domain simplifies all the computations, while it seems probable that most of the asymptotic results will be the same as for flows in a bounded layer or half-space.) According to Chimonas, if $\xi = x - bzt$, then in the case of unbounded flow $|w(\xi, z, t)| \propto t^{2\mu-1}$ at large values of t and fixed values of ξ and z. Chimonas explained the difference between his result and that found by Case by the fact that Case, contrary to him, determined the asymptotic behavior of $|w(x, z, t)|$ at fixed values of x and z by assuming that the initial disturbance was of bounded extent in x. It is clear that then the disturbance velocity w (x, z, t) must fall off at a fixed point (x, z) more rapidly with t than at fixed values of (ξ, z) and that here the physically most interesting behavior is that at fixedξ and not at fixed x. In addition Chimonas also determined the asymptotic behavior as $t \to \infty$ of the horizontal velocity, density and pressure disturbances $u(x, z, t)$, $\rho'(x, z, t)$ and $p'(x, z, t)$. He found that at Ri $< 1/4$, $|p'|$ decays as $t^{2\mu-1}$, but both $|u|$ and $|\rho'|$ grow with t as $t^{2\mu}$ (i.e., without limit). Proceeding from this, Chimonas asserted that at 0 < Ri < 1/4 exponentially-stratified inviscid Couette flows are unstable. This assertion contradicted the results of the other available investigations of the same topic and therefore from the very beginning seemed to be dubious; later an error in Chimonas' analysis was indicated by Brown and Stewartson (1980).

Correct asymptotic relations replacing those suggested by Chimonas (and also results found by Eliassen et al. Case, and Kuo) were published by Hartman (1975). His results incorporate also the earlier results by Phillips (1966), Chap. 5, and Booker and Bretherton (1967) relating to development of internal waves in stably-stratified ocean or atmosphere; so it is reasonable to begin with some conclusions from the latter two publications. To investigate the influence of the velocity shear on the evolution of oceanic internal waves, Phillips used a model example of wave development in an exponentially-stratified inviscid Couette flow filling an unbounded space. In this respect he considered particular solutions of Eq. (2.63′) for small disturbances, of the form

$$w(x, y, z, t) = W(t) \exp[i(k_1\xi + k_2 y)] = W(t) \exp[i(k_1 x + k_2 y - k_1 btz)], \tag{3.32}$$

where $\xi = x - bzt$ and $b = \partial U/\partial z = U'(z)$ is a constant velocity shear. (Phillips explained that his model, in which the dependence of the velocity shear U'(z) and Brunt-Väisälä frequency $N(z) = (-g\rho'/\rho)^{1/2}$ on z was neglected, is appropriate only for wave motions of small vertical scale; therefore the dependence of these motions on z was also neglected. However the possible dependence on y was taken into account, in contrast to all the work considered above in this subsection except that of Kuo.) Phillips noted that the general solution of the second-order differential equation for $W(t)$ implied by Eq. (2.63′) can be expressed in terms of hypergeometric functions, but in his book only the case of strong stability (or weak shear), where Ri $= N^2/b^2 \gg 1$, was studied at length. He showed that in this case the solution obtained represents a three-dimensional wave motion whose amplitude, wave numbers and direction of propagation are slowly changing with time. According to this solution, the amplitude and the wavelength (which is inversely proportional to the length of the three-dimensional wavenumber vector) are continuously decreasing and the direction of propagation is approaching the vertical axis. These features of the wave motions considered agree more or less satisfactorily with some features of real oceanic internal waves, but for us here the most important discovery by Phillips is his finding that in the case of very strong stability (i.e. for Ri $\gg$ 1)$W(t) = |w(x, y, z, t)|$ is asymptotically proportional to $t^{-3/2}$ as $t \to \infty$. Wave amplitudes for the horizontal velocity components $u(x, y, z, t)$ and $v(x, y, z, t)$ were also found to decrease, but more gradually, only as $t^{-1/2}$. Hence the motion becomes more and more horizontal with time and its mean kinetic-energy density per unit mass $T^*(t)$ decreases asymptotically as t^{-1}. However, the rapid decrease of the vertical scale leads to unlimited increase of the vertical gradients of u and v and hence also of horizontal components of the vorticity field, ζ_1 and ζ_2.

Booker and Bretherton (1967) further developed Phillips′ approximate theory. They were primarily interested in atmospheric internal waves, and paid most attention to wave motions near the critical height z_{cr} where the undisturbed velocity $U(\mathrm{z_{cr}})$ coincides with the phase velocity c of the wave. For the present discussion it is important that these authors also analyzed the general solution of the initial-value problem for the wave-like vertical-velocity disturbance $w(x, z, t)$ (variable y is absent here since only two-dimensional waves were considered), for a flow model which included a layer where both $U'(z)$ and $N(z)$ took constant values. The results obtained included the discovery of some particular solutions which are valid within this layer (and in the case of exponentially-stratified Couette flow in the whole space too, a fact mentioned by the authors in passing), under the condition that $\mathrm{Ri} = N^2/U'^2 > 1/4$. Asymptotically (i.e., for large enough values of t) these solutions have the forms of damped waves

$$w(x, z, t) = W_1(z)t^{\mu-3/2}e^{ik(x-bzt)} + W_2(z)t^{-\mu-3/2}e^{ik(x-bzt)} \tag{3.33}$$

where, as usual, $\mathrm{b} = U'(z) = \mathrm{const.}$ and $\mu = (1/4-\mathrm{Ri})^{1/2}$ (hence μ has a purely imaginary value here). Corresponding solutions for the horizontal velocity component have the forms

$$u(x, z, t) = U_1(z)t^{\mu-1/2}e^{ik(x-bzt)} + U_2(z)t^{-\mu-1/2}e^{ik(x-bzt)}, \tag{3.34}$$

showing that horizontal velocity also decays, but more slowly. These equations represent a slight refinement (concerning the admissibility of the amplitude dependence on z) of the asymptotic laws found by Phillips, but Booker and Bretherton discovered that these laws are valid not only for Ri $\gg$ 1 but for any Ri exceeding 1/4.

Hartman (1975) considered only the idealized model of exponentially-stratified Couette flow in an unbounded space, and found a simple form of the general solution to the initial-value problem for an infinitesimal two-dimensional disturbance $\{u(x, z, t), 0, w(x, z, t)\} = \mathbf{u}(x, z, t)$. Instead of the vertical velocity w or the stream function ψ chosen as the dependent variables in many previous studies, Hartman solved the initial-value problem for the non-zero vorticity component $\zeta = \partial u/\partial z -$

$\partial w/\partial x = -\nabla^2\psi$. Using the convected "spatial coordinates" $(\xi, z) = (x - btz, z)$ he found that the dependence on the variable t of the two-dimensional Fourier transform $\hat{\zeta}(k_1, k_2; t)$ of $\zeta(\xi, z, t)$, with respect to coordinates (ξ, z), can be determined from a second-order differential equation, whose solution for the given initial values of ζ and $\partial\zeta/\partial t$ at $t = 0$ can be simply expressed through standard hypergeometric functions. This solution was then skillfully used for the investigation of wave-packet propagation in unbounded stratified Couette flow under the condition that $\mathrm{Ri} > 1/4$ supplemented by comparison of the results obtained with those of Booker and Bretherton (1967). It was also mentioned that this solution can be applied to the determination of the behavior of localized disturbances in unbounded stratified Couette flow at $0 < \mathrm{Ri} < 1/4$, but this specific application is omitted from Hartman's paper. However, he described the asymptotic behavior of his solutions at large values of t (more correctly, of the dimensionless time $T = U'(z)\,t = bt$) and these results are most interesting for the present discussion.

According to Hartman the main terms of the asymptotic expressions for the general solution of the initial-value problem have the following forms:

$$\hat{\zeta}(k_1, k_2; t) \approx a_1 t^{\mu+1/2} + a_2 t^{-\mu+1/2}, \quad \text{if} \quad \mathrm{Ri} \neq 1/4, \tag{3.35a}$$

$$\hat{\zeta}(k_1, k_2; t) \approx a_3 t^{1/2} \ln t, \quad \text{if} \quad \mathrm{Ri} = 1/4, \tag{3.35b}$$

at $T = bt \gg 1$, where the coefficients a_1, a_2 and a_3 depend on k_1, k_2 and the initial conditions. We see that vorticity is growing without limit at any Ri. Now, using the simple relationship between the Fourier transforms of the vorticity ζ and the stream function ψ given by Hartman, it is easy to find the main terms of asymptotic expressions for the Fourier transforms $\hat{w}(k_1, k_2; t)$ and $\hat{u}(k_1, k_2; t)$ of the vertical and horizontal velocity components $w(x, z, t)$ and $u(x, z, t)$. Obtained in this way, asymptotic relations at $\mathrm{Ri} \neq 1/4$ have the form, recalling that $\mu = (1/4 - \mathrm{Ri})^{1/2}$

$$\hat{w}(k_1, k_2; t) = c_1 t^{\mu-3/2} + c_2 t^{-\mu+3/2}, \tag{3.36}$$

$$\hat{u}(k_1, k_2; t) = d_1 t^{\mu-1/2} + d_2 t^{-\mu+1/2}, \tag{3.37}$$

while for $\mathrm{Ri} = 1/4$ (i.e., $\mu = 0$) an additional logarithmic factor must be included in the right-hand parts. These equations imply that T^* $(k_1, k_2; t)$, the averaged density per unit mass of the kinetic energy for a wave-component of the disturbance with wave numbers (k_1, k_2), satisfies the relationships:

$$\begin{aligned} &T^*(k_1, k_2; t) \propto t^{2\mu-1} \quad \text{if} \quad \mathrm{Ri} < 1/4, \\ &T^*(k_1, k_2, t) \propto t^{-1} \quad \text{if} \quad \mathrm{Ri} > 1/4. \end{aligned} \tag{3.38}$$

Later Farrell and Ioannou (1993a) supplemented Hartman's results (3.35–3.38) by similar results for Fourier transforms of the density disturbance, $\hat{\rho}(k_1, k_2; t)$, and of the averaged total energy density (per unit mass), $E(k_1, k_2; t)$, for the disturbance component with wave numbers (k_1, k_2). (The total energy density can be represented as $E(k_1, k_2; t) = T^*(k_1, k_2; t) + V(k_1, k_2; t)$ where $V(k_1, k_2; t)$ is the density of the potential energy disturbance for fluid elements of variable density $\rho(\boldsymbol{x},\ t)$ in a gravitational force field.) The asymptotic relationships are:

$$\hat{\rho}(k_1, k_2; t) = e_1 t^{\mu - 1/2} + e_2 t^{-\mu - 1/2}, \tag{3.39}$$

$$E(k_1, k_2; t) \propto t^{2\mu - 1} \quad \text{if} \quad \text{Ri} < 1/4,\ E(k_1, k_2; t) \propto t^{-1} \quad \text{if} \quad \text{Ri} > 1/4. \tag{3.40}$$

Recall that wave numbers (k_1, k_2) correspond to waves of the form exp $[i\{k_1(x - bzt) + k_2 y\}]$; therefore in the material space the decay laws (3.35–3.40) are related to asymptotic behavior at fixed $(x - bzt, z)$ and not at a fixed point (x, z).

Equations (3.36) and (3.37) clearly agree with the results found by Phillips (1966) and Booker and Bretherton (1967) for decay laws relating to waves in an unbounded stably-stratified Couette flow. However, Phillips derived these laws only for the case where $\text{Ri} \gg 1$, and Booker and Bretherton generalized them to the wider class of stratified Couette flows with $\text{Ri} > 1/4$. Now we see that these results are true for these flows with any value of Ri, positive, zero, or negative, the only exception being $\text{Ri} = 1/4$ exactly, where there is a degeneracy (the merging of two solutions) which produces the appearance of a slight correction. According to Eqs. (3.38) and (3.40), the growth or decrease of energy of wave-like disturbances in an unbounded exponentially-stratified Couette flow is always algebraic, not exponential. This proves that for this flow the Taylor-Goldstein equation has no discrete eigenvalues at any value of Ri (as explained earlier, this statement is incorrect in cases where the Couette flow considered has one or two solid boundaries).

Brown and Stewartson (1980) also considered the question of the precise form of decay laws for waves in an unbounded exponentially-stratified Couette flow. They did not mention the paper by Hartman (1975) and apparently did not know about it, but their main result is the same as that found by Hartman: it consists in the confirmation of Eq. (3.33), proved by Booker and Bretherton (1967) for the case where $\text{Ri} > 1/4$, supplemented by the proof that this result is in fact true for any Ri (the slight correction needed at $\text{Ri} = 1/4$ was unnoticed here). Brown and Stewartson also indicated the error in the derivations by Case (1960b) and Chimonas (1979).

We have already mentioned the paper by Farrell and Ioannou (1993a). This paper is also devoted to the study of development of small two-dimensional disturbances to an inviscid, exponentially-stratified Couette flow with $\text{Ri} \geq 0$, filling either an unbounded space or a layer between two parallel walls. However, here the authors pay most attention, not to asymptotic results for $t \to \infty$, but to transient development of disturbances during finite time intervals. First of all they are interested in the possibility of considerable growth of disturbances during the early stage of their development, as first discovered, for the case of a Couette flow of constant-density fluid, by Orr (1907).

Farrell and Ioannou remark that, according to results by Phillips (1966) and Hartman (1975), in the case of an unbounded exponentially-stratified Couette flow, the time-dependent amplitude $\hat{\zeta}(k_1, k_2; t) \equiv Z(t)$ of a Fourier component

$Z(t)\exp[i(k_1\xi + k_2z)]$ of the vorticity field can be explicitly expressed in terms of the standard hypergeometric functions, and for given initial conditions it can be accurately determined by numerical integration of the relevant second-order differential equation. When $Z(t)$ is known, it is easy to determine also the Fourier amplitudes $\Psi(t)$, $W(t)$, $U(t)$ and $R(t)$ of the disturbance stream function ψ, the velocity components w and u, and the density ρ'' that satisfy the asymptotic relationships (3.35–3.37) and (3.39). Farrell and Ioannou do not consider the asymptotics further, but pose a question about the initial conditions which yield greatest growth of the total disturbance energy in a specific time $T_{opt} = (bt)_{opt}$ (the corresponding disturbance is then called the *optimal disturbance* for the time T_{opt}). However, to make this question fully definite it is necessary to clarify what is meant in this case by "initial conditions."

For unique determination of the development of a flow disturbance it is necessary to give the initial values of two of its independent fluid-dynamic fields, e.g., fields of the vorticity ζ and its time derivative $\partial\zeta/\partial t$ (as Hartman did), or of the stream function ψ (uniquely determining two velocity components u and w) and the density ρ' (this second choice was made by Farrell and Ioannou). Since the finding of the optimal value for two fields $\psi(x,z,0)$ and $\rho'(x,z,0)$ is a complicated task, and plane waves form an orthogonal basis in the functional space of functions in an unbounded space, Farrell and Ioannou restricted themselves to determination of only the optimum plane-wave initial values of ψ and ρ'. In this case the initial conditions are given by the initial values Ψ_0 and R_0 of amplitudes for the stream-function and density waves, and by the corresponding wave numbers k_1 and k_2; the relative energy growth $G(t) = E(t)/E(0)$ is dependent only on the wave-number ratio $s = k_2/k_1$ and not on k_1 and k_2 individually. So, for the determination of the optimal wave disturbance, it is only necessary to find the maximum value of a function of three variables Ψ_0, R_0 and s.

At fixed value of t, the maximal value $G_{\max}(t)$ of $G(t)$, and the values of the three variables corresponding to its maximum, clearly depend on the choice of the time t. In Sect. 3.1 we referred to Orr (1907) to mention that the maximal growth $|w|_{\max}/|w|_0$ of the vertical velocity for a plane-wave disturbance in an inviscid plane Couette flow of constant-density fluid can be made as large as desired, if it is permitted to increase indefinitely the time t_{opt} at which this growth is reached. Farrell and Ioannou calculated the values $G_{\max}(t)$ for different values of t and Ri; the results obtained are shown in Fig. 3.5. It was found that for small values of t (measured in the shear units b^{-1}, i.e., given by values of $T = bt$) the function $G_{\max}(T, Ri)$ is practically independent of Ri and only slightly exceeds unity, but later on it begins to increase and becomes significantly dependent on both variables, increasing indefinitely with T and decreasing with Ri. Farrell and Ioannou remarked that, in real geophysical flows, ambient fluctuations usually impose a time scale beyond which the growth of disturbances according to the theory is definitely disrupted. Therefore the computations of the function $G_{\max}(T, Ri)$ for very large values of T are practically useless. As a reasonable representative value they selected $T_{\text{opt}} = 6$ in their study, noting that other choices for T_{opt} usually do not change the results qualitatively. In Fig. 3.6 the values of $G(T) = E(T)/E(0)$ for the optimal plane wave corresponding to $T_{\text{opt}} = 6$ are plotted against T for a number of nonnegative values of Ri. We see that for zero and small positive values of Ri the value of $G(T_{\text{opt}})$ for $T_{\text{opt}} = 6$ is in the range from

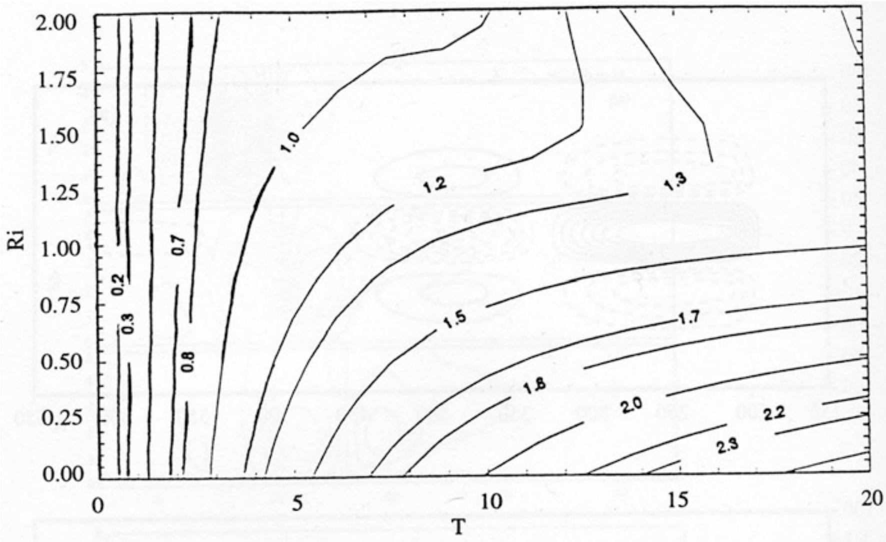


Fig. 3.5 Maximal energy growth $G_{max}(T, \mathrm{Ri})$ for plane-wave disturbances with optimal values of Ψ_0, R_0, and s in a stratified Couette flow, as a function of the time of maximal growth $T = bt$ and Ri. The contour values are those of $\log_{10} G_{max}$ (T, Ri). (After Farrell and Ioannou (1993a))

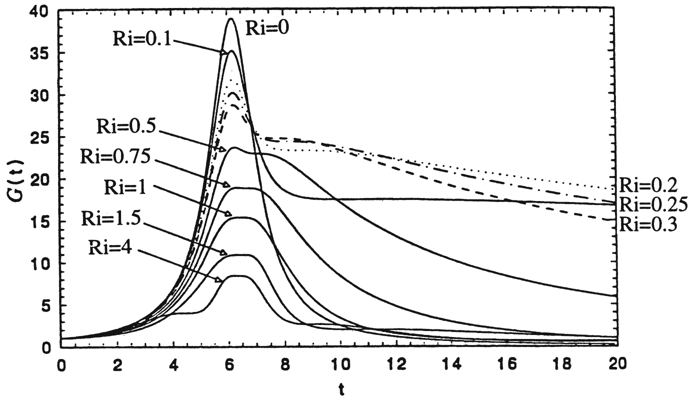


Fig. 3.6 Dependence on $T = bt$ of the normalized energy $G(T) = E(T)/E(0)$ of a plane-wave disturbance in a stratified Couette flow having maximal possible energy growth at $T = 6$, for different values of Ri. (After Farrell and Ioannou (1993a))

25 to 40, while at $\mathrm{Ri} = 4$, it is close to 7. Note the appearance of near-persistence of $G(T)$ (i.e., of the total energy $E(T)$) at $T > T_{opt}$ when Ri is in the range $0.1 < \mathrm{Ri} < 0.3$; such persistence can have some practical importance. Similar graphs of the functions $T^*(t)/T^*(0)$, and $V(t)/V(0)$, showing the dependence of kinetic and potential energies of the optimal disturbance on t and Ri, are also presented by Farrell and Ioannou, together with some data characterizing the optimal initial values and discussion of the results obtained.

In the case of a stratified Couette flow in a layer between two parallel solid walls (for example, in a channel of finite depth) the problem proves to be more complicated, since here there are no closed-form solutions of the dynamic equations satisfying the required boundary conditions. The corresponding T-G equation has in this case a continuous spectrum of eigenvalues filling the real-axis interval $U_{\min} \le c \le U_{\max}$ (exactly as in the case of a constant-density flow in a channel; see Sect. 2.82) supplemented, for Ri > 1/4, by an infinite number of real discrete eigenvalues. Farrell and Ioannou approximately determined the optimal disturbances in this case, by replacing the differential equations in the interval $0 \le z \le H$ by finite-difference equations in a domain with a sufficiently large number N of mesh points. The algebraic finite-difference equations then have only a finite number $2N$ of discrete eigenvalues c_j, and the general solution can be represented by a linear combination of the corresponding eigenvectors multiplied by $\exp[ik(x - c_j t)]$. The energy $E(t)$ takes the form of a Hermitian positive-definite quadratic form of $2N$ variables, with coefficients depending on t. Finding of the maximum value for $G(t) = E(t)/E(0)$ is now a more difficult problem than in the case of an unbounded Couette flow, but this problem is also accessible to modern computers. Computations made for different values of N showed that results for $N = 30$ are as a rule sufficiently accurate in this case. Using this approximate method, the authors presented two examples of optimal initial disturbances (corresponding to $T_{\mathrm{opt}} = 6$, Ri = 0.2 and 0.75, and fixed k) and of their forms at $T = T_{\mathrm{opt}} = 6$ and $T = 15$, supplemented by graphs of the same type as in Fig. 3.6 showing the dependence of the total, kinetic, and potential energies of the optimal disturbance (again for $T_{\mathrm{opt}} = 6$) on Ri and T.

In this subsection, only studies of hydrodynamic stability of exponentially-stratified Couette flows have been discussed so far. Moreover, with the exception of the works by Kuo and by Phillips, only two-dimensional disturbances were considered in these studies. Note in this respect that the first applications of the normal-mode stability investigations to various inviscid stratified flows were made very early (more than a hundred years ago by Kelvin (1871), who studied the case of an unbounded flow having very simple discontinuous velocity and density profiles, and then by Taylor (1931) and Goldstein (1931), who considered several more complicated examples of profiles $U(z)$ and $\rho(z)$). Later many other normal-mode stability studies of inviscid stratified flows, involving a great number of new examples, were carried out. However, until now only a few papers have appeared in the physics and engineering literature on applications of the initial-value method to stability studies for stratified non-Couette flows. True, Chimonas (1979), in addition to the main example of unbounded exponentially-stratified Couette flow, presented some general results relating to unbounded flows with arbitrary profiles $U(z)$ and $\rho(z)$, but the derivation of these results had the same defect which invalidated Chimonas' results for Couette flows. Moreover, Brown and Stewartson (1980) noted in passing that in the case of arbitrary smooth profiles $U(z)$ and $\rho(z)$ it is only necessary to replace the constants $U'(z) = b$ and Ri in the main terms (3.33) of the asymptotic expansions for $w(x, z, t)$ (or $\psi(x, z, t)$) by functions $U'(z)$ and Ri(z) depending on z, but apparently no proof for this assertion has been published.

However, in the above remark about the rarity of papers on the subject in the physics and engineering literature the mention of "physics and engineering" was meaningful. It was stressed at the beginning of Sect. 2.83 that atmospheric and oceanic flows represent the most important practical examples of stratified fluid flows. Therefore, it is not surprising that stability of stratified flows is discussed more often, and usually at greater length, in the geophysical literature. In fact, the above mentioned works of Taylor, Dikii, Case, Dyson, Eliassen et al., Phillips, Booker and Bretherton, Farrell and Ioannou, and Hartman all originated from geophysical problems. And in the geophysical literature many other publications can be found where the initial-value method of stability investigations is applied to some particular inviscid atmospheric and oceanic flows, for determination of either the asymptotic laws of the disturbance evolution or its transient development. As typical examples we can mention the papers by Pedlosky (1964), Burger (1962), Yamagata (1976a, b), Farrell (1982, 1988b, 1989), Tung (1983), and Farrell and Moore (1992). However, the flows considered in these papers are not so simple as the exponentially-stratified Couette flows and often involve some additional geophysical factors (e.g., the baroclinity or the Coriolis force) which require additional space for description and discussion of the corresponding examples. Unfortunately, space limitations make it impossible to include this material in the present book.

3.3 The Initial-Value Problem for Viscous Parallel Flows

In Sect. 3.2 the fluid was assumed inviscid, but any real fluid (with the sole exception of liquid helium in the state of superfluidity) has viscosity $\nu \neq 0$, and this can significantly affect the flow. Therefore, inviscid fluid mechanics is an approximation, which in cases where Re $\gg 1$ often (but, of course, not always) has relatively high accuracy. It was noted in Sect. 3.22 that, according to Landahl (1993a, b), in studies of the evolution of weak localized disturbances in the near-wall region of a boundary layer, viscous effects can be neglected only during an initial time interval of duration $t \ll t_\nu$ where t_ν is the so-called viscous-interaction time scale determined by the values of the viscosity ν, the mean-velocity shear dU/dz, and the streamwise length scale of the disturbance, L. In general, the role of viscous effects can be determined only by comparison of the deductions from inviscid theory with the results of more complete theory which takes viscosity into account.

Let us recall that in the first attempt by Kelvin (1887a) to solve the initial-value problem for a weak flow disturbance (which proved to be unsuccessful but nevertheless led to discovery of some important new results) the flow considered was a viscous Couette flow, and that later Orr (1907) also considered development of disturbances with given initial values in such viscous flow. Orr showed that there exist initial values which lead to very great growth of disturbances during the beginning stage of their evolution and proposed on this basis the important concept of "practical instability." It has already been indicated, in Sect. 3.21, that the early results by Kelvin and Orr only began to attract attention many years after their appearance, when they stimulated a significant new development of the initial-value approach to flow-stability investigations. Now we will discuss the present situation concerning the initial-value problems for disturbances in viscous flows.

3.3.1 Wave-Packet Approximations for Solutions of the Initial-Value Problem

The first new attempts to solve the initial-value problem for disturbances to some viscous laminar flows, which appeared after Kelvin's (1887a) and Orr's (1907) old papers, had no relation to these early works but were made with the purpose of explaining the results of observations of flow instabilities collected during the 1940s and 1950s. The normal-mode method of linear stability theory, well known at that time, connected the instability of a plane-parallel flow with the appearance of the so-called Tollmien-Schlichting (for short, T-S) waves-two-dimensional plane waves growing exponentially with time. It was indicated in Sect. 2.92 that at first the T-S stability theory seemed to be unsuitable as an explanation of real flow instabilities, since the available boundary-layer observations did not confirm the existence of T-S waves. Later such waves were observed in the brilliant wind-tunnel experiments by Schubauer and Skramstad, and their development was found to agree excellently with the theoretical predictions by Tollmien and Schlichting. However, in these experiments a plane wave was artificially excited in the upstream part of the flow, while for other shapes of initial disturbances transition to turbulence usually began with the appearance of "turbulence spots," which grew, coalesced with each other, and gradually filled up all the flow domain (cf. Sects. 2.91 and 2.92). Thus, although the Schubauer-Skramstad experiments proved the accuracy of the T-S theory, questions nevertheless arose about the reconciliation of known theoretical results with experimental data, because the latter showed that flow instabilities are in most cases not accompanied by the appearance of T-S waves.

It seems natural to try to explain this phenomenon by supposing that at supercritical Reynolds numbers (even slightly supercritical, i.e., when $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ but $(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})/\mathrm{Re}_{\mathrm{cr}} \ll 1$) the whole group of different T-S waves is usually simultaneously excited and forms a wave packet where individual waves are masked and thus are hardly observable. Note that at any $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ there exists one most unstable mode, with the maximal value of $\Im\mathrm{m}\,\omega = \omega^{(i)}$ where $\omega = k_1 c$ is an eigenvalue of the Orr-Sommerfeld (O-S) eigenvalue problem (2.41–2.42). (According to Sect. 2.81, the eigenvalue $\omega = \omega_j\ (k_1,\ k_2)$ at given Re depends on two wave numbers, k_1 and k_2, and the integer j; therefore max $\omega^{(i)}$ must be taken with respect to all possible values of k_1, k_2, and j.) For any k_1 and k_2, let $j = 1$ correspond to the most unstable mode (or, if there are no unstable modes, then to the least stable), so that $\max_j \omega_j^{(i)}(k_1, k_2) = \omega_1^{(i)}(k_1, k_2)$. Variations of the function $\omega_1^{(i)}(k_1, k_2) = s(k_1, k_2)$ with k_1 and k_2 are as a rule smooth and gradual, and hence its maximum in a wave-number plane is rather broad. Therefore, at $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ there exists a number of different unstable normal modes with nearly the same growth rate, and an unstable initial disturbance can include many of them. This makes convincing the above-mentioned explanation of the non-observability of individual T-S waves in the majority of experiments on boundary-layer instability.

The arguments given above are qualitative and they naturally stimulate a number of related quantitative studies of the initial-value problems for disturbances in slightly supercritical steady viscous flows. One of the first such studies was due to Benjamin (1961) (see also Drazin and Reid (1981), Sect. 47.1) who used the representation of the solution for the initial-value problem in terms of normal modes. It was explained in Sects. 2.81 and 2.91 that in a plane-parallel flow of a finite thickness there exists, for any values of horizontal wave numbers (k_1, k_2) and Reynolds number Re, an infinite set of normal modes for the vertical velocity disturbance w, having the form

$$w_j(\boldsymbol{x}, t) = W_j(z) \exp\left[i(k_1 x + k_2 y - \omega_j t)\right], \quad j = 1, 2 \ldots \tag{3.41}$$

(similar expressions are also valid for the horizontal velocity disturbances u and v and pressure disturbance p'). Here $\omega_j = \omega_j^{(r)} + i\omega_j^{(i)} = \omega_j^{(r)} + is_j$ are complex eigenvalues of the O-S problem and $W_j(z)$ are the corresponding eigenfunctions, ω_j and $W_j(z)$ both depending on k_1, k_2 and Re. If the set of eigenfunctions $W_j(z)$ is complete in the space of admissible vertical profiles of w, the general solution of the initial-value problem for the vertical-velocity disturbance can be presented in the form

$$\begin{aligned} w(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^{\infty} w_j(k_1, k_2) W_j(z) \exp\left[i\{k_1 x + k_2 y - \omega_j^{(r)}(k_1, k_2) t\}\right. \\ \left. + s_j(k_1, k_2) t\right] dk_1 dk_2 \end{aligned} \tag{3.42}$$

where $w_j(k_1, k_2)$ are coefficients in the expansion of the two-dimensional Fourier transform $\hat{w}(k_1, k_2, z)$ of the initial value $w(x, 0)$ in a series of terms proportional to eigenfunctions $W_j(z)$. However, the form (3.42) of the required solution is too cumbersome to be useful.

To simplify this result, Benjamin noted that, at $\text{Re} > \text{Re}_{\text{cr}}$ there is usually not more than one unstable mode for given k_1 and k_2, while all the other modes are stable. Let us again assume that, for any k_1 and k_2, $j = 1$ corresponds to the most unstable (or, if all modes are stable, to the least stable) normal mode, with the greatest value of $\omega_j^{(i)} = s_j$. Making the natural assumption that the main contribution to the asymptotic behavior of $w(x, t)$ as $t \to \infty$ is due to the most unstable modes, it is now possible to omit all terms with $j \neq 1$ from the right-hand side of (3.42). Such neglect simplifies Eq. (3.42), but not enough to make it easily applicable to real fluid flows.

It is however natural to suppose that the most unstable normal mode with the greatest rate of growth (i.e., the wave with the maximal value $s = \max_{k_1,k_2} s_1(k_1, k_2)$ of the imaginary part of the eigenvalue ω_1), together with a group of modes with $j = 1$ and wave numbers k_1 and k_2 close to those for the most unstable mode (and hence corresponding to values of $s_1(k_1,\ k_2)$ close to s), will after some initial time fully dominate all the other modes. Relying on this assumption Benjamin (1961) (and also Criminale (1960) and Criminale and Kovasznay (1962); see below) proposed to replace the functions s_1 (k_1, k_2) and $\omega_1^{(r)}(k_1, k_2)$ in equations of the form of

Eq. (3.42) by their Taylor's expansions in the neighborhood of the point $(k_1^{(0)}, k_2^{(0)})$ where $s_1(k_1, k_2)$ takes its maximal value s, and to preserve in these expansions only terms not higher than second order. Recall now that according to Watson's result (1960), mentioned in Sect. 2.81, the most unstable normal mode is necessarily two-dimensional for a substantial range of $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$, so that $(k_1^{(0)}, k_2^{(0)}) = (k_0, 0)$. Using such value of $(k_1^{(0)}, k_2^{(0)})$ in the above-mentioned Taylor's expansions, and assuming that the initial disturbance $w(x,0)$ was localized near the point where $x = 0$ and $y = 0$, Benjamin derived from Eq. (3.42) the following asymptotic result:

$$w(x, t) \approx W(x, y, z)t^{-1}e^{st} \quad \text{as} \quad t \to \infty. \tag{3.43}$$

In Eq. (3.43) the dependence of the amplitude W on z is determined by the eigenfunction corresponding to the most unstable mode, while for given z this amplitude takes the maximal value at $x = U_g t$, $y = 0$ (where $U_g = (\partial\omega_1^{(r)}/\partial k_1)_{k_1=k_0,k_2=0}$ is the group velocity of the most unstable wave) and is negligibly small outside of an ellipse in the (x, y)-plane with the center at the point $(U_g t, 0)$ and semiaxes proportional to $t^{1/2}$ (i.e., with the area proportional to t). We see that a localized disturbance is convected downstream at the group velocity U_g in the form of an expanding elliptically-shaped perturbed region. Note also that in this case the wave-packet amplitude does not grow exponentially with t, as do the amplitudes of individual normal modes, but, due to interference of wave-packet components, as $t^{-1}e^{st}$.

The theory sketched above is approximate and its degree of accuracy cannot be easily determined. However Benjamin (1961) showed that the results obtained describe, satisfactorily enough, some results of his experiments on a slightly unstable film flowing down in inclined plate. He also mentioned the possibility of applying the same approach to study boundary-layer instabilities, and referred in this respect to a lecture by Criminale, which was later published as a report (see Criminale (1960)) and still later was used as the basis of the interesting paper by Criminale and Kovasznay (1962).

Criminale and Kovasznay considered the initial-value problem for localized disturbances in a plane-parallel boundary-layer flow with the Blasius velocity profile $U(z)$. In such flow there is no infinite family of discrete eigenvalues ω_j determining a set of eigenfunctions $W_j(z)$ complete in an appropriate function space—as it was explained in Sect. 2.92, only a few discrete eigenvalues exist here, but they are supplemented by a continuous spectrum. Therefore, the form (3.42) of the general solution for a vertical-velocity initial-value problem is inapplicable in this case. However, this form is not needed in arguments based on the assumption that the contribution of all higher normal modes (corresponding to either a discrete or a continuous spectrum) to a disturbance development is negligibly small in comparison to the contribution of the most unstable modes. Since at any values of Ri, k_1, and k_2 in a Blasius boundary layer, there exists "the first mode" with the greatest value of $\omega^{(i)}$, Eq. (3.42) can be applied to this flow too, if the equality symbol in this equation is replaced by symbol $\approx$, index j is replaced by 1, and the summation symbol in the integrand is omitted.

The approach to the initial-value problem used by Criminale (1960) and Criminale and Kovasznay (1962), coincides with that used by Benjamin (1961), but their investigation is much more comprehensive. The authors used the data of previous computations of the O-S eigenvalues and eigenfunctions corresponding to the unstable two-dimensional waves in a Blasius boundary layer, and supplemented them by some new numerical results relating to unstable three-dimensional (oblique) waves. As the initial condition for vertical velocity disturbance at a level z_1 in the outer portion of the boundary layer (the exact value of z_1 is of small importance according to the results obtained) Criminale and Kovasznay considered an axisymmetric Gaussian pulse with small enough standard deviation. Generating reasonably-truncated Taylor's expansions of $\omega^{(r)}(k_1,k_2)$ and $\omega^{(i)}(k_1,k_2) = s(k_1,k_2)$ around the point of maximum amplification, the authors analytically determined the behavior of $w(x,y,z_1,t)$ at small and large values of dimensionless time $\tau = Uk_0t$ (where U is the free-stream velocity and k_0 is the streamwise wave number of the most unstable wave). For intermediate values of τ, some numerical computations were presented. The main terms of the asymptotic equations found for $\tau \to \infty$ agreed with Eq. (3.43) and with the conclusion obtained by Benjamin (1961), that at large t the distribution of vertical velocity of the disturbance in planes $z = \text{const.}$ has the shape of an expanding elliptical wave packet traveling downstream at the group velocity.

Benjamin's and Criminale and Kovasznay's papers stimulated a number of subsequent investigations of developments of weak localized disturbances in slightly-supercritical plane-parallel flows of viscous fluid. In particular, Tam (1967) applied to this problem the general method used by Case (1960a) and Dikii (1960b) for study of disturbance development in plane-parallel inviscid flows. Case and Dikii looked for the general solution of the two-dimensional Eq. (3.3), while Tam instead considered the three-dimensional viscous equation (2.38) of the form

$$\left[\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right]\nabla^2 w - U''(z)\frac{\partial w}{\partial x} - \nu\nabla^4 w = 0 \tag{3.44}$$

where $0 \le z \le H$ and where $w = w(x,y,z,t)$ and $\partial w/\partial z$ vanish at $z = 0$ and $z = H$. Therefore Tam replaced the simple Fourier-Laplace integral (3.4) by the triple Fourier-Laplace integral

$$\hat{w}(k_1,k_2,p;z) = \int_0^\infty e^{-pt}dt \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_1x+k_2y)}w(x,y,z,t)dxdy. \tag{3.45}$$

Equations (3.44) and (3.45) imply the following equation for the Fourier-Laplace transform $\hat{w}(k_1,k_2,p;z)$:

$$\left[\{p + ik_1U(z)\}\left(\frac{\partial^2}{\partial z^2} - k^2\right) - ik_1U''(z) - \nu\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2\right]\hat{w}(k_1,k_2,p;z)$$
$$= w_0(k_1,k_2;z) \tag{3.46}$$

where

$$k^2 = k_1^2 + k_2^2, w_0(k_1,k_2;z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_1x+k_2y)}\nabla^2 w(x,y,z,0)dxdy \tag{3.47}$$

(and hence $w_0(k_1, k_2; z)$ depends only on the initial value $w(x, y, z, 0)$). It is easy to see that the left-hand side of Eq. (3.46) coincides with the O-S equation (2.41) where $-ik_I c = -i\omega = p$. According to (3.45–3.47), the solution of Eq. (3.44) corresponding to the above boundary and initial conditions can be written as

$$w(x, y, z, t) = \frac{1}{8\pi^3 i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{H} \int_{\gamma-i\infty}^{\gamma+i\infty} G(z, z'; k_1, k_2, p) w_0(k_1, k_2; z) e^{i(k_1 x + k_2 y) + pt} dp dz' dk_1 dk_2. \quad (3.48)$$

Where $G(z, z'; k_1, k_2, p)$ is the Green function of Eq. (3.46) for the indicated boundary conditions (cf. Eqs. (3.6) and (3.7)). Similarly to the case of inviscid flow, the Green function G can be explicitly constructed if four linearly independent solutions of homogeneous Eq. (3.46) are known. Moreover, again similarly to the inviscid case, the asymptotic behavior of the solution (3.48) for $t \to \infty$ is fully determined by the singularities of the function G of the complex variable p laying on the right-hand half of the p-plane, and the only substantial singularities are poles at points corresponding to eigenvalues $c = ip/k_1$ of the corresponding O-S eigenvalue problem.

In the case of a slightly supercritical flow only a pair of simple poles of G lies in the right-hand half of the p-plane, and these poles exist only for $\boldsymbol{k} = (k_1, k_2)$ lying within some small vicinity of the point $\boldsymbol{k}_0 = (k_0, 0)$ corresponding to the most unstable normal mode. Therefore, only the values of $\boldsymbol{k}$ within this vicinity of $\boldsymbol{k_0}$ make significant contributions to the asymptotic behavior of $\boldsymbol{w}(\boldsymbol{x}, t)$, and for any such k only the contributions of the complex values of $p = -\, i\omega$ which correspond to the poles of G need to be taken into account. To estimate contributions of these poles to the integral on the right side of (3.48), Tam used the general equation determining the functional form of $(p - p_0)G$ in the vicinity of the pole p_0 of G, and the trinomial Taylor-series approximation of the functions $s(k_1, k_2) = \Re e p(k_1, k_2)$ and $\omega^{(i)}(k_1, k_2) = -\Im m p(k_1, k_2)$ for normal modes with wavenumber vectors $\boldsymbol{k} = (k_1, k_2)$ close to $\boldsymbol{k}_0$. Substituting all these equations into Eq. (3.48), Tam derived from it the same asymptotic equation for the behavior of disturbance $w(x, y, z, t)$ as $t \to \infty$ as was found by Benjamin (1961) and Criminale and Kovasznay (1962). In conclusion, he also noted that the asymptotic shape of an expanding ellipsoidal disturbance is similar to the shape of turbulent spots often observed in laminar flows at the beginning of their transition to turbulence.

A more detailed analysis of the solution of the initial-value problem for an initially localized disturbance to plane-parallel viscous flow with $\mathrm{Re} < \mathrm{Re}_{cr}$ was performed by Easthope and Criminale (1992), for the case of a model boundary layer with the piecewise linear velocity profile shown in Fig. 3.1a. The authors applied the Fourier transform with respect to horizontal coordinates to all terms of Eq. (3.44), then, using the simplicity of the profile $U(z)$, found an analytical equation describing with good accuracy the dependence of w on t and z, and finally determined the other velocity components and pressure and inverted the Fourier transform numerically. They plotted the function $w(x, y, z, t)$ for $z \approx 0.1H$ and several values of t, and also showed the spreading of the wave packet with time. The results obtained agreed well with the asymptotic predictions by Benjamin (1961) and Criminale and Kovasznay (1962) and describe in more details the initial stage of packet development. In particular, these results duplicated some of the experimental findings (corresponding to relatively small values of t) by Gaster and Grant (1975) (whose experiments will be discussed below), although they were insufficient to explain the data obtained by these authors for larger values of t. Note also that, according to Easthope and Criminale's results, the vertical velocity w rises rapidly at small values of t but then begins to decay, while the components u and v (and the vertical vorticity ζ_3) continue to increase at least linearly with t. Therefore, the computations confirmed Landahl's (1980) general predictions based on quite different arguments.

General solutions of the initial-value problem for initially localized disturbances to the boundary-layer flow were considered also at first by Gustavsson (1979) and Hultgren and Gustavsson (1981) and then by Brevdo (1995a, b). These authors replaced real boundary layers with gradually growing thickness by a model plane-parallel flow in the half-space having the Blasius velocity profile. Gustavsson, and Hultgren and Gustavsson obtained some particular results (which will be considered in Sect. 3.32 below) about the stability properties of disturbances at such flow at subcritical values of Re. Brevdo studied in detail asymptotic behavior of wave packets in the considered model flow at supercritical values of Re; he proved, in particular, that for a wide range of Re values exceeding $\mathrm{Re}_{\mathrm{cr}}$ the plane waves and wave packets in this flow can be only convectively, but not absolutely, unstable (cf. Sect. 2.93 in Chap. 2 where, in particular, a similar result by Deissler (1987) relating to a plane Poiseuille flow was indicated).

Another approach to the asymptotic analysis of approximate wave-packet solutions of the initial-value problem for localized weak disturbances was proposed by Gaster (1968a) who applied it first to the Blasius boundary-layer flow and then, jointly with Davey (see Gaster and Davey (1968)), to the highly unstable two-dimensional wake in unbounded space having a Gaussian velocity profile. Here the wave-packets produced by an initial pulsed disturbance in a point in the fluid were represented by the following equation

$$w(x, y, z, t) = \int_{c_1}\int_{c_2} \hat{w}(k_1, k_2; z) \exp\left[i\left(k_1\frac{x}{t} + k_2\frac{y}{t} - \omega\right)t\right] dk_1 dk_2 \qquad (3.49)$$

where $\hat{w}(k_1, k_2; z)$ is the two-dimensional Fourier transform of the initial value $w(x, y, z, t, 0)\omega = \omega(k_1, k_2)$ is the complex eigenvalue of the O-S problem corresponding to the most unstable wave with the wave-number vector $\boldsymbol{k} = (k_1, k_2)$, and the integration paths c_1 and c_2 in the complex k_1- and k_2-planes are obtained from the horizontal axes $-\infty < k_j < \infty, j = 1, 2$, by continuous deformations placing them above all the singularities of the integrand.

To find the asymptotic behavior of $w(\boldsymbol{x}, t)$ for $t \to \infty$, it is necessary to determine the asymptotic of an integral whose integrand includes the exponential of a function $\Psi(k_1, k_2)$ multiplied by a large factor t. Gaster (1968a) followed his analysis for the vibrating ribbon problem (presented in Gaster (1965), cf. Sect. 2.92) by applying the method of steepest descent to evaluation of the integral on the right-hand side of

(3.48). For this it was necessary to expand the function Ψ about its stationary saddle points where

$$c_{gx} = \frac{\partial \omega}{\partial k_1} = \frac{x}{t}, c_{gy} = \frac{\partial \omega}{\partial k_2} = \frac{y}{t} \tag{3.50}$$

and c_{gx} and c_{gy} are group velocities of the wave packet in the x and y directions. However the precise location of the saddle points in the complex $(\boldsymbol{k}, \omega) = (k_1, k_2, \omega)$ space is a tricky problem requiring some complicated computations. At first both Gaster (1968a) and Gaster and Davey (1968) did not take into account all the complications involved, and therefore the asymptotic wave-packet shapes found by them proved to be distorted by some spurious contributions. Later Gaster's result for boundary-layer wave packets was corrected by the author himself (see Gaster (1979, 1981, 1982a) and also Craik (1981, 1982)), while Gaster and Davey's results for wave packets in an unbounded plane wake were corrected by Gaster's student Jiang (1991). At the same time some other methods of the wave-packet shape evaluation were also developed by Gaster (1975); Landahl (1972, 1982); and Craik (1981, 1982).

A simple theoretical model of early stages of wave-packet development in the Blasius boundary layer was proposed by Gaster (1975) as an explanation of experimental data collected at the same time by Gaster and Grant (1975). These data were from careful hot-wire-anemometer measurements of the development of a disturbance produced by a short acoustic pulse injected in the flat-plate boundary layer over a large plate in a wind tunnel through a small hole near the leading edge of the plate. According to the results obtained, the resulting wave packet is roughly elliptic in plane view at small distances from its origin, but further downstream the disturbed region spreads out (roughly in proportion to elapsed time t) and distorts, becoming distinctly bowed into a crescent shape. Gaster's model represents a wave packet as a superposition of a large number of wave-like normal modes of the ordinary form

$$u(x, t) = u(z) \exp\left[i(k_1 x + k_2 y - \omega t)\right] \tag{3.51}$$

(cf. Eq. (3.41); now u is used instead of w since only the streamwise velocity of disturbance was measured by Gaster and Grant). However, contrary to earlier wave-packet models, Gaster supposed that k_2 and ω are two given real constants while the streamwise wavenumber $k_1 = k_1(k_2, \omega)$ is a complex function of two variables equal to the most unstable eigenvalue (i.e., that having the numerically-greatest negative imaginary part) of the corresponding spatial O-S eigenvalue problem. Hence, a wave packet was considered as a superposition of spatially (and not temporally) growing waves. Moreover, all higher normal modes and also the continuous spectrum of eigenvalues in the O-S problem in a half-sapce were ignored by Gaster, exactly as was done in the other studies mentioned above. That is, he used the following approximation to the general solution of the initial-value problem for the streamwise velocity,

$$u(\boldsymbol{x},t) \approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(k_2,\omega;z)\exp\left[i\{k_1,(k_2,\omega)x + k_2 y + \omega t\}\right]dk_2 d\omega \qquad (3.52)$$

where $u(k_2,\omega;z)$ is the Fourier transform, with respect to y and t, of the value $u(0,y,z,t)$ of the streamwise velocity disturbance at $x=0$.

Note also that in fact Gaster introduced in his paper of 1975 (and studied at greater length in Gaster (1982b), for two-dimensional packets but for a much wider range of x-values) some corrections of the simple Eqs. (3.51) and (3.52), which approximately described the influence on the packet development of the slow growth of boundary-layer thickness δ or displacement thickness δ^* (and or Re) with streamwise distance x. However, consideration of the proposed corrections would take too much space here, so only a simplified version of the arguments given in the original publications will be presented.

To evaluate the right-hand side of (3.52), Gaster (1975) replaced the double integral by an appropriate integral sum. First of all, he carefully calculated a great number of eigenvalues $k_1(0,\omega)$ for two-dimensional waves with $k_2=0$ and different values of the wavenumber k_1, also varying the Reynolds number $\mathrm{Re}_{\delta^*}=U\delta^*/\nu$ (Where δ^* is the displacement thickness of the boundary layer) within a range corresponding to the streamwise locations of the measurements of Gaster and Grant (1975). The eigenvalues corresponding to three-dimensional (oblique) modes with $k_2\neq 0$ were computed from the eigenvalues for two-dimensional waves with the aid of Squire's transformation, described in Sect. 2.81, which can also be applied to spatial formulation of the eigenvalue problem. Gaster and Grant's measurements include time records of the streamwise velocity $u(\boldsymbol{x},t)$ at a fixed value of z and various values of x and y. The amplitudes $u(k_2,\omega;z)$ entering the integrand in Eq. (3.52) represent the two-dimensional Fourier transform of the function $u(0,y,z,t)$ with respect to y and t, for fixed value of z. For a disturbance produced by a narrow acoustic pulse of short duration the dependences on both y and t must be close to Dirac δ-functions, and therefore it was natural to assume that the initial (ω,k_2)-spectrum must be quite flat in both frequency ω and spanwise wavenumber k_2. Gaster and Grant found that this assumption agreed well enough with frequency-wavenumber spectral data at different distances x from the hole in the plate where the disturbance was introduced, and therefore this assumption was used by Gaster (1975) as a reasonable first approximation.

The amplification (or attenuation) of various normal modes was determined from the measured values of the (ω,k_2)-spectra for values of $u(x,y,z,t)$ at the selected value of z and various x. At the same time this amplification could also be calculated by the linear theory of hydrodynamic stability, determining the most unstable values of $k_1(k_2,\omega)$ for any given values of k_2,ω, and Re. According to Gaster (1975) the measured and calculated values of the amplification agreed well, giving additional confirmation of the satisfactory accuracy of the linear stability theory and of the approximation (3.52). The calculated shapes of the disturbed regions, i.e., of the regions where the relative amplification of the wave-packet power exceeded some appropriate threshold value, also agreed well with the experimental data by Gaster and Grant at not-too-large values of x. In other words, Gaster's theoretical model predicted with satisfactory accuracy the observed variations of the overall shape of the disturbed region and the way it expands as the wave packet traveled downstream. Note, however, that good agreement between the results of normal-mode summation and the observations was found only for the early stages of wave-packet development. For larger values of x the observed shape of the packet was found to be more distorted and Gaster (1975) attributed these definite discrepancies between theory and experiment to nonlinear effects.

Another explanation of these discrepancies is the possibility that the summation of normal modes is insufficient for the determination of long-time behavior, the main contribution to which is due to saddle points of the dispersion relation $\omega = \omega(k_1, k_2, \mathrm{Re})$ for complex wavenumbers k_1 and k_2. It has been already noted above that the asymptotic saddle-point analysis was first applied to wave-packet development by Gaster (1968a) and Gaster and Davey (1968) but since the exact form of the dispersion relation is usually unknown and cannot be easily determined, these attempts were not wholly successful. Later Gaster (1979, 1982a) thoroughly analyzed the applications of the saddle-point method to the evaluation of long-time development of two-dimensional (2D) wave packets (composed of 2D waves) in a plane-parallel Blasius boundary layer. He found that the method gives accurate results for all but very short times after the generation of the disturbance, and he also theoretically estimated the errors for various simplified asymptotic representations of 2D packets. For three-dimensional (3D) wave packets Gaster (1981) considered some approximate asymptotic representations (including that used by Benjamin and by Criminale and Kovasznay), and estimated their accuracies (which was found to be sufficiently good) by comparison of the results obtained with those given by numerical integration. Independently Craik (1981, 1982) proposed evaluating the developments of wave packets generated by short-term localized 3D disturbances in unstable plane-parallel flows by a saddle-point method, using simplified algebraic models of the 3D dispersion relations. For models containing enough free numerical parameters it is possible to achieve good agreement with the available results of computations of the O-S eigenvalues, and Craik then showed that the saddle-point method leads to conclusions which also agree well with the results for wave-packet development. In particular, Craik's models imply the initial elliptic shape of a packet, its subsequent bending to a crescent shape, and the expected behavior of packets in cases where Re considerably exceeds $\mathrm{Re}_{\mathrm{cr}}$.

One more method for evaluation of the wave packet development was first used for a special purpose by Landahl (1972) who later (in 1982 and 1985) developed it further and applied it to representation of evolution for rather general localized 3D packets of waves growing both in space and time. This new method is based on the kinematic wave theory by Whitham (1965, 1974) (see also the short presentation by Landahl and Mollo-Christensen (1992), Cap. 6). Whitham's theory in its original version dealt only with conservative waves; therefore, its application by Landahl (1972) to waves with small dissipation at first gave rise to some criticism. However, later formal extension of kinematic wave theory to the case of such waves was developed by several authors (in particular, by Jimenez and Whitham (1976) and

Chin (1980)). This extension confirmed Landahl's results of 1972 and showed (see Landahl (1982, 1985)) that the kinematic wave theory leads to results equivalent to those given by the saddle-point method and can be used also for representation of some nonlinear features of wave-packet propagation.

Advances in computer technology led, during the last decade, to a number of investigations of wave-packet developments in laminar boundary-layer flows by direct numerical simulations (DNS), i.e., by numerical solutions of Navier-Stokes equations with the initial and boundary conditions corresponding to a steady boundary-layer flow disturbed at the instant $t = 0$ by a strongly localized small disturbance. Some examples of such simulations were described, in particular, by Lenz (1986), who considered only two-dimensional disturbances, and by Fasel et al. (1987) and Konzelmann (1990). Note that the modern development of computational methods makes it unnecessary to linearize the navier-Stokes equations with respect to flow disturbances, and hence permits combined DNS study of the nonlinear stage of wave-packet development and its initial linear stage. Moreover, the boundary-layer development can also be numerically simulated independently of the computation of disturbances, and the influence of the growth of the boundary-layer thickness is automatically taken into account. We will not consider here details of the available DNS results for wave-packet development, but simply note that the results of the above-mentioned authors show excellent qualitative and fully satisfactory quantitative agreement with the experimental results obtained by Gaster and Grant (1975).

3.3.2 Resonance and Degeneracy Growths of Disturbances in Subcritical Flows

Section 3.31 was devoted mostly to consideration of disturbance development in supercritical laminar flows, with $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$, and to discussion of some possible reasons for the so-called *by-pass transition* to turbulence (see Sect. 2.92), where no T-S waves are observed. In this and the following sections, the main case to be discussed will be that of subcritical flows, with $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$, and a possible explanation will be given of the *subcritical transitions* that are frequently observed (e.g., in plane Couette and circular Poiseuille flows, where $\mathrm{Re}_{\mathrm{cr}} = \infty$, and in plane Poiseuille flows, where $\mathrm{Re}_{\mathrm{cr}} \approx 5770$, while transition to turbulence usually occur at $\mathrm{Re} \approx 1000$; see Sects. 2.1 and 2.9). For this purpose it will be necessary to pay more attention to general solutions of the initial-value problems for localized disturbances in laminar flows.

The general solution (3.48) of the initial-value problem for the disturbance vertical velocity $w(x, y, z, t)$ in a laminar channel flow was considered above in the discussion of the paper by Tam (1967). The same solution was later studied by Gustavsson (1979) with application to subcritical disturbance development in a Blasius boundary layer (strictly, a plane-parallel flow in the half-space $0 \le z < \infty$ with the Blasius velocity profile). Gustavsson used a slightly different, but equivalent, form of the integrand in Eq. (3.48); he expressed the Green's function G explicitly, in terms of four linearly independent solutions of the homogeneous Eq. (3.46). Like Tam, he was mainly interested in the asymptotic behavior of the vertical velocity $w(\boldsymbol{x}, t)$ as $t \to \infty$, which is determined by the contribution to w made by singular points of the integrand in the complex p-plane. However, in the case of a Blasius boundary layer, the simple poles of the integrand, at values of $ip = \omega$ equal to discrete eigenvalues ω_j of the corresponding O-S equation, are supplemented by a branch point. Therefore in this case the path of integration in the p-plane must be deformed around the branch cut ending at the branch point. As in the case of a channel flow, the poles at points $p_j = -i\omega_j$ contribute to the log-time behavior of w, the summands proportional to $e^{-i\omega_j \psi}$, which describe the asymptotic behavior of the T-S wave components of the vertical velocity, depending exponentially on time. Now, however, the loop enclosing the branch cut also makes a non-zero contribution to the value of w, which determines the component of w generated by the continuous spectrum of the O-S eigenvalues. Gustavsson showed that this component also eventually decays exponentially, but at small values of t its dependence on time is algebraic (with exponent depending on the shape of the initial disturbance). He also showed that the duration of the initial period of algebraic variation increases, and the rate of exponential decay as $t \to \infty$ decreases, with increase of the disturbance length scale.

More detailed analysis of Gustavsson's solution of the initial-value problem for a disturbance in a Blasius boundary layer was carried out by Hultgren and Gustavsson (1981) for the special case of disturbances having a streamwise length scale l very-large compared to the boundary layer thickness δ. (It was explained by the authors that since the parallel-flow assumption was used, l had to satisfy the double inequality $\delta \ll l \ll \delta \,\mathrm{Re}$.) In this case $k_1\delta \ll 1$, and it is easy to show that then the integrand to the inverse Laplace transform in Eq. (3.48) does not possess any poles, and hence only a branch cut in the p-plane (describing the continuous-spectrum contribution) must be taken into account. Applying some algebraic manipulations to Gustavsson's solution for $w(\boldsymbol{x}, t)$, Hultgren and Gustavsson found that if $l/\delta \gg 1$ (so that the dependence of w on x can be neglected in the first approximation), then inside the boundary layer $w(y, z, t) \approx w(y, z, 0) = w_0(y, z)$ (i.e., the vertical velocity essentially remains constant) for small times $t \ll \delta^2/\nu$, but $w \propto U_0(t\nu/\delta^2)^{-2}$ (i.e., it decays as $t-{}^2$) for large times $t \gg \delta^2/\nu$. The results for $w(y, z, t)$ were then used to find the asymptotic behavior of horizontal velocity disturbances. Using Eq. (2.35) for the streamwise velocity disturbance u, and the continuity equation (2.36), and neglecting the dependence of u and the pressure disturbance p on x, Hultgren and Gustavsson obtained for u and the spanwise velocity v the following two equations

$$\frac{\partial u}{\partial t} - \nu\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u = -U'w, \tag{3.53}$$

$$\frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}.$$

Here the vertical velocity disturbance $w(y, z, t)$ was known from the above results; therefore Eqs. (3.53) could be used for finding the horizontal components of the disturbance velocity. The second Eq. (3.53) allows the spanwise velocity component $v(y, z, t)$ to be found quite easily; it leads to obvious results which will be omitted here. The first Eq. (3.53) has the form of a heat-conduction (or diffusion) equation with a source term. Hultgren and Gustavsson showed that the solution of this equation corresponding to a given initial value $u(y, z, 0) = u_0(y, z)$ can be written as a sum of two terms T_1 and T_2, the first of which depends on $u_0(y, z)$ and the second on the solution for $w(y, z, t)$. For an arbitrary value of t, this solution must be evaluated numerically, but its asymptotic behavior at small and large times is given by simple analytic expressions. At small times $t \ll \delta^2/\nu$ the term T_1 differs from the initial value $u_0(y, z)$ only by a small viscous correction, while the main part of T_2 has the form $-U'(z)w_0(y, z)t$ (again with a small viscous correction).Thus the inviscid result of Ellingsen and Palm (1975) (see Eq. (3.21)) was recovered (with a viscous correction and a viscous limit of validity) directly from the solution of the initial-value problem. At times $t \gg \delta^2/\nu$, it was found that the streamwise velocity disturbance $u(y, z, t)$ decays as t^{-2} in both the boundary layer and the free stream.[4]

Thus, the algebraic initial growth of $u(\boldsymbol{x}, t)$ was derived by Hultgren and Gustavsson (1981) from a solution of the initial-value problem for disturbances with a large streamwise scale. More complete analysis of the considered general solution of the initial-value problem was carried out by Brevdo (1995a, b); these papers were already mentioned in Sect. 3.31. Note now that there are also many other special cases where solutions of the initial-value problem imply the algebraic growth of disturbance velocities; see, for example, the discussion of Easthope and Criminale's paper (1992) in Sect. 3.31. In Sect. 3.2, when discussing the inviscid Eq. (3.9), it was noted that for two-dimensional disturbances, of the form $\{u(x, z, t), 0, w(x, z, t)\}$, the only dynamical equation needed is that for the vertical velocity disturbance w; in this case, once w is known, the streamwise velocity disturbance u follows from continuity (and, if needed, the pressure disturbance p can also be determined easily). However, if general three-dimensional disturbances are considered, then to find the whole velocity field, at least one more dynamic equation is needed. In Sect. 3.2, the inviscid Eq. (3.10) for the vertical vorticity component $\zeta_3 = \partial v/\partial x - \partial u/\partial y$ was recommended as such a supplementary equation.

The same considerations clearly apply to viscous flows. In this case, Eq. (3.9) for the vertical velocity disturbance $w(\boldsymbol{x}, t)$ in a steady plane-parallel flow must be replaced by Eq. (3.44), which includes a viscous term. Also, Eq. (3.10) for ζ_3 must now also be supplemented by an additional viscous term turning it into

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\zeta_3 - \nu\nabla^2\zeta_3 = U'\frac{\partial w}{\partial y}. \tag{3.54}$$

[4] These results were found for the plane-parallel model of the Blasius boundary layer. In reality the thickness of a boundary layer increases with x and this must lead to gradual weakening of the influence of viscosity. This effect was studied by Luchini (1996) who found that in the model of a boundary layer with the thickness depending on x a three-dimensional disturbance can exist which algebraic growth produced by the lift-up effect overcomes the viscous damping. Therefore, within the limits of the linear stability theory and of the model of a boundary layer of infinite streamwise extent, this disturbance is growing at all times.

In the case of purely horizontal velocity disturbances, where $w(x,t) \equiv 0$, Eq. (3.54) for the vorticity ζ_3 has the same form as the equations for both horizontal velocity components $u(\boldsymbol{x}, t)$ and $v(\boldsymbol{x}, t)$ given for this case in Sect. 2.81 (before Eqs. (2.46)). It was explained there that in steady plane-parallel flows these equations for u and v describe some normal modes having the form of horizontal-velocity waves, in addition to the better-known T-S waves. These new modes (sometimes called the *Squire modes* in contrast to the more ordinary *Orr-Sommerfeld modes* where w satisfies the O-S equation) always decay with time and therefore may be ignored when the normal-mode approach to the linear theory of hydrodynamic stability is used. It can also be shown (see, e.g., Reddy et al. (1993); Reddy and Henningson (1993); Henningson et al. (1994); or Schmid and Henningson (2001)) that, according to linear stability theory, the energy of any horizontal-velocity disturbance $\{u(x, y, z, t),\ v(x, y, z, t),\ 0\}$ always decays monotonically with time (in contrast to the case of vertical velocity disturbances $w(\boldsymbol{x}, t)$ where the possibility of very large initial growth had been proved already by Orr (1907)). Hence, one might think that infinitesimal disturbances with zero vertical velocity can also be omitted in stability studies using the initial-value-problem approach. However, this conclusion is incorrect, since it does not follow from the above-mentioned results that horizontal velocity components and vertical vorticity of a disturbance are irrelevant in the general case where all components of the velocity vector $\boldsymbol{u}(\boldsymbol{x}, t) = \{u(\boldsymbol{x}, t), v(\boldsymbol{x}, t), w(\boldsymbol{x}, t)\}$ differ from zero.

In the general case, Eqs. (3.44) and (3.54) form a closed system of two equations, with two unknowns w and ζ_3. Equation (3.44) may be solved independently from Eq. (3.54), and the solution obtained for w then substituted into Eq. (3.54). Let us consider the normal modes of disturbance which are proportional to $\exp\left[i(k_1\xi + k_2\eta - \omega\tau)\right]$ (where, as in Sects. 2.81 and 2.92, $\xi = x/H, \eta = y/H$, and $\tau = tU_0/H$ are dimensionless horizontal coordinates and time, H and U_0 being appropriate length and velocity scales). Then the dimensionless vertical-velocity amplitude $W(\zeta)$ (where $\zeta = z/H$) will satisfy the O-S equation (2.41) (with $c = \omega/k_1$), while the dimensionless vertical-vorticity amplitude $Z(\zeta)$ will satisfy an equation having a left side of the same form as in Eq. (2.46) (again with $c = \omega/k_1$), but the non-zero term $ik_2U'W(\zeta)$ on the right side.

In Chap. 2 it was explained that in the case of the O-S eigenvalue problem (i.e., for the O-S equation with the appropriate boundary conditions) there corresponds, to any values of k_1, k_2, and Ri, an infinite (in the case of flows in channels of finite depth) or finite (for plane-parallel flows in an unbounded or semibounded space) set of eigenvalues $\omega_j(k_1, k_2, \mathrm{Re})$ determining a set of O-S waves. Another set of eigenvalues $\omega_j^0(k_1, k_2, \mathrm{Re})$ corresponds to the Squire (briefly, Sq) eigenvalue problem (i.e., to the Sq equation (2.46) with the appropriate boundary conditions), and determines a set of Sq waves. As was said above, the Sq waves always decay, since $\Im\mathrm{m}\, \omega_j^0(k_1, k_2, \mathrm{Re}) < 0$ for any values of $j,\ k_1,\ k_2$ and Re (for information about the eigenvalues ω_j^0 see, e.g., Davey and Reid (1977) where the same eigenvalue problem appeared in a different context). However, $\Im\, m\omega_j(k_1, k_2, \mathrm{Re})$ is negative for any j, k_1 and k_2 only if $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$.

Let us assume that $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$; then all O-S and Sq waves decay as τ (i.e., t) tends to infinity. Therefore the flow is stable from the standpoint of the normal-mode approach to linear stability theory. Note, however, that the Sq waves represent vorticity waves corresponding to "free oscillations" of the vorticity and horizontal velocity fields, while Eq. (3.54) contains on the right side a "force" $U'\partial w/\partial y$.[5] Here, therefore, "forced," not "free," solutions of the Sq problem must be considered.

For $\mathrm{Re} < \mathrm{Re}_{cr}$ viscous effects lead to damping of all wave-like disturbances as $t \to \infty$. It is however known that even forced oscillations that eventually die out can be strongly amplified initially in the case of a resonance, i.e., when a frequency of free oscillations of (say) a mechanical structure coincides with a frequency of the applied force. Therefore, it is natural to investigate whether a resonance can occur in forced excitation of vertical-vorticity waves, and if so what will be its consequences.

Solutions of the homogeneous Eq. (3.54) for a wide range of conditions can be expanded into Sq waves, while a force can be represented by a series of O-S waves. Therefore a resonance is possible here if values of k_1, k_2 and Re exist, such that $\omega_j(k_1, k_2, \mathrm{Re}) = \omega_i^0(k_1, k_2, \mathrm{Re})$ for some integers j and i. Apparently Gustavsson and Hultgren (1980) were the first to formulate the resonance problem of linear hydrodynamic stability theory, and to study it for the case of a plane Couette flow. They began with numerous computations of the Couette-flow O-S and Sq eigenvalues belonging to the first four eigenvalue modes. Instead of complex frequencies ω they used the complex phase velocities $c = \omega/k_1$, which depend on two variables $k = (k_1{}^2 + k_2{}^2)^{1/2}$ and $k_1\mathrm{Re}$ (while in case of the Sq eigenvalues, $c + ik^2/k_1\mathrm{Re} = c$' depends only on $k_1\mathrm{Re}$). The computations showed that for any value of k at least two values of $k_1\mathrm{Re}$ exist, such that $c(k,\ k_1\mathrm{Re}) = c^0(k,\ k_1\mathrm{Re})$ (where c and c^0 are the Couette-flow O-S and Sq eigenvalues) for either the first or the second eigenalue mode. If $c = c^0$, then clearly $\omega(k_1, k_2, \mathrm{Re}) = \omega^0(k_1, k_2, \mathrm{Re})$ for $k_2 = (k^2 - k_1{}^2)^{1/2}$. Thus, for corresponding values of k_1, k_2,Re (and maybe also for some other still-unknown values of these variables), a resonant excitation of the vertical vorticity (and horizontal velocity) waves can occur in a plane Couette flow. In such cases the corresponding wave-like solutions of Eq. (3.54) (and the related horizontal-velocity waves too) will include a resonance term depending on time as $\tau e^{-i\omega\tau}$. Since $\Im\mathrm{m}\,\omega = \omega^{(i)} < 0$, this term will eventually die out, but at first, as long as $-\omega^{(\mathrm{i})}\tau \ll 1$, it will grow linearly with time. Gustavsson and Hultgren found that the slowest exponential decay (and hence the longest period of the resonance growth of a disturbance, and the largest value of an amplitude at the end of this period) are usually reached for $k \approx 2$ (but at $k \approx 2$ there exists a broad wave packet consisting of waves with similar growth properties). At large values of Re the structures which grow most significantly in the initial period have a streamwise elongated shape, and the duration of their period of growth increases with Re. In the limit $\mathrm{Re} \to \infty$, the linear resonant growth is real-

---

[5] The physical mechanism of the "force effect" is rather simple: If $U' \neq 0$, the vertical velocity w leads to vertical displacements of fluid particles transferring their original streamwise velocity to a new height with different mean velocity U, i.e., producing additional disturbances of the streamwise velocity (Landahl's lift-up effect mentioned in Sect. 3.2). If $U' \neq 0$, and $\partial w/\partial y \neq 0$, then the lift-up effect will vary with the span wise coordinate y creating regions of non-zero derivative $\partial u/\partial y$ and hence acting as a source of vertical vorticity. Note also that the first Eq. (3.53) describes forced streamwise velocity oscillations where the force on the right side represents the lift-up effect.

ized at all times (in full accordance with Ellingsen and Palm's (1975) and Landahl's (1980) results considered in Sect. 3.22).

An investigation of possible resonances in a subcritical plane Poiseuille flow was performed by Gustavsson (1981, 1986) (see also Benney and Gustavsson (1981)). Since there were no data to determine whether the coincidences $c = c^0$ are possible or impossible in this case, Gustavsson calculated anew a number of the corresponding O-S and Sq eigenvalues c and c^0. He found that, contrary to the case of a plane Couette flow, in the case of a plane Poiseuille flow resonances can occur only for certain isolated points $(k, k_1\mathrm{Re})$. A number of such Poiseuille-flow "resonance points" (where $c = c^0$) was indicated in Gustavsson's papers (1981, 1986) where it was also stated that their number is apparently infinite.

It has been mentioned above that resonant excitation of disturbances is usually responsible for only a part of the total values of $u(\boldsymbol{x}, t)$, $v(\boldsymbol{x}, t)$, and $\zeta_3(x, t)$. As a rule, initial disturbances include many different Fourier components, and generate wave packets in which waves corresponding to resonant values of k and $k_1\mathrm{Re}$ are masked by all the other waves. However, even for the Fourier component of a disturbance with such wave numbers k_1 and k_2 that $((k_1{}^2 + k_2{}^2)^{1/2}, k_1\mathrm{Re})$ is a resonance point in the $(k, k_1\mathrm{Re})$-plane, the wave amplitude is not exactly proportional to $\tau e^{-ik_I c\tau}$. According to Gustavsson's general solution of the corresponding initial-value problem, the resonant term (i.e., the contribution of the resonance pole of the integrand in Eq. (3.48) in the complex p-plane) has the form $[r_1 + r_2 k_1\tau]\exp[i(k_1\xi + k_2\eta - k_1 c\tau)]$ where r_1 and r_2 are complex numbers (depending on initial conditions and the parameters ζ, k_1, k_2, and Re). Hence the time-dependent wave amplitude, $R(\tau)$, is equal to $(r_1 + r_2 k_1\tau)\exp(k_1\Im\mathrm{m}\, c\tau)$ where $c = c(k,\ k_1\mathrm{Re})$ is the joint O-S and Sq eigenvalue. We see that the amplitude $R(\tau)$ includes two terms, the first of which decreases exponentially (since $\Im\mathrm{m}\, c\tau < 0$ in a subcritical flow) while the second at first grows linearly (this is just the resonance growth) and only later begins to decay. It is easy to see that the general character of the time evolution for the wave disturbance considered depends on the sign of the difference $\Re e(r_2/r_1) - \Im\mathrm{m}\, c$; only if it is positive will the amplitude $R(\tau)$ grow initially and decay at later times (see, e.g., Shanthini (1989)).

Gustavsson (1986) computed time-dependent amplitudes $R(\tau)$ of the resonant vertical-vorticity waves for a number of resonance parameters (k_1, k_2, Re) and initial values $w_0(k_1, k_2;\ z)$ (defined by Eq. (3.47)). It turned out that all the computed amplitudes decay monotonically with time. This shows that here the contributions of a monotonically-decreasing term with coefficient r_1 usually dominate the disturbance development. Of course, Gustavsson's computations covered only a limited range of conditions but, nevertheless, his results cast doubt on the assumption that the resonance mechanism is the main cause of the observed transient growth of flow disturbances.

Studies of the possible resonances in the case of plane-parallel boundary-layer flows were carried out by Benney and Gustavsson (1981), who investigated three examples of a velocity profile $U(\zeta)$ but published only results for the Blasius profile. For boundary layers only a finite number of discrete eigenvalues c (or $\omega = k_1 c$) exists, so here there are not too many choices for possible direct resonances. Calculations of both O-S and Sq eigenvalues suggested that in boundary-layer flows no exact resonances (i.e., no values of k and k_1Re such that $c(k,\ k_1\mathrm{Re}) = c^0(k,\ k_1\mathrm{Re})$) occur for any of the considered velocity profiles. (Later Jang et al. (1986) discovered that an exact resonance, which may be physically important, exists in a turbulent boundary layer, where the velocity profile is quite different from that in laminar layers. However, this topic is beyond the scope of this chapter of our book.) Moreover, Benney and Gustavsson found that in the case of laminar boundary layers some near-resonances exist, i.e., here the difference $c - c^o$ can take quite small absolute values. The authors stated that such near-resonances can also produce substantial growth of disturbances, which can lead to important consequences when the nonlinear mechanisms of the disturbance development, also considered in their paper, are taken into account. However, we have no space to discuss this matter here.

Gustavsson (1989) also studied the forcing mechanisms and resonances occurring in disturbed Poiseuille flow in a circular tube. Equations for small disturbances of an axisymmtric laminar flow were given in Sect. 2.84, where cylindrical coordinates, r, ϕ, x were used instead of rectangular coordinates $x,\ y,\ z$. Gustavsson showed that in a tube flow, resonances can occur only in the case of non-axisymmetric disturbances (depending on ϕ). (Recall that in plane-parallel flows resonances are possible only for three-dimensional disturbances, depending on the spanwise coordinate y.) Therefore, Gustavsson considered only the normal modes with the azimuthal wave number $n \neq 0$.

Gustavsson used Eqs. (2.73) to obtain a system of four homogeneous differential equations for the r-dependent amplitudes g, $f^{(r)}$, $f^{(\phi)}$ and $f^{(x)}$ of normal modes corresponding to reduced pressure p/ρ and three components (u_r, u_ϕ, u_x) of disturbance velocity. (This system differs from Eqs. (2.74) by viscous terms which were omitted in Sect. 2.84.) Then he eliminated all unknowns except g from the system, and thus found a sixth-order homogeneous differential equation for the pressure amplitude $g(r)$. The dimensionless form of this equation utilizes the normalized radial coordinate r/R instead of r (a dimensional radial coordinate will not be used below and therefore r will later denote just the normalized radial coordinate) and includes the following dimensionless parameters: $k, n, c = \omega/k$, and $Re = U_0R/\nu$, where R and U_0 are the tube radius and the centerline velocity, k is the streamwise wave number multiplied by R, and c is the phase velocity of the modal pressure wave divided by U_0. The boundary conditions require that all three velocity components vanish at the tube wall (i.e., at $r = 1$) and are finite at the tube axis; they allowed Gustavsson to obtain six boundary conditions for the function $g(r)$. The boundary conditions, together with the equation for g, form an eigenvalue problem determining a set of eigenvalues $c_j(k,\ n,\ \mathrm{Re})$ for the given values of $k,\ n$ and Re.

The third Eq. (2.73) relating to streamwise disturbance velocity u_x leads to the following dimensionless u_x-amplitude equation

$$\tilde{\nabla}^2 f^{(x)} - ik\,\mathrm{Re}\,(U - c^0) f^{(x)} = \mathrm{Re}\,(U' f^{(r)} + ikg),$$

$$\tilde{\nabla}^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - k^2 - \frac{n^2}{r^2}, \qquad (3.55)$$

where now $U(r) = 1 - r^2$ is the velocity of the circular Poiseuille flow divided by U_0, a prime denotes the derivative *d/dr* and c^0 is the dimensionless streamwise velocity of the u_x-mode. The homogeneous version of Eq. (3.55), where the right-hand side is replaced by zero, together with boundary conditions requiring that $f^{(x)} = 0$ for $r = 1$ and tends to a finite value when $r \to 0$, form the u_x-mode eigenvalue problem determining the set of eigenvalues $c_i^0(k, n, \mathrm{Re})$. The right side of Eq. (3.55) contains a linear combination of functions g and $f^{(r)}$, and it is easy to deduce from Eqs. (2.73.) that $U' f^{(r)} = i\tilde{\nabla}^2 g/2k$. Therefore, the inhomogeneous Eq. (3.55) can be regarded as an equation describing forcing of the streamwise velocity component by a specific force linearly depending on $g(r)$ (i.e., in fact by the pressure force). A resonance occurs here when, for some values of $k, n \geq 1$, and Re, a pressure eigenvalue $c_j(k, n, \mathrm{Re})$ coincides with some streamwise-velocity eigenvalue $c_i^0(k, n, \mathrm{Re})$.

To find resonances, Gustavsson calculated values of discrete eigenvalues c_j and c_i^0 for a great number of combinations of values for k^2, kRe and n, where k was assumed to be complex. He found that resonances are very numerous (apparently there are infinitely many of them) and listed 36 resonances presenting the corresponding values of k, n, and Re together with the values for the phase velocity and damping rate of resonance waves. However, no attempt to investigate the possible resonance growth of velocity disturbances was made.

Above, we considered a number of studies of the initial algebraic growth of small disturbances in steady parallel viscous flows. Special attention was given to resonance effects, which play an important part in many mechanical problems. Let us now recall that in Sect. 2.5, in the introductory discussion of the normal-mode approach to linear theory of hydrodynamic stability, it was indicated that initial algebraic growth of disturbances can also be caused by the degeneracies of the frequency spectra, i.e., by coalescences of frequencies for some pairs of normal modes. Some references to papers on this topic were given in Sect. 2.5; here we shall briefly consider only the papers by Koch (1986), Jones (1988), and Shanthini (1989) devoted to investigations of degeneracies in the Orr-Sommerfeld spectra of some standard steady plane-parallel flows.

Let us, however, begin with a general remark. According to the above discussion, resonant growth of small disturbances is due to a coalescence of two eigenvalues belonging to spectra of two different fluid-dynamic fields, while degeneracy growth is due to a coalescence of two eigenvalues belonging to the spectrum of one such field. These two mechanisms clearly have some internal similarity; therefore, it is sometimes said (see, e.g., Koch's paper) that degeneracy growth is caused by a resonance between two normal modes of the same fluid-dynamic field. At the same time it is also possible to look at this matter the other way round. The two equations (3.44) and (3.54), with the appropriate boundary conditions, form a two equation system for the unknown vector field $\{w(\boldsymbol{x}, t), \zeta_3(\boldsymbol{x}, t)\} = q(\boldsymbol{x}, t)$ determining (together with the equation of continuity (2.7b)) all three components of the disturbance velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. The normal modes of the vector field $\boldsymbol{q}$ are the solutions of the system which are proportional to exp $\{i(k_1 x + k_2 y - \omega t)]$. It is clear that for any given values of k_1 and k_2, the finding of such solutions is reducible to solution of a two-equation eigenvalue problem determining the spectrum of the admissible eigenfrequencies ω_j.

It was shown by Henningson and Schmid (1992) (see also Eq. (3.66) in Sect. 3.33) that this spectrum of the field $\boldsymbol{q}$ consists of all the O-S eigenvalues $\omega_j, j = 1, 2, \ldots$, and all the Sq eigenvalues $\omega_i^0, i = 1, 2, \ldots$ (recall that in the case of a channel flow both the O-S and Sq frequency spectra are discrete and infinite). Hence the resonance condition: $\omega_j = \omega_i^o$ for some j and i, determines a part of the degeneracies in the combined spectrum of the two-equation eigenvalue problem relating to the vector field $\boldsymbol{q}(\boldsymbol{x}, t)$, while degeneracies in the O-S and Sq spectra determine two other parts of the set of all such degeneracies. This shows that the distinction between resonances and degeneracies is in fact even smaller than it seems at first. The situation with the tube-flow resonances studied by Gustavsson (1989) is completely similar to that in the case of channel flows: here also resonances form a part of the degeneracies in the spectrum of eigenfrequencies of the vector field $\{p(x,t)/\rho, u_x(\boldsymbol{x}, t)\}$ (see Schmid and Henningson (1994)).

Now we will pass to consideration of the O-S-spectrum degeneracies. Koch (1986) and Shanthini (1989) both used Gustavsson's (1979, 1986) general solution of the initial-value problem for a small disturbance in a steady plane-parallel viscous flow to determine the contribution to the disturbance development of a double eigenvalue of the corresponding O-S eigenvalue problem. To such an eigenvalue there corresponds a double pole of the integrand in Eq. (3.48) in the complex p-plane. The contribution of such a pole to the inverse Laplace transform in this equation is of the same form as the resonant contribution considered above,$(r_1 + r_2 k_1 t)\ \exp(-i\omega_o t)$ where ω_o is the complex double eigenvalue. (Now we return to the dimensional variables and assume that the O-S eigenvalue problem is formulated for the unknown eigenfrequencies ω. If phase velocities $c = \omega/k_1$ are the sought-for eigenvalues, then, of course, ω_o must be replaced by $k_1 c_o$. Moreover, in the case of spatial formulation of the O-S eigenvalue problem, the Laplace transform must be carried out with respect to x, so that here $p = ik_1$, where $k_1 = k_1(\omega, k_2, \mathrm{Re})$ are the unknown eigenvalues, and the contribution of the double pole$p_o = ik_1^o$ in the p-plane to the disturbance amplitude has the form $(r_1 + r_2 x)\ \exp(ik_1^o x)$).

Note that, in contrast to the resonant growth which is possible only for three-dimensional disturbances, the degeneracy growth can take place for either three-dimensional or two-dimensional disturbances. For two-dimensional disturbances $k_2 = 0$ and the O-S eigenvalue depend on only two parameters, $k_1 = k$ and $k\mathrm{Re}$ (in the case of temporal formulation of the problem or ω and Re) (in the case of spatial formulation). Jones (1988) studied multiple eigenvalues of the temporal O-S eigenvalue problem for a plane Poiseuille flow, considering only symmetric two-dimensional normal modes of disturbances $\{u(x, z, t),\ 0,\ w(x, z, t)\}$. (Since the velocity profile of a Poiseuille flow in a channel bounded by walls at $z = 0$ and $\mathrm{z} = H$ is symmetric with respect to the channel midplane $\mathrm{z} = H/2$, the vertical-velocity amplitude $W(\mathrm{z})$ of a normal mode is always eilther symmetric or antisymmetric with respect to this plane, i.e., is represented by either an even or an odd function of $z_1 = z - H/2$. According to this, the normal modes fall into symmetric and antisymmetric ones; the most unstable mode is always symmetric.) Jones recomputed the Poiseuille-flow O-S eigenvalues and found 16 double eigenvalues c_i for symmetric normal modes in flows with values of $\mathrm{R} = k\mathrm{Re}$ in the range $0 < \mathrm{R} < 6000$. All the double eigenvalues c_i found have negative imaginary parts (i.e., correspond to damped modes) and real parts close to each other. However, Jones made no attempt to estimate the possible transient amplitude growth for the degenerating modes he found.

Koch also paid great attention to double O-S eigenvalues in plane Poiseuille flow, although he stated that his prime object was to study spectral degeneracies in boundry-layer flows. Like Jones, he considered only symmetric normal modes but concentrated on the spatial eigenvalues k_1 (ω, k_2, Re) where the values $k_2 \neq 0$ were also permitted. He began with accurate computation of a number of eigenvalues $k_1(\omega)$ for the case where $k_2 = 0$ and $\mathrm{Re} = 10^4$; here no double eigenvalues were found. Then he studied the more general case where both variables k_1 and ω take complex values and, following Gastser (1968b) and Gaster and Jordinson (1975), he determined six branch points in the complex k_1-plane which correspond to modal degeneracies (i.e., to singularities of the "dispersion relation" $\mathrm{D}(k_1, \omega) = 0$; cf. Sect. 2.93 above). After this he began to vary the value of Re (keeping k_2 zero) in the hope that $\Im\mathrm{m}\omega$ would vanish for some of the singular points (k_1, ω) However, in the range $10^3 \leq \mathrm{Re} \leq 2.10^4$, no singular points on the realω-axis were discovered. The next step was to vary the spanwise wave number k_2; then at least one degeneracy of the spatial O-S modes was found at real values of ω, k_2 and Re, but at a high value of Re which in practice would definitely correspond not to laminar but to fully turbulent plane Poiseuille flow. Hence, Koch discovered no degeneracies of the spatial O-S spectrum which might possibly induce certain growth of disturbances.

However, application of the same procedure to a Blasius boundary-layer flow (beginning with computations at $k_2 = 0$ and $\mathrm{Re} = 580$, where $\mathrm{Re} = (U_o x/\nu)^{1/2} = \mathrm{Re_x}^{1/2} = \mathrm{Re}\,\delta^*/1.72$, with subsequent varying of values for Re and k_2) was found to be more fruitful. Here again no coalescences of spatial eigenvalues $k_1(\omega, k_2, \mathrm{Re}\,)$ were obtained for $k_2 = 0$ and real values of ω and Re, although the complex singular points (k_1, ω) were numerous here, Then the value of k_2 was varied, still at $\mathrm{Re} = 580$, and the double complex eigenvalue $k_1 \approx 0.21 + 0.07i$ was found for $\omega = 0.1$ and $k_2 = 0.283$ (all the variables are given in dimensionless form). The spatial spectral degeneracy found was easily traced to other Reynolds numbers (degeneracy values of k_1, k_2, and ω for five values of Re up to $\mathrm{Re} = 2200$ were presented). Koch also indicated the variation of the degeneracy frequency ω and spanwise wave number k_2 with the parameter β characterizing the family of the Falkner-Skan velocity profiles (where $\beta = 0$ corresponds to the Blasius boundary layer, see Sect. 2.92). Thus, it was shown that in the case of boundary-layer flows there are some double spatial O-S eigenvalues which possibly can induce certain growth of disturbances.

Shanthini (1989) studied the degeneracies of the temporal O-S eigenvalues in plane Poiseuille flow for two-dimensional and three-dimensional O-S modes of both symmetric and antisymmetric types. He discovered several new double eigenvalues, and thoroughly investigated the first six degeneracies (four for symmetric and two for antisymmetric modes) to determine which of these degeneracies can produce growth of flow disturbances and what maximal amplitude can then be reached. It was found that growth is possible only in the cases of the first symmetric and first antisymmetric degeneracies, where amplitudes can become at most seven and two times larger that their initial values, respectively. The unexplored degeneracies of higher orders seemed to Shanthini to be unpromising as growth sources; moreover, the majority of them correspond to supercritical values of Re and hence cannot play any role in the "by-pass" transition process. Nevertheless, later Reddy and Henningson (1993) examined the maximal energy growths $G^* = \max_{t>0}[E(t)/E(0))$ (an exact definition of this quantity will be given in the next subsection) for disturbances corresponding to four other degeneracies of the Poiseuille-flow O-S eigenvalues, and found that in these cases the values of G^* are in the range from 1.00 to 5.15. Similar calculations were made by Reddy and Henningson for four degeneracies of O-S eigenvalues they found in plane Couette flow; here $1.00 \leq G^* \leq 1.30$.

Summing up the results of this subsection, we may say that they confirm that resonances and degeneracies can produce some contributions to the often-observed transient growth of flow disturbances. However, none of the investigators whose work was considered above found a resonance or degeneracy growth rate large enough to explain the numerous experimental and computational data showing very significant transient growth of flow disturbances. (See in this respect the discussion of the inadequacy of resonance and degeneracy mechanisms of disturbance energy growth in the papers by Butler and Farrell (1992), p. 145, and Reddy and Henningson (1993), Sect. 6.2, which will be considered later in this book.) Note also, that among the works discussed above the greatest growth was found by Hultgren and Gustavsson (1981) who did not refer to resonance or degeneracy mechanisms at all (they considered the case where the O-S equation has only continuous eigenvalue spectrum), but investigated some special solutions of the general initial-value problem. Therefore it seems natural to think that some other mechanism of disturbance growth must exist which is more universal and more effective than resonance and degeneracy mechanisms. To seek such a mechanism, the general solutions of the initial-value problems for flow disturbances will be considered at greater length in the next subsection.

3.3.3 *Complete Solutions of the Initial-Value Problem and Transient Growth of Disturbances in Plane-Parallel Flows*

Tam's (1967) general solution (3.48) of the initial-value problem for the vertical velocity $w(\boldsymbol{x},t)$ of a small disturbance to a steady plane-parallel flow, and its more explicit form given by Gustavsson (1979), were mentioned several times in Sects. 3.31–3.32. However, the general solution was used by these authors only to find the part of the complete vertical-velocity field which determines the asymptotic behavior of $w(\boldsymbol{x},t)$ as $t \to \infty$. Now we shall consider some studies in which the complete solutions of the initial-value problem for three-dimensional disturbance velocity $\boldsymbol{u}(\boldsymbol{x},t) = \{u(\boldsymbol{x},t), v(\boldsymbol{x},t), w(\boldsymbol{x},t)\}$ (or, what is equivalent, for $\{w(x,t),\ \zeta_3(x,t)\}$ where $\zeta_3 = \partial v/\partial x - \partial u/\partial y$) were applied to investigation of the behavior of small disturbances in the initial stage of their evolution.

Gustavsson's (1981, 1986) representation of the exact solution to the initial-value problem for the vertical velocity w of a three-dimensional disturbance in plane

Poiseuille flow was rewritten at greater length by Shanthini (1989), while an equally detailed solution for the vertical vorticity ζ_3 in plane Couette flow was given by Gustavsson and Hultgren (1980). Both these explicit solutions were given for the three-dimensional Fourier-Laplace transforms of the fields studied, and in the cited papers they were used only for evaluation of the disturbance growth caused by spectral degeneracies or resonances (see Sect. 3.32). Later Gustavsson (1991) combined his previous results to obtain, for the case of plane Poiseuille flow, a complete solution of the initial-value problem for both fields $w(\boldsymbol{x},t)$ and $\zeta_3(\boldsymbol{x},t)$, which fully determine the three-dimensional velocity $\boldsymbol{u}(\boldsymbol{x},t)$ of a disturbance. He applied this solution to the special case where a single O-S normal mode of the vertical velocity $w(\boldsymbol{x},t)$, excited the vorticity field $\zeta_3(\boldsymbol{x},t)$. In this case $w(\boldsymbol{x},t) = W(z)\exp[i(k_1x + k_2y - \omega\mathrm{t}))$ and this allows the solution of Eq. (3.54) for ζ_3 to be simplified. The initial value $\zeta_3(\boldsymbol{x},0)$ was set equal to zero in this paper, and hence only the induced vertical vorticity was included in the solution of initial-value problem. The given values of the O-S mode and the derived values of $\zeta_3(\boldsymbol{x},t)$ were used by Gustavsson to determine the kinetic energy of a disturbance

$$T^*(t) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{H}(u^2 + v^2 + w^2)dxdydz. \tag{3.56}$$

Using a Fourier representation with respect to horizontal coordinates, Eq. (3.15) allows us to rewrite Eq. (3.56) in the general case as

$$T^*(t) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{H}\frac{1}{k^2}(|\hat{\zeta}_3|^2 + |\hat{w}'|^2 + k^2|\hat{w}|^2)dk_1dk_2dz \tag{3.56$'$}$$

where $k^2 = {k_1}^2 + {k_2}^2$, while the circumflex and prime respectively denote a Fourier transform and a derivative with respect to z. According to (3.56′), the energy density in the wave-number plane, $E(k_1, k_2; t)$, is given by

$$E(k_1, k_2; t) = \int_{0}^{H}\frac{1}{2k^2}\left(|\hat{\zeta}_3|^2 + |\hat{w}'|^2 + k^2|\hat{w}|^2\right)dz. \tag{3.57}$$

Gustavsson considered the case of subcritical Reynolds numbers, $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$, (where $\mathrm{Re}_{\mathrm{cr}} \approx 5772$) and carried out the energy computations for a single O-S mode of the vertical velocity, in the hope that results for the least-damped mode would indicate the possible maximal rate of transient growth of energy. Following his paper, we will non-dimensionalize the spatial coordinates, time, wave numbers, and flow variables by the use of the Poiseuille-flow maximum (centerline) velocity U_o and the channel half-depth $H_1 = H/2$ as units of velocity and length, and define $\mathrm{Re} = U_oH_1/\nu$. The dimensionless amplitude of the forcing O-S mode was chosen by Gustavsson so that $E(k_1, k_2; 0) = 1$ (where k_1 and k_2 are the wave numbers of the exciting w-mode and $\zeta_3(0) = 0$). He then showed that the dimensionless energy density of the induced normal vorticity

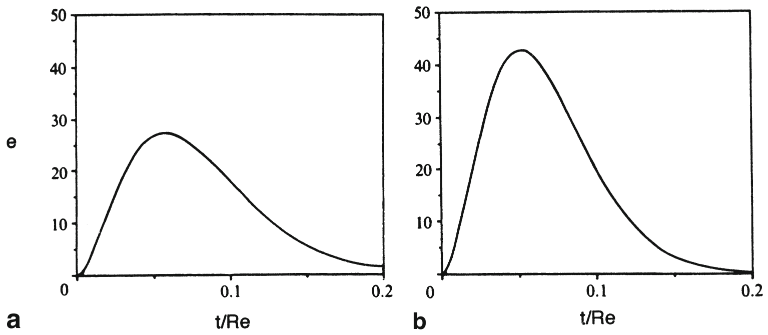


Fig. 3.7 Dependence on t/Re (where $t = t^* U_0/H_1$ is the dimensionless time) of the energy density $e(t)$ for normal vorticity induced in a plane Poiseuille flow by the least-damped symmetric **a** and antisymmetric **b** O-S normal modes of the vertical velocity with energy density $E(0) = 1$, in the case where $\mathrm{Re} = 1000$, $k_1 = 0.1$, $k = (k_1{}^2 + k_2{}^2)^{1/2} = 1$. (After Gustavsson (1991))

$$e(k_1, k_2; t) = \frac{1}{2k^2} \int_0^2 |\hat{\zeta}_3|^2 dz$$

depends only on k_1, k_2 (or k) and $t/\mathrm{Re} = t^* \nu / H_1{}^2$ (where t^* is the dimensional time)

The computations showed that when Re is not exceptionally small, the energy density $e(t)$ at first grows with time much faster than the energy of the forcing O-S mode decays due to viscosity. Therefore, the total kinetic energy density $E(t)$, which includes $e(t)$, grows rapidly with time and can greatly exceed the initial energy density $E(0) = 1$. Some of the results obtained for $\mathrm{Re} = 1000$, $k_1 = 0.1$ and $k = 1$ and the least-damped symmetric and antisymmetric (with respect to the channel midplane) O-S vertical-velocity modes are presented in Fig. 3.7. (Note that in a plane Poiseuille flow the symmetric vertical-velocity mode excites the antisymmetric vertical vorticity mode and vice versa). In Fig. 3.8, again for $\mathrm{Re} = 1000$, contours in the (k_1, k_2)-plane are shown for $e(k_1, k_2, t)$, the energy density of the normal vorticity excited by the least-damped symmetric O-S mode, at $t = 80$ (which is close to the time when the maximum value of $e(k_1, k_2, t)$ is reached for $k_2, \approx 2,\ k_1 = 0$). According to Fig. 3.8, at the highly-subcritical Reynolds number $\mathrm{Re} = 1000$, the kinetic energy density of the induced disturbance can take values which are almost two hundred times greater than the initial energy $E(0)$. Gustavsson also showed that at other values of Re, $e(k_1, k_{2,}\ t)$ in the region of the $(k_1, k_{2)}$-plane with substantial energy growth is approximately proportional to Re^2, if k_1 and t are rescaled in proportion to Re^{-1} and Re, respectively. Hence at subcritical values of Re higher than 1000 the ratios $e(t)/E(0)$ and $E(t)/E(0)$ can considerably exceed 1000. If the forcing O-S mode of the vertical velocity is not the least-damped symmetric or antisymmetric mode, then the growth of $e(t)$ is not so great but, nevertheless, computations show that at $\mathrm{Re} = 1000$ second and even third symmetric and antisymmetric modes can also produce substantial transient growth of disturbance energy. Figure 3.8 also shows that the main growth is achieved for small values of k_1 (corresponding to streamwise wave-lengths much greater that the full

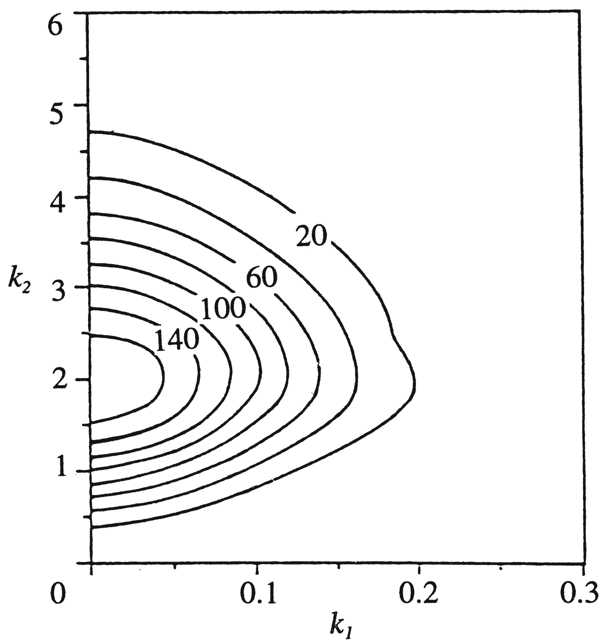


Fig. 3.8 Contours in the (k_1, k_2)-plane for the energy density $e(k_1, k_2; t)$, at $t = 80$, of the vertical vorticity ζ_3 induced in a plane Poiseuille flow with Re = 1000 by the least-damped normal O-S mode of the vertical velocity with $E(k_1, k_2; 0) = 1$. The labels are values of the energy density; the adjacent contours correspond to increments of 20 in the energy-density values. (After Gustavsson (1991))

height of the flow and the span wise wavelength of a disturbance). Hence the forcing most effectively generates streamwise-elongated flow structures, whose amplitudes can reach rather high values before viscous decay becomes appreciable. This is, in fact, true also for supercritical Poiseuille flows at not-too-high values of Re, where an unstable O-S mode exists but in many cases grows much more slowly during the initial stage of evolution than the induced vertical vorticity ζ_3 (see, e.g., Farrell (1988a), p. 2094, and Reddy and Henningson (1993), Fig. 9).

Gustavsson's student Diedrichs (1996) computed the energy growth in a plane Couette flow for a disturbance with $\zeta_3(x, 0) = 0$ and $w(\boldsymbol{x}, 0)$ corresponding to the least-damped O-S mode of the vertical velocity, either symmetric or antisymmetric with respect to the channel midplane. On the basis of results by Gustavsson (1991) and Butler and Farrell (1992) showing that the greatest growth is most often obtained for structures infinitely elongated in the streamwise direction, Diedrichs confined his study to x-independent disturbances with $k_1 = 0$. Hence the O-S modes considered were of the form $w(\boldsymbol{x}, t) = W(z)\ \exp[i(ky - \omega t))$ (such modes are sometimes called the Stokes modes). According to Diedrichs' computations for Couette flow with Re = 1000 (where the channel half-depth H_1 and the half-difference of the wall velocities U_o are taken as the length and velocity scales), in the case of vertical vorticity forcing by the least-damped symmetric Stokes mode, the maximum growth of energy density is obtained for $k = 1.66$, where $E(t)/E(0) \approx 1157$ for $t \approx 139$. Considerably smaller growth occurs when the least-damped antisymmetric Stokes mode induces the growth of vertical vorticity; here the maximum value of E(t)/E(0) is close to 116 (and is reached at $t \approx 46$ for the optimal spanwise wave number $k \approx 2.72$).

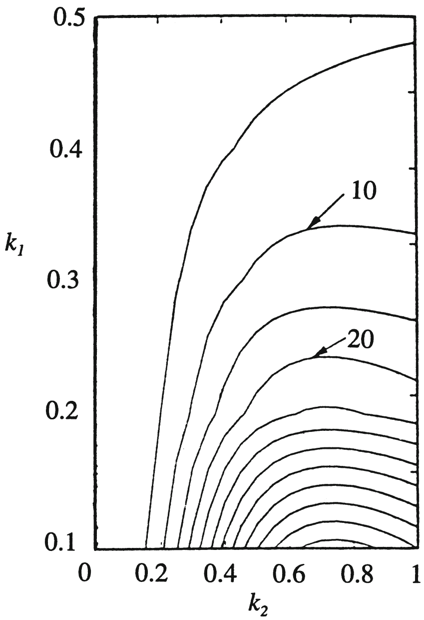


Fig. 3.9 Contours in the (k_1, k_2)-plane for the maximum energy growth $G(k_1, k_2) = \max_t [E(k_1, k_2, t)/E(k_1, k_2, 0)]$ of the plane-wave disturbance induced in a Blasius boundary layer with $\mathrm{Re} = U_0\, \delta^*/\nu = 500$ by the least-damped O-S mode of the vertical velocity, under the condition that the initial vertical vorticity is equal to zero. The adjacent contours correspond to increments of 5 in the G-values. (After Breuer and Kuraishi (1994))

These results agree quite well with the results of Butler and Farrell (1992) which will be considered later. Diedrichs also investigated the disturbance growth in some other channel flows with more complicated velocity profiles (either "Couette-like" or "Poiseuille-like"), where even greater growth of energy can be achieved than in ordinary Couette and Poiseuille flows; however, these results will not be considered here.

Transient growth of small disturbances in various boundary-layer flows was studied by Breuer and Kuraishi (1994). Following Gustavsson (1991), these authors also paid most attention to disturbances with fixed horizontal wave numbers k_1 and k_2, having initially-zero vertical vorticity ζ_3 and a vertical velocity w corresponding to the least-damped O-S mode; however, the case of a localized initial disturbance with the shape depicted in Fig. 3.2 was also briefly considered. The boundary layers investigated were mostly three-dimensional (i.e., with non-zero cross-stream velocity $V(z)$ as well as a velocity component $U(z)$ parallel to the free-stream velocity outside the boundary layer, as is typical for boundary layers over swept wings) and also often had non-zero pressure gradient. Such more complicated boundary layers will not be considered in this series; therefore, only some of Breuer and Kuraishi's results for a simple two-dimensional Blasius boundary layer are shown in Fig. 3.9. In this figure the contours in the (k_1, k_2)-plane of the maximum energy growth $G(k_1, k_2) = \max_t > 0[E(k_1, k_2; t)/E(k_1, k_2; 0))$ are presented for the case of a disturbance imposed at $t = 0$, with $\mathrm{Re}\,\delta^* = U_o\delta^*/\nu = 500$ (here U_o and δ^* are the free-stream velocity and boundary-layer displacement thickness, respectively, and k_1 and k_2 are made dimensionless with δ^*). We see that, as in the case of plane Couette flow, the maximum possible energy growth is substantial and occurs for disturbances with quite small values of k_1 (thus, strongly streamwise-elongated) and finite k_2 (in the range from 0.6 to 0.8). The greatest numerical values of energy growth in Fig. 3.9 are considerably smaller than in Fig. 3.8, but the computation procedure used by Breuer and Kuraishi did not permit reliable estimates of $G(k_1, k_2)$ for $k_1 < 0.1$, while in Fig. 3.8 the greatest growth rates do correspond to very small values of k_1.

Let us now assume that, in a given plane-parallel channel flow, $w(x,0) = W(z)$ $\exp[i(k_1 x + k_2 y)]$ where $W(z)$ is an arbitrary function. If the growth of disturbance energy has already been computed for cases where the forcing is due to a single O-S mode of vertical velocity, then to find the disturbance development in the case of arbitrary $W(z)$ we need only expand this $W(z)$ into O-S eigenfunctions and then superpose the solutions corresponding to normal-mode components of $W(z)$. More general results, related to behavior in the real time-space of the vorticity $\zeta_3(x, t)$ induced by arbitrary vertical-velocity disturbance $w(\boldsymbol{x}, t)$, can be obtained by expanding the initial value $w(\boldsymbol{x}, 0)$ in a two-dimensional Fourier integral, applying the above results to individual Fourier components, and then carrying out the inverse Fourier transformation. Some computations of this type were performed for the case of a localized initial disturbance in a plane Poiseuille flow by Henningson (1991) and Henningson et al. (1993), whose results will be considered later in this subsection. A simpler approach was used by Criminale et al. (1997) who investgated the problem of the transient growth of disturbances in plane Couette and plane Poiseuille flows, based on the direct numerical solution of Eqs. (3.44) and (3.54) for $w(\boldsymbol{x}, t)$ and $\zeta_3(x, t)$ with given initial values $w(\boldsymbol{x}, 0)$ and $\zeta_3(x, 0)$. The initial values were assumed to be represented by two-dimensional Fourier integrals, but the subsequent expansion of individual Fourier components into O-S and Sq eigenfunctions was not used in this paper. Dimensionless forms of Eqs. (3.44) and (3.55) (where all independent and dependent variables were again non-dimensionalized by using the undisturbed velocity at the channel midplane, U_o, and the channel half-depth, H_1, as units of velocity and length) imply the following equations for two-dimensional Fourier transforms, $\hat{w}(k_1, k_2; z, t)$ and $\hat{\zeta}(k_1, k_2; z, t)$, of $w(x, y, z, t)$ and $\zeta_3(x, y, z, t)$:

$$\left[\left\{\frac{\partial}{\partial t} - ik_1 U(z)\right\}\left(\frac{\partial^2}{\partial z^2} - k^2\right) + ik_1 U''(z)\right]\hat{w} - \frac{1}{\mathrm{Re}}\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2 \hat{w} = 0, \quad (3.58)$$

$$\left[\frac{\partial}{\partial t} - ik_1 U(z)\right]\hat{\zeta} - \frac{1}{\mathrm{Re}}\left(\frac{\partial^2}{\partial z^2} - k^2\right)\hat{\zeta} = -ik_2 U'(z) w. \quad (3.59)$$

By shifting the origin of z to the channel midplane, the vertical extent of the flow can be transformed to the segment $-1 \le z \le 1$. Then the dimensionless undisturbed velocity $U(z)$ becomes equal to $1 - z^2$ or z for plane Poiseuille or Couette flow, respectively, while boundary conditions take the form:

$$\hat{w}(-1, t) = \hat{w}(1, t) = \hat{w}'(-1, t) = \hat{w}'(1, t) = 0, \hat{\zeta}(-1, t) = \hat{\zeta}(1, t) = 0. \quad (3.60)$$

In Eq. (3.58–3.60), as always, $k^2 = k_1^2 + k_2^2$ and primes denote derivatives with respect to z.

Criminale et al. solved the differential equations (3.58), (3.59) by a finite-difference method on a uniform grid. The results were verified by grid-independence checks and by recomputing known results for O-S eigenvalues and eigenfunctions. It was found that the computational scheme allowed the values of the energy density $E(k_1, k_2; t) = E(t)$ for both Couette and Poiseuille flows to be determined relatively swiftly for any given values of k_1, k_2 and Re, and arbitrary initial values $\hat{w}(k_1, k_2; z, 0) = W_o(z)$ and $\hat{\zeta}(k_1, k_2; z, 0) = Z_o(z)$. The authors published some results of computations for various combinations of five forms of the function $W_0(z)$ (viz. $W_0 = A_o(1 - z^2)^2 \equiv W_o^{(1)}(z), (A_o/n\pi)(\cos n\pi - \cos n\pi z) \equiv W_0{}^{(2)}(z)$, $W_o^{(2)}(z)(4\pi\lambda)^{-1/2}e^{-z^2/4\lambda} = W_o^{(3)}(z)$, and two other forms which were anitsymmetric in z) with three forms of the function $Z_o(z)$ (viz. $Z_o = 0$, $A_1 \cos[(2n-1)\pi z]/2$, and $A_1 \sin n\pi z$), where A_o, A_1, and λ are positive constants and n is an integer. At first they considered the case of two-dimensional disturbances with $k_2 = 0$. Here, results were obtained for a disturbance with $k_1 = 1.48$ in plane Poiseuille flow with Re = 5000 and a disturbance with $k_1 = 1.21$ in plane Couette flow with Re = 1000. (The choice of values for k_1 was motivated by the results of Butler and Farrell (1992), considered later in this section.) If $n = \lambda = 1$ and $A_o = A_1$, then, for all combinations of initial values $W_o(z)$ and $Z_o(z)$ considered, the disturbance decays monotonically with time in Poiseuille flow, while in Couette flow some combinations of the initial values W_o and Z_o lead to minor disturbance growth, increasing the energy density $E(t)$ by less than a factor of two. These conclusions seemed somewhat strange since, according to the above-mentioned paper by Butler and Farrell, substantial growth can occur for two-dimensional disturbances in both Poiseuille and Couette flows. Therefore Criminale et al. continued their study by considering the case of the second above-mentioned form for $W_o(z)$ with $n > 1$, corresponding more closely to disturbances for which Farrell (1988b) and Butler and Farrell (1992) found maximum growth. Then it was found that in plane Poiseuille flow the second form of $W_o(z)$, together with the assumption that $\zeta_3(\boldsymbol{x}, 0) = 0$ (i.e., the first form of $Z_o(z)$), leads to maximal growth of disturbance energy for $n = 7$, when the maximum value $E(t)/E(0) = 12$ is reached at $t = 14.1$. Similarly, in the Couette flow the greatest growth was found for the same combination of functions $W_o(z)$ and $Z_o(z)$ if $n = 3$, when $E(t)/E(0)$ reaches a maximum value of 4.8 at $t = 7.8$. The calculated values of the times when the maximum growths are reached agree well with values obtained by Butler and Farrell (1992) for quite different initial conditions but the maximum growths are appreciably smaller than those found in the latter paper.

In the case of three-dimensional disturbances, with $k_2 \neq 0$, much greater growth can occur. Again on the basis of the results of Butler and Farrell (1992), Criminale et al. gave special attention to consideration of disturbances with $k_1 = 0$, $k_2 = 2.044$ in plane Poiseuille flow with Re = 5000, and disturbances with $k_1 = 0$, $k_2 = 1.66$ in plane Couette flow with Re = 1000. All possible combinations of the above-mentioned forms for vertical velocity and vorticity profiles $W_o(z)$ and $Z_o(z)$ were studied, with a number of values for n, λ, A_0 and A_1, but results were given in the paper only for $n = \lambda = 1$ (leading to maximal disturbance growth in the cases considered) and $A_1 = A_0$, if $\zeta_3(\boldsymbol{x}, 0) \neq 0$. In Fig. 3.10 the functions $G(t) = E(t)/E(0)$ are shown for three-dimensional disturbances in plane Poiseuille and Couette flows

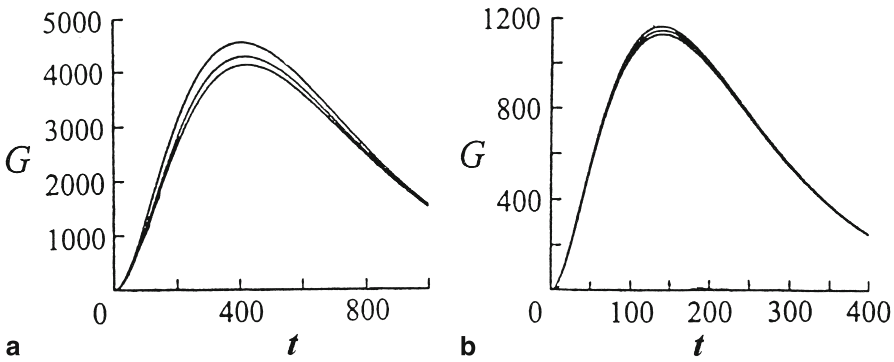


Fig. 3.10 Dependence on dimensionless time t of the energy-growth function $G(t) = E(t)/E(0)$ for a plane-wave disturbance with $k_1 = 0$ and $k_2 = 2.044$ in plane Poiseuille flow with Re = 5000 **a**, and a plane-wave disturbance with $k_1 = 0$, $k_2 = 1.66$ in plane Couette flow with Re = 1000 **b**, for various initial values of the vertical velocity and zero initial value of the vertical vorticity. The three curves correspond to the values $W_{o(1)}(z)$, $W_{o(2)}(z)$, and $W_{o(3)}(z)$ of the initial vertical-velocity amplitude $W_o(z)$. (After Criminale et al. (1997))

having the above-mentioned values of k_1 and k_2, zero initial vertical vortilcity $\zeta_3(x, 0)$ and the three forms of the initial vertical-velocity profile $W_o(z)$ indicated above. We see that the three forms of $W_o(z)$ lead to very similar forms of the function $G(t)$, with maximum values close to 4500 or 1150 in the case of Poiseuille or Couette flow, respectively. These maximum values of $G(t)$, and also the times when the maxima are reached, proved to be only slightly different from values found by Butler and Farrell (1992) for the "optimal initial conditions", which will be considered later. However Criminale et al. found that for two $W_o(z)$ profiles that were antisymmetric in z, the disturbance growth for $\zeta_3(\boldsymbol{x}, t) = 0$ was much smaller than that shown in Fig. 3.10.

Non-zero initial vertical vorticity $\zeta_3(x, 0) = Z_o(z) \exp[i(k_1 x + k_2 y)]$ also considerably reduces the values of $G(t)$. This can be explained by the fact that the initial (non-induced) vertical vorticity contributes to the value of $E(0)$ but decays rapidly with time, while only the induced vertical vorticity grows with time and produces growth of $E(t)$. It was shown by Criminale et al. that, in the case of the first $W_o(z)$ profile introduced by them, the function $G(t)$ nevertheless grows quite appreciably with time for both non-zero forms of $Z_o(z)$ with $A_1 = A_o$, while for the other four forms of $W_o(z)$ considered, the maximum of $G(t)$, in cases where $Z_o(z) \neq 0$, proves to be many times smaller than that for the first form.

Computations by Criminale et al. also confirmed the conclusion of Gustavsson (1991) that $G^* = \max_{t>0} G(t)$ is almost exactly proportional to Re^2, while the time t^* when this maximum is reached is proportional to Re. Since, according to Fig. 3.10, $G^* \approx 4540$ for Poiseuille flow with Re = 5000 and ≈ 1150 for Couette flow with Re = 1000, we deduce that, for the initial conditions considered, $G^* \approx 180$ for Poiseuille flow with Re = 1000 (in good agreement with the result by Gustavsson relating to another initial condition), while $G^* \approx 29000$ for Couette flow with Re = 5000.

Let us now consider some other approaches to investigation of transient disturbance growth. Henningson (1991) (see also Henningson et al. (1994); Henningson and Alfredsson (1996); and Schmid and Henningson (2001)) used the general Eqs. (3.44) and (3.54) somewhat differently. Studying the disturbance development in a plane Poiuseuille flow, he also assumed that the horizontal wave numbers k_1 and k_2 of disturbance vertical velocity and vorticity w and ζ_3 are fixed (and thus $w(x,t) = \hat{w}(k_1,k_2;z,t)\ \exp[i(k_1x+k_2y)], \zeta_3(\mathbf{x},t) = \hat{\zeta}(k_1,k_2;z,t)\ \exp[i(k_1x+k_2y)]$) where $\hat{w}(k_1,k_2;z,t)$ and $\hat{\zeta}_3(k_1,k_2;z,t)$ satisfy Eqs. (3.58) and (3.59)). However, instead of solving these equations, he expanded the functions $\hat{w}(k_1,k_2;z,t)$ and $\hat{\zeta}_3(k_1,k_2;z,t)$ in eigenfunctions of the O-S and Sq equations (2.41) and (2.42) corresponding to wave numbers k_1 and k_2. The possibility of expanding $\hat{w}(k_1,k_2;z,t)$ in the O-S eigenfunctions follows from the completeness of the system of these eigenfunctions, proved by Schensted (1960) and Di Prima and Habetler (1969) (see Sect. 2.5); the general solution $\hat{\zeta}(k_1,k_2;z,t)$ of Eq. (3.59) may be represented as the sum of the general solution of the corresponding homogeneous equation with zero right-hand side (this summand may be expanded in Sq modes) and some particular solution of the inhomogeneous equation. Henningson showed that such a particular solution can be constructed rather easily for the case where $\hat{w}(k_1,k_2;z,t)$ is represented by single O-S mode or by a given linear combination of such modes. Therefore, his approach avoids using the complicated Fourier-Laplace-transformation technique for determination of the general solution of the initial-value problem.

In the case where $\hat{w}(k_1,k_2;z,t)$ is given by a single O-S normal mode (say the first, i.e., the least stable one) and hence $\hat{w}(k_1,k_2;z,t) = \hat{w}(z,t) = AW_1(z)e^{-i\omega_1 t}$ (where $W_1(z)$ and ω_1 are the first O-S eigenfunction and eigenfrequency, and A is an arbitrary coefficient), Henningson's solution has the form

$$\zeta_3(\mathbf{x},t) = \left[\sum_{j=1}^{\infty} C_j e^{-i\omega_j^o t} + \sum_{j=1}^{\infty} D_j^{(1)} \frac{e^{-i\omega_1 t} - e^{-i\omega_j^o t}}{\omega_1 - \omega_j^o}\right] \zeta_{3j}(z) e^{i(k_1x+k_2y)}, \qquad (3.61)$$

where $\zeta_{3j}(z)$, ω_j and ω_j^0 denote the j^{th} Sq eigenfunction and O-S and Sq eigenfrequencies, respectively, while C_j and $D_j^{(1)}$ are coefficients in the expansions of the functions $\hat{\zeta}(k_1,k_2;z,0) = \hat{\zeta}(z,0)$ and $-iAk_2U'(z)W_1(z)$ into Sq eigenfunctions $\zeta_{3j}(z), 1 \le j < \infty$. (This solution, where all the frequencies and coefficients are complex, can be checked easily by direct substitution into Eq. (3.59).) In the more general case where

$$\hat{w}(k_1,k_2;z,t) = \sum_m A_m W_m(z) e^{-i\omega_m t} \qquad (3.62)$$

(i.e., $\hat{w}$ is given by a sum of O-S modes), the solution (3.61) clearly takes the form

$$\zeta_3(x,t) = \left[\sum_j C_j e^{-i\omega_j^o t} + \sum_{m,j} D_{mj}^{(1)} \frac{e^{-i\omega_m t} - e^{-i\omega_j^o t}}{\omega_m - \omega_j^o}\right] \zeta_{3j}(z) e^{i(k_1x+k_2y)}. \qquad (3.63)$$

A Taylor series expansion of the right-hand side of (3.63) in powers of time gives, for small values of *t,* the result

$$\zeta(\mathbf{x},t) = \sum_{j=1}^{\infty} [C_j(1 - i\omega_j^o t) - i\sum_m D_{mj}^{(1)} t + O(t^2)]\zeta_{3j}(z)e^{i(k_1x+k_2y)} \tag{3.64}$$

(where O symbolizes order of magnitude), This shows that the normal vorticity (and hence, by virtue of (3.15), the horizontal-component velocities u and v also) initially grow linearly with time. Moreover, Henningson noted that, according to an asymptotic expansion given by Drazin and Reid (1981), p. 159 at small k_1Re and large Re the eigenfrequencies ω_m and ω_j^0 are inversely proportional to Re. Hence, at high Reynolds numbers and small enough values of k_1, all the O-S and Sq eigenfrequencies take values close to zero and they coalesce as Re $\to \infty$. This means that at high Reynolds numbers and small streamwise wave numbers a number of near-resonances and near-degeneracies necessarily exists. This circumstance can explain the substantial growth of disturbances elongated in the streamwise direction in flows with large Re.

Summing all the terms in the right-hand side of (3.62) which do not tend to zero as Re $\to \infty$, Henningson (1991) (see also Henningson et al. (1994); Schmid and Henningson (2001)) found that at small values of k_1Re and large Re

$$\hat{\zeta}(z,t) = \hat{\zeta}(z,0) - ik_2U'(z)\hat{w}(z,0)t + O\left(\frac{t}{\mathrm{Re}}\right). \tag{3.64$'$}$$

According to (3.15) this equation coincides, when Re $\to \infty$, $t = O(1)$, and $k_1 = 0$ with the known inviscid result (3.21) due to Ellingsen and Palm.

Henningson also calculated a number of values for the coefficients $D_j^{(1)}$ in Eq. (3.61) (note that in the case of a non-self-adjoin eigenvalue problem, the eigenfunctions of the adjoint problem are needed for the computation of the expansion coefficients; see, e.g., Schensted (1960); Eckhaus (1965); Betchov and Criminale (1967); Joseph (1976); or Schmid and Henningson (2001)). It was found that values of $D_j^{(1)}$ are sometimes many tens (or even many hundreds) times greater than the maximum amplitude of the driving O-S mode. Henningson then used Eq. (3.61) for calculation of the time development of the normal-vorticity amplitude in plane Poiseuille flow, disturbed by a superimposed vertical velocity represented by the least-stable O-S mode with the maximum of the amplitude $AW_1(z)$ equal to one. He assumed that $\zeta_3(\boldsymbol{x}, 0) = 0$, and either Re = 3000, $k_2 = 1$ with varying k_1, or $k_1 = 0$, $k_2 = 1$ and varying Re. In all these cases significant initial transient growth of vorticity amplitude was observed, increasing with decreasing k_1 (i.e., increasing streamwise wavelength) and increasing Re. For small values of k_1 and not-too-small subcritical Re (equal to or exceeding 3000), it was found that the maximum amplitude of the normal vorticity can be fifty or more times larger than the amplitude of the initial normal-velocity wave. These results clearly agree well with those obtained by Gustavsson (1991).

Solution (3.63) was applied by Henningson to the study of the normal vorticity development in plane Poiseuille flow, produced by a localized initial disturbance of the shape shown in Fig. 3.2. The results obtained were compared with direct numerical simulation of the same vorticity development (i.e., with numerical solution of the corresponding Eqs. (3.44) and (3.54)). These results of Henningson (1991) were developed further by Henningson et al. (1993), whose paper will be considered later in this section.

Strong renewed interest in transient disturbance growth, arising in the late 1980s and early 1990s, led several authors to pay special attention to the "optimal initial conditions" providing maximal growth of the initial disturbance. One of the first investigations of this type was due to Farrell (1988a) who studied the development of two-dimensional disturbances in plane Poiseuille and Couette flows. Since $\zeta_3 \equiv \partial v/\partial x - \partial u/\partial y = 0$ for a two-dimensional disturbance, where $\boldsymbol{u}(\boldsymbol{x},t) = \{u(x,z,t), 0, w(x,z,t)\}$, only Eqs.(3.44) and (3.58) (where $\partial w/\partial y$ and k_2 are equal to zero) are of importance in this case. Farrell based his analysis on consideration of a Fourier component $\psi(z,t)e^{ikx}$ of the stream function $\psi(x,z,t)$. (Then$-ik\psi(z,t)e^{ikx}$ will be the corresponding Fourier component of the vertical velocity $w(x,z,t)$; hence it makes no difference whether stream function of vertical velocity is considered.) To simplify the computations, he approximated differential equation on the interval $0 \le z \le H$ (here we use dimensional independent and dependent variables again) by finite-difference equations with a sufficiently high number N of mesh points (cf. in Sect. 3.23 the description of a similar approach used by Farrell and Ioannou (1993a)). Thus, the function $\psi(z,t)$ and the O-S eigenvalue problem (relating to the case where $\psi(z,t) = \psi(z)e^{-i\omega t}$) were replaced, respectively, by the vector-function $\psi(t)$ and by the finite-difference version of the classical O-S problem dealing with a system of N linear algebraic equations. The value of N was chosen to be 100, giving results practically indistinguishable from exact ones. The algebraic eigenvalue problem has N discrete eigenvalues ω_j (or $c_j = \omega_j/k$) and N eigenvectors ψ_j. The N-dimensional vector $\psi(0)$ corresponding to the Fourier amplitude $\psi(z,0)$ of the initial stream function $\psi(x,z,0)$ (such correspondence will be denoted below as $\psi(z,0) \Rightarrow \psi(0)$) can be expanded in eigenvectors ψ_j of the discretized O-S equation. Let this expansion be

$$\psi(0) = \sum_{j=1}^{N} a_j \psi j;$$

then the evolution in time of the initial disturbance is described by the equation

$$\psi(z,t)e^{ikx} \Rightarrow \psi(t)e^{ikx} = \sum_{j=1}^{n} a_j \psi j e^{ik(x-c_j t)}. \tag{3.65}$$

The initial conditions are now given by the vector $\boldsymbol{a}$ with components, $a_j, j = 1, \ldots, N$. To make the concept of "optimal initial conditions" definite, we must introduce a measure for the disturbance magnitude and determine what specific meaning is given to the word "optimal". Farrell considered two different measures of disturbance magnitude: the simplest L_2 measure, corresponding to the norm

$$||\psi||_L = \left[\int_0^H |\psi(z,t)|^2 dz\right]^{1/2},$$

and the energy measure based on the norm

$$||\psi||_E = \left\{\frac{1}{2}\int_0^H [k^2|\psi(z,t)|^2 + |\psi'(z,t)|^2]dz\right\}^{1/2} = \left\{\frac{1}{2}\int_0^H [|u|^2 + |w|^2]dz\right\}^{1/2}$$

where a prime again denotes a derivative with respect to z. Both the measures (squares of the corresponding norms) were approximated by positive-definite quadratic forms of variables a_j, with coefficients depending on t. As to the meaning of the "optimality", it must be chosen in accordance with the optimization problem being solved. Two such problems were investigated by Farrell: i) Determination of the minimum initial disturbance exciting a chosen normal mode of unit measure (for example, the least stable or, if $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$, the unstable mode), and ii) Determination of the shape of the initial disturbance that produced the maximum growth of the disturbance magnitude over a fixed reasonably chosen time interval. For the two measures of disturbance magnitude given above, both problems can be reduced to the relatively simple problem of finding a conditional maximum (or minimum) for a quadratic form of N variables. The most striking feature of the solution of the first problem (relevant to the best way to produce the most persistent wave) was the discovery that the optimal initial conditions for generation of a given mode are very different from this mode itself. Therefore, it is most advantageous here not to put the energy available at the time $t = 0$ directly into the mode which one wants to excite, but to use this energy quite differently.

Farrell's problem ii) is more interesting to us, being much closer to the topic of this subsection. It represents an attempt to estimate maximal possible transient growths of a disturbance over various finite time intervals (cf. again the discussion of the paper by Farrell and Ioannou (1993a) in Sect. 3.23). Farrell showed that the required initial disturbance is given by the eigenvector of some specific $N \times N$ matrix corresponding to its greatest eigenvalue, while the eigenvalue itself determines the relative growth of the disturbance magnitude. Then he illustrated this general result by two examples. The first of these concerned the development of the disturbance with $kH_1 = 1$ that grows maximally in the L_2 or energy norm (both were considered) over the first 20 time units (i.e., from $t = 0$ to $t = 20\,\mathrm{H_1/U_o}$ where, as usual, H_1 and U_o are the channel half depth and the maximum undisturbed velocity) in a plane Poiseuille flow with supercritical Reynolds number $\mathrm{Re} = 10^4$. The precise form of the optimum disturbance depends on the choice of the norm but many general features of this disturbance are common in both cases and differ strikingly from the features of the unstable normal mode with the same value of k. However, in course of its development the form of the optimal disturbance approaches the form of the unstable mode. The maximal kinetic energy density of the disturbance which can be reached over 20 time units is 61 times greater than the initial disturbance energy density;

and the transient growth rate found was nearly two orders of magnitude greater than the rate of the exponential growth of the unstable mode at $\mathrm{Re} = 10^4$. The second example considered dealt with a disturbance having the same wave number k as above and growing maximally in the energy norm over 12 time units in Couette flow with $\mathrm{Re} = 10^3$ (now the velocity U_o entering the definition of Re is the half-difference of wall velocities). Here the growth rate was smaller than in the first example, and the maximal energy density exceeded the initial energy density only by a factor of about 11.

Later some supplementary results concerning maximally growing two-dimensional disturbances in plane Poiseuille and Couette flows were mentioned in passing by Butler and Farrell (1992) who gave their main attention to development of three-dimensional disturbances. In particular they stated that, in the case of a Couette flow with $\mathrm{Re} = 1000$, comparison of the optimal two-dimensional disturbances found for various dimensionless wave numbers $\kappa = kH_1$ and growth periods $\tau = tU_o/H_1$ shows that the maximal transient growth of the kinetic energy density $E(t) = ||\psi(x,z,t)||_E^2$ occurs for a disturbance with $\kappa = 1.21$ where $E(t)/E(0)$ reaches the maximal value 13 at $\tau \approx 9$ but then begins to decrease. In the case of a plane Poiseuille flow a similar investigation was carried out for $\mathrm{Re} = 5000$ (i.e., for a subcritical value, half the supercritical Re used in Farrell's paper (1988a)). It was found that here $\kappa = 1.48$ for the most strongly growing two-dimensional disturbance, whose energy density $E(t)$ reaches a value close to $46E(0)$ at $\tau \approx 14$ but then also starts to decrease. These results stimulated Criminale et al. (1997) to choose the same values of κ and Re in a subsequent study of two-dimensional disturbance growth in plane Couette and Poiseuille flows. It was noted above that the maximum energy growths found in this paper for the initial conditions considered was appreciably smaller than those obtained by Butler and Farrell for optimal initial conditions.

The maximum energy growth found by Farrell and by Butler and Farrell for two-dimensional disturbances in a plane Poiseuille flow were much smaller than those found, for example, by Gustavsson (1991) and Henningson (1991), who analyzed the evolution of some particular disturbances but did not solve optimization problems. Recall that the last-mentioned two authors both considered three-dimensional disturbances and stressed that the growth mechanisms studied by them were in principle three-dimensional. In fact, two-dimensional disturbances produce no forcing of the vertical vorticity (and horizontal velocities) by vertical velocity, which plays such an important part in the dynamics of three-dimensional disturbances. The principal growth mechanism for two-dimensional disturbances is based on the extraction of the energy from the sheared mean flow by disturbances through the action of the "Reynolds stress of a disturbance" uw having a sign opposite to that of the velocity shear of the undisturbed flow dU/dz (see in this respect the energy-balance equation (3.74) in Sect. 3.4 and the papers by Farrell and Butler and Farrell cited above). However, according to all available data, this mechanism is much less efficient than the mechanism of vertical-velocity forcing of vertical vorticity.

The above discussion makes it clear that Farrell's paper (1988a) must be considered only as an introduction to the main part of the paper by Butler and Farrell (1992) devoted to application of Farrell's variational procedure to the study of the development of small three-dimensional disturbances in plane-parallel Couette, Poiseuille, and Blasius boundary-layer flows. In this case Eqs. (3.44) and (3.54) (or, if the horizontal Fourier components of disturbances are studied, Eqs. (3.58) and (3.59)) must be solved simultaneously. Butler and Farrell looked for normal-mode solutions, i.e. assumed that $w(\boldsymbol{x},t) = W(z)\exp[i(k_1x + k_2y - \omega t)]$, $\zeta_3(\boldsymbol{x},t) = Z(z)\exp[i(k_1x+k_2y-\omega t)]$. These expressions were substituted into Eqs. (3.44) and (3.54) leading, together with the usual wall boundary conditions, to an eigenvalue problem determining the spectrum of complex eigenvalues ω_j at given real values of k_1 and k_2. For plane Couette and Poiseuille flows the set of eigenvalues is discrete and the system of corresponding eigenfunctions is complete in the Hilbert space of vector functions {W(z), Z(z)} equipped with the energy norm. For a Blasius boundary layer the situation is more complicated but this circumstance is immaterial for the work considered here, where the differential equations are in all cases replaced by finite-difference ones. In fact, it the functions $W(z)$ and $Z(z)$, where $0 \le z \le H$, are replaced by N-dimensional vectors **W** and **Z**, representing the values at N finite-difference locations, then the resulting algebraic eigenvalue problem, relating to a $(2N \times 2N)$-matrix, will always have $2N$ eigenvalues. Now any vector pair (**W**,**Z**) may be expanded in the corresponding eigenvectors. Hence, the initial disturbance may be represented as

$$\{\mathbf{W}(0), \mathbf{Z}(0)\} = \left\{\sum_{j=1}^{2N} a_j\mathbf{W}_j, \sum_{j=1}^{2N} a_j\mathbf{Z}_j\right\},$$

and at time t this disturbance will be transformed into

$$\{\mathbf{W}(t), \mathbf{Z}(t)\} = \left\{\sum_{j=1}^{2N} a_j\mathbf{W}_j e^{-i\omega_j t}, \sum_{j=1}^{2n} a_j\mathbf{Z}_j e^{-i\omega_j t}\right\}.$$

According to Eq. (3.56′), the square of the energy norm for three-dimensional disturbances has the form

$$||(w,\zeta_3)||_L^2 = \frac{1}{2}\int_0^H \left\{|w|^2 + k^{-2}\left(|\partial w/\partial z|^2 + |\zeta_3|^2\right)\right\} dz.$$

Butler and Farrell showed that at time t the expansion coefficients $(a_1, \ldots, a_{2N})$ of the “optimal initial disturbance” coincide with the components of the first eigenvector (i.e., that corresponding to the greatest eigenvalue) of some specific $2N \times 2N$ matrix, which depends on t, while its greatest eigenvalue is equal to the optimal-disturbance energy gain $E(t)/E(0)$ (where the kinetic energy density $E(t)$ is given by the square of the energy norm). Thus, finding the optimal initial disturbances and computing the corresponding energy growths is reduced to solving some tedious but quite standard problems of linear algebra. Some results obtained in this way are presented in Butler and Farrell’s paper.

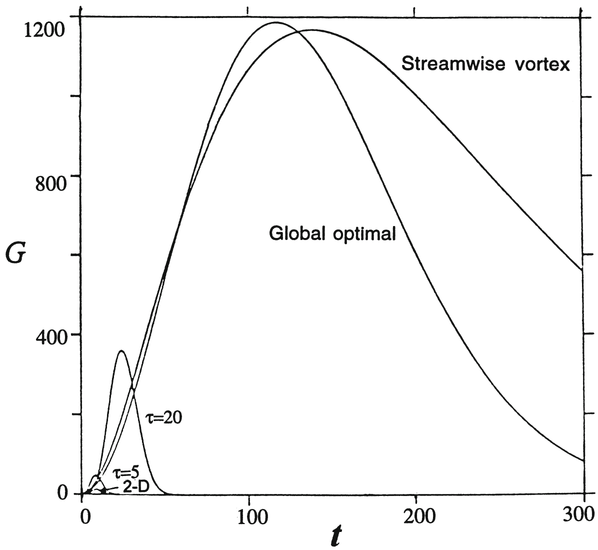


Fig. 3.11 Plot of the energy-growth function $G(t) = E(t)/E(0)$ for the globally optimal, x-independent optimal (streamwise vortex), and two-dimensional (2-D) optimal disturbances, and for disturbances which grow the most in time $t_{\text{opt}} = \tau$ equal to 5 and 20 time units, in Couette flow with Re = 1000. (After Butler and Farrell (1992))

Results related to the optimal (i.e., most strongly growing) wave-like two-dimensional disturbance to plane Couette flow with Re = 1000 were described above. They were computed simply for comparison with similar results relating to three-dimensional disturbances. Since the previous results of other authors (e.g., those by Gustavsson shown in Fig. 3.8) showed that three-dimensional disturbances which do not vary in the streamwise direction (i.e., those with $k_1 = 0$) apparently grow more strongly than all the others, Butler and Farrell first studied the development of such disturbances in the same case of Couette flow with Re = 1000. They found that maximum energy growth is achieved for a disturbance with $k_2 H_1 = 1.66$. (Only non-dimensional wave numbers kH_1 and times tU_o/H_1 will be used later in this subsection, so they will be denoted below simply by symbols k and t. Also the frequencies ω and coordinates x, y, z will now always be assumed to be non-dimensionalized.) For a disturbance with $k_2 = 1.66$, the energy density $E(t)$ increases up to $1166E(0)$ at $t = 138$. Such large energy growth is utterly surprising for a flow which was so long considered to be absolutely stable. Note, however, that it is achieved for a streamwise-unbounded disturbance (having the form of an infinite streamwise vortex) and is reached after a rather long development time.

Butler and Farrell then arranged a search for a globally optimal disturbance among those with any values of k_1 and k_2 in order to check whether streamwise-vortex disturbances were in fact the most strongly growing. They found that this is not so, since the most strongly growing wave-like disturbance in a Couette flow with R = 1000 depends on all three coordinates, has the wave numbers $k_1 = 0.035$ and $k_2 = 1.60$, and reaches its maximum energy $E_{\max}$ (equal to $1185E(0)$ at $t = 117$. This disturbance is also strongly elongated in the streamwise direction (since $k_1 \ll 1$) and requires a long time to reach the maximal energy (since $t \gg 1$), and its maximum growth is only slightly larger than that for the streamwise vortex. However, as shown in Fig. 3.11, the streamwise vortex decays considerably more slowly, after reaching maximum energy, than the globally optimal disturbance. In Fig. 3.11 the dependence of the energy density on time is also shown for the optimal two-dimensional disturbance and for disturbances achieving the maximal energy admissible at $t = 20$ and at $t = 5$. In the latter two cases the rates of energy growth at small values of time are greater, but the maxima are much lower than those for the optimal streamwise vortex or the globally optimal disturbance.

Results by Butler and Farrell relating to globally optimal disturbances in Couette flows with Re varying from 31 to 4000 are presented in Table 3.1. We see that the values of the spanwise wave number k_2 are nearly the same for all these disturbances. At the same time the streamwise wavelength $2\pi/k_1$ (which at all values of Re is much greater than the flow thickness $2H_1$ or the spanwise wavelength $2\pi/k_2$) and the time t for maximum growth to occur both increase as Re, and the maximum energy growth $E_{\max}/E(0)$ increases as $(\mathrm{Re})^2$, at high Reynolds numbers.

In the case of plane Poiseuille flow it was shown that the globally optimal disturbance has the form of a streamwise vortex with $k_1 = 0$. At $\mathrm{Re} = 5000$, maximum energy growth was determined to be $E(t) = 4897E(0)$, reached at $t = 379$, for a spanwise wave with $k_2 = 2.04$ and with stream function $\psi(y, z, t)$ antisymmetric in z (where it is assumed that $-H_1 \leq z \leq H_1$ and the stream function is defined by equations $-\partial\psi/\partial y = w$ and $\partial\psi/\partial z = v$). Unlike Couette flow (where nothing similar occurs), Poiseuille flow also contains a second set of strongly-increasing disturbances independent of x, whose energy growths are about half of those for globally optimal disturbances but whose stream function is symmetric in z (and the disturbances themselves are optimal in the set of all disturbances with symmetric stream functions). At $\mathrm{Re} = 5000$ the symmetric optimal disturbance has the spanwise wave number $k_2 = 2.64$ and its energy grows from $E(0) = 1$ up to the maximum value $E(t) \approx 2819$ at $t \approx 270$. Note that Gustavsson (1991) and Diedrichs (1996), who studied the growth of the vertical vorticity induced by the least-damped symmetric or antisymmetric O-S modes in plane Poiseuille and Couette flows, both found maxima of $E(t)/E(0)$, and times t needed to reach these maxima, which are rather close to values obtained by Butler and Farrell for corresponding optimal disturbances. Moreover, Criminale et al. (1997), who studied some non-modal disturbances with $k_1 = 0$ and optimal values of k_2 (found by Butler and Farrell for x-independent disturbances), also obtained, for both Poiseuille and Couette flows, values for maxima of $E(t)/E(0)$ and for times t which are very close to those given by Butler and Farrell for optimal disturbances. These facts lead one to believe that the transient turbulence growth is not very sensitive to the shape of the initial disturbance.

As to the boundary-layer flow, it was modeled by a plane-parallel flow in the half-space $0 \leq \zeta < \infty$ with the Blasius velocity profile $U(z)$. (See, however, footnote 4 on p. 64 relating to the influence of the gradual thickening of a boundary layer.) Recall now that results found for Poiseuille flow show that the most strongly growing disturbances are confined to the shear regions near the walls. Since the velocity shear $U'(z)$ decreases rapidly with z in a Blasius boundary layer, it seemed reasonable to assume that, to find the most strongly growing disturbances, the boundary-layer flow may be replaced by a synthetic channel flow with the Blasius velocity profile and with the upper wall at a height z where the velocity $U(z)$ is practically independent of z (i.e., indistinguishable from the free-stream velocity U_0). Just such a model was used by Butler and Farrell; its adequacy was verified by comparison of a few first O-S eigenvalues computed for this model (more precisely, for its finite-difference approximation) with those computed for a semi-infinite boundary layer with the smooth Blasius velocity profile. (Of course, changing to this model greatly changes the spectrum of O-S eigenvalues: for a real Blasius boundary layer it consists of a few discrete eigenvalues supplemented by a continuous spectrum, while in a channel flow the O-S spectrum is discrete and infinite, but in the case of a finite-difference approximation it is finite. However, the form of the O-S spectrum does not, as a rule, significantly affect the transient growth of disturbances and the forms of the most strongly growing structures; see, e.g., Farrell and Ioannou (1993c).) Butler and Farrell defined the Blasius boundary layer by the Reynolds number based on the free-stream velocity U_0 and the displacement thickness δ^*, $\mathrm{Re}_{\delta^*} = U_o\delta^*/\nu$; U_0 and δ^* were also used to make wave numbers k and times t' dimensionless.

The globally optimal disturbance in this synthetic-channel model of a boundary layer was found to be a streamwise vortex independent of x. At $\mathrm{Re}_{\delta^*} = 1000$ this disturbance has spanwise wave number $k_2 = 0.65$ (which is close to the optimal value of k_2 in Fig. 3.9), and its energy density grows from the value $E(0)$ up to a value $E_{\max} = 1514E(0)$, reached at $t = 778$, while for the optimal two-dimensional disturbance with $k_2 = 0$ in this flow, $k_1 = 0.42$, and the energy gain $E_{\max}/E(0) = 28$ at $t = 45$. At other values of Re_{δ^*} the globally optimal disturbances have the same value of k_2, but the time t when the maximum energy is reached and the value of the maximum energy growth, $E_{\max}/E(0)$, are proportional to Re_{δ^*} and $(\mathrm{Re}_{\delta^*})^2$, respectively. Later Butler and Farrell (1993) tried to apply their method of finding optimally growing small disturbances to a turbulent boundary layer, with a mean velocity profile quite different from the Blasius one. The idea was to seek an explanation of some well-known, but until now inexplicable, features of near-wall regions in turbulent boundary layers (cf. also the work by Jang et al. (1986), referred to in Sect. 3.32, which had a similar purpose). However, discussion of the 1993 paper by Butler and Farrell must be postponed until after a general introduction to near-wall turbulent flows.

Some other methods for determination of the optimally growing wave-like disturbances with given horizontal wave numbers k_1, k_2 in plane Poiseuille and Couette flows, and computation of the corresponding growth functions $G(k_1, k_2; t) = E(k_1, k_2; t)/E(k_1, k_2; 0)$ (which also depend on Re), were proposed by Reddy and Henningson (1993) and Criminale et al. (1997). Reddy and Henningson used their method, which will be described shortly, to obtain a number of new results relating to characteristics of the optimally growing disturbances and values of their maximal growth $G^*(k_1, k_2) = \max_{t>0} G(k_1, k_2; t)$ for various values of k_1, k_2 and Re. In particular, they plotted growth contours $G^*(k_1, k_2) = \text{const.}$, in the (k_1, k_2)-plane for several values of Re, and, for two-dimensional disturbances with $k_2 = 0$, plotted growth contours of $G^* = G^*(k_1, \mathrm{Re})$ in the (k_1, Re)-plane. (Recall that a disturbance is called "optimally growing." or, more precisely, "optimally growing from $t = 0$ till time t" if it produces the greatest value of $G(k_1, k_2; t)$; this disturbance clearly depends on t. However, the "globally optimal" disturbance, for which $G(k_1, k_2; t)$ as a function of t has the highest maximum, does not depend on t.) In their computations Reddy and Henningson used the expansion of functions $\hat{w}(k_1, k_2; z, t)$ and $\hat{\zeta}(k_1, k_2, z, t)$ in the O-S and Sq eigenfunction, which follows from Eq. (3.63). In fact, according to this equation, the general solution $\{\hat{w}(k_1, k_2; z, t), \hat{\zeta}(k_1, k_2; z, t)\}$ of Eqs. (3.58), (3.59) can be represented as

$$\{\hat{w}(k_1, k_2; z, t), \hat{\zeta}(k_1, k_2; z, t)\} = \sum_{m=1}^{\infty} A_m \mathbf{q}_m(z) e^{-i\omega_m t} + \sum_{j=1}^{\infty} B_j \mathbf{p}_j(z) e^{-i\omega_j^o t} \quad (3.66)$$

where $\mathbf{q}_m(z) = \{W_m(z), V_m(z)\}$ and $\mathbf{p}_j(z) = \{0, k\zeta_{3j}(z)\}$ are definite two-dimensional vector-functions of z. These vector-functions are simply the eigenfunctions of the two-dimensional eigenvalue problem arising from Eqs. (3.58–3.59) when the derivatives $\partial/\partial t$ are replaced by $-i\omega$, where ω is the unknown eigenvalue (see, e.g., Henningson and Schmid (1992)). Therefore, Eq. (3.66) represents the expansion of the solution of Eqs. (3.58–3.59) in the corresponding two-dimensional eigenfunctions. (For the sake of simplicity it is assumed here that all eigenvalues ω' s are distinct.) Reddy and Henningson proposed arranging the terms on the right-hand side of Eq. (3.66) in order of decreasing imaginary parts of the corresponding eigenvalues ω_k (either ω_m or ω_j^o), and preserving only a limited number N of terms, taking into account that $\Im\text{m}\,\omega_k \ll 0$ for large values of k so that the corresponding terms of (3.66) would be negligibly small for practically any $t > 0$. The truncation of the series (3.66) was combined with a special vertical discretization (differing from the simplest uniform grid discretization) which reduces the computation of unknown eigenvalues and eigenfunctions to problems from linear algebra. All this makes the numerical procedure somewhat different from that used by Butler and Farrell (the latter is less economic in number of arithmetic operations needed).

The results of the computations were presented by Reddy and Henningson in a number of figures and tables. As examples, in Figs. 3.12a–3.14 contours of $G^*(k_1, k_2) = G^*(k_1, k_2;\ \text{Re})$ are shown for Poiseuille flow, in the $(k_1,\ \text{Re})$-plane for the case of two-dimensional disturbances, having $k_2 = 0$, and in the (k_1, k_2)-plane for the case where $\text{Re} = 1000$; also, contours of the normalized function $k^2 G^*(k_1, k_2)/(k_2\text{Re})^2$ are presented in the $(k, k_1\text{Re})$-plane, where $k = (k_1^2 + k_2^2)^{1/2}$, for both Couette and Poiseuille flows with two different values of Re. Figure 3.14 confirms the conclusion by Reddy and Henningson that, at all not too small values of Re, $k^2 G^*(k_1, k_2)/k_2\text{Re})^2$ is almost a function of k and $k_1\text{Re}$ alone, especially at low values of $k_1\text{Re}$. Therefore the curves in Fig. 3.14 make possible the determination of the values $G^*(k_1, k_2)$ for plane Couette and Poiseuille flows at any not-too-small value of Re. In particular, Trefethen et al. (1993) used this result, together with some data of other authors, to compute the contours of $G^*(k, \text{Re}) = \max_{k_1^2+k_2^2=k^2} G^*(k_1, k_2, \text{Re})$ for Poiseuille flow. Their results presented in Fig. 3.12b supplement Fig. 3.12a and show to what extent the growth of three-dimensional disturbances can exceed the growth of two-dimensional ones. The contours of $G^*(k_1, 0,\ \text{Re})$ and $G^*(k,\ \text{Re})$ for plane Couette flow are similar to those in Fig. 3.12 (the first of them are presented in the paper by Reddy and Henningson) but they contain no shaded regions, since no unstable normal modes exist in a plane Couette flow at any value of Re. Some other results of Reddy and Henningson's paper will be briefly considered in Sect. 3.4.

Criminale et al. (1997) proposed to use expansion of the values $\{\hat{w}(k_1, k_2; z, 0), \hat{\zeta}(k_1, k_2; z, 0)\}$ in a Fourier series with respect to the vertical coordinate z and then to solve Eqs. (3.58), (3.59) with the initial conditions represented by a properly truncated series, and to find the optimal growth function $G(k_1, k_2; t)$ by solving the variational problem relating to the values of Fourier coefficients. According to these authors, such a procedure has an advantage over that based on the expansion (3.66) used by Reddy and Henningson, but only its application to the study of the optimal two-dimensional disturbance in a plane Poiseuille flow at Re = 5000, which was found by Butler and Farrell (1992), briefly outlined in their paper.

Schmid et al. (1994) and Lundbladh et al. (1994) (these two papers strongly overlap) also considered the problem of transient growth for small disturbances in a plane Poiseuille flow, but for the spatial evolution of disturbances (in the streamwise x-direction), rather than the temporally-evolving case discussed above. Therefore, it was assumed that $w(\boldsymbol{x},t)$ and $\zeta_3(\boldsymbol{x},t)$ are proportional to $\exp\{i(k_1x + k_2y - \omega t)\}$, where k_2 and ω are given real values while k_1 is the unknown complex eigenvalue. Then Eqs. (3.44) and (3.54) imply the known O-S equation (Eq. (3.58) with $\partial/\partial t$and $\hat{w}$ replaced by $i\omega$ and W, respectively) for the vertical-velocity amplitude $W(z)$, and an inhomogeneous equation for the amplitude $Z(z)$ of the vertical vorticity ζ_3 (Eq. (3.59) with the same replacements as above and $\hat{\zeta}$ replaced by Z). These two equations, supplemented by the usual boundary conditions at the walls $z = 0$ and $z = H$, form an eigenvalue problem in which the eigenvalue k_1 appears in powers up to the fourth. This complicates the solution, but the authors showed that this eigenvalue problem can be transformed, with the aid of some manipulations, to a standard linear eigenvalue problem for the system of three differential equations, to which known numerical techniques can be applied. Then the authors discretized the eigenvalue problem in the vertical z-direction, transforming it into an algebraic eigenvalue problem, expanded the velocity field in terms of the eigenvectors found, and truncated this expansion to reduce the number of arithmetic operations needed. Using the appropriate definition of the local "energy density" $E(k_2, \omega; x) = E(x)$ (introduced by Henningson and Schmid (1994)), they reduced the determination of the maximal energy amplification $G(x) = \max_{\boldsymbol{u}(0)} E(x)/E(0)$ (where the maximum is taken with respect to all "initial" velocity disturbances $\boldsymbol{u}(0, y, z, t)$ with given values of k_2 and ω) to the solution of an algebraic variational problem very similar to that studied by Reddy and Henningson (1993). In Fig. 3.15a the results given in both indicated papers for the function $G^*(k_2, \omega) = \max_{x>0} G(k_2, \omega; x)$ at Re = 2000 are presented as contours in the (ω, k_2) -plane. They show that, at this Re, the maximum value of $G^*(k_2, \omega)$, which is close to 100, is reached at $\omega = 0$ (i.e., for a steady disturbance) and $k_2 \approx 2$ (as usual for a plane-Poiseuille primary flow, the wave numbers, frequencies, coordinates, and flow variables are made dimensionless by using the half-depth H_1 and the maximum Poiseuille-flow velocity U_0 as unit length and velocity). In Fig. 3.15b graphs of the function $G(2, 0; x)$ are given for Re = 500, 1000, and 2000. These graphs show that the decrease of Re leads to a significant decrease of $G^* = \max_x G(x)$ but has less effect on the streamwise coordinate $x_{\max}$ where the

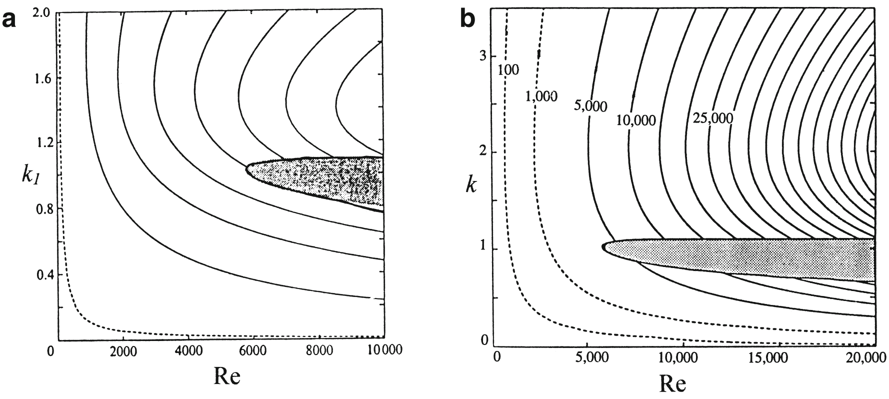


Fig. 3.12 **a** Contours of $G^*(k_1, 0, \mathrm{Re})$ in the (k_1, Re)-plane for plane Poiseuille flow (after Reddy and Henningson (1993)). Dotted line: $G^* = 1$; *solid lines*, from left to right: $G^* = 10, 20, 30, \ldots, 70$; in the shaded region exponentially growing O-S modes exist and hence unbounded energy-growth is possible. **b** Contours of $G^*(k, \mathrm{Re}) = \max_{k_1^2+k_2^2=k^2} G^*(k_1, k_2, \mathrm{Re})$ in the (k, Re)-plane for plane Poiseuille flow (after S. Reddy, whose results were published in slightly different form by Trefethen et al. (1993), and in the form presented here in the book by Panton (1996)). Two *dotted lines*: $G^* = 100$ and 1000, the left-most solid curves correspond to increments of 5000 in the G^*-vlaues; the shaded region has the same meaning as in Figure **a**

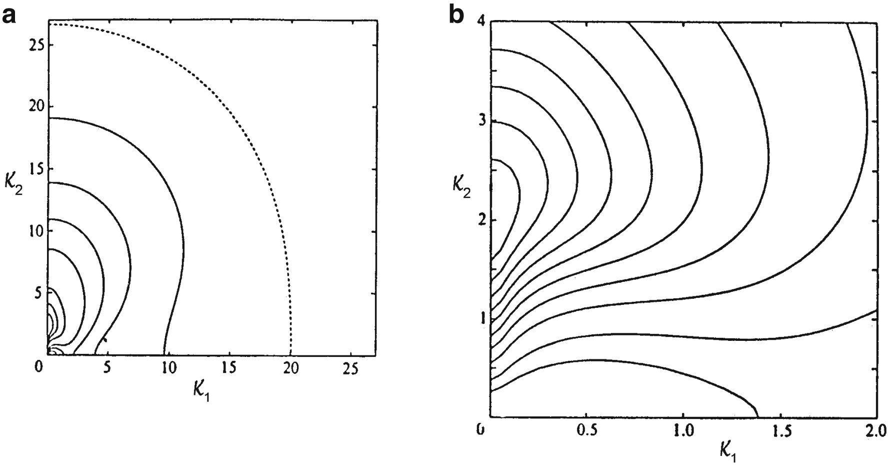


Fig. 3.13 Contours of $G^*(k_1, k_2, \mathrm{Re})$ in the (k_1, k_2)-plane for plane Poiseuille flow with Re = 1000 (after Reddy and Henningson (1993)). **a** *Dotted line*: $G^* = 1$; *solid lines*, from outer to inner: $G^* = 2$, 5, 10, 20, 60, 100, 140. **b** Lower-left part of (**a**); lines from outer to inner correspond to $G^* = 10$, 20, 40, ..., 140, 160, 180

value G^* is reached. The cusps on the curves reflect the dependence of the optimal disturbance on x: at cusp points the shape of the optimal w-disturbance switches from symmetry to antisymmetry with respect to the channel midplane.

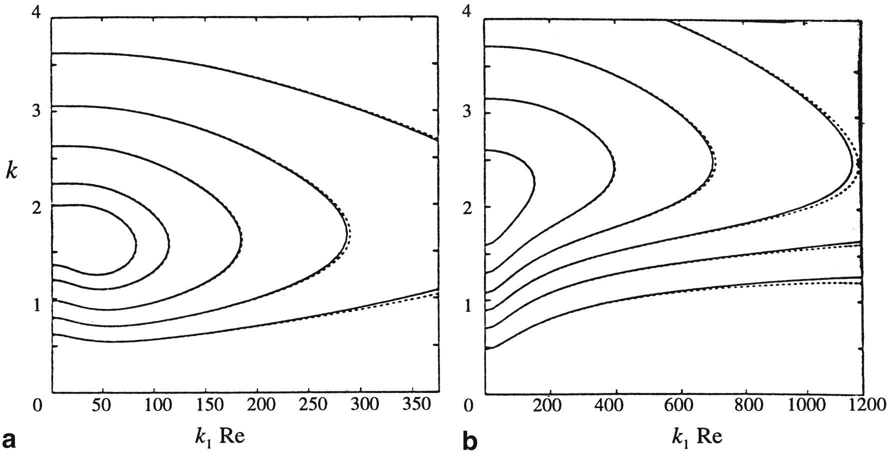


Fig. 3.14 Contours of $k^2G^*(k_1, k_2, \mathrm{Re})/k_2{}^2\mathrm{Re}^2$ in the (k, k_1 Re)-plane for plane Couette and Poiseuille flows (after Reddy and Henningson (1993)). **a** For Couette flows with Re = 1000 (*solid lines*) and Re = 500 (*dotted lines*, sometimes merging with solid ones); the contours, from outer to inner, correspond to values: 0.4, 0.6, 0.8, 1.0, 1.1 ($\times 10^{-3}$). **b** For Poiseuille flows with Re = 3000 (*solid lines*) and Re = 1500 (*dotted lines*); the contours from outer to inner correspond to values: 0.3, 0.6, 0.9 1.2, 1.5, 1.8 ($\times 10^{-4}$)

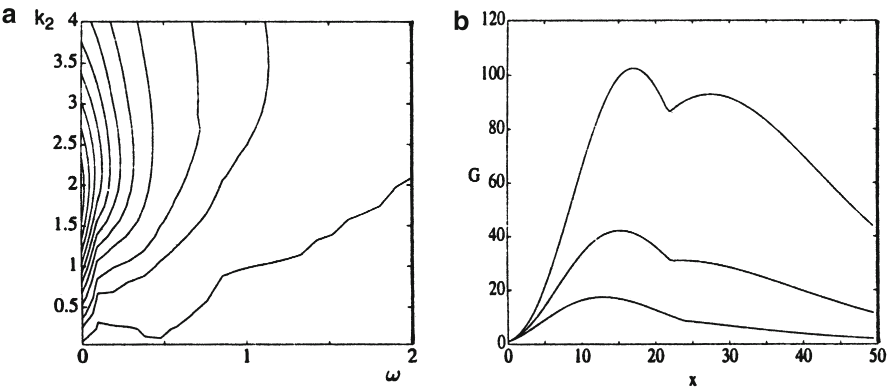


Fig. 3.15 Spatial growth of disturbance energy in plane Poiseuille flows (after Schmid et al. (1994) and Lundbladh et al. (1994)). **a** Level contours of $G^*(k_2, \omega, \mathrm{Re})$ in the (k_2, ω)-plane for Re = 2000. The contours, from outer to inner, corresponds to levels: 5, 10, 20, ...,100. **b** Plots of the functions $G(k_2, \omega, \mathrm{Re}; x)$ for $k_2 = 2$, $\omega = 0$, and various values of Re; the curves correspond. (From *bottom* to *top*) to Re = 500, 1000, and 2000

Spatial development of small disturbances in a plane Poiseuille flow was also studied by Criminale et al. (1997), by the method of direct numerical simulation, i.e., by numerical solution, at Re = 5000, of the complete Navier-Stokes (N-S) equations for the disturbed velocity $\boldsymbol{U}(\boldsymbol{x}) + u(\boldsymbol{x}, t)$ and the deduced pressure

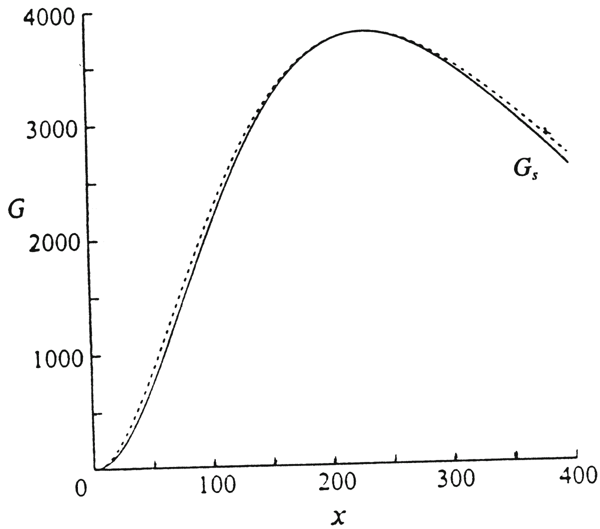


Fig. 3.16 Computed spatial energy-growth curve $G_s(x)$ for disturbance with near-optimal (with respect to temporal growth) initial conditions in plane Poiseuille flow with Re = 5000. The dashed line represents the temporal energy-growth curve $G(t)$ computed for the same Re and initial values, and rescaled in both coordinates to make the maxima of spatial and temporal growth curves equal and corresponding to the same abscissa. (After Criminale et al. (1997))

$[P(\boldsymbol{x}) + p(\boldsymbol{x}, t)]/\rho$. Here ρ is the density, $\boldsymbol{U}(\boldsymbol{x}) = \{1 - z^2, 0, 0\}, -1 \le z \le 1$, and $P(\boldsymbol{x})/\rho = P_o/\rho - 2\boldsymbol{x}/\text{Re}$ are Poiseuille-flow velocity and deduced pressure, while $\boldsymbol{u}(\boldsymbol{x}, t)$ and $p(\boldsymbol{x}, t)$ are velocity and pressure fluctuations of a small disturbance (all variables are assumed to be non-dimensionalized). The boundary conditions are the spatially-initial ("inflow") condition $\boldsymbol{u}(0, y, z, t)$ at $x = 0$ and the usual solid-surface conditions at $z = \pm 1$. The authors chose time-independent (steady) inflow conditions: $\zeta_3(0, y, z) = 0$ and $w(0, y, z) = A_0(1 - z^2)^2 \cos 2y$. This model $W_o^{(1)}(z)$ of the vertical-velocity amplitude was used in this paper when the temporal disturbance development was studied, with wavenumber values $k_1 = 0$, $k_2 = 2$. These wavenumbers correspond to temporal disturbance growth close to the maximum possible at Re = 5000 (see Table 3.1 and Fig. 3.10a above). From the continuity equation, the horizontal velocity components at $x = 0$ are $u(0, y, z) = 0$, $v(0, y, z) = 2A_o z(1 - z^2)^2 \sin 2y$. The initial amplitude A_0 was taken to be 10^{-7}; this small value was chosen to guarantee that the growing disturbance remains small so that the results agree with those implied by linearized dynamic equations. The computed spatial growth curve $G_s(x) = E(x)/E(0)$ for the given inflow conditions is shown in Fig. 3.16. This growth curve has a shape similar to those shown in Fig. 3.10a for temporal growths of some fixed disturbances in Poiseuille flow at the same Reynolds number. The $G_s(x)$-curve has a much higher peak, and is considerably smoother, than the upper $G(x)$-curve in Fig. 3.15b, which corresponds to a lower Reynolds number Re = 2000 and concerns not a fixed disturbance but a family of disturbances depending on x (and is in fact composed of two separate growth curves, for symmetric and antisymmetric w-disturbances). To illustrate graphically the similarity between spatial and temporal growth curves, Criminale et al. also computed the temporal growth curve $G(t) = E(t)/E(0)$ in Poiseuille flow at Re = 5000, for the disturbance having an initial velocity $\boldsymbol{u}(x, y, z, 0)$ equal to the inflow velocity

$\boldsymbol{u}(0, y, z, t)$ defined above (this was possible since this $\boldsymbol{u}(0, y, z, t)$ was chosen to be independent of t). This new growth curve was then rescaled; abscissa t was replaced by $x/0.561$, and ordinate G by $G_T = 0.831G$ (the factors were chosen to make the maxima of the spatial and temporal growth curves $G_s(x)$ and $G_T(x)$ coincide in magnitude and location in x). The function $G_T(x)$ obtained is also shown in Fig. 3.16; as can be seen, it differs only insignificantly from $G_s(x)$.

Farrell and Ioannou (1993b) studied the optimally growing temporally-evolving three-dimensional disturbances of infinite plane. Couette flow having constant shear $U'(z)=b$ in the unbounded space—$\infty < z < \infty$. For such a flow the O-S equation has no discrete eigenvalues and hence the O-S spectrum is purely continuous. However, here the investigation of the disturbance development is simplified by the existence in this case of a complete set of analytic solutions which are orthogonal (in the inner products corresponding to both the L_2 and energy norm) and which have the form of plane waves, with constant horizontal wave numbers k_1 and k_2 and vertical wave number k_3 depending lineary on time. (Recall that such solutions were in fact first found by Kelvin (1887a) and Orr (1907) but then were forgotten for a long time; see Sect. 3.1, and particularly Eqs. (3.1) and (3.2) where on the right-hand sides the last term in (3.1) and the last two terms in (3.2) must be omitted in the case of infinite Couette flow without walls.)

The initial value problem for disturbances in an infinite couette flow was solved in full generality (with inclusion of an external force and a mass source affecting the flow) by Criminale and Smith (1994). However, here the imposed generality of the problem statement and the search for fundamental solutions and universally applicable generalized Green's function led to rather complicated equations, and restricted the authors mainly to the inviscid solution (providing the correct leading term in cases where the time t is not too long and the disturbance length scale l is not too small). In contrast to this, Farrell and Ioannou considered only some simple particular solutions where the initial values $w(\boldsymbol{x},0)$ and $\zeta_3(\boldsymbol{x}, 0)$ both have plane-wave or checker-board shape (i.e., are proportional either to $\exp\{i(k_1x + k_2y + k_3z)\}$ or to $\cos(k_1x)\cos(k_2y)\cos(k_3z)$) If the optimal growth is interpreted as the maximal energy-density growth $G(t) = E(t)/E(0)$ attainable at some time t by appropriate choice of the values for W_0, Z_0, k_1, k_2, k_3 (i.e., for amplitudes of initial plane-wave or checkerboard structures and for three wave-number components), then the optimal values of $G(t)$ will be the same for viscous and inviscid fluid and will increase infinitely with time t. (This is so since the influence of viscosity can be arbitrarily decreased by a large enough increase of disturbance length scale l determined by the wave numbers k_i.) Therefore, Farrell and Ioannou (1993b) first computed the maximal growth $G(T_{\text{opt}})$ attainable in an inviscid fluid in a specific non-dimensional time $T_{\text{opt}} = (bt)_{\text{opt}}$ (cf. the discussion of the related paper by Farrell and Ioannou (1993a) in Sect. 3.23). The values of $G(T_{\text{opt}})$ were found to be different for single-plane-wave and checkerboard initial conditions. However, for large values of T_{opt}, of the order of several tens or a hundred, they usually take high values of the order of 10^3 or even 10^4.

To take into account the effect of viscosity, it was enough to consider only the initial conditions constrained to have a fixed value of k_1. Then Re may be defined as $bl^2/\nu =$

$bk_1^2/\pi^2\nu$, the optimal values of k_2/k_1 and k_3/k_1 may be uniquely determined for any given value of T_{opt}, and the value of $G(T_{opt})$ will be a function of Re. For example, if $k_1 = 1$, then for Re = 100, 1000, 10000 and checkerboard initial conditions, it was found that maximum growth $G(T_{opt})$ is achieved at $T_{opt} = 7$, 15, 30, and is equal to 12.5, 109, 707, respectively. It was also noted that in real flows the ambient turbulent fluctuations usually interfere with the regular growth of a velocity disturbance by disrupting the corresponding flow structures; therefore, the results for large values of T_{opt} are often of no physical meaning. Bearing this in mind, Farrell and Ioannou presented a contour plot of $G(T)$ in the (k_1, k_2)-plane for Re = 1000 and checkerboard initial conditions, with a moderate value of $T_{opt} = 10$. It was found that in this case $G_{max}(10) \approx 115$, which can be reached for practically all $k < 1$ if $\tan^{-1}(k_2/k_1) \approx 63°$. In another paper, Farrell and Ioannou (1993c) showed that the contour plot of $G_{max}(T)$ for $T = 10$ in their paper (1993b), is similar in many respects to contours of G_{max} for plane Couette and Poiseuille flows in channels of finite height. This similarity reflects the similarity in the flow structures responsible for maximal growth of disturbance energy in the three flows considered.

Practically all specific computations of transient disturbance growth considered above in this subsection concerned horizontally unbounded disturbances (in most cases plane waves with given horizontal wave numbers, but also unbounded checkerboard structures). However, real disturbances appearing in natural, engineering, and laboratory flows of fluids are most oft en due to some factors affecting only a finite fluid volume and producing a localized initial disturbance. Note that in Sects. 3.22 and 3.23 much attention was given to papers by Henningson (1988), Breuer and Haritonidis (1990), and Landahl (1980, 1990, 1993a, 1996) devoted to study of evolution of localized disturbances in an inviscid fluid, while in this section the papers by Henningson (1991) and Henningson et al. (1993) on the same evolution in viscous fluids have been mentioned already. Now it is the time to dwell at somewhat greater length on the results of the last-mentioned paper.

The major part of this paper was devoted to numerical investigation of the development of a localized disturbance in a plane Poiseuille flow. The case of a disturbance initially consisting of two pairs of counter-rotating vortices (see Fig. 3.2) filling the height of the channel and symmetric with respect to its midplane was investigated in most detail. (Recall that this is just the case that was earlier studied by Henningson (1988) and (with a change of the z-dependence to one appropriate for the boundary-layer flow) by Russell and Landahl (1984); and Breuer and Haritonidis (1990); and was also considered by Henningson et al. (1990); and Henningson (1991)). However, in the paper by Henningson et al. (1993) an initial disturbance consisting of the same two pairs of eddies, but rotated around the vertical z-axis by some angle θ, was also considered (such rotation clearly changes the wave-number composition of the initial disturbance). Moreover, to investigate the sensitivity of the results obtained to changes in the primary flow or initial disturbance, the authors also considered a quite different initial disturbance shape (used only in the study of nonlinear effects which will not be discussed here) and added some remarks on comparison with the related boundary-layer results (corresponding to a disturbance having similar general shape but different z-dependence). Developing further the approach by Henningson

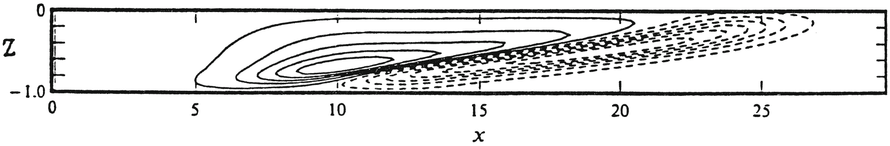


Fig. 3.17 Contours in the (*x, z*)-plane of streamwise velocity $u(x, y, z, t)$ at $y = 0$ and $t = 25$ of a localized disturbance, sketched in Fig. 3.2, in plane Poiseuille flow with Re = 3000. *Solid* and *dashed lines* correspond to positive and negative *u*-values. (After Henningson et al. (1993))

(1991), Henningson et al. (1993) computed the time evolution of the disturbances by two different methods. The first was the method of direct numerical simulation (DNS), i.e., of numerical solution of the N-S equations for the case of small enough amplitude of the initial disturbance. The second method used a truncated eigenfunction expansion of the Fourier-transformed initial data $\{\hat{w}(k_1, k_2; z, 0), \hat{\zeta}(k_1, k_2; z, 0)\}$ (cf. The description above of the paper by Reddy and Henningson (1993)) combined with the subsequent application of Eq. (3.15) and inverse Fourier transformation of the resulting Fourier components of velocity and vorticity.

The DNS results given by Henningson et al. (1993) for Re = 3000 show strong initial growth of the streamwise velocity component during a few tens of time units (as usual, length and time units are set equal to H_1 and H_1/U_0) with formation of the familiar wave-packet structure at the rear (later time) of the disturbance and the streaky structure at the front (see also Fig. 3.3 and the experimental data by Klingmann and Alfredsson (1991) and Klingmann (1992), revealing similar features). According to these results, the streamwise-velocity amplitude quickly becomes about one order of magnitude larger than the amplitude of the vertical velocity, although the streamwise velocity at $t = 0$ was zero. The contours of constant streamwise velocity in the (*x, z*) plane, shown in Fig. 3.17 for the lower half of the channel, clearly demonstrate the stretching of the disturbance structure in the streamwise direction (the initial horizontal spread of the disturbance was close to two units), with generation of alternating high-speed and low-speed regions and formation of inclined shear layers between them (cf. again Fig. 3.3b). Figure. 3.18a shows the dependence of the disturbance energy on time for a localized initial disturbance similar to that shown in Fig. 3.2, and for the same disturbance rotated by the angle $\theta = 10°$, 20° and 45°. (This figure was based on the DNS data but the eigenfunction-expansion computations gave practically the same results.) We see that some initial energy growth is present at any θ, but for $\theta = 0$ it is significant only at small values of t and is quickly replaced by decay, while increasing the value of θ increases the energy growth. (This is because rotation of the disturbance in the physical space rotates the wavenumber spectrum in the (k_1, k_2)-plane and increases the contribution of the region of small values of k_1. Recall also Landahl's result of 1990, illustrated with Fig. 3.4, that breaking of the symmetry of the initial disturbance with respect to a plane $y =$ const. increases the transient growth and leads to generation of streaky structures corresponding to small values of k_1.) Computation of the percentage contribution to disturbance energy at different values of t from three components of velocity, and from the vertical vorticity as such, showed that at $\theta = 0$ the percentage contributions of both the vertical and spanwise velocities w and v decrease monotonically with time, while the contributions of u and ζ_3 increase monotonically and at $t \geq 15$ the contribution of ξ_3 fully dominates all the others; essentially the same results are valid also for the cases where $\theta \neq 0$. These results agree well with the assumption that forcing of the vertical vorticity by the vertical velocity plays the main part in the transient growth of small disturbances. Similar results were obtained by Henningson et al. for the development of disturbances of the type shown in Fig. 3.2 in a Blasius boundary layer (see Fig. 3.18b, where some results for a boundary layer with $\mathrm{Re}\,\delta^* = 1000$ are presented; time is measured here in δ^*/U_0 units).

The study of the transient growth of disturbances in plane parallel flows will be continued in Sect. 3.4. However, now we will passs to consideration of the very important case of circular Poiseuille flow (flow in a round tube).

3.3.4 Transient Growth of Small Disturbances in the Poiseuille Flow in a Round Tube

In Chap. 2, in the very beginning of Sect. 2.9.4, it was pointed out that the problem of stability of Poiseuille flow in a round tube is apparently the most challenging and mysterious problem in the theory of hydrodynamic stability. It was also noted there that this problem has been studied repeatedly by a number of first-rate scientists but nevertheless is still far from being solved. However, in Sect. 2.94 only applications of the classical normal-mode method to this problem were considered. Now we will consider studies of transient disturbance growth in circular Poiseuille flow in a tube, which represent a natural extension of the investigations described in the previous subsection.

Just as in the case of plane-parallel flows, the first attempt to consider this topic was made by Orr (1907). He could not overcome the analytical difficulties appearing in the general case and was forced to simplify the statement of the problem. Therefore, in Chap. II of Part I of his paper, he confined himself to the study of a flow of inviscid fluid in a tube of radius R, where a disturbance which was small (*i.e.*, satisfying the linearized equations) and axisymmetric, having the velocity $\boldsymbol{u}(\boldsymbol{x},t) = \{u_r(r,\ x,\ t),\ 0, u_x(r,\ x,\ t)\}$ (*i.e.*, no circumferential component u_ϕ) was superimposed on the primary steady axisymmetric Poiseuille flow with the velocity $\boldsymbol{U}(\boldsymbol{x}) = \{U(r),\ 0,\ 0\},\ U(r) = A(R^2 - r^2)$. As was indicated in the concluding part of Sect. 3.1, Orr considered solutions of the initial-value problem for the velocity $\boldsymbol{u}(\boldsymbol{x},t)$ corresponding to several initial values $u_r(r,\ x,\ 0)$ of the radial velocity u_r (since $u_\phi = 0$, the axial velocity u_x can be easily determined from values of u_r with the help of the equation of continuity given in Sect. 2.84). He found a class of initial conditions (indicated in Sec 3.1) which for some values of the parameters implies a very great increase of the disturbance kinetic energy with time before the ultimate decay as $t \to \infty$. Therefore, he concluded that the circular Poiseuille flow of an inviscid fluid is *practically unstable* (cf. again Sect. 3.1). In the beginning of

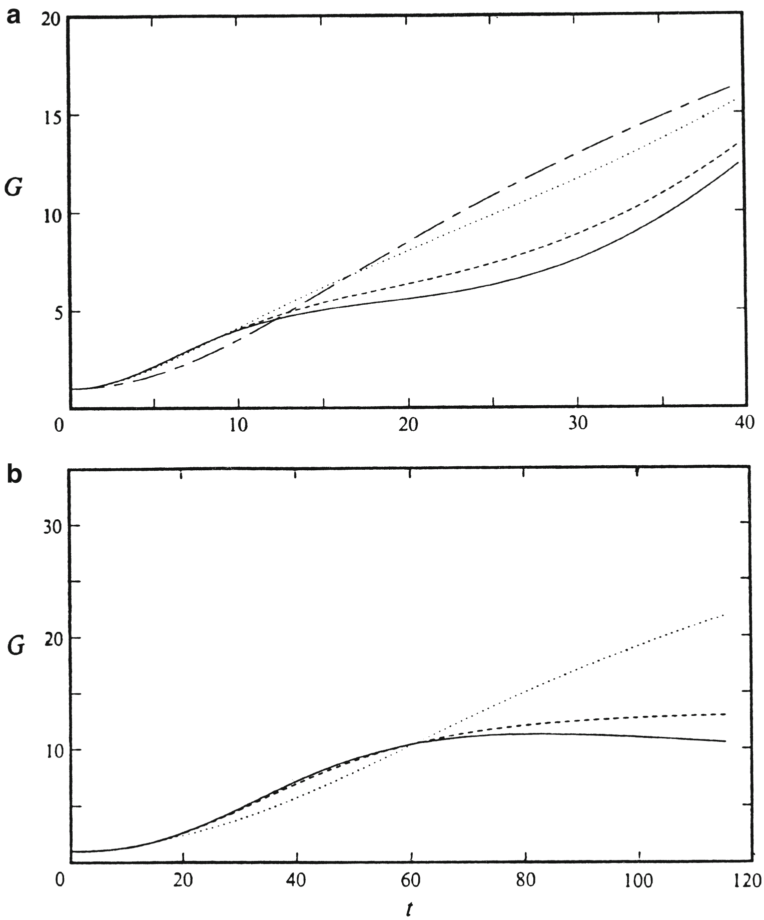


Fig. 3.18 **a** Energy-growth function $G(t) = E(t)/E(0)$ for some localized disturbances in plane Poiseuille flow with Re = 3000. Each disturbance either had the initial shape sketched in Fig. 3.2 (*solid line*) or its initial shape was obtained from that in Fig. 3.2 by rotation around the z-axis by the angle $\theta = 10°$, 20°, or 45° (*dashed*, *dotted*, and *chain-dashed lines*, respectively). **b** Energy-growth function $G(t)$ for localized disturbances in the boundary-layer flow with $\mathrm{Re} = U_0\,\delta^*/\nu = 1000$ having an initial shape either similar to that shown in Fig. 3.2 (*solid line*) or obtained from it by rotation by the angle $\theta = 10°$ or 45° (*dashed* and *dotted lines*, respectively); time is measured in δ^*/U_0 units. (After Henningson et al. (1993))

Part II of his paper Orr also noted that although he could not obtain similar results for viscous flow in a circular tube, he considered a proof of instability of the inviscid flow as also a proof of instability for the case where viscosity is not exactly zero but is small enough. However, these results by Orr had been completely forgotten when, about 80 years later, the subject was investigated anew by several scientists.

The first new study of the circular-Poiseuille-flow instability by a method going beyond the limits of the traditional normal-mode approach was due to Boberg and Brosa (1988). They tried to develop a nonlinear theory describing all stages of transition of a tube flow to turbulence. However, a very important part was played in their investigation by a study of a purely linear mechanism of extracting the kinetic energy from the mean flow, by transiently-growing small nonaxisymmetric disturbances with azimuthal wave number $n = \pm 1$. Boberg and Brosa carried out a numerical simulation of the corresponding process, based on the expansion of the disturbed velocity field into the so-called "Stokes functions" which are in fact the eigenfunctions of the N-S equations corresponding to very slow motion of fluid. Their severe truncations of the expansions used made many of their results only qualitatively relevant to the real onset of turbulence in a tube, although some important features of real disturbance development were described with satisfactory accuracy. The paper by Boberg and Brosa was also quite important as the trigger for many subsequent investigations of instability and transition by non-standard methods.

Slightly later, the paper by Gustavsson (1989) was entirely devoted to transient disturbance growth in circular Poiseuille flow. In this paper the important role of nonaxisymmetric disturbances in the disturbance-growth mechanism was revealed, and the equation which describes the forcing of the streamwise velocity of a disturbance by its pressure was derived. However, Gustavsson gave his main attention to the search for resonances between disturbance pressure and streamwise velocity, assuming that such resonances can make substantial contribution to algebraic growth of disturbances. This was the reason why his paper was considered in Sect. 3.32, devoted to resonances and degeneracies. However, later it became clear that resonance contribution does not play a substantial part in the disturbance growth (see the discussion of this topic in Sect. 3.33). Therefore, Gustavsson's paper of 1989 cannot now be considered as of great importance.

Later Gustavsson's student Bergström (1992) investigated the development of small nonaxisymmetric, x-independent, disturbances in Poiseuille tube flow. This work was stimulated by results by Ellingsen and Palm (1975) and Hultgren and Gustavsson (1981) who found that x-independent disturbances often grow algebraically with time in plane-parallel flows (see Sects. 3.22 and 3.32 above). It was also taken into account that the absence of x-dependence considerably simplified the stability analysis.

In fact, neglecting all x-derivatives in Eq. (2.73) (and hence assuming that the normal modes of a disturbance are proportional to $e^{i(n\phi-\omega t)}$) it is easy to obtain a simple fourth-order differential equation (with coefficients depending on n and $\mathrm{Re} = U_0 R/\nu$, where U_0 is the centerline velocity) for the amplitude $f^{(r)}(r)$ of the radial-velocity normal mode. This equation, with the physically-obvious boundary conditions for $f^{(\mathrm{r})}(r)$ at $r = 0$ and $r = R$, forms the radial eigenvalue problem, which was found to be easily solvable in terms of Bessel functions. In particular, it was shown that the dimensionless eigenvalues ω_j, $j = 1, 2, \ldots$ (below, all independent and dependent variables will be non-dimensionlized by appropriate combinations of R and U_0) are given by the equation $\omega_j = -i\hat{J}^2_{n+1,j}(\mathrm{Re})^{-1}$ where $\hat{J}_{n+1,j}$ is the j^{th} zero of the Bessel function J_{n+1}. Since all ω_j have negative imaginary and vanishing real parts, the normal modes derived are non-oscillating and damped. When u_r is known, the azimuthal velocity u_ϕ can be found from the continuity equation (the last of Eqs. (2.73)); in particular, $u_\phi = (i/n)\partial(ru_r)/\partial r$ for velocity disturbances with azimuthal wave number n. Proceeding further, one may obtain, for the amplitude $f^{(x)}(r)$ of the streamwise-velocity mode, a second-order inhomogeneous differential equation with the term $\mathrm{Re}U'f^{(r)}(r)$ on the right. This term describes forcing of x-independent stream-wise velocity disturbances by a radial velocity disturbance $u_r(r, \phi, t)$. Below, most attention will be given to development of the streamwise velocity $u_x(r, \phi, t)$ induced by a normal mode of the radial-velocity field.

Bergström fixed the value of $n \geq 1$ of the azimuthal wave number, assuming that $u_x(r, \phi, t) = u_x\ (r, t)\ e^{in\phi}$ and that $u_r(r, \phi, t) = f_j^{(r)}(r)e^{i(n\phi - \omega_j t)}$ is represented by a single (j^{th}) radial-velocity normal mode corresponding to this wave number. Moreover, he also accepted the initial condition $u_x(r, \phi, 0) = 0$, thus considering only the component of the streamwise velocity u_x which is induced by the radial-velocity disturbance. He showed that it is possible to find an explicit expression for the Laplace transform

$$\hat{u}_x(p, r) = \int_0^\infty e^{-pt} u_x(r, t)dt$$

of the function $u_x(r, t)$ and then to determine this function itself by inverse Laplace transformation. Normalizing the initial radial-velocity disturbance $u_r(r, \phi, 0)$ by the condition that $E_r(0) = 1$ (where the "radial kinetic energy" $E_r(t)$ is defined as the integral of $(u_r(r, \phi, t))^2/2$ over the tube cross-section), Bergström calculated the values of the "streamwise kinetic energy" $E_x(t)$ (defined similarly to $E_r(t)$) for a number of values for n, j, Re, and t (where t is measured in the R/U_0 units). Since it was shown that $E_x(t)/(\mathrm{Re})^2$ depends only on t/Re but not on t and Re separately, only the dependence of $E_x(t)/(\mathrm{Re})^2$ on n, j, and $t/\mathrm{Re} = T$ had to be determined.

According to the above results, $E_r(t) = \exp(-|\omega_j|t) = \exp(-\hat{J}^2_{n+1,j}T)$ falls off exponentially with t. Since $u_\phi \propto \partial(ru_r)/\partial r$, it follows that the "azimuthal kinetic energy" $E_\phi(t)$ (the integrated valued of $u_\phi^2/2$) also decays as $\exp(-|\omega_j|t)$. However, the streamwise kinetic energy $E_x(t)$ behaves differently. $E_x(0) = 0$ since $u_x(r, \phi, 0) = 0$, but when t increases, $E_x(t)/(\mathrm{Re})^2$ at first grows algebraically with time, reaches a maximum value $E_0 = E_0\ (n, j)$ at some value $T_0\ (n, j)$ of t/Re, and then begins to decay (asymptotically at the same rate as the energies $E_r(t)$ and $E_\phi(t)$). Results of computations (partially presented in Fig. 3.19) clearly showed that the disturbance component with $n = 1$ and $j = 1$ dominates the total growth of kinetic energy in the initial stage of disturbance development. Note that since the maximal values of $E_x(t)$ is proportional to $(\mathrm{Re})^2$, it increases rapidly if Re increases (*i.e.*, ν decreases). According to Bergström's computations, if $\mathrm{Re} = 1000$ and $E_r(0) = 1$, then $\max_{t>0} E_x(t) \approx 167$ for $n = 1, j = 1$, but this maximum value decreases quickly with n and even quicker with j (see Fig. 3.19).

Subsequent computations of disturbance development in Poiseuille tube flow, relating to the general case of disturbances depending on three coordinates x, r, and ϕ, were carried out by Bergström (1993a), Schmid and Henningson (1994) and O'Sullivan and Breuer (1994). In this case normal modes are proportional

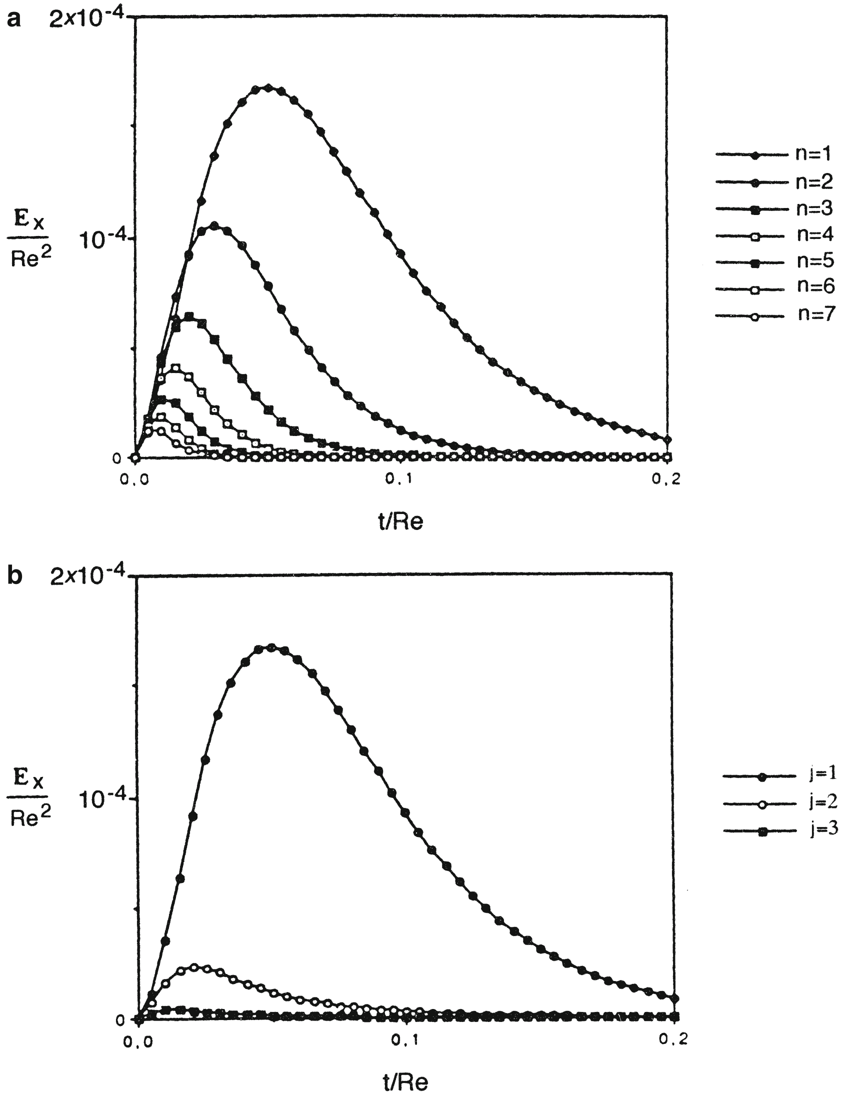


Fig. 3.19 The normalized energy-growth function $E_x(t)/(\mathrm{Re})^2$ for the streamwise kinetic energy $E_x(t)$ of an x-independent disturbance in circular Poiseuille flow induced by the radial-velocity disturbance represented by the jth normal mode with azimuthal wave number n having initial kinetic energy $E_r = 1$, for **a** $j = 1$, $n = 1, 2, \ldots, 7$, and **b** $n = 1$, $j = 1$, 2 or 3. (After Bergström (1992))

to $e^{i(kx+n\phi-\omega t)} = e^{i[k(x-ct)+n\phi]}$; therefore, the wave number k also appears, and frequency ω can be replaced by phase velocity $c = \omega/k$, Bergström (1993a), like Gustavsson (1989), began his investigation with the derivation, from dynamic equations (2.73), of a sixth-order homogeneous differential equation for the pressure-mode amplitude $g\,(r)$. This equation and the boundary conditions appropriate to it form the pressure-field eigenvalue problem, with coefficients depending on k, n and Re. This problem determines the set of eigenvalues ω_j or c_j, where $j = 1, 2, \ldots$ However, contrary to Gustavsson's approach, Bergström did not supplement the equation for $g(r)$ by the inhomogeneous differential equation (3.55) for the streamwise amplitude $f^{(x)}(r)$. Instead of this he showed that when $g(r)$ and c are known, $f^{(x)}(r)$ and the radial-velocity amplitude $f^{(r)}(r)$ can be determined with the help of equations of the form

$$f^{(r)}(r) = \mathrm{L}g(r), f^{(x)}(r) = Mf^{(r)}(r) + Ng(r), \tag{3.67}$$

where L, M, and N are three linear differential operators (with coefficients depending on k, n, Re and c), while $f^{(\phi)}(r)$ can be determined from values of $f^{(r)}(r)$ and $f^{(x)}(r)$ with the help of the equation of continuity. Therefore, according to his analysis, solution of the pressure-field eigenvalue problem allows the normal modes of all velocity components to be found as well.

Bergström solved the pressure-field eigenvalue problem numerically and found a number of complex eigenvalues c_j and eigen-function $g_j(r)$. Using Eqs. (3.67) and (2.74) he also evaluated the amplitudes $f_j^{(x)}(r)$, $f_j^{(\phi)}(r)$, and $f_j^{(r)}(r)$ of normal velocity modes. Then he considered disturbances represented by finite linear combinations of normal modes

$$\mathbf{u}(x, r, \phi, t) = \sum_{j=1}^{N} A_j \mathbf{u}_j(r) e^{\iota[k(x - c_j t) + n\phi]}, \mathbf{u}_j = \left\{ f_j^{(x)}, f_j^{(r)}, f_j^{(\phi)} \right\}. \tag{3.68}$$

The kinetic energy density $E\,(k, n; t) = E(t)$ of such a disturbance is given by a certain positive-definite quadratic form of coefficient A_j, and hence the method of Butler and Farrell (1992) can now be applied to determination of the maximum possible value of the energy growth $G(t) = \max_{\mathbf{A}} E(t)/E(0)$, where $\mathbf{A} = \{A_1, \ldots, A_N\}$ determines the shape of the initial velocity field $\mathbf{u}(\boldsymbol{x}, 0)$. The value of $G(t)$ depends on k, n and Re, and can also depend on the number N of normal modes considered, and the selection of these modes. However, if the modes are numbered in order of decreasing imaginary parts of the eigenvalues c_j (*i.e.*, in order of increasing mode-damping), the computational results show that an increase of N above some not-too-high lmiting values leaves the value of $G(t)$ practically unchanged. Therefore, the dependence on N is immaterial, if N is not too small. The dependence of $G(t)$ on t is shown in Fig. 3.20a for Re = 1000, $n = 1$, and several values of k. This figure shows that the maximal possible growth $G^* = \max_t >_0 G(t)$ increases with decreasing k (*i.e.*, here again the greatest growth occurs for streamwise-elongated disturbances), although at small values of time the disturbances with smaller streamwise wavelength can grow faster than strongly-elongated structures. Note that even at $k = 0.01$ (wavelength $l \approx 600R$) the largest energy amplification $G^* = 72.4$ is considerably smaller than the value $\max_t >_0 E_x(t)/E_r(0) \approx 167$ found by Bergström in 1992 for a non-optimal

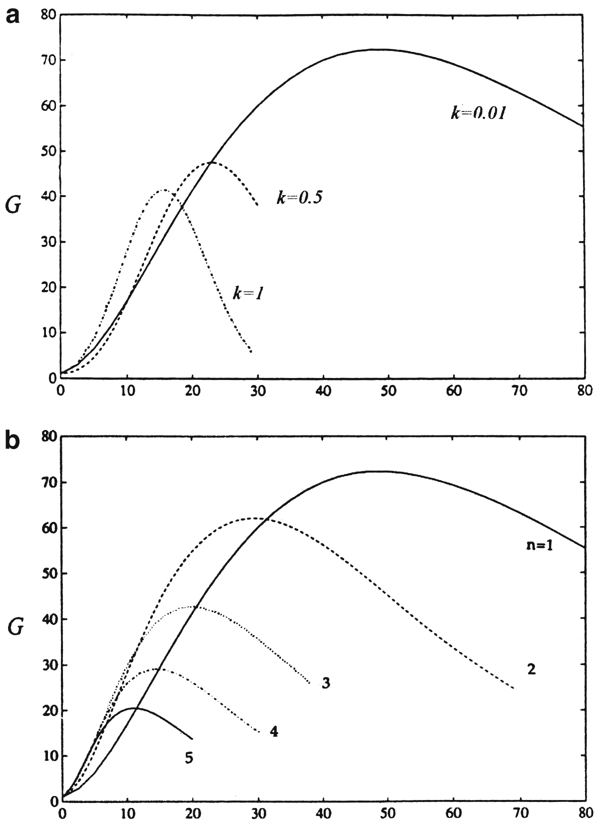


Fig. 3.20 The growth functions $G(t)$ for optimally-growing disturbances in circular Poiseuille flow with $\mathrm{Re} = U_0\, R/\nu = 1000$ having wave numbers **a** $n = 1$, $k = 0.01$, 0.5, 1; and **b** $n = 1, 2, \ldots, 5$, $k = 0.01$. (After Bergström (1993a))

x-independnet disturbance with $k = 0$ (see Fig. 3.19). Dependence of $G(t)$ on n is shown in Fig. 3.20b for the case where $\mathrm{Re} = 1000$ and $k = 0.01$; according to this figure, $G^* = \max_t >_0 G(t)$, at small k, decreases with increase of n, but at small values of t the disturbances with greater azimuthal wave numbers can grow faster than that with $n = 1$. (At larger values of k the disturbances with $n \geq 1$ can sometimes grow more than that with $n = 1$; see Fig. 3.21 below.) It was also shown by Bergström that the energy of the streamwise velocity component usually dominates the growing energy, especially for streamwise-elongated disturbances with small values of k.

Schmid and Henningson (1994) and O'Sullivan and Breuer (1994) employed a form of the linearized dynamic equations which differs from that used by Gustavsson and Bergström. In both these 1994 papers, the authors followed Bergström's paper of 1993 by considering the normal modes with fixed wavenumbers k and n and unknown frequency ω or phase velocity $c = \omega/k$. However, in these papers the four equations (2.73), with the unknown functions u_x, u_r, u_ϕ and p, were replaced by two equations, for the radial velocity u_r and the radial vorticity $\zeta_r = \partial u_\phi/\partial x - \partial u_x/\partial r\phi$, as was suggested long ago by Burridge and Drazin (1969). The definition of the vertical vorticity and the equation of continuity make it easy to express the streamwise and azimuthal velcotiy components u_x and u_ϕ in terms of u_r and ζ_r; in particular, for disturbances proportional to $e^{i(kx+n\phi)}$ we obtain

$$u_x = \frac{ik}{rK^2}\frac{\partial(ru_r)}{\partial r} + \frac{in}{rK^2}\zeta_r, \quad u_\phi = \frac{in}{r^2K^2}\frac{\partial(ru_r)}{\partial r} - \frac{ik}{K^2}\zeta_r \tag{3.69}$$

where $K^2 = k^2 + n^2/r^2$. Equations (3.69) are similar to Eqs. (3.15), which express horizontal velocity components of a normal-mode disturbance in a plane-parallel flow in terms of vertical velocity and vorticity; thus, these new equations may be considered as representing the form taken by Eqs. (3.15) in cylindrical coordinates. Burridge and Drazin (1969) showed that for disturbances proporiotnal to $e^{i(kx+n\phi)}$ the linearized dynamic equations (2.73) can be reduced to the following system of two equations for functions $-iru_r = \phi$ and $i\zeta_r/K^2r = \psi$ associated with u_r and ζ_r:

$$\left[\left(\frac{\partial}{\partial t} + ikU\right)\Xi - ikK^2r\left(\frac{U'}{K^2r}\right)'\right]\phi = \frac{1}{\mathrm{Re}}[\Xi^2\phi - 2kn\,\Xi\psi],$$
$$\left(\frac{\partial}{\partial t} + ikU\right)\psi - i\frac{nU'}{K^2r^3}\phi = \frac{1}{\mathrm{Re}}\left[\Theta\psi + \frac{2kn}{K^4r^4}\Xi\phi\right] \tag{3.70}$$

Here K has the meaning indicated above, a prime denotes differentiation with respect to r, and Ξ and Θ are the following second-order differential operators:

$$\Xi = \frac{1}{r^2} - \frac{1}{r}\frac{d}{dr}\left(\frac{1}{K^2r}\frac{d}{dr}\right), \Theta = K^4r^2 - \frac{1}{r}\frac{d}{dr}\left(K^2r^3\frac{d}{dr}\right). \tag{3.70$'$}$$

In Eq. (3.70) it may be assumed that the unknown functions $\phi = \phi\ (r,\ t)$ and $\psi = \psi$ (r, t) depend only on r and t, and represent "amplitudes" preceding the factor $e^{i(kx+n\phi)}$. Assuming now that the disturbance depends exponentially on time, one may replace $\partial/\partial t$ by $-i\omega$ (or $-ikc$) and consider only the stationary amplitudes $\phi(r)$ and $\psi(r)$. We now supplement Eqs. (3.70), (3.70′) by the boundary conditions at $r = 1$ and $r = 0$. At $r = 1, \phi = \phi' = \psi = 0$, while at $r = 0$ we have $\phi = \phi' = 0$ if $n = 0$; or $\phi = \psi = 0$, with ϕ' finite, if $n = \pm 1$; or $\phi = \phi' = \psi = 0$ if $n \geq 2$. We thus obtain the general vector eigenvalue problem of the linear theory of tube-flow stability (see, e.g., Schmid and Henningson (1994)). This problem determines the sets of eigenvalues ω_j (or c_j) and eigenfunctions $\phi_j,\ \psi_j$ allowing the normal modes of all velocity components to be found.

Schmid and Henningson (1994) solved the eigenvalue problem numerically, and found a great number of eigenvalues ω_j and eigenfunctions f_j and ψ_j corresponding to various combinations of wavenumbers k and n. Following Reddy and Henningson, they numbered the eigensolutions in order of increasing mode-damping and then, fixing the values of k and n, expanded vector $\mathbf{q}(r,t) = \{\phi\ (r,\ t),\ \psi\ (r,\ t)\} = \{-iru_r(r,\ t),$ $i\zeta_r(r,\ t)/K^2r\}$ into the eigensolutions

$$\mathbf{q}(r,t) = \sum_{j=1}^{N} A_j\mathbf{q}_j(r)e^{-i\omega_j t}, \mathbf{q}_j(r) = \{\phi_j(r), \psi_j(r)\}. \tag{3.71}$$

(the integer N determines the degree of the series truncation). Using Eqs. (3.69) it is easy to show that the energy density $E(k, n; t) = E(t)$, which also determines the energy norm $||\mathbf{q}(r, t)||_E = [E(t)]^{1/2}$ in the space of vector-functions $\mathbf{q}(r; t)$, is given here by the equation

$$E(k, n; t) \equiv E(t) = \pi \int_0^1 \left[\frac{|\phi'|^2}{K^2 r^2} + \frac{|\phi|^2}{r^2} + K^2 r^2 |\psi|^2 \right] r dr. \qquad (3.72)$$

Equations (3.71) and (3.72) allow the energy $E(t)$ to be represented as a positive-definite quadratic form of the coefficients A_j. Hence here again the variational method of Butler and Farrell (1992) may be used for determination of the optimally-growing disturbance and the maximum energy growth $G(t) = \max_{\mathbf{A}} (E(t)/E(0))$ where $\mathbf{A} = \{A_1, \ldots, A_N\}$. As above, $G(t)$ depends here on the values of k, n, and Re, while the dependence on N was found to be immaterial if N is not too small (it was indicated by Schmid and Henningson that for $N > 7$ the results vary with N by less than 1.2 %).

In Fig. 3.21 the calculated functions $G(t)$ are presented for $k = 0$, 0.1, and 1, $n = 1$, 2, 3, and 4, and Re = 3000. For $k = 0$ it was proved by Bergström (1992) (and was confirmed by Schmid and Henningson (1994) and by O'Sullivan and Breuer (1994)) that $G(t)/(\text{Re})^2$ depends only on t/Re; therefore the data in Figs. 3.21a and 3.21b refer to any values of Re. In Figs. 3.21c and 3.21d, values of time on the abscissa are also divided by Re but here this does not make the data sufficient for determination of the energy growth at any Re. For $n = 0$ (axisymmetric disturbances) some transient growth was found under the condition that $k\text{Re} > 370$ but it proved to be quite small (here $\max_{t>0} G(t) \le 3$ in all cases investigated). Note also that Bergström (1992) appeared to find a larger energy growth for disturbances with $k = 0$ than that indicated in Fig. 3.21a, but he normalized $E(t)$ by the initial energy of only one radial velocity component. Therefore, data in Fig. 3.19 must be rescaled for comparison with data in Fig. 3.21a (cf. Fig. 4 by O'Sullivan and Breuer (1994) who performed the necessary rescaling and found that it brings Bergström's results much closer to those by Schmid and Henningson). As to the results of Bergström (1993a) presented in Fig. 3.20, they do not contradict those in Fig. 3.21.

Contours of $G^*(k, n, \text{Re}) = \max_{t>0} G(t)$ in the (Re, kRe)-plane were also given by Schmid and Henningson separately for $n = 0$, 1, 2, and 3; they show that for $n > 0$ the maximum energy amplification increases with decreasing k and that $G^*/(\text{Re})^2$ becomes practically independent of Re at large values of the Reynolds number. Some examples of the initial velocity fields leading to the maximum energy growth are presented in Schmid and Henningson's paper, together with some other results which will be discussed in Sect. 3.4.

O'Sullivan and Breuer (1994) also made computations related to those performed by Schmid and Henningson (1994). In both these papers, Eqs. (3.70) and (3.70′) with the unknown functions $\phi(r, t)$, $\psi(r, t)$ were used to find the eigenvalues ω_j

or $c_j = \omega_j/k$ and the eigenfunctions $\phi_j(r)$, $\psi_j(r)$. However, these results were applied by O'Sullivan and Breuer only to the study of the tube-flow eigenvalues and for evaluation of some specific initial conditions. They followed Gustavsson (1991) and assumed that $u_r(\boldsymbol{x}, 0)$ corresponds to some normal-mode solution of the linearized stability equations (3.70) and (3.70′) while the initial radial vorticity $\zeta_r(\boldsymbol{x}, 0)$ is equal to zero. Two other velocity components, $u_x(\boldsymbol{x}, 0)$ and u_ϕ $(\boldsymbol{x},0)$, can then easily be determined with the help of Eq. (3.69) and the initial pressure $p(\boldsymbol{x},0)$ can be found from dynamic equations (2.73). The resulting initial conditions are clearly non-modal since infinitely many normal modes are needed to annihilate the non-zero radial vorticity entering the full normal-mode solution. However, such initial conditions seem to be interesting since they guarantee strong transient growth of radial vorticity $\zeta_r(\boldsymbol{x}, t)$. Moreover, comparison of Gustavsson's results of 1991 with those of Butler and Farrell (1992) suggests that, in the case where $u_r(\boldsymbol{x}, 0)$ corresponds to the least-stable normal mode, the energy growth must be close to the optimal one.

O'Sullivan and Breuer did not try to expand the chosen initial values in normal modes $\{\phi_j, \psi_j\}$, $j = 1, 2, \ldots$, but used direct numerical simulation, *i.e.*, numerical solution of the N-S equations in the cylindrical domain with given initial and boundary conditions. The numerical factor entering the normal-mode solution representing $u_r(\boldsymbol{x}, 0)$ was chosen to be small enough to make nonlinear effects unimportant even when the velocity disturbances grow by more than four orders of magnitude. The DNS results are shown in the paper for a number of initial normal modes of radial velocity, several values of n, and values of k varying from 0 to 0.5. The results showed considerable transient growth of disturbances in almost all cases considered, in some cases comparable with the optimal growth determined by Schmid and Henningson. It was also shown that growth curves $G(t)$ at $\mathrm{Re} = 1000$ and $\mathrm{Re} = 2000$ are practically the same if the doubling of Re is accompanied by the halving of k.

There were only a few experiments on tube flows which gave data that can be compared with the above theoretical conclusions. However, Bergström's (1993b, 1995) results are worth mentioning in this respect. In the first of the indicated papers, results of the measurements by the laser-doppler anemometer of the spatial developments of disturbances in a tube water flow were presented. The non-axisymmetric localized disturbances (corresponding to azimuthal wave numbers $n = 1$ and 5) were introduced in flows with several values of Re through 60 small holes made in the tube wall at a fixed value of the streamwise coordinate x. Then the disturbance amplitudes were measured at a number of downstream positions. In the experiments described in the second paper, localized initial disturbances of an air flow in a tube were produced, at several values of Re, by two jets induced radially into the tube by diametrically opposed loudspeakers and then amplitudes of the streamwise disturbance velocity were measured by a hot-wire anemometer at different radial and axial positions. Transiently-growing disturbances (growing at first and later beginning to decay) with theoretically reasonable streamwise velocity were detected in both experiments; especially conclusive results were obtained in the second set of experiments, where the evolution with x of the peak amplitude and spatial distribution of the streamwise velocity were investigated in detail. It was shown that the disturbances having most significant transient growth correspond to $n = 1$ and are

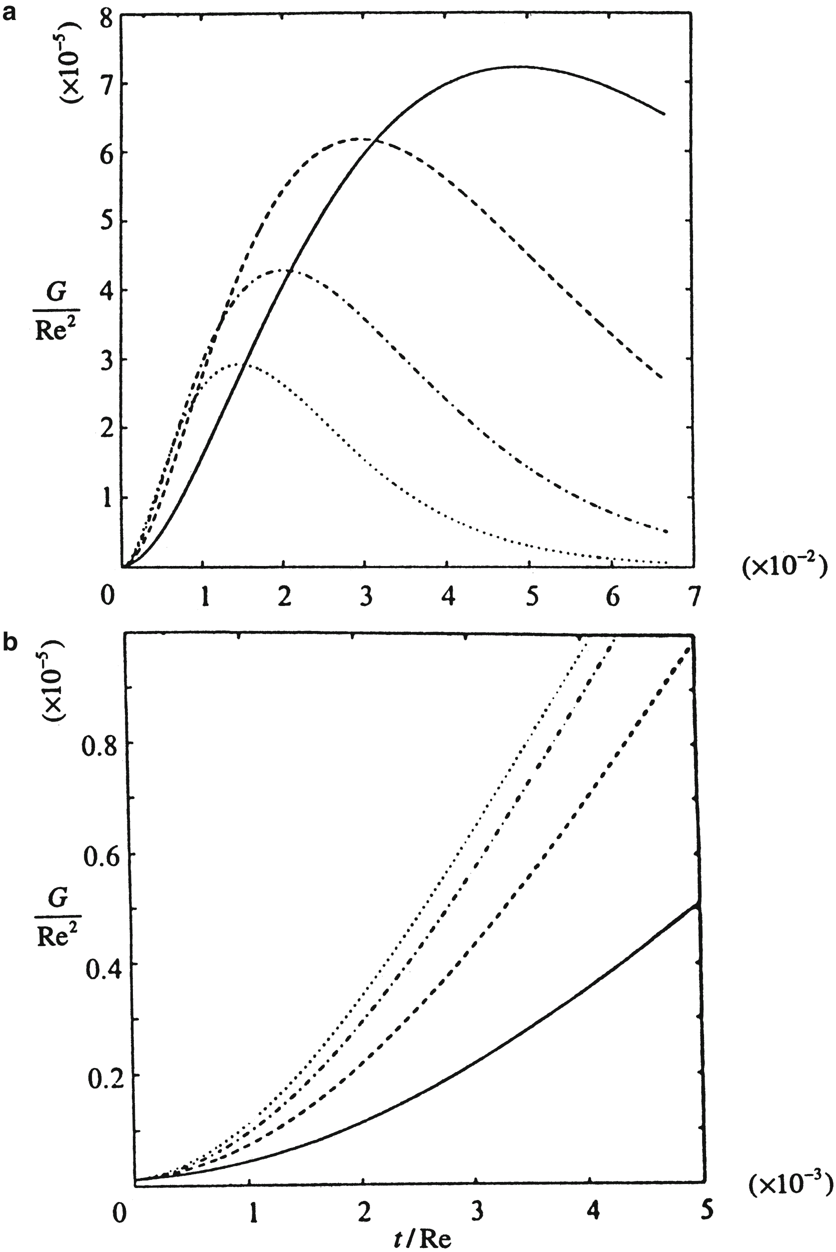
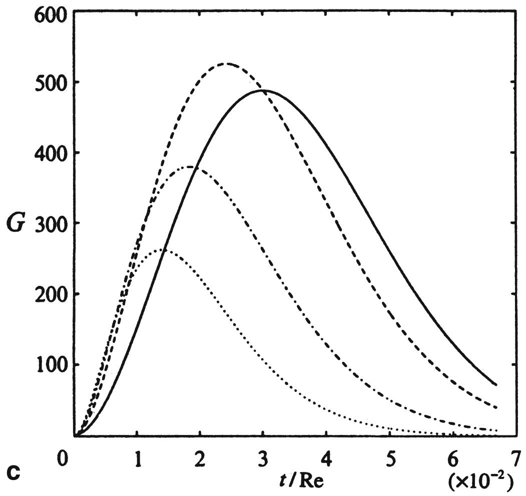
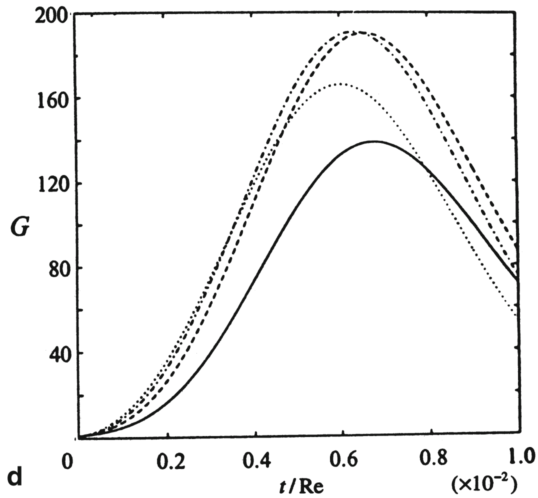


Fig. 3.21 The growth functions $G(t)$ for optimally-growing disturbances in circular Poiseuille flow with $\mathrm{Re} = 3000$ having wave numbers $n = 1$, 2, 3, or 4 and **a** $k = 0$; **b** $k = 0$, data for small times; **c** $k = 0.1$; and **d** $k = 1$. The *solid*, *dashed*, *chain-dashed*, and *dotted lines* correspond to $n = 1$, 2, 3, and 4, respectively; in **a** and **b** the scaling of G by $(\mathrm{Re})^2$ and t by Re makes the graphs correct at any value of Re. (After Schmid and Henningson (1994))

Fig. 3.21 (Continued)





streamwise elongated; these results agree well with available theoretical predictions. The maximal growth of the disturbance energy was found to be increasing with the Reynolds number, but in all cases this growth was considerably smaller than that calculated for the case of optimally growing disturbances. However, this is only natural since the produced initial disturbances were clearly far from optimal. In any case, it was important that the reality of the transient disturbance growth was experimentally confirmed. As to the above theoretical results, indicating the possibility of the energy growth of tube-flow disturbance by three or even more orders of magnitude, they definitely indicate a mechanism that can affect the transition of tube flows to turbulence. Note again that such transition, which was first described by Hagen in 1839 and then was carefully investigated by Reynolds in 1883, remains up to now unexplained.

3.4 Some General Remarks About Transient Growth of Small Disturbances in Parallel Fluid Flows

Results of Sects. 3.33–3.34 show that in a viscous fluid with a subcritical value of Re, weak disturbances, whose evolution is described by linearized dynamic equations, often grow very significantly with time at first, and only later begin to decay. For $\mathrm{Re} > \mathrm{Re}_{cr}$ the situation is not too much different since here the rate of transient growth of small disturbances often greatly exceeds the rate of growth of the unstable normal mode (having the form of the O-S wave in the case of a plane-parallel flow). As a result the nonlinear interactions of transiently growing weak disturbances can become quite important, not only in subcritical flows with $\mathrm{Re} < \mathrm{Re}_{cr}$ but also in supercritical flows at times when the unstable normal mode is still very weak and practically unobservable. Thus, in both these cases the nonlinear interactions can lead to the so-called *bypass transition* to turbulence where the normal modes play no role at all (see Sect. 2.92).

The above arguments may produce the impression that transition studies must be based mainly on the nonlinear theory while the linear development of small disturbances is here only of secondary interest. However, in fact the solutions of the linearized initial-value problems are here also of primary importance since they provide the initial conditions for the subsequent nonlinear development. Moreover, some authors even suggested that the exact form of non-linear interactions of not-too-small disturbances is in fact of secondary importance. According to these authors, the only requirement limiting the form of nonlinear interactions producing transition to turbulence is the following one: they must lead to not-too-late breakdown of growing disturbances, which transfers the accumulated energy, prior to its appreciable viscous decay, to numerous small disturbances subjected again to transient growth in accordance with the linear stability theory (see, e.g., the remarkable early paper by Boberg and Brosa (1988) and the recent survey by Baggett and Trefethen (1997)). This suggestion (which is not taken for granted by everybody) stimulated the appearance of a number of quite different simple low-dimensional models of nonlinear interactions leading, as a rule, to relatively similar conclusions which do not contradict the available data; see, in particular, the two above-mentioned papers and the papers by Trefethen et al. (1993), Gebhardt and Grossmann (1994), Baggett et al. (1995), and Grossman (1996), which will be considered at greater length in Chap. 4.

Henningson and Reddy (1994) (see also Henningson (1995, 1996) and Schmid and Henningson (2001)) stressed that the possibility of transient growth of small enough disturbances obeying linear dynamic equations is necessary for the transition of a given flow to turbulence. This conclusion follows from consideration of the energy balance of disturbances. Let an incompressible fluid fill a spatial domain V, which is either finite or bounded by solid walls or is bounded in some directions but unbounded in the directions of one or two coordinate axes. Assume further that there is a flow in domain V with the velocity and pressure fields $\boldsymbol{U}(\boldsymbol{x}) + u(\boldsymbol{x}, t)$ and $P(\boldsymbol{x}) + p(\boldsymbol{x}, t)$. Here $\{U(\boldsymbol{x}), P(\boldsymbol{x})\}$ is a steady solution of the N-S equations satisfying the "no-slip" boundary conditions at solid walls and independent of the coordinates with respect to which the domain V is unbounded, while $\{\boldsymbol{u}(\boldsymbol{x},t),\ p(\boldsymbol{x},t)\}$ are the velocity and pressure of a disturbance (of an arbitrary size) which is periodic in directions of unboundedness. Consider the kinetic energy of the disturbance

$$E(t) = \frac{1}{2}\int_{V'}\sum_{j=1}^{3} u_j^2 dx \tag{3.73}$$

where $d\boldsymbol{x}$ is an element of volume in the three-dimensional space of points $\boldsymbol{x}$ and V' is the whole domain V, if it is bounded in all directions, or the part of V' bounded by one disturbance period in directions of unboundedness. Then using the N-S equations it is easy to obtain for $E(t)$ the following equation for the energy balance:

$$\frac{dE(t)}{dt} = -\int_{V'} u_j u_i \frac{\partial U_j}{\partial r_i} d\mathrm{x} - \nu \int_{V'} \sum_{j,i=1}^{3}\left(\frac{\partial U_j}{\partial r_i}\right)^2 dx \tag{3.74}$$

where, as always, summation is carried out over three values of indices occurring twice in the same term (see, e.g., Monin and Yaglom (1971), Sect. 2.9, or Joseph (1976), Sect. 3). Equtaion (3.74) was first derived by Reynolds (1894) and was then used by Orr (1907); at present it is usually called the *Reynolds-Orr* (or R-O) *equation*.

Both terms on the right-hand side of Eq. (3.74) are of the second order with respect to the disturbance velocities. The second of them describes the dissipation of the disturbance energy due to viscosity, and is always negative, while the first term, which describes the exchange of energy between the undisturbed flow and the disturbance, can be of any sign but, as a rule, is positive (the transfer of energy is usually directed from the undisturbed flow to the disturbance). If so, then the relative value of the two considered terms will determine whether the energy of the disturbance decreases or increases. If we transform Eq. (3.74) to dimensionless quantities, measuring distance, velocity, and time in units of characteristic length L, velocity U, and time L/U, respectively, then the dimensional coefficient ν in the second term on the right-hand side will be transformed into the dimensionless coefficient $\nu/UL = 1/\mathrm{Re}$. Hence, if the Reynolds number Re is sufficiently small, the negative second term on the right-hand side will always dominate the positive first term, and the energy of any disturbance will be damped, *i.e.*, the flow will be stable to disturbances of any shape and amplitude. Equation (3.74) in principle makes it possible to obtain certain estimates from below $\mathrm{Re}_{\mathrm{cr\ min}}$, which bounds the range of "sufficiently small" Reynolds numbers, within which the energy of any disturbance can only decrease. This remark is due to Reynolds, who tried to use Eq. (3.74) for estimation of $\mathrm{Re}_{\mathrm{cr\ min}}$ in his paper of 1894 where the quantities Re and $\mathrm{Re}_{\mathrm{cr\ min}}$ first appeared. Later this equation was used many times for the same purpose by a number of authors; some of them will be indicated below, partly in this section and partly in Chap. 4, Sect. 4.1. In cases where the flow is unbounded in some directions and the disturbance is periodic with respect to the corresponding coordinates (say, to x and y with wavelengths $\lambda_1 = 2\pi/k_1$ and $\lambda_2 = 2\pi/k_2$, respectively) the domain V' depends on k_1 and k_2 and hence here the Reynolds number below which the right-hand side of (3.74) is necessarily negative also depends on k_1 and k_2. Let us denote this value by $\mathrm{Re}_1(k_1, k_2)$. Then at $\mathrm{Re} < \mathrm{Re}_1(k_1, k_2)$ the energy density $E(k_1, k_2; t)$ of any disturbance with horizontal wave numbers (k_1, k_2) cannot grow with time (thus, $E(k_1, k_2; t) \le E(k_1, k_2; 0)$ for any $t > 0$) and hence if $\mathrm{Re} < \min_{k_1,k_2} \mathrm{Re}_1(k_1, k_2) = \mathrm{Re}_{\mathrm{cr\,min}}$ we may be sure that the energy $E(t)$ of *any* disturbance will be damped.

In the above-mentioned paper by Henningson and Reddy (see also Reddy and Henningson (1993); Schmid and Henningson (1994); and Henningson and Alfredsson (1996)) the importance of the fact that the nonlinear terms of the N-S equations for the disturbance velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ drop out, when Eq. (3.74) is derived, was especially emphasized. (These terms play no role since they produce a divergence term in the integrand of the first integral on the right-hand side of Eq. (3.84) and thus by virtue of the boundary conditions this term disappears after application of the Gauss theorem.) Therefore, Eq. (3.74) preserves its form when the full N-S equations are replaced by the linearized Eq. (2.7) describing the time evolution of very small disturbances. This implies that if the energy of some disturbance of any amplitude grows with time in a flow satisfying the above conditions (and such growth is necessary for transition to turbulence since otherwise $E(t) < E(0)$ for any disturbance and any $t > 0$), then certainly the energy of an infinitesimal disturbance of the same shape will be also growing, at least for small values of t. Since for $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$ only transiently growing infinitesimal disturbances can exist and the disturbances appearing in real flows are usually quite small at the beginning, it is natural to assume that subcritical transitions to turbulence of laminar flows encountered in engineering and nature begin, as a rule, with the transient algebraic growth of randomly arising small disturbances.) It is easy to show that subcritical transition is impossible for flows where the transient growth of infinitesimal disturbances cannot occur; see Henningson and Reddy (1994)).

According to results by Busse (1969) and Joseph and Carmi (1969) for plane Poiseuille flow, and similar results by Joseph (1966) for plane Couette flow and by Joseph and Carmi (1969) for circular Poiseuille flow, $\mathrm{Re}_{\mathrm{cr\,min}} \approx 49.6$, 20.7, and 81.5, respectively, for these three flows (for more details see Sect. 4.1 in Chap. 4 of this series). Hence in these flows, at values of Re smaller than the given values of $\mathrm{Re}_{\mathrm{cr\,min}}$, energy of a disturbance of any size will decrease monotonically with time, while at $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr\,min}}$, disturbances of any size will exist whose energy will grow with time, at least for not too large values of t. Farrell and Ioannou (1993b) carried out a similar computation for the case of an unbounded Couette flow and showed that for disturbances with the given horizontal wave numbers k_1 and k_2 no growth is possible if $\mathrm{Re} = bl^2/\nu \equiv b\pi^2/k^2\nu < 2\pi^2 \approx 19.7$, where b is the velocity shear and $k = (k_1^2 + k_2^2)^{1/2}$. Thus, in this case a range of "sufficiently small" values of ν exists within which the energy of any disturbance with a given value of k decays, but "sufficiently small-scale" disturbances can grow here at any value of viscosity.

In Sects. 3.2–3.3, where transient growth of small disturbances in laminar plane-parallel flows was discussed, two physical mechanisms of such growth were indicated: the direct transfer of the kinetic energy of the undisturbed laminar flow with velocity $U(z)$ to streamwise disturbance energy $u^2/2$ occurring when $uwdU/dz$ takes negative values; and the forcing of the vertical vorticity of disturbance by spanwise variations of vertical velocity w closely related to Landahl's lift-up effect, which produces streamwise high- and low- velocity streaks. Butler and Farrell (1992) suggested the term "vortex tilting" for the second, most effective, growth mechanism since here the growth is due to spanwise tilting of the vertical vorticity. There is, however, also a mathematical explanation of the growth mechanism, which is related to some special features of linearized fluid dynamics equations which have been mentioned briefly in Sect. 2.5.

Let us rewrite the lienarized dynamic equations for the disturbance velocity $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ in the operator notations:

$$\frac{\partial \mathbf{u}}{\partial t} = -iL\mathbf{u}. \tag{3.75}$$

where L is a time-independent linear operator (easily derived from the N-S equations) in the space of divergence-free vector functions $\boldsymbol{u}(\boldsymbol{x}),\ \boldsymbol{x} \in \boldsymbol{V},$ satisfying the boundary and periodicity conditions indicated above. (The factor $-i$ on the right-hand side is added to make the eigenvalues of L equal to the complex frequencies ω of the normal modes entering Eq. (2.8) in Sect. 2.5.) To use the highly developed mathematical theory of linear operators in Hilbert spaces we must extend Eq. (3.75) to the space of complex vector-functions $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{t})$ satisfying the above conditions and having finite norm $||\boldsymbol{u}(\boldsymbol{x},\ \boldsymbol{t})|| < \infty$, where

$$||u(x,t)||^2 = \tfrac{1}{2}\int\limits_{v'} |\mathbf{u}(\mathbf{x},t)|^2 d\mathbf{x} = E_u$$

is the kinetic energy (or, for unbounded flows, kinetic energy density) of a disturbance. Then the disturbance velocity at any value of t will belong to the Hilbert space H of vector functions $\boldsymbol{u}(\boldsymbol{x})$ with the scalar product

$$(\mathbf{u}, \mathbf{v}) = \tfrac{1}{2}\int\limits_{v'} [\mathbf{u}(\mathbf{x}) \cdot \mathbf{v}^*(\mathbf{x})] d\mathbf{x}$$

where the asterisk denotes complex conjugation. This allows us to consider the evolution operator L on the right-hand side of Eq. (3.75) as an operator in H, and define for it the adjoint operator L^* as a linear operator satisfying the condition $(L\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, L^*v)$ for any $\boldsymbol{u}$ and $\boldsymbol{v}$. The best studied are the *self-adjoint* operators L satisfying the condition $L^* = L$. According to the spectral theory of self-adjoint operators, any such operator L has a real spectrum (which can be discrete, continuous, or mixed) and every element of H can be expanded into eigenvectors of L (or "generalized eigenvectors" corresponding to points of a continuous spectrum) where the eigenvectors corresponding to different eigenvalues are orthogonal to each other. This important property of eigenvector orthogonality is valid also for the wider class of the *normal* linear operators in a Hilbert space H having the property that $LL^* = L^*L$ (see, e.g., Dunford and Schwartz (1958, 1971); Rudin (1973); Kato (1976, 1982) and Pazy (1983) for more information about this topic[6]). For normal operators the

[6] Note that the properties of self-adjointness and normality of an operator L depend not only on the operator itself but also on the norm (and scalar product) introduced in the space H. Thus, both these properties can be lost (or gained) when the norm is changed.

spectral representation theorem is also valid and according to it every element of H can again be expanded into reciprocally orthogonal eigenvectors ("generalized" for points of a continuous spectrum) of L, but the spectrum of eigenvalues in this case is placed in the complex plane. However, the *evolution operator* L entering the linearized N-S equations written in the form(3.75) is very often not only non-self-adjoint but also non-normal (in particular, it is non-normal in all cases of parallel shear flows considered in this chapter[7]). Nevertheless, as was indicated in Sect. 2.5, the existence of a complete system of eigenfunctions and the corresponding eigenfunction expansion theorem has also been proved for all these cases (and for some more general cases too), in particular by Di Prima and Habetler (1969), Yudovich (1965, 1984) and Herron (1980, 1982, 1983). It was also noted in the indicated section that for non-normal operators eigenvectors corresponding to different eigenvalues are as a rule not orthogonal. This last circumstance is very important and it has direct relation to the strong transient growth of small disturbances in parallel shear flows.

Let us normalize all the eigenvectors making their norms equal to one. Then in the expansion of a vector $\boldsymbol{u}$ of the Hilbert space H into orthogonal eigenvectors the squared norm $\|\boldsymbol{u}\|^2$ (*i.e.*, the energy, if in accordance with the above agreement the "energy norm" is used) is equal to half the sum of the squares of all the expansion coefficients. Therefore, the square of any coefficient is here definitely not greater than twice the energy. However, when the eigenvectors are not orthogonal, the energy is given by some complicated positive-definite quadratic form of the coefficients and in this cases individual coefficients can take very large values. This is especially so when the operator L is "strongly non-normal," which means that its eigenvectors corresponding to different eigenvalues are not only non-orthogonal but in some cases even nearly linearly dependent. Then only small parts of some new eigenvectors will provide really new contributions to the linear combination of all previous eigenvectors. Suppose for definiteness that at $t = 0$ the initial condition $\boldsymbol{u}(\boldsymbol{x}, 0)$ is a vector-function of unit norm so that $E_{\boldsymbol{u}}(0) = 1$. Let us now represent the solution $\boldsymbol{u}(\boldsymbol{x}, t)$ of Eq. (3.75) in the form of eigenfunction expansion

$$\boldsymbol{u}(x,t) = \sum_j a_j \boldsymbol{u}_j(\boldsymbol{x}) e^{-i\omega_j t}$$

where a_j are expansion coefficients, while $\boldsymbol{u_j}(\boldsymbol{x})$ are the normalized eigenvectors and ω_j the eigenvalues of operator L. If L is "strongly non-normal" then coefficients a_j corresponding to nearly linearly dependent eigenvectors must be quite large to provide significant values of the "really new contributions" from these eigenvectors. At the same time, these coefficients must lead to the near-cancellation of linearly dependent parts of various vectors to yield the initial value $\boldsymbol{u}(\boldsymbol{x}, 0)$ of unit norm

[7] The situation is different in the case of stability problems four Couette-Taylor flow between two rotating coaxial cylinders and for convection in a layer of fluid heated from below. This fact leads to a fundamental difference between these two problems and stability problems for parallel shear flows, and means that the initial-problem approach is not useful in the case of the former two problems.

in spite of large values of a_j. For moderate values of t, the sum still consists of large terms, even if $\Im\text{m}\ \omega_j < 0$ for all subscripts j and hence all the exponentials are decaying functions. However, since the time-dependent factors $\exp(-i\omega_j t)$ differ from each other, the cancellation that occurs at $t = 0$ need not occur later and therefore the norm of the solution $\boldsymbol{u}(\boldsymbol{x}, t)$ at positive but not too large values of t can exceed considerably its initial value $\|\boldsymbol{u}(\boldsymbol{x}, 0)\| = 1$. At still greater values of t, the decaying factors $\exp(-i\omega_j t)$ begin to play the main part making the growth of the norm of $\boldsymbol{u}(\boldsymbol{x}, t)$ only transient.

Given here the "mathematical explanation" of the transient growth of small disturbances, whose evolution is described by non-normal dynamic operators, was given by Henningson (1991) who first found that the coefficients of an eigenfunction expansion of a small disturbance in a subcritical plane Poiseuille flow sometimes take surprisingly high values (so, some of the expansion coefficients for the initial disturbance with $E_u(0) = \text{O}(1)$ in a flow with $\text{Re} = 3000$ were found to be of the order of 10^3). Later Reddy et al. (1993) developed a method for crude estimation of the order of expansion coefficients and showed that this order grows very rapidly as Re increases: according to their estimates, for two-dimensional disturbances to a plane Poiseuille flow with $E_u(0) = 1$ the coefficients may have the order of 10^8 if $\text{Re} = 10000$, of 10^{10} if $\text{Re} = 15000$, and of 10^{16} if $\text{Re} = 40000$. Very large expansion coefficients definitely show that the corresponding evolution operator is "strongly (even enormously, if Re is relatively high) non-normal" and hence must produce very great transient growth of disturbance energy. On the other hand, rapid transient growth of small disturbances shows by itself that the corresponding evolution operator L must be strongly non-normal. The given "mathematical explanation" of the reason of the rapid transient growth of small disturbances clearly has no relation to the search for the physical mechanisms producing such growth; it only explains how these mechanisms affect the form of the evolution operator L.

Let us emphasize in conclusion that the study of the possibility of large transient growth of small disturbances is only a part of the comprehensive investigation of transition of laminar flows to turbulence. In the majority of papers considered above, most attention was given to "optimally growing disturbances" or, at least, to disturbances whose growth is "nearly-optimal". However, most often the disturbances appearing in flows encountered in real life are rather far from optimal ones. So, in the future it will be quite desirable to combine the study of transient growth with the study of the sources producing disturbances in real flows (of the flow *receptivity* to various disturbing factors in the terminology of Morkovin (1969); cf. Sect. 2.9.2 in Chap. 2) and of characteristics of the resulting disturbances.

Let us now consider briefly (omitting all technical details) some mathematical tools used in studies of growth potential for a given non-normal evolution operator L. (For proofs of the statements given below and the additional details see, *e.g.*, the books by Kato (1976, 1982), Pazy (1983), and the accounts by Reddy et al. (1993) and Trefethen (1996)). As is known, the range of possible rates for exponential growth or decay of disturbances as $t \to \infty$ is given by the spectrum Λ of the evolution operator L. In particular, the exponential growth is impossible (*i.e.*, the flow is stable according to the normal-mode stability theory) if and only if all points of Λ are in the lower half-plane of the complex-variable plane **C** (*i.e.*, have non-positive imaginary parts). The spectrum Λ can also be defined as the set of complex numbers z such that the resolvent $R_z = (zI - L)^{-1}$ of the operator L (here$I$is the identity operator) has an infinite norm. (The norm $\|A\|$ of a linear operator A in the Hilbert space H is defined as $\sup_{u \in H, ||u||=1} ||A\mathbf{u}||$ where sup denotes the maximum or, if it does not exist, the least upper bound.) However, in fact, behavior of the solutions $\boldsymbol{u}(\boldsymbol{x}, t)$ of Eq. (3.75) is not determined by the spectrum Λ of L alone but depends also on the region in complex plane **C** where the norm $\|R_z\|$ of the resolvent is "very large". Therefore it is natural to consider, parallel with the spectrum Λ of L, the greater set of complex numbers Λ_ε which includes not only Λ but also all such numbers z that $||(zI - L)^{-1}|| > \varepsilon^{-1}$ (or, what is the same, that $\|\Lambda \boldsymbol{u} - z\boldsymbol{u}\| \leq \varepsilon$ for some $\boldsymbol{u} \in H$ with $\|\boldsymbol{u}\| = 1$), where ε is a given (usually small) positive number. The set Λ_ε is called the *ε-pseudospectrum* of L; under wide conditions it can also be defined as a set of eigenvalues of all the operators of the form $L + K$ where K is a "small" operator with $\|K\| < \varepsilon$. The concept of a pseudospectrum was independently introduced in the 1970s and 1980s by a number of authors who often used different names for it, and at present the applications of this concept are very numerous and diverse. For more details about this topic, see recent surveys by Trefethen (1992, 1996) containing many examples and additional references; this author is also now writing a book on this subject.

If L is a normal operator, its pseudospectra have simple shape since here

$$||R_z|| \equiv ||(zI - L)^{-1}|| = \frac{1}{\text{dist}\{z, \Lambda\}} \tag{3.76}$$

for any $z \notin \Lambda$, where Λ is the spectrum of L and dist $\{z, \Lambda\}$ is the distance from the point z of the plane **C** to the set Λ. Therefore, in this case the ε-pseudospectrum is simply the union of disks of radius ε centered at all points of Λ. However, if L is non-normal, then the equality in (3.76) is replaced by $\geq$, and the norm of the resolvent R_z may be quite large even if the point z is far from the spectrum Λ. Therefore, the figure showing the pseudospectra of L also gives some information about its degree of non-normality, which is characterized by the excess of the depicted pseudospectra over their shapes given by Eq. (3.76).

The pseudospectra of the evolution operator L characterize its "degree of non-normality" and thus have definite relation to the transient growth possible for the solutions of Eq. (3.75). Another set in the complex plane **C** also useful for estimation of this growth is the *numerical range* of L which is the set $\Gamma$of all complex numbers which can be represented as scalar products $(L\boldsymbol{u}, \boldsymbol{u})$ where $\boldsymbol{u} \in H$ and $\|u\| = 1$. It is clear that all the eigenvalues of L are contained in Γ. Moreover, under some wide conditions (which are fulfilled for evolution operators L of steady parallel fluid flows and below will be always assumed to be satisfied) the spectrum Λ lies as a whole in the closure of the set Γ (consisting of the points of Γ and all limits of sequences of such points). The ε-pseudospectra Λ_ε also lie not far from the numerical range Γ; it may be proved that Λ_ε lies in the set $\Gamma + \Delta_\varepsilon$ formed by the union of disks of radius ε with centers in all points of Γ.

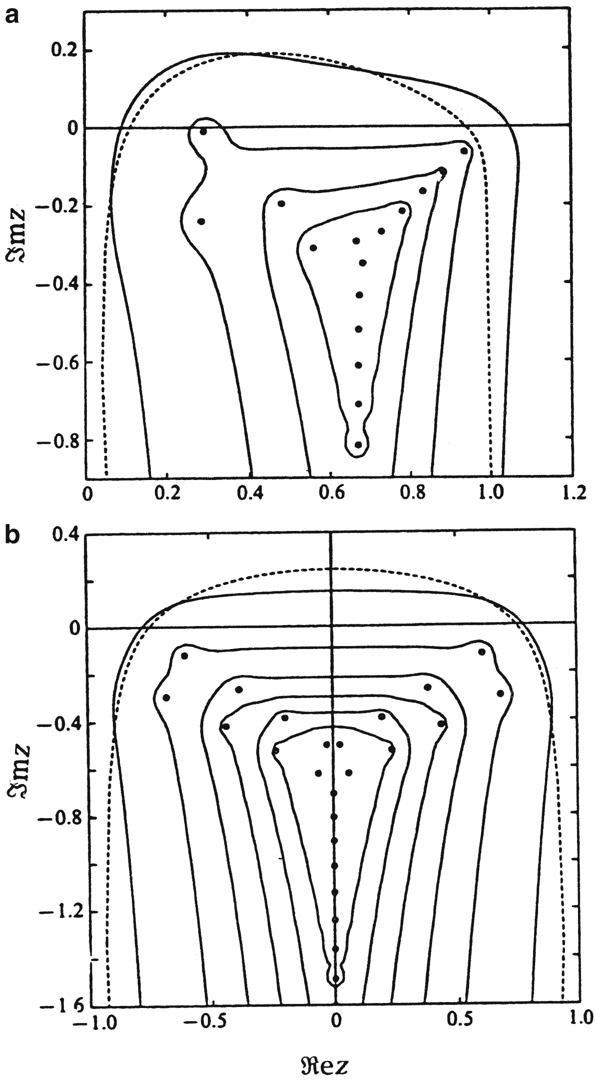


Fig. 3.22 Pseudospectra and the numerical range for operators L determining the evolution of antisymmetric wave disturbances with $k_1 = 1$ and $k_2 = 0$ in plane Poiseuille flow with Re = 3000 (**a**), and plane Couette flow with Re = 1000 (**b**) (after Reddy et al. (1993) and Reddy and Henningson (1993)). • - eigenvalues (in the case of ε-pseudospectra they are represented by discs of radius ε); *dotted lines* represent boundaries of the numerical ranges; *solid lines*, from *outer* to *inner* are the boundaries of the ε-pseudospectra for $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$ in Figure (**a**) and for $\varepsilon = 10^{-1}$, $10^{-2}, \ldots, 10^{-6}$ in Figure (**b**)

The importance of the numerical range of L for the analysis of behavior of disturbances $\boldsymbol{u}(\boldsymbol{x}, t)$ satisfying Eq. (3.75) is due to the following *Hill-Yosida theorem* (see, e.g., Pazy (1983)): $\|\boldsymbol{u}(\boldsymbol{x}, t)\| \leq \|\boldsymbol{u}(\boldsymbol{x}, 0)\|$ for all initial values $\boldsymbol{u}(\boldsymbol{x}, 0)$ and all $t \geq 0$ if and only if Γ lies in the closed lower half-plane of $\mathbf{C}$ (*i.e.*, $\Im\mathrm{m}\, z \leq 0$ for all $z \in \Gamma$). The last condition may also be formulated in terms of ε-pseudospectra; namely, it is equivalent to the condition that $\Im\mathrm{m}\, z \leq \varepsilon$ for all $z \in \Lambda_\varepsilon$ and any $\varepsilon \geq 0$. It is easy to show that the conditions given here of absence of any disturbance growth are equivalent to the condition following from the energy-balance equation (3.74).

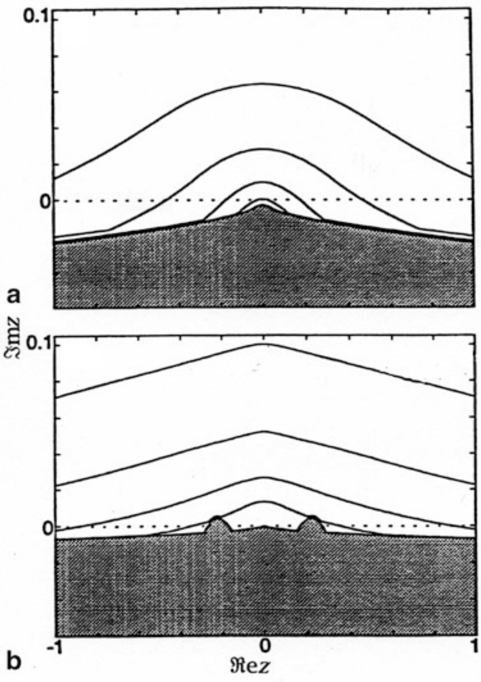


Fig. 3.23 The spectra (represented by shaded regions) and the upper boundaries of the ε-pseudospectra (the *solid lines*, from *outer* to *inner*, correspond to $\varepsilon = 10^{-2}$, $10^{-2.5}$, 10^{-3}, and $10^{-3.5}$) for plane Poiseuille flow at Re = 1000 **a**, and Re = 10000 **b**. (After Trefethen et al. (1993))

The estimation of the constant $C = \max_{\mathbf{u(x,0)},t\geq 0}[||\mathbf{u}(\mathbf{x},t)||/||\mathbf{u}(\mathbf{x},0)||]$, which determines the maximum possible growth of the disturbance energy, is more difficult than the determination of conditions guaranteeing that $C = 1$. A definite estimate can be obtained if norms $||R_z^k|| = ||(zI - L)^{-k}||$ are known for all integers $k > 0$ and all z lying in the upper half-plane of **C** (see, *e.g.,* Pazy (1983) and Reddy et al. (1993)), but this estimate is rather complicated and it will not be presented in this book. It can also be proved that if $C' = \max_{\varepsilon>0}[\max_{z\in\Lambda_\varepsilon}\Im z/\varepsilon] > 1$, then $C > C'$, but this estimate of C is not simple enough and moreover is less precise than the numerical estimates of the maximum possible growth of the disturbance energy described in Sects. 3.33 and 3.34.

Many graphs showing the shapes of computed pseudospectra Λ_ε and numerical ranges Γ for dynamic operators L corresponding to various steady parallel flows were published, together with some comments about connection of pseudospectra with transient growth of disturbances, by Trefethen et al. (1993), Reddy et al. (1993), Reddy and Henningson (1993), A. Trefethen et al. (1994), Schmid and Henningson (1994), and Trefethen (1996). Some typical examples of such graphs are presented in Figs. 3.22 and 3.23. In Fig. 3.22 the pseudospectra Λ_ε and numerical ranges Γ are shown for the evolution operators L determining the development of two-dimensional disturbances, antisymmetric with respect to the channel midplane and with dimensionless wave number $k_1 = 1$, in plane Poiseuille flow with Re = 3000 and plane Couette flow with Re = 1000. The figure makes it clear that the operator

L is here non-normal (the pseudospectra are much greater than the union of disks centered at the eigenvalues). Note that the greatest excess in pseudospectrum areas is observed near the intersection of eigenvalue branches; this makes it clear that the eigenvectors corresponding to eigenvalues in this region are especially far from being reciprocally orthogonal. In Fig. 3.23 the upper boundaries of the spectrum and several pseudospectra are depicted for plane Poiseuille flows with $\text{Re} = 1000$ and 10000. The eigenvalues $z = \omega_j$ and complex numbers z belonging to pseudospectra clearly depend on the disturbance wave numbers k_1 and k_2 ; in Fig. 3.23 they are collected for all values of wave numbers and therefore in place of discrete spectra shown in Fig. 3.22 we have here continuous spectral regions symmetric with respect to the axis $\Re e z = 0$. These regions lie wholly in the lower half-plane if $\text{Re} < \text{Re}_{\text{cr}}$, but they have bumps extended into the upper half-plane if $\text{Re} > \text{Re}_{\text{cr}}$.

Acknowledgments This work could not have been accomplished without much help from a number of individuals and institutions. Therefore I am happy to have the possibility to express here my deep gratitude to Massachusetts Institute of Technology (MIT) which gave me an office for everyday work at its Department of Aeronautics and Astronautics (Dept. Head Prof. E.F. Crawley) and to the Center for Turbulence Research (CTR) at Stanford University and NASA Ames Research Center (Director Prof. P. Moin) which provided the main part of the financial support for the work on the series and arranged the publication of its separate chapters as CTR Monographs. Substantial support for my work was also given by the U.S. Army Research Laboratory (Dr. R.E. Meyers); Cambridge Hydrodynamics, Inc.; and by Dr. J.W. Poduska, Sr.

I am particularly indebted to Prof. P. Bradshaw of Stanford University, who read the whole manuscript with great care, corrected many defects of my English and made a number of suggestions leading to the improvement of the text, and to Prof. M.T. Landahl (MIT), who also read the manuscript and made some valuable suggestions. (Of course, for thau remaining deficiencies of language, style and scientific content I take full responsibility.) Many important remarks and useful written material related to the contents of Chap. 3 were received from K.S. Breuer, F.H. Busse, W.O. Criminale, L.A. Dikii, B.F.Farrell, S. Grossmann, F. Hussain, M.V. Morkovin, S.C. Reddy, and L.N. Trefethen. Moreover, K.S. Breuer, B.F. Farrell, and S.C. Reddy also helped me in preparing figures for this chapter. Debra Spinks of CTR regularly helped me with many administrative matters and problems arising in the preparation of the written material for publication, and (Ms.) Ditter Peschcke-Koedt made many improvements to my typing. To all the persons and organizations mentioned I express my sincere thanks.

References

Arnold, V. I. (1972). Notes on the three-dimensional flow pattern of a perfect fluid in the presence of a small perturbation of the initial velocity field. *Prikladnoi Mathematik Mekh, 36,* 255–262 (Trans.: *Appl. Math. Mech., 36,* 236–242, 1972).

Baggett, J. S. & Trefethen, L. N. (1997). Low-dimensional models of subcritical transition to turbulence. *Physics of Fluids, 9,* 1043–1053.

Baggett, J. S., Driscoll, T. A. & Trefethen, L. N. (1995). A mostly linear model of transition to turbulence. *Physics of Fluids, 7,* 833–838.

Benjamin, T. B. (1961). The development of three-dimensional disturbances in an unstable film of liquid flowing down an inclined plane. *Journal of Fluid Mechanics, 10,* 401–419.

Benney, D. J. & Gustavsson, L. H. (1981). A new mechanism for linear and nonlinear hydrodynamic instability. *Studies in Applied Mathematics, 64,* 185–209.

Bergström, L. (1992). Initial algebraic growth of small angular-dependent disturbances in pipe Poiseuille flow. *Studies in Applied Mathematics, 87,* 61–79.

Bergström, L. (1993a). Optimal growth of small disturbances in pipe Poiseuille flow. *Physics of Fluids, 5,* 2710–2720

Bergström, L. (1993b). Evolution of laminar disturbances in pipe Poiseuille flow. *European Journal of Mechanics-B/Fluids, 12,* 749–768.

Bergström, L. (1995). Transient properties of a developing laminar disturbance in pipe Poiseuille flow. *European Journal of Mechanics - B/Fluids, 14,* 601–615.

Betchov, R. & Criminale, W.O. (1967). *Stability of Parallel Flows,* New York: Academic Press.

Boberg, L. & Brosa, U. (1988). Onset of turbulence in a pipe. *Zeitschrift fur Naturforschung A: Journal of Physical Sciences*, *43,* 697–726.

Bogdat'eva, N. N. & Dikii, L. A. (1973). Note on the stability of three-dimensional flows of an ideal fluid. *Prikladnoi Mathematik Mekh, 37,* 557–558 (Trans.: *Appl. Math. Mech.*, *37,* 536–537, 1973).

Booker, J. R. & Bretherton, F.P. (1967). The critical layer for internal gravity waves in a shear flow. *Journal of Fluid Mechanics, 27,* 517–539.

Breuer, K. S. & Haritonidis, J. H. (1990). The evolution of localized disturbances in a laminar boundary layer. Part 1. Weak disturbances. *Journal of Fluid Mechanics*, *220*, 569–594.

Breuer, K. S. & Kuraishi, T. (1994). Transient growth in two-and three-dimensional boundary layers. *Physics of Fluids, 6,* 1983–1993.

Brevdo, L. (1995a). Initial-boundary-value stability problem for the Blasius boundary laeyr. *Zeitschrift fur Angewandte Mathematik und Mechanik, 75,* 371–378.

Brevdo, L. (1995b). Convectively unstable wave packets in the Blasius boundary layer. *Zeitschrift fur Angewandte Mathematik und Mechanik, 75,* 423–436.

Brown, S. N. & Stewartson, K. (1980). On the algebraic decay of disturbances in a stratified linear shear flow. *Journal of Fluid Mechanics, 100,* 811–816.

Bun, Y. & Criminale, W. O. (1994). Early-period dynamics of an incompressible mixing layer. *Journal of Fluid Mechanics, 273,* 31–82.

Burger, A. P. (1962). Instability associated with the continuous spectrum in a baroclinic flow. *Journal of Atmospheric Science*, *23,* 272–277.

Burridge, D. M. & Drazin, P. G. (1969). Comments on "Stability of pipe Poiseuille flow". *Physics of Fluids, 12,* 264–265.

Busse, F. H. (1969). Bounds on the transport of mass and momentum by turbulent flow between parallel plates. *Zeitschrift fur Angewandte Mathematik und Physik, 20,* 1–14.

Butler, K. M. & Farrell, B. F. (1992). Three-dimensional optimal perturbations in viscous shear flow. *Physics of Fluids: A, 4,* 1637–1650.

Butler, K. M. & Farrell, B. F. (1993). Optimal perturbations and streak spacing in wall-bounded turbulent shear flow. *Physics of Fluids: A, 5,* 774–777.

Case, K. M. (1960a). Stability of inviscid plane Couette flow. *Physics of Fluids, 3,* 143–148.

Case, K. M. (1960b). Stability of idealized atmosphere. I. Discussion of results. *Physics of Fluids, 3,* 149–154.

Chimonas, G. (1979). Algebraic disturbances in stratified shear flows. *Journal of Fluid Mechanics, 90,* 1–19.

Chin, W. C. (1980). Effect of dissipation and dispersion on slowly-varying wavetrains. *Journal of AIAA, 18,* 149–158.

Craik, A. D. D. (1981). The development of wavepackets in unstable flows. *Proceedings of the Royal Society of London Series A, 373,* 457–476.

Craik, A. D. D. (1982). The growth of localized disturbances in unstable flows. *Journal of Applied Mechanics, 49,* 284–290.

Craik, A. D. D. & Criminale, W. O. (1986). Evolution of wavelike disturbances in shear flows: a class of exact solutions of the Navier-Stokes equations. *Proceedings of the Royal Society of London Series A*, 406, 13–26.

Criminale, W. O. (1960). Three-dimensional laminar instability, *AGARD Rep. No. 266.*

Criminale, W. O. & Drazin, P. G. (1990). The evolution of linearized perturbations of parallel flows. *Studies in Applied Mathematics, 83,* 123–157.

Criminale, W. O. & Kovasznay, L. S. G. (1962). The growth of localized disturbances in a laminar boundary layer. *Journal of Fluid Mechanics, 14,* 59–80.

Criminale, W. O. & Smith, F. I. P. (1994). Fundamental solutions in perturbed shear flows. *Fluid Dynamics Research, 13,* 167–195.

Criminale, W. O., Jackson, T. L. & Lasseigne, D. G. (1995). Towards enhancing and delaying disturbances in free shear flows. *Journal of Fluid Mechanics, 294,* 283–300.

Criminale, W. O., Jackson, T. L., Lasseigne, D. G. & Joslin, R. D. (1997). Perturbation dynamics in viscous channel flows. *Journal of Fluid Mechanics, 339,* 55–75.

Criminale, W. O., Long, B. & Zhu, M. (1991). General three-dimensional disturbances to inviscid Couette flow. *Studies in Applied Mathematics, 85,* 249–267.

Davey, A. & Reid, W. H. (1977). On the stability of stratified viscous plane Couette flow. Part 1. Constant buoyancy frequency. *Journal of Fluid Mechanics, 80,* 509–525.

Deissler, R. J. (1987). The convective nature of instability in plane Poiseuille flow. *Physics of Fluids, 30,* 2303–2305.

Diedrichs, B. (1996). Three-dimensional disturbances; considering starting profiles and optimal profiles in Couette and Poiseuille flow. *Physics of Fluids, 8,* 1149–1158.

Dikii, L. A. (1960a). Stability of plane-parallel flows of an inhomogeneous fluid. *Prikladnoi Mathematik Mekh, 24,* 249–257 (Trans.: *Appl. Math. Mech.*, *24,* 357–369, 1960).

Dikii, L. A. (1960b). Stability of plane-parallel flows of an ideal fluid. *Doklady Akademii Nauk SSSR, 135,* 1068–1071 (Trans.: *Sov. Phys. Dokl.*, *5,* 1179–1182, 1960).

Dikii, L. A. (1960c). The roots of the Whittaker functions and the Macdonald functions with a complex index. *Izvestia Akademii Nauk SSSR, ser. Matem, 24,* 943–954.

Dikii, L. A. (1976). *Hydrodynamic Stability and Dynamics of the Atmosphere*, Gidrometizdat, Leningrad (in Russian).

Di Prima, R. C. & Habetler, G. J. (1969). A completeness theorem for non-selfadjoint eigenvalue problems in hydrodynamic stability. *Archive for Rational Mechanics and Analysis, 34,* 218–227.

Drazin, P. G. & Howard, L. N. (1966). Hydrodynamic stability of parallel flows of inviscid fluid. *Advances in Applied Mechanics, 9,* 1–89.

Drazin, P. G. & Reid, W. H. (1981). *Hydrodynamic Stability*. Cambridge: Cambridge university Press.

Dunford, N. & Schwartz, J. T. (1958, 1971). *Linear Operators*. Vols. 1–3, New York: Wiley-Interscience.

Dunford, N. & Schwartz, J. T. (1971). *Linear Operators*. Vols. 1–3, New York: Wiley-Interscience.

Dyson, F. J. (1960). Stability of idealized atmosphere. II. Zeros of the confluent hypergeometric function. *Physics of Fluids, 3,* 155–158.

Easthope, P. F. & Criminale, W. O. (1992). An alternative approach to disturbances in boundary layers. In: T. B. Gatski, S. Sarkar, & C. G. Speciale (eds). *Studies in Turbulence* (pp. 582–602) New York: Springer.

Eckhaus, W. (1965). *Studies in Non-Linear Stability Theory,* Berlin: Springer.

Eliassen, A., Høiland, E. & Riis, E. (1953). Two-dimensional perturbation of a flow with constant shear of a stratified fluid. *Institute for Weather, Climate Research*, Oslo, Publ. No. 1, 1–30.

Ellingsen, T. & Palm, E. (1975). Stability of linear flow. *Physics of Fluids, 18,* 487–488.

Engevik, L. (1966). *On the Stability of Plane Inviscid Couette Flow*, Rep. No. 12, The Department of Applied Mathematics, Bergen University.

Faddeev, L. D. (1972). On the theory of the stability of stationary plane-parallel flows of an ideal fluid. *Report of Scientific Seminars Leningrad Branch Mathematical Institute, 21,* 164–172 (English Trans.: *J. Soviet Math., 1,* 518–525, 1973).

Farrell, B. F. (1982). The initial growth of disturbances in a baroclinic flow. *Journal of Atmospheric Science, 39,* 1663–1686.

Farrell, B. F. (1987). Developing disturbances in shear. *Journal of Atmospheric Sciences, 44,* 2191–2199.

Farrell, B. F. (1988a). Optimal excitation of perturbations in viscous shear flow. *Physics of Fluids, 31,* 2039–2102.

Farrell, B. F. (1988b). Optimal excitation of neutral Rossby waves. *Journal of Atmospheric Science, 45,* 163–172.

Farrell, B. F. (1989). Optimal excitation of baroclinic waves. *Journal of Atmospheric Science, 46,* 1194–1206.

Farrell, B. F. & Ioannou, P. J. (1993a). Transient development of perturbations in stratified shear flow. *Journal of Atmospheric Science, 50,* 2201–2214.

Farrell, B. F. & Ioannou, P. J. (1993b). Optimal excitation of three-dimensional perturbations in viscous constant shear flow. *Physics of Fluids, 5,* 1390–1400.

Farrell, B. F. & Ioannou, P. J. (1993c). Perturbation growth in shear flow exhibits universality. *Physics of Fluids, 5,* 2298–2300.

Farrell, B. F. & Moore, A. M. (1992). An adjoint method of obtaining the most rapidly growing perturbation to oceanic flows. *Journal of Physical Oceanography, 22,* 318–349.

Fasel, H. F., Rist, U. & Konzelmann, U. (1987). Numerical investigation of the three-dimensional development in boundary-layer transition. *AIAA Paper No. 87–1203*.

Gaster, M. (1965). On the generation of spatially growing waves in a boundary layer. *Journal of Fluid Mechanics, 22,* 433–441.

Gaster, M. (1968a). The development of three-dimensional wave packets in a boundary layer. *Journal of Fluid Mechanics, 32,* 173–184.

Gaster, M. (1968b). Growth of disturbances in both space and time. *Physics of Fluids, 11,* 723–727.

Gaster, M. (1975). A theoretical model of a wave packet in the boundary layer on a flat plate. *Proceedings of the Royal Society of London Series A, 347,* 271–289.

Gaster, M. (1979). The propagation of linear wave-packets in laminar boundary layers—asymptotic theory for nonconservative wave systems. *AIAA Paper No.* 79–1492.

Gaster, M. (1981). Propagation of linear wave-packets in laminar boundary layers. *AIAA Journal, 19,* 419–423.

Gaster, M. (1982a). Estimates of the errors incurred in various asymptotic representations of wave packets. *Journal of Fluid Mechanics, 121,* 365–377.

Gaster, M. (1982b). The development of a two-dimensional wavepacket in a growing boundary layer. *Proceedings of the Royal Society of London Series A, 384,* 317–332.

Gaster, M. & Davey, A. (1968). The development of three-dimensional wave-packets in unbounded parallel flows. *Journal of Fluid Mechanics, 32,* 801–808.

Gaster, M. & Grant, I. (1975). An experimental investigation of the formation and development of a wave packet in a laminar boundary layer. *Proceedings of the Royal Society of London Series A, 347,* 253–269.

Gaster, M. & Jordinson, R. (1975). On the eigenvalues of the Orr-Sommerfeld equation. *Journal of Fluid Mechanics, 72,* 121–133.

Gebhardt, T. & Grossmann, S. (1994). Chaos transition despite linear stability. *Physical Review E, 50,* 3705–3711.

Gel'fand, I. M. & Shilov, G.E. (1958). *Generalized Functions. Vol. 3. Theory of Differential Equations,* Moscow: Gostekhizdat (Trans.: Academic Press, New York, 1967).

Goldstein, S. (1931). On the stability of superimposed streams of fluids of different densities. *Proceedings of the Royal Society of London Series A, 132,* 524–548.

Grossmann, S. (1995). Wie entsteht eigentlich Turbulenz? *Physikalische Blätter*, *51*, 641–646.

Grossmann, S. (1996). Instability without instability? In: J. Parisi, S. C. Müller, & W. Zimmermann, (eds.). *Nonlinear Physics of Complex Systems. Current Status and Future Trends*, (pp. 10–22), Berlin: Springer.

Gustavsson, L. H. (1978). On the evolution of disturbances in boundary layer flows, *Tech. Rep. TRITA-MEK 78–02, Royal Institute of Technology*, Stockholm.

Gustavsson, L. H. (1979). Initial-value problem for boundary layer flows. *Physics of Fluids, 22,* 1602–1605.

Gustavsson, L. H. (1981). Resonant growth of three-dimensional disturbances in plane Poiseuille flow. *Journal of Fluid Mechanics, 112,* 253–264.

Gustavsson, L. H. (1986). Excitation of direct resonances in plane Poiseuille flow. *Studies in Applied Mathematics, 75,* 227–248.

Gustavsson, L. H. (1989). Direct resonance of nonaxisymmetric disturbances in pipe flow. *Studies in Applied Mathematics, 80,* 95–108.

Gustavsson, L. H. (1991). Energy growth of three-dimensional disturbances in plane Poiseuille flow. *Journal of Fluid Mechanics, 224,* 241–260.

Gustavsson, L. H. & Hultgren, L. S. (1980). A resonance mechanism in plane Couette flow. *Journal of Fluid Mechanics, 98,* 149–159.

Hagen, G. (1839). Über die Bewegung des Wassers in engen zylindrischen Röhren. *Pogg. Annales, 46,* 423–442.

Hartman, R. J. (1975). Wave propagation in a stratified shear flow. *Journal of Fluid Mechanics, 71,* 89–104.

Henningson, D. S. (1988). The inviscid initial value problem for a piecewise linear mean flow. *Studies in Applied Mathematics, 78,* 31–56.

Henningson, D. S. (1991). An eigenfunction expansion of localized disturbances. In: A.V. Johansson and P. H. Alfredsson, (eds). *Advances in Turbulence* 3, (pp. 162–169), Berlin: Springer.

Henningson, D. S. (1995). Bypass transition and linear growth mechanism. In: R. Benzi (ed) *Advances in Turbulence,* 5, (pp. 190–204), Kluwer, Dordrecht.

Henningson, D. S. (1996). Comments on "Transition in shear flows. Nonlinear normality versus non-normal linearity". *Physics of Fluids, 8,* 2257–2258.

Henningson, D. S. & Alfredsson, P. H. (1996). Stability and transition, Chap. 2 In: M. Hallbäck, D. S. Henningson, A. V. Johansson, & P. H. Alfredsson, (eds). *Turbulence and Transition Modelling*, (pp. 13–80), Dordrecht: Kluwer.

Henningson, D. S. & Reddy, S. C. (1994). On the role of linear mechanisms in transition to turbulence. *Physics of Fluids, 6,* 1396–1398.

Henningson, D. S. & Schmid, P. J. (1992). Vector eigenfunction expansions for plane channel flows. *Studies in Applied Mathematics, 87,* 15–43.

Henningson, D. S. & Schmid, P. J. (1994). A note on measures of disturbance size in spatially evolving flows. *Physics of Fluids, 6,* 2862–2864.

Henningson, D. S., Gustavsson, L. H. & Breuer, K. S. (1994). Localized disturbances in parallel shear flows. *Applied Scientific Research, 53,* 51–97.

Henningson, D. S., Johansson, A. V. & Lundbladh, A. (1990). On the evolution of localized disturbances in laminar shear flows. In: D. Arnal & R. Michel (eds). *Turbulent-Laminar Transitions* (Proc. 3rd Int. Symp.,), (pp. 279–284) Berlin: Springer.

Henningson, D. S., Lundbladh, A. & Johansson, A. V. (1993). A mechanism for bypass transition from localized disturbances in wall-bounded shear flows. *Journal of Fluid Mechanics*, *250,* 169–207.

Herron, I. H. (1980). A completeness observation on the stability equations for stratified viscous shear flows. *Physics of Fluids, 23,* 836–837.

Herron, I. H. (1982). A completeness theorem for the linear stability of nearly parallel flows. *Journal of Mathematics Analysis and Applications, 86,* 168–175.

Herron, I. H. (1983). Expansion problems in the linear stability of boundary layer flows. *Advances in Applied Mathematics, 4,* 260–297.

Howard, L. N. (1961). Note on a paper of John W. Miles. *Journal of Fluid Mechanics, 10,* 509–512.

Hultgren, L. H., & Gustavsson, L. H. (1981). Algebraic growth of disturbances in a laminar boundary layer. *Physics of Fluids, 24,* 1000–1004.

Jang, P. S., Benney, D. J., & Gran, R.L. (1986). On the origin of streamwise vortices in a turbulent boundary layer. *Journal of Fluid Mechanics, 169,* 109–123.

Jiang, F. (1991). Asymptotic evaluation of three-dimensional wave packets in parallel flows. *Journal of Fluid Mechanics, 226,* 573–590.

Jimenez, J., & Whitham, G. B. (1976). An averaged Lagrangian method for dissipative wavetrains. *Proceedings of the Royal Society of London Series A, 349,* 277–287.

Jones, C. A. (1988). Multiple eigenvalues and mode classification in plane Poiseuille flow. *Quarterly Journal of Mechanics and Applied Mathematics, 41,* 363–382.

Joseph, D. D. (1966). Nonlinear stability of the Boussinesq equation by the method of energy. *Archive for Rational Mechanics and Analysis, 22,* 163–184.

Joseph, D. D. (1976). *Stability of Fluid Motions,* Vols. 1–2, Berlin: Springer.

Joseph, D. D. & Carmi, S. (1969). Stability of Poiseuille flow in pipes, annuli and channels. *Quarterly of Applied Mathematics, 26,* 575–599.

Kato, T. (1976). *Perturbation Theory for Linear Operators*. Berlin: Springer.

Kato, T. (1982). *A Short Introduction to Perturbation Theory of Linear Operators*. Berlin: Springer.

Kelvin, Lord (Thomson, W.) (1871). Hydrokinetic solutions and observations. *Philosophical Magazine*, (*4*)*, 42,* 362–377.

Kelvin, Lord (Thomson, W.) (1887a). Rectilinear motion of viscous fluid between two parallel planes. *Philosophical Magazine,* (*5*) *24,* 188–196.

Kelvin, Lord (Thomson, W.) (1887b). Broad river flowing down an inclined plane bed. *Philosophical Magazine*, (*5*) *24,* 272–278 (also in *Mathematical and Physical Papers*, Vol. IV, pp. 69–85, 321–330, and 330–337, Cambr. Univ. Press, 1910).

Kline, S. J., Reynolds, W.C., Schraub, F.A. & Runstadler, P.W. (1967). The structure of turbulent boundary layers. *Journal of Fluid Mechanics, 30,* 741–773.

Klingmann, B. G. B. (1992). On transition due to three-dimensional disturbances in plane Poiseuille flow. *Journal of Fluid Mechanics, 240,* 167–195.

Klingmann, B. G. B. & Alfredsson, P. H. (1991). Experiments on the evolution of a point-like disturbance in plane turbulent flow into a turbulent spot. In: A. V. Johansson & P. H. Alfredsson, (eds.) *Advances in Turbulence 3,* (pp. 182–188). Berlin: Springer.

Koch, W. (1986). Direct resonances in Orr-Sommerfeld problems. *Acta Mechanica, 59,* 11–29.

Konzelmann, U. (1990). *Numerische Entwicklung dreidimensionaler Wellenpakete in einer Plattengrenzschichtströmung,* Dr.-Ing. Dissertation, Institut A für Mechanik, Germany: Universität Stuttgart.

Kuo, H. L. (1963). Perturbations of plane Couette flow in stratified fluid and origin of cloud streets. *Physics of Fluids, 6,* 195–211.

Landahl, M. T. (1972). Wave mechanics of breakdown. *Journal of Fluid Mechanics, 56,* 775–802.

Landahl, M. T. (1975). Wave breakdown and turbulence. *SIAM Journal on Applied Mathematics, 28,* 735–756.

Landahl, M. T. (1980). A note on an algebraic instability of inviscid parallel shear flows. *Journal of Fluid Mechanics, 98,* 243–251.

Landahl, M. T. (1982). The application of kinematic wave theory to wave trains and packets with small dissipation. *Physics of Fluids, 25,* 1512–1516.

Landahl, M. T. (1985). The growth of instability waves in a slightly nonuniform medium. In: V.V. Kozlov, (ed) *Laminar-Turbulent Transition* (Proc. 2nd Int. Symp.), (pp. 133–140). Berlin: Springer.

Landahl, M. T. (1990). On sublayer streaks. *Journal of Fluid Mechanics, 212,* 593–614.

Landahl, M. T. (1993a). Model for the wall-layer structure of a turbulent shear flow. *European Journal of Mechanics - B/Fluids, 12,* 85–96.

Landahl, M. T. (1993b). Role of algebraic instability in transition and turbulence. In: Th. Dracos & A. Tsinober, (eds.) *New Approaches and Concepts in Turbulence*, (pp. 251–263). Basel: Birkhäuser.

Landahl, M. T. (1996). Role of non-wave-like disturbances in transition, In: J. Grue, B. Gjevik & J. E. Weber, (eds.) *Waves and Nonlinear Processes in Hydrodynamics*, (pp. 255–266). Dordrecht: Kluwer.

Landahl, M. T. (1997). private communication.

Landahl, M. T. & Mollo-Christensen, E. (1992). *Turbulence and Random Processes in Fluid Mechanics,* 2nd edition, Cambridge: Cambridge University Press.

Landau, L. D. & Lifshitz, E. M. (1958). *Fluid Mechanics*. London: Pergamon Press (1st English edition).

Landau, L. D. & Lifshitz, E. M. (1987). *Fluid Mechanics*. London: Pergamon Press (revised English edition).

Lenz, K. (1986). *Numerische Simulation des Ausbreitung von Wellenpaketen in einer laminaren Grenzschichtströmung*, Dissertation. Universität Stuttgart, Germany.
Luchini, P. (1996). Reynolds-number-independent instability of the boundary layer over a flat surface. *Journal of Fluid Mechanics, 327,* 101–115.
Lundbladh, A. (1993). Growth of a weak localized disturbance in inviscidly stable shear flow, paper 4 in: Lundbladh, A., *Simulation of Bypass Transition to Turbulence in Wall Bounded Shear Flows* (Doctoral Thesis), pp. 1–21, *TRITA-MEK Tech. Rep. 1993:8,* Sweden: Royal Inst. Technol., Stockholm.
Lundbladh, A., Schmid, P. J., Berlin, S. & Henningson, D. S. (1994). Simulation of bypass transition for spatially evolving disturbances. *AGARD Conference Proceedings*, CP 551, pp. 18.1–18.13.
Marcus, P. S. & Press, W. H. (1977). On Green's function for small disturbances of plane Couette flow. *Journal of Fluid Mechanics, 79,* 525–534.
Miles, J. W. (1958). On the distorted motion of a plane vortex sheet. *Journal of Fluid Mechanics, 4,* 538–552.
Moffatt, H. K. (1967). The interaction of turbulence with strong wind shear, In: A.M. Yaglom and V.I. Tatarsky, (eds.) *Atmospheric Turbulence and Radio Wave Propagation*, (pp. 139–156), Moscow: Nauka.
Monin, A. S. & Yaglom, A. M. (1971). *Statistical Fluid Mechanics, the Mechanics of Turbulence,* Vol. 1, MIT Press, Cambridge, Mass.
Morkovin, M. V. (1969). On the many faces of transition. In: C.S. Wells (ed.) *Viscous Drag Reduction* (pp. 1–31). London: Plenum.
Orr, W. M. F. (1907). The stability of instability of the steady motions of a perfect liquid and of a viscous liquid, Parts I and II. *Proceedings of the Royal Irish Academy, A27,* 9–68 and 69–138.
O'Sullivan, P. L. & Breuer, K. S. (1994). Transient growth in circular pipe flow. I. Linear disturbances. *Physics of Fluids, 6,* 3643–3651.
Panton, R. L. (1996). *Incompressible Flow*, 2nd edition New York: Wiley.
Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Berlin: Springer.
Pedlosky, J. (1964). An initial value problem in the theory of baroclinic instability. *Tellus, 16,* 12–17.
Phillips, O. M. (1966). *The Dynamics of the Upper Ocean*, 1st edition. Cambridge: Cambr. Univ. Press.
Prandtl, L. (1925). Bericht über Untersuchungen zur ausgebildeten Turbulenz. *Zeitschrift fur Angewandte Mathematik und Mechanik, 5*, 136–139.
Rayleigh, Lord (1880). On the stability, or instability, of certain fluid motions, Parts I–III. *Proceedings of the London Mathematics Society, 11,* 57–70.
Rayleigh, Lord (1887). On the stability, or instability, of certain fluid motions, Parts I–III. *Proceedings of the London Mathematics Society, 19,* 67–74.
Rayleigh, Lord (1895). On the stability, or instability, of certain fluid motions, Parts I–III. *Proceedings of the London Mathematics Society, 27,* 5–12.
Rayleigh, Lord (1892). On the question of the stability of the flow of fluids. *Philosophical Magazine*, (*5*) *34,* 59–70 (also in *Scientific Papers*, Vols. I–VI, Cambr. Univ. Press, 1899–1920).
Rayleigh, Lord (1894). *The Theory of Sound,* Vols. I–II, 2nd ed. London: Macmillan.
Reddy, S. C. & Henningson, D. S. (1993). Energy growth in viscous channel flows. *Journal of Fluid Mechanics, 252,* 209–238.
Reddy, S. C, Schmid, P. J. & Henningson, D. S. (1993). Pseudospectra of the Orr-Sommerfeld operator. *SIAM Journal on Applied Mathematics, 53,* 15–47.
Reynolds, O. (1883). An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and the law of resistance in parallel channels. *Philosophical Transactions of the Royal Society, 174,* 935–982 (also in *Scientific Papers*, Vol. II, pp. 51–105, Cambr. Univ. Press, 1901).
Reynolds, O. (1894). On the dynamical theory of incompressible viscous fluids and the determination of the criterion. *Philosophical Transactions of the Royal Society, 186,* 123–161 (also Scientific Papers, Vol. II, pp. 535–577).

Rosen, G. (1971). General solution for perturbed plane Couette flow. *Physics of Fluids, 14,* 2767–2769.

Rudin, W. (1973). *Functional Analysis,* New York: McGraw-Hill.

Russell, J. M. & Landahl, M. T. (1984). The evolution of a flat eddy near a wall in an inviscid shear flow. *Physics of Fluids, 27,* 557–570.

Schensted, I. V. (1960). *Contribution to the Theory of Hydrodynamic Stability*, Ph.D. Thesis, University of Michigan, Ann Arbor.

Schmid, P. J. & Henningson, D. S. (1994). Optimal energy growth in Hagen-Poiseuille flow. *Journal of Fluid Mechanics, 271,* 197–225.

Schmid, P. J. & Henningson, D. S. (2001). *Stability and Transition in Shear Flows*. Series Applied Mathematical Sciences, vol. 142. Springer.

Schmid, P. J., Lundbladh, A. & Henningson, D. S. (1994). Spatial evolution of disturbances in plane Poiseuille flow. In: M. Y. Hussaini, T. B. Gatski & T. L. Jackson, (eds.) *Transition, Turbulence and Combustion,* Vol. I (pp. 287–297). Dordrecht: Kluwer.

Shanthini, R. (1989). Degeneracies of the temporal Orr-Sommerfeld eigenmodes in plane Poiseuille flow. *Journal of Fluid Mechanics, 201,* 13–34.

Shepherd, T. G. (1985). Time development of small disturbances to plane Couette flow. *Journal of Atmospheric Sciences, 42,* 1868–1871.

Tam, K. W. (1967). A note on disturbances in slightly supercritical plane Poiseuille flow. *Journal of Fluid Mechanics, 30,* 17–20.

Taylor, G. I. (1931). Effect of variation in density on the stability of superposed streams of fluid, *Proceedings of the Royal Society of London Series A, 132,* 499–523 (also in *Scientific Papers*, Vol. II, 219–239, Cambr. Univ. Press, 1960).

Trefethen, A. E., Trefethen, L. N. & Schmid, P. J. (1994). Spectra and pseudospectra for pipe Poiseuille flow, Rep. No. 94–16, Interdisciplinary Project Center for Supercomputing, ETH, Zürich.

Trefethen, L. N. (1992). Pseudospectra of matrices. In: D.F. Griffiths & G.A. Watson, (eds.) *Numerical Analysis 1991*, (pp. 234–266) Harlow: Longman (Essex, UK).

Trefethen, L. N. (1996). Pseudospectra of linear operators. In: K. Kirchgässner, O. Mahrenholtz & R. Mennicken, (eds.) *ICIAM 95* (Proc. 3rd. Int. Congr. Industr. Appl. Math.,) pp. 401–434, Berlin: Akademie Verlag.

Trefethen, L. N., Trefethen, A. E., Reddy, S. C. & Driscoll, T. A. (1993). Hydrodynamic stability without eigenvalues. *Science, 261,* 578–584.

Tung, K. K. (1983). Initial-value problems for Rossby waves in a shear flow with critical level. *Journal of Fluid Mechanics, 133,* 443–469.

Watson, J. (1960). Three-dimensional disturbances in flow between parallel planes. *Proceedings of the Royal Society of London Series A, 254,* 562–569.

Whitham, G. B. (1965). A general approach to linear and non-linear dispersive waves using a Lagrangian. *Journal of Fluid Mechanics, 22,* 273–283

Whitham, G. B. (1974). *Linear and Nonlinear Waves,* New York: Wiley-Interscience.

Willke, H. L. (1972). A distribution-theoretic approach to a stability problem of Orr. *Studies in Applied Mathematics, 51,* 85–98.

Yamagata, T. (1976a). On the propagation of Rossby waves in a weak shear flow. *Journal of Meteoritics Society Japan, 54,* 126–128.

Yamagata, T. (1976b). On trajectories of Rossby wave-packets released in a lateral shear flow. *Journal of the Oceanographical Society of Japan, 32,* 162–168.

Yudovich, V. I. (1965). Stability of steady flows of viscous incompressible fluids. *Doklady Akademii Nauk SSSR, 161,* 1037–1040 (Trans. in *Soviet Phys. Doklady. 10,* 293–295).

Yudovich, V. I. (1984). *The Linearization Method in Hydrodynamic Stability Theory,* Izd. Rostov Gosud. Univ., Rostov-on-Don (Trans. publ. by Amer. Math. Soc., Providence, R.I., 1989).