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Becoming a Mathematician

An international perspective

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Chapter 1
Introduction: How Does a Person Become a Mathematician?

What This Book Is About

People understand and experience the world in different ways. Some people believe that love, family and friendship are the most important aims in life, others would argue that that art, music and culture are most important, and yet others believe that science or football are critical for their existence. Mathematics pervades human experience and makes an impact in nearly all spheres of life. For some people, mathematics is only a passing inconvenience and they may pay very little attention to it at all. For others – and these are the people this book is about and for – mathematics, mathematical thinking and mathematical practices play a vital role in their personal and professional lives. This book is about how a person, often a student, goes through the process of becoming a mathematician, and how he or she comes to think of themselves as a mathematician. For some people, their identity is, or can be, aligned with the ideas and approaches of mathematics in an essential way; but this is not true of everyone. We explore this idea of mathematical identity from the perspective of students of mathematics, with an aim of learning from them and identifying the pedagogical approaches that foster a mathematical identity.

Over the course of more than a decade, with a number of colleagues in university education, we have grappled with various aspects of mathematics education. We have investigated our students' learning and our own pedagogical approaches, the curriculum we design and the mathematics that we present, the utility of mathematical knowledge and processes for learners' careers, and the concept of mathematical identity and what this implies for the process of becoming a mathematician. In the course of our investigations we have discovered (as many others have before us) that there are no simple solutions to the problems of how to help students to learn mathematics and to develop a mathematical identity. In this book we describe some of our approaches to these problems and some of the results we have obtained. And a key feature of these approaches and results is that they are firmly based on the experiences of mathematics learners, related to us in interviews and surveys in their own

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words, and included in most of our chapters in the form of verbatim quotes. In this way, our book also includes the voices of learners of mathematics and recent graduates from degrees in the mathematical sciences.

Why Is Mathematics Important?

A recent report on mathematics and statistics in Australia talks of *"the critical nature of mathematical sciences"*:

> The mathematical sciences are fundamental to the well-being of all nations. They drive the data analysis, forecasting, modelling, decision-making, management, design and technological principles that underpin nearly every sector of modern enterprise. Mathematics is the pre-eminent 'enabling science' that empowers research, development and innovation in business and industry, science and technology, national security and public health. Enabling sciences are, by their nature, often invisible to the wider community but without them, modern societies would cease to function. (Australian Academy of Science 2006, p. 18)

There is a fairly clear, if utilitarian, message here – and one that applies equally to other countries and will remain relevant well into the future. But we can compare this with an earlier description of the important aspects of mathematics, written by the mathematician G.H. Hardy a lifetime ago:

> A mathematician, like a painter or a poet, is a maker of patterns. If his [sic] patterns are more permanent that theirs, it is because they are made with *ideas*. A painter makes patterns with shapes and colours, a poet with words. … The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. (Hardy 1940, section 10, pp. 13–14)

Our experience and use of mathematics in the early twenty-first century could be seen, in some respects, to be quite different from the situation in previous times. In the contemporary era of mass education, developing technologies, rapid communications and knowledge work, mathematics may be benefiting from a broader appreciation of its utility – and maybe also from its beauty, though it is likely that only a minority of people would have appreciated Hardy's view even when it was written. A look at the declining numbers of school and university students choosing to study mathematics – in many other countries as well as Australia – suggests that for many people mathematics is not a glorious and fascinating subject. Nevertheless, most people will agree that mathematics lies at the heart of contemporary communication technologies, it plays an essential role in the world of finance, and it is indispensable in modern medicine. Exponents of mathematics would say that it lies at the very heart of oneself; and this is the point that causes educators the most concern. Despite the critical nature of mathematics, very few people consider mathematics to be central to their own identity. More than a decade of research with students has shown us that very few learners come to understand mathematics in this intimate way, and the same research shows that those who *don't* often have a very hard time trying to work out how mathematics could be used for anything!

This brings us to a conundrum. Just why is mathematics important, to whom and for what? And how could we communicate this importance to learners of mathematics? A brief examination can illuminate some cultural attitudes towards mathematics. Firstly, populations usually only have access to the forms of knowledge about mathematics that are immediately applicable to their situations. For instance, early humans used physical objects to represent mathematical thinking and communication. A collection of rocks in a bowl could communicate how many sheep were in a field, or how many of them were to be exchanged for something else. Early literate societies could represent numbers with objects that were close to hand, such as stones, bones, ceramic pieces, notches in wood or knots in strings. The use of the phrase 'early literate' is deliberate in this sense and pertains to a form of mathematical literacy that is both conceptual and practical. The material of mathematics could include fingers or other body parts, attached to a single person or in groups of many persons, or the materials could be manufactured specifically to represent a concept of number.

Progressions in mathematical literacy are often generated by practical need, and each cultural group may respond in quite different ways to a similar need. In the Inca empire, financial and demographic information was recorded in the knotted strings called khipu, prepared by the expert 'khipukamayuqs' (see http://khipukamayuq.fas.harvard.edu) – maybe the original statisticians, and hence mathematical scientists. In fifteenth-century Europe, one focus of mathematics was bookkeeping, using the talents of some clever 'computers', people who were able to carry out arithmetical operations. The mathematician Luca Pacioli wrote a short treatise on the subject in 1494, introducing symbols for plus and minus. Our contemporary professions of accountant and actuary continue this line, keeping track of economic and financial transactions and the associated risks. The 'Songlines' of the Australian Aboriginals allowed them to navigate over large distances by referring to the words of the song, describing the geometry of the land by referring to locations of various landmarks. From the fifteenth century onwards, to support European voyages of exploration, mathematicians devoted much effort to the complex construction of tables which would help people to determine their location on the surface of a sphere; the problem was not completely solved until John Harrison completed the first true chronometer in 1735. Nowadays, GPS or Global Positioning System – a satellite-based navigation system – enables users to easily obtain their location anywhere on Earth, and Google Earth allows anyone with access to a computer to 'visit' any spot with equal ease.

As these examples indicate, mathematics seems to have a habit of infiltrating human activity, maybe in a form that is recognisably mathematical, but maybe just sitting behind the practical face of a pressing problem. In his communication of the architectural needs for the Colonia Güell chapel, the Spanish Catalan architect Antoni Gaudí utilised 'funicular models' (see Collins 1963, or the photos at http://www.gaudidesigner.com/uk/colonia-guell-construction.html) of dome stresses using string, shotgun pellets and gravity. Basically the dome of the chapel was hung upside down using string to 'draw' the catenary-shaped lines that were then followed by the masons. This simple idea was then used for the much-larger Sagrada Familia cathedral where the spaces were not regular. This architectural design approach was revolutionary and an example of implicit mathematical thinking (though the mathematical basis was very clear). Nevertheless, Gaudí's techniques were based on traditional Catalan tile-vaulting methods that stretched back to Roman times, or even Mesopotamian brick vaults, and they anticipated late-twentieth-century shell-vaulted methods of architecture. The results seem to combine the functional and aesthetic aspects of mathematics that are exemplified in the quotes earlier in this section.

It seems that people have always had a relationship with mathematics, and that the relationship is built on culture, interest and opportunity. One such opportunity arose in Europe in the nineteenth century when compulsory primary education resulted in a population that was literate. This led to a demand for different sorts of reading material: novels, newspapers, cartoons, letters, pamphlets and so on. In amongst these new texts, entertaining mathematics started to crop up. The logical puzzles of Lewis Carroll and Henry Ernest Dudeney, the mathematical games of Martin Gardiner in *Scientific American*, the statistical diversions of Petocz and Sowey (e.g., 2011) in *Teaching Statistics*, numerous mathematical competitions and Olympiads, and the contemporary love affair with Sudoku are all examples of this of this form of opportunity. By now we have come to an idea that mathematics can be in some way entertaining and enjoyable, as well as useful and beautiful.

In stark contrast to these examples of the importance of simple mathematics for everyday living, there have always been a few exceptional mathematicians who have been able to imagine, extend, speculate, experiment, deduce and create new forms of mathematics just for the sake of it – though it so often happens that applications are found even for such mathematics. G. H. Hardy thought he was on secure ground when he wrote: *"No one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years"* (Hardy 1940, section 28, p. 44). However, the development of cryptography after the second World War made extensive use of number theory, and contemporary public-key systems depend on Hardy's favourite branch of mathematics. All electronic financial interactions are now based on strong security measures that are purely mathematical, essentially the multiplication of extremely long primes. Most ordinary people simply accept that there must be some tricky maths going on somehow, but trust the system to work (in the same way that they would expect their car to work). Yet the designers of the systems are entranced by developing even more complex mathematical models to deal with security protocols and breaches. This in turn delights the mathematically inclined hackers who see it as a wonderful puzzle to solve that also destabilizes established hegemonies. This latter form could be seen as an instance of 'malevolent creativity', but perhaps it could also be seen as an emancipatory act.

Apple (1992) suggests that there are two forms of mathematical literacy, the functional and the critical. Functional mathematical literacy pertains to the applications of mathematics required in practical contexts, while critical mathematical literacy has the capacity to change the ways of thinking and even challenge society. The examples above show many instances of the functional, but also of the critical. In some senses, even the functional have the ability to change society because the job itself has a social effect. Take, for instance, the bookkeeper of eighteenth-century industrial Europe, a person who had an individual capacity for numeracy. However, today's accountancy profession embeds analysis of elements of financial risk, ethics, quality, financial growth and communication. Essentially, the bookkeeper's job has led to a change in society's understanding of risk and insurance. An individual can also develop a critical personal understanding of mathematics that challenges society. Compiling data provided by the United Nations, Hans Rosling created the online resource *Gapminder* (see http://www.gapminder.org). This investigative tool enables people to explore the world and 'mind the gap' between poverty and quality-of-life indicators for all nations. In this way, we can see the impact of a single mind on society, but also the intention to change society and individuals within it through the power of statistical graphics.

The potential of the mathematical sciences to raise awareness of social problems and maybe even change society for the better is another valid reason for the importance of mathematics. The Brazilian mathematics educator Ubiritan D'Ambrosio introduced the notion of 'ethnomathematics' to describe and legitimise the mathematics practiced among identifiable cultural groups that is often distinct from mainstream mathematics. As an exponent of 'critical mathematics education', he summarises his broad vision of the role of mathematics:

> No one will deny that the *most universal problem* is survival with dignity. Many people claim that mathematics is the *most universal mode of thought*. I believe that to find the relation between these two universals is an inescapable result of the claim of the universality of mathematics. (D'Ambrosio 1998, p. 68, original italics)

Much has been written on gender and the dominance of male views in mathematics (for example, the ICMI study edited by Hanna 2002), and also about the role of power and equity in mathematical endeavour (for example, the collection edited by Valero and Zevenbergen 2004). In some instances, mathematics can act as a form of liberation in which individuals or groups of people are able to transcend social expectations and excel in the discipline. This has been true of males for many years, and the past 50 years has allowed women to move into mathematics in increasing numbers. However, there are still barriers to becoming a mathematician due to gender and socioeconomic status and mathematics is poorer for it. Increasing the opportunities for talented mathematics students to become professional mathematicians will enhance mathematics itself.

This leaves us with several questions: Is it possible for people to move from one way of thinking about mathematics to another? Is such movement always in a positive direction? How can students move from a functional towards a critical approach to mathematical literacy? Is understanding of mathematics tied to potential job opportunities, or is it only possible through an interest in the mathematics itself? How can students of the discipline develop identity as mathematicians?

How People Come to Work with Mathematics

In the early twenty-first century, the majority of people in the ‘developed’ world now live in situations where technology is ubiquitous and mathematics forms a central component of a wide range of jobs and entertainments. It seems that most students entering formal higher education expect to be working in the area in which they studied (Reid et al. 2011). In Australia, where the authors of this book reside, the government’s strategic goal for education is that by 2035 at least 40% of the population should have a tertiary degree, a recognition that higher education is needed for twenty-first-century life. There is an awareness that ‘knowledge workers’ are required in every field, and this knowledge work includes sophisticated understanding of mathematics for a whole range of professions. We firmly believe that the mathematical sciences are important to society, including industries, governments and individuals. Mathematics in all its guises of quantitative skills, analytic skills, logical thinking, quantitative problem solving and specific content, is a core capability for the majority of graduates, and high-level capability in mathematics is important for a smaller number of graduates.

The three international authors of the report *Mathematics and Statistics: Critical skills for Australia’s future* explain the same point in this way:

> Mathematics and statistics permeate the complex fabric of developed societies, and mathematicians distinctively shape its texture. At once a domain of knowledge itself and the quantitative language for other fields, mathematics constantly evolves through research driven by the interplay of internal and external questions. As but one example, mathematicians give form and voice to ideas that help structure and deploy the flood of information transforming all of commerce, technology, medicine and daily life. The ever-expanding role of the mathematical sciences, and the strategic need for support of the associated research and training work, has been well recognised in our home countries [France, United States] and in other vibrant and developing economies. (Australian Academy of Science 2006, p. 1)

Comparable reviews of mathematics in the UK and Canada find that mathematics is critical to economic prosperity and that it seems to be under-utilised in industry (Hoyles et al. 2002; Fields Institute Annual Report 2005). Rubinstein (2006), in an article entitled *The crisis in maths in Australia*, also describes the importance of mathematical ideas to industry, adding that:

> The paradox is that although there is no ‘mathematical industry’ similar to the chemical industry or earth sciences in mining, it is also true that there is no non-mathematical industry. Every area requires or can benefit from mathematicians and statisticians to increase efficiency.

This illustrates a difficulty with the mathematical sciences. As ‘enabling sciences’, they can contribute to a wide range of industries and occupations. However, the form of the contribution is not as clear as, say, a mining engineer to a mining project. It is harder to form a mathematics community and more difficult for others to see what you do professionally as a mathematician.

The authors of the Australian report conclude that:

> Nurturing the health and securing the future of Australia’s mathematical sciences hinges on: ensuring an adequate supply of properly qualified mathematics teachers for all levels of schooling; mathematics and statistics at universities being taught by qualified mathematicians and statisticians. (Australian Academy of Science 2006, p. 58)

To this we would add the initial problem of attracting students to a discipline that many of them do not find attractive for a variety of reasons, including the lack of visibility of the profession of mathematician, the supposed impersonal nature of the discipline, and the perceived (and sometimes actual) discrimination shown towards groups such as women. This report, like many others, does not seem to address these aspects at all. Our studies, reported in this volume, aim to investigate these perceptions and to make the transition to becoming a mathematician more explicit.

While the previous discussion has had a largely Western focus, we suggest that the views put forward potentially have a much broader application. The empirical research that supports the ideas contained in this book hails from several nations. The students (and researchers) involved in the research are members of several different ethnic and national groups. Mathematics is like that: for serious theoretical mathematicians, national difference and spoken language form no barrier to their mathematical work, in the same way that musicians with a musical score in front of them can immediately work with each other to produce meaning and pleasure for listeners – regardless of their ethnic and language backgrounds. Mathematicians have the advantage of a 'super-language' that enables cross-cultural interactions. Yet, at an essential level, learners will encounter mathematical ideas mediated through language (as Barton 2008, demonstrates), society, culture, work and opportunity. In this sense, the experiences related in this book are necessarily limited to the students who have participated in the research studies: and while they represent collectively a wide range of countries, languages and ethnic groups, their very status as university students in reasonably wealthy countries means that they cannot represent the experience of all learners.

A feature of the early twenty-first century is that education is becoming increasingly international and globalised. International, in that many students and teachers have a mobility and choice over the location of their activity and can benefit from co-learners who have a diverse range of learning experiences. For mathematics learners, internationalisation allows learners from many different nations to experience different approaches to, uses for, and thinking about mathematics. Globalisation, at the same time, extends the reach of education to a greater diversity of topics. As Hans van Ginkel, rector of the United Nations University, observed in his introduction to a conference considering the changing role of higher education:

> The globalizing, knowledge society brings into focus new themes in education and research well beyond the regular discipline-based programmes: ICT, bio-and nano-technology; ethics and values; ageing and migration; issues of cultural diversity; dialogue and integration; intercultural leadership and entrepreneurship; climate change and sustainable development; disarmament, reconciliation and peace-building; among others. (UNESCO/UNU 2008, p. 35)

Such new themes incorporate discipline-specific knowledge, as well as how that knowledge is placed socially and politically. Later in the conference report, there is discussion of a curriculum of 'reconciliation' wherein ideas of internationalisation and globalisation collide (Workshop 2 report: HE and Peace, Democracy and Dialogue, p. 213). This collision is a metaphor for the mathematical worlds in which our students may find themselves, worlds where mathematics and its use are bound by cultural and economic imperatives.

Learning Mathematics

What does it take to learn mathematics? It is possible to learn with specific objects in mind, or only in a less directed or more generic way? Mathematics educators in higher education continually grapple with such questions. Recent mathematics education research has focused on aspects of these problems. For example, Durrani and Tariq (2009) provided the results of a survey of students that showed that there were some relations between learners' ideas of mathematics, their approach towards learning mathematics and their attitudes towards the discipline. In one sense, this could be seen as a regular form of investigation in higher education, based on the premise of finding out what students know or do not know, and how they go about the process of learning. Alternatively, it could be seen as a sophisticated form of student-centred learning, attempting to understand the essence of people's experience, that is, how a person thinks internally about something and then how this experience mediates other associated activity. Cano and Berbén (2009) report an investigation along the same lines, but with a more psychological approach focusing on motivation. In their view, determining what people want as an outcome for their learning is seen as the precursor for an approach to learning. Others, such as Bolden et al. (2009), have explored the relation between mathematics and specific ideas (in their case, creativity) and suggest that these more generic ideas form the core of mathematical and learning activity. Still others indicate that an exploration of overarching ideas will not lead to learning mathematics, but that attention should be paid to the difficulties of specific mathematical problems. This approach was undertaken in a phenomenographic study of the notion of equivalence relations (Asghari and Tall 2005) and concluded that it is through the development of confidence in specific mathematical ideas that learning can take place. Clearly, there are many conflicting ideas about learning mathematics.

An aim of higher education is the promulgation of knowledge through the development and activity of its participants. Specific learning outcomes are usually identified at a discipline curriculum level, while the more generic learning outcomes are often specified more broadly as part of a university mission statement. Contemporary learners expect that they should be able to take the formal results of their learning and use them for entry into other learning environments or work contexts. The Bologna system in Europe is built on this premise, that there is a generic quality of learning that can be related to length of study and course outcomes, and that these qualities enable a quick recognition of student knowledge and experience and hence facilitate student movement from one learning environment to another.

Disciplines, however, see knowledge in a rather more specific way. In any discipline (including mathematics) there is a body of material that has been built, tested, refined or refuted, understood and integrated into a whole. It is this specific knowledge that is central in the academy, built from history and the experience of academics. Reid et al. (2011) show how this form of knowledge may be related to students' views of their potential professional work and their own identity as practitioners in that field. Mathematics, however, currently falls into a dangerous situation. While it was once a stable part of higher education, as far back as the quadrivium of the mediaeval universities, it has now become a component part of a plethora of disciplines, with many students undertaking mathematical studies as only a small part of their overall learning. This means that the academic knowledge claims of the mathematical sciences compete with those of other disciplines. In this way, mathematics as a field of endeavour has become diffused as it interacts and even merges with other disciplines. It may be that 'becoming a mathematician' is not the right way to look at the process, but rather, becoming a computer systems analyst, or a radiographer, or a chemical engineer, or an artist who can appreciate the role of mathematics in their specific career. Certainly, many students view the profession of mathematician in this 'diffuse' way.

An area in which mathematics teachers are less inclined to engage is students' development of generic capabilities, including areas of skills such as communication and teamwork, as well as dispositions such as ethics and sustainability. From one viewpoint, they are separate from a specific mathematics curriculum, and yet from another view, they form an integral part of such a curriculum. It will be argued later in this book that these generic elements constitute the means by which knowledge is transformed from the arcane to the practicable in the workplace. A problem for teachers in higher education is deciding where in the curriculum these generic capabilities could be experienced, practiced and assessed. Depending on the context, such capabilities may be directly addressed within specific mathematics courses, or simply assumed to be assimilated during the complete course of education. There is an argument that the demonstration of these capabilities at university level increases the employability of graduates. Consequently teachers and students are encouraged to consider the relationships between generic capability and discipline knowledge and activity. However, apart from some rather bland graduate surveys (Graduate Careers Australia 2010), there is very little known about how new graduates experience generic capabilities in the workplace. In the context of the mathematical work, it would seem that these skills and dispositions may represent the bridge between university-acquired knowledge and the professional workplace.

Learning mathematics, therefore, seems a little complex. Taking a broad perspective, there is knowledge about mathematical concepts and propositions, knowledge about how to carry out mathematical procedures, and knowledge for working with mathematics: together, these form the basis of students' learning needs. This complexity can be mediated by teachers, textbooks, media, friends or personal experience. But at the centre of it all, is the notion of a transformation: a student has to *become a mathematician* (and they usually have to become quite a lot of other things as well). The philosopher Charles Taylor (2004) points out that there are several different sceptical positions that people can use to rationalise the stances that they take to work and living. At one level, there are those who are happy to accept a specific position if it originates with some form of authority, such as a teacher; then there are those who wish to test and reflect on a position, and this becomes part of their own questioning and curiosity-driven identity; there are also those for whom both of these positions happily co-exist, each coming into prominence as the situation allows; and finally there are those who have a continuously-developing identity, constantly changing through different contacts and contexts.

While we focus in this book on the notion of becoming a mathematician, we would be remiss if we failed to consider students' desire to become one. Some students who take degrees majoring in an area of the mathematical sciences may already identify themselves as mathematicians. However, students with mathematics as a minor component of their studies in some other discipline may feel less inclined to identify with, or want to 'become a mathematician'; they may be happier to 'become a competent user of mathematics'. We recognise, however, that identification with a specific field is only one component part of a person's entire sense of being. When a student makes the transfer into the work environment, their identity as learners (and as student mathematicians) shifts to that of a worker. Billett and Somerville (2004, p. 309) point to the *"relational interdependency between the individual and work that can act to sustain or transform both self and their work."* From the evidence that we present in later chapters, we can see how students move, easily or perhaps uneasily, through different internal processes as their identity as a learner is superseded by their identity as a worker.

Barnett (1994) suggests an alternate way of thinking about the knowledge that potential workers bring to the workplace, by considering the differences between academic and operational competence. In this sense, he recognises that students may learn to negotiate academic learning spaces with a competence that demonstrates their ability to undertake group learning, examinations, creative problem-solving and so on. This is often the case for student mathematicians, who become familiar with the knowledge and work of being a student through years of being in a university environment. Operational competence is somewhat different. While at university, students have developed a picture of what work may be like (and our own research supports this), but in the context of work, knowledge now has to be tightly aligned with the mission of the workplace. The skill of operational competence is to be able to recast previous knowledge in a form that is suitable for the new situation. Slowey and Watson (2003) extend beyond the issue of academic or operational competence as they investigate the important connection between higher education, which they see as a form of socialisation, and work, which has a predominantly economic function.

In the twenty-first century, higher education is a vehicle for worldwide mobility, and hence must be inclusive of gender, ethnicity, opportunity and different cultural perspectives. In the research included in this book, we try to show how students from different national backgrounds encounter learning mathematics. While our research includes an aspect of national differences (in that students were located in different places, and some have travelled to different places for their education), we are aware that professional work including the use of mathematics is now located in a world-wide domain. While universities can claim that they prepare students for particular professions, those professions are now located in a complex international world. Increasingly, students are becoming aware that their learning colleagues and future professional co-workers will have a wide variety of different educational, cultural and world experiences – all of which contribute to students' identity as mathematics learners and as future professional mathematicians and users of mathematics.

Despite the national locations of our research, we know that most tertiary students in ‘Western’ countries study in cohorts consisting of students from many countries, unlikely to share cultural and pedagogical values. There is an expectation that all students will need to adjust to different learning cultures and negotiate language in diverse social and academic contexts. Rizvi (2000) points out that internationalisation must consider the ‘global-local relationship’, that is, the situatedness of knowledge. In the modern professional market, graduates will discover that their knowledge forms a commodity that can be sold. However, the very idea that knowledge can be marketed emphasises the notion that it is finite and unchanging, and is sustained by educational practices that encourage memorisation and recall of facts. Our discussion in Chap. 4 shows how the situation of learning impacts on students’ conceptions of mathematics.

Researching the Experience of Learning and Working with Mathematics

Teachers of mathematics at all levels will have had the experience of admitting their profession in a social situation, and then being regaled with stories of their social partners’ inadequacy in mathematics: a common phrase is ‘oh, I was never any good at maths at school’. It seems that a general distaste for mathematics is generated early in many people’s lives and very quickly admitted. By contrast, there is a smaller group who confess that ‘I always loved mathematics’. It does seem, however, that almost all people have had an experience of mathematics and are able to explain what they think about it. Most of these explanations focus on affect – I really love maths, or I really hate it – and some of them on the content of mathematics – it’s so great to have a mobile phone because I don’t have to add any more, or (even more rarely) it’s so great to have a mobile phone because it has a graphics application and you can just see a mathematical equation come to life. Feelings about mathematics can play a significant role for a person’s motivation to learn mathematics. Many of the students involved in our studies would start their interviews with a statement positioning them as lovers, haters, or users of mathematics.

There seem to be about as many views on mathematics as there are people and situations. So, how is it possible to make sense of this wide variation in people’s experience in a way that will help educators work with students? In the chapters that follow, we will describe and investigate mathematics students’ and graduates’ views of mathematics, learning mathematics and working with mathematics. Our participants also have a diverse range of understandings of mathematics, and their views represent an invaluable source to help us think about our pedagogical task of helping students in their learning of mathematics. It is one of the significant features of our book that we make maximum use of our participants’ ideas, often letting them speak in their own words through extracts from their interviews, and allowing us, as tertiary educators of mathematics, to learn from them.

Earlier in this chapter we pointed out that some people see mathematics as an essential aspect of themselves, while others view mathematics as a confusing annoyance. Our research teases out the reasons for this from the perspective of learners and early-professional graduates. It is important to emphasise at this stage that it is our view, built from the evidence contained in the research, that those people who have expansive, integrated, holistic views of mathematics and its use also have a way of living with mathematics as a core element of their personal identity. In this sense, their focus as students is on becoming a mathematician, and their focus as early-career professionals is on being a mathematician. These elements represent a 'sense of being' (Reid and Solomonides 2007). Such a sense of being implies both a transformative perspective, including a willingness to learn, and a sense of affiliation with a group or profession. This is an ontological view of the process and outcomes of learning. In the process of learning, all students will be transformed in some ways, and will come to identify strongly with areas that become part of their personal identity. This is true in informal as well as formal learning situations: a keen clarinettist will come to think of herself as a musician, an enthusiastic soccer play will develop a view of himself as an athlete, and someone who spends many hours growing plants will develop an identity as a gardener. The key issue is the strength of the relationship between their social situation and their concept of themselves. As educators, we try to help students identify strongly, or create a personal identity, that belongs to a specific profession: we give a student undertaking a degree in statistics opportunities to become a statistician. While we know that people will identify with various aspects of their personal context, we see it as essential for mathematics educators to provide as many opportunities as possible to help students of mathematics identify happily with the idea that they are mathematicians.

A first step in doing that is to investigate the different ways that people experience, understand and use mathematics, and our research program has used a variety of different techniques to do this. We started from the premise that not everyone experiences mathematics in the same way and that their experience plays a role when they are learning or using mathematical approaches. We aimed to uncover the variety of qualitatively different ways that people experience mathematics. Building on an investigative approach called phenomenography (Marton and Booth 1997), we interviewed students from different mathematically-oriented degrees with the aim of finding the full range of variation. Earlier studies in the field of education that used this method (e.g., Marton and Säljö 1976) described two distinct approaches to learning – a 'surface' approach, in which students simply do what is asked with no attempt to understand the importance or meaning of the material, and a 'deep' approach, in which students focus on the essential meaning of the activity. This notion of surface and deep approaches to learning is now widely known, though its genesis in phenomenographic studies is often forgotten. The two approaches to learning would be familiar to mathematics educators, maybe most commonly in the 'drill and practice' exercises of many school mathematics textbooks, which tend to foster a surface approach through the overuse of meaningless examples. However, this early research focused on the *approach* that students took to assessment tasks. A broader understanding of learning was obtained when it was found that learning approaches were integrally tied to peoples' *conception* of the phenomenon – of the phenomenon of learning itself (Marton et al. 1993) or of some aspect of the mathematics being learned, such as the experience of number (Neuman 1987) or recursion in programming (Booth 1992). It is this focus on conceptual understanding that our research team pursued in the context of mathematics learners.

To find the range of ways that students understand something, mathematics or learning or working with mathematics, we interviewed them. An important aspect of interviewing for research purposes is the clear definition of the main research questions combined with the flexibility to follow ideas and examples presented by the participants. In our research interviews, we made a particular point of encouraging participants to reflect deeply on their own ideas. A participant's response to an initial question might be followed by general questions such as *"How interesting, could you tell me more about that?"*, or more specific questions such as *"You just said 'problem solving', could you explain to me what you mean by that – could you give me an example?"* In all our interviews, we continued questioning participants about their responses until we felt that they had exhausted the topic.

Once the entire interview process was completed, the research team commenced the analysis of the data. The phenomenographic approach focuses on exploring the difference or variation in a group's view of a phenomenon. As analysts, we are constantly asking: What has made this person's view different from that one's? What is the characteristic of that difference? Is there evidence of more than one way of seeing the phenomenon in this transcript? Every transcript was read by all members of our research team, and each person kept a record of their understanding of the essential differences and commonalities, and quotations that exemplified them. This was followed by a discussion between all members of the team. Through several iterations, the group identified the major areas of qualitative difference between one way of seeing the phenomenon and another. By now, the analysis is not focusing on a rich account of a single person's experience, but rather a description of variation that takes account of the experiences of the whole participant group. The next stage of the process is showing the relationships between the different ways of experiencing, or 'conceptions'. The conceptions and the relationships between them are mapped out in an 'outcome space'. Finally, each conception is provided with a brief description, and evidence supporting it in the form of direct quotations taken from the interview transcripts. These quotations may be taken from different interviews, though occasionally one transcript may provide evidence of more than one of the conceptions. This then may give evidence of a hierarchical relationship – a common phenomenographic structure – between different conceptions.

Although our underlying methodological approach was phenomenographic, we did not hesitate to extend this and support it with a range of alternative methodologies when they seemed appropriate for investigating various aspects of our research questions. At various times in the project we have used content analysis, focusing on the way in which particular words and phrases are used in different contexts, and discourse analysis, an approach to analysing spoken or written language, or 'communicative events' in general. Where necessary, we have included quantitative approaches for the statistical analysis of data that we have collected. Our overall approach was to keep the research questions foremost and tailor our methodological approaches as appropriate.

Dramatis Personae

The research process described above provided the basic groundwork for our long research project. As usual in such an extended investigation, many people participated in the research team. The initial project included the three authors of this book: Leigh Wood is a mathematics educator with a background in mathematics and engineering, currently Associate Professor and Associate Dean of Learning and Teaching in the Faculty of Business and Economics at Macquarie University; Peter Petocz is a statistician and statistics educator, Associate Professor in the Department of Statistics at Macquarie University; and Anna Reid is a specialist in higher education and qualitative research, initially developed in her discipline area of music, currently Professor and Associate Dean of Learning and Teaching at the Sydney Conservatorium of Music, a department of the University of Sydney. Our initial team also included Dr Geoff Smith, a mathematician in the Department of Mathematical Sciences at the University of Technology, Sydney (UTS). The initial interview studies with undergraduates were carried out at UTS in 2001–2004 (at that stage, we were all colleagues working there). Some of the undergraduates were enrolled in the degree in mathematical sciences, or a joint degree in mathematics and finance; others were studying mathematics or statistics as part of degrees in engineering or sports science and tourism. While we conducted some of the interviews ourselves, most of them were carried out by our research assistants Kate Henderson, an environmental scientist, and Emma Dortins, a historian. In all, over 70 undergraduate students participated in interviews with members of our team, and these interviews form the empirical basis of our writing. In particular, a series of 22 interviews with undergraduate mathematics major students provides most of the raw material for the two following chapters: in Chap. 2 we investigate students' conceptions of mathematics itself, and then in Chap. 3 we explore students' intentions for learning, their approach to learning, and then their outcomes of learning. The other interviews, including those with students majoring in other areas, showed a consistent range of views, providing a validation of our results.

We quickly recognised that although we had a participant group located in Australia, the problems of conceptual understanding in mathematics were international. We then extended our study to include undergraduate students from South Africa (under the direction of Professors Ansie Harding and Johann Engelbrecht), Northern Ireland (under Professor Ken Houston), Brunei (under Dr Gillian Perrett) and Canada (under Professor Joel Hillel). The second stage of the research included all of our collaborators and was carried out by undertaking an open-ended, qualitative survey of over 1,000 students using students from five countries, either majoring in mathematics or undertaking mathematical studies as part of degrees in engineering, computing science, teacher education and various other disciplines. The aim of the survey was to see if the range of variation amongst the international group was similar to that in the Australian group. We utilised a content analysis of students' written responses to our questions. Content analysis looks at the way words and phrases are used in specific contexts. How and where specific phrases occur in a text can help researchers understand the importance and meaning of those words to the participant. The results of this phase of our research are reported in Chap. 4: we were able to extend the conceptual categories of our interview studies, confirm their prevalence in different student cohorts, and examine differences due to location and other factors.

In Chap. 5 we move to the final empirical phase of our research with undergraduate students – the development of a closed-form survey that can be administered to larger groups. Again, our international team played a key role, ably assisted by our research assistant, Glyn Mather. Lecturers of mathematics can use this survey to find out how any particular group of students understands mathematics, and to track their progress during the course of their studies. A first version of the survey was trialled by another international sample of over 500 students, and a revised and shortened version is provided in the book (see Appendix 1). In this chapter we use some standard quantitative methods to analyse the data that we collected and refine our survey instrument. Information from this survey can be very useful for teachers in order to establish learning environments that help students to move from narrow and atomistic to broader and more holistic conceptions.

Concurrent with these studies, we began an exploration of graduate mathematicians' early experience of work, and their views of the role of their formal studies within their new work place. We used two sources of data: an initial series of interviews with recent graduates from the UTS mathematical science degrees, and a later series of interviews with recent mathematics graduates from UTS and Macquarie University (carried out by Leigh Wood in 2004–2005), working in a range of professional workplaces. In Chap. 6 we use the complete set of (over 30) interviews with graduates to investigate the contribution of mathematics studies to their professional working life, focusing on their discussion of the role of mathematics in developing their skills as problem solvers and their development of an identity as mathematicians.

As we suggested in the opening sections of this chapter, the situation in which people find themselves makes an impact on the way that they use mathematics and hence the way they think about themselves. Wood examined the diverse workplaces in which mathematicians find themselves, and the various materials that they use. Her participants showed her examples of mathematics in their workplace. Not surprisingly, these materials looked quite different from the mathematics they encountered during formal studies. The materials utilised a combination of mathematical symbols, written language and images, with in each case the graduate mathematician playing the role of interpreter for the rest of their work team. This study of graduates used a combination of research methodologies that enabled a neat triangulation of experience for interpretation. Phenomenography was used to analyse interviews, content analysis to look at the ways in which graduates negotiated their new workplaces, and discourse analysis for the interpretation of mathematical artefacts. Discourse analysis investigates forms of communication, in this case mathematical communication and texts. The important thread running through the study was the importance of mathematical communication in professional contexts. In Chap. 7 we investigate graduates' conceptions of professional mathematical communication and how it is learned.

In our two final chapters we stand back from our empirical results and reflect on two key aspects of mathematics pedagogy. In Chap. 8 we draw on our research with undergraduates and graduates in order to investigate the tertiary mathematics curriculum. We present an argument for a 'broad' curriculum, one that focuses on the use of mathematics as a way of investigating, understanding and even changing the world. Such a broad curriculum incorporates the context of mathematics as well as an explicit aspect of generic skills such as communication and teamwork and generic dispositions such as ethics and sustainability. We contrast this with a 'narrow' curriculum, one that focuses only on the mathematics itself. We believe that with a narrow curriculum students will learn mathematics, with a broad curriculum they will also learn to become mathematicians. An essential part of our argument is that in order to achieve such a broad curriculum, we as mathematics educators will have to carry out our pedagogical job in a broader way. In Chap. 9 we discuss appropriate ways in which to carry out the professional development that would allow for such an approach to mathematics education. We present several models of professional development, each applicable to a variety of contexts.

In our final chapter, Chap. 10, we summarise and reflect on our project as a whole – a project that we have carried out over more than a decade with a team of academic colleagues and research assistants, and with the help of many of our students and graduates. We give our thanks to them all, acknowledging their contributions, but accepting our responsibility for the final form in which we have presented our results. We also acknowledge the funding the provided by the University of Technology, Sydney, for the interview studies that we carried out there, and by the Australian Learning and Teaching Council for the professional development framework reported in Chap. 9.

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Chapter 2
How Do Mathematics Students Think of Mathematics? – A First Look

Introduction

In this chapter we begin our investigation of the process of becoming a mathematician. The first step in this inquiry is based on a series of interviews that we carried out a decade ago with a small number of students studying mathematics as a major at an Australian university. In these interviews we asked students about their ideas of mathematics as a discipline, their learning of mathematics, and their future use of mathematics in their studies and professional life: in this chapter we will focus on the first of these aspects. We found that students' views could be described in three levels – from narrowest to broadest, focusing on the techniques and components of mathematics, mathematical models and mathematics as an approach to life. These levels showed a hierarchical relationship: students who viewed mathematics as an approach to life were also aware of the modelling aspects and the technical components of the discipline. However, those who viewed mathematics in terms of its techniques seemed unable to appreciate the broader views, the modelling and the approach to life. In this chapter, we describe the research that we carried out to arrive at these conclusions, and illustrate the levels that we discovered using our students' own words taken from the interview transcripts. We discuss the place of these results in our overall project and put forward some initial thoughts about their implications for our teaching and our students' learning.

We started our project by looking at students' conceptions of mathematics itself, and we did this in the most direct way possible – by asking them to talk about their ideas in response to a few general initial questions. At this stage in the investigation, we preferred to talk at length with a small number of students, in order to explore their ideas in depth. The students were a convenience sample of volunteers from the mathematical sciences course at the university where we were working. Of course, this could limit the generalisability of these results to the larger group of students studying mathematics. However, the information that we obtained was corroborated by views from students who were not majoring in mathematics, and then utilised – with

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some modifications – in later stages of our project when we surveyed much larger groups of students.

We believe that it is of fundamental importance to know how our students view mathematics in order to help them to become successful students of mathematics, and later, successful mathematicians or professional users of mathematics. Some of our students may think about mathematics in the same way that we ourselves think about it, but there is much evidence to show that most of them think about the discipline in quite different ways to their teachers. If we are unaware of how our students view mathematics, our pedagogical efforts are less likely to have the desired impact. It is a relatively straightforward task to present a particular mathematical technique to our students and help them learn how to carry it out, first in the artificial situation of an exercise, then in the context of a textbook problem, and then maybe in a real problem – though even these steps are not always successful. It is harder to help students to develop the ability to see when the technique should be applied, when it should be modified – and how to do this – and when it should be used as one step in a complex investigation. It is even harder to encourage students to see the world through the lens of mathematics – to become aware of and then internalise the mathematical way of thinking, to see the mathematical aspects of any problem or situation in one's professional and even personal life.

The process of becoming a mathematician involves more than simply learning about the elements of mathematics, and more than developing the ability to use mathematical methods to address and solve problems. There is widespread acknowledgement of a mathematical way of thinking, a mathematical way of viewing the world around us, that seems to be an important component of being a mathematician. We would hope that those of our students who do become mathematicians have developed an awareness of such aspects of mathematics. This is so whether they actually take up professional positions as mathematicians (under whatever name they are referred to) or whether they find themselves in positions where they have to make professional use of mathematics. We will see from our interviews with mathematics students that some of them, those who viewed mathematics as an approach to life – the broadest level that we identified, have already developed such a broad view of the discipline. Other students had not yet reached this broadest level, and viewed mathematics in terms of modelling aspects of reality, or maybe only as a collection of technical components. Our essential task as teachers is to help such students broaden their conceptions to enable them to become mathematicians.

Previous Investigation of Views of Mathematics

Before we describe and discuss the results of our investigations, we will review some previous work that has looked at the various ways in which people view mathematics. The majority of writing in this area is based on mathematicians' and/or mathematics educators' views of mathematics. There are classic works such as G.H. Hardy's (1940) essay, *A Mathematician's Apology*, in which the author, a successful professional mathematician in his 60s, wrote about the experience of being a creative mathematician, and the aesthetics of the discipline itself, particularly the most beautiful part of mathematics – that which has no practical application, the so-called 'pure mathematics'. Hardy presents a view of mathematics as the most abstract and beautiful part of life itself. A more recent exposition is *The Mathematical Experience* by Davis and Hersh (1981), two professors of mathematics in the United States. The authors present their book in the form of short essays on various aspects of the substance, history and philosophy of mathematics. They highlight the human aspects of mathematics, pointing out that everyone is a mathematician (when they buy or sell or measure) and even a philosopher of mathematics (when they state that numbers don't lie) to some extent. The small number of people who are professional mathematicians need to explain to the rest of the world what they are doing – and this is what Davis and Hersh aim to do in their book. A second edition of their book (Davis et al. 1995) contains extra material that enables its use as a text for future teachers and those interested in the ideas of mathematics.

These expositions, and many others like them, offer the views of mathematics of their authors, usually professional mathematicians. An empirical study carried out by Grigutsch and Törner (1998) investigated the views of mathematics of 119 German-speaking university mathematicians. The research was carried out using a questionnaire in which respondents indicated their agreement with various statements about mathematics. A factor analysis was used to identify four dimensions of views, labelled by the researchers as process (thinking about problems and achieving realisations), formalist (strictness, abstraction, formal logic), application (practical use, relevance to society), and schema (procedures and rules, algorithms, toolbox). The university mathematicians agreed most strongly with the process view, and only little with the schema view – the other two were intermediate. A fifth dimension, labelled 'Platonic', represented a view of mathematics as *"aesthetic divine games"* (p. 17), but this view received little support. Grigutsch and Törner contrast the static views (formalist, schema) of mathematics as a fixed system with the dynamic views (process, application) of mathematics as the activity of contemplating problems, making connections and constructing knowledge. They point out that the university mathematicians have most sympathy for a scientific view of a process of creating and developing mathematics, yet tend to teach it as a formal collection of complete knowledge.

Burton (2004) carried out a study in the UK of university research mathematicians' ideas about mathematics and coming to know mathematics. The study was based on interviews with 35 female and 35 male mathematicians. On the basis of previous research, Burton postulated an epistemological model that emphasised the human aspects of mathematics – the heterogeneous ways in which people learned about the subject, the strong connections with feelings, emotions and aesthetics, and the group nature of mathematical activity. This is in contrast to the accepted view of mathematics as objective knowledge, codified and transmitted inertly, and separated from the people who learn and do mathematics. The results showed that the mathematicians' views of their own mathematical work supported her conjecture: in general, they did research and 'learn' mathematics in the ways described by the model. Significantly, their views of teaching were more traditional, recalling the conclusions of Grigutsch and Törner, and on this basis, Burton argued that mathematics pedagogy could be improved by treating learners as researchers. We will see later in this chapter that the views of some of her university research mathematicians are consistent with those of some of our students.

In terms of empirical results on students' views of mathematics, we turn to a study carried out by Crawford et al. (1994) in Australia. Utilising a phenomenographic approach, Crawford's team investigated students' conceptions of mathematics and how it is learned. They surveyed a large first-year mathematics class of around 300 students with open-ended questions designed to assess their conceptions of mathematics and also their approaches to learning mathematics. They were instructed: *Think about the maths you've done so far. What do **you** think mathematics is?* Their responses were organised into five hierarchical categories: the narrowest was *"maths is numbers, rules and formulae"*, while the broadest was described as *"maths is a complex logical system which can be used to solve complex problems and provides new insights used for understanding the world."* The team focused on the distinction between fragmented conceptions (the first two) and cohesive conceptions (the other three) of mathematics. Those students with a fragmented conception of mathematics were more likely to use a surface approach to learning in the subject ('learning for reproduction'), while those who displayed a cohesive conception of mathematics were more likely to use a deep approach ('learning for understanding').

The Crawford team's seminal study produced some important insights, demonstrating that students viewed mathematics in qualitatively different ways, showing the relationship between fragmented or cohesive conceptions and surface or deep approaches to learning mathematics, and developing a questionnaire to measure students' conceptions. Nevertheless, the study had some limitations. Among them was the fact that the students comprised a mixed first-year class, only a minority of whom would have continued with their mathematics study. Another limitation was the fact that the conceptions were derived from relatively short written responses to questions, though the team carried out a small number of confirmatory interviews to validate the results. Finally, the team's focus on fragmented versus cohesive conceptions of mathematics was taken up by following researchers, and may have diluted the impact of the identification of five conceptions.

Nevertheless, the work of Crawford's team was an important advance, and provides a useful comparison for our own results. Their results have provided a theoretical basis for a series of more recent studies investigating fragmented and cohesive conceptions of mathematics and their relationship to other factors such as achievement goals (Cano and Berbén 2009), developing numeracy skills (Durrani and Tariq 2009) and affective aspects of service mathematics courses (Liston and O'Donoghue 2009, 2010). Indeed, most studies of tertiary students' views of mathematics have referred to the work of Crawford's team or used it as a theoretical basis. One exception is Saleh (2009), who investigated the views of mathematics and problem solving shown by secondary school teachers in Malaysia. Most of her small sample viewed mathematics as a formal system of numbers, symbols and operations, and focused on the algorithmic aspects to enhance their students' abilities to solve examination questions. Similarly narrow views of mathematics, and beliefs that there was little room in mathematics teaching for creativity, were shown by the primary school teachers in a project carried out by Bolden et al. (2009).

Our Study of Students' Conceptions of Mathematics

Our investigation was based on a series of in-depth interviews with second to fourth-year students majoring in an area of the mathematical sciences (statistics, mathematical finance or operations research) at an Australian university, the University of Technology, Sydney. The students had various backgrounds, and some of them were working professionally as well as studying. Students in relevant mathematics classes were introduced to the research program and its aims, and were invited to participate: the project had approval from the relevant ethics committee of the university. We remain grateful to the 22 students who did volunteer to participate in the study and provided the key data for our project. They were interviewed by a member of our research team who was not involved in teaching them, transcripts were then prepared by the interviewer, and each respondent was allocated a pseudonym. Interviews generally took between a half and one hour, with a few lasting even longer. The transcripts (almost 92,000 words) form the raw material of our study. A further 14 graduates of the same mathematics program were also interviewed in the same way, and the information from these interviews (a further 62,000 words) was used later to validate the results from the undergraduate participants, as well as for further investigations: we will discuss these interviews in Chap. 5.

The interviews were 'semi-structured'; that is, we started with a small number of key questions and then posed further questions depending on the direction of the ensuing discussion. The interviews were carried out using a mathematically naïve approach. Our interviewer was experienced in collecting oral histories and carrying out interviews in the social sciences; she has written in detail about the process of producing shared meaning from the interactions between interviewer and interviewed (Dortins 2002). However, she was not a mathematician, and did not pretend to be one, so students were encouraged to provide explanations of any mathematical aspects of their responses. The initial question posed to all respondents was: *What do you think mathematics is about?* Depending on the response, different probing questions were used to establish and continue a dialogue about the student's conception of mathematics. Some of these questions were general, such as *Could you give me an example of what you mean?* while others were more specific, such as *How do people apply maths to the real world?* or *How is mathematics related to beauty?* The interviews also had two further initial questions designed to investigate students' approaches to learning mathematics (*How do you go about learning mathematics?*) and their expectations of future professional work (*What do you* *think it will be like to work as a qualified mathematician?*). The resulting responses will be discussed in later chapters.

The data were analysed using a phenomenographic approach (Marton and Booth 1997) aiming to discover the qualitatively distinct ways in which students in this context experience, understand and ascribe meaning to the phenomenon under question, in this case, the nature of mathematics. A large number of phenomenographic studies, from the earliest Swedish work on ways of reading a text (Marton and Säljö 1976) to contemporary studies (e.g., Gordon et al. 2010), have demonstrated an unexpected fact: when a group of people experience a specific situation, they will view it in a small number of qualitatively different ways, as opposed to a continuum of views. Some people will share a particular way of experiencing a phenomenon in the surrounding world, while others will understand the same phenomenon in quite different ways. The outcome of a phenomenographic analysis is a map of these distinct ways of experiencing a phenomenon, from the narrowest and most limited to the broadest and most inclusive, and the logical relationships, usually hierarchical, between the categories. This is referred to as the 'outcome space' for the research. Phenomenography defines aspects that are critically different within a group involved in the same situation: these differences that make one way of seeing mathematics qualitatively different from another. The categories are defined by their qualitative difference from the other categories, and reported in order of their inclusivity and sophistication.

In practical terms, each transcript was read multiple times by each member of the research team independently. At subsequent team meetings, categories describing the variation in students' conceptions of mathematics were suggested, refined and checked by repeated reading. The final categories were confirmed by identification of appropriate quotes in the transcripts, and their structure was confirmed to be hierarchical; that is, the broader or more expansive categories included the narrower or more limited ones. This resulted in the outcome space for the phenomenon that is described in the following section and illustrated by quotes. Although each quote is labelled with the appropriate pseudonym, the aim of the analysis was not to put any particular student into a particular category – indeed, quotes from the same student can be used to illustrate more than one category. Each individual quote is not necessarily a summary of the meaning of each category, but rather is supportive, and the richness of the category is defined by the whole set of transcripts, and illustrated by the several quotes that we have selected.

Conceptions of Mathematics

Our analysis of the transcripts identified three qualitatively different ways in which students understand mathematics, ranging from the narrowest conception of 'mathematics is about components', through a broader conception of 'mathematics is about models' to the broadest conception of 'mathematics is about life'. We will describe these conceptions in more detail and illustrate them with quotes from our respondents. The hierarchical relationships between the conceptions indicate that students are able to appreciate and utilise features of the narrower conceptions if they feel they need them. However, they do not have access to the broader conceptions, and for this reason we sometimes describe the narrower conceptions as 'limiting'.

(1) Mathematics Is About Components

In this conception, students see mathematics as made up of individual components or techniques. They focus their attention on disparate mathematical activities or aspects of mathematics, including the notion of calculation, interpreted in the widest sense. Several students put forward such views in their initial response to the question *What do you think mathematics is about?* – they put forward a 'definition' that focused on the various elements or components of the discipline. George presents the briefest summary of this conception, while Brad, Candy and Andy point out that the components or techniques have various characteristics – they are logical, can be widely understood and can help to solve problems.

> *George*: The study of numbers, the relationships between variables I suppose, not much else really, just the study of numbers I suppose, that's the way I sort of view it, sort of see it, yeah.

> *Brad*: It's almost like, maths is like a lot of small tools, and you can use those to do lots and lots of things, the more you know, the more you can use them, so you don't just have to go out and accept what the rest of the world says is the right way or the wrong way or whatever, you have actually got the tools to test to see whether that is the best or not or whether that actually, you know, logically follows on from where it is supposed to.

> *Candy*: To me, mathematics, there's a, there's a lot of things involved. Actually I haven't really thought about that too much. Basically I think it involves, I just think it just helps us to calculate, or not calculate, to solve, whatever problem it is we have at hand, and it's a way of doing it so that everybody can understand./…/To me mathematics is a whole lot of numbers and symbols, which basically can say, anyone in any background from anywhere can understand.

> *Andy*: A set of procedures, a set of, I don't know, a set of I guess methods which we can use to solve problems in the real world, some models we can use to have a look at phenomena or that sort of thing./…/I guess numbers and just using them, working with them using these sort of methods, things like regressions or hypotheses or something to test assumptions you might make about a particular system or something, a company maybe. Yeah, it's a bit of a toolbox really, that's about it I think, I can't think of anything else.

Andy's quote makes mention of models and *"problems in the real world"*, pointing towards the more expansive conceptions: however, he returns firmly to the component view of mathematics. Some students expanded their view of mathematics upon further questioning, and some of these more expansive quotes are used to illustrate aspects of the broader conceptions (for example, George's quote in the next section). However, others such as Monique remained focused on the disparate elements, components or techniques of the discipline.

> *Monique*: Mathematics it's about, it's not just about calculating and computing, but also the application that will be useful in the work environment, you know, when we got to work, when we graduate and we find the work, we can use those maths techniques and, yeah the

> maths techniques and all those things that we learned in maths we can use that in the work environment. Yeah and yeah, as I said, it's not just about calculating, it's about applying those formulas to practical terms./…/Yeah, in my first year I learnt about calculus and, and at first I thought it is just, you know, calculating, but then it has an application to, to say oh, I forgot the example, we learned some practical issues relating to calculus, yeah. Do you want me to give examples?/…/Okay, let's just forget about calculus and just get another example. Let's say statistics, yeah it is really useful in the real world, I worked in Bureau of Stats before and I could see how the theory works in practical. Yeah, because we had the census and then after they processed the census, they, after they process it, they, I think they used the stats formulas, like they find the correlation, and those things that relate to statistics and then, yeah and then they use those information to decide, to decide on the, on the society, on the how, for example the budget of the state. Yeah, after they statistically analysed the information, then they will find, they will learn how to allocate the budget to different states.

Monique's more extensive quote demonstrates the basic features of this conception. She finds difficulty identifying components of her calculus course, aside from the notion that it is about calculating: her statistics example lists the disparate components of a statistical investigation. (We have used the symbol/…/to indicate statements that we have omitted, which may show that the sections of the quote are even taken from responses to different questions.)

(2) Mathematics Is About Models

In this conception, students see mathematics as being about building and using models, translating some aspect of reality into mathematical form. This is qualitatively different from the previous conception as the focus is on the models themselves, rather than the component activities or techniques of mathematics. In some cases, such models are representations of specific situations, such as a production line or a financial process. In other cases, the models being considered are universal principles, such as the law of gravity. The first two quotes give brief illustrations of this conception. Grant and Hsu-Ming both describe mathematics as models without actually using the term:

> *Grant*: It's just observing the world, explaining it in terms of numerical values and coming up with formulas to sort of predict behaviours and explain what's going on.

> *Hsu-Ming*: What mathematics is about? [pause] Generally I guess it would be universal laws and principles, yeah, very broadly and short and sweet./…/I don't understand them all yet, still being a student, I guess. But that's my understanding of, that's my perception of mathematics is that, yeah, universal laws and principles, you know, that govern day to day things and life in general sort of.

By contrast, George describes the specific models that he is using in his work in the finance industry and uses them to illustrate his particular conception of mathematics. This amplifies his earlier description of mathematics as "the study of numbers".

> *George*: Well I sort of, yeah, at the moment my course is sort of directed at my work, so I'm doing it basically to get an overall understanding of what I do at work. I'm a currency options trader at a bank, and that has a lot of underlying models that they trade off, and that

> sort of thing, and I want to be able to understand those models and how they are derived basically. That's why I am trying to do this degree./…/Well, basically I work in foreign exchange and I work on the currency options desk which is a derivative of, of an asset price which is the Aussie dollar, so as that varies you get these derivative products which change price, and the prices of these products are sort of based on a certain model which was derived by Black and Scholes and I just want to understand the mathematics that underlie that model, to actually understand my job a lot better, so that's why I am sort of trying to get all the background knowledge through this course.

The final quotes show two students, Elly and Richard, focusing on a variety of models. Elly discusses models in general, talks about specific models in OR (operations research) and shows her awareness of models around her in everyday life (maybe giving a hint of the broadest conception).

> *Elly*: Okay, I mean if it wasn't for maths we wouldn't have buildings and bridges and things like that, and doing a double major I've just realised, especially with my OR, that I remember my lecturer mentioning something about he was involved in talks with the, about the Sydney Olympics, that, that they were running a simulation on the train timetable, how often trains should come and, you know, there's an expected amount of this many people and stuff, and people don't realise I think that maths is really all around us. I mean people think, oh maths is one plus one, but it's not, it's really, it's all around and I don't think that many people realise that, that there are, I mean I've just realised now that I'm doing a maths degree as well about how much maths is involved in everyday life.

> *Richard*: Okay, I think mathematics is largely about the building of models. I guess, I really should start, well I think it's about abstract thought, which then leads to the building of models, that's my own personal opinion on this. And these models can be completely a thought construct, completely abstract in which one would question perhaps their, at the time at which the model was developed, their practical application to a world, a real world situation. But still, it's the product of someone's intellectual efforts. I think that's a good thing, I like that. On the other, the other direction which I believe mathematics can be looked at is the application or the development of a model which closely aligns itself to a real-world situation, with a view to bringing to someone's enlightenment, a solution to a given problem, be it, you know, something to do with the vibration of the mechanics of a car or something like this, so say, okay, this part wears out often, what can we do to fix this? Can we analyse this mathematically? Yes, we can, here are some opportunities for improvement, and, but whichever way you go it is still a model, in the latter description I gave I guess it more closely aligns itself to someone's particular requirements at the time, an application. That's my thinking on that one.

Richard states explicitly his belief that mathematics is concerned primarily with models, and he explains his distinction between 'pure' abstract models, and models that are 'applied' to the real world – a classical distinction in the world of mathematics that was mentioned by several other respondents.

(3) Mathematics Is About Life

In this broadest and most holistic conception, students view mathematics as an approach to life and a way of thinking. They believe that reality can be represented in mathematical terms and their particular way of thinking about reality is mediated by mathematics. They make a strong personal connection between mathematics and their own lives. This is qualitatively different from the previous conception, as the focus is on the broad notion of the relationship between reality and mathematics, and the ontological notion of being a mathematician, rather than on the idea of individual models of aspects of reality. Students holding this conception express it in ways illustrated by the following quotes. Ian describes his growing awareness of the universal applicability of mathematics, while Yumi – somewhat hesitantly – gets to the ontological core of becoming a mathematician:

> *Ian*: I guess I didn't realise how much everything in life and in nature is described, or can be described, or seems to be able to be described by mathematics and there's, rules, rules and processes which we can describe using some sort of language which we can understand, you know.

> *Yumi*: I never really thought about it and, I don't know, I just think that it helps me every day, like even if I go shopping or if I …, I just see the world in a different way to people who don't study mathematics, and I don't know how to describe it, but I do.

In a more extensive extract, Dave makes an obvious personal connection with mathematics, and alludes to the changes that he has noticed in his approach to life as a result of studying mathematics.

> *Dave*: Um, about a way of thinking about things, a thorough way of thinking about things./…/Well I think it gives me a sense of clarity about things, it gives me a bit more confidence that if I'm trying to make a judgement about an issue, a concept, a political debate, anything that you can, that there is a sort of fall-back way of thinking about things that can quite, quite reliably help you form a conclusion, or let you know that a conclusion might not be appropriate./…/Ploddy. Taking things step by step and not sort of jumping, not jumping to, it's anti-intuitive, I guess that's how I see it, it's not intuitive, you can combine the two together, you can have an intuitive idea about something and think that seems to be a sort of natural instinctive way that you might look on something, or a natural type of decision that you might take, and then you can, using your sort of mathematical training, you could sort of deconstruct that thought and take it apart and think, 'well is this, is this an appropriate way of thinking, an appropriate way of deciding about something?', and you can be a little bit surprised now and again.

Vitali also displays an obvious personal connection with the discipline, and expresses his view that mathematics is a language for dealing with the world, and a way of developing thought processes.

> *Vitali*: Mathematics is describing the world in front of us, just in a different language, the same thing as we have words, spellings, sayings, it's for something else, I mean like vocabulary of something else, within and without our cosmology. Mathematics is great. Mathematics is a set of thinking and it's something that you can develop in your life./…/I think mathematics, mathematics, everybody has to go through that, even if you're, what's the, you are not able to, I don't know, discover something new or formulas or anything, you have to just go through the course to start your mind thinking and just ability to look at the things and just basically solve them, that's what mathematics is.

Finally, Eddie talks of mathematics as a comprehensive way of describing all aspects of the real world. Although he gives many individual models as examples, he is focused on the overall idea of describing reality.

> *Eddie*: Oh jeez, oh I don't know, that's a very hard question. Describing the real world, I suppose, yeah, I mean that would be my answer./…/You know, when we wanted to predict

> when the, you know, when the seasons would come, you know there was a bit of hit and miss there and crops went, crops went belly up and people starved until the basic relationships were hit upon. You know, if I want to throw something up in the air and predict, you know, predict where it is going to come down or if I'm firing a cannon ball or something at someone and there's some, a guy called Newton described all that mathematics that we still use today and in fact we use to, that we use to take pot shots at the moon. I mean when they launched the astronauts to the moon, I mean the rocket didn't go all the way there, it only went for the first few minutes, after that it was just a really good shot and that's, and I mean, exactly the same, you know like, exactly the same set of assumptions, you know, the same underlying relationships rule everything. I mean there's not one piece of mathematics that works here on earth and another piece of mathematics on the moon, I mean it all works. If it works, it works everywhere and I know that the latest developments in quantum theory seem to be at odds with that, but quantum theory is an incomplete theory, so. But, yeah, I think it's about describing the real world, I mean you can describe the real world in general terms with something like statistics or you can describe the real world in specific terms with something like, something like mathematical physics, or you can describe the, you can describe interactions in the real world, I mean psychological things such as, you know, people running businesses and trying to make money and behaviour on the stock market and that's also modelled pretty well mathematically.

Eddie first states that mathematics is about *"describing the real world"*, but it becomes clear with his further discussion that he means all aspects of the real world rather than just the specific models to which he refers. His obvious ease of moving from the mathematics of one aspect of reality to another underlines his overall conception of mathematics as a significant aspect of life.

Discussion

The group of students that we interviewed seem to have overall quite sophisticated ideas about the nature of mathematics. This is not unexpected. They are specialising in the area; they have been studying mathematics all through high school, and for at least a year at tertiary level. Nevertheless, even in this group of students, there is a range of conceptions about their subject. Whereas they are all familiar with the notion of using mathematical techniques to describe aspects of the world around them, they show different levels of focus. At the narrowest level is a focus on mathematical components, and at a broader level there is a focus on the mathematical models built and used to describe real-life situations. The importance of modelling in mathematics, and its unifying nature in the world of professional mathematics has been noted by mathematicians themselves (for example, Houston 2001). The broadest conception we identified shows a focus on the role of mathematics in life in general and the particular way of thinking and being that implies. In a related discipline, statistics, we have obtained broadly similar outcomes (Reid and Petocz 2002). The narrowest view of statistics saw it as isolated technical components, a broader view focused on statistics as analysis of sets of data (corresponding to the modelling conception), and the broadest view regarded statistics as a way of personally interpreting and understanding the world.

The transcripts give empirical evidence of the hierarchical nature of the outcome space. The narrower conceptions are referred to as 'limiting' to indicate that students who subscribe to such views seemed unable to describe any characteristics of the more integrated and expansive views, a feature that limits their approach to the subject. Such students may only be able to focus their attention within their learning environment on fragmented and isolated aspects of mathematics. Monique's transcript is one example: despite several reformulations of the basic question, her responses remained focused on the 'components' conception, indicating an underlying fragmented view of the nature of mathematics. On the other hand, students who show evidence of the broader conceptions can also discuss mathematics using the narrower ones: for instance, George discusses the notion of financial models, but then also describes their components. Those students who can describe the broadest, most expansive view are able to make use of characteristics throughout the whole range to display their understanding of mathematics. An example is Eddie's quote given under the 'life' conception: sections of this quote could be used to illustrate the modelling conception, though it is quite clear that Eddie's overall conception of mathematics is at the broadest level.

To some extent, the views of our students are consistent with the categories identified in the study by Crawford's team (1994). Their 'fragmented' conceptions could be seen as instances of our 'components' conception: for instance, *"Maths is the study of numbers, and the application of various methods of changing numbers"* (p. 335), and Sally's quote (p. 339): *"I feel mathematics is the process of using different techniques to solve various problems. In my experience this has only involved numbers yet I feel it may be helpful for a wider range of future subjects."* Their 'cohesive' conceptions could be compared with our modelling conception, though the two approaches do not seem to be quite equivalent. For instance, one of their respondents wrote: *"Mathematics is the study of logic. Numbers and symbols are used to study life in a systematic perspective and requires the mind to think in a logical and often precise manner"* (p. 335). However, Crawford's team gave no instances that could be interpreted as our 'life' conception in the small number of verbatim quotes that they presented. This could be due to the nature of their sample group – students in a general first-year mathematics course – or to the fact that they obtained their data as short responses to written questions.

Burton's (2004) study of university research mathematicians also contains some verbatim statements that support our hierarchy of conceptions. Not surprisingly, none of her respondents put forward a view of mathematics as isolated components. However, the statement *"I see mathematics as a way of describing physical things using formulae that work, modelling the real world and having a set of rules, a language, which allows you to do that"* (p. 47) seems to be consistent with our 'modelling' conception (and possibly broader, though it is difficult to tell from such a short extract). Several statements by the university mathematicians seem to show elements of our 'life' conception, emphasising the personal and affective aspects of mathematics, for instance:

> When it works, it feels great. It is very exciting. It doesn't happen that often but when it does it makes it all worthwhile. You write something down that you are pretty sure has never been written down before, it is very exciting, and that excitement is shared by other people as well. (p. 84)

> Anything that I am working on I would want to look elegant enough that it could actually describe nature … If something were inelegant or not beautiful, too contrived or wrong, I swould dismiss it. (p. 116)

Burton's sample group was comprised of people who had actually become successful mathematicians (though interestingly some of them did not regard themselves as such), and so would be expected to display strong ontological connections with their discipline.

Implications for Teaching and Learning

We will say more about the pedagogical aspects of our findings in later chapters: here we aim to put forward our first thoughts about the implications for teaching and learning. The most obvious lesson is that this range of qualitatively different conceptions of mathematics exists in students studying mathematical sciences at university level. Moreover, the full range of conceptions was present in all years, implying that the categories are not solely developmental but simply a feature of students' experience. In the earlier study of students' conceptions of statistics, carried out with students of the same mathematical sciences degree course, we also found that the full range of conceptions, from narrowest to broadest, was present in first-year and final-year students (Reid and Petocz 2002). This finding seems to be generally true in a variety of disciplines that we have studied (for an example in legal education, see Reid et al. 2006).

As mathematics educators, we need to cater for this variation in terms of our pedagogical methods and materials. We cannot assume that students think of mathematics in the same way that we, their teachers, do, but neither can we assume that all our students (or all students in one of our classes) share a uniform view of mathematics. An important part of our job as teachers is to encourage our students towards the broader conceptions of our discipline. As a first step, we can help students become aware of the range of variation in conceptions of mathematics. This may be enough in itself to encourage some students towards broader views. Any activity that allows students to explore the nature of their thinking about mathematics and compare it to others' views can encourage such awareness. For instance, the usual initial discussion surrounding assessment tasks can be directed towards an exploration of the different ways that students may tackle such tasks. Asking students to work in groups, inside or outside the lecture class, also increases their access to different conceptions of mathematics. Research on using phenomenographic outcomes, such as those presented in this study, has shown that simple awareness of difference can be a catalyst for change (Reid 2002). Note also that the hierarchical nature of the conceptions implies that it is not the case that our students view mathematics as components *or* as models *or* as life: rather, those students aware of the broadest and most inclusive views will see mathematics as life *and* as models *and* as components.

Further, we can provide learning activities and assessments that encourage students towards the broadest levels of understanding of mathematics, and away from the narrowest levels. Assessment that is based on learning and reproducing mathematical theory will tempt even those students with the broadest conceptions of mathematics towards the narrower views. A series of exercises on aspects of one particular technique, for instance, triple integrations, will engage students at the narrowest conception of mathematics. If this is the only type of activity with which they come into contact, those who view mathematics in terms of components will be confirmed in their ideas, while those with broader conceptions will find themselves unsatisfied by the demands of the course. By contrast, a real (or realistic) problem situation requiring use of triple integrations will allow students to practice their basic technique, while at the same time developing an awareness of the modelling aspects of mathematics. If this problem incorporates common professional expectations, such as writing a summary report to a manager who is unfamiliar with the technical aspects of the mathematics, it can reinforce the student's appreciation of the role of mathematics in their future professional (and personal) lives, and lead them to a growing awareness that they are becoming a mathematician.

And What About Students Who Are Not Mathematics Majors?

Our initial investigations of students' conceptions of mathematics were undertaken with mathematics majors – those who were undertaking a degree in the mathematical sciences, focusing on statistics, mathematical finance or operations research. A question that arose later was whether students who were studying mathematics as a component of a professional course in some other discipline would show different conceptions of mathematics. To answer this question and extend our results, we carried out a later series of interviews with students studying mathematical subjects as part of an engineering course (16 students) or a sports science/tourism course (14 students). These interviews were carried out using the same approach as the interviews with the mathematics major students, and included the same initial question: *What do you think mathematics is about?*

We analysed the results once again using a phenomenographic approach. While we naturally had in mind the previous outcome space identified with mathematics major students, we fully expected to find different results for the service mathematics students. However, there did not seem to be anything in this second series of interviews concerning conceptions of mathematics that did not fit into our previous model: service mathematics students' conceptions of mathematics could be described using the same three conceptions of mathematics as 'components', 'models' and 'life'.

Here are some illustrative quotations to justify this possibly-surprising conclusion. Scott and Kylie illustrate the 'components' conception of mathematics from engineering and tourism respectively. Each of them lists different techniques or areas of mathematics to explain their view.

> *Scott* (Mechanical engineering): Mathematics – calculus, integration and differentiation, acceleration and velocities./…/We always use maths in general, addition, subtraction, division. That is common knowledge. Geometry is pretty well common knowledge as well./…/Just calculations which determine the motion of a particle, motion of the object.

Kylie (Arts and tourism management): Gosh, I wouldn't know. You have got your textbook maths I guess that you have learnt at school. Algebra, calculus, trig, stuff I really can't remember how to do. And then also you have got your numbers, putting numbers together and number crunching. To get some answers to, or to find out ways to solve problems or, I haven't done that in so long, so I guess that would be my best explanation of it.

Paul and Ben explain the 'modelling' conception of mathematics, the first in an abstract way, the second using a specific example from his area of sports science.

Paul (Mechanical engineering): Modelling the real world, modelling things that you can't make a physical model for./…/Well that is your modelling, you model something, you can give it certain variables in certain ranges, and then basically you can run your model and see if that's what happened. You might have something that has happened in the physical world, and you go, well, why did that happen? You can set up a model that you think suits that and then run it within these constraints and see if it actually gives you something that sort of semi fits. And when more information becomes available you can sort of rearrange a model to sort of suit.

Ben (Sports management/International relations): Like if you want to help someone run faster, you could use maths to work out how to minimise the time their feet are touching the ground and maximise flight time and stuff, so you can help use maths to work out how they can accelerate quicker and what they need to do to improve their technique. We did this thing where you study the centre of gravity of people and stuff like that, and I guess we used maths there.

Finally, two extracts that show the broadest, 'life' conception of mathematics. Matt seems well aware of the place of mathematics in his life. Omid has an unusual way of expressing the same idea about the essential role of mathematics.

Matt (Human movement): I don't know, it seems to be the universal base of everything, but I try my damnedest to pretend that that is not true. Even because, when I like art as well, there seems to be no escaping it. You know of piece of work in art works because it has a composition which you can work out mathematically, if you have meant it or not. Through to maths being finding out how fast someone has run over a certain distance because some theory and some formula says if you put the right numbers in it works. So I guess it's important to everything.

Omid (Telecommunications engineering): My definition of mathematics is more philosophical. I think all the part in the world including spiritual part or material part, all of them are related to each other and mathematics is the definition of those relations./…/For example the attraction between materials, if you get for example consider two materials, if they are in long distance the attraction is going to be decreased mathematically and it is true in spirituality too. If two parts, two souls are a very long distance the attraction is going to be decreased. So everything can be defined and explained by mathematics, but we just up to the level where we define the physical things. We express the physical relations with mathematics, but I think those are true in the spiritual part too.

These short extracts justify our assertion that service mathematics students show the same range of conceptions of mathematics as do mathematics major students, and in this case we have deliberately given examples from two quite different disciplines. Further, the conceptions showed the same hierarchical relationship as they did with students majoring in mathematics. We will continue our investigation of students who learn mathematics as a component of their studies in Chap. 4.

Summary and Looking Forward

We began our investigations by describing the first stage of our research project – a series of interviews with later-year university students who were studying degrees in the mathematical sciences. One focus of our discussions with them was their ideas about the nature of mathematics. We found that we could identify three quite different ways in which our students viewed mathematics, with a focus broadening from components or techniques of mathematics, to the idea of mathematics as models of real-world phenomena, both particular and general, to the notion that mathematics was a way of looking at and thinking about life, and an essential aspect of professional and even personal life. Further interviews with students studying mathematics as a component of other degrees identified the same three conceptions. The interview data confirmed that these conceptions were hierarchical, so students with the 'life' conception of mathematics were also able to discuss mathematics in terms of models and were aware of the component view. However, this did not occur in the other direction; students who viewed mathematics in terms of components did not seem to be aware of the modelling or life conceptions. In an initial discussion about the pedagogical implications, we put forward a general principle – that an essential aspect of our role as teachers is to help students become aware of the full range of views about mathematics, and with appropriate learning materials and teaching approaches, to encourage them towards the most inclusive 'life' conception.

In the following chapter, we will continue our analysis and discussion of the series of interviews with our undergraduate mathematics students. We move our focus from their conceptions of mathematics itself to their thoughts about the nature of learning mathematics. We carry out a quite detailed analysis of their comments about their own learning and use this as a basis for further comments on the implications of our findings for teaching and learning mathematics. Then we are ready to start extending these results to larger and more diverse groups of students.

Note: Some of the material in this chapter was previously published in Reid, A., Petocz, P., Smith, G. H., Wood, L. N., & Dortins, E. (2003). Maths students' conceptions of mathematics. *New Zealand Journal of Mathematics*, *32* (Suppl. Issue), 163–172.

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Chapter 3
How Do Mathematics Students Go About Learning Mathematics? – A First Look

Introduction

In this chapter we investigate students' views about learning mathematics. We continue our analysis of the series of interviews that we conducted with 22 undergraduate students studying mathematics as a major at an Australian university. In the previous chapter we explored their ideas about the nature of mathematics itself; here we move the focus to their ideas about how they go about studying mathematics. We present a theoretical model based on our research findings, aiming to build on and expand earlier descriptions of students' learning approaches, such as the surface and deep approach of Marton and Säljö (1976a) and the 3P (presage-process-product) model of Biggs (1999). As for our results in the previous chapter, the small convenience sample that we have used for our interviews could limit the generalisability of our findings to the larger group of mathematics students as a whole. However, our results seem consistent with previous literature on student learning, and also with our findings about students' conceptions of mathematics.

When we carried out analyses of our interview transcripts we found that students' conceptions of learning mathematics could be described using three hierarchical levels. At the narrowest level, students learned by focusing on the disparate techniques of mathematics. A broader level was represented by a focus on the subject of mathematics itself. In the broadest and most comprehensive view, students considered the role of mathematics in their lives and talked about developing a mathematical way of thinking and viewing the world, satisfying their intellectual curiosity and helping them to grow as a person. Again, the hierarchical nature implies that students with the broadest 'life' conception of learning mathematics were also aware of the 'subject' and 'techniques' aspects, but those with the narrowest techniques conception did not seem to aware of the broader ideas. There seems to be an obvious parallel between these three conceptions of learning mathematics and the three conceptions of mathematics itself that we identified in the previous chapter.

During their interviews, students commented on various aspects of their mathematics learning. Supported by previous theoretical discussion, we distinguished

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three separate aspects of their learning: their intentions for learning, their approach to learning, and the outcomes of their learning. We illustrate these three aspects at each of the three levels of conceptions with quotations taken from the interviews – that is, we present our results in the words of the learners themselves. This is a deliberate decision; we believe that teachers of mathematics can benefit from becoming aware of the views of learning held by their students, who are, after all, the people carrying out the learning. Indeed, some of these students show a very sophisticated appreciation of their own learning and the various problems that they have to face and overcome during the process of learning. Towards the end of the chapter we summarise the pedagogical implications of our results, and we return to this point in more detail in Chap. 8 of the book.

Early phenomenographic studies of students' conceptions of learning were carried out in the context of asking students to read sections of a textbook and then questioning them about it (e.g., Marton and Säljö 1976a, b). From these studies, five hierarchical conceptions of learning were identified, in two groupings. A narrower view of learning as reproducing comprised views of learning as: accumulating knowledge, memorising and reproducing, and applying to other situations. A broader view of learning as primarily seeking meaning included: learning as understanding a situation, and learning as seeing something in a different way. A later study added a sixth, and broadest, conception – learning as changing as a person (Marton et al. 1993) – resulting in the classical outcome space for conceptions of learning (see Marton and Booth 1997, Chap. 3). The well-known distinction between 'surface approach' and 'deep approach' to learning, that is, between aiming for reproduction and aiming for meaning, was developed in the context of these results.

An interesting feature of these early studies is that the broadest and most holistic levels of student learning seem to include an explicit recognition of the ontological aspect of learning. The broadest conceptions of learning – as 'seeing something in a different way' and 'changing as a person' – are exemplified by statements from students about their learning: *"Opening your mind a little bit more so you see things (in the world) in different ways"* and *"I think any type of learning is going to have to change you … you learn to understand about people and the world about you and why things happen and therefore when you understand more of why they happen, it changes you."* (Marton et al. 1993, pp. 291–292). These general results concerning students' views of learning were known at the stage when we commenced our study with students of mathematics. We aimed to look at students' ideas about learning mathematics in particular, and to investigate how these ideas compared with the general views of learning developed from studies in the context of reading (and writing) texts.

Previous Investigation of Views of Learning Mathematics

Research on students' learning in mathematics at the tertiary level is a wide-ranging and active field, and its overall directions have been summarised in various handbooks (such as Holton 2001; Bishop et al. 2003; Skovsmose et al. 2009) and reports of international conferences (for example, the ICME – International Congress on Mathematical Education – conferences held every 4 years). Bishop's team (2003) divide their comprehensive handbook into four main themes of contemporary relevance: policy dimensions (social, political and economic), responses to technological developments (calculators and computers), issues in research (ethical practice, impact of research on practice and the role of teachers as researchers) and professional practice (teacher education).

One feature of research on learning mathematics at the tertiary level and its translation into curriculum is that it often focuses on the ways that lecturers understand teaching and learning, and the nature of mathematics itself, such as ideas about the precision and rigour of mathematics, the cumulative nature of the subject, and the importance of mathematical skills (see, for example, Thomas and Holton 2003; Burton 2004; Nardi 2008). This is based on the view that lecturers are best placed to make changes to the learning environment, and the underlying assumption that changes and developments in teaching practice will result in changes – hopefully improvements – in learning. Lecturers of mathematics believe, not unreasonably, that they know what is critical for students to learn from their own experience of being learners and mathematicians. With this belief it would seem that developmental efforts could focus on current and early-career lecturers' conceptions of mathematics. A curriculum in mathematics, particularly in specialist mathematics degrees, is developed primarily from this knowledge base of the lecturers, combined with the strategic requirements of the university and the demands of relevant industries (Bowden and Marton 1998); the ways that students understand learning in the discipline are often assumed.

Nardi has carried out more than a decade of research on tertiary mathematics education that included interviews with students, mathematics lecturers and mathematics educators. She has summarised her results (Nardi 2008) about learning mathematics in the form of a series of dialogues between a university mathematician and a researcher in mathematics education; although the student does not participate in the dialogues, her voice is heard implicitly, and samples of her work are discussed. Nardi's mathematician shows some of the characteristic attitudes of his colleagues, but he seems genuinely interested in the insights into learning that he is offered in the form of samples of student work and discussions with the mathematics educator. He talks of the importance of conception of mathematics and its relation to learning mathematics:

> Getting to see how each one of us understands the subject is useful: and communicating our own misconceptions amongst each other as mathematicians before having a go at innocent students! (p. 263) … A lot of the problems you have to deal with when you meet our students is that they have a very singular view of mathematics, a rather poor view of mathematics. (p. 262) … May I say that it is in these discussions exactly that these sessions have proved enormously valuable already. There are things I will teach differently. There are things that I feel like I understand better of mathematics students than I did before. (p. 260)

We discussed in the previous chapter Burton's (2004) study of university research mathematicians' ideas about mathematics and coming to know mathematics. Based on the 70 interviews that she carried out, she concluded that *"the mathematicians' experiences, as learners, are relevant to less sophisticated learners in schools, and in universities"* (p. 178), and used this position to argue for *"a pedagogical approach to mathematics that treats learners as researchers"* (p. 183). Burton's findings support a view that, in at least some important respects, mathematicians' experience researching in mathematics parallels students' learning of mathematics. Houston (1997) considered the way of life of professional mathematicians, and argued that there were three corresponding ways in which students learn mathematics. The first step is learning the methods or tools of the discipline; these tools are used to understand models of the world that have been created by others; finally, this understanding enables the student to engage in the creative activity of building models, so modelling – and mathematics – becomes a way of life. Houston's three steps are reflected in our identification of three levels of conceptions of learning mathematics.

Looking specifically from the students' viewpoint, we return to the study carried out by Crawford et al. (1994). They investigated students' conceptions of learning mathematics, asking them to respond to the survey question: *How do you usually go about learning some maths?* They identified five conceptions of learning mathematics: the narrowest was *"learning by rote memorization with an intention to reproduce knowledge and procedures"*, and the broadest was *"learning with the intention of gaining a relational understanding of the theory and looking for situations where the theory will apply."* They distinguished between learning for reproduction (the first two categories) and learning for understanding (the other three) – the classical distinction between surface and deep approach to learning. Students who aimed to learn for reproduction were likely to have a fragmented conception of mathematics, while those who aimed to learn for meaning were likely to hold a cohesive conception. Meyer and Parsons (1996) reported on a quantitative investigation of student learning in mathematics using a questionnaire developed from an investigation of students' qualitative approaches to their study of mathematics. They identified two major factors: an association of desirable features (such as strategic problem solving, deep approach to learning, incorporation of group work and explaining to others, intrinsic motivation and confidence) and another association of undesirable features (such as a memorising approach, single-strategy problem solving, insecurity and fear of failure). However, relatively little has been investigated from the viewpoint of students who plan to be professionals in the mathematical sciences. The studies mentioned in this paragraph were carried out in large first-year mathematics classes that contained few students planning to specialise in mathematics.

Our Study of Students' Conceptions of Learning Mathematics

The empirical basis for our investigation of students' ideas about learning mathematics consists of the transcripts of a series of interviews that we carried out with 22 undergraduate students majoring in the mathematical sciences. In terms of their views about learning mathematics, we asked them the key questions: *What do you aim to achieve when you are learning in mathematics?*, *How do you go about learning mathematics?* and *What do you think you want to take with you from your learning of maths?* These were followed by further probing questions that depended on the student's responses – general questions such as *Can you give me an example of that?* and specific questions such as *What part does logic play in mathematics?* and *How does doing the exercises help your learning?*

As a first step in the analysis, all members of the research team read through the interview transcripts several times to get an overall idea of their content. This was followed by several days' exploration of various aspects of the transcripts by the whole team – here we focus on the aspects pertaining to learning mathematics. Although our initial approach was based on phenomenography, and the results are reported in the form of a classical phenomenographic outcome space, we extended the approach in several ways. Early in the analysis, we identified a framework of 'intention, approach, outcome' (IAO) that could be used to analyse the information concerning learning mathematics. This was not only suggested by the group of transcripts themselves, but also receives theoretical support from previous work (see for instance Dahlgren 1997, and other articles in Marton et al. 1997). We took 'intention' to mean statements where students referred to future plans or aims; 'approach' statements included general descriptions of their methods of learning or specific details of what they did as learners; and 'outcome' statements were clearly oriented towards skills, both procedural and conceptual, or attitudes that they had developed. To some extent, our three main questions could be interpreted as directing students towards IAO, but our aim was simply to encourage students to talk about all relevant aspects of their learning of mathematics. Moreover, we (and other researchers) have used similar questions in previous studies of learning and the IAO framework has not emerged (for example, Petocz and Reid 2001).

Having decided on this analytic approach, we utilised the qualitative research package NVivo (QSR International 2007) to go through each transcript and code any statement about learning mathematics under one of these three aspects. This was done by two of the team independently (Anna Reid and Peter Petocz) and any differences were then discussed and resolved. Statements about 'intention', for example, were often made in response to the question *What do you aim to achieve in learning mathematics?*, but this was by no means the case all the time. NVivo made it easy for us to extract and investigate all the statements for 'intention', 'approach' or 'outcome' separately, coding them as 'free nodes' without any predetermined structure, appropriate for such an emergent analysis. This then allowed us to identify qualitatively different conceptions for each of the IAO aspects, and to place them in a tentative hierarchy. The parallel nature of the outcome spaces, particularly for the 'intention' and 'outcome' aspects, became quite clear. Each conception was examined separately and tested against students' transcripts to determine its place in the hierarchy; the student quotes given in the following section are all taken from this level of the coding. The analysis showed that the conceptions for each IAO aspect could be grouped into three hierarchical orientations which we have labelled 'techniques', 'subject' and 'life'.

Individual participants' statements were then used to categorise their conceptions of each aspect of IAO; this was particularly easy since the NVivo codings were immediately accessible. The first two transcripts that we examined showed a consistency across IAO, suggesting that we investigate the consistency across aspects for each participant. Such a classification of individual transcripts takes the analysis beyond a standard phenomenographic one; however, our aim was not to categorise individual students, but rather to use such classification as a method to extended the analysis. We have used this approach before in a comparison of individual students' conceptions of statistics and learning statistics (Reid and Petocz 2002) and an investigation of the relationship between their ideas of learning statistics and their views of their teacher's role (Petocz and Reid 2003).

Conceptions of Learning Mathematics

Students' conceptions of learning mathematics can be considered from three aspects, intention, approach and outcome (IAO) and can be organised into an outcome space for each aspect. The conceptions can be grouped into three broad orientations for each aspect that we have called 'techniques', 'subject' and 'life'. Table 3.1 displays the outcome space for students' conceptions of learning mathematics and shows that the conceptions forming each of the three aspects are broadly comparable, the intention and outcome showing almost parallel conceptions.

In common with other phenomenographic outcome spaces, these conceptions are hierarchical and inclusive (Marton and Booth 1997). The narrowest level focuses on the extrinsic and technical attributes, the middle level is essentially concerned with the subject of mathematics itself, and the broadest level looks beyond the mathematics to its place in students' lives. Those students who describe the narrower, more limiting views of learning mathematics seem unable to appreciate features of the broader, more expansive views. However, those students who describe the more holistic views are aware of the narrower views, and are able to integrate characteristics of the whole range of conceptions to further their own understanding of learning mathematics. It is for this reason that we as educators value the broader, more holistic conceptions.

The conceptions are now described and their aspects are illustrated with succinct quotes from the students' transcripts, each labelled with the student's pseudonym. Each individual quote is not necessarily indicative of the meaning of the category, but merely supportive, and the richness of each category is defined by the whole set of transcripts. Quotes from a particular student may appear in several conceptions, and at more than one level, illustrating the hierarchy discussed previously.

(1) Techniques Orientation

These conceptions are concerned with the extrinsic and practical features of learning mathematics, and where mathematics itself is explicitly involved the focus is on

Table 3.1 Students' conceptions of learning mathematics

	Aspect →		
Orientation↓	Intention	Approach	Outcome
Techniques	To pass the subject or course	Focus on course requirements and expectations	A pass, degree or qualification
	To get a [better] job, status, money		A [better] job, status, money
	To acquire mathematical tools and skills		Acquire mathematical tools and skills
Subject	To understand mathematics, practice, theory, applications	Focus on mathematical elements	Understanding mathematics, practice, theory, applications
	To help others with mathematics		Help people using mathematics
Life	To acquire a mathematical way of thinking or philosophy	Focus on understanding beyond mathematics	A mathematical way of thinking or philosophy
	To open one's mind, to satisfy intellectual curiosity		Satisfaction of intellectual curiosity, personal growth

the tools and component skills rather than the mathematics itself. In terms of intention, Brad's quote puts forward the idea of passing the subject or course, often mentioned in a jocular way by students. Dave is aiming for a job, and Grant for a better job – this is an important feature for some students and is in the background for others:

> *Brad*: [What do you think it's important for you to learn about mathematics?] Is 'whatever I think's going to be on the final exam' the wrong answer or …?

> *Dave*: [What do you aim to achieve through learning mathematics?] I, well I, a job, fairly straightforwardly, yeah.

> *Grant*: I basically have had a bit of a soul-searching time over the past few months, because like I just thought, oh yeah, I'll just sort of get this degree and be more employable and earn lots of money kind of thing.

Marios and Candy talk explicitly about acquiring the tools and skills of mathematics, but their intention for learning is a fragmented one:

> *Marios*: I see it as it's probably going to be a tool that I use, so I'm going to be doing something else and all of a sudden I'm going to need this tool and 'oh okay, yes I've learnt that, so I'm going to be able to use that tool to solve whatever problem I have in front of me.' So that's how I see I'm probably going to use the maths that I'm studying now, yep.

> *Candy*: [What do you think are the important things that you need to learn about maths while you are here?] Basic things like probably like calculating, the fundamentals of maths are really important, yes, but I think a lot of the theory that comes with mathematics, most students seriously don't understand it, and I can say I don't understand it either.

In terms of the way they actually go about learning, students focus on course requirements and expectations. This conception describes an approach to learning mathematics that emphasises doing the set work and completing the subject or course requirements. This approach is put forward by Candy, and then more fluently described in Dave's extract:

> *Candy*: Let's just say I do what is required in the course, I probably don't do as much study as I should. It's a matter, I think a lot of people just do enough to be able to pass and understand the basics of it, that is basically how I go about it.

> *Dave*: And most of, when you say, you know, 'how have you focussed on learning maths?', most of it has simply been getting the basic work done, getting the assignments in and just hoping that you have enough time to revise for the exams. And it's just been, I've felt on the back foot throughout, I haven't had that much trouble getting assignments in on time, and I reckon I probably revise a little bit more than average for most of the exams. But, so relatively speaking my performance has been above average, my marks are quite high, but from a personal basis I do feel I could have done better.

The outcomes described by students seem to be broadly parallel to the intentions. First, there are the extrinsic outcomes of qualifications and jobs, important for some and in the background for others. Candy explains this viewpoint:

> *Candy*: [What do you aim to achieve while you are learning mathematics?] Well, my degree, hopefully. Some, it's more, to be quite honest, I think most uni, most people who go to university to get university degrees today just are into it for the recognition of having a degree, and to be quite honest when it comes into the workplace, whatever, just say I have a mathematics degree right now, I'm a graduate with a mathematics degree, I can go into some sort of workplace or work area that's just totally unrelated at the moment. It's just, I think to a lot of people, it's just having a degree means you have some sort of level of achievement or intelligence that employers will recognise.

When mathematics is explicitly mentioned, the outcome focuses on acquiring the tools and skills of mathematics, rather than on the mathematics itself. As an example, Candy sees the skill of using a statistical package as an outcome of her mathematical learning:

> *Candy*: [What is it that you want to hold on to and remember?] I want to be able to remember how, how things are done, so just say I'm doing, I'm doing the statistics right now, yeah I take away from the course, so that's actually very practical because it uses a lot of computer packages and they teach how to calculate the statistics using these packages, so that is something I can actually take away with me and be able to use.

(2) Subject Orientation

These conceptions are concerned with learning the actual subject of mathematics itself. The intention is to understand all aspects of the mathematics being studied, the practical and theoretical, the pure and the applied. In the two following quotes Heather focuses on understanding the theory while Ian is more interested in the application of mathematics to his area of finance:

> *Heather*: [What do you aim to achieve when you are learning in mathematics?] A deeper understanding of the use of formulas and techniques of mathematics, so not just rote learning a whole lot of formulas and saying 'okay, this is such and such theorem and this is

what it means', but more 'how has he come to this conclusion and why?'/…/So you've got to understand the actual techniques, not just how to apply them. That's usually my aim because I don't really care about the applications much, as understanding, because I figure that if you understand then you can easily apply it, usually.

Ian: My objective is to learn what areas, and in finance and yeah I can learn more about them and at higher levels using the mathematics and understanding the mathematics of how I guess financial markets and that sort of thing, and prices in financial markets and basically just the dynamics and pricing of assets in the financial markets I'm primarily interested in.

As part of this conception, while retaining the focus on the mathematics itself, some students talk about an intention to use their mathematical skills and abilities to help other people in a variety of situations. Paulo and Ashleigh both express this notion in general or specific ways:

Paulo: Yeah, so in some ways you can help the world when it has a problem and you have this background in applied maths and you can say, 'oh, I can help'. That's good.

Ashleigh: [What do you think it will be like to work as a qualified mathematician?] It will be good for me because, you know, I would know everything, and I think it's different because not many people know it, so I would be able to help them with a lot of things, like mathematical things, yeah./…/For example if I was to work as an analyst, with using stats packages and things, I would be able to create for the company, different strategies and improvements.

With this conception, students describe an approach to learning mathematics that focuses on various aspects of the mathematics itself. This includes selecting, connecting and applying particular aspects of mathematics, learning from theory or from examples and applications. There is an underlying idea of the cohesive nature of the mathematics being studied (as opposed to the fragmented ideas in the 'techniques' conception). Elly and Sujinta explain the view:

Elly: [What kind of things would you say you focus on when you are learning in maths?] I guess I try and understand what I'm learning first and then try to put it into practice, that's the way I go about it. [What do you mean by understand?] If I'm given a theory I'm not just going to use it blindly, I'm going to try not to. I'm going to try and see where the theory comes from, what it relates to and then be able to use it, yeah.

Sujinta: So I'll look at the theory and sort of understand that and then actually see numbers changing, so if you are using a different, well, oh okay, if you are using one method perhaps, seeing the starting numbers and seeing all that at the start and then seeing someone work through, seeing how the numbers change as you work through the process and getting to the final number, and then I can, if I don't understand anything in the theory, then what I can do is check that back with the numbers and play with the numbers and go 'oh okay, this number came from this one divided by that one' or something, and look at the theory and then I understand it more there because I'm getting the practical side of it.

The parallel outcome is an understanding of the practice, theory and applications of mathematics. In this conception, the outcomes are described in terms of the aspects of mathematics that students are able to use. Ian refers to financial mathematics while Yumi talks about the use of her statistical learning:

Ian: [What do you want to take with you from your learning of mathematics?] Oh, I just, well for me in particular I guess the most important for me is a deeper or more technical or involved understanding of financial markets and the dynamics of that sort of field.

Yumi: [What do you think you will be taking with you from your learning of maths into work?] Oh, just the ability that I can apply things, especially with the, I think statistics is really, really helpful and that's really important I think in the workforce as well to have that sort of knowledge, how to use it, because you are always looking to improve things in the company and if you have got historical data, then you can perhaps analyse it to see what areas are, do you know what I mean?

Some students are aware of an outcome that allows them to help other people using their mathematical skills and abilities. Joseph has found that he is able to help school students with a subject that they find difficult, while Andy has a more general idea of being a mathematical 'trouble-shooter':

Joseph: [What is it that you liked about teaching mathematics?] Oh, because like most of the students are in trouble with mathematics, you know, so if I explain to the students about the concepts and then they are happy, very, they are happy straight away and then I am happy teaching them. So I want to get people to get more happy, you know, I want to help as much as I can do to people.

Andy: [What do you think that work will involve, the work that you are going to do?] Well, I think the ability to, well being able to give advice to certain people as to how to go about solving a problem that they might have, a bit like a trouble-shooter maybe, or actually doing a lot of the analysis myself, like one or the other or both I think.

(3) Life Orientation

In these conceptions, students go beyond the actual discipline to focus on the role of learning mathematics in their personal and professional lives, and the way it changes their view of the world around them. Students may talk explicitly about their intention to develop a mathematical way of thinking and looking at the world – Richard and Hsu-Ming explain this view:

Richard: My personal aim in learning in mathematics is to strive for clarity of thought more than anything, and by saying that, I guess I lean towards the more abstract models because it requires a greater effort intellectually to grasp what it is that is being proposed.

Hsu-Ming: Maybe I want to have an understanding of the world around me and I believe that with mathematics, with the principles of mathematics, I feel as though I will be more able to, I guess yeah, more able to do that, to understand, with yeah, with the basic concepts, just that, that line of thought will be able to help me understand the world around me better.

Some students talk about being intrigued by mathematics and its place in the world, and describe an intention to learn mathematics to broaden their mind and satisfy their intellectual curiosity. Eddie describes his feelings of wonder about learning mathematics:

Eddie: There are some things that you can study that are, that are intellectually so beautiful it's gratifying to study it for its own sake, I mean when you can see it you think 'oh yes, this, this has to be right'. And it's just such a beautiful and elegant chain of reasoning like I, I sometimes hear people talking about murder mysteries or something or where the inspector, Inspector Poirot, you know, sort of engages in this, or Sherlock Holmes or whatever, engages in this beautiful piece of deductive logic to determine who did something. Well that's sort of about one percent of what you can be talking about with, with some mathematics.

The corresponding approach to learning mathematics involves going beyond formal studies of the subject and using that as a way to understand mathematics itself. Grant tries to explain this approach and its motivation of satisfying his intellectual curiosity. Hsu-Ming explicitly utilises his broad life experience in his learning project:

> *Grant*: I just find maths sort of fascinating as a subject, so it's just a matter of sort of finding out, you know, because I have a lot of sort of questions that I sort of, I'm wondering is it possible to do this or, you know, how, or what's the basis behind that or whatever, and it's just interesting finding out, you know, how these various theories have come about to sort of allow you to do things when you kind of wondered if it was possible to get the answer to certain questions and, you know, then finding out that you can. So yeah, it was just a matter of sort of satisfy my own curiosity I guess.

> *Hsu-Ming*: Well I'm a mature age student, I had to do something between leaving high school and I had a few years until I was back in the TAFE college and then quite a few years and now I'm here, so I know from those times that I'm not in an educational facility, that nothing stops me from learning and we have to learn every day. Again, it's just that knowledge base, this gives me a broader knowledge base to start from, and that's, and my starting point is much broader now, yeah my understanding of things, or if I wish to learn something, and I'll have the confidence to go out and do it too, I guess.

In terms of outcomes, students in this conception are aware of having acquired a mathematical philosophy, or approach to life, or way of thinking. Richard, Dave and Hsu-Ming express this awareness of the outcomes of their mathematics learning.

> *Richard*: My learning of maths, I think that is fairly straightforward, when I say that, in my mind what I mean is that my learning of maths gives me, hopefully, a robust framework, clarity of thought in which to apply that framework to future problems.

> *Dave*: But overall, as I said, that doesn't detract from the style of thinking that maths gives you, which I think is brilliant.

> *Hsu-Ming*: Now, I wanted to develop my mathematical skills to, as I said, that would certainly enhance that field, with the under…, the knowledge that I'm gaining. However, I'm finding that, back to the original goal, I'm finding it, the knowledge that I'm gaining, more adaptive to the broader spectrum of the world rather than just in a particular situation, so in that sense, that's how it's diverging. That field is specific whereas I'm finding more general, generalisation occurring.

And some students, such as Julia, describe the personal growth and intellectual satisfaction, the awareness of the broad role of mathematics in the world, that they have acquired in the course of their learning:

> *Julia*: I think it's, it's mind dazzling how much maths accounts for. I mean, to the naked eye maths is one, two, three plus four, have to do this subject and let's get out of school, but if you sit there and look a little bit further, you find maths in all sorts of areas and that, I think, is wonderful. It's like you've learnt this, you've got this knowledge that can be applied to so many different areas, that's got to count for something. It's a powerful thing actually, it's the same theory that engineers use, it's the same theory that financial managers and leaders use, it's exquisite. /…/ Yeah, it explains so much in life, technology. It's amazing how many things are explained by maths and applied, I mean I don't even know all the applications and I've done three years, maths has got a whole lot of applications in our life, in our world, which you wouldn't be aware of until you scratch the surface and try to find out. /…/ So the more you learn, the more you find out there is to learn.

Discussion

Early results from the NVivo coding encouraged us to investigate the consistency between IAO in individual transcripts. Since IAO are three aspects of learning, it would seem reasonable that a particular student's conception of their own intention, approach and outcome would display the same general orientation. So if a student viewed mathematics learning in terms of 'techniques', then their conceptions of IAO would show consistency across the technical orientation. This is the case with Candy, for example, and we have given quotes from her under all three aspects. Importantly, no sections of her transcript were coded at any of the broader conceptions. On the other hand, if a student viewed mathematics as an integral part of their life orientation, then their conceptions of IAO would show consistency across the life orientation: Hsu-Ming presents an example with quotes under all three aspects. However, due to the hierarchical nature of outcome spaces, we might also expect to see evidence of the narrower conceptions at various points in her transcript, and indeed this was the case.

The majority of our respondents showed this sort of consistency in their transcripts: 14 out of 22 were easily in this group, and six of the others could have been included but for a single section, often in a summary of their views, where they expressed broader conceptions of one of the aspects. We believe that the discussion in the interview had the effect of encouraging them towards articulating broader conceptions, and we have experienced this feature in previous interview studies. We also noticed that two mature-aged students expressed narrower intentions but broader outcomes. In each case, it seemed that they had started their study of mathematics with very pragmatic intentions (getting a better job), and were somewhat surprised at the broader outcomes (a mathematical way of thinking). Dave's quotes in the previous section illustrate this situation – he noticed the 'brilliant' style of thinking that he developed through his study of mathematics. Another respondent, from a non-Australian background, showed an interesting transcript that combined the narrowest conceptions focusing on the pragmatic concerns of job, status and money with the broadest conceptions exploring the philosophical and intellectual excitement of learning mathematics, but with nothing in between these extremes.

The one distinct difference between students' generally parallel discussions of intention and outcome is concerned with a group of professional skills and dispositions. These include personal qualities such as hard work, persistence and patience, technical abilities such as computer skills, social skills such as teamwork, and intellectual skills of decision making and problem solving (which comes close to 'mathematical way of thinking', but isn't the same). These were not mentioned by any of our respondents in terms of intention, but were discussed by several of them as outcomes of their learning. It seems that students were not expecting to develop these professional skills as part of their mathematics learning, but some of them were surprised to see how they had in fact acquired them. Here are some examples from the transcripts:

> *Ashleigh*: I think like so far it's helped me a lot, like yes to help, it's like I've actually learnt over the years like doing maths, how to, you know, get myself involved, more involved in like teamwork, yeah I've noticed, especially with stats, so, it's interesting.

Monique: Thinking back over the subjects that I have done, they're, they're contributing to the kind of work that I want to, so not only the subjects, the things that I've learnt in the subjects, but also the skills that I've learned while doing the subject. And not only the skills but also the qualities, like being patient and having to persevere in studying and doing projects and I think that will be useful too when I go to work force.

Yumi: Into work, yeah, so I've said, applying my knowledge to certain problems and discipline. I think just the skills you learn in general just by going to university. How to deal with people as well, how to use, you know, a computer and processes and packages and things like that.

Gabrielle: Seeing patterns in things and analysing things, analysing data and breaking it down and looking at problem solving obviously as well. So not only in a mathematical context, but I think it's given me analytical, not powers but reasoning in other areas as well, yeah and problem solving.

We will discuss this outcome of students' learning in mathematics Chap. 5, when we report on recent graduates' comments on their previous mathematics learning. It seems that when students start a degree in mathematics, such professional skills do not generally form part of their expectations. As their course progresses, they become aware of learning a broader range of skills and dispositions than they had previously expected.

Implications for Teaching and Learning

Our findings concerning students' conceptions of learning mathematics have some immediate implications for mathematics pedagogy, and in this section we will discuss some of these. Our aim is to indicate how the research results that we have obtained can be utilised in the process of teaching and learning mathematics. The first point follows from our discussion of conceptions of mathematics in Chap. 2. Students are usually unaware of the range of variation in thinking about their learning, and tend to assume that their fellow students share their own views. Yet students sitting in the same class can have very different ideas about the nature of learning in mathematics and these ideas contribute to the approach that they adopt in any learning situation. Introducing them to the full range of conceptions seems to be an effective initial step in helping them develop broader views. Students could be asked to think about how they view learning in mathematics, and a short description of the results in Table 3.1 could be presented, maybe when their first assessment task is introduced or as part of an activity in an early tutorial class. Telling students about the range of variation in ideas about learning mathematics will not, of itself, broaden their conceptions, but it will make it easier for students to think more deeply about their own views (Reid and Petocz 2003 gives a case study of such an approach in the context of a course in regression analysis).

When students are aware of the range of conceptions of learning mathematics, they will be more likely to discuss their learning with their colleagues, especially when opportunities are made for them in the form of group work in classes or laboratories, and group assessment. Students can have quite sophisticated views of their own learning, and can be strong advocates for a deeper approach. Julia's quote expresses the situation better than many teachers could:

> *Julia*: There's monkey learning and there's proper learning. Monkey learning is finding out what you need to learn for the exam to get through, proper learning is finding out what's behind the numbers that you are writing down so that you know for yourself. There are people that do very well in a subject because they learn what they need to know for the exam, but you ask them three or four weeks later and they couldn't tell you. There are people that won't do that well in their marks, but you ask them three years down the track and they will be able to explain to you how that matrix works or whatever you are talking about. There's always a difference. And it takes a lot more time to learn the background than the 'what you need to know'.

Another important strategy is to arrange learning situations that encourage students towards the broadest conceptions of mathematics learning and away from the narrower conceptions. For instance, a class environment that presents a course in terms of a sequence of definitions, theorems and proofs, and rewards students in examinations for rote learning them, encourages students to focus only on passing the course and acquiring the appropriate techniques. In such a situation, even those students who are aware of broader conceptions of learning will be encouraged to work using the more limited ones. On the other hand, laboratory work and assignments that ask students to analyse the solutions to a differential equation, or to carry out an analysis of a set of statistical data, and then explain the meaning of the analyses to the people involved (clients, colleagues, readers of a professional publication) immediately expand students' focus. So too do assignments or projects that allow students to select material based on their own interests and explicitly ask them to think about and discuss their own learning (see Viskic and Petocz 2006, for example). Another option for teachers is to develop learning materials that try to engage students at a broader level with an expanding notion of learning mathematics. One example is the book *Reading Statistics* (Wood and Petocz 2003), which asks students to read and engage with research articles in a variety of areas of application, and to communicate the statistical meaning in a range of professional situations. Using such materials and pedagogy, we can set up learning situations that afford scope for students to become aware of the broader role that mathematics can have in their studies and their professional lives. Our experience is that many students will take up these opportunities.

Summary and Looking Forward

We continued the description of the first stage of our research project by reporting on students' conceptions of learning mathematics. Early in our analysis, we identified the IAO – intention, approach and outcome – framework, and used this to guide our analysis. We found that students viewed learning mathematics in three quite different ways. At the narrowest level, which we have termed 'techniques', they focused on the extrinsic and atomistic aspects, including obtaining a pass or a qualification, and learning isolated technical skills. At the next 'subject' level, they broadened their focus to the subject of mathematics itself. At the broadest and most inclusive 'life' level, they appreciated the contribution of mathematics learning to their personal and professional lives. These three levels of conception were identified in statements about intention, approach and outcome, with a particularly strong parallel between intention and outcome, though with some unexpected results in terms of professional skills for some students. Our identification of the IAO aspects of learning builds on the work of previous researchers, particularly the notion of surface and deep approaches to learning (Marton and Säljö 1976a) and the 3P (presage-process-product) model (Biggs 1999). Our 'techniques' and 'subject' orientations are loosely linked with surface and deep approaches to learning, but our broadest 'life' orientation seems to extend the notion of deep approach to learning. Biggs implicitly acknowledges the importance of intention and outcome in students' views of learning: we have provided a description of the qualitative differences in these components, as well as the process or approach component, specifically in the discipline of mathematics.

In discussing the pedagogical implications of these results, we again pointed out the importance of helping students to become aware of the full range of views about learning mathematics, and of using appropriate pedagogy and learning materials to encourage students towards the broadest and most inclusive 'life' conception. There is much evidence to indicate that this is contrary to the common practice in many school classes and even university lectures. Wiliam (2003, p. 475) states this clearly: *"If one observes practice in mathematics classrooms all over the world or looks at textbooks, the predominant activity seems to be the repetition of mathematical techniques through exercises."* Classes of this type direct students strongly towards the narrowest conception of learning mathematics and of mathematics itself. Students are likely to see little relevance of learning mathematics to their own studies, professions and life situations. With a clearer understanding of students' ideas of mathematics and learning mathematics, we are in a better position to develop a pedagogy of mathematics that will encourage students towards the broadest views of their subject and its uses in their professional lives.

In the next chapter, we begin the process of extending our results, obtained from a small group of mathematics undergraduates, to a larger group of students studying mathematics in a wider variety of contexts and in a number of countries with different educational systems. Having obtained information from in-depth interviews, we are now in a position to make use of it to design and carry out a project to investigate the views of a larger group of students. For obvious practical reasons, we have to collect their views using a less labour-intensive approach, replacing interviews with open-ended survey questions.

Note: Some of the material in this chapter was previously published in Reid, A., Smith, G. H., Wood, L. N., & Petocz, P. (2005). Intention, approach and outcome: University mathematics students' conceptions of learning mathematics. *International Journal of Science and Mathematics Education*, *3*(4), 567–586.

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Chapter 4
What Do Mathematics Students Say About Mathematics Internationally?

Introduction

In this chapter we continue our investigation of students' views about mathematics. Here we move to the second phase of our project, involving a much larger and more diverse group, over 1000 undergraduate students from five universities in five different countries – indeed, five different continents. The previous chapters have investigated the views of a small number of undergraduate students studying mathematics as a major. Information on their ideas about mathematics, learning mathematics and using mathematics was collected from a small number of semi-structured interviews and analysed using a phenomenographic approach. By extending our research to an international undergraduate group, we are able to check whether the results that we obtained with our 22 mathematics major students in an Australian university can be supported with a much broader sample group.

In order to achieve this generalisation, we have to change some features of our earlier study. The most obvious one is that it would not be practicable to carry out so many interviews, so instead we have gathered students' views using written, open-ended questions. We are still interested to see what new ideas may come up from the broader group in the study, so we give them the opportunity to write down their thoughts, yet we realise that for practical reasons we can ask them only a few questions about their views of mathematics, and their responses may only be fairly short written statements, making it more difficult to identify their conception of mathematics. However, we are not starting from scratch – we already have a categorisation for conceptions of mathematics from our small group of undergraduates, and our aim is to investigate whether those categories can be used to classify the ideas from this much larger group.

Another feature of this larger sample is that they are not all mathematics majors: the group includes students studying mathematics as a 'service' subject in areas such as engineering, commerce and computer science, as well as some who are studying mathematics as part of a teaching qualification. For such students, the use of mathematics

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is one component of their professional life, rather than its centre. This might result in a narrower range of conceptions of mathematics amongst groups of students for whom mathematics was a component rather than the focus of their professional preparation – though this is not our general experience in phenomenographic studies. We have shown in Chap. 2 that engineering and sports science/tourism students showed the same range of conceptions of mathematics as did mathematics major students. In previous research, we have shown that this situation holds also for statistics, with majors and service students showing the same range of conceptions of the discipline, from narrowest to broadest views (Petocz and Reid 2005). Further, in a quite different field, we have found that primary and secondary school children show the same range of conceptions of the notion of 'environment' (Loughland and et al. 2002), and that the views of adults can be adequately classified using the same sample space (Petocz and et al. 2003).

Our results show that the original sample space can be applied successfully to the larger group, though we have made two modifications. Firstly, we identified a conception of mathematics ('number') that was narrower than the 'components' conception. The two conceptions share a fragmented view of mathematics, though the narrower 'number' conception lacks the notion that 'components' are part of a coherent mathematical investigation. When we analysed the results from the interviews with mathematics major students, we did not distinguish this narrowest conception as the sample did not contain any student who viewed mathematics exclusively at this level. Secondly, we decided to separate the intermediate 'modelling' conception into applied and pure aspects, referred to here as 'modelling' and 'abstract' respectively. This was foreshadowed in our earlier analysis, and in the team's early discussion of the interview transcripts, when we referred to models as being representative of specific situations or universal principles.

In the next sections of this chapter we report on the categorisations of students' conceptions of mathematics and the relationships between these categories, supported by quotes from their written responses to the open-ended questions. We use standard quantitative techniques to investigate the relationships between the conceptions of mathematics and other variables such as university, year and area of study. We summarise students' views about their future use of mathematics, obtained from the responses to two of the survey questions. Finally, we discuss the implications of these results for teaching and learning university mathematics. By the end of the chapter, we are in a position to assess the contribution of adding an international perspective to our overall aim of investigating the process of becoming a mathematician.

The Methodology of Our Study

In this second phase of our study of students' ideas about mathematics, we broadened our focus from a small group of undergraduate mathematics majors at an Australian university to a much larger sample of students studying mathematics as a major or as a service component in courses such as engineering, commerce and computer science, or as part of a mathematics education degree. With our international team, we developed an open-ended questionnaire that was completed by 1182 students from five universities on five continents – the University of Technology, Sydney (UTS) in Australia; the University of Pretoria, South Africa; the University of Ulster, Northern Ireland; Concordia University, Montreal, Canada and the Universiti Brunei Darussalam, Brunei. All these countries have historical links to the British Empire, and hence their educational systems derive in some part from the British system. Nevertheless, their present situations are quite different, and include for the majority a non-English national language.

Students in relevant classes were informed of the aims of the research, were invited to participate and, if they agreed to do so, completed the survey in class. While the survey questions were presented in English, the language of instruction in the class, students were told that they could respond in their preferred language, and some students from the South African and Canadian samples replied in Afrikaans and French respectively. Although participation was voluntary, the majority of students in each class supported the research program by completing the survey. The survey consisted of general demographic questions, asking students their age, sex and language background, and their degree and year of studies; these were followed by three open-ended questions. The first and most important question was: *What is mathematics?* The responses to this question form the focus of the first part of the present chapter. The two additional questions were: *What part do you think mathematics will play in your future studies? … in your future career?* A summary of the responses to these questions is presented later in this chapter.

To analyse the data from the first question, we used an approach based on a knowledge of the previously-developed phenomenographical categories of conceptions of mathematics (as we have described them in Chap. 2). These categories were used as a starting point, with the possibility of modifying or augmenting them if necessary. Student responses from the different institutions were randomly rearranged, making it impossible to determine to which institution, sex or study field the student belonged. The responses were labelled so that they could later be ‘un-randomised’ and linked to the demographic information for quantitative analysis. Four members of our international research team (Leigh Wood, Ansie Harding, Johann Engelbrecht and Geoff Smith) studied each response separately and suggested categories describing the variation in students’ conceptions of mathematics. These categories substantiated and in some cases extended those found in the previous interviews undertaken with the Australian undergraduates, and they were refined and checked by repeated reading and discussion. The final categories were confirmed by identification of appropriate illustrative quotes from the questionnaires. Following this, each analyst coded the responses into the categories identified and compared results to check the consistency of the coding. Where there was a difference of opinion, we were usually able to reach a consensus after further discussion. On those occasions (fewer than 4% of cases) where consensus could not be reached, the response was omitted from the analysis.

For the second and third questions a content analysis (Weber 1990) seemed more appropriate than a phenomenographic analysis, particularly as we had no previously-developed outcome space on which to base analysis. We returned to the original interview study, which contained some relevant comments about future use of mathematics, and identified various themes that were then used in the content analysis of the survey responses. The unit of analysis was the sentence (or occasionally, group of sentences) that each student wrote in the survey concerning their future use of mathematics. Again, the same four team members independently classified the students' responses according to the themes identified and compared results. All differences were resolved by discussion by the team.

To some extent, the form of the survey inclined students to write less rather than more, and hence their statements were in some cases too brief to accurately determine their conception of mathematics, or their views about its future use. However, there were many students who wrote more lengthy responses to one or more of the questions – over 100 words in some cases, although the average was around 20 words per question. Although we used the students' responses to determine their categorisation, it is an unavoidable limitation of our methodology that such short responses could have resulted in wrong allocations. During interviews, such problems can be resolved by further questioning, so that even if a student starts with a short 'definition' of mathematics as *"a subject involving numbers"*, or a brief statement that mathematics will play *"a major part"* in their future professional work (quotes from the survey responses), their underlying ideas can be explored in greater depth.

Results from the Survey – Conceptions of Mathematics

In this section we describe the categories that we found, illustrate them with quotes from students and summarise the relationships between them. Some of the responses to this question consisted of short statements without details, or longer but evasive statements, and so we were unable to categorise them into conceptions. Some examples are: *"a language"*, *"a science"* and *"such a question could require a very complex answer which I feel I am not both educated and wise enough to answer"*. Other responses gave information on students' attitudes to rather than conceptions of mathematics, for example, *"hard stuff"*, *"a waste of time"* and *"my first, my last, my everything"*. The majority of responses, however, could be attributed to one of the conceptions of mathematics identified in the previous interview study, or a conception derived from those. We identified five qualitatively different categories (the numbering has been chosen to highlight the consistency with the previous results from the interviews).

(0) Mathematics Is About Numbers

In this conception, students consider mathematics to be connected with numbers and calculations. Mathematics is manipulation with numbers with no essential advance beyond elementary arithmetic. People mention numbers, calculations, sums and basic operations, usually fairly briefly, as the following examples show.

- Mathematics is where we do calculations and deal with numbers.
- The study of numbers in different formats.
- It is a study of the science of numbers and manipulation of numbers.
- Mathematics is the understanding of numbers and calculations.
- Mathematics is a subject that is set up and designed to help your knowledge of arithmetic. It is a very important subject for everybody who is situated at a school or college as it is a compulsory unit which deals with numbers and multiplication which is needed in future life.

The last quote is more lengthy, and talks about the importance of mathematics, though it retains a focus on number (or arithmetic).

(1) Mathematics Is About Components

Here, students view mathematics as a toolbox to be dipped into when necessary to solve a problem, or a disparate collection of isolated techniques unrelated to real-world applications. Students may list some of the components, or mention formulas, equations and laws.

- Mathematics is a collection of tools that you can use to solve problems.
- Maths is a selection of theorems and laws, which help solve equations and problems.
- Mathematics is a study involving analysis, statistics, algebra and other calculations.
- Mathematics is the use of numbers and letters to solve equations and problems so that objectives can be achieved.
- Mathematics is the use of numbers to solve problems and engineering tasks through the use of processes and formulas.
- It is the field where we learn to add, subtract, multiple and divide correctly. We also learn how to put data onto graphs and, well I don't know, I think it also has got to do something with logic, and probability. Since people or rather mathematicians said maths is the universal language, I was interested in know[ing] more about it.

This last and most extensive quote also illustrates the hierarchical nature of the conceptions, containing obvious reference to the idea of mathematics as number.

(2a) Mathematics Is About Modelling

In this conception, students link mathematics to the world using the notion of modelling. They make strong connections between mathematics and the physical world, which can be described, perhaps imperfectly, by mathematics. Mathematics is seen as a human endeavour invented to describe the world. Here are some illustrations of the conception:

- The attempt to explain the physical laws, patterns of the physical world by algebraic and numerical means.
- Mathematics, and especially actuarial mathematics, is the model set up to analyse and predict real world events.
- Mathematics is a way to solve problems presented by physics, chemistry, finance and many other fields. It is a way to model the world, so we can understand it better.
- Mathematics is the study of modelling real life issues. Because we are extremely uncertain about the future, the models we derive in mathematics [are] actually a relief for future uncertainties.
- A modelling tool. We use it to predict and determine things such as how a beam will react under stress, e.g. it would be less costly to predict and model the above on paper rather than building first.
- It is the study that provides you with the knowledge to explain things in a form of an equation, for example, the number of accidents in one place can be represented by a graph that is defined by a mathematical equation, and of course simple arithmetic.

Again, the last quote gives a hint of the hierarchy, referring to *"simple arithmetic"*.

(2b) Mathematics Is About the Abstract

The emphasis in this conception is on mathematics as a logical system or structure, the 'pure' models that are a counterpart to the 'applied' models in the previous conception. Applications and modelling techniques may be recognised but are regarded as secondary to the abstract structure of the mathematics. Mathematics is the 'other', perhaps even a kind of game of the mind, somehow pure and abstract. The following quotes illustrate the conception:

- Conceptual thought and logical development of ideas.
- An intellectual pursuit. Conceptual understanding and the application of techniques.
- Maths is a way to describe a perfect world. A way to put something on paper that is impossible in the real world. Thus, maths is something that only exists in theory. A very happy place.
- It is the way in which an abstract problem gets turned into a logical numerical analysis and from which a logical conclusion can be made. It's the logical way of solving an abstract problem through various methods.

- Mathematics is not so much a subject as a way of thinking. It is based on logical thought used to solve complex problems. It is neither just creative nor just methodical but rather a combination. It has little to do with numbers and much to do with principles.
- The abstract yet the most fundamental science in the world. The basic tools for analysing other subjects. Though other subjects will study the real thing (theorems, phenomena, particles …), mathematics focuses on providing logical and numerical ways for your research in others.

(3) Mathematics Is About Life

In this broadest conception, students view mathematics as an integral part of life and a way of thinking. They believe that reality can be represented in mathematical terms in a more complete way than the modelling conception. Mathematics mediates their way of thinking about reality. They may make a strong personal connection between mathematics and their own lives. This is what some students have written illustrating this conception:

- A way of life – [I] don't have a definition handy at present though.
- Mathematics is the language of nature. It is the way in which nature is ruled by God.
- Mathematics is a way to approach life in an analytical manner as to support and formalise natural processes. In a sense it is a way to understand how life works.
- Mathematics is life. Without it life would be a misery. It helps us understand many concepts in life. it even helps in science subjects, especially physics.
- Mathematics is an essential part of everyone's life. It is used either directly or indirectly (was done by someone else) in every facet of our lives. Most technology uses mathematics either in developing the strategy or in implementing a task. Maths is what makes the world function.
- Mathematics is the key factor involved in almost all lines of work, buildings, banking, accountancy, etc. Maths helps society to build in units and provides society with a backbone to all careers. It is a combination of numbers and equations which helps find calculations and answers to lengths, statistics, etc.

The first couple of quotes are very short, maybe even glib, but they give a clear flavour of the 'life' conception. The others develop the idea of the universal aspect of mathematics and its connection with professional and personal life. Interestingly, the last quote includes reference to the 'numbers' and 'components' conceptions. And here is a final interesting quote; it also includes reference to the narrowest conceptions, as well as illustrating aspects of mathematics as modelling. Yet it contains both a range of modelling situations (reminiscent of Eddie's quote from Chap. 2

Table 4.1 Distribution of conceptions of mathematics

Conception	N	%
0. Number	109	9
1. Components	515	44
2a. Modelling	235	20
2b. Abstract	165	14
3. Life	71	6
(Missing/uncoded)	87	7
Total	1,182	100

illustrating the 'life' conception) and also a reference to the role of mathematics in *"everyday life"*:

- It is series of calculations and formulas that aims to achieve business needs and everyday needs. It is an art because there are many ways to solve a problem. It is a language because you have to learn a lot of jargon. Research, we use mathematics to determine the optimal solution to companies' problems. In additional to that, mathematics has also played a vital role in our everyday life. Housewives use mathematics to do household accounting, budgeting. Students use it to pass their examinations.

These conceptions have a clear hierarchical relationship, inherited from the outcome space found in the earlier interview study, with two modifications. The narrowest conception is 'number', which was not identified as a separate conception in the interview transcripts (though our previous studies of students' views of statistics, which could be viewed as a component of mathematics, did include a 'numbers' conception – Reid and Petocz 2002). This is followed by the 'components' conception, then the 'modelling' and 'abstract', and finally the broadest 'life' conception. We regard the 'modelling' and 'abstract' conceptions to be at the same hierarchical level: one describes modelling applied to the real world, while the other refers to abstract (mathematical) structures and ideas. Neither seems to include the other logically, and the empirical evidence – both from these surveys and the previous interviews – leads us to place them at the same level, as two aspects of the same idea.

The distribution of these conceptions is shown in Table 4.1. Due to the larger numbers of students involved in the open-ended survey, we are able for the first time to make meaningful quantitative comment about the distribution of conceptions. The 'components' conception is the most common in this group, followed by 'modelling', and the 'life' conception is the least common.

Results from the Survey – Future Use of Mathematics

The framework for our content analysis of students' views about their future use of mathematics is grounded in comments that students made in our original interview study. Although they were not specifically asked about the role of mathematics in their future, many of the interviews did include some responses that seemed to relate to their careers. Surprisingly, some students said that they had no idea what role mathematics would play in their future careers, even though they were majoring in an area of mathematics. Our content analysis of these comments could be grouped into four categories: unsure; procedural skills ('knowing how'); conceptual skills ('knowing that'); and professional skills ('knowing for'). We have found these skills categories useful in assessing the development of generic skills in the context of undergraduate business education (Wood et al. 2011). The categories allow us to describe the range of ideas that students had, and form a basis for checking the validity of the later (and much briefer) questionnaire responses.

Unsure: Some students had little idea of what they might be doing as a mathematician. It was sometimes even difficult for them to see themselves as a mathematician or know what they could offer an employer with their mathematical skills.

> *Candy*: I'm not exactly sure myself, so I can't really imagine what it will be like to work as a mathematician, or be recognised as a mathematician until I graduate, and a lot of people wouldn't even realise, they will be probably thinking 'what can I do, what can a mathematician, like, offer me?', in a sense, if you know what I am saying. It's not like, oh, accountant, lawyer, like that's just straight away 'oh, I need one of those', but like with a mathematician, 'what can I do with a mathematician, what do I need one for?', you know. So I'm not exactly sure, because right now that's what I think as well.

Procedural skills: Some students had the idea that mathematics could be used as a toolbox of procedures from which to select as needed in their further career. The tools may be simple or complex, but they remain tools, and only certain isolated skills or techniques are regarded as relevant.

> *Sujinta*: It's like a toolbox, you are getting a lot more tools and in, like if you are a carpenter or something, like you did, before you had a maths degree, you just had a screwdriver, where you come back and now you have a screwdriver, a Phillips head, you've got pliers to do different things, you've got saws to shorten things, so like simplifying things, you've got a lot more tools there to play with, so it makes you a lot, well, when you get into the workplace, you are much more of an attractive sort of employee to have, yeah.

Conceptual skills: Some students focused on the idea that studying mathematics develops conceptual skills such as the logical thinking associated with the mathematical approach – solving mathematical problems and the notion of mathematical proofs.

> *Hsu-Ming*: [What are you aiming to achieve through learning mathematics?] Nothing specific, I guess it's more of a way of thinking, a thought process, rather than anything specific, or I can't come out saying, I've not come into this particular course wanting to learn one specific thing or many specific things. So I guess I'd have to say, and its rather vague, a generalised thought process.

> *Brad*: I think it's one of the fundamental things, because mathematics is all based on proof. We, somebody, notices something happens in this particular case and then they sit down and establish whether it will happen in every single case, and all those proofs are based on logic.

Professional skills: Some students pointed to the generic benefit of studying mathematics, rather than to any specific role for the mathematics. These included generic problem solving (as opposed to the skill of solving specific mathematical problems), analytical and communication skills.

> *Monique*: The problem solving, the analytical skills, the, the decision making and some, be able to use like, be able to use for example the statistical packages that, and we learn that at uni, we are learning it right now, so I think it will be useful when I find work.

> *Dave*: I guess statistical consulting is something that interests me and it seems for that the skills you need are relatively narrow. I guess first and foremost you need the communication skills to, you know, figure out what is going on and relate it to your, whoever's employing you, your customer.

The responses in the survey addressed the role of mathematics in students' future studies and in their future careers. These were obviously seen by students as related, and many students gave the same or similar responses to both questions. Using the framework of unsure, specific procedural or conceptual skills and generic professional skills, we now present some quotes from the survey responses illustrating students' comments on their future use of mathematics. The framework is extended by a final category where students wrote of the essential role of mathematics, though without any detail.

Unsure: Some students felt uncertain, or even said that they had no idea at all, about their future use of mathematics.

- I don't know what career I am going to pursue so I do not know.
- Do I look like a fortune teller?
- Dunno, get back to me in about 10 years.
- To be honest I'm not entirely sure. Originally I thought it would have minimal, since I thought I would concentrate on the IT part of my studies. But I'm now open to any job opportunities math may open for me.

Procedural skills: This is the view of mathematics as a toolbox. Students believed that they would select from a range of mathematical procedures as they needed them, and that only certain isolated skills or techniques were relevant.

- Mathematics will provide a way to find values and prices of various financial instruments.
- I will use it to measure materials and work out dimensions of materials.
- A large part as calculations have to be carried out in order to carry out safety checks.
- It will be important for calculating stresses, loads and forces on structures.
- For calculating money in business if I have one. Also when I need to calculate currency exchange rate in overseas. Basically for the calculation of the daily use products.

Conceptual skills: These statements are based on the idea that mathematics develops logical thinking and mathematical problem-solving skills, which are useful for deepening understanding in other disciplines or professions.

- I will be required to assess the financial positions of funds and also be able to analytically make decisions based on a firm foundation.

- Quick thinking, ease with tackling difficult life problems, ability to think on an intense level.
- It will help us to think more logically and therefore able to solve problems more efficiently.
- Logic, since I'm taking computer science.
- The methods of solving problems, the many ways to derive to a solution, the discovering of new methods of solving them will help me enhance my mind to work faster as I go along to further my study in maths.

Professional skills: Here students see the importance of mathematics as providing a range of generic skills for their future profession, whatever that may be. They refer to communication and presentation skills, high-level numeracy skills, and generic problem-solving skills. Also included here are comments about the generic usefulness of mathematics in studies or the professional workplace, though without any specific details, as illustrated by the last quote.

- It will allow me to make better judgements based on a sound knowledge of various types and forms of data. My presentation skills may also be improved through the use of graphs, charts, etc.
- Maths helps me to become a good teacher and a better person, not left behind by much more educated people.
- It is important that I have a sound understanding of maths as it will allow me to evaluate and communicate ideas and problem solving throughout my future career.
- Huge! But I fear that my limited knowledge of IT programming and hard math as well as the increasing capabilities of PCs will render my maths skills too rudimentary and expensive to be of any real use. I hope to take a sort of reasoning or method of problem solving away from this rather than a bag of tricks and clever manipulations.

Necessary: Another group of responses talked about the necessary role of mathematics in a student's future studies or professional life, though without giving any further detail. This category seems to be an artefact of the survey methodology, asking students to respond in a few words or sentences to the questions. Follow-up questions (in an interview) would have been able to decide why such students thought that mathematics had an essential role.

- A very large part as I plan on being a statistician.
- As an actuary, I think it is obvious that it will be essential.
- It will play a major role in my professional career and it is essential in most, if not all, professional jobs. Therefore it is essential to have.

Uncoded: Finally, some of the students wrote responses that did not fit into the framework. Some of these just stated 'none', 'a small part' or 'a large part' with no further elaboration. This includes a small group of students who believed that they will not use much mathematics as it will be superseded by technology, and this may be a result of their experience with computer programs such as Maple and Mathematica.

Table 4.2 Distribution of views of future role of mathematics

Role of mathematics	Studies		Career	
Unsure	83	7%	55	5%
Procedural skills	327	28%	347	29%
Conceptual skills	82	7%	69	6%
Professional skills	264	22%	182	15%
Necessary	120	10%	243	21%
(Missing/uncoded)	306	26%	286	24%
Total	1,182		1,182	

- Not very big. Computer programs will do the work and applications for you.
- I don't think it will be as much as how important it is now in my studies because computers nowadays are capable of doing most things to do with engineering.

The group of students who participated in the survey had a range of views about the role that mathematics would play in their future studies and career, and these have been classified using categories that were consistent with statements made by the smaller group of students in the interview study. Again, the results can be presented quantitatively (Table 4.2).

There are some similarities in students' views of the role of mathematics in their future studies and career; indeed, many students gave the same or similar response to both questions. It is interesting that a small but not negligible group stated that they didn't know what role mathematics would play in their future. The most common response, of more than one-quarter of students, described a procedural role of mathematics, which seems consistent with a fragmented view of the discipline, though a substantial proportion of students focused on professional skills or described the necessary role of mathematics in their future.

Factors Associated with Broader Conceptions of Mathematics

With information collected from more than 1000 students, the survey gave us the opportunity to carry out quantitative as well as qualitative analyses. This section presents quantitative analyses of the responses to the first question, *What is mathematics?*, from which students' conceptions of mathematics were identified. The demographic variables collected in the survey (personal information on age, sex and language background, and academic information on degree, year of study and university) were categorised as shown in Table 4.3. We were interested to examine whether any of these variables were correlated with broader as opposed to narrower conceptions of mathematics. In particular, we wanted to check whether later-year students were more likely to show broader conceptions than were students at earlier stages in their studies. In this context, we defined the 'modelling', 'abstract' and 'life' conceptions

Table 4.3 Variables used in the quantitative analyses

Variable	Levels
Conception	0 = number, components; 1 = modelling, abstract, life
Age	0 = up to 25 years; 1 = mature aged (over 25)
Sex	0 = male; 1 = female
Language	0 = English; 1 = non-English speaking background (nesb)
Degree	1 = mathematics, 2 = other 'service' (computing, commerce, etc.); 3 = mathematics education; 4 = engineering
Year	1 = first year; 2 = second year; 3 = third year; 4 = fourth year
University	1 = Ulster; 2 = Pretoria; 3 = UTS; 4 = Brunei; 5 = Concordia

as the broader ones, and 'number' and 'components' as the narrower ones. The analysis was carried out using two complementary statistical techniques – logistic regression and classification trees. However, we do not believe that each individual student's conception of mathematics could be unambiguously determined from a relatively short written response to a question. This results in lack of precision of measurement and contributes to the residual variability in the models.

In logistic regression, we model the odds of a student showing a broader rather than a narrower conception of mathematics in terms of the other factors. Technically, odds is defined as the ratio of the probability that an event occurs (in this case, a broader conception) to the probability that it doesn't occur (in this case, a narrower conception). While probabilities are by definition constrained to be between zero and one, the odds scale uses any non-negative number to measure chances. The potential explanatory factors are checked in the model for statistical significance – that is, for a relationship stronger than one that could be attributed to chance alone. The methods used are described in standard books such as Agresti (1996), and the fact that this was an observational study rather than an experiment implies that those factors found significant should be interpreted as relationships rather than causes. The analysis was carried out using the statistical package SPSS version 18 (IBM SPSS 2011), and the results of the model that examines the joint effects of all the factors on the response is shown in Table 4.4. Standard statistical tests indicated that the overall fit of the model was adequate and a significant improvement on the null model (the one without any demographic explanatory variables), and it was able to correctly classify over two-thirds of the students (68%) in terms of their conceptions of mathematics on the basis of their demographic information.

The results (see Table 4.4) show that university and year were the most significant factors (each with $p < 0.001$), followed by more marginal effects of language background and sex. The odds ratios show the multiplicative effect of each factor on the odds of a broader rather than a narrower conception of mathematics. Because odds are measured on an open scale (from zero upwards), multiplying odds by any positive number will result in another value of odds (this would not work with probabilities on a zero to one scale). Compared to Ulster, Brunei is not significantly different, but students at Pretoria are almost 12 times as likely to describe a broader

Table 4.4 Results from binary logistic regression analysis

Factor	Interpretation	Odds ratio(s)	95% Confidence interval	p-value
Age	Mature aged compared to young	1.43	(0.85, 2.40)	0.18
Sex	Females compared to males	0.74	(0.55, 1.00)	0.048
Language	nesb compared to English	0.65	(0.47, 0.91)	0.011
Degree	Compared to mathematics …			0.27 (overall)
	Other 'service'	0.68	(0.44, 1.06)	
	Mathematics education	1.35	(0.49, 3.75)	
	Engineering	0.74	(0.48, 1.13)	
Year	Compared to first year …			<0.001 (overall)
	Second year	1.49	(0.91, 2.43)	
	Third year	2.33	(1.49, 3.64)	
	Fourth year	2.50	(1.32, 4.72)	
University	Compared to Ulster …			<0.001 (overall)
	Pretoria	11.71	(6.83, 20.08)	
	UTS	4.67	(2.40, 9.09)	
	Brunei	1.09	(0.39, 3.05)	
	Concordia	3.23	(1.93, 5.41)	

conception of mathematics, and those at UTS and Concordia about five times and three times as likely, respectively. Compared to first-year students, those in second year are 50% more likely (that is, 1.5 times as likely) to describe a broader conception of mathematics (though this is not significantly more so), and those in third and fourth year are 2.3 and 2.5 times as likely, respectively. This seems to support the idea that students are more and more likely to describe broader conceptions of mathematics in later years of study, with the largest increase between second and third year.

There is also a marginal indication that non-English speaking background (nesb) students are less likely to describe broader conceptions of mathematics, possibly due to students having trouble reading or expressing themselves in English (note that there was no problem translating those responses that were written in Afrikaans or French, since our team included members who are fluent in those languages). There is also a marginal indication that females are somewhat less likely than males to describe broader conceptions of mathematics. An interesting observation is that age and degree are not significant explanatory variables, although the odds ratios seem to be in the 'expected' directions, higher for mature-aged students and those studying mathematics education, lower for those studying mathematics as a service subject.

This model could be refined by the deletion of non-significant factors (degree and age) and the addition of possibly significant interactions between factors (none was found). When we investigated such refinement, the overall results did not change. Only one observation might be useful: removing age and degree, and combining the (small number of) fourth-year students with those from third year showed that students in second-year were 1.7 times as likely, and those in third/fourth-year were 2.8 times as likely as first-year students to describe broader conceptions of mathematics (and in each case the figure is statistically significant).

An alternative approach is afforded by techniques of data mining, particularly methods using classification trees. In this approach, the set of data is progressively sub-divided based on the values of the explanatory factors into groups that are more homogeneous in terms of conceptions of mathematics. The relative importance of each factor is assessed in terms of how much it contributes to splits into more homogeneous subgroups. This data mining approach is independent of distributional assumptions about the data, deals with missing values using 'surrogate' variables, uses automatic 'cross validation' of models, and is able to investigate local rather than global structure to identify important interactions. CART (Classification and Regression Trees) models were introduced by Breiman et al. (1984), and this analysis was carried out using the package based on their work, CART version 6 (Salford Systems 2011).

The input to the data mining investigation was the same set of response and explanatory variables (shown in Table 4.3) as for the logistic regression. The results of the analysis are presented in the form of a classification tree (Fig. 4.1) that shows which factor and which values were the most important separators at each step of the process. The first split was on the basis of university – Ulster, Brunei and Concordia (with overall 33% broader conceptions) were separated from Pretoria and UTS (with overall 67%), and this latter group was left untouched by further splits. In the first group of universities, first-year students (with overall 28% broader conceptions) were then separated from higher-year students (with overall 44%). This latter group was split on the basis of degree, with mathematics students (overall 51% broader conceptions) separated from the others (overall 42%). In this group, the younger age group was selected out, and then further broken down in terms of degree (mathematics education versus the service courses including engineering) and by language background.

With this classification tree, CART was able to classify 67% of the students in terms of their conception of mathematics, an overall result almost identical to that from the logistic regression. Yet the different methodology showed some important differences of focus. It identified university as the most important factor, followed by year of study, in exactly the same way as the logistic regression analysis, but it indicated only a minor role for the other variables, including language background that was identified as marginally significant in the logistic regression. The similarity in these results increases our confidence in the analysis. In terms of the overall results, Pretoria and UTS were identified as universities with a high proportion of students showing broader conceptions of mathematics, and the *local* analysis focused then on the three other universities, Ulster, Brunei and Concordia, identifying locally-important variables that correlated with broader conceptions. However, when the data from Pretoria and UTS were analysed separately, in order to examine the possible effects of factors on that branch, the most important factor was age, followed by year of study and language background (younger students in first and second years, particularly

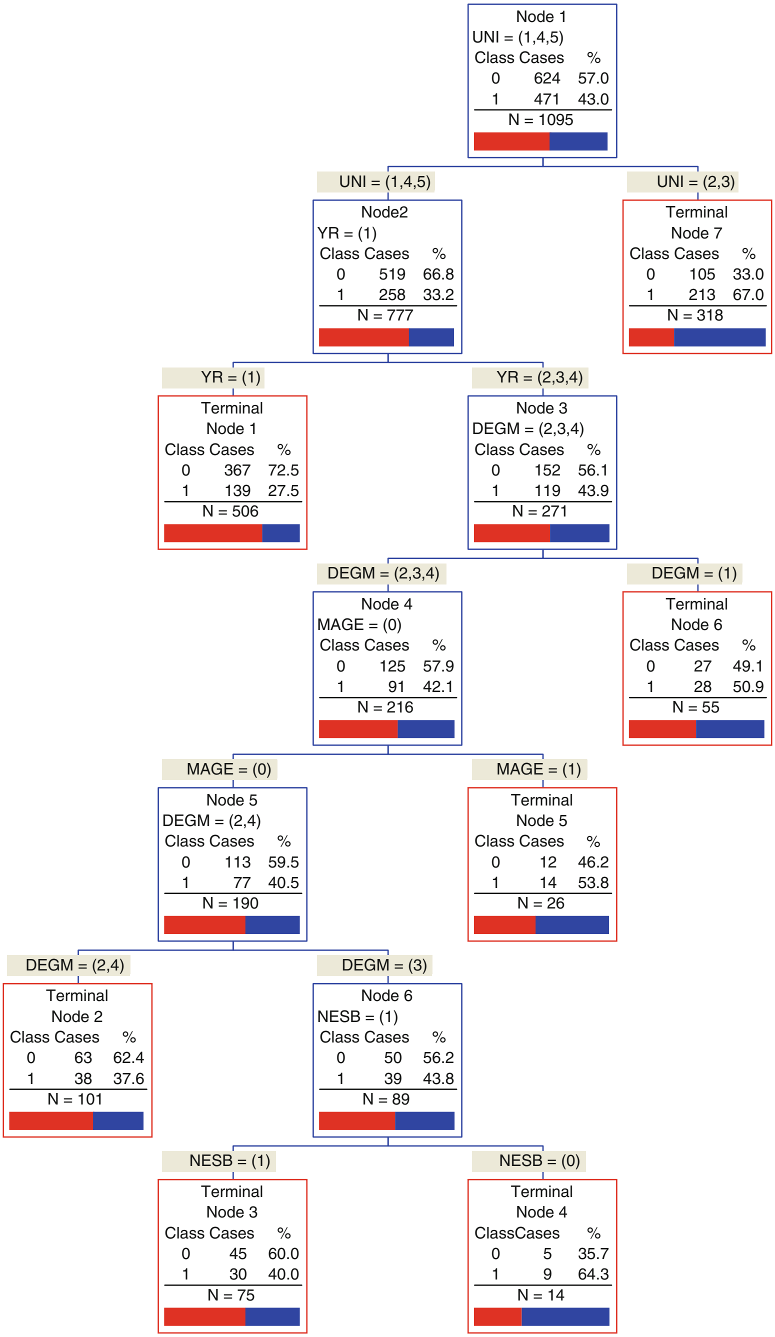

Fig. 4.1 Results from the CART modelling

those with nesb background, were less likely to show the broader conceptions of mathematics). This may explain the higher importance placed by the logistic regression model, which dealt with all the data globally, on language background.

Discussion

The analysis of responses to the open-ended survey has added some important information to our overall investigation of students' views of mathematics and their ideas about the role that mathematics will play in their future. The responses to the first question, *What is mathematics?*, have confirmed and extended our outcome space for conceptions of mathematics with a much larger and more diverse sample of students, including many for whom mathematics is only a component of their professional preparation. Although students who major in mathematics are often seen as the more important group, mathematically speaking, those students who study mathematics as a professional component form the much larger group. One change in the outcome space is that the intermediate conception of mathematics as modelling has been separated into two conceptions, 'modelling' and 'abstract', that comprise the practical models of specific aspects of reality and theoretical models of the logical and abstract nature of mathematics. This division corresponds to some extent with the classical notion of applied and pure mathematics, and was discussed – though not formalised – during the analysis of the interview transcripts. We regard these two conceptions as being at the same level in the hierarchy. The other change is the identification of the 'mathematics is about numbers' as a narrower conception than 'components', and the most limited conception in the hierarchy. We did not identify this 'numbers' conception in the analysis of the interviews, possibly because the participants were (a small number of) later-year students majoring in mathematics, and possibly because in such interviews a brief statement of the 'number' conception would be amplified in further discussion.

We also classified students' views about the role of mathematics in their further studies and future professional life. In this case, there did not seem to be any hierarchical structure in the phenomenographic sense, so we utilised a content analysis that suggested a framework of procedural, conceptual and professional skills, as well as groups of students who were unsure about the role of mathematics, and those who were convinced about its utility but gave no details about how they would use mathematics. We were able to identify some appropriate quotes from the interviews to illustrate students' use of the various skills, and then use the framework to classify the majority of the much briefer survey responses.

As well as confirming and extending the outcome space for conceptions of mathematics, and identifying a categorisation for utility of mathematics, the survey results allow us to investigate the distribution of views about mathematics and its use. Over 90% of responses could be classified as to students' conception of mathematics. The 'components' view was the most common, and together with the 'number' view, just over half of the students (53%) showed fragmented conceptions of mathematics. Only a small group of students (6%) seemed to hold the broadest ‘life’ conception. In terms of the future use of mathematics, we were able to categorise only around three-quarters of responses, and over half of the responses gave evidence for one of the groups of skills. Of these, the reference to procedural skills was most common, mentioned by more than half the students who wrote about one of the skills, with smaller numbers pointing to professional or conceptual skills. A small group of students (5–7%) wrote that they were unsure about their future use of mathematics, while a larger group was convinced that it would be essential, though they did not give any reason for this view. Thus, we have evidence that a significant proportion of students are unsure of the uses to which they will put the mathematics that they are studying, or at least, are unable or unwilling to articulate their views on this topic.

While we increased our knowledge of students’ views, the survey methodology has some limitations. Some students wrote over 100 words in response to individual questions, but the average response was much lower (around 20 words) and there were students who wrote only one or two words. This, and the impossibility of follow-up questions, means that there is a degree of imprecision in our categorisations, even though our procedures ensured reliable coding of responses. Nevertheless, since we were able to classify the majority of responses, we carried out further quantitative investigation of students’ conceptions of mathematics, identifying the factors that were associated with the broader conceptions (‘modelling’, ‘abstract’ and ‘life’) rather than the narrower conceptions (‘number’ and ‘components’).

The results from modelling using logistic regression and classification trees point out that there seem to be substantial differences between universities represented in the study. Around two thirds of the students at Pretoria and UTS indicated broader conceptions of mathematics, while at the other universities (Ulster, Brunei and Concordia) only around one third of students indicated the broader conceptions. This could be due to national differences, since each university was located in a different country. Such differences include societal views about the nature, usefulness and status of mathematics and its applications: a recent European study (Dahlgren et al. 2007) found such national differences in disciplines including psychology and political science. They may also be reflected in different entry requirements for the various mathematics programs. Different views about appropriate roles for males and females in mathematical professions may also be pertinent. For instance, the largest group at the Universiti Brunei Darussalam was studying mathematics education, and about four-fifths of them were women. By contrast, the largest group at the University of Ulster was studying Engineering, and more than four-fifths of them were men. It is sometimes suggested that students’ views about mathematics derive from their lecturers’ ideas. While lecturers’ conceptions may have an effect on their students’ views, the situation is much more complex. Students in later years would have had several different lecturers, and even those in first year would have been exposed to different teachers during their schooling.

Maybe the most important result from the quantitative modelling is the strong indication that there are higher proportions of students with broader conceptions of mathematics at later years of study (although as this was not a longitudinal study, we cannot conclude that *individual* students are broadening their conceptions). Students in second year are 50% more likely than those in first year to show broader conceptions of mathematics, and those in third and fourth year are almost three times as likely to do so. This is surely what we would like to see; that during the course of their university studies our students are broadening their view of the discipline of mathematics. However, we should note that at each university, in each year level and in each degree, students indicated the full range of conceptions of mathematics, from ‘number’ to ‘life’. The hints of sex, language background and degree differences, and the national and cultural differences between universities, suggest areas for further research, since our results are not extensive enough to allow more definite conclusions.

Implications for Teaching and Learning

The results from the open-ended survey reinforce the teaching and learning strategies suggested in the two previous chapters. Because of the hierarchical nature of the conceptions, we would prefer our students to hold broader conceptions of mathematics, at least by the time they finish an undergraduate degree. This may be achieved by designing curricula and using pedagogical approaches that will allow students to develop deeper understandings. This should include the strategy of introducing our students to the full range of conceptions about mathematics and giving them opportunities to discuss the nature and implications of these conceptions. While this is important for all students, it is particularly important for students of mathematics education, as they will be educating the next generation of students.

The quantitative results send a strong message that our students in every year group and degree program hold widely differing ideas about the nature of mathematics and its use in their studies and future careers. While our results show that there is a pleasing increase in the proportion of students who have broader conceptions of mathematics in later years of study, the full range of variation is present in classes at all levels; it is simply not true that all students enter university with the narrowest ideas about mathematics and leave with the broadest ideas. By the time they enter university, students have already developed conceptions of mathematics, which may be reinforced or modified during tertiary study. With appropriate teaching methods and learning materials, we can encourage students towards the broader conceptions – and the hierarchical nature of the conceptions implies that they will then be aware of the full range and be able to make use of them as appropriate.

As an example, if we focus solely on the techniques of mathematics and regularly ask rote and procedural questions, we will encourage a ‘component’ (or even narrower) view of mathematics. Such an approach may even lead to the sorts of views about mathematics being a ‘waste of time’ that some of the students in our study seemed to hold. On the other hand, if we focus on the broader role of mathematics in real or realistic contexts, we will encourage students towards the broader conceptions. Such an approach will benefit from making explicit connections between students' courses and the world of professional work. As mathematics lecturers, we can design learning tasks that model the way that mathematicians work in industry and academia in order to give students an idea of the way mathematics is used in their future professions. For example, arranging a student conference on some mathematical topic, with students carrying out the planning, writing, reviews and presentations, can introduce academic aspects of the discipline. A film clip presenting some aspect of professional work as an engineer or a statistician with follow-up discussion and questions (for example, Wood et al. 2000; Petocz et al. 1996) can model professional roles and highlight the diverse range of skills (much more than mere technical competence with the mathematics) needed in a typical work situation.

The significant differences between universities in terms of the proportion of students holding the broader conceptions of mathematics sound a positive note in this endeavour. These may be the result of national differences in the way that students are prepared for studying mathematics at university, or they may be due to the practices and pedagogical approaches at the different universities. While these factors may be hard to change, the fact that there are such substantial differences implies the possibility of changing pedagogical and institutional approaches to emphasise the broadest views of and roles for mathematics.

Summary and Looking Forward

In this chapter we have described the results of the second stage of our investigations: a survey of students' views of mathematics and its future use in their studies and profession, undertaken with a sample of over 1000 students from five different universities in five different countries. Although we were unable to get detailed information from a sample of this size, we have balanced that by obtaining a much larger amount of information that is still in the form of responses to open-ended questions, supported by demographic details for each student. This has allowed us to confirm and extend our previous qualitative results concerning conceptions of mathematics, and then investigate these results using quantitative methods – logistic regression and classification trees.

The quantitative analysis has revealed two important aspects. Firstly, there seem to be substantial differences between universities in terms of the proportion of students holding the broadest conceptions of mathematics. Allowing for differences in year, sex, language, degree and age – which the logistic model does – three of the universities have one third, and the other two universities have two thirds of their students indicating the broader conceptions, and this result is quite unlikely to have occurred by chance. Such differences seem likely to result from some feature of the national context of each university, or the specific approach taken by each university to their mathematics education. In either case, it affords the possibility of making positive changes. Secondly, the analysis shows that, allowing for differences in the other variables, students in later years are more likely to hold broader conceptions of mathematics: about 50% more likely in second year, and almost three times as likely in third or fourth year, than in first year. This is certainly a good sign, and something that we would hope to result from our mathematics courses. We need to remember, however, that only a longitudinal study could establish that students were broadening their views as they progressed through their degrees, and also that almost half of final-year students are currently finishing their courses with the narrower views of mathematics as 'number' or 'components', some of them majors in mathematics education or mathematics itself.

In the following chapter, we continue our studies of undergraduate students and report on the construction of a survey instrument, based on the statements that we have obtained from students in interviews and open-ended surveys, that can be used as a tool to track the development of the broader ideas that are so important in their future professional lives.

Note: Some of the material in this chapter was previously published in Petocz, P., Reid, A., Wood, L. N., Smith, G. H., Mather, G., Harding, A., Engelbrecht, J., Houston, K., Hillel, J., & Perrett, G. (2007). Undergraduate students' conceptions of mathematics: An international study. *International Journal of Science and Mathematics Education*, *5*, 439–459 and in Wood, L. N., Mather, G., Petocz, P., Reid, A., Engelbrecht, J., Harding, A., Houston, K., Smith, G. H., & Perrett, G. (2011). University students' views of the role of mathematics in their future. *International Journal of Science and Mathematics Education*, *9*(1). Online at http://springerlink.com/content/a255j102v01653xh/

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Chapter 5
How Can We Track Our Students' Progress Towards Becoming Mathematicians?

Introduction

In earlier chapters we have described our investigations of students' ideas about mathematics and learning mathematics. We developed an outcome space for conceptions of mathematics through the analysis of data from interviews with a small number of undergraduate students, and then supported and extended this using information from a large open-ended survey. We found that the narrowest, atomistic views were of mathematics in terms of numbers or components, broader views described mathematics in terms of modelling or abstraction, and the broadest and most holistic views held that mathematics was an integral aspect of life, personal as well as professional. We maintain that becoming a mathematician consists of the process of moving towards this broadest conception of mathematics – the inclusive nature of the conceptions ensures that this will contain the narrower conceptions. So students will become mathematicians as they acquire skills with the technical components of mathematics, gain experience with constructing and using mathematical models, both applied and abstract, and develop an appreciation for the essential role of mathematics not only in their future workplace but also in other aspects of their life. Interviews with recent graduates confirm this from the point of view of people who have, to a greater or lesser extent, made the transition to becoming mathematicians.

In this chapter we consider the question of how mathematics lecturers in classes that may be small, but are often quite large, can assess and track the views that their students have about mathematics and its uses. To answer the question, we will describe the third stage in our overall project: the development and utilisation of a closed-form survey of students' conceptions of mathematics. Our development of this survey was informed by the results of the previous two stages – the initial interviews and the open-ended survey. As well as structuring the questions around the conceptions we identified, we were able to make use of the actual words that our respondents had spoken or written. We present an initial closed-form survey consisting of 67 questions, assess its statistical properties and use these to construct a shorter version, comprising a subset of the original questions, that can be

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utilised as a valid and reliable survey of students' conceptions of mathematics and its uses. We are happy to make this survey available to mathematics educators through this book (see Appendix 1).

Such a survey can be used for various pedagogical purposes, in addition to its continued use as a research tool. Maybe the most important of these is for assessing and then tracking the views of groups of students in the typical tertiary mathematics lecture class. The survey can be administered by a lecturer early during a course to gauge the views of his or her students, and thus to plan more effective ways of helping them towards broader conceptions. The survey can be used again during or at the end of the course, and a comparison with the previous results can indicate the extent of any change in views. Such repeated assessment is helped by the compact nature of the survey, and can also be expedited by using current technology for administration – in which case it will be equally easy with a small class of 30 students as with a large lecture group of 250. Further, presenting the results to the students concerned and discussing the actual questions with them can be used as one way of introducing them to the full range of conceptions of mathematics.

Questionnaire-Based Studies of Students' Conceptions of Mathematics

There have been many studies that have surveyed university students about some aspects of their views about mathematics and their mathematical education. Survey methodology is widely used in a range of disciplines, particularly in psychology (and psychological aspects of areas such as education and business). In particular, the quantitative methodology is more in tune with the basic nature of mathematics, and so seems to be more frequently used than the qualitative, interview-based approach with which we began our investigations. The quantitative studies often have qualitative components: for instance, in some cases they utilise previous interview results to help construct appropriate questions. Here we restrict our attention to surveys that ask university students about their ideas, beliefs or conceptions of mathematics, rather than other aspects such as their anxiety, ability or actual performance in mathematics. We give brief descriptions of some relevant studies.

A good place to start is with the work of Crawford and her team. We have described in Chap. 2 their identification of five hierarchical conceptions of mathematics, ranging from "maths is numbers, rules and formulae" to "maths is a complex logical system which can be used to solve complex problems and provides new insights used for understanding the world" (Crawford et al. 1994). These they divided into fragmented conceptions (the narrowest two) and cohesive conceptions (the broadest three). The results were used to develop a 19-item *Conceptions of Mathematics Questionnaire* (Crawford et al. 1998a) that assessed the extent to which a student held fragmented or cohesive conceptions of the discipline. The questionnaire was used in a research project with a group of 300 Australian university students enrolled in a first-year mathematics course in the faculties of science or engineering, including some students who were planning to take a major in mathematics (Crawford et al. 1998b). The researchers surveyed approaches to learning (surface approach focusing on reproduction, or deep approach focusing on understanding) using an *Approaches to Learning Mathematics Questionnaire* (Crawford et al. 1998b), a modification of Biggs' *Study Process Questionnaire* (Biggs 1987). They utilised a modified version of the *Course Experience Questionnaire* (Ramsden 1990) to assess participants' mathematical experiences (perceptions of good teaching, clear goals, workload, assessment and independence of learning). The students' matriculation results and end-of-course examination results were also obtained. Differences in students' conceptions of mathematics were found to be related to their approaches to learning mathematics, their experience of studying mathematics and their examination performance. A fragmented conception of mathematics was linked with a surface approach to learning, perceptions that assessment is measuring reproduction and that the workload is excessive, and lower examination performance. On the other hand, a cohesive conception of mathematics was linked with a deep approach to learning, positive perceptions of the learning environment, and higher achievement in examinations.

Somewhat beyond the limits of our discussion is Grigutsch's survey of a large sample (1650) of German secondary school ('gymnasium') students (Grigutsch 1998). He used a range of questionnaires to investigate students' views of mathematics (the same survey previously used with university mathematicians, Grigutsch and Törner 1998), their experiences in mathematics education, and their self-concept as learners of mathematics, including their estimated diligence, achievement and pleasure in the subject. The students' latest results were also recorded. Pupils with a 'schema' view of mathematics (focusing on procedures, rules and algorithms) took less pleasure in their mathematics study, had a more negative self-concept and obtained poorer results than those pupils with a 'process' view (tackling problems by discovering and understanding relevant mathematics) or an 'application' view (focusing on practical use and relevance to society). The 'formalist' view of mathematics (focusing on precise thinking, argument and proof) seemed of less importance (it is likely that fewer school students than university mathematicians would view mathematics in this way). Although the survey was carried out with secondary school students, including some in their final year, the link with views of mathematics expressed by university mathematicians gives it some relevance here.

Meyer and Parsons (1996) investigated learning behaviour that seemed to be specific to the discipline of mathematics and depended to some extent on students' beliefs about the nature of mathematics. First-year students (321) studying mathematics in South Africa were surveyed using the *Extended Approaches to Studying Inventory* plus questions asking about internal or external causal attribution for academic success and failure. A factor analysis on the approaches to studying questions identified the expected three underlying dimensions corresponding to 'deep', 'surface' and 'strategic' approaches. When the causal attribution variables were included, the first three factors represented 'surface' and 'strategic/deep' approaches, as well as an external causal (luck and circumstances) dimension. However, a fourth dimension was also needed, and this represented a *"pure 'deep' structure of variation that appears to be independent of any causal attribution for success or failure … that may be peculiar to the study of mathematics"* (p. 744). Since the sample was taken from those who had already passed a stringent mid-year examination, Meyer and Parsons conclude that *"among mathematically successful students there is thus … [a] dimension of variation in learning behaviour that is purely intrinsic to the love and enjoyment of the subject and that is unfettered by concerns of relevance, utility or other external concerns"* (p. 745). This dimension seems to represent a particular view of the beauty and abstract structure of mathematics.

One of our original interview group expressed a similar view very clearly, though it was set in the context of describing the wide range of applications of mathematics to problems and phenomena in the real world:

> *Eddie*: To me mathematics is sort of bit like art, I don't know why I like it but I do. [Why do you think it is?] Because there are some things that you can study that are, that are intellectually so beautiful it's gratifying to study it for its own sake, I mean when you can see it you think 'oh yes …this, this has to be right.' And it's just such a beautiful and elegant chain of reasoning./../And the fact that it might not have a, have a use in the real world, to me it doesn't matter, I mean because it's a beautiful, elegant thing to look at for its own sake.

In a subsequent study, Meyer and Eley (1999) investigated affective sources of variation in student learning specific to the discipline of mathematics. They aimed to elaborate an affective component that was specific to learning mathematics – a pleasure in working with purely mental constructs, along the lines that Eddie's quote illustrates. The authors developed an *Experiences of Studying Mathematics Inventory*. A set of questions was derived from interviews with mathematicians, and this was followed by several cycles of trialling and modification with students in South Africa and Australia. The final questionnaire contained five scales representing the constructs of enjoyment of mathematical activities (enjoyment), seeing mathematical entities as objects of beauty (beauty), preferring certainty and integrity in decision rules (truth), preferring systematic and logical approaches to problems (procedures), and extra-curricular interests in mathematics (recreation). The 'enjoyment', 'beauty' and 'recreation' subscales (but not 'truth' and 'procedures') were shown to have different profiles with distinct groups of students – majors, engineers, generalists and students studying computing.

A more recent study was carried out by Cano and Berbén (2009) in Spain. They used a questionnaire approach with 680 first-year mathematics students to investigate the relationships between students' achievement goals – their reasons or purposes for studies – and their approaches to learning – the way they actually go about their learning. The 'achievement goals' perspective is a prominent theory in contemporary research on student motivation and learning in higher education, complementing the 'student approaches to learning' perspective. The *Achievement Goal Questionnaire* (Elliot and McGregor 2001) contains four scales representing combinations between absolute (intra-person) or normative (comparative) competence, and positive (aiming for success) or negative (avoiding failure) motivation. They surveyed students' approaches to learning using the *Approaches to Learning Mathematics Questionnaire* (Crawford et al. 1998b) which comprised dimensions for surface and deep approaches, focusing specifically on the discipline of mathematics. They also used the *Conceptions of Mathematics Questionnaire* (Crawford et al. 1998a) to assess the extent to which students showed fragmented or cohesive conceptions of the discipline. The researchers found a consistent pattern of relationships between the two theories. For example, those students who viewed mathematics in a cohesive way and saw their academic environment as aimed at their successful mastery of material tended to use a deep learning approach (and avoid a surface approach). Cano and Berbén distinguished four clusters of students depending on their motivation and approach and showing different pathways to learning and experience in learning mathematics. They recommended that in order to enhance their teaching, mathematics lecturers should consider not only the mathematical content but also the ways in which their students view mathematics and experience learning.

Liston and O'Donoghue (2009) carried out a study that focused specifically on the role of affective variables, conceptions of mathematics and approaches to learning in 'service' mathematics at an Irish university. As in many parts of the world, Irish students are often held to be under-prepared mathematically for university courses that include mathematics as a component. The researchers used a questionnaire that contained questions on attitudes, beliefs and self-concept: overall like or dislike of mathematics, the extent of usefulness of mathematics, whether one is good or bad at mathematics, and belief in the ability to perform in mathematical situations. They also included questions about conceptions of maths (fragmented versus cohesive), and approaches to learning maths (surface versus deep), incorporating the *Conceptions of Mathematics Questionnaire* (Crawford et al. 1998a) in the overall survey. The survey was administered to 607 first-year students studying engineering, science or technological mathematics, and students' early diagnostic scores and examination results were also collected at the end of semester. Liston and O'Donoghue confirmed the positive relationship between cohesive conceptions of mathematics and deep approaches to its learning, though they saw no evidence that these notions influenced examination results (which may be a comment on the examination itself!). Statistically significant relationships were found between the affective aspects of enjoyment, value and self-concept and cohesive conceptions of mathematics, and weaker (though still significant) relationships with deep approach to mathematics learning. These were explored in more detail in a follow-up qualitative study (Liston and O'Donoghue 2010).

The studies that we have described in this section illustrate the use of the survey-based approach in investigations of students' views about mathematics and the connection of these views to their approaches to studying mathematics and their beliefs about the value of the discipline. They show in detail the benefits of using quantitative approaches in untangling the detailed web of relationships concerning views of mathematics that comprise the world of students who are studying the subject at university. There are many other survey-based studies that focus on some other aspect of mathematics learning, for example, students' anxiety about mathematics (Walsh 2008).

Our Questionnaire Study

Our aim was to develop a questionnaire that would allow us, and other researchers, to investigate students' conceptions of mathematics and their views about their use of mathematics in their further studies and their future professional work. A questionnaire seemed the only feasible option for a tool to track the changing views of sometimes large classes of students in a typical university situation. Previous conceptions of mathematics surveys (such as the one developed by Crawford's team 1998a) focused on the distinction between fragmented and cohesive views of mathematics. We wanted a questionnaire that was in line with the results from our interview and open-ended questionnaire studies, based on the three broad conceptions of 'number' and 'components', 'modelling' and 'abstract', and 'life'. In particular, we wished to include the ontological aspect of becoming and being a mathematician that was highlighted in the broadest, 'life' conception that we identified, and that was not part of the earlier questionnaires. We also wanted to include items that investigated students' perceptions of their future use of mathematics in studies and work, built on the comments from our original interview participants and the statements made by students in their responses to the open-ended survey.

We (and other members of our team, particularly Glyn Mather) used the previous material that we had gathered to develop a questionnaire with items focusing on three areas: *What is mathematics? Mathematics is …*; *What part do you think mathematics will play in your future studies? In my future studies, mathematics will …*; *What part do you think mathematics will play in your future career? In my future career, mathematics will …* . We included background demographic questions asking about students' university, age, sex, home language, degree program, year (or stage) of study, full or part-time mode of study, and major (and stream if they were undertaking a mathematics major). The items consisted of statements (for example, *Mathematics is … a set of rules and equations*, or *… a theoretical framework that describes reality*) and students indicated their agreement or disagreement on a five-point Likert scale (1 = *strongly disagree*, 2 = *disagree*, 3 = *neutral/don't know*, 4 = *agree*, 5 = *strongly agree*). The text of the items was taken directly, or with minor modifications, from the interview statements or written responses to the open-ended survey. The first version of the questionnaire contained a total of 67 (25 + 20 + 22) items.

The paper-based questionnaire was filled in by 562 students from three participating universities: University of Ulster, Northern Ireland (20%); University of Pretoria, South Africa (61%); and Macquarie University, Australia (19%). The majority of students were male (61%), and in their first year of study (77%). The largest group comprised engineering students (45%), followed by other 'service' students (36%) and then mathematics majors (19%). The surveys were administered during lecture times, preceded by a brief explanation of the aims of the project and an assurance that participation was voluntary and would not affect students' course enrolment or results; ethics approval was obtained from each participating institution. Although 562 students filled in the forms, there were several who did not rate all the items (the survey was rather long, and some students missed the last page of questions) or did not give all their demographic details. In the following statistical analyses, the effective sample size for the various procedures was between 500 and 550, depending on the techniques used and the variables included.

We analysed the results following standard statistical and psychometric methods. As a first step, we carried out a factor analysis on each set of items – conceptions of mathematics, future use in studies and future use in career. Factor analysis is a statistical method for dimension reduction: each set of 20–25 questions is summarised by a small number of 'factors', independent linear combinations that between them contain most of the information in the original responses to all the questions in the set. There are various ways of carrying out the process mathematically, and rules for determining the number of factors to use; we used the 'principal components' method (other methods gave very similar results) and selected three factors in each case on the basis of the shape of the 'scree plot'. Each of the original items contributes to each factor, and the factors are usually 'rotated' (we used varimax rotation) to make these contributions or 'loadings' as high or as low as possible, to help in interpreting the meaning of the factor. We then constructed scales based on the average results from a number of items that seemed to be measuring a consistent idea. The scales were checked for validity – to see whether they were measuring what they should have been – and reliability – to examine how consistently the measurement process was carried out. We investigated the relationships between these scales and the demographic variables using the statistical technique of multivariate analysis of variance. This allowed us to see which of the demographic variables were related to students' scores on the scales. More information on these statistical aspects can be found in any standard textbook of multivariate methods (for example, Johnson and Wichern 2007) or measurement (for example, Trochim 2006; Furr and Bacharach 2008). The final result was a shorter and tighter questionnaire comprising 46 items, which is now available for further use. All the statistical calculations were carried out using the statistics package SPSS version 18 (IBM SPSS 2011).

Results – Conceptions of Mathematics

The factor analysis of the items investigating students' views of mathematics identified three dimensions that seemed to correspond with conceptions 'life' (L), 'number'/'components' (N/C), and 'modelling'/'abstract' (M/A), in that order. This implies that students can be distinguished most easily by their different responses to items relating to the three groups of conceptions. Together, the three factors accounted for 41% of the variability in the original items, a proportion that is adequate though not high; however, there seemed no indication that more components should be extracted. The items and their loadings are shown in Table 5.1 (loadings less than 0.3 have been suppressed).

The items indicated (*in italics*) are the ones that have the highest factor loadings (at least 0.57 in the first two factors and 0.50 in the third) and negligible loadings on

Table 5.1 Items and factor loadings for conceptions of mathematics

What is mathematics? Mathematics is …	3(L)	1(N/C)	2(M/A)
A set of models used to explain the world	*0.60*		
An infinite universe of possibilities	0.45		0.26
A way of analysing ideas and problems			*0.58*
A system used to analyse and predict real world events	0.55		
A set of rules and equations		*0.68*	
An abstract concept defined by a small number of laws	0.28	0.29	−0.40
Basic knowledge for all scientific fields			*0.50*
No use to me at all			*−0.72*
A way to solve problems in my life	*0.58*		
A tool that can be applied in various fields	0.28		*0.58*
Capable of being used in every aspect of everything	0.54		
Figuring out problems using numbers		*0.58*	0.29
Using formulas to get results		*0.77*	
A way to give humans a more advanced life	*0.59*		
A lot of different methods which can be used to solve the same problem	0.26	0.47	
The language of nature	*0.72*		
Calculations		*0.72*	
A whole way of thinking	0.37	0.28	0.33
Numbers being processed		*0.73*	
A theoretical framework that describes reality	*0.71*		
Based on logical thought	0.33	0.28	0.32
An attempt to explain physical laws and patterns	0.35	0.36	0.36
A set of tools and techniques	0.33	0.45	0.34
The study of numerical concepts		*0.66*	
A way to generate new ideas	*0.57*		

the other factors. These are used to construct the 'conception scores' for each student, by taking averages of their responses for each of the indicated items, multiplied by 10 (so the scale goes from 10 to 50). Note that one item (*Mathematics is … no use to me at all*) in the third factor has a negative loading, and so is reverse scored for the purposes of constructing the score for conception 2 ('modelling'/'abstract'): this essentially constructs an equivalent reversed item (*Mathematics is … of use to me in some way*). Thus, the scales for the 'number'/'component', 'modelling'/'abstract' and 'life' conceptions (con1 N/C, con2 M/A, con3 L) comprise averages of six, four and six items respectively.

The validity of these scales is supported by the fact that the content (and often even the wording) of the items was developed from students' own words, from the interviews or the open-ended surveys. There were a few surprises in the original questionnaire. For example, we expected *"a set of tools and techniques"*, *"an attempt to explain physical laws and patterns"* and *"a whole way of thinking"* to relate to the 'number'/'component', the 'modelling'/'abstract' and the 'life' conceptions, respectively. Yet each of these items loaded fairly similarly on each factor and so does not help us to distinguish between the different views. On the other hand, the items selected by the factor analysis did seem to be logically valid for each

Table 5.2 Mean scores for the conception scales overall and by subgroups

Subgroup		con1 (N/C)	con2 (M/A)	con3 (L)
Overall		40	41	36
University ($p < 0.001$)	Ulster	40	40	33
	Pretoria	41	43	37
	Macquarie	39	40	36
Sex ($p = 0.022$)	Female	41	41	35
	Male	40	42	36
Year of study ($p = 0.002$)	First	40	41	33
	Second	41	42	36
	Third	38	41	38
Group ($p = 0.61$)	Mathematics	40	41	35
	Other servicing	40	41	35
	Engineering	40	42	37

Scale from 10 to 50, all standard errors are 1.1 or lower

of the three conceptions: even *"no use to me at all"* makes sense when it is reversed. The mean results for con1, con2 and con3 were 40, 41 and 36 respectively, suggesting that students' agreement with the items on the 'life' scale was somewhat less than for the other scales, though still above the midpoint of 30.

The reliability of the scales was checked by standard methods. Cronbach's alpha is a measure of reliability in terms of internal consistency of the items making up the scale: values above 0.8 are regarded as good, while values between 0.6 and 0.8 are regarded as acceptable. The values for the three scales were 0.81 (con1), 0.60 (con2) and 0.74 (con3), indicating that all three scales were acceptable or better, with the scale for the 'modelling'/'abstract' conceptions just on the borderline. We were able to carry out this analysis using over 95% of our sample group.

Students' scores on each of the scales (con1, con2 and con3) were used as dependent variables in a multivariate analysis of variance, with the demographic variables (university, sex, year of study, group, language background and age) as explanatory variables. Only the first three variables were significant. Removing language background and age from the list of explanatory variables confirmed the statistical significance of university, sex and year of study, and the non-significance of group. The mean values (model based) for these variables are shown in Table 5.2; the actual differences are quite small (due in part to the use of averages for constructing the scales). The most important result is the significant rise in scores for con3, representing the 'life' conception, across years of study, from 33 in first year to 36 in second year and 38 in third year.

Results – Future Use of Mathematics

The factor analyses of items investigating students' future use of mathematics in their studies and in their career each identified three dimensions. In both cases, these dimensions seem to represent coherent underlying ideas of the use of mathematics.

Table 5.3 Items and factor loadings for future use of mathematics in studies

What part do you think mathematics will play in your future studies? In my future studies, mathematics will …	1(prac)	2(gen)	3(know)
Be necessary because it is compulsory for my degree	*0.55*		
Help me develop logical thinking	*0.67*	0.28	
Give me new ideas about other things	*0.65*		
Help me solve problems in my other subjects	*0.70*	0.27	
Help me get a job	0.54		
Help me do calculations in my other subjects	*0.60*		–0.35
I do not know how mathematics is useful for my degree			*0.81*
Be the basis of my future studies		*0.70*	
Help me create math models to enable me to understand the applications studied	0.36	0.51	
Be of no use as computers and calculators can do all the work			*0.79*
Help me solve mathematical problems	0.47		–0.39
Be used in all my subjects	0.37	*0.58*	
Help me analyse ideas	0.50	0.50	
Be my major/main/primary subject		*0.69*	
I do not know how I will use it in my future studies			*0.74*
Be the foundation of all my other subjects		*0.74*	
Be used in most of my future		*0.73*	
Give me tools to use in my other subjects	0.52	0.47	
Give me a whole way of thinking	0.46	*0.58*	
Help me do calculations	0.49	0.26	

For future studies, the first dimension (prac) represents the practical use of mathematics in calculations, solving problems, logical thinking and analysis; the second dimension (gen) represents the generic and essential use of mathematics as a basis of future studies; the third dimension (know) represents (lack of) knowledge about or appreciation of the role of mathematics in future studies. Together, these three factors account for 52% of the variability in the original items, most of which comes from the first two dimensions (total 39%). For future career, the dimensions have a very similar interpretation: the first dimension (gen) represents generic and essential use of mathematics – now as a basis of jobs, career and even life; the second dimension (prac) represents practical as before; the third dimension (know) again represents (lack of) knowledge about or appreciation of the role of mathematics – now in future career. The three factors explain 55% of the variability in the original items, and again the first two are most important (total 43%). The different order of the first two factors does not seem to be important, as they each explain very similar amounts of variation after rotation. The items and their loadings are shown in Tables 5.3 (studies) and 5.4 (career); again, loadings less than 0.3 have been suppressed.

We constructed scales (stu1, stu2, stu3 and car1, car2, car3) in the same way as previously (leading to values between 10 and 50), based on the items with highest factor loadings and without high loadings on the other factors (14 and 16 items for

Table 5.4 Items and factor loadings for future use of mathematics in career

What part do you think mathematics will play in your future career? In my future career, mathematics will …	2(gen)	1(prac)	3(know)
Give me new ideas	*0.60*	0.30	
Be my future career	*0.77*		
Be a building block for any job	*0.72*		
Help me get a job because employers know it is useful	*0.69*		
Give me a whole way of thinking	0.56	0.35	
Be of no use as computers and calculators can do all the work			*0.75*
I don't know how I'll use it as I don't know what I'll be doing			*0.79*
Help me with calculations		*0.71*	
Help me with analysing problems	0.29	*0.74*	
Provide me with mathematical models to help me to understand problems	0.35	*0.68*	
Help me solve everyday problems in my life	*0.64*	0.34	
Give me foundation skills for my future career	*0.60*	0.47	
Help me think logically	0.35	*0.66*	
Be useful for work in many fields	0.51	0.61	
Help me with analysing data	0.33	*0.72*	
Help me get a good job	0.55	0.27	
I don't know how I'll use it as I don't know the industry			*0.82*
Give me tools to use		*0.63*	
Help me solve mathematical problems		0.61	
Not be any use in my future career			*0.78*
Help me in my whole life	0.59	0.29	
Help me solve a wide range of work problems	0.56	0.49	

studies and career respectively), indicated in the tables (*by italics*). The scoring for the knowledge scales (stu3 and car3) was reversed as the items are phrased in a negative sense. The means for the practical scales (40 for stu1 and car1) were higher than those for the other scales (between 34 and 36), showing that students agreed most strongly with the items referring to practical uses of mathematics, though agreement with the generic and knowledge items was still above the midpoint.

The items forming the scales were again taken from students' own words in interviews or responses to open-ended surveys, giving evidence for their validity. Further, the scales seem to be related to the categories that we identified in our concept analysis of responses from the open-ended survey (Chap. 4). The validity of the scales seemed reasonable, with Cronbach's alpha values of 0.76 (stu1), 0.83 (stu2) and 0.76 (stu3) for the studies scales, and 0.85 (car1), 0.82 (car2) and 0.80 (car3) for the career scales. Again, these analyses utilised over 95% of our sample group.

Using multivariate analyses of variance, we investigated the effects of the demographic variables on the studies and the career scales. Language background and age were once again found to be non-significant, so only the variables university, sex,

Table 5.5 Mean scores for the studies and career scales overall and by subgroups

Subgroup		stu1 (prac)	stu2 (gen)	stu3 (know)
Overall		40	36	36
University ($p<0.001$)	Ulster	40	34	35
	Pretoria	42	40	38
	Macquarie	39	35	35
Group ($p<0.001$)	Mathematics	41	39	36
	Other servicing	39	33	35
	Engineering	41	37	37
		car1 (prac)	car2 (gen)	car3 (know)
Overall		40	34	36
University ($p<0.001$)	Ulster	39	34	35
	Pretoria	41	36	37
	Macquarie	39	32	35
Year of study ($p<0.001$)	First	40	36	36
	Second	41	37	38
	Third	38	30	33
Group ($p=0.003$)	Mathematics	40	36	36
	Other servicing	39	33	35
	Engineering	40	34	36

Scale from 10 to 50, all standard errors are 1.5 or lower

year of study and group were retained. For the studies scales, university and group were statistically significant, but not sex and year. For the careers scales, university, group and year of study were statistically significant, but not sex. The (model-based) mean values are shown in Table 5.5 for the significant variables; again, the actual differences are small. The most important results include the following: universities have quite different means on the generic scales, and to a lesser extent on the knowledge scales, with Pretoria the highest; mathematics majors have higher means on the generic scales than the other groups. Other servicing students have a particularly low mean on the generic studies scale (stu2). The low means on the generic and knowledge career scales (car2 and car3) for third-year students seem to be unexpected; we would hope that final-year students would have a higher appreciation of such aspects, and a greater knowledge of role of mathematics in their future careers.

Discussion and Implications for Teaching and Learning

Our closed-form survey of students' views of mathematics and their ideas about its future use in their studies and their careers is based firmly on the previous qualitative information that we have collected, using interviews in the first instance, and then an open-ended survey. We have trialled the survey with a large international sample of over 550 students, and on the basis of the results we have constructed a shorter closed-form survey with better statistical properties. Thus, the quantitative results presented in this chapter have added to our overall outcomes in two ways: first, by corroborating our previous qualitative results, and second by the construction of a better and more compact questionnaire.

The factor analyses of the three sections of our original survey identified quantitative scales that correspond to our earlier qualitative results. In the first section of the questionnaire, the scales corresponded to the three broad levels of conceptions of mathematics – 'number' and 'components', 'modelling' and 'abstract', and 'life'. Using this section of the short questionnaire (the first 16 questions only), we can investigate the extent to which our students are aware of and in sympathy with these three levels of conceptions. The inclusive nature of the conceptions implies that students could score highly on all three of these dimensions, rather than just one of them, and we can see this in the quantitative results. In the other sections of the questionnaire, the scales that we have identified correspond broadly with the results of our content analysis of students' statements about their future use of mathematics. The practical scales – focusing on calculations and analysis, as well as solving problems and logical thinking – seem to have a connection with the procedural skills that we identified in the content analysis of comments from the open-ended survey. The generic scales – including the overall mathematical way of thinking, and also statements about the essential role of mathematics in studies and career – seem to have a connection with the conceptual and professional skills from the content analysis. The knowledge scales – representing knowledge (or lack of knowledge) about the role of mathematics in studies and career – seem connected to the earlier comments made by students who did not know how they would use mathematics, or believed they would not use it at all, and those who stated that it was essential but gave no reasons for their view.

The connection between these scales and the previous content analysis is not completely clear. For instance, the practical scales seem to include some notion of problem solving (mathematics will *"help me solve problems in other subjects"* and *"help me with analysing problems"*), amongst items dealing with calculations and data analysis. By contrast, in our content analysis we included methods of problem solving amongst conceptual skills and generic problem solving as part of professional skills. Maybe this is another manifestation of the central position in mathematics of problem solving – it can be discussed in terms of specific techniques, or a way of thinking mathematically, or as a generic approach to life's challenges.

The shorter version of the questionnaire (given in Appendix 1) can be used by a lecturer to appraise and monitor a lecture group of any size, even large introductory classes of several hundred students. Any of the three sections of the questionnaire could be used separately – for instance, the first set (16 items) could be used to investigate students' conceptions of mathematics, or the third set (16 items) could be used to assess students' perceptions about their future professional use of mathematics. Unusually low mean values, such as a low average on the knowledge scales, could be addressed in later classes. If the questionnaire is given early in the semester and then again towards the end, it can be used by the lecturer to see how students' ideas have changed, maybe in response to particular pedagogical strategies.

As well as a useful tool for the lecturer, the items can be used as the basis of debate about aspects of mathematics – conceptions of mathematics or its use in students' studies or careers – that do not usually form part of mathematics lectures. With the availability of interactive student response technology in the form of 'clickers' (see, for example, Patry 2009; Haeusler and Lozanovski 2010), students can indicate their agreement with survey items immediately, and the overall results can be displayed to the whole lecture group and used to prompt class discussion. Although such surveys are more usually used for content-based questions, they seem to be equally effective in allowing students to express their perceptions or opinions (Bode et al. 2009). Monitoring and discussing students' ideas and beliefs allows lecturers to stay more in touch with students' beliefs and expectations about mathematics, and to include discussion of a type that rarely occurs in mathematics lectures.

Summary and Looking Forward

In this chapter we have changed our focus to quantitative approaches to investigating students' ideas about mathematics and its uses. As mathematicians, we are always pleased to examine things quantitatively, to utilise the very nature of our discipline. Yet here we have built a quantitative investigation on our previous qualitative results – the original interviews with undergraduate mathematics majors and the open-ended survey results from a large group of students studying mathematics. Our analysis of questionnaire data from another international group of mathematics students has corroborated the previous qualitative results. At the same time, we have been able to develop a compact survey asking students about their conceptions of mathematics and their perceptions of its future use in their studies and careers.

We have suggested that this compact survey can be used by lecturers to appraise and monitor their classes, whether they are small groups or one of the large groups of several hundred students that has become more common in first-year mathematics classes. The survey can be used once to give a snapshot of class opinions, and it can be repeated later in the semester to assess changes in students' ideas. Even more importantly, the results of the survey can be used as a focus for class discussion about mathematics and its uses. This is particularly powerful if students can take the survey and see the class results immediately. It does not seem to be common to include such discussion in lecture classes. Most mathematics syllabuses are focused on technical material, and sometimes dominated by this to such an extent that there is little room for any overview or talk about the nature of the discipline itself.

In the following chapter, we will return to our interview studies and introduce some of the material that has not so far been utilised in our exposition. We take a look at the conversations that we had with recent graduates of mathematical sciences degrees at Australian universities. We asked them what they thought that their mathematics studies had contributed to their professional work. One notion that the graduates expressed was that studying mathematics had developed their ability to solve problems and think logically, though their interpretation of these terms was quite varied. We will investigate this outcome of a mathematical education and show the part that it plays in the process of becoming a mathematician.

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Chapter 6
What Is the Contribution of Mathematics to Graduates' Professional Working Life?

Introduction

We now change the focus of our investigations to students who have made the transition to becoming mathematicians. We report on aspects of two interview studies that we carried out with recent graduates from degrees in the mathematical sciences. We investigate the contribution that mathematics has made to their professional working lives and even to their personal lives. While many of them are working in a range of mathematical areas – including finance and banking, academia and research, business and IT, biostatistics and statistical forecasting – some of them are doing quite different work – for example, one of them is a police officer and another is a jazz musician (though since the interviews she has got a job as a lecturer in mathematics). Each of these graduates has successfully completed a degree in mathematics and is now qualified mathematically, even if they are not currently working with mathematics. The analysis of the interviews that we carried out with these graduates allows us to look back at the development of professional aspects of mathematics from the viewpoint of people who have made this transition. Their experiences and thoughts provide a significant viewpoint, and one that is vital in the investigation of becoming a mathematician. In this way, we extend our previous research where we focused on what students learn from undertaking a degree in mathematics, and what they take with them from their studies as they move into their workplace. Research in higher education rarely moves beyond the institutional learning situation, but here we explore what graduates have taken from their previous learning and how they interpret what they have learned in their new work contexts.

Recent mathematics graduates are in an intermediate position between students and professionals. On the one hand, their experiences as mathematics students are still fresh in their minds; on the other, they have started to develop an appreciation of the role that mathematics plays in their working life – and even those who are not working with mathematics can make comments about this. In our studies with graduates, we have found that they identify a range of outcomes from their mathematics education.

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Mathematics Education Library 56, DOI 10.1007/978-94-007-2984-1_6,

These include the obvious technical skills in mathematics itself as well as more generic skills, the awareness of a range of personal characteristics such as self confidence and persistence, and the development of a professional identity. One notion that was expressed by almost all of the graduates was that studying mathematics has developed their ability to solve problems and think logically, although their interpretation of these terms was quite varied.

In this chapter we present recent graduates' views of the contribution of mathematics to their professional life. We begin by identifying the range of technical, personal and professional skills that the graduates talked about, and illustrate them with quotations from the interviews. Then we narrow our focus to one specific aspect, a key characteristic of mathematics from the viewpoint of mathematicians and mathematics educators, as well as from students and recent graduates – the notion of problem solving. We explore in detail their experience of mathematics as problem solving and the way this links with their view of their professional role as mathematicians. Continuing our methodology from the previous interview studies, we analyse graduates' ideas about problem solving using a phenomenographic approach. The result is a hierarchy of conceptions of the notion of problem solving in the context of the professional life of mathematical scientists.

Studies of Transition from Study to Professional Work from the Viewpoint of Graduates

Recent graduates are in an intermediate position between students and professionals, in that they have completed their studies but have not yet taken on their full professional role – they could be regarded as novice rather than expert practitioners (Rosenfield 2002; Wenger 1998). They are in a unique position to comment on aspects of their studies and the transition to their profession. However, there is not a long history of research utilising graduates' views on these topics. More commonly they are asked about their overall rating of their universities, job prospects, career progression and earning power (Johnston 2003; for some current Australian examples see the surveys on the Graduate Careers Australia website 2010).

In the British context, Knight and Yorke (2003, 2004) describe views of the relationship between higher education and employability. Firstly, higher education can be a preparation for a profession, so employability can be defined as how well students are prepared for that profession. Secondly, higher education can prepare students for any job by developing generic achievements, so that employability is enhanced by the development of excellent generic skills. Problems arise when assessment is overloaded with expectations of promoting employability as well as discipline achievements. An interesting aspect of their investigations was a pilot study of unemployed graduates (2004, pp. 15–16) that identified unrealistic aspirations, poor career planning and a mismatch between study and work as reasons for their unemployment.

In Australia, Scott and colleagues (Scott and Yates 2002; Rochester et al. 2005; Vescio 2005) have carried out investigations focussed on successful graduates. For a variety of fields of study, several employers were selected and asked to nominate a group of their most successful recent graduates. These graduates and their supervisors were then interviewed in depth to ascertain the attributes that had contributed to the graduates' success. Scott's research points out that it is when things go wrong, when an unexpected or troubling problem emerges, that professional capability is most tested. At such times, the individual must use the combination of a well-developed emotional stance and an astute way of thinking to 'read' the situation and, from this, to figure out or 'match' a suitable strategy for addressing it, a strategy that brings together and delivers the generic and job-specific skills and knowledge most appropriate to the situation. As an example, if a professional is unable to remain calm and work with staff when things go wrong, then their disciplinary knowledge and skills are quite likely to be irrelevant.

Also in Australia, Crebert et al. (2004) discussed graduates' and employers' perceptions of the role of the university, work placements and subsequent employment in developing generic skills, including problem solving. Data were collected via a survey of graduates who had participated in work placements during their studies and through two small focus groups. There was some disparity in views. Employers felt that graduates were not adequately prepared for the realities of work, while some employees believed that they were constrained by limitations in the workplace (including the views of their employers). In terms of the area of problem solving, employers felt that graduates should have more experience of open-ended problems that need a solution within the constraints of limited time and resources. This study takes the investigation of generic skills into the contexts in which they will be used, beyond the more common investigations that focus on the topic from the university viewpoint (for example, Barrie 2004, 2007).

Johnston's (2003) call for further studies of graduates' experiences in their early employment years has been addressed by several research projects (including our own). The European 'Journeymen' project (Dahlgren et al. 2007) investigated the experiences and understanding of learning and work of cohort groups in several disciplines in four universities, one each in Sweden, Germany, Poland and Norway. Two of the disciplines, psychology and political science, were common to all four universities, while engineering, education, information technology and law were each studied at one university. The overall purpose was to investigate the extent to which tertiary education provides adequate preparation for working life. The information was obtained by interviewing participants – the 'journeymen' – in their passage through the culture of higher education and into professional working life. The study found distinct differences in the views of students between countries, disciplines and programs. Those programs that are closely integrated with a professional discipline (such as the psychology program at Linköping University in Sweden) promote a strong development of students' professional identity (Abrandt Dahlgren et al. 2006). This contrasts with programs such as political science at the same university, where the more traditional humanistic pedagogy and the absence of clear role models leave students vague about the profession and with a more diffuse professional identity (Johansson et al. 2008). Karseth and Solbrekke (2006) discuss the somewhat different experience of the psychology students at the University of Oslo, Norway, highlighting the role played by program characteristics; Linköping has a problem-based psychology course, while Oslo has a more traditional structure.

A particular feature of the Journeymen project was its longitudinal aspect; each group of participants was interviewed more than once at different stages in their studies and career. Nyström (2009) has reported on Swedish students interviewed towards the end of their studies, then after a year and then almost 3 years into their professional career. She represents the transition from formal studies into working life as a 'trajectory'. This theoretical concept indicates that the movement from education to working life is not self-evident, but a course that can be influenced by a number of factors. The participants in her longitudinal study were able to re-consider their formal learning in the light of their new professional lives.

Unfortunately, none of the studies in the previous paragraphs investigated any of the disciplines that comprise the mathematical sciences. The engineers in the Linköping sample in the Journeymen project may be the closest, though their focus was the profession of mechanical engineering, with mathematics studied as a professional component only (Abrandt Dahlgren et al. 2006). Yet the situation in mathematics is quite different in various ways, particularly in the mathematical view of problem solving. In most disciplines, this is discussed as a 'generic skill', but in mathematics it occupies a specific and more central position.

From the point of view of mathematicians, the notion of solving problems is an essential defining component of mathematics. Many 'classic' books have been written about the process of problem solving (for example, Pólya 1957, 1962; Schoenfeld 1985; Zeitz 1999). The general viewpoint is expressed by Zeitz (1999, p. ix) when he writes: *"Problems and problem solving are at the heart of mathematics. … Someone who learns to solve mathematical problems enters the mainstream culture of mathematics."* Pólya (1962, p. ix) introduces his book with a broad definition: *"Solving a problem means finding a way out of a difficulty, a way around an obstacle, attaining an aim which was not immediately attainable."* He presents a theory of 'heuristics', the study of means and methods of problem solving, that can be applied to problems in mathematics and beyond. He paraphrases Descartes' (1628) universal method for the solution of problems as: reduce any problem to a mathematical one, then to a problem in algebra and finally to the solution of a single equation. Schoenfeld (1992) investigated mathematicians' and students' approaches to solving problems, concluding that important factors for success include cognitive resources, heuristics, control or metacognition and beliefs. These 'cognitive resources' can been seen in action in Burton's (2004) reports on interviews with research mathematicians discussing their approach to problem solving. All the mathematicians claimed to be working on problems all the time, anywhere from 1 to 20 at the same time, using one or more of three characteristic styles of thinking: visual (in pictures, possibly dynamic), analytic (in symbols, formalistic) and conceptual (in ideas, classifying).

Leonhard Euler, one of the greatest mathematicians of all time, expressed the close connection between mathematics and problem solving in a letter concerning his investigation of the Königsberg bridge problem sent in 1736 to Carl Gottlieb Ehler:

> Thus you see, most noble Sir, how this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on

> any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others. (quoted in Sachs et al. 1988, p. 136)

Professional mathematicians would agree with this sentiment even today, and later in this chapter we will see some of our graduates expressing very similar ideas.

Our Studies with Graduates

Our investigations with graduates were carried out in two distinct phases. As part of our original interview-based study with undergraduate students, we also interviewed a small group of 14 recent graduates of the same mathematical sciences degree at the University of Technology, Sydney, a 3 or 4-year degree including a wide range of mathematically-based subjects, and allowing specialisations in statistics, operations research, mathematical finance and applied mathematics. Most of our participants were within 1 year of graduation. We interviewed them at convenient locations (including the university, the workplace, and even a local café). Two volunteers were in different parts of the world (one in the US, the other in the UK), so we carried out the interviews by e-mail. The raw material of this study consisted of over 62,000 words of transcripts. As opposed to students in a contained learning environment, graduates are situated in a range of work contexts that provide different opportunities for undertaking mathematical work. Our graduates included people working in finance, insurance, IT and statistics, as well as people still in the academic arena, two undertaking doctorates (one after a period of work in the finance industry) and one who was working as a lecturer, having completed a doctorate.

As with the undergraduates, we asked them about their views of mathematics, their previous learning of mathematics and the contribution of the mathematics that they had studied to their current work situation. We found that their conceptions of mathematics and learning mathematics could be described by the same categories that we had obtained from analysis of the undergraduate data – which provided a useful confirmation of the original outcome spaces described in Chaps. 2 and 3. New information was provided by our respondents' answers to the questions: *What is it like to work as a qualified mathematician?* and *What do you feel you brought from learning mathematics at university to your work?* While respondents referred to various outcomes of their mathematics study, an interesting feature was that they all discussed the fact that mathematics had enhanced their abilities at 'problem solving' (using this or related language), although they seemed to interpret this term in a variety of ways.

A second (and later) interview study was carried out by one of us (Leigh Wood) with another group of recent graduates from the same degree or from a mathematics degree from a second, nearby university, Macquarie University. The prime focus of this second study was on the graduates' views of mathematical communication in the workplace (and this theme will be explored in detail in Chap. 6). This group comprised another 18 graduates, with no overlap of participants between the two studies. Again, there was a range of employment situations, including people with jobs in banking and finance, government and industry, research and academia; two of them were not currently working in any mathematics-related field (one was a jazz violinist, though she had a doctorate in mathematics and was looking for academic work, another was a police officer). While the focus of their interviews was mathematical communication in their workplace, each of these graduates was also asked the question: *In what way has studying mathematics at university level prepared you for work?* Again, they identified a range of outcomes of their mathematics education, and many of them included some discussion of problem solving.

As a result, we have combined their responses to this question with the information that we obtained from the earlier study. We read through the complete set of transcripts several times and identified common themes concerning the contribution of mathematics to working life. In particular, we noted the theme of problem solving that seemed to be present in some form in all the transcripts of the first study and many of those from the second. We then collected together the material under the themes of technical skills, generic skills, personal qualities and identity. Finally, we focused on the problem-solving theme, and identified a phenomenographic outcome space representing the range of ways that our participants viewed the notion of problem solving. We present our analyses in the following sections.

Contributions – Technical and Generic Skills, Personal Characteristics and Identity

We identify and illustrate several themes that were commonly expressed by the graduates in our studies in answer to the question of what they had brought from their mathematics courses to the workplace. The first of these themes is an obvious one: graduates stated that their study of mathematics had given them a range of technical skills, and attested to the importance of this in their current work situation. Nathan and Adele express this in fairly general terms, while Matt indicates how he uses the technical knowledge that he has acquired.

> *Nathan* (mathematics/IT graduate, working freelance as a web programmer): But what I did learn at uni which I couldn't have really got elsewhere was that the skills set where I could sort of lay a foundation and, you know, put my foot out in the door and, you know, sort of go into the workplace and move from there.

> *Adele* (mathematics and finance graduate, working as an IT consultant): Okay, a good fundamental of maths knowledge, because somehow outside people still think 'oh maths, yuk!' and that I could do it really make a difference./…/So I'd say that's what I got most out of uni, I'm not scared of numbers, if people give me numbers, I'm actually interested to work it out, 'what's the pattern, what's going on with the numbers?', I think that's what I got out of uni.

> *Matt* (statistics graduate, working as a biostatistician): I got a great grounding in maths basics and theory. It really helps me to understand what's going on with any method we are trying to use, what might cause a violation of an assumption and how to go about fixing

> it./…/ We learnt not just when to use a test and how to plug the numbers in but also how the equations were derived. Plus basic algebra, and calculus theory which are used to derive these equations. Thus, it made it a lot easier to develop new methods based on old ones, understand why an assumption was important and ideas on what to do if it was violated.

Roger explains the importance of learning the technical details of mathematics, even if it is at a more advanced level than he expects to use in his work.

> *Roger* (pure mathematics graduate, working in research and development in geological exploration): You have to know mathematics on a much deeper level than you will ever really use. So these third, these analysis courses with epsilons and deltas, whereas in reality you will never use an epsilon and a delta because naturally such things don't exist in real-world measurements. Nevertheless it is worthwhile to know it, because then you understand why many algorithms involve approximations and so on, so you can understand. And so instead of just remembering one or two algorithms and uses, you can perhaps even invent your *own* one. And you will be able to understand why.

By contrast to the technical skills, some of our participants referred to generic skills that they had acquired during their mathematics studies and that they were finding useful in the workplace. However, others identified that they had not picked up these skills to the required extent. In the next quotes, Linda highlights the importance of workplace communication, while Thi believes that she has not yet developed adequate skills in this area.

> *Linda* (mathematics and finance graduate, undertaking a Ph.D.): To be honest, if you are not very good at maths but you are a very good communicator, you'll go better in the workplace than if you are very good at maths and you are a poor communicator. It's more in, in the workplace my experience is that it's more important how you get along with people, your level of communicating things, and that can be a subset of just communication in general with other people and also communicating technical knowledge to non-technical, people with non-technical backgrounds.

> *Thi* (engineering and mathematics graduate, running her own finance business): I think we should be able to know how to communicate our knowledge to others so that they can comprehend it in a way that's easy and I don't think we know how to do that yet.

In a similar way, Leo talks of the importance of group-working skills in his workplace, while Christine suggests that it would have been more 'realistic' to have more group assignments in her course.

> *Leo* (mathematics and finance graduate, working in statistical forecasting): Well it's the whole working in groups and stuff like that, that's been, that's really useful because no matter what workforce you sort of go into, there's always groups, you know, team, it's all about team effort and stuff so, you know, you've got to be able to interact with other people, do things as a group. So it was good how we did lots of projects that involved group work, you know, distributing the work evenly between everyone, making sure everyone was pulling their own weight and that kind of stuff, so that was good.

> *Christine* (mathematics graduate, then police academy, working as a police officer): I hate group work, and I hate to encourage group work, but probably group work'd actually, because being mathematics you can do an assignment the night before it's due if you have to, and you *can* not go to bed if you have to, but if you've got group work then you've got people that are counting on you which is probably more realistic.

Another theme that was apparent in students' comments about the mathematics that they had studied was the range of personal characteristics that such study had helped to develop or reinforce. Such characteristics included confidence (Angie and Paul), motivation (Leo) and determination (Thi).

> *Angie* (mathematics and finance graduate, working in a bank preparing loans documents): Well, I think it gives – finishing a course like Mathematics and Finance gives you a great boost in confidence and a big edge.

> *Paul* (mathematics and finance graduate, working in finance): Well, it gives you a background in something and a confidence in your background so you know that you've got confidence in yourself and it gets you into a problem-solving mind-set where you want to take on a problem, 'oh this is interesting' and it's stretching my mind a little bit, so you want to take on that kind of work.

> *Leo*: You've sort of got to be motivated to do the work and do it on your own, because it's not like high school where, you know, everyone is pushing you 'do your work, do your work, da-da, da-da', so its self motivated which you have to be in a job as well, so being able to push yourself and, you know, wanting to exceed, to accelerate because you want to do well, so yeah.

> *Thi*: One of the most important traits that people have, something that they have within themselves is, to be successful, in my opinion, is the willingness to work and work and work and just continue doing it and not giving up. I think it goes into everything that you do in life: uni; friends, even; networking; business. Just striving and working and continuing to work until you're satisfied that this is all that you can do and you cannot do any more and just doing that. Just actually going for it.

These personal characteristics are highlighted by Christine's experience. She tells us of the determination that she needed to overcome an early set-back, determination that she sees as strongly related to the non-trivial nature of the mathematics that she was studying.

> *Christine*: I probably wouldn't say that going to university helped me as far as work is concerned, it was more failing for the first year that really hit home that I had to work for what I wanted, so that, that helped me. But the fact that mathematics is not the sort of subject you can just, you know, figure it out in the last week of semester, read a book and you'll be right, because of the type of subject it is, you need to be working throughout the whole thing, so by the time I eventually finished my degree, I suppose that sort of taught me – up until a point I had just managed to coast through but then it just got too hard so then I had to work for what I was getting. So, that – university in general could have taught me that, but because I was doing mathematics it was more, more of an effort, so …

A more subtle but more essential contribution of undertaking a degree in mathematics was the development of a mathematical identity. Several of the graduates referred to this outcome of their studies with pride, as Christine's next quote shows, sometimes drawing boundaries around themselves and their colleagues, as compared to people who 'haven't got it', as David puts it.

> *Christine*: I'm very proud about it [my degree] and I'll sometimes, you know, flaunt it and people look at me strangely, and I love that look! And so I think it makes me who I am.

> *David* (mathematics and finance graduate, working in a bank treasury): It is, it's very subtle, it's often in the job we do, you sometimes see examples of – it's very difficult to explain – but people who've got it and people who don't.

On the other hand, Vida indicates that her essential mathematical identity may be what prompted her to study mathematics and work as a mathematician.

> *Vida* (mathematics graduate working in financial modelling): I think that what you study or where you decide to work has a lot to do with who you are and what you are like, so there may be, what I'm saying is that there is a possible relationship between what you study and, say, working in the field of maths, with what you are and the way you are thinking and things like that. I mean I see myself to be a very logical and organised type person and I can't really see myself doing something, you know, too far from maths.

It is also interesting to note that some of the mathematics graduates do not feel that they have progressed far enough to take on the identity of a mathematician, even if they are working mathematically. This interchange with Penny illustrates that she is still uncomfortable with this aspect of her identity.

> *Penny* (mathematics and finance graduate, working for a large bank): [What is it like to work as a qualified mathematician?] Well, I'm not! [But you've got a mathematics degree.] Yeah, but I don't classify myself as a mathematician. [Why not?] To me it's, it's a very honourable title, you know when you see someone like 'mathematician' that's kind of respectful, you know. But no, I mean I just think that gaining an honours degree from uni is not good enough to be a mathematician, I have a lot more respect for mathematician than just an honours degree. [How would you describe yourself?] I would describe myself as having a mathematical background, working in a financial institute and I did maths and finance which was a great combination.

The quotes in this section have illustrated graduates' views of the benefits that they had obtained from undertaking a degree in the mathematical sciences, and that they were using in their professional workplaces. The benefits included the technical skills of mathematics – a basic outcome, but mentioned by many respondents – and generic skills such as communicating and working in groups. Graduates highlighted a range of personal characteristics that they felt were developed or reinforced by their studies, and they referred to their development of a mathematical identity – an aspect that was still in progress for some of them.

Contributions – Problem Solving and Logical Thinking

In the first set of interviews with graduates, all our respondents put forward the idea that studying mathematics had helped them in the general area of problem solving, either using this specific term or an alternative wording, often as the first response they made, but sometimes elsewhere in the course of the interview. Many of the graduates in the second study made similar statements. The next quotes from Kath, Leo and Lucy are examples of these responses.

> *Kath* (mathematics and finance graduate, working in a modelling unit in the public service): [What do you think mathematics is about?] I think it's about learning techniques to apply to problems, it's about the way you approach problems and think about, yeah, think about techniques needed to solve the problems.

Leo: [Okay, so could you just describe that way of sort of tackling things?] It's just the whole problem solving thing, like, with maths, you know what I mean, being able to tackle a problem, going through it step by step, you know, in how to solve it, you can apply those steps in any other situation in life, so working through problems, problem solving, being able to sort of quickly analyse something or yeah stuff like that.

Lucy (mathematics graduate, currently undertaking a Ph.D. in financial mathematics): [What do you think mathematics is about?] Problem solving, for me it's problem solving, you've got a problem and I would think about things in a very logical fashion, which I think mathematics teaches you to do, and is a very reasonable area in that it's all based on reasoning. I'm not making any sense, that's what I think, for me it's problem solving.

Using a phenomenographic approach to the analysis, we identified an outcome space, a hierarchy of three qualitatively distinct conceptions of problem solving displayed by these mathematics graduates. From the narrowest to the broadest, we summarise these conceptions as 'specific mathematical problems', 'problems in a work context', and 'problems in life generally'. In the following paragraphs, we describe these three conceptions in more detail, and illustrate them with brief quotations from the transcripts.

(1) Specific Mathematical Problems

Problem solving refers to specific mathematical problems, often in a school or university context, possibly the sort of problems that occur in textbooks. Linda makes this explicit, Gavin focuses on the specific mathematical technique of independence, while Vida links 'questions' and 'problems' in a study context.

Linda: [What do you think mathematics is about?] For me, yeah, maths is about solving problems. That's, I mean if I go back to high school mathematics, which is probably what got me interested in maths, especially in the last year or so, it's just, actually I like solving problems, and maths is one way to solve problems I guess. [Could you tell me a bit more about what you mean by solving problems?] Well I guess, I don't know, the, I mean the example I always think of in my head is I remember in HSC [Higher School Certificate, secondary school matriculation examination] three-unit maths, is a problem where you've got to manipulate, I guess it's just algebraic manipulation, pretty simple stuff, and it's when you are introduced to factorials and stuff, and you've actually got to do a bit of manipulation and follow a method until you can sort of see what answer you are trying to find, and you are going down a path and you've got to find the solution and it's just finding different ways to get to the end.

Gavin (applied mathematics graduate, undertaking a Ph.D. in climate modelling): Another useful idea that I found from maths that I use a lot is the idea of, given a particular, it's just the idea of independence and dependence of variables in problems. I often think of, you know, things like in this act of having a problem that you put into a context, and cut down, a really important thing is to say, you know, can you establish dependence relationships between this and this, or is that independent of that? I guess that's one of the ways you cut it down.

Vida: Well, maths is one of those things where it's very practical, you've got to actually go in and do the questions, as opposed to say just reading about it. I mean you can read all you like but you won't know how to do questions or you won't know how to solve problems until you actually go and have a go at it.

(2) Problems in a Work Context

Problem solving refers to problems in a work context generally, including but not limited to mathematical problems. In the quotes below, all four respondents set their discussion in terms of general workplace problems. Kath and Heloise talk in general terms, Paul faces a particular problem, how to price a 'swaption', while Vida talks about the problem of understanding work reports.

> *Kath*: I think what I've derived from mathematics more than a set way that people use mathematics in the workplace, I've actually derived more of an idea that it's a framework to think about problems and not, you don't see calculus in the workplace or anything like that, well I haven't in my particular position, I mean I'm not generalising it and saying that no one else will, but I think as far as the workplace has been for me it's just, it's been a starting point for me to think about problems and for me to analyse things.

> *Paul*: A couple of weeks ago I was given a new swaption, this weird derivative to price, and how do you do it? And I kind of sat down and had a think about it and worked through, pulled out an old derivatives securities textbook and worked through it like a problem, that was it. So that's another way uni had helped

> *Heloise* (operations research graduate, working as a logistics analyst): I think what really helped me was, not so much remembering every theory and every formula that I'd learnt, it was just the way of thinking that kind of helped me – 'look at everything, here's a problem, translate it to mathematical problem, break it down, try and get a solution and then translate it back'. Just that way of thinking has really helped.

> *Vida*: Okay, my job doesn't really rely heavily on maths or mathematical knowledge, but what I use, or what I perceive to be using are the skills that I've picked up, you know the analytical skills, the research skills, the ability, you know, like I said to pick up something, a report or whatever it is that you have got and see whether it does make sense. Although other people, although other professionals write them, sometimes it's not always obvious or clear what they are trying to say or trying to do, so I think that's where the logic and the analytical skills come into it. I'm able to sort of look at something and piece the puzzle if you like.

(3) Problems in Life Generally

Problem solving refers to an approach to problems in life generally, not necessarily limited to the workplace, or to obviously mathematical problems. In the following quotes, all four respondents are explicit about their problem-solving abilities being applied beyond the boundaries of their professional work.

> *Matt*: So logic and reasoning were ingrained into my family. I guess that's part of the reason I like statistics and research. It's very much solving real-world problems. Statistics also attracted me as it could be applied to answer important questions. In the medical area where I use my stats, I got into it because it could be used to help people by helping to solve important medical questions.

> *Adele*: [What do you think mathematics is about?] It's about learning how to be logical, how to do problem solving, so in addition to learning all the formulas and stuff, I would say it is an approach to help us to structure our thinkings and thoughts, so that structure allows

us to deal with problem solving in daily life./…/I mainly see mathematics as a way to help me learn how to think in problem solving.

Gavin: I guess it, if you were gonna have any kind of success in mathematics as an individual, you need to develop the skills to recognise where the boundaries of a particular problem are, what's involved in a problem, how do you split the problem up, so that I suppose, at least that attitude carries over into problems in everyday life. That I definitely use, over and over again, and I think I've learnt that through mathematics.

Vida: [Would you say that learning maths has changed you?] Studying maths for three years, has it changed me? Might've, it may have changed the way I approach certain projects or certain obstacles in my life. I don't know, it's not something you think of or think about, but I find myself, or you try to be a bit more objective with problems that you face, I don't know if that's just me or if it's something that I've learnt, it's hard to answer that one.

In common with other phenomenographic outcome spaces, the conceptions that we have identified and described are hierarchical and inclusive. Those graduates who describe the narrower views of problem solving seem unable to appreciate features of the broader, more expansive views. On the other hand, those graduates who describe the broader, more holistic views are aware of the narrower views, and can discuss and use them when necessary; Vida's quotes at all three levels illustrate this. This is the reason why we as educators value the broader, more holistic conceptions.

Our participants also offered specific comments on their methods of problem solving. A common theme involved the classical technique of breaking a problem down into components, which could then be attacked in various ways. Both Adele and Alistair illustrate this technique.

Adele: Problem solving, it's more like okay you've got a problem in front of you with maybe restrictions, constraints and stuff like that, and you will need to look at how you come to a solution, maybe you see a problem in the middle and you've got to just eliminate say, look outside the box in a way, so you come over a problem and you can say 'okay, let's try to solve it doing what you were taught or what you experienced too', but you come and get stuck and you have got to say, you know, how do you fix that up.

Alistair (mathematics graduate, working in IT in the finance industry): Like I don't use that calculus, but I definitely use the same methodology, like I'd break it down into bits, I'd work it out, like I'd work from the middle or something and work my way out or something like that, or even vice versa, you work from the end sometimes and just, you know came back.

Another theme was the notion of quantifying a problem, translating it into equations which could then be solved. This sounds very like Descartes' (1628) universal method, as the quotes from Phil and Matt show:

Phil (mathematics and finance graduate, working in financial risk management): When you are trying to solve these problems, you look at the problem that you are faced with, a particular problem and you try and isolate what are the key variables, what are the key factors that are driving the problem, and if they drive the problem, they probably drive the solution in some way as well. So what you need to do is arrange these variables, if you will, into some sort of relationship.

Matt: So, by thinking about the maths behind something, it makes more sense to me. The easiest example I can think of reducing to an equation is think of a pedalling a bike. We all

> know that the faster you pedal the faster the bike will go. But why that's true is based on maths, you use this to work out how much 'pedal power' you would need to go at a certain speed./…/ So, basically maths can help you answer questions even in nature, which would be one of the reasons you would want to reduce something to an equation. Also, the problem never looks as difficult to me once it's in an equation.

One graduate, Linda, raised an interesting point about the influence of the workplace on the actual method of problem solving, both in terms of the pressure of time and the necessity for communicating the solution.

> *Linda*: [Okay, did communicating non-technically about maths, does that change the way you see maths itself?] It can change the way you approach problem solving. For example, there may be technically sort of, there may be two ways to solve a problem and one may be very technical and sort of the optimal way of solving a problem and the other one may be a little more rough guess or more an estimate, and quite often you haven't got the time to do the optimal method anyway in the workplace, certainly in the environment that I worked in, so you've only got the time to do the estimate and also when you do do that sort of thing it's easier to work through and show people what you have done, if you've taken, if you've taken a simplistic method.

It is an interesting counterpoint to the many discussions of problem solving by experienced professional mathematicians that recent graduates seem to be able to talk fluently and perceptively about this important mathematical activity.

Discussion

The graduates whom we interviewed identified a range of outcomes of their study of mathematics that they were able to take into their professional work situations. To some extent, these or similar outcomes would be found after completing any course of university study, particularly one aimed at a professional field. Technical, discipline-specific skills and generic skills are developed in every university course, combined in different ways in different disciplines. The Journeymen results point out that tertiary study of each discipline seems to combine substantive and generic knowledge in characteristic ways: the former refers to discipline-specific skills, contextually situated and clearly relevant for specific work situations, the latter refers to skills applicable across a range of disciplines and transferrable between different contexts. For example, the Swedish political science course emphasised the generic skills, while the psychology course contained a larger component of substantive skills. In contrast to both of these 'rational' aspects of knowledge, some disciplines included 'ritual' knowledge – where the connection to context was lacking, and the reason for inclusion was unclear to the people studying, for example, the early mathematics component of the mechanical engineering course at the same university (Reid et al. 2011, Chap. 4). As opposed to undergraduates, our mathematics graduates did not talk about ritual knowledge, for two likely reasons: first, their studies focused on mathematics and hence they were more likely to understand the reason for inclusion of various components, and secondly they were already in a work situation where they could see the uses for the skills that they had acquired.

The enhancement of personal characteristics such as confidence, motivation and determination can be observed as a result of university studies in a wide variety of disciplines, not only mathematics. The same is true of the development of professional identity, particularly in university courses that have some professional focus – and this is generally the case in contemporary tertiary studies. Our graduates' comments on their increasing professional competence and confidence could be mirrored by statements from graduates of other disciples (see, for example, Nyström 2009). The notion of problem solving, however, plays an essential role in mathematics well beyond the notion of generic skill, as it is viewed in many other disciplines. Several of the graduates' comments indicate the important role in their professional identity formation that is played by the idea of problem solving. For instance, Alistair characterises his identity as a mathematician in words that are reminiscent of Euler's:

> *Alistair*: Like if something's hard you don't just sort of drop it and let it go, you work out how to do it, and that's what you have to do if you want to, I suppose, succeed or excel. You have got to solve the problems that other people can't.

In this context, we are looking specifically at professional identity, a relational and fluid concept, open to influence from social interactions and contexts. Wenger (1998, pp. 189–190) considers the notion of professional identity formation as a tension between investment in various forms of belonging or membership and the ability to negotiate meaning in those contexts. Our graduates' comments about their developing identity as mathematical scientists are linked to features of the discipline of mathematics. During their university studies, they encountered the world view and social practices of mathematics, and they gradually appropriated aspects of them in the process of building a professional identity and becoming a mathematician. As mathematics students, they became part of a community of practice (Wenger 1998), but now they are widely dispersed in various companies and various professional areas – and some of them not even working in a mathematical area. So while they show awareness of membership of a community of mathematicians (as shown by their comments about mathematical knowledge or ways of thinking) they are also developing membership of other communities of practice, most particularly, their specific work group. They are often isolated from their mathematical community, except for connections through professional societies or conferences. In the process of negotiating meaning in their work environment, their communication skills are critical – and we will return to this point in the following chapter. They utilise their more or less developed mathematical identity to negotiate their professional workplace identity – often successfully, but sometimes less so.

Implications for Teaching and Learning

Graduates' reflections on their studies give an insight into the pedagogical approach that they have experienced, and a useful source of suggestions for strengthening the teaching. This is particularly the case in mathematics, as graduates of degrees in the mathematical sciences have not often been given opportunities to critique the teaching that they have experienced. Our data are particularly useful since the graduates have been allowed to explain their views in the context of semi-structured interviews. On the basis of the results that we have summarised earlier, various pedagogical ideas emerge.

Several of our respondents pointed out the importance of their technical mathematical skills in their workplace, though others claimed that generic skills such as communication and teamwork, and personal qualities such as motivation and determination were as important, and sometimes more so. This sends a strong message that the development of generic skills should be incorporated into mathematics courses, alongside the technical material that is the more traditional focus. The importance of personal qualities such as determination could also be discussed, maybe in the context of assessment tasks. While many students will give up on a mathematical problem if they can't solve it in a few minutes, their approach is only reinforced by setting them straightforward technical tasks that can be dealt with quickly. And such an approach is most likely to occur if students hold the narrowest views of mathematics as technical components, and of learning mathematics as focusing on those techniques. By contrast, setting more extended real (or realistic) problems, maybe in the form of projects, is likely to encourage students towards the broadest conceptions of mathematics and learning – conceptions that include a specific ontological component that promotes reflection on personal characteristics.

Several of our respondents pointed out that employers may only have a limited understanding of the benefits of their employees' mathematics background. On the one hand, this gives a mathematics graduate a certain status in the world of work:

> *Alistair*: When you are looking for a job to be able to say that you've got a degree, a science degree in mathematics, straight away they open the door, no matter what the job is./…/ People really listen to you when you are just starting out, even without the experience they are like 'well you have got no experience but, you know, you've been studying pure mathematics, you must have, you must know what you are doing', which is not always true, but they don't realise that, which is great.

However, such lack of understanding resulted in some employers being unable to get the full benefit from the skills and abilities of their recently appointed graduate employees, a finding also noted by Crebert and colleagues (2004). The frustration of mathematically-skilled graduates comes out clearly in these quotes:

> *Leah* (statistics graduate, working with survey administration and quality control): It's actually really easy to see areas in industry where you think you could make a difference, it's extraordinarily difficult to get past that recruitment filter and to actually find yourself in one of those jobs.

> *Lucy*: So my experience of working mathematically hasn't been that great, mainly because I think they thought they needed a maths person, and they did, but they weren't giving someone with the maths anything to do, so yeah that was my feeling anyway./…/It just felt silly that these people thought that this was mathematics and it was just odd, I just didn't know how to explain to them that there is a lot more than a t-test, but anyway.

As mathematics educators, we should prepare students for such professional experiences, positive and negative, by incorporating in our classes a more robust discussion of their future professional role as mathematical scientists. For example, we can include guest lectures from employers or recent graduates, practice in interpreting technical knowledge in a professional context, and consideration of professional applications of mathematics as part of assessment tasks. Again, this argues for a clear connection with students' future professional lives, which is most likely to happen under the broadest conceptions of discipline and learning.

Writing about statistics teaching, Sowey (1995) paraphrased an aphorism of the psychologist B.F. Skinner that *"education is what remains when the facts one has learned have been forgotten"*. Sowey claimed that in his experience, and the experience of his students, what remains when the facts have been forgotten is a sense of the structure and 'worthwhileness' of the subject, in his case, the discipline of statistics. It seems quite clear from our interviews that for many of the mathematics graduates in our study what has remained is an ability to solve problems and an identity as a problem solver:

> *Lucy*: In my undergraduate, I don't know what I would have described maths as, but it probably wouldn't have been the problem solving aspect of it. For me now, that's what we do, that's what I do and the people I work with do, we go and we have a problem and we attempt to solve it and we do that by using techniques available to us or creating new ones.

Such a view supports a pedagogical approach where specific mathematical techniques are set in the context of overall ideas about the nature and utility of mathematics, and the particular mathematical approach to problem solving, not only in disciplinary contexts but also in life in general. This highlights the connection between the conceptions of problem solving and conceptions of mathematics in general: the narrowest levels focus on mathematical techniques, a broader level on the applications of mathematics, and the broadest level contains the professional and personal connections that indicate the ontological aspects. Again, Leah illustrates this broadest mathematical approach:

> *Leah*: My whole way of dealing with [problems with people] is to wait until I see a pattern. If I don't see a pattern – I never get stressed about the occasional dummy-spit or – something can seem really off in terms of the way people behave but if it is a one-off then I just don't engage at all. Unless a pattern emerges I don't, I just don't go there.

Summary and Looking Forward

In this chapter we have presented the views of recent graduates about the benefits of their mathematics education in their professional (and sometimes personal) lives. Each of these graduates has gone through the process of becoming a mathematician, developing an identity as a mathematician, though some of them do not regard it as completed. They make valuable comments on their mathematical education and what they took from it into their workplaces. As well as the obvious technical skills they also refer to generic skills such as an ability to communicate with and work with people. Maybe the most important result that they identify of studying mathematics at university is their ability to solve problems – not only mathematical problems, but more general workplace problems, and even problems in life in general. In this, they concur with the views of famous mathematicians and writers about mathematics. They define themselves, in their own words, as people who *"solve the problems that other people can't"*.

In the following chapter, we will continue our investigations of the views of graduates of degrees in mathematics. We focus on their experiences with and ideas about communicating mathematically in their workplaces, an area in which many of them felt that their university preparation was inadequate. The graduates in our study showed a range of experiences with different discourse practices. They were well aware of the importance of communicating mathematically with colleagues, bosses and clients, and although some of them had developed skills in this area, many of them experienced frustration with their communication abilities. Overall, they believed that they could have benefitted from a greater range of opportunities to learn how to communicate mathematically in their profession.

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Chapter 7
What Is the Role of Communication in Mathematics Graduates' Transition to Professional Work?

Introduction

In this chapter we continue our investigations of students who have made the transition to becoming mathematicians. We report on our study with 18 graduates of degrees in mathematical sciences (carried out by Leigh Wood) that focused on their views of the role of communication in their transition to professional work. We used the interviews with these graduates in our discussion in the previous chapter concerning the contribution of mathematics to professional working life, and most particularly, to the aspect of mathematics as problem solving. Here, we use the notion of mathematical communication as another lens to investigate the move from study to work. Communication is a key professional (and indeed personal) skill, and the discipline of mathematics places particular demands on this skill and engenders particular approaches to its utilisation. We will see that recent graduates are keenly aware of the importance of mathematical communication and attribute their professional successes or problems in large part to their abilities or lack of abilities for effective communication. Their experiences and reflections will be valuable for any student who is aiming to become a mathematician, and any teacher who is helping students in that endeavour.

Communication skills are an essential employment requirement and a key indicator in the successful transition to professional work. From the point of view of an organisation, an employee who can communicate well makes a more valuable contribution to that organisation. From the perspective of the individual, the power of language choices can help to get a position, adapt to the requirements of the job and progress in a professional career. Reflecting this essential importance, communication skills are listed in graduate capabilities statements for every undergraduate degree in Australia. This is the case in every professional field, but in mathematics

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the notion of communication seems to go further. In *The Language of Mathematics*, a earlier contribution in this series, Barton (2008) discusses aspects of mathematical communication and concludes:

> Mathematics is created by communicating, that is, mathematics is created in the act of communication about the QRS [quantitative, relational or spatial] aspects of our world. A corollary of this is that mathematics is both enabled and restricted by the conventions of communication. … Learning mathematics, and doing mathematics, involves talking mathematics: the more we talk mathematics, the better we will learn it and do it. (pp. 173–174)

Barton puts forward a strong message. Not only is mathematics disseminated by communication, it is also *created* and *learnt* by communicating. On the basis of our studies with tertiary students and recent graduates, discussed in earlier chapters, we would agree with these findings. Experiences of mathematical and more general academic 'discourse' – a more inclusive term than communication – form the basis of the skills that are implicitly and explicitly embedded within mathematics curriculum. By examining how mathematics is used in workplaces, and listening to the experiences of recent graduates talking about the communication aspects of their mathematical work, we can use the results from this chapter to guide our approach to mathematics pedagogy in a way that will enhance the transition to the workplace for recent mathematics graduates. In this chapter, we will not focus on the more general aspects of interpersonal communication, such as working in teams, though it is clear in the following analyses that graduates were taking their communication cues from others in the workplace. We concentrate on the descriptions by the graduates of how they use mathematical communication in their employment and on their reflections of how they learnt that communication.

Mathematical Discourse

Communication comprises the giving or sharing of information, thoughts or opinions by speech, writing or signs. Here we will be more precise and use the term 'discourse'. Discourse can be defined *"… in the broad sense of a 'communicative event', including conversational interaction, written text, as well as associated gestures, facework* [communication designed to avoid 'loss of face'], *typographical layout, images and any other 'semiotic' or multimedia dimension of signification"* (van Dijk 2001, p. 98). The term is used to emphasise *"… interaction between speaker and addressee or between writer and reader, and therefore processes of producing and interpreting speech and writing, as well as the situational context of language use"* (Fairclough 1992, p. 3). Discourse can also refer to the type of language used to construct some aspect of reality from a particular perspective, for example, the liberal discourse of politics (Chouliaraki and Fairclough 1999, p. 63), a system of discursive practices that constitute their objects of knowledge (Howarth 2000, p. 68), and a relational identity whose existence depends on differentiation from other discourses (Howarth 2000, p. 102).

Communicating in or about the field of mathematics involves taking part in mathematical discourse, whether by reading, writing, listening or speaking. Discourse is a broader concept than language because it also involves all the activities and practices that make up the mathematical profession. The discourse of mathematics includes all the ways that mathematics is done: through language, textbooks, blogs, emails, computer packages, seminar presentations, mathematicians talking to each other and to a wider public, and through popularisation and application of mathematical knowledge. Mathematical discourse is distinct from other types of discourse because mathematics is distinct from other disciplines, and many activities that mathematicians carry out are distinct from activities in other academic or professional areas. Mathematicians often require specific methods and forms of communication. Mathematical discourse is frequently highly symbolic and has been developed because of advantages over natural language: it has a precision and economy, it encapsulates concepts in a way that clarifies connections, it facilitates manipulation of mathematical objects, and it is less open to ambiguity and multiple interpretations.

When we asked one of the graduates in our study whether mathematical communication was different to other forms of communication, he replied:

> *William* (actuarial studies graduate, working in IT): Absolutely, because it's symbolic representation; because of its concise and precise meanings; because of its generally quantitative nature; because of the level of complexity involved and the depth of conceptual understanding that is related to, sometimes, one particular symbol or word could be much more than in, … other domains.

Mathematical symbolic language is not universal and unchanging; it has been invented and developed by mathematicians for a particular purpose. In his famous volumes, *The History of Mathematical Notations*, Cajori (1928, 1929) catalogues these developments. Two examples can indicate the power of notation: the adoption of Hindu-Arabic numerals in the late Middle Ages (Cajori 1928, pp. 74–99) and the systematic development of notation by Leibniz (Cajori 1929, pp. 180–196), which allowed mathematics to develop into such a powerful discipline. Of course, there have been many changes since 1929, particularly with the development of computers, but it is often considered that Cajori described the foundations of mathematical notation. Symbolic language is easier to follow when it is expressed in ways that follow the conventions of natural language. Mathematicians use natural language to help other people to understand mathematics; they mix symbolic language with English (or another language) in different proportions and in different ways depending on what they think their audience will find comfortable.

Are our mathematics graduates able to demonstrate fluidity with mathematical discourse? Are they able to choose appropriate communication strategies for a professional situation? Were they taught explicitly to communicate mathematically? In this chapter we will investigate these questions using the transcripts of our interviews with 18 recent graduates of the mathematical sciences.

Becoming a Communicator of Mathematics

Academic teaching staff go through a long apprenticeship to become researchers in mathematics. They complete an honours or masters degree then a Ph.D. and perhaps several post-doctoral fellowships. They generally have good publication records by the time they are appointed as an academic. There is considerable pressure to continue to publish throughout an academic career. Each of these career stages helps to develop an understanding of mathematical discourse within a very specific, academic context. As pointed out by Burton (2004), research mathematicians do not receive formal training in writing and the expectations of writing are not clear. Other authors agree:

> Even though the ability to write for publication is a key skill for an academic staff member to possess, most staff will not at any stage of their career, whether as a student or as a staff member, be directly taught how to write for publication in refereed literature. In most cases, it is expected that they will have already attained a medium level of written communication, and will be able to learn on-the-job the more specific academic writing skills needed. However, this is not always the case and some universities have introduced writing courses, believing that their staff will benefit by attending these. (McGrail et al. 2006, p. 24)

This lack of formal teaching of mathematical writing flows into all undergraduate mathematics education, so graduates are often under-prepared for writing both for research and for work situations. To increase the quality and quantity of academic writing, several ideas have been implemented, generally after the person has been employed as an academic. For example, McGrail et al. (2006) published a review of interventions to increase academic publication rates. These interventions characteristically provided participants with information about the writing and publication process. Most of them required the development of a draft manuscript during the course. A few universities offered a professional writing coach for staff to assist with one-to-one development of writing. In all cases, resulting increases in publication rates were noted. But we would argue that such interventions are too late. How many of our graduates would have been able to make a contribution to mathematics but did not receive any appropriate development of writing (or general communication) skills in their undergraduate education?

Becoming a communicator of mathematics is a subset of becoming a mathematician. Table 7.1 shows the several transitions that are experienced as one moves from school through university into the professional workforce, and an idealised view of mathematics during these stages. As we learn, our ideas of mathematics expand and become more structured (though as we have seen, many university students seem unaware of the structure of mathematics) until, as professionals, we are able to adapt and apply mathematics in a variety of contexts and discern the contributions that mathematics can make to professional situations. Our learning becomes self-directed and our identity moves from being a student to being a professional. It is a particular and essential skill for a graduate to be able to use discipline-specific discourse in a way that communicates effectively with their

Table 7.1 Student transitions and contexts (Adapted from Wood and Solomonides 2008)

	School	University	Professional
Mathematics	Disconnected calculations	Structure; abstract and modelling	Flexible application in context
Learning	Very structured	Structured	Self-directed
Identity	Student	Student	Professional
Discourse	Limited academic	Academic	Discipline-specific

audience, whether this is the manager, a shareholder, a client, a colleague or the general public. Later in this chapter (see Table 7.4 and also Appendix 2), we will indicate the types of capabilities that a mathematics graduate should demonstrate for effective mathematical discourse.

We now turn to what the graduates in our study said about how they used mathematical discourse in their work and their reflections on how they learnt to communicate mathematically.

The Methodology of Our Study

We have already described the selection of our participants in the previous chapter. They were all graduates of degrees in mathematical sciences from two Sydney universities. All but two of them were working in mathematically-related fields – the exceptions were a police officer and a jazz violinist. The graduates were interviewed using a semi-structured format in which we started from a small number of questions and allowed the interview to broaden out depending on the participants' responses. We used the term 'communication' to investigate graduates' interpretations of their experience with discourse. After questions relating to their use of mathematics in the workplace, graduates were asked the following questions specifically relating to mathematical discourse: *In what ways do you use mathematics to communicate ideas?*, *Is mathematical communication different to other forms of communication?* and *How did you learn these communication skills?* Each of these questions was followed by further questions that probed the responses and allowed the interviewees to expand on particular aspects of mathematical communication in which they were interested.

The methodology used for a particular investigation depends on the research questions being asked, and different methodologies are necessary in different contexts. To investigate graduates' use of mathematical communication and their reflection on their experiences of learning to communicate in mathematics, we return to phenomenography (Marton and Booth 1997), the approach that we used earlier in this book to investigate students' conceptions of mathematics and learning mathematics. Using a phenomenographic approach, we aimed to uncover the particular different ways in which recent graduates experience the phenomenon of mathematical communication, and the qualitatively different ways in which they learn how to communicate mathematically. The outcome space of our investigation would consist of a set of categories, related logically and/or empirically – usually in a hierarchy from the narrowest and most limited conception to the broadest and most holistic one. Phenomenography identifies the aspects of mathematical communication and its learning that are distinctly different between different members of the group being considered, in this case, recent graduates of degrees in mathematics. The questions posed are designed to encourage the participants to think about how they understand, experience and make sense of the phenomenon – mathematical communication. In particular in this investigation, the interplay between the participants' experiences and how they describe their experiences gives valuable insight into the development of meaning in communication.

The Importance of Communication for Recent Graduates

We begin with some overall views from our recent graduates. They were unequivocal about the importance of communication in their professional work:

> *Evan* (mathematics and finance graduate, working in IT solutions): [As a student] You can … not talk to anyone for 3 years, and you'll be fine. Do a presentation now and again, you don't really need to talk to anyone. Or be nice to anyone. Or be tactful. Or know how to teach someone something. I'm just calling it the way I see it. As a student, you don't need that. At work, you'd be out the door quicker than anything, doesn't matter how good you are, if you are not tactful, or don't know how to talk to someone, you can't have a, provide a client-style consultative relationship with, with people you work with, doesn't matter how good you are. It's, it's secondary, far secondary. I've learnt that first-hand, you know, that you need to be able to communicate and have other skills beyond the technical skills. It's so much more important.

> *Nathan* (mathematics/IT graduate, working freelance as a web programmer): The reality is, out there in industry, if you don't communicate well and if you don't have those sort of skills to go along with all the rest of your programming or your analytical skills you're not going to make it, it's as simple as that.

Being able to communicate with colleagues, clients and managers made the transition to work a positive experience. For many graduates, their manager was the key to the transition to work:

> *Heloise* (operations research graduate, working as a logistics analyst): I think the biggest help was the people I work with. When you're comfortable, especially with your manager, the person you report to, you're comfortable with that person, it makes a big difference, I think. I just … I've taken a step back and had a look sometimes and, if I'd had a different manager I don't think I could have gotten as far as I have and I've only been there seven months. He's very open.

Part of our role as university educators is to prepare graduates for life after university so that they are able to use their mathematical knowledge and skills. This should be one of our important outcomes. We asked our graduates what they perceived were the differences between mathematical communication and other forms of communication, in order to tease out their conceptions of mathematical discourse. Leah seems very clear about the aspects of mathematical communication:

> *Leah* (statistics graduate, working with survey administration and quality control): I think that communicating any technical or non-average lifestyle thing is different to communicating my feelings, my emotions, … I think that something that has a difficult concept to grasp does take a special type of communication, and I think that it really takes you understanding that other person and where they are at because if you are communicating a concept or a mathematical idea, you need to know where that other person's knowledge, or understanding, is at before you start communicating that, and I think that just the number of mathematical and statistical seminars that I've been to and not understood is testament to the fact that, yes, it does require extra communication skills, just to recognise what type of information you are trying to transmit.

There was a clear message from our group of recent graduates that mathematical communication was a distinct form of communicating, essential in a professional workplace. We now present the outcome space showing the qualitatively-different types of mathematical communication that these graduates articulated in their interviews.

Conceptions of Mathematical Communication

When communicating with mathematicians, or others who were 'analytic', graduates used a range of communication types. They wrote on whiteboards, read and discussed mathematical papers in seminars and used symbolic notation and jargon freely.

> *Roger* (pure mathematics graduate, working in research and development in geological exploration): Fortunately there both of the bosses were PhDs in maths themselves, … so they understood. So you could just leave everything in the mathematical form, you didn't have to first interpret it back into some hand-waving thing, you could just leave the equations, and then they could understand that.

With people who were not mathematicians, there is a basic problem communicating the nature of the discipline.

> *Melanie* (mathematics graduate and Ph.D., then diploma in music, working as a jazz violinist): You've kind of somehow got to convey to somebody who thinks that maths is just like, a couple of, like, numbers and equations maybe, and a tick or a cross, like something's right or wrong, … like it's kind of trying to get over that hurdle and say, OK, well the maths that you know about isn't actually what maths is.

When communicating with non-mathematicians, including colleagues, managers or clients, we identified three qualitatively different ways in which recent graduates used mathematical communication. At the narrowest level was a conception in terms of 'jargon and notation', at a broader level there was a conception that focused on 'concepts and thinking', and the broadest and most inclusive conception, referred to as 'strength', made explicit use of the power of mathematics in the professional context.

(1) Jargon and Notation

Mathematical discourse is about using technical vocabulary. When participants explain mathematics to non-mathematicians they avoid technical vocabulary. Graduates who describe this conception see the components of mathematical 'texts' as being the barrier to communication, and ensure that they communicate effectively by changing their vocabulary, removing the 'jargon and notation', and perhaps changing their semiotic representation, maybe by using a picture or a graph. All graduates moderated their use of jargon and avoided mathematical notation in their dealings with non-mathematicians. This is different to talking with mathematicians, when you do not have to change your vocabulary and can talk using equations.

> *Paul* (mathematics and finance graduate, working in finance): What I find is that the majority of people are not really maths literate to the point of being maths grads, then, as soon as you start talking, as soon as you drop a technical term, whether what you are saying is technical or not, if you drop anything that sounds technical then straight away they close up, 'I don't understand, too hard for me'. I guess that's what I was getting at when explaining things in layman's terms, removing anything technical so that a kid could understand what you are saying.

> *Thi* (engineering and mathematics graduate, running her own finance business): Talking to colleagues we can go into a more technical discussion about our field and they'll understand the lingo. Predominantly, it's the lingo, it's the things like LVR and rates and how it affects somebody. So, they'll understand compound interest, they'll understand cross-collateralising. Just words. … Whereas, if I was to start an in-depth conversation with a client I'd have to explain to them what every word meant first … because if you go into too technical an explanation you lose them and that's it, you have to start all over again.

What was the graduates' intention in doing this? They wanted to communicate with their audience and also not waste their own time. It is career limiting not to be able to communicate with your colleagues or your clients, and sometimes even more so with your managers. These graduates' response is to remove any technical terms, *"it's the lingo"*. They had learnt from experience that these technical terms caused their audience to *"close up"*. Notice that in this conception graduates react to communication situations, rather than control the communication exchange. This changes in the next conception, as they take control of the process of explaining mathematical aspects of a situation.

(2) Concepts and Thinking

In this conception, mathematical discourse is about difficult concepts and thinking differently. Some graduates realised that it was this way of thinking and the mathematical concepts themselves that formed the barrier to understanding for their audience, including but going beyond the jargon and notation. This is the reason that mathematical discourse differs from communication about other areas of life.

With this conception of mathematical communication, graduates were able to explain how they changed their discourse for non-mathematicians by simplifying and explaining the mathematical ideas.

> *Angie* (mathematics and finance graduate, working in a bank preparing loans documents): Because mathematics is quite technical and you need to have the ability to understand it and then the ability to come down to a normal level and write about it normally without being all technical so that other people know what you're talking about. For instance, if we're doing our statistical report, you don't talk about 'the p-value was .05', you've got to explain to them in the report what that means, rather than just giving them the output. So there … it takes a bit more of a sophisticated communication approach.

Several graduates expressed frustration because their audience had little idea about the nature of mathematics itself and, while the graduates would have liked to convey their understanding of mathematics to the audience, they believed that the leap to mathematical understanding was beyond their audience. These graduates could see that it was more than jargon and notation that prevented their audience understanding mathematics; rather, it was the ideas of mathematics itself that were the barrier to communication.

> *Gavin* (applied mathematics graduate, undertaking a PhD in climate modelling): You've kind of somehow got to convey to somebody who thinks that maths is just like, a couple of like, numbers and equations maybe, and a tick or a cross, like something's right or wrong, there's nothing mysterious about it, there's nothing unknown, like everything's right or wrong, like it's kind of trying to get over that hurdle and say, OK well the maths that you know about *isn't* actually what maths is like.

Why do graduates feel that they need to simplify or explain the mathematical concepts, as opposed to merely changing the jargon and the notation? Because it matters in the workplace. In the large bank where Evan is working, he needs his specialist IT colleagues to understand the mathematics so that they can create correct programs. He himself has the mathematical knowledge and ways of thinking; his intention is to *teach* his colleagues to ensure an accurate outcome.

> *Evan*: If they don't know they make a mistake and … you can't afford mistakes. In the big picture the mistakes cost us money, and it can be serious amounts of money. And so it's always better to give that little bit too much information and wait for them to glaze over, than give them not enough information, have them make a mistake. I've found that's very important, that, giving the purpose of what you're doing is, and the implications of not doing it correctly give people an actual incentive to do it right, … if I said to someone, tell me how to price an American option using Monte Carlo, … so if you don't do this right, every option you trade, if you're out, that price is out by two cents, over the course of a year, we're down a hundred grand. That's your disincentive – you don't learn it, we lose that money.

In this extract, Evan is talking about explaining, even teaching, mathematical ideas to his colleagues, motivating them by pointing out the consequences of mistakes. Included in this extract is a hint of the next and broadest conception, that brings to the fore the power of mathematics – the implication that if you are *not "out by two cents"* you will be up *"a hundred grand"*.

(3) Strength

In this conception, mathematical discourse is viewed in terms of the power of mathematical knowledge and the benefits of being able to communicate mathematical ideas and approaches. Those who hold this conception strive to present mathematical arguments accurately and with integrity; this includes explaining the jargon and notation, and even the concepts and thinking of mathematics, but goes beyond that. They realise that their powerful mathematical knowledge and thinking is only released by being able to communicate it. As the mathematician in an organisation, they consider their ethical responsibilities to use mathematics and put forward justified arguments, explaining the mathematical consequences so that management (or others) can understand the risks of various courses of action and make informed decisions.

In his interview, David gave details of a particular problem that he had identified in his work as a dealer arranging short-term finance for his bank. The various amounts of money that the bank had raised from promissory notes had been historically summarised using an Excel sheet and standard Excel functions – straight averages. However, this resulted in bias when the results of a large and short-term borrowing were averaged with those of small or long-term ones. David explains the context in which he is working:

> *David* (mathematics and finance graduate, working in a bank treasury): One of the problems we see with management information is the lack of real information that it often contains. Should someone see a statistic merely because there is a readily available Excel function that produces it? The answer is no. The reality is that most managers outside the specialist quantitative fields do not understand anything beyond basic descriptive statistics. Even the concept of standard deviation is alien to many. … [My manager's] an intelligent man, but he's not, he doesn't have, his background's not quantitative, but he's quite comfortable with quantitative ideas. But I don't think I'd be able to take him to the … calculus, that's not his cup of tea.

Having identified the problem, he needed to find a solution – relatively straightforward – and then explain or 'sell' this solution to management:

> *David*: It's very simple, but in order to get there what you have to do is you have to question, well, what am I doing, am I just simply filling in the numbers on the spreadsheet that goes off to the Board, what am I doing, what is it that I'm actually trying to do, what is the Board about? They're about managing funding risk for the bank. What is it that I'm actually giving them? Well, I'm giving them this, but when you look at this, and when you look at that, the two, there is a bit of a disconnect there which the statistics are driving. So you actually have to come up with a solution. What is the solution for the problem? Weight it.

These extracts show David working using the strength conception of mathematics. He used his mathematical knowledge (in this case, fairly simple statistics) to come up with the idea of using a weighted mean in order to convince management of a mathematically-correct solution to the problem. In the process, he was considering carefully his ethical responsibilities to show the true situation to management in a transparent way.

Table 7.2 Outcome space for mathematical communication

Conception	Intention	Action
(1) Jargon and notation	To avoid the 'glaze over' effect	Omit technical terms or mathematical notation Write in natural language
(2) Concepts and thinking	Understanding in the audience Win an argument or contract	Simplify the ideas (or avoid them completely) Explain (or even teach) concepts carefully Inspire, sell the idea
(3) Strength	Justify a viewpoint Influence the decision process	Flexible use of mathematical discourse Ethical responsibility as a professional

James, another graduate also working in banking, summarised the principles of this approach, indicating the importance of a *"bullet-proof argument"* in the context of professional responsibility with his mathematical expertise. Notice how James also refers to previous, narrower conceptions of mathematical communication when he talks about the need to *"communicate in general and in jargon"*, illustrating the hierarchical nature of the conceptions.

> *James* (mathematics and finance graduate, working in the banking section of a large insurance company): You can do a bullet-proof argument without being subjective you can just quote the facts. … For example if you work in a bank if you look at the issues right at the moment that are around its losses, if you have someone who is an expert in that field, they are going to be a maths person no doubt, I'm sure everyone in management is asking, 'what's going on here?' You need to be able to communicate in general and in jargon.

The outcome space showing the three levels of conceptions of mathematical communication, and the corresponding intentions and actions of the graduates, is shown in Table 7.2. As we move from the narrowest to the broadest conception, the locus of influence moves from the mathematics itself to the professional context of the mathematics. With the narrowest 'jargon and notation' conception, participants change the way that the mathematics itself is presented – they speak or write without using technical vocabulary or mathematical notation. At the broader 'concepts and thinking' conception, they change the way that mathematical concepts are presented – they simplify the ideas (or maybe even avoid them) so that their audience will achieve some understanding. At the broadest level, the 'strength' conception, they use the power of mathematical concepts and approaches to influence and justify decisions in a professional context; and with this comes an awareness of their professional responsibilities. At this level, these graduates conceive that mathematics is no longer external to the communication and the professional situation. They are able to use discourse in the constitution of the social world including social identities and relations (Jørgensen and Phillips 2002). The internalising of the mathematics, the mathematical communication in the professional context, is one of the transitions to becoming a mathematician.

Learning Mathematical Communication Skills

None of the graduates recalled that they had been taught mathematical communication in their undergraduate studies, or at least, if the topic had arisen during their studies, it was not made explicit to them. In fact, some of them believed from their experience that communication skills and mathematics were incompatible:

> *Roger*: Those sort of people skills I do not think, one certainly cannot learn them at the maths department!

However, it may be that communication had been mentioned in their classes, but they had forgotten (though this would also be a comment on its teaching), as Evan explains:

> *Evan*: I've learnt myself, and that could be a combination of, ah, I don't remember specifically being taught it, but it could also mean, maybe I wasn't ready to be taught it or wasn't listening at the time, or! And so, I would have to say I don't think I learnt it at uni, whether it was taught or not!

Indeed, it seems that several graduates had found difficulty getting into appropriate jobs, so their communication skills let them down even before they started in the workforce. Angie and Leah, both looking for improved professional employment that makes more use of their skills, illustrate this point:

> *Angie*: I've got the skills. Given the opportunity I'm more than capable of doing the job. However, having the piece of paper itself hasn't helped me get a job. So, that's a negative problem. Once I get the job I've got plenty of theory and knowledge and skills to go ahead and do it but it has been a very real hurdle trying to get the position.

> *Leah*: I've had some times when I've felt a bit sorry for myself but I've often thought well at least I wasn't one of those poor kids from the Western Suburbs [of Sydney] with atrocious verbal communication skills and I often wondered what happened to them and I don't know whether … I mean, I really do, I wonder what happened to them.

In his work in a large financial organisation, Paul is now on the other side of the fence. He sees graduate applicants who have good mathematics skills but are totally unprepared for the interview process. His comments may throw some light on the experiences of the graduates in the earlier quotes:

> *Paul*: One of the important things in the bank, that certainly drives our decisions these days, in terms of recruitment, is behavioural-based interviewing, and behavioural-based techniques, … What's more important now is, 'Give us an example of a time when you had to solve a difficult problem; how did you go about solving that problem?', 'Give us an example of a time when you had to face a difficult manager; how did you overcome that?' What they're trying to do is, they're trying to see what, how does this person react. And you cannot overcome that sort of thing with b.s. [bullshit] really. It's the truth, your real graduates are just … They won't have any idea what's hit them. They're all prepared to answer questions about …, their weekend pastime, and mathematics and, can you do this sort of mathematics, what can you tell us about Black-Scholes' formula? That's not gonna even be a factor. It's not even going to be asked.

As none of the graduates believed that they had explicitly learnt mathematical communication in their undergraduate studies, we turn to how they believe that they developed mathematical communication skills in the workplace. The graduates showed three conceptions of learning mathematical communication: trial and error; mediated by others and outside situations; and active, detached observation.

(1) Trial and Error

Those using trial and error observed the reaction of those around them and used this to work out the right communication technique for any particular situation, moderating their discourse accordingly. Heloise, working in a company where she and her boss are the only 'quants', explains her approach:

> *Heloise*: [So, how did you learn to talk differently to different people?] I don't know. I just kind of picked it up. Mainly I'd respond by the way they spoke to me. If they speak analytically to me … with my manager. … So, I know that I can speak like that to him, type thing, and then again with the sales lady and with my director, he's a director, he just wants the bottom line, as all directors want. [But how did you figure that out?] Just kind of picked it up.

Evan is a successful young mathematics graduate working under high pressure in a team developing IT solutions for his bank. He is explicit about his use of this approach:

> *Evan*: Trial and error. It's usually that glaze-over effect! When they start glazing over it's time to stop! Trial and error, very much so. [So nobody taught you how to do this?] I'd have to say that it's been a lot of, ah what's the word, internal hit and miss.

The importance of this quote (and others like it) is that this graduate was successful in his transition to work and has been able to adapt to the work situation. There may be many other graduates who were not able to secure professional employment or progress in their careers because they were unable to adapt in the way that Evan has done. Employers also have responsibilities also to ensure that their staff are able to work and communicate effectively in the workplace, and this leads to next approach to learning about mathematical communication.

(2) Mediated by Others and Outside Situations

Some graduates have benefited from having guidance in the development of their mathematical communication skills. Those being mediated generally have a manager who is assisting the graduate to develop appropriate discourse. Again, we turn to Paul for an explanation of how this comes about:

> *Paul*: Experience. When you first come out of uni and you try to explain, you're really keen, oh I've got this great idea and here, blah blah blah, and your boss just looks at you blankly and says that's too technical, I don't understand; or if you get asked to write a report for someone and you run it past your boss, who says it is way too technical, take all the technical stuff out of it and just give them what they can understand, write for the audience. If you hear that and get told that often enough, then sooner or later it changes the way that you deal with people at work, so it just is trial and error, and going through the process often enough.

Table 7.3 Outcome space for learning mathematical communication skills

Conception	Control
(1) Trial and error	Reacting to situations, coping
(2) Mediated by others and outside situations	External influences (e.g. manager)
(3) Active, detached observation	By graduate themselves

Note that Paul also mentions trial and error as a technique, illustrating the hierarchy of these conceptions. A few graduates were able to utilise a more scientific approach to learning about mathematical communication, representing the third level of learning.

(3) Active, Detached Observation

These graduates used their mathematical deductive reasoning to observe then model their mathematical discourse based on the behaviour of those around them. Roger relates his experience as a pure mathematics graduate in the mining industry:

> *Roger*: Oh, well, the, from the people that, like, there were two kinds that stood out. Firstly, those that stood out in a positive way, and those that stood out in a negative way! So then the ones that stood out in a negative way, they were exactly the sorts of people you would expect to come out of the maths department almost, very dry and tetchy [technical], … and then on the other hand the interesting people had a different style, you know, they spoke about interesting things, … not just in more general human things but even in the context of their work, they'd always try to approach it from the more general perspective. … Yeah, it was quite a learning curve, I must say. So, because when I went there I didn't know, obviously it was the first time I'd worked in the so-called real world, which is not as real as people think.

This level 3 approach allowed graduates to be in control of their communication needs, as opposed to the external influences of level 2, and the simple reactive coping mechanisms of level 1. However, none of the graduates showed formal 'critical language awareness' (Jørgensen and Phillips 2002) which could have given them insight into the discursive practice in which they participate. No graduate had explicitly learnt principles of communication either in their studies or in their workplace, and thus they were unable to link action with theory to develop greater facility for finding creative solutions to problems of communication. The outcome space is shown in Table 7.3.

Discussion

There are several implications of the results that we have presented in this chapter. Firstly, in terms of communication, mathematics graduates do not seem to be adequately prepared for professional employment. Although their mathematical capabilities are good, generally more than required for their positions, their communication skills are not sufficiently developed for them to be able to use the strength of their mathematical knowledge in the workplace. Some of the comments from these graduates showed that their other 'soft skills' were not well developed:

> *Evan*: The point, in most cases, is to get a degree that allows you to go and start off a career and move up. Well, that career is only going to flourish if you have all the skills. Now, if you've only learnt half of the skills at university and the other half to do with, you know, communicating, whatever that … you learn on the job, aren't you going to be better off if you've got all the skills when you finish university? So I think it should be part of the degree.

Secondly, mathematical discourse is different to other forms of discourse, and graduates were able to distinguish between natural language and mathematical communication. They were aware that at times they were 'dumbing down' and 'hand waving' when talking to non-mathematicians. Sorting out these different forms of discourse proved difficult for new graduates when they began professional work, and sometimes communication problems prevented them securing appropriate employment using their mathematical qualifications. Some graduates were able to adapt and became good mathematical communicators – they also developed an ethical perspective towards communication. They were able to use their mathematical knowledge to communicate the risks associated with various decisions to people such as clients or managers who did not have that knowledge. Evan explains the problem and once again highlights the vital role of communication skills:

> *Evan*: The people that are in a position to make a decision on, to spend with so much money because they have a budget and, they're high up, … government, educational, corporate institute, … none of them are technical and often … the people that have the skills, they don't have the decision-making abilities. So, all of a sudden you'd be in a meeting where you have two people who are in a position of power, they are talking about a concept they don't really understand it yet they're the ones making the decisions on what to promise, how much money to spend and that always causes problems because once they've made the decision they then push the project down … and eventually it falls into the hands of the technical people who do the project and so often it's out of scope, or there's problems, because the people who made the decisions in the first place never really understood the technologies. But, that's the perfect example of those people getting to where they are because they have good business and communication and negotiation skills. Not necessarily technical skills at all.

Thirdly, graduates report that they were not taught mathematical discourse skills during their undergraduate courses, maybe with the exception of being asked to write a report or two in some statistics classes. Essentially, they have learnt whatever mathematical discourse capabilities they have in the workplace by trial and error, assistance from their managers or their own observation and deduction. Each of these approaches is less than satisfactory. Professionals in the mathematical sciences need to be aware of their language and communication choices to be able to make optimal use of their discipline knowledge.

It seems that mathematical discourse skills are critical to graduates' success in professional work, yet their undergraduate education has not delivered these skills. To what extent is it the role of university mathematicians to teach such discipline-specific discourse? It is clear that the lack of communication expertise affects our graduates' ability to use their mathematical capabilities in the workplace, and may even prevent them getting appropriate employment in the first place. This reflects badly on their studies and could discourage students from undertaking mathematics degrees. Thus, we believe that it *is* one of the roles of the discipline to prepare students for professional work, including capabilities such as communication in the mathematics curriculum. Even more importantly, we believe that this enhances the learning of the discipline content: talking and writing about mathematics is how mathematics is learnt and created, listening to and reading mathematics is how mathematics is disseminated. Graduates with good mathematical discourse skills will simply become better mathematicians.

From our viewpoint as mathematicians and mathematics educators, there is a further step beyond the conceptions of mathematical discourse to the *utilisation* of such discourse – communication that comprises a new contribution to the field, such as the development of a new notation that simplifies a problem, the demonstration of a proof of a previously-unproved result, or the development of computer algebra and geometry systems. Such acts of communication would represent the 'strength' conception of mathematical communication applied to the broadest 'life' conception of mathematics to developing new and original contributions to the field. None of our sample of recent graduates discussed this theme, although it is one that we should keep in mind in our teaching.

We conclude this chapter by considering the standards of mathematical discourse that we would ideally like graduates of the mathematical sciences to demonstrate when they leave university. On the basis of recent work with students towards the conclusion of their studies (see Wood et al. 2011), we have constructed a framework of standards of mathematical discourse capabilities for graduates. (see Table 7.4). Our framework is organised into conceptual, procedural and professional aspects, an idea adapted from Billett (2009). Conceptual knowledge is discipline-specific and/or skill-specific knowledge of concepts, facts and propositions (Glaser 1989); it can be summarised by the expression 'knowing that'. Procedural knowledge is discipline-specific knowledge that is specific to strategic procedures, (Anderson 1993), summarised as 'knowing how'. Professional knowledge consists of values, attitudes and capabilities related to professional practice (Perkins et al. 1993); it is represented by the phrase 'knowing for'. The context of professional knowledge includes capabilities needed by those who wish to proceed to research and further study opportunities. Several writers (such as Billett 2001) have called our professional dimension a dispositional, attitudinal or values dimension, but here we choose to use the term 'professional' to encapsulate the concept of 'becoming a professional' upon completing an undergraduate degree. Table 7.4 shows some representative outcomes for the 'pass' level, illustrating the differences between the three aspects – the complete framework is given in Appendix 2.

The outcomes are listed in qualitatively different levels aligned to the typical reporting of grades in Australian universities and can be adapted for other higher education systems. Indeed, there are strong connections between these standards and those from the European *Assessment of Higher Education Learning Outcomes*

Table 7.4 A framework for mathematical communication outcomes for graduates

	Aspect→		
↓Level	Conceptual	Procedural	Professional
High Distinction			
Distinction			
Credit			
Pass	Demonstrates the ability to describe and define the basic concepts of mathematical discourse	Demonstrates knowledge of basic communication strategies and can practice the rules of a given communication strategy	Demonstrates a basic understanding of the significance of mathematical communication in professional practice
Fail			

(AHELO) project (OECD 2011) and the Australian *Learning & Teaching Academic Standards* project (ALTC 2011) investigating 'threshold learning outcomes'. A student may achieve different levels for each of the aspects, so reporting would be most useful for each of the levels separately.

Summary and Looking Forward

In this chapter we have used information from a study with recent graduates to investigate the role of communication in their transition to professional work. We have pointed out the specific nature of mathematical discourse, encompassing all aspects of mathematical communication using reading, writing, listening or speaking, but also including other 'communicative events'. We first looked at graduates' ideas of mathematical communication, specifically with people who were not mathematically trained, and identified three conceptions: 'jargon and notation', 'concepts and thinking' and 'strength'. We also identified three conceptions of learning mathematical communication: 'trial and error', 'mediated by others and outside situations' and 'active, detached observation'. In each case, these conceptions are presented from narrowest and most limited to broadest and most inclusive, in the usual phenomenographic approach.

While the graduates had not learned (or at least were not aware of learning) mathematical communication during their undergraduate studies, we concluded our investigation by considering what standards of mathematical discourse we would like our graduating students to have achieved. The mathematical communications outcomes for graduates that we indicated in Table 7.4 (and Appendix 2) represent a first step towards explicitly including communication skills in a mathematics curriculum.

In the following chapter, we will examine the mathematics curriculum itself in an effort to determine how it can be arranged in a way that helps students to not only learn the techniques of mathematics, but also gain an appreciation of the role of mathematics in their future professional lives. Such a curriculum should incorporate a real acknowledgement of the different ways in which students view mathematics and its uses. In this way, we can help each student in the process of becoming a mathematician.

Note: Some of the material in this chapter was previously published in Wood, L.N. (2012). Practice and conceptions: Communicating mathematics in the workplace. Educational Studies in Mathematics. *79*(1), 109–125

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Chapter 8
What University Curriculum Best Helps Students to Become Mathematicians?

Introduction

In this chapter we draw on our research with mathematics students and recent graduates to investigate the tertiary mathematics curriculum. We present an argument for a particular vision of such a curriculum, which we refer to as a 'broad' as opposed to a 'narrow' curriculum. A narrow curriculum looks inwards and focuses primarily on the mathematics itself – the mathematical techniques that will be used by students of mathematics in specific situation. Traditional courses in mathematics have relied on such a curriculum for a long time. A broad curriculum looks outwards and focuses on the uses of mathematics as a way of investigating, understanding, and even changing the world. Such a curriculum positions students as citizens of the world first, future professionals in a variety of areas second, and mathematicians or users of mathematics third. Fewer mathematics courses are built on such an approach. Our arguments are based firmly on our students' and recent graduates' voices, as shown in transcripts of research interviews, and are consistent with ideas advocated by mathematics educators, in groups such as critical mathematics education and calculus reform, and with more general views of curriculum. We believe that the difference between a narrow and a broad curriculum is the difference between simply learning mathematics and actually becoming a mathematician.

Curriculum for tertiary mathematics has often been conceived pragmatically as a means of presenting specific mathematical content and techniques (computational, algebraic, graphical, logical) that will be used by students in specific situations. This limits the focus of course units and programs to particular 'essential' topics, and these quickly fill (and even over-fill) the available time. This approach to mathematics has resulted in many students developing lack of interest in or even distaste for the discipline, and hence avoiding it as much as possible. A curriculum that is outward looking, a broad curriculum, incorporates the use of mathematics as an approach to life, a way of thinking, acting and being. By making explicit connections with the world of the students, it brings a range of benefits in terms of relevance, engagement and enthusiasm. A broad curriculum is equally appropriate for mathematics major

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students and those students taking a mathematics course as a component of their studies. Mathematics major students will support a broad study of mathematics with subjects focused on specific aspects of the discipline and the profession of mathematician. Students taking a service course in mathematics have other subjects focused on their chosen profession, whether it be engineering, environmental sciences, psychology, information technology or health sciences.

Research into students' conceptions of mathematics and learning mathematics, and their perceptions of their future professional role, indicates clearly what they recognise as important for mathematics curriculum. Research into graduates' early professional experiences and their views of their previous mathematics education reinforces these ideas and grounds them in the reality of the professional workplace. Such research indicates that a curriculum acknowledging and incorporating students' and graduates' ideas can assist in increasing their effectiveness as future professionals – as mathematicians or expert users of mathematics. A curriculum that incorporates such views can include important aspects of professional formation such as communication and interpersonal skills, appreciation of a creative approach to problems, awareness of issues of sustainability and ethical considerations. A key aspect of our job as mathematics educators is to use this body of research to develop curricula, focusing on our students' experience of learning in their discipline area, and our recent graduates' experience of making use of that learning.

The Philosophy of Tertiary Mathematics Curriculum

The notion of curriculum can be understood in various ways. At its most expansive, a curriculum can be seen as an overall philosophy of pedagogical engagement, where learning approaches and practices are based on a particular view of the world. Curriculum theorists would maintain that all learning and teaching is situated within specific philosophical, cultural and social frameworks (Gable 2002) that provide the curriculum creators with a range of supporting and critical epistemologies. Broader curriculum perspectives such as justice and democratisation, feminism, or history and politics may be used to underpin the practice of curriculum development. Each of these perspectives leads towards the development of particular approaches to teaching and learning. Here are some specific examples in the field of mathematics.

In her interview-based study of university research mathematicians' ideas about mathematics, Burton (2004) conjectured that *"coming to know mathematics is a product of people and societies, that it is hetero- not homogeneous, that it is inter-dependent with feelings especially those attached to aesthetics, that it is intuitive and that it inter-connects in networks"* (p. 13). The mathematicians' views supported this conjecture, in opposition to accepted ideas about the objective, fixed and impersonal nature of mathematics. Burton found, rather, that mathematics is embedded in its socio-cultural and historical setting, is acknowledged for its beauty and elegance, is approached intuitively as well as formally, is studied using a variety of thinking styles, and above all is viewed as a personal and connected endeavour, emphasising collaborative learning and links with other areas of life. She also found evidence of significant discrimination against some groups of learners, most particularly women. She concluded that *"the mathematicians' experiences, as learners, are relevant to less sophisticated learners in schools, and in universities"* (p. 178), and used this position to argue for *"a pedagogical approach to mathematics that treats learners as researchers"* (p. 183). A curriculum that emphasised such human aspects of mathematics would help to remove current discrimination. Burton's study overlaps with the 'women in mathematics' movement that has been addressing the notion of gender equity in mathematics education; an international study edited by Hanna (2002) presents a wide range of views on the problem. The feminist movement has played a similar role in other areas of technology where a male-dominant culture has been related to specific choices of teaching and learning activities, texts and examples, languages and work expectations. In the field of engineering, for instance, deconstruction of these ideas has led to an increase in female participation, a change in teaching and work practices, and a change in the discourses surrounding the discipline (Cater-Steel and Cater 2010).

A different, though not incompatible, philosophical position informs the work of Barton (2008), who has advocated for many years the inclusion of 'language in mathematics' (see, for instance, Barton et al. 2004). From the fundamental insight that different languages express mathematical ideas – even basic ones such as number – in quite different ways, Barton concludes that the language in which mathematics is carried out has an influence on the mathematics itself, and in particular, the dominance of English leaves its imprint on the very structure of mathematics. This recognition that mathematics is not fixed and unchanging, or free of human aspects, leads to a particular view of curriculum that incorporates more language activities in mathematics teaching, as well as more teaching about the nature of mathematics itself. Barton's view is that we should include more abstract activity and undirected 'play' at all levels of mathematics in order to elucidate the relationships between mathematics and language and to experience the process of constructing mathematics. He concludes that *"a proper understanding of the link between language and mathematics may be the key to finally throwing off the shadow of imperialism and colonisation that continues to haunt education for indigenous groups in a modern world of international languages and global curricula"* (p. 9).

Keitel and Vithal (2008) have investigated mathematics education from the viewpoint of political power. In the chapter referred to here, they start from a statement made by mathematics educator Alan Bishop:

> … that the current mathematics education situation in most countries is non-democratic, that the curriculum is a mechanism of governmental control rather than educational enlightenment, that the commercial textbook 'business' community controls the materials available to teachers to an extent that teachers are slaves of the textbook rather than autonomous and enlightened users, and that assessment is still primarily and predominantly a mechanism for selecting the mathematical elite only. (Keitel and Vithal 2008, p. 167)

They point out that, although mathematics is traditionally seen as objective and impersonal, it is *“deeply implicated in the distribution and enactment of political power”* and in its gate-keeping role *“distributes access to high status and high paying professions and positions”* (p. 173). As a challenge to this view, they explore mathematics as a human activity from a historical perspective. They point out the important role of developments in information technology in broadening access to mathematics and its products, and the consequent rise in importance of ‘mathematical literacy’ as a means of allowing people to engage in the democratic process. Keitel and Vithal put forward a vision of a mathematics curriculum that places high importance on developing such mathematical literacy and using it to engage in political and social debate. Their writing continues an important thread of discussion focusing on the social justice implications of the mathematical sciences, including, for example, similar discussions of ‘statistical literacy’ (Lesser 2007). The implications for curriculum include a focus on the context of mathematics and a challenge to objective and impersonal views of the discipline.

The field of ‘critical mathematics education’ focuses on the role of mathematics in raising awareness of social problems and inequalities, and directing mathematics education as an active and progressive social force. Amongst many authors, the writings of D’Ambrosio have been particularly influential, particularly in his identification of ‘ethnomathematics’ (1985) and the importance of global ethical concerns in mathematics curriculum (1998). The tenor of his views is shown by this summary statement:

> But if one does not accept, very clearly and unequivocally, that her/his professional commitments are subordinated to a global ethics such as the proposed ethics of diversity, it will be difficult engage in a deeper reflection of her/his role as mathematics educator. I see my role as an educator and my discipline, mathematics, as complementary instruments to fulfil these commitments. (D’Ambrosio 1998, p. 71)

Skovsmose (1994, 2009) has consistently argued for such a critical position and described elements of a critical mathematics pedagogy. In his view (Skovsmose 2009), modern mathematics education reflects the underlying abstract assumptions of progress (the notion that society moves forward by the application of rationality), neutrality (the idea that mathematics and science are value neutral) and epistemic transparency (the view that mathematics is a deductive system that proceeds from axioms). All three of these assumptions can be challenged to move towards a ‘critical professionalism’. For education, as both a practice and a field of research, to be critical it must incorporate discussion of basic conditions for obtaining knowledge, and it must explicitly acknowledge and address societal problems. Skovsmose finds various types of curriculum more appropriate to such a view of mathematics. He argues for problem-based learning as one of the key approaches, privileging as it does the whole and loosely-defined problem, including its human and ethical aspects, and the group processes required to address it, over the formal and objective view of mathematics itself.

The Practice of Curriculum in Tertiary Mathematics

By contrast with the philosophical positions described in the previous section, the practice of tertiary mathematics curriculum is most often based on pragmatic ideas, viewed as a series of discrete units making up a program of study. Such an approach is commonly reinforced by university accreditation requirements, and results in an approach focusing on the detailed mathematical content of these units of study and their sequencing. The reality seems to be that most tertiary mathematics classes are content focused and carried out with traditional pedagogy (Burton and Haines 1997), accurately described by Thomas and Holton (2003, p. 351): *"For many years now the majority of teachers and lecturers have been presenting the subject as if it was just a set of rules that needed to be learnt."* The consequence of this pragmatic approach is that the curriculum documentation and teaching activities come to represent a form of disciplinary knowledge and reinforce specific attitudes in the learners and teachers.

At times, influential groups in the mathematics education community get together to produce philosophically-coherent statements on tertiary curriculum. In the United States, over a decade of discussion of 'calculus reform' culminated in a comprehensive *Curriculum Guide* (Mathematical Association of America 2004) listing the possibilities for curriculum change in the mathematical sciences, both for students majoring in an area of mathematics and for the far larger group of students taking some service mathematics. The calculus reform movement grew up as a reaction to mathematics courses that were predominantly content-based, focused on the acquisition of skills and techniques, reliant on the authority of a textbook and free of significant real-life applications. The *Curriculum Guide* recommends that departments and lecturers should: get to know their students and the world they live in; develop their mathematical thinking and communication skills; show them the breadth of the mathematical sciences and its interconnection with other disciplines; and utilise technology as a natural component of teaching and learning. More detail and background is given by Ganter and Barker (2004). Although statistics is included in the *Curriculum Guide*, the specific situation with 'statistics reform' is summarised by Garfield et al. (2002) and the recommendations of the *Guidelines for Assessment and Instruction in Statistics Education* (American Statistical Association 2010). Taken as a whole, these developments give a significant impetus to mathematics and statistics departments to move towards broader curricula.

The last recommendation of the *Curriculum Guide*, to utilise technology in teaching and learning, is an important one, both practically and theoretically. On the practical side, the developments in digital technologies even since 2004 have been enormous, and for the most part they are being incorporated into pedagogy in large numbers of tertiary mathematics courses. Indeed, in affluent countries, it seems impossible not to utilise such technologies. On the theoretical side, however, there is continued discussion about the effects of such technological changes on the mathematics curriculum, with many voices pointing to the benefits of such changes (see, for example, the reports in Hoyles and Lagrange 2010), while others resist the use of graphical calculators, computer algebra systems, online learning or the use of 'tweets' – whatever is the current latest development. Artigue (2010) discusses the historical development of graphical technology, pointing out how the ability to show graphically the solutions of differential equations led to a new curricular balance between qualitative, algebraic and numerical methods of solution. She points out the problems of introducing digital technologies into the curriculum without consideration of the resultant changes in our ideas of mathematics itself. We should be aware that technology affects what is learned and the way in which it is learned, a key aspect of the current developments in collaborative and social aspects of learning.

A feature of curriculum for tertiary mathematics is that it usually focuses on the ways that lecturers and discipline experts understand teaching and learning, and on their views of the nature of mathematics itself. These include, for example, ideas about the precision and rigour of mathematics, its cumulative nature and the importance of component mathematical skills. Such curriculum may place particular importance on specific mathematical topics that are related to teachers' own research interests. Lecturers of mathematics believe that they know what is critical for students to learn from their own experience of being learners and mathematicians, though we have seen in Burton's (2004) study that they may actually miss the fact that they are doing their own mathematics in quite a different way from their teaching. Curriculum in mathematics, as in many areas, is usually developed primarily from this knowledge base of the teachers, possibly incorporating the strategic requirements of the university and the demands of relevant industries (Bowden and Marton 1998; Toohey 1999).

Exploring how students report understanding mathematics, and then using this as evidence for curriculum development is a less common approach. Gable (2002) notes that the voices of students are often missing, and the ways they understand the discipline are often assumed, despite the fact that the curriculum is prepared for their learning. Further, there are inevitable differences between the curriculum that is intended from the point of view of course designers or lecturers and that actually experienced by the learners. Students can and do set their own agendas for their learning, often depending on what they think will be useful for their future professional careers (Reid et al. 2011), and the university curriculum is only one aspect of their learning about mathematics. In the early twenty-first century, learners are able to utilise contemporary technology such as the internet to investigate topics in which they have a particular interest, irrespective of their appearance or not on a university curriculum. The voices of current and recent students, including their experience, their expectations for learning and their preparation for their future, can provoke us to rethink our approach to curriculum. Within the context of mathematics, a broad curriculum such as we are proposing might be one that acknowledges and makes full use of these perspectives.

The Message from Undergraduate Learners of Mathematics

We have described in earlier chapters our interviews with undergraduate students in which we asked them about their ideas about mathematics and their learning of mathematics. In the course of these interviews, students also talked about their perceptions of their future use of mathematics. Using the information from the open-ended survey completed by over 1000 students, we were able to refine these conceptions of mathematics and obtain further information about students' ideas about how they would use the mathematics that they were learning. Here we summarise the implications of these results for a broad mathematics curriculum. Rather than suggesting a completely new approach, the conclusions that we reach allow us to support a range of ideas about curriculum, ideas that have been part of many previous discussions. We did not specifically ask our respondents about mathematics curriculum, but rather asked them about their experiences in participating in the learning of mathematics. Yet from our studies we obtain a justification of these approaches taken from the point of view of the learners themselves, and as we have pointed out this is not a common approach in the development of mathematics curriculum.

The results from our study of students' conceptions are quite clear. In terms of mathematics, there are the narrowest conceptions as 'number' or 'components', intermediate conceptions as 'modelling', both applied and abstract, and the broadest 'life' conception. In terms of learning mathematics, the narrowest conception is in terms of 'techniques', an intermediate conception focuses on the 'subject' of mathematics itself, and the broadest conception is oriented towards 'life'. In each case, the 'life' conceptions are the most holistic and inclusive, and from our point of view as educators, the most desirable, not the least since their hierarchical nature implies that they include the more limited conceptions. Students who are aware of the 'life' conceptions of mathematics and learning mathematics are also able to understand, appreciate and utilise the narrower conceptions when they need to. So the first major conclusion from our results is that we should aim at a mathematics curriculum that highlights the aspects of the broadest 'life' conceptions, making explicit connections between learners' personal situations, viewpoints and interests. This conclusion is quite consistent with many of the philosophical and practical approaches to curriculum that we described earlier in this chapter, for instance, the recommendations from the US *Curriculum Guide* to get to know our students and the world in which they live.

We have frequently heard the view that a course in mathematical sciences should focus at its earlier stages on the techniques of mathematics (or statistics, or any other branch of the mathematical sciences). The view is that students in first-year courses should be studying only mathematical techniques, an extension of the 'basic skills' and 'drill-and-practice' exercises that were commonly presented to them at the school level. In later-year classes, students can be introduced to the modelling aspects of mathematics, and towards the end of their studies to the unifying ideas and principles of the subject and its ways of thinking. On the basis of our research results, we would fundamentally disagree with this view. The presentation of the subject at the narrowest level at the beginning of mathematical studies results in those students who see mathematics and mathematical learning in the narrowest way having their views reinforced by the pedagogy with which they are faced. Later, they are less likely to be able to move towards broader conceptions. On the other hand, those students who start university studies with broader conceptions of mathematics and learning will be encouraged to drop these broader views, and to see their formal study of mathematics from the narrowest point of view (though they may retain a broader view of mathematics outside the institutional context). They will find this unsatisfying, and may even conclude that a university mathematics course is not for them. The contrast is shown clearly in this pair of quotes from interviews with two students (from an earlier study reported in Reid and Petocz 2002). Chris is encouraged towards broader conceptions, while Danny is discouraged from them by the pedagogical approach:

> *Chris*: Our lecturer really made it as if, treated us as if we were working, like, giving us real life problems. Because our lecturer is a consultant, and he gives us problems that he gets from clients, to us as assignments, so that helps./…/Lecturers create [assignments] in such a way that you have to understand your work before you answer, because most of the assignments that we're given are actually real situations, they're not just made up scenarios, so that's why you have to understand your work first.

> *Danny*: [Why would you want to rote learn things?] People do, and they do really well. [Well why is that?] Because if you are doing a lot of maths stuff and you have to reproduce proofs they just learn it all and write it all out. They don't know how to do anything with it, but they can write it all out. [And you think that is superior to your attempt to understand the stuff?] No, no. It's not superior; I would rather understand it, but you can get better marks for rote learning.

The 'life' conceptions should be our guide to an appropriate curriculum for learners that engages them from their very first mathematics classes. This suggests that we should include those aspects of mathematics that encourage students to make the link between the subject they are studying and their own lives and the world around them. We can present mathematics as a human endeavour by including elements of its history and the people who made that history. We can help students investigate the different ways that mathematics can be used in various real situations of interest to them, and other real situations with which they may have no experience. We can encourage them to develop mathematical ways of thinking and approaching problems, and highlight the importance of mathematical communication. We should explicitly discuss with them the process of becoming a mathematician, and the world view implied by being a mathematician. This does not mean that we must avoid the theoretical aspects – we should complement the applied models that we utilise with abstract models, and encourage students to appreciate the beauty of mathematics. Nor does it mean that we must avoid the techniques of mathematics; but we should aim to situate such techniques in a context that makes sense to our students.

We need to keep in mind the interview comments of those students who showed the broadest ways of thinking about mathematics and learning mathematics. For instance, as part of longer quotes that we have presented earlier, Yumi noted that *"I just see the world in a different way to people who don't study mathematics"*, Dave commented that *"I think it gives me a sense of clarity about things"* and Julia stated that *"it's mind dazzling how much maths accounts for."* Another student expressed the connection between mathematics and life in this way:

> *Hsu-Ming*: Maybe, maybe I want to have an understanding of the world around me and I believe that with mathematics, with the principles of mathematics I feel as though I will be more able to, I guess yeah, more able to do that, to understand, with yeah, with the basic concepts, just that that line of thought will be able to help me understand the world around me better.

Some of our students will agree with these comments and maybe express similar ones. For them, we require a broad curriculum that allows them to continue to develop an understanding of the world through mathematics. Other students will hold narrower views about the nature of mathematics and their learning in the discipline. For these students, a broad curriculum may come as a surprise but it will enable them to establish links between their mathematical studies and their personal and professional lives. Our evidence shows that every group of learners at every year level is likely to include students with the narrowest views of mathematics and learning, and also those with the broadest views.

Nor should we forget that in the absence of explicit discussion, many learners will believe that all their colleagues think about mathematics in the same way that they do. An important part of a broad mathematics curriculum is to challenge students to articulate their ideas about the discipline and their learning, and then to compare them to the ideas of their fellow students, maybe in class discussions or in an assessment activity carried out in groups. Learning activities and assessment tasks, particularly those undertaken in conjunction with their peers, can encourage students with more limiting views to expand their idea by reflecting on and even assimilating different ways of thinking. Such an approach can help students broaden their conceptions of mathematics and take their first steps towards becoming a mathematician.

Broadening the View of Professional Skills and Dispositions

In our original interview studies, one of the questions asked students about their expectations of using mathematics in their professional life: *What do you think it will be like to work as a qualified mathematician?* Their responses gave us information about students' perceptions of their future use of mathematics, and allowed us to investigate its relationship to their ideas about mathematics and learning. Our open-ended survey asked students: *What part do you think mathematics will play in your future studies? … in your future career?* These questions added information about future use of mathematics from a much larger sample group. Not surprisingly, tertiary students look beyond their classes and curriculum towards their future professional life. Indeed, we found that their perceptions of their future professional use of mathematics are linked with their ideas of mathematics itself, and the ways that they approach learning mathematics. This is an instance of a relationship that we have investigated in a variety of disciplines, including music, law, design and statistics, and referred to as the 'professional entity' (Reid et al. 2011).

The professional entity is a unifying way of thinking about students' understanding of professional work, organised in a hierarchy of three levels. At the narrowest level, students describe a perception that professional work in mathematics comprises technical components utilised as needed in the work situation. It corresponds to a fragmented view of mathematics as 'number' or 'components', and an approach to learning that consists of gathering mathematical or statistical techniques for use in different situations. At an intermediate level, students hold that professional work in mathematics is about understanding the important characteristics of the professional discipline itself – finding meaning in numbers, equations or datasets, and using them to build models of aspects of reality. It corresponds to a 'modelling' view of mathematics and a view of learning that focuses on the subject itself. At the broadest level, students perceive that their professional work is related to their own personal and professional lives, creating and modifying views of the world based on mathematical and statistical evidence. This is achieved using the 'life' conceptions of mathematics and learning mathematics, focusing on the place of mathematics in personal and professional life, and the development of mathematical ways of thinking and interacting with the world.

So each of the levels of thinking about professional work in mathematics corresponds with a specific approach to mathematics itself, and to learning mathematics. Importantly, encouraging students to broaden their conception of their discipline and profession seems to give them access to a broader range of approaches to learning (see Reid 2000, in music, and Reid and Petocz 2002, in statistics). The implications for curriculum are immediate; we should utilise pedagogical approaches that help students to develop their appreciation of professional uses of mathematics, and their conceptions of mathematics and approaches to learning mathematics will be correspondingly broadened.

In the context of the education of future professionals in a diverse range of areas, mathematics could be viewed as a higher-level numeracy, one of a range of key professional skills, including communication and interpersonal skills, and dispositions such as creativity, sustainability and ethics – and of course these are equally important for professionals in the mathematical sciences. When students graduate, their employers will expect them to be able to communicate clearly and work effectively in teams, to demonstrate creativity in finding and solving problems, to be able to contribute to their company's sustainable development processes and to have an awareness of the ethical dimensions of their professional work. Knowing that students express a range of different ideas about their future work, and that this is related to their learning whilst at university, gives us the opportunity to set up learning situations where students who habitually express the narrower, more technical conceptions may be provoked to move beyond those ideas towards broader and more inclusive views. Mathematics curriculum has the potential to gain synergies from such components of professional formation. For instance, the explicit inclusion of professional aspects increases the relevance and interest of the course for students and provides situations where they may start to consider that those skills and dispositions are important for their future. By contrast, a curriculum that focuses exclusively on mathematical techniques may constrain students' development of a range of professional attributes and ignore an opportunity to make the course more relevant to them.

Mathematics students may have little idea that creativity, sustainability or ethics (or even communication and team work) will play a part in their professional lives or their further mathematical studies, and they may have difficulty interpreting these important notions. Many students have only a vague idea of the role of creativity in mathematics, and indeed some of them will happily tell us that they are 'not creative' (Petocz et al. 2009). Few learners believe that they can create mathematics themselves, and the notion is not often discussed in university mathematics classes. Although academic mathematicians are well aware of the role of creativity in mathematics (Burton 2004), they may not address it explicitly in their teaching, beyond linking it with notions of problem solving (Reid and Petocz 2004). The United Nations *Decade of Education for Sustainable Development* (2005–2014) aims to promote *"development that is environmentally sound, socially equitable, culturally sensitive and economically just"* by *"rethink[ing] educational programmes and systems … that currently support unsustainable societies"* (UNESCO 2011). However, students and academics alike hold a wide range of ideas about the meaning of these terms and there is little evidence of the inclusion of notions of sustainability in mathematics education (Petocz and Reid 2003; Reid and Petocz 2006; Reid et al. 2009). Ethical aspects are an essential component of any professional activity, and mathematical sciences are no exception. Students show a range of conceptions of ethics, with many of them holding only naive views about 'right and wrong' (Reid et al. 2011). However, ethics in mathematics or statistics is often neglected as students and their lecturers focus on the technical content of a course.

It seems that from the point of view of many lecturers in mathematics such aspects of professional formation are something that can or should be picked up elsewhere. Even if a lecturer is willing to discuss the issues, their students may perceive them to be merely an 'add-on' to their syllabus unless they develop deeper, more realistic notions about their future professional lives (de la Harpe and Radloff 2000). Although these aspects of professional dispositions may seem to be peripheral to mathematical studies, they are an important component of a broad curriculum that addresses the human dimension of mathematics and links the subject clearly with aspects of the learners' world – and helps students to develop a critical view of that world and the possibilities of changing aspects of it. It would seem that a mathematics course that acknowledges and specifically includes students' views of their future professional lives as part of the core content could benefit students through an increased possibility of engagement with their study of mathematics.

The Message from Mathematics Graduates

In the two previous chapters we have described some aspects of our interviews with recent mathematics graduates. In the studies that we carried out, we asked questions concerning respondents' views of their university study of mathematics and how it had prepared them for their professional work; questions such as: *What did you find useful from your learning of mathematics at university, that you use now? In what ways has studying mathematics at university level prepared you for work? What do you think should be in a university course to help you make the transition to professional work?* In this section, we draw together recent graduates' comments (and sometimes robust criticisms) about their undergraduate mathematics education and their suggestions for changes to the curriculum on the basis of their recent work experiences.

Few of the graduates believed that they used all the technical aspects of mathematics that they had learned at university in their current workplace. Of the 18 graduates in the second study (carried out by Leigh Wood), only four were using the third-year level mathematics that they had studied. In general, graduates appreciated the extra confidence that this gave them, and some of them were able to use this to learn further mathematics that they needed. Some others were frustrated about having more extensive technical knowledge that they were not using at work, particularly if their bosses were unaware of their mathematical potential. There is a role here for professional mathematical societies to educate future employers about the benefits of employing mathematics graduates. On the other hand, graduates often commented that the computing aspects of their studies were inadequate, particularly in relation to 'standard' tools such as Excel, Visual Basic and statistical packages like SAS. James' comment is a typical example:

> *James* (mathematics and finance graduate, working in the banking section of a large insurance company): As far as transition for work, everywhere uses standard products like Excel, and if you come out of a maths degree, I wasn't really taught to use Excel all that much here [at university] and I think it's really a tool of the trade.

A university degree will never be able to cover all the technical content required for the range of workplaces of all future graduates. Work requirements vary, the tools used to carry out mathematical processes change, and mathematics itself may develop and change. Yet a common viewpoint from our graduates was that with their focus on high-level mathematics they had missed out on learning to use some of the standard yet very useful tools such as Excel. The fact that graduates had to learn to use such tools, and in some cases some new technical areas of mathematics, points out the benefits of incorporating learning that is self-directed without formal instruction into the undergraduate mathematics curriculum as a way of preparing students for such situations.

Another interesting feature is that although most graduates were using mathematics on a daily basis, only a few of them seemed to believe that they themselves could push the boundaries and create new mathematics. Kath and Matt are exceptions, while a more common view is expressed by Roger:

> *Kath* (mathematics and finance graduate, working in a modelling unit in the public service): I'm doing a bit of research at the moment with another guy at work and we are kind of doing something that's quite new to the type of work that we were doing beforehand, so that's quite interesting for me.

Matt (statistics graduate, working as a biostatistician): The other part is stats research. A lot of this is based on data to start with too. We get data that can't really be answered well with existing methods, and so I get to think up new ways of analysing stuff, new stats/maths methods.

Roger (pure mathematics graduate, working in research and development in geological exploration): … you could perhaps even invent your *own* one [algorithm].

Roger even seems a little doubtful whether he could do this, but he is one of the few respondents to mention creating new mathematics. Aside from the few graduates who were involved with doctoral studies, most others did not even mention it as a possibility. This suggests that an undergraduate curriculum would benefit from including situations where students were explicitly asked to create new mathematics.

Previously (in Chap. 7) we have investigated graduates' mathematical discourse, and the various difficulties that they faced when communicating with non-mathematicians. For the most part, graduates had not been taught about mathematical communication in any formal way, though there were some learning situations that included practice in such skills:

David (mathematics and finance graduate, working in a bank treasury): And lab is vital, I think, having the labs is absolutely vital in statistics. I certainly think that one of the more valuable learning experiences I did have was labs in statistics and computing.

The type of laboratory classes to which David is referring ask students to carry out practical investigations using a computer and then to explain the meaning of various results that they have obtained, sometimes to specific people such as a colleague who has not studied statistics, or a manager who requires an 'executive summary' (see examples in Prvan et al. 2002). Many graduates said that they had appreciated such opportunities; it would be only a small change to make the communication aspects more explicit.

Related to the communication theme were requests from many graduates for a more interactive and 'human' learning environment. Group tasks were strongly advocated as an important way of preparing for the workplace:

Nathan (mathematics/IT graduate, working freelance as a web programmer): Maybe do more presentations or group assignments, something that's going to give them more confidence and be able to step up to explaining in a group./…/I mean most of them would be so scared to do that but I think it would do them a world of good if you had that as part of the study.

Leo (mathematics and finance graduate, working in statistical forecasting): Well, it's the whole working in groups and stuff like that, that's been, that's really useful because no matter what workforce you sort of go into, there's always groups, you know, team, it's all about team effort and stuff so, you know, you've got to be able to interact with other people, do things as a group. So it was good how we did lots of projects that involved group work, you know, distributing the work evenly between everyone, making sure everyone was pulling their own weight and that kind of stuff, so that was good.

Some graduates recommended including opportunities for peer teaching as way of enhancing their learning, an approach that is consistent with many studies on collaborative learning (D'Souza and Wood 2003). Thi and Matt highlight the importance of such activity and comment on the benefits:

Thi (engineering and mathematics graduate, running her own finance business): I think we should be able to know how to communicate our knowledge to others so that they can comprehend it in a way that's easy and I don't think we know how to do that yet.

Matt: Explaining to others makes you think about things in more than one way. Not everyone learns something the same way, and so you need to find another way of explaining the thing to them. You also have to really know the basics, what you're trying to explain, in order to clearly explain it to someone else.

Some graduates made a plea for more interaction, to counter isolation in mathematics classes, rather than launching straight into the content-driven curriculum:

Christine (mathematics graduate, then police academy, working as a police officer): The classes aren't big, you've got maybe 20 students in a class, but I couldn't have told you a single name of someone in some of those classes. I've just, there was no interaction with other students, for a lot of the time,/…/so perhaps if you start up with a bit of the touchy-feely stuff, and just getting to know everyone, and I don't know if other people do that sort of thing, but that perhaps would have been a bit nicer. Just to make, encourage more discussion in the class,'cos I think there's so much to get through in the course that it's just, day 1, right, this is my name, this is my contact details and all that sort of thing, right let's begin. And then it's just scrambling through it for the rest of the semester.

However, this was not everyone's experience:

Josef (mathematics graduate working in business risk management): It was good at [the university] because it's quite a small department and there was only a few of us doing maths and so we were all quite close and we'd all help each other out.

And one graduate put forward a novel idea for overcoming the problem (and maybe the person she refers to could actually be the lecturer):

Melanie (mathematics graduate and Ph.D., then diploma in music, working as a jazz violinist): This is a left field idea as well, but like if the very first lecture of any course was given by somebody who wasn't actually going to be the lecturer of that course, but just somebody who had good communication skills and could actually put the whole course in context.

Melanie's comment points out another area which graduates identified as needing improvement – the lack of overall coherence of their mathematical studies and a failure to link areas of mathematical knowledge with each other. Many graduates perceived the various mathematical subjects they studied as unrelated and could not see how they fitted into the overall structure of mathematics. Gavin's quote implies that the problem may be wider than in mathematics, while Boris describes how it can take a long time to make such connections unaided:

Gavin (applied mathematics graduate, undertaking a Ph.D. in climate modelling): That's the problem with most university courses is that you're not introduced to the philosophy of the course, you know, you're not introduced to the motivation of the course, you just go straight onto the content.

> *Boris* (engineering and mathematics graduate working as a cryptographer in security research): In my honours year, there were honours subjects and in the beginning you get the idea that they are starting to deal with each other, and at the end of them it's actually have idea why they have something to do with each other, I think that in the middle of the next semester you actually realise why they have actually something to do with each other.

As well as coherence within mathematics, graduates felt that connections should be made to real-world situations where such mathematics was utilised:

> *Nathan*: I think that you could probably, if you had a lot more exposure to the 'real world' as part of your learning process, I think you can't go wrong.

> *William* (actuarial science graduate, then mathematics teacher training, designing educational software): A graduated approach, where you might start by learning some theory, then be working a bit with, say, a lecturer in a mock team situation on a realistic project, then having industry experts come and work with you to maybe do a real project.

William suggests a gradual incorporation of real context in mathematics learning, while other respondents proposed structural changes to the courses:

> *Heloise* (operations research graduate, working as a logistics analyst): Maybe an option, to do a six-month formal work experience, but if you didn't want to do a full six months of it, maybe a few weeks/…/I just found it [university study] very theoretical.

> *Paul* (mathematics and finance graduate, working in finance): Probably more putting it in a professional context/…/so making something more work relevant would be useful, so maybe a report writing course or something like that, but for business rather than for academics.

> *James*: Maybe like a transition to employment, like a third-year subject, whether it's a subject or something else.

James' suggestion of a 'transition to employment' subject towards the end of a degree course seems very sensible, though advice about the professional career aspects of mathematical studies needs to be accurate and up-to-date, as Paul points out:

> *Paul*: To be honest, in first year we were told that this course is designed to send everyone into dealing rooms and that stuff. What I have found is that maybe two or three people in each year will get a job in a dealing room, so managing the expectations of everyone in the course could be done more effectively. Probably by not over-selling things early. Trying to be more realistic early. Like, it's good to get everyone excited about what they are studying, but not so excited that they expect that when they graduate they are going to be this, because a lot of people will end up disappointed or frustrated that they don't get to do that.

In many respects, the views of the graduates are consistent with those of the undergraduate students, with extra insights obtained from their workplace experience. Overall, the message from recent graduates is to modify the undergraduate mathematics curriculum to include more computing tools, maybe at the expense of some technical mathematical content, and to spend more time showing the overall structure of the mathematical sciences, the way that individual topics fitted in to a global view, and the ways they could be used in real life. We should give students explicit opportunities to create mathematics, utilise interactive and communicative approaches to learning in a more social and human context, and make clear connections to students' future professional working lives. This view confirms many aspects of the broad curriculum that we are proposing.

Discussion

The previous sections have highlighted various aspects of our research with students and recent graduates to draw out some features of a broad curriculum for mathematical sciences. There are some general principles. Students sitting in the same mathematics class can have very different ideas about the nature of mathematics and mathematics learning, and these ideas contribute to the approach that they take towards their studies. Those with the broadest ‘life’ conceptions view mathematics in a holistic way as an approach to life and a way of thinking, and learning mathematics as a way of making sense of, and maybe even changing, the world. Learners with the ‘life’ conceptions make a strong personal connection between mathematics and their lives. Further, students at tertiary level, both mathematics majors and those studying mathematics as a component of other disciplines, have a range of perceptions about their future work, and these perceptions provide a focus for their learning whilst at university. All mathematics students would benefit from a curriculum that could enable them to take maximum advantage of the time that they spend learning at university in preparation for work. Overall, our approach towards a broad curriculum is based on the characteristics of the ‘life’ conceptions, which are educationally most desirable, not least because they include the elements of the narrower conceptions. In order to show how these principles can be enacted, we can consider the process of curriculum development and review from several perspectives – the institution, program and unit level, the students’ professional preparation and learning needs, and the learning and teaching strategies used in the classes.

From an institutional viewpoint, most universities have entrenched traditions surrounding mathematics education. Typically, curriculum changes in response to external developments in the nature of professional work, characteristics of the students and their teachers, and government funding and quality assurance initiatives. Yet, these changes may be very slow; formal course reviews may be undertaken once or twice a decade and may focus more on bureaucratic rather than pedagogic requirements. For instance, the recent interest in mathematics courses in areas such as mathematical finance or climate change has resulted in new units of study, focus for doctorates and appointment of staff with research interests in the area. But the relationship between the content and activities of such units may not be integrated with other units that students have available for study. Curriculum is also affected by internal relationships between departments involved in the service mathematics and statistics components of courses such as engineering, business and psychology. Such relationships are often driven by financial rather than pedagogical concerns, and the functional concerns of the serviced department, which often direct the curriculum towards narrower rather than broader views of mathematics (they may believe, for instance, that a small number of lectures should be adequate to teach engineering or psychology students to carry out triple integrals or statistical inference, respectively).

When individual academics take possession of a specific unit, they may focus on their choice of the important content of the unit, neglecting how the unit fits in with other units and the overall aims of the program of professional study (Watts 2000). The content of subjects may be revised frequently, but the learning and teaching methods, styles of assessment tasks and broader generic skills are less frequently considered, and usually left to the discretion of the individual lecturer. Yet the development of curriculum at the individual unit level can benefit from the overall course aims, particularly in terms of the professional aspects that are important for that course. Skills of mathematical discourse are an obvious example, and one that should be integrated with the mathematical content of a sequence of units. As another example, a professional capability such as creativity can be introduced, developed and expanded (rather than simply repeating an introductory level of engagement, see Fallows and Steven 2000, pp. 17–31). In a first-year unit, it may be appropriate to examine the mathematically creative thinking of others using examples from a range of professional perspectives, and to discuss the meaning of creativity in mathematics. In a second course, students can be encouraged to try multiple forms of analysis and to discuss the appropriateness of different models for different outcomes. In a final-year course, creativity may be demonstrated by students' capacity to first find and refine a problem, then develop appropriate mathematical models and finally communicate the results of the analyses to a third party. Other professional dispositions such as ethics and sustainability can be incorporated in a similar way under a broad view of mathematics curriculum.

An important component of such an approach is the development of learning materials that engage students at a broader level with such an expanding notion of relevant professional skills in the mathematical sciences. Two examples explicitly incorporating the results of our research are *Advanced Mathematical Discourse* (Wood and Perrett 1997), a textbook for a first-year course in Mathematical Practice – thinking, communicating and working mathematically (more details are given in Petocz and Reid 2008), and *Reading Statistics* (Wood and Petocz 2003), a book which asks students to read and engage with research articles in a variety of areas of application, and to communicate the statistical meaning in a range of professional situations. Another example in a different format is the video *Terror, Tragedy and Vibrations* (Wood et al. 2000), which investigates the uses of mathematics in the context of the work of an engineer in various professional contexts. The video points out that these components of professional formation exist, can be learned, and are integrally tied to the work of a mathematical scientist. Of course, there are many more materials of this type prepared by others. The imaginary conversation with Florence Nightingale about the use of statistics in evidence-based medicine (Maindonald and Richardson 2004) would be an ideal resource for a medical statistics class. The *Gapminder* website (www.gapminder.org) contains outstanding interactive graphics (*Gapminder World*) for investigating the relationship between variables such as countries' income and life expectancy, and in *The Joy of Stats* video, Hans Rosling demonstrates a range of professional and social uses of statistics.

Another important component of a broad curriculum is the pedagogical approach needed to engage students with their mathematical studies. A more social and collaborative approach than is common in mathematics classes, utilising interactions between students as well as between students and lecturers, is an essential part of the approach. Learners will make connections between their mathematics studies and their own personal life, and future professional lives, when they work mathematically on a range of relevant problems. Problem and discovery-based learning methods, carried out in groups, will be much more effective in this regard than traditional lecturing on technical aspects of mathematics. Students can be given opportunities to prepare and make presentations, to take part in discussions, debates and conferences, and even to research and prepare learning materials for their peers. Using such pedagogy, and corresponding learning materials, we can set up learning situations that afford scope for students to become aware of a broader range of issues related to their studies as they move on the path towards becoming a mathematician.

Summary and Looking Forward

In this chapter we have presented an argument for a 'broad' curriculum for the study of mathematical sciences. A broad curriculum looks beyond the discipline of mathematics itself, beyond its techniques and components, even beyond the mathematical models of real and abstract situations, to focus on the role that mathematics plays in the personal and professional lives of the students who are learning mathematics, helping to describe, explain and even change the world they live in. By presenting our subject to our students, whether they are mathematics majors or those studying mathematics as a component of another discipline, in this holistic way we encourage them towards the broadest forms of learning and enhance their preparation for their future work with the inclusion of professional skills such as communication and teamwork, and professional dispositions such as creativity, ethics and sustainability. A broad curriculum helps learners to become mathematicians, as well as learning about mathematics.

The notion of a broad curriculum is based firmly on learners' views about the nature of mathematics and learning, and their perceptions of their future professional lives. We have integrated information from our interviews and surveys carried out with undergraduate students, and interviews with recent graduates, participants who are in an excellent position to reflect on their recent mathematical courses and the benefits and shortcomings that these studies had for their current professional roles. The views of learners and recent learners comprise an essential input into the design of a curriculum that allows students to develop their full potential as professional mathematicians or users of mathematics.

We do not claim that a broad curriculum is a new idea, though we believe that our theoretical justification for it is novel. Much of what we are recommending has appeared in previous discussion of curriculum and pedagogical approaches to mathematics, though it seems that such discussion has changed only occasionally the standard way of *"presenting the subject as if it was just a set of rules that need to be learned"*, as Thomas and Holton (2003, p. 351) put it. In order to have a greater effect on mathematics pedagogy, mathematics educators need to critically consider their role in guiding students and helping them in their learning. In the following chapter, we reflect on the variety of ways in which this could be achieved.

Note: Some of the material in this chapter was previously published in Petocz, P., & Reid, A. (2005). Re-thinking the tertiary mathematics curriculum. *Cambridge Journal of Education*, *35*(1), 89–106.

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Chapter 9
How Can Professional Development Contribute to University Mathematics Teaching?

Introduction

In this penultimate chapter we examine the role of mathematics academics in the teaching and learning of university mathematics courses. We consider various ways in which university mathematics educators could enhance their professional work by guiding students towards not only learning mathematics, but also becoming mathematicians. This aim arises from a view that our professionalism as educators can always be improved by reflection on our current practice, rather than from a view that we are not doing our job properly at present. Yet there is a need for such reflection. In mathematics, as in many university disciplines, lecturers are appointed on the basis of their discipline expertise in mathematics itself, and their pedagogical skills are assumed to follow from this. However, their individual teaching approaches may be based on their previous experience as (often atypical) students and the approaches used by their own lecturers. Burton's (2004) study with university mathematicians showed that they were well aware of the human aspects of mathematics in their own research work, but fell back upon traditional approaches when they were teaching. Our students and graduates have certainly made many comments about the problems caused by such an approach.

Professional development is the term that is usually used to refer to a range of activities such as study, travel, research, workshops, courses, sabbaticals, internships, apprenticeships, residencies or work with a mentor. Although it is sometimes viewed as an activity that is carried out at the beginning of an academic career or on taking up a new position, and is usually seen as the responsibility of an individual academic, the whole notion can be considered in a broader frame, regarded as usual practice for professionals and viewed as appropriate for academic teams as well as individuals. Essentially, professional development is all about learning, and in particular, about considering professional tertiary teachers as learners. As well as the ontological aspect of students becoming mathematicians, we can consider the ontological dimension of mathematicians becoming tertiary mathematics teachers.

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From such a viewpoint, professional development should be regarded as usual practice for professionals, and this includes development of research, teaching, and wider general development. Appropriate professional development can have the effect of improving teaching, and hence learning, which can then lead to better outcomes for graduates. An international study of quality teaching from 29 higher education institutions across 20 countries took the view that better quality designs, policies and practices *"improve the quality of their teaching, and thereby, the quality of their graduates"* (OECD Institutional Management in Higher Education 2009, p. 4). The report found a direct link between teaching quality and quality of graduates. One of their main findings was that teaching *matters* in HE institutions, and *"initiatives (actions, strategies, policies) aimed at improving the quality of teaching are spreading"* (p. 4).

A key question is what approach to professional development is most effective in improving the pedagogical effectiveness of university mathematics educators, and the ultimate aim of enhancing our students' learning. It is likely that there are a range of effective approaches, and that some will work better than others in different situations. In this chapter we will consider several models of professional development for mathematics teaching at the university level. These include models based on action research processes, models that consider learning from peers, a model based on professional standards and quality assurance processes, and finally, models that have their genesis in a sociological approach to mathematics education. The professional standards model was developed recently by a team headed by one of us (Leigh Wood, see Brown et al. 2010) and supported by the Australian Learning and Teaching Council.

What Is Professional Development?

From the viewpoint of tertiary mathematics educators, it is useful to explain some background to the notion of professional development. Academic development and professional development are terms that are often used synonymously, though there are subtle differences. Academic development orients the development activity towards persons involved primarily in tertiary education and describes a holistic concept in which an academic is perceived to be a part of a large enterprise and involved in a range of activities, including research, teaching, administration, and community activity (Reid 2002). Professional development has a more defined focus, the development of skills related to specific professional activities (Dall'Alba and Sandberg 2006). It is a term borrowed from the world of industry and business, moved initially into vocational education contexts and then to tertiary education more generally. Despite these origins, professional and academic development have now become interchangeable descriptors for activity intended to improve some aspect of practice.

In the 1990s, universities in the UK and Australia were revolutionised as they welcomed into their ranks professions that were previously considered vocational.

Vocational colleges (for instance, education, music, nursing, design or chiropractics) amalgamated to form new universities or became part of established ones. Prior to this, universities employed people as teachers who had come through the academy with a qualification (such as Ph.D.) that demonstrated superior disciplinary knowledge. Classes were generally small, and drawn from an economic, social and intellectual elite. With the amalgamations, the universities gained academic staff who had developed their professional capacities through experience. To ensure an integrated and equitable workplace, they were encouraged to improve their qualifications (to get a Ph.D.) and professional skills (through workplace links). Academic development units were set up to provide support for these activities. Hicks (1999) identified various approaches to academic development in terms of their location: central – in which a central unit was responsible for development activity; dispersed – where academic developers worked in departments; and mixed/integrated – where generic and local activities were combined. Each approach utilised a menu of activities for participants, and provided opportunities for them to meet with others who were keenly interested in teaching development. The overall approach was managerial, with things done to and provided for lecturers. A perennial problem was that non-participants were not caught in the developmental enthusiasm.

Ling et al. (2009) surveyed 20 years' worth of academic development activity and noted differences between a 'teacher-focused' approach, intended to give individual teachers the know-how to teach and assess students' work in different contexts, and a 'learner-focused' approach, with primary attention on the relation between student learning and the manner in which that learning was supported. The learner-focused approach was often based on the outcomes of phenomenographic research, on which we also draw in this volume. The quality assurance drive that commenced in the late 1990s generated an 'organisation-focused' approach. In this, the connection was made between university mission statements and teaching and learning on the ground. This *"squarely places the responsibility for academic development onto academic leaders"* Reid (2002, p. 2).

By contrast, Boud (1999) suggested that:

> … most academic development takes place in locations where academics spend most of their time: departments, professional settings and research sites. It takes the form of exchanges with colleagues, interacting with students, working on problems, writing and associated activities. It is informal and not normally viewed as development. (p. 3)

This observation is completely at odds with formal models of development as it allows individuals autonomy over the improvement of their own practice. When these ideas are distilled to their essence, development becomes a moral imperative. Adult learning respects the autonomy of learners and emphasises learning by consent. This approach fits well into the view that learning is emancipatory and is based on the individual's willingness to participate, and to understand and appreciate the development of their own academic identity. In the context of academic reform, Henkel (2005) found that the most important components of such an identity were the discipline itself and the notion of academic freedom. While recognising academic autonomy as crucial for identity formation, she also acknowledges the limitations of the professional academic context, including the various pressures of teaching, research and management. The various models of professional development for mathematics that we describe later in this chapter should be considered in the light of such background factors.

Professional Development in University Mathematics

Suggestions for approaches to professional development for university mathematics teachers have been made from the viewpoint of mathematicians and mathematics educators, and also from experts in the professional development area. The former group has the advantage of intimate contact with the discipline of mathematics itself, and the legitimacy that comes from setting professional development in its disciplinary context. The latter group has the advantage of being able to consider a broader range of approaches, some arising from quite different disciplines yet still providing novel directions that may be very effective in the field of mathematics. In this chapter, we will consider some representative ideas from each group.

Kline (1977) published his sweeping critique of undergraduate liberal arts education in his book, *Why the Professor can't Teach*. He based his ideas on five decades of academic work, and claimed that an over-emphasis on research was leading to a neglect of pedagogy in the (United States) higher education system – though the same pressures existed, and still exist, in most other systems. More than two decades later, when Krantz (1999) published *How to Teach Mathematics*, he was able to report that US academics were paying much more attention to their teaching duties than before, and indeed, a dozen of his colleagues wrote appendices exploring and discussing pedagogical issues for the second edition of his book. Nevertheless, *"tired, disillusioned instructors … exist in virtually every mathematics department"* and provide *"a poor role model for the novice instructor"* (p. xvii). Krantz's book sets out the principles and practice of university mathematics teaching, including the philosophical issues and advice on how to handle difficult matters. It is still a useful document for lecturers starting to teach mathematics in universities.

The early years of this century saw the publication of increasing numbers of books and articles on mathematics pedagogy that can be used as a basis for professional development for university mathematics teachers. One example is the collection by Kahn and Kyle (2002) that contains articles on a range of issues, including the transition to higher education, assessment in mathematics, developing active learners, designing mathematics courses, service teaching of mathematics, developing transferrable skills and teaching proof and reasoning. These are still current issues in tertiary mathematics education. One of the chapters in this book (Davies 2002) focused on learning and teaching statistics, and led to the development of a certificate in Teaching Statistics in Higher Education (Davies and Barnett 2005; Royal Statistical Society 2011), the first such course aimed at a specific area of mathematics pedagogy.

We discussed in Chap. 3 Nardi's (2008) investigations of tertiary mathematics teaching and learning, presented in the form of a series of dialogues between a mathematician and a mathematics education researcher. These conversations give a deep insight into the problems of assisting students to learn mathematics, and can be used as a basis for professional development in the discipline. The protagonists of the dialogue represent the two most important influences on the experience that students have as mathematics learners. In discussing the benefits of engaging with mathematics education research, the mathematician states:

> I would like to say that these discussions are already beginning to influence the way I think about my teaching. [In what ways?] I think discussing the examples is a very good starting point, and a well-structured one. By seeing these often terrifying pieces of writing I am faced with the harsh reality of the extent of the students' difficulties. Too often I see colleagues who are in denial and opportunities like this are poignant reality checks! (Nardi 2008, pp. 261–262).

While the views of mathematicians and educators are sometimes in opposition, it is heartening to see the negotiation of common ground in these dialogues. The various models of professional development that we describe in the following sections provide examples of how this process can be carried out.

An Action Research Model of Professional Development

The simplest approach to professional development is the pragmatic one. A university, a department or a lecturer is faced with a problem, and they find some way to deal with it. For instance, a large influx of students with weaker backgrounds in mathematics from their secondary studies may lead to the establishment of university preparatory courses in mathematics. Staff develop the skills (and the associated curriculum) to help these students to strengthen their backgrounds, or alternatively, teachers with the appropriate background may be hired to teach the course. This situation has arisen over the past decade in Australian universities as fewer students take higher-level mathematics courses at secondary level, caused in part by universities lowering their mathematics pre-requisites to attract larger enrolments (Australian Academy of Science 2006, pp. 53–54).

Such approaches can result in increases to the professional skills of university academics, but a better approach to professional development results from more systematic identification of problems, development of solutions and evaluation of their effectiveness. Action research is an appropriate methodology for such a process, as it is built on the notion of participatory researchers aiming to make changes to their practice (McNiff 2002a; Haggarty and Postlethwaite 2003). It involves a cycle of deliberate action steps – planning, acting, observing and reflecting – that are incremental, practicable, inclusive and re-iterated. Action research focuses on a critical approach to practice, undertaken by practitioners within their own practice rather than by researchers outside the practice. McNiff (2002b, p. 3) writes: *"research is as much about the process of answering questions as it is about the answers themselves."*

This process can be carried out by an individual academic facing a specific problem, but its effectiveness is enhanced when it is set up as a team process. In this way, the individual action research projects undertaken by academics can be nested inside a larger action research project. We have set up and directed such a team, the *Learning Excellence and Development Team* (LEAD), at Macquarie University (Wood and Petocz 2008), and the project has been through three complete cycles. Individual or small groups of academics proposed projects to address pedagogical problems that they had identified. While the location of the team was a business faculty, several of the component projects involved statistics and mathematics pedagogy. Examples include the development of online assessment for large undergraduate classes in operations research, relationships between cultural background and approach to learning statistics, and an evaluation of a statistics learning package. The overall LEAD project was addressed at enhancing the quality of teaching in the faculty, including the scholarly aspects of such teaching. The ultimate aim was to improve the learning experiences of our students, and project proposals were asked to demonstrate how this would happen.

The LEAD team met monthly to discuss general aspects of running projects, such as action research methodology, project management, working with research assistants, preparing ethics applications and writing pedagogical research articles. An important component of any pedagogical change is the evaluation of such change, and we included this as a mandatory aspect of the LEAD team at individual and overall levels. Each individual project had a modest budget and was supported by a more experienced academic. The final outcome from each project was a researched and evaluated solution to the original problem; the outcomes from the overall project included an increased awareness in the faculty of scholarly approaches to teaching and the publication of a volume of a journal containing articles on each individual project and on the overall outcomes (Wood and Petocz 2008, gives an overview, and the individual articles follow).

As a model of professional development, such an action research project has several advantages. It encourages academics to identify problems and propose, research and evaluate solutions. It develops the skills of participants in the research process associated with their teaching, sometimes referred to as the scholarship of teaching. It gives the participants a supportive group environment for such investigations, and provides them with records of obtaining small grants and academic publications. One disadvantage is that the benefits are only obtained by the group of participating academics. However, for a department or faculty, the interest generated and the solutions achieved can make a noticeable difference.

Peer Tutoring as a Model of Professional Development

Another approach to professional development is peer tutoring, sometimes referred to as peer-assisted learning (PAL), peer-assisted study sessions (PASS), or supplemental instruction (SI). This approach has become an important part of many transition-to-university programs. In general, peer tutoring consists of senior students assisting more junior colleagues with their learning, and with settling in to university studies. It has been used for assistance in mathematics and also other discipline areas. There are several benefits of peer tutoring in university mathematics: to improve the supply and quality of tutors for mathematics subjects (Oates et al. 2005); to improve the learning of those who are doing the peer teaching (Griffin and Griffin 1998; Congos and Schoeps 1993); to prepare students for teaching situations in the workplace (Wood and Smith 2007); and, more pragmatically, to provide services that are outside the university or department budget.

Peer tutoring is a way of providing educational and social support to students when they start university. At the same time, it develops skills within the peer tutors, who gain a greater appreciation of the breadth and depth of their chosen discipline of mathematics. They gain confidence and self knowledge, and begin to develop a more mature approach to life. As we have shown in Chap. 7, graduates need to demonstrate good skills in mathematical communication in their professional workplaces, and peer tutoring is an excellent preparation for work situations where they are required to teach their colleagues and to present succinct information to their managers. The wider exposure to mathematics and ways of learning mathematics encourages peer tutors towards the development of the broadest 'life' conceptions of mathematics and learning.

There are several examples of successful peer learning programs. In Australia, the University of Wollongong has a comprehensive PASS program with training for leaders and coordinators (details can be viewed at http://www.uow.edu.au/student/services/pass/index.html). Macquarie University runs a large PAL program in its business faculty – an evaluation of its effectiveness was one of the component projects in the LEAD initiative described in the previous section (details can be accessed at http://www.businessandeconomics.mq.edu.au/new_and_current_students/undergraduate/bess/peer_assisted_learning and the evaluation is described by Dobbie and Joyce 2008). The University of Pretoria in South Africa runs a successful SI program that is facilitated by senior students; this has been found to be a cost-effective way of delivering support, particularly for disadvantaged students. Bidgood et al. (2010) and Hammond et al. (2010) describe some examples of PAL programs from the UK. All these programs report significant improvement in the results of the participants and considerable personal and professional development in the tutors. Some universities have introduced peer tutoring courses for undergraduate credit. The University of Auckland, New Zealand, has a course entitled 'Tutoring in Mathematics' (details are given at http://www.math.auckland.ac.nz/uoa/home/about/our-courses/stage-2-courses), while Macquarie University has a similar unit, 'Learning and Teaching in Business' (see http://www.handbook.mq.edu.au/2012/Units/UGUnit/FBE204); such courses prepare students for tutoring in a variety of areas.

Peer tutoring as a model of professional development is aimed at participants at the earliest level of academic life – students who are interested in working as voluntary or paid tutors for their fellow learners. Yet there are other academics involved with the peer tutoring process, leading training courses or workshops, and helping to oversee the peer tutoring program. Students who participate as peer tutors usually receive very positive feedback in student appraisals, creating a climate where other academic staff are encouraged to enhance their own pedagogical skills.

The approach can be broadened in several directions. Students need only a little encouragement to prepare their own materials for peer learning, and some of these are at least as effective as materials that their teachers prepare. As an example from a quite different discipline, music education, students at the Sydney Conservatorium of Music have prepared videos explaining how they learn from each other (two of these are available online at http://www.youtube.com/watch?v=TJ9DApyuHMQ and http://www.youtube.com/watch?v=RXkmm7dfhbY). In a chapter about student engagement, Kay et al. (2012) report a project where second-year students in a biosciences degree researched and prepared a *Biosciences essay writing guide, written by students for students* for their first-year colleagues, having identified a need for such a guide on the basis of their own experience. The feedback from the first-year users was very positive.

Another direction is professional development based on 'peer observation' of teaching, in which professional educators observe each other's classes and offer support by discussing the features of the pedagogy. In this model, lecturers learn from other lecturers, in the same way that students can learn from other students. Paterson et al. (2011) describe a project where a small group of university mathematicians and mathematics educators formed a group to re-examine their lecturing practice, videoing and then discussing segments of lectures. While peer observation (or 'critical friendship') is often used as a resource by beginning teachers, it can be effective at any level; Schuck (2011) talks about using this approach to 'disrupt' her assumptions about teaching, and to challenge the complacency that can come with experience. These variations show that peer teaching at any level can form the basis of effective professional development in mathematics.

Professional Development Based on Professional Standards

Another approach to professional development is to use a framework of professional standards as the basis of a comprehensive structure for a categorisation of developmental activities. Such standards indicate professional expectations about quality and consistency, facilitate discussion of plans and policies at an institutional level, provide a basis for accreditation and recognition of courses, and form a guide to professional learning. Adopting professional standards for teachers in higher education is increasingly advocated by educators and government. In Britain, the Higher Education Academy has developed a generic Professional Standards Framework (Higher Education Academy 2006) for teaching and supporting learning at the tertiary level.

This framework is presented in three domains: areas of activity, core knowledge and professional values, each broken down into sub-domains. 'Areas of activity' comprises planning learning activities and programs, teaching, assessment and feedback, developing effective learning environments, integrating the results of relevant scholarship, and evaluation of practice. 'Core knowledge' includes the subject material, ways of teaching it, how students learn, use of appropriate technologies, ways of evaluating teaching effectiveness, and implications of quality requirements. 'Professional values' embraces respect for individual learners, incorporating the results of relevant research, development of learning communities, acknowledging diversity, and continuing development and evaluation of practice. There are obvious connections between the three domains; for example, core knowledge of the subject material underpins the planning of learning activities and programs, and the professional commitment to incorporating the results of relevant research finds its expression in the corresponding area of activity. Example standards are given at three levels: for staff new to higher education with no prior qualifications or experience, for staff with a substantive role in learning and teaching, and for experienced staff with an established record of work in learning and teaching.

A recent Australian project, supported by the Australian Learning and Teaching Council and carried out by a team of mathematics academics led by one of us (Leigh Wood), has modified and developed this generic framework (Brown et al. 2010). The *Australian Professional Development Framework* seeks to articulate indicators of good practice for teaching and professional development in the mathematical sciences at tertiary level. The 'values' domain has been augmented by an aim to advance the discipline itself, and the three levels are identified as teaching classes, coordinating units (or groups of classes), and leading programs – roughly corresponding to the levels in the UK framework. Each cell in the resulting grid of sub-domain by level contains various indicators; an example is given in Table 9.1 (the full matrix can be seen in Appendix 3).

The next step is to contextualise the indicators for the discipline of mathematics (Wood et al. 2011). Continuing the assessment and feedback example, at level 1 the focus is on what happens in the classroom, and the predominant mode of assessment is formative. Different types of assessment could include written solutions to mathematical problems, verbal explanations in response to questioning, contributions to class discussion of a proof, and formulation of new problems to illustrate a particular concept in statistics. Teachers should provide opportunities to students to monitor their own progress and understanding by providing constructive feedback during classes, supplying a variety of sample solutions, encouraging student discussion, and presenting a range of problem-solving approaches. These activities can be used as indicators for the two points in the second column of table.

At level 2, course coordinators may have overall responsibility for formative assessment, and they would usually design the summative assessment for their classes – which could include reports, presentations or (most commonly) examinations. Such assessments should be constructed so that they are aligned with the learning objectives in the syllabus. The range of tasks can be checked against a taxonomy that ensures that high-level outcomes can be demonstrated by the most able students. Smith et al. (1996) present a taxonomy for examination questions in three levels: factual knowledge, comprehension and routine procedures; application of knowledge to new situations; justifying, interpreting, evaluating, conjecturing. The authors point out that examinations in mathematics rely heavily on the first level, with occasional questions at the second.

Table 9.1 Excerpt from the Australian framework (From Brown et al. 2010)

Areas of activity	Teaching classes	Coordinating units	Leading programs
Assessment and giving feedback to learners	Using different types of assessment Providing effective and timely feedback to individuals	Designing effective and aligned assessment tasks Employing a range of assessment tasks Providing clear instructions and marking criteria Moderating between markers	Ensuring appropriate variety and balance of assessment tasks across the program Calibrating levels of difficulty between units Leading the implementation of current assessment practices

Activities from the third level, such as supplying a justification for a step in a proof or making a conjecture about a sequence of numbers, are mostly ignored. The marking (grading) of summative assessment tasks can begin by discussing criteria and standards in the form of a marking scheme with the group of markers, and checking that they are in agreement by multiple marking of a few example submissions, a process that can be repeated at intervals during the marking. These activities become indicators for the points in the third column of the table.

Finally, at level 3, program leaders are responsible for balancing assessment across a whole program so that the program goals and associated graduate capabilities can be demonstrated. Curriculum mapping can be used to check that students have developed the whole range of mathematical skills and generic capabilities. Calibrating the levels of difficulty between units can be most effectively carried out by a peer review team. Program leaders can engage with professional associations to determine mathematical requirements and design professional development to keep up to date with trends in assessment methods and tools in the mathematical sciences. Again, the activities carried out can be used as indicators for the points in the fourth column of the table.

In this way, professional standards can be used to set out the pedagogical activities that are required at different levels of teaching responsibility. An advantage of this approach is that it forces academics to remain aware of the full range of the tasks that are necessary in the process of teaching classes, coordinating units and leading programs. It is very easy to focus on some 'obvious' areas (such as preparing classes) and neglect others (such as evaluating practice). On the other hand, such an approach can encourage a response of 'ticking boxes' at the expense of the holistic view of pedagogical quality.

Sociological Models of Professional Development

Some models of professional development are based on the social context of education, and the desire to address particular social problems by enacting changes. Such a social context could consist of a single class, course or institution, but it could also be broadened to a national or even global concern. The resulting professional development has a holistic nature. We have seen several examples where professional development is aimed at institutional change, as opposed to the traditional development of individual capacities and practices. Borrowing ideas from mathematics (as well as educational change research), Reid and Marshall (2009) applied complexity theory to a large-scale professional development program for the development of research supervision at an Australian university. Standard approaches would utilise professional development for individual supervisors to improve their capacity and quality, and a result would be that only the willing would be 'developed'. Yet, individuals are strongly influenced by internal factors – such as their conceptions of research and the supervisory role – and external factors – such as the structural, political and cultural aspects of the institution (Senge 1994). This suggests that a combination of individual and institutional development processes might drive successful change.

Reid and Marshall's program married the institutional quality assurance agenda with individual agency that welcomed, and even encouraged, criticism, argument, confrontation, collaboration and cohesion. The overall focus was the improvement of learning outcomes for research students, but the program took a second-order perspective by focusing on the range of experiences, expectations, practice and knowledge of supervisors. In this model, disciplinary differences were strongly acknowledged and formed the basis of robust discussion of established and imagined practices; for instance, investigating differences between the supervision of doctorates in the mathematical sciences and those in creative arts. Overall, the most critical aspect of the program was to allow dissent, autonomy and discussion in an environment that gave persons involved on the ground (at disciplinary level) and at the top (university executive) space to contribute meaningfully in the areas which they felt to be important to their enterprise. A basic feature of Reid and Marshall's model is the acknowledgement that academics are professional and expert learners.

Professional development need not be limited to a single institution; in some cases it encompasses an entire national education system. An example from tertiary education in Malaysia illustrates this model (details are given in Dahan et al. 2008). With the shift from a production-based to a knowledge-based economy, the Malaysian government articulated a plan to improve the effectiveness of its higher education institutions to develop *"quality human capital"* (p. 284), holistic graduates in terms of both capabilities and character. The desired graduates with *"first class mentality"* (p. 288) would have knowledge attributes – discipline specific and general, including current affairs and at least three languages, interpersonal attributes – including communication and group skills, and personal attributes – they will be adaptable, innovative and autonomous learners and problem solvers, with a sound basis in moral ethics. In order to achieve this, curriculum must treat content and process, personal and inter-personal, holistically and the learners (and also the educators) must be spiritually grounded in a belief *"in God, the Almighty"* (p. 289). The national higher education system and all those within it are encouraged to move towards a pedagogy that focuses on deep learning, presents content and processes in authentic situations, relevant to learners' backgrounds, and is committed to multi-disciplinary and trans-disciplinary perspectives.

In this endeavour, the key disciplines are science and technology, with mathematics as a basic supporting subject for both. To this is added the humanistic approach that aims to enhance the moral and spiritual qualities of the participants; this can be seen as driving a change in the very conception of mathematics and its role. One practical avenue for professional development is promoting the use of information technology, including the establishment of the *Malaysian Lecture Hall*, a repository of online courses and learning materials that can be utilised throughout the country. Another is the setting up of broad curriculum teams including representatives from industry and professional bodies, as well as academics. The Malaysian initiative leans heavily on the establishment of a holistic curriculum, supported by contemporary information technology, in its aim of developing all participants in the educational endeavour. Applied to mathematics, the description given by Dahan and colleagues seems consistent with the broadest 'life' conceptions of the discipline and learning.

Some developments in mathematics education can be seen as instances of global professional development, affecting large numbers of mathematics educators and their students. One such example is the field of ethnomathematics. The term was introduced by D'Ambrosio to describe the mathematical practices of diverse cultural groups, and broadened to refer to the relationship between mathematics and culture generally (D'Ambrosio 1999). Three decades ago, the term ethnomathematics was known to very few mathematicians, and the mathematical ideas and approaches of different cultures were seen only as a curiosity. In the intervening time, such ideas have moved through being seen as 'enrichment material' for mathematics courses, particularly at the school level (Begg 2001), to an acceptance as an essential element of mathematics, the study of mathematical aspects of specific groups and activities. As Barton puts it:

> An ethnomathematician's task is to explore – in the present – the consequences of different worlds for mathematics: first to understand where they were/are leading, and then to reflect on them mathematically. (Barton 2008, p. 136)

And contemporary mathematics educators would agree when D'Ambrosio talks of his *Program Ethnomathematics* and says:

> It can bring new light into our understanding of how mathematical ideas are generated and how they evolved through the history of mankind. It is fundamental to acknowledge the contributions of other cultures and the relevance of the dynamics of cultural encounters. (D'Ambrosio 2007, p. 201)

Indeed, teachers in indigenous communities can effectively communicate the role and purpose of mathematics using the (ethno)mathematical system of the community (Barton 2008, p. 170), and in a more general setting, D'Ambrosio offers guidance on a pedagogical approach:

> I would like to finish by reporting on my practices with children, teenagers, college and graduate students and in-service courses for teachers. The problem is always the same: we have to awaken them for more reflective thinking, even when they do mathematics. I propose questions as 'What do you think of [a current event or a philosophical question]?', let some discussion follow and then come with another question 'What does mathematics have to do with this?' Of course, the educator must be prepared to move into a subject not very familiar to him [or her]. (D'Ambrosio 1998, pp. 71–72)

In this way, professional development can be extended to the whole community of mathematics educators, including those at tertiary level.

Discussion

The essence of professional development is the notion that mathematicians teaching mathematics at university level are also learners – certainly learners of mathematics pedagogy, and often also learners of mathematics. Like their students, they also hold a range of conceptions about their discipline, and about teaching and learning, and like their students, they can move towards broader conceptions, and thus increase their effectiveness as teachers. In the same way that mathematics students can move towards becoming and being mathematicians, so too can mathematicians lecturing at university level move towards becoming (better) teachers – skilled professionals who help other people to learn effectively.

Previous studies have shown that university mathematicians have a range of conceptions about mathematics itself, for instance, the survey carried out by Grigutsch and Törner (1998). They also have a range of conceptions about their own teaching. Gordon et al. (2007) studied a group of mathematical scientists – university teachers of service statistics courses – and showed that their conceptions of such teaching represented three hierarchical views. The narrowest conception placed the teacher themselves in the central position; the students were passive recipients, statistics was seen as a collection of techniques, and the serviced discipline was seen as peripheral. A broader conception placed the subject itself in the centre; statistics was seen in terms of this subject material, which was made relevant by the serviced discipline, and the teacher's job was to illuminate the statistical techniques and ideas. The broadest conception placed the student at the centre of the picture; the student's world was most important, while the teacher remained in the background presenting statistics as an approach and a way of thinking that would form a professional component in the serviced discipline. These results were based on e-mail interviews with 36 participants, experienced teachers working at all levels, from pre-degree to postgraduate. Their views of teaching service statistics courses are consistent with conceptions of teaching that have been explicated in more general contexts (see, for example, the review by Kember 1997) on scales from 'teacher focused' to 'student focused'.

Teachers with the broadest conceptions of mathematics, learning and teaching are in a good position to help students broaden their views and come to see mathematics as an approach to life, and an essential part of their professional and personal being. Unfortunately, a teacher who holds the narrower conceptions of mathematics and pedagogy can also influence students – confirming the views of those with narrow views, and maybe even encouraging those with broader views to approach their mathematics learning in more limited ways. The importance of professional development is to ensure that we as teachers have access to the broadest views of discipline and pedagogy, and hence can encourage our students in this direction.

The various models that we have described in this chapter have been presented separately, but in reality the distinctions between them are blurred – they can overlap and be used in combination. In some situations, interacting with peers, including learning from them and helping them to learn, is a powerful approach to professional development. Under other conditions, problematic aspects of the learning context will yield to individual or collective action research processes. In yet other circumstances, a framework based on professional standards and quality management will prove to be most effective. And behind all these approaches is the broader view of mathematics as a social force for giving people voice, allowing them to take control of their conditions and helping them to change the world they inhabit.

Summary and Looking Forward

In this chapter we have investigated the idea of professional development for mathematics academics, describing its historical background, and its role in mathematics pedagogy. At its most fundamental level, professional development is about acknowledging that teachers are also learners, and that the process of helping learners move towards broader conceptions of their discipline and their learning is mirrored by encouraging teachers to move to broader conceptions of their discipline and their role. We have given several models of how this can be carried out. In the final chapter of this book, we will summarise the overall ideas and conclusions from our decade of research on the process of becoming a mathematician.

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Chapter 10
Conclusion: Becoming a Mathematician – Revisited

Introduction

In this concluding chapter we begin by summarising the main ideas and arguments of the previous chapters, based on the research that we have carried out over more than a decade with many students and graduates of the mathematical sciences, both majors and non-majors. In particular, we point out the importance for our work as educators of learning from our students in their role as (often expert) learners. Throughout the book, we have presented the views of many of these students and some recent graduates, mostly in fairly short quotations focused on the specific topic under discussion. In this chapter, we will allow two students to present their context, goals and ideas about their mathematical studies in a more extensive way – one is an engineer studying mathematics as a professional component, the other is an honours student majoring in statistics. We use their stories to illuminate the important aspects of becoming and being that have been central themes throughout the book.

A Summary of the Main Ideas

Becoming a mathematician involves not only learning mathematics, but also becoming aware of the mathematical way of looking at the world, and the role of mathematics in professional and personal life. This is true not only for mathematics majors but also for students who will use mathematics as a professional component of their work.

Students see mathematics and learning mathematics in a small number of qualitatively different ways. The narrowest views are based on techniques and components; broader views are based on the notion of models, applied or abstract, that are central to the discipline of mathematics; the broadest view is a strong connection between mathematics and personal and professional life. The hierarchical nature of

L.N. Wood et al., *Becoming a Mathematician: An international perspective*, Mathematics Education Library 56, DOI 10.1007/978-94-007-2984-1_10,

Table 10.1 Students' conceptions of mathematics and learning mathematics

	Mathematics is …	Learning is …	
Number/components	Individual components or techniques, including numbers, formulae, calculations, etc.	Acquiring the tools, skills and techniques of mathematics	Techniques
Models/abstract	Building and using applied or abstract models of aspects of reality	Understanding practical and theoretical aspects of the discipline itself	Subject
Life	An approach to life, a way of thinking, an integral part of personal and professional life	Changing view of the world, developing a way of thinking, seeing the role of mathematics in personal and professional life	Life

Table 10.2 Graduates' conceptions of problem solving and communication

	Problem solving …	Mathematical communication (with non-mathematicians) …	Learning to communicate mathematically …
Narrowest	Specific mathematical problems	Jargon and notation	Trial and error
	Problems in a work context	Concepts and thinking	Mediated by others and outside situations
Broadest	Problems in life generally	Strength	Active and detached observation

these conceptions implies that the broader views include the narrower ones – ideas are added on to narrower views, rather than replacing them, so students gain by moving to broader conceptions. We have presented evidence for these ideas in Chaps. 2 and 3, and they are summarized in Table 10.1. Again, this is true for students from other disciplines, in the same way as for mathematics majors, as we indicated in Chap. 2, and detailed in Chap. 4.

Some students, those who already view mathematics at the broadest levels, can have very sophisticated insight into their own ideas, learning and future professional use of mathematics. Others, whose views are still at the narrower levels, need some help and encouragement to get there. We can track their progress during the course of their studies, as we showed in Chap. 5.

Graduates have corresponding narrow to broad conceptions of various aspects of their professional work, including problem solving, mathematical communication and learning mathematical communication. These views are based on their workplace experiences, but could be addressed more explicitly in their undergraduate courses. Evidence for this is presented in Chaps. 6 and 7, and the ideas are summarised in Table 10.2.

Mathematics curriculum can be aimed at the broadest conceptions of mathematics and learning, inclusive of a range of graduate skills and dispositions, and mathematics teachers can usefully direct their own learning to the broadest conceptions of mathematics, learning and teaching. We gave arguments to support these statements in Chaps. 8 and 9.

As educators, we should utilise these points to enhance our ability to help our students not only to learn mathematics but also to become mathematicians. Essentially, this occurs when students move towards the broadest conception of mathematics and learning.

Learning from Our Students

University students and graduates are expert learners. By the time they have completed their tertiary education they have spent the greater part of their lives coming to understand and appreciate their chosen field of study. Part of this expertise is learning how to understand specific and diverse contexts of learning, and part is coming to understand the complexities of a certain field. Most books on mathematics education take the view of mathematics experts, those who work with mathematics at university, those who teach it, or those who have used it often in professional situations. But in this book, we have focused on the voices of expert learners, to look at how students and recent graduates see their chosen discipline and their own learning in mathematics, and to understand the goals that they have for this learning.

During their pre-university lives, students have undertaken studies that have been completely mediated by mathematics professionals – those who write primary and secondary school texts, and those who deliver those texts in a class room. Twelve years' (or more) experience of such texts and lessons has often had the effect of reducing mathematics to a series of rules and technical components. At the conclusion of compulsory schooling, many potential mathematics professionals have come to understand that the world of mathematics is presented in these very specific (and limiting) ways. These early years of study and experience provide the starting point for higher education.

University study, however, is somewhat different, particularly in the choice that students have – choice in the discipline that they will study, and even a choice as to whether they will continue formal study. In some sense, in a tertiary environment students *volunteer* to engage in their studies (though some will have enrolled as a result of other pressures). At this stage it is likely that students are able to muster a real motivation for their work of learning. They may be expecting that their studies will provide them with a degree that is an entry qualification for work, or they may see their degree as a means of developing their understanding of the world. In both cases, tertiary education provides learners with a range of opportunities. As we have seen from our student and graduate participants, many have developed a view that mathematics consists of numerical and technical aspects. Even during their tertiary studies, they strongly identify with the notion that mathematics has something to do with numbers or equations, and that learning is about getting the numerical examples correct. They may believe that mathematics has little relevance to their personal or professional lives (and for some of them it may indeed be true). Happily, other students and graduates have a broader view of mathematics, and have found that it plays an integral role in their personal and professional lives. We present the stories of two such students in the following sections.

What is the difference between these groups? And how can we encourage students from the former group towards the broader views of their discipline and their learning? An important goal of this book has been to present the perspectives of expert learners on mathematics and learning mathematics in a way that can help educators appreciate students' learning goals and facilitate them most effectively. The experiences of Paul and Heather that we present in the following sections are examples.

Paul's Interview

Paul is more than half-way through a 4-year degree in mechanical engineering, studying part time while he works in a printing company. As part of his degree, he has taken three mathematics and statistics subjects, the most recent of which is called 'uncertainties and risks'. When we interviewed him, it was to ask him about his ideas of statistics and mathematics, and their role in his studies and his professional life. In response to an early question, he gave a detailed description of a work problem and the way he had tried to deal with it:

> [The first question then is, what is statistics?] Can I throw in what I do? Because I am trying to use this at work to prove a point./…/Basically, we have an ink pumping system to pump coloured ink for newspapers, right, and they were just continually falling apart. I started remodelling it and then I started basically checking against the sheet – we just kept a written sheet on a board. And then I thought, oh, this is no good, because I don't actually have any room there to write down what the failure was, because you would forget after time, and you know I think that one was a seal failure or I think one was mechanical. And then I just started writing that down, and I'm the type of bloke that will just get carried away so it goes on weeks it runs for, and then I just put all the statistics across the top because I was doing it here (at university). [Is that where you got your statistics knowledge from?] Yeah, from Maths 2, because I brought up this problem with the lecturer, same problem early on, about a year ago. Since then, I have done a lot more data collection and that's the thing I have too.

Although Paul seems naturally to have a mathematical approach to his problem, the theory that he was studying has obviously given him a broader range of options for his inquiry. The application of what he was learning during classes to his work situation seems to motivate his continued investigation of the problem and give him an appropriate language to communicate his results. He continues with more details:

> OK, we had a problem with a particular pump. Right, so before we don't keep any statistics on this pump. How long it runs for, how many days between when I install it and when it fails, we have never done that. I've been keeping statistics on that, or how long it runs for and also how many cycles it does in that period. To see that basically any changes I make

better the system./…/Also if someone comes down and says 'well I don't think it is any better', and I can say 'well here is all my information to back up what I am saying'. And I can sort of brush the challenge aside, sort of thing. If it is a genuine sort of question, I can also use the statistics to say 'well previously this is what was going on with the older style and now that I have done this, we are now averaging, you know, a hundred and fifty days, and I can pick it within sixty days either side of that.' So it is easier to justify when I say I want to do this to it, and I can justify it by saying what I've done is working.

And the link between the theory that he learned in classes and its application in his workplace seems to have a positive effect in both directions:

[From your experience in your workplace, does it affect how you learn statistics in that you might be able to apply it?] I can see probably just in those two components, the one component we played with for eight years, there is a million and one things we could do there with statistics to prove, because there becomes urban myths involved in machinery, why things don't work. And you could shoot them down quite easily by just doing some little experiments and proving with data. This is what happens, that is what happens, and then you could move on to try and find out what the root cause of something is. I can see that it is a great tool, it really is.

It seems that this two-way connection between theory and practice is having a strong influence on Paul's process of becoming a mathematician. Indeed, we might say that he is experiencing the process of being a mathematician while he grapples with and comes up with solutions to his workplace problem. The next question, though, reveals a quite unexpected reality:

[So obviously statistics plays a huge role in your workplace.] What I do normally, nothing! I'm a maintenance supervisor, so it doesn't play any part in it, this is just – I took this job on 'above and beyond'.

Paul's formal position at his company is not the problem-solving mathematical scientist that his earlier responses suggested, but rather a more limited role as an engineer-in-training supervising maintenance of his company's printing machines. We can only hope that his employers appreciated his initiative and the development of his mathematical and statistical capacities. In addition to becoming an engineer, Paul is also becoming a mathematician, even if it is in the context of professional work as an engineer.

After his specific and detailed answer to the question 'what is statistics?', Paul gave a more general response to the parallel question about mathematics. We can see him taking a step backwards from his discussion of the printer pump problems and considering the nature of mathematical modelling:

[So what do you understand about maths to be about?] Problem-solving./…/Modelling the real world, modelling things that you can't make a physical model for. Do you know what I mean, you can numerically model them./…/Finding an answer to something, or sort of taking the uncertainty out of what you are doing./…/You model something, you can give it certain variables in certain ranges, and then basically you can run your model and see if that's what happened. You might have something that has happened in the physical world, and you go well why did that happen, you can set up a model that you think suits that and then run it within these constraints and see if it actually gives you something that sort of semi fits. And when more information becomes available you can sort of rearrange a model to sort of suit.

And his discussion of the differences between the statistical and mathematical approaches reveals his thinking about different types of models and the information on which they are built:

> [What do you sort of see as the key differences between stats and maths, if any?] Just different branches really. You are getting your data, you are getting real-world data in with statistics I guess and you are trying to work out information from the real-world data. I mean you can do that by basically trying to work out the equation of the data. But I mean with maths sometimes you don't have real world data, you are working the other way. You are working from like the virtual to the real world, whereas the other way you are working from the real-world information back to a point.

If we considered only these last two quotes, we may have decided that Paul's conception of mathematics (including statistics) was at the modelling level; indeed, we used one of these extracts to illustrate the modelling conception in Chap. 2. However, his earlier discussion shows clearly that he is aware of a professional and personal connection with mathematics and that he is able to use the mathematical approach to develop his way of thinking about the world around him. Although Paul is primarily a student of engineering rather than mathematics, his conception of mathematics is at the broadest 'life' level, and his academic and professional experiences have allowed him to progress a long way towards becoming a mathematician.

Heather's Interview

Heather is in her honours (fourth) year of a degree in mathematical sciences. She has studied a wide range of subjects in mathematics, statistics, mathematical finance and operations research, and is specialising in statistics in her honours year. When we asked her about her learning in mathematical sciences, it was clear that her focus during the early stage of her degree was at the narrowest level:

> [What would you say that you focus on in learning?] Well I guess at the university level, number one you focus to pass unfortunately, so you learn what you have to, you learn how to apply it, but then if you have a bit more time and you are studying, hopefully this learning how to apply will somehow become how to, to apply, but how to learn about it, as I was saying before, how it works. So I guess when I'm studying it, firstly to pass, secondly to actually understand what I'm doing and why I'm doing it.

Her current desire to understand the material, to learn at a deeper level, was stimulated by the fact that she is now also in the position of a teacher, helping other students to learn. The need to communicate mathematics not only helped her understanding but also made the subject more human and interesting to her:

> If I don't understand it then, I tend to understand it when I go back and tutor the subject. So that's my way of covering up my first year of rote learning, in a sense. I'm actually going back now, and since you've got to explain it to students, you've actually got to understand it. [So you're understanding more about the maths because of the tutoring?] Oh definitely, only because I actually have to communicate it. I mean you can understand it yourself but

> you may have no idea how to communicate it, and when you think about ways of doing it, you have to simplify it, and that's actually getting to the basics of what these things are, which is good, I think everyone should be made to do it.

Whereas previously she had focused on the components of mathematics – she lists several subjects by their name – she was starting to broaden her aims, becoming interested in a mathematical way of thinking and being, based particularly on a historical assessment of topics such as probability:

> [What would you say are your overall aims in learning mathematics?] In learning the mathematics, maybe not so much the numeric sort of mathematics, but I've come up to a barrier and I'm wondering why people use mathematics, because everyone seems to use these maths and stats figures in advertising, but why do they play this important role. I think that's my main aim this year. Before that I guess it was just learning more about the subject area in terms of quantitative views,/…/just learning different sorts of mathematics, differential equations, calculus, integration, that sort of thing, more the actual rules and application side of it, but now I'm interested more in the why this field or, I guess a philosophical view of it, area has changed quite dramatically. [Could you tell me some more about what your philosophical aims are?] Well, I'm having a bit of a problem with that at the moment. I've sort of looked at why the mind works this way and looking at what is explanation, and looking back at the history of statistics in particular and trying to figure out why it all came about. I mean people have used probability for years, but they never put it down on paper, gamblers especially have always used it, they always think 'okay, no I don't want to gamble on this particular game', but why do they think like that?

Like many students, Heather seems to gravitate towards the human aspects of mathematics, and the possibility of communicating and helping people via the discipline. Some of her university experiences have suggested possible avenues of professional work that she might enjoy:

> [What do you think it will be like to work as a qualified mathematician?] No idea really, I haven't done it yet, I don't know. I have this fear of sitting behind a computer all day in an office and not talking to anyone and I know a few of my friends are doing that right now and they are qualified statisticians, so that's very scary./…/I liked, we had a whole subject called statistical consulting, so that gave us real-life experience about what we should expect. It was only a basis, it was only a grounding, it was obviously not that difficult compared to what you could get, but it certainly gave me a liking for that area of study, so I think that I might head for that one day, not easy to get into. [What is it that you like about it?] I still think it's the people, that I can actually talk to someone and help them instead of just sitting in front of the computer, that's probably the only, that's probably the main thing I liked actually.

She reveals a broad interest in learning and knowledge. She obviously sees a large role for mathematics in her life, but one that is not exclusive of other areas. She shows an interesting progression of studying mathematics because she was 'good at it', then finding that she liked it and specialising in an area of it, but nevertheless not wanting to base her life around it:

> [So you've talked a lot about interest, would you say that was a driving force in your learning?] That's probably the main factor, yeah definitely, because I read books on all different subjects. I'm not just interested in maths, in fact it wasn't really my main interest when I left school anyway, I just did it, yeah, probably because I was good at it. I liked it, but I was good at it and that, unfortunately, is a good reason to do something these days, in a lot of people's minds. I realise now that it was probably wrong, but I've ended up liking it, so

> it doesn't matter./…/I like knowing about all sorts of fields, not just maths. There's not enough in maths just to like maths, it is a tool of science, it's no more really. I mean you can love the subject, sure, but you can't, I don't think that you should base your life around it, if you know what I mean.

Nevertheless, she seems to view mathematics in the broadest 'life' conception – maybe despite herself. She expresses this in an idiosyncratic way that shows the broad view but at the same time keeps aspects of the narrowest view, that mathematics is numbers:

> [Could you tell me a bit more about the things that you find intrinsically interesting in the maths?] Intrinsically interesting in the maths? [pause] I guess the way that you can solve problems, you can better people's way of life, better our technology, that sort of thing, all due to these little numbers that you play with, and it's amazing that numbers can have such a powerful effect when that's all they are, little things on paper. I guess that's my main interest.

Her final comment is a plea for at least some chance to look at mathematics from a broad, philosophical viewpoint during a university course, to complement the focus on the more specific, the practical applications:

> [Is there anything else that you'd like to say about maths, or learning or working in maths?] Only that learning maths at university I think may be aimed in the wrong direction. They basically just look at how you apply mathematics to problem solving and decision making, I think they should definitely in the earlier years, look at why you study maths, I think it's quite important. I don't think anyone really thinks about it at all, I know my friends didn't, most of the course probably didn't, you just don't have time. So it would be nice if they had one subject that would reflect this.

Heather retraces her steps in becoming a mathematician, from an initial practical focus on learning the components for the purpose of passing examinations, through aiming to understand the subject based on her desire to communicate with students and clients, to a desire to explore the role of mathematics in her personal and professional life. Her success in this process is shown by her current status as an honours student. Her interview contains many useful ideas for mathematics educators.

Being and Becoming

Reid and Solomonides (2007) described students' 'sense of being' – their ideas and feelings about themselves in relation to other things around them. Entwined with a sense of being is their 'sense of transformation' – the way in which they adapt and change their views of self through the process of learning. Although this was identified in the context of design education, it is just as applicable to mathematics. A person may have a sense of being a student of mathematics and a sense of becoming a successful mathematical problem solver. But these aspects of identity are not exclusive. The same person may also have a sense of being a tennis player, a girl friend, a Christian and a part-time employee. They may see themselves as becoming a competitive athlete, a mother, a youth group leader and an entrepreneur. From our viewpoint as mathematics educators, we are most interested in their progression towards becoming a mathematician, as Heather's story illustrates so well, culminating in a mature identity as a mathematician. We accept that this will not be the case for all our students, particularly those who are studying mathematics as a professional component – though some of these, such as Paul, will become mathematicians as well as engineers (or members of some other profession). Yet our job is to give students the opportunity and encouragement to include mathematics in their core identity.

The research results that we have described in earlier chapters, and many other phenomenographic studies (see Marton and Booth 1997; Reid et al. 2011) in mathematical sciences and other discipline areas, conclude that the important ontological aspects of personal change and connection with personal and professional life are components of the broadest conceptions of discipline and learning. That is, the broadest conceptions of mathematics and learning mathematics acknowledge of the importance of becoming a mathematician (or a person who uses mathematics and thinks mathematically), as opposed to simply learning about mathematics. Barnett (2007) points out that *"a 'deep' orientation towards her studies is a personal stance on the part of the student in which she invests something of herself* ***as a person****; in a 'surface' orientation, by contrast, the student lacks such a will and subjects herself passively to her experiences. That is, underlying the apparently cognitive level on which the 'deep'/'surface' distinction works, is an ontological substratum"* (p. 18, emphasis in the original). The hierarchical and inclusive nature of the conceptions we have identified implies that we are considering the ontological in addition to (rather than instead of) the epistemological aspects; that is, one learns mathematics *and* becomes a mathematician.

As we discussed in Chap. 8, the inclusion of a range of professional dispositions such as creativity, sustainability and ethics, into the mathematics curriculum can increase the relevance and interest of a mathematics course for majors or service students alike. This can increase students' engagement with their studies, contributing to their sense of transformation, while allowing them to develop and practice such professional dispositions. An important aspect of such professional dispositions is that at their broadest levels they too contain a strong ontological aspect – a student aims (or can be pedagogically provoked) to *become* a creative problem finder and solver or an ethical practitioner rather than simply learning *about* creativity and ethics. When our students are working with the broadest conceptions of mathematics, learning mathematics and their professional roles, their sense of transformation will be greatest. The result, as Barnett summarises: *"The degree in [mathematics] confirms that the student has become, if only embryonically, a [mathematician]"* (2007, p. 50, with our small modification).

What We Can Do

Throughout this book, we have heard from our students about their views of mathematics and their approaches to learning, and their thoughts about their future professions – and from graduates about their previous mathematics education and their current professional use of mathematics. Earlier in this chapter we presented the experiences and thoughts of two students in more detail. We do not believe that our students are unique in being able to talk to us in this way. Any mathematics educator discussing these topics with their own students would find a similar range of experiences and would gain useful insights into their students' lives, their goals and the role that mathematics plays for them. Maybe our most important recommendation is to take some time regularly to converse with students, incorporate their ideas into pedagogy and curriculum, and keep them at the centre of our teaching practice.

When we do talk with our students, we will find that in every class at every level, some will view mathematics and learning in the broadest ways, making implicit or even explicit connections with their personal life, their role as learners and their professional aspirations. For such students we only need to keep doing our work in such a way that their conceptions are supported and they continue along the path to becoming mathematicians. On the other hand, some students will see mathematics and learning in the narrowest and most limiting ways – as numbers, calculation and components, carried out to satisfy course requirements and obtain a pass. The challenge with these students is much greater. We need to use appropriate learning materials and pedagogy to make them aware of the full range of conceptions, and encourage them to move towards the broadest and most holistic ones, to come to appreciate the role of mathematics in their lives – either in terms of practical application or intellectual and aesthetic fascination.

And it is essential that this is done at a local level. We saw in Chap. 4 that there are significant differences between universities in the proportions of students who view mathematics in the broadest way. These universities were located in different countries, all of which have different approaches, methods and values surrounding mathematics education; and the prospects for working as a mathematician are different in each country. We do not know the reasons for these differences, but we see them as a positive sign. In some way, the educational context and what we do as educators can make a difference. Pedagogical solutions to the problems of teaching mathematics are unlikely to apply across the board, to all countries, contexts and times. Rather, investigation into the best ways of supporting students in their learning of mathematics must be undertaken at specific times and places. Universities and university educators need to continually reassess how they can best help each student in his or her process of becoming a mathematician.

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Appendices

Short Form of Conceptions of Mathematics Survey

We would now like to ask you a few questions about your opinions. For each statement, please choose how much you agree or disagree.

1. What Is Mathematics?

Mathematics is:	Strongly disagree	Disagree	Neutral/don't know	Agree	Strongly agree
A set of models used to explain the world	❍	❍	❍	❍	❍
A way of analysing ideas and problems	❍	❍	❍	❍	❍
A set of rules and equations	❍	❍	❍	❍	❍
Basic knowledge for all scientific fields	❍	❍	❍	❍	❍
No use to me at all	❍	❍	❍	❍	❍
A way to solve problems in my life	❍	❍	❍	❍	❍
A tool that can be applied in various fields	❍	❍	❍	❍	❍
Figuring out problems using numbers	❍	❍	❍	❍	❍
Using formulas to get results	❍	❍	❍	❍	❍
A way to give humans a more advanced life	❍	❍	❍	❍	❍
The language of nature	❍	❍	❍	❍	❍
Calculations	❍	❍	❍	❍	❍
Numbers being processed	❍	❍	❍	❍	❍
A theoretical framework that describes reality	❍	❍	❍	❍	❍
The study of numerical concepts	❍	❍	❍	❍	❍
A way to generate new ideas	❍	❍	❍	❍	❍

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2. What Part Do You Think Mathematics Will Play in Your Future Studies?

In my future studies, mathematics will:	Strongly disagree	Disagree	Neutral/don't know	Agree	Strongly agree
Be necessary because it is compulsory for my degree	❍	❍	❍	❍	❍
Help me develop logical thinking	❍	❍	❍	❍	❍
Give me new ideas about other things	❍	❍	❍	❍	❍
Help me solve problems in my other subjects	❍	❍	❍	❍	❍
Help me do calculations in my other subjects	❍	❍	❍	❍	❍
I do not know how mathematics is useful for my degree	❍	❍	❍	❍	❍
Be the basis of my future studies	❍	❍	❍	❍	❍
Be of no use as computers and calculators can do all the work	❍	❍	❍	❍	❍
Be used in all my subjects	❍	❍	❍	❍	❍
Be my major/main/primary subject	❍	❍	❍	❍	❍
I do not know how I will use it in my future studies	❍	❍	❍	❍	❍
Be the foundation of all my other subjects	❍	❍	❍	❍	❍
Be used in most of my future	❍	❍	❍	❍	❍
Give me a whole way of thinking	❍	❍	❍	❍	❍

3. What Part Do You Think Mathematics Will Play in Your Future Career?

In my future career, mathematics will:	Strongly disagree	Disagree	Neutral/don't know	Agree	Strongly agree
Give me new ideas	❍	❍	❍	❍	❍
Be my future career	❍	❍	❍	❍	❍
Be a building block for any job	❍	❍	❍	❍	❍
Help me get a job because employers know it is useful	❍	❍	❍	❍	❍
Be of no use as computers and calculators can do all the work	❍	❍	❍	❍	❍
I don't know how I'll use it as I don't know what I'll be doing	❍	❍	❍	❍	❍
Help me with calculations	❍	❍	❍	❍	❍
Help me with analysing problems	❍	❍	❍	❍	❍
Provide me with mathematical models to help me to understand problems	❍	❍	❍	❍	❍
Help me solve everyday problems in my life	❍	❍	❍	❍	❍
Give me foundation skills for my future career	❍	❍	❍	❍	❍
Help me think logically	❍	❍	❍	❍	❍
Help me with analysing data	❍	❍	❍	❍	❍
I don't know how I'll use it as I don't know the industry	❍	❍	❍	❍	❍
Give me tools to use	❍	❍	❍	❍	❍
Not be any use in my future career	❍	❍	❍	❍	❍

Mathematical Communication Outcomes

Table A2.1 Mathematical communication outcomes for graduates (After Wood et al. 2011)

Aspect	Conceptual	Procedural	Professional
Level 4 HD	The discourse is linked and integrated with other communication strategies, resulting in a new pattern of communicating. The depth and breadth of the mathematical concept is understood in such a way that the individual is inspired to reorganise other concepts, and motivated to make creative and innovative applications	Demonstrate the capacity to create/develop new valid ways of communicating mathematics or communication strategies are applied in novel ways, or to new audiences derived from deep understanding of mathematics and mathematical discourse	Demonstrates a strategic view to communicating innovative outcomes in complex situations with a range of audiences
Level 3 D	The understanding of mathematical discourse is broadened, appreciated from different angles, and this elaboration reflects in the ability to communicate mathematics in other contexts and from different perspectives	Demonstrates the ability to select appropriate communication in a given context. Communication strategies no longer need to be given	Demonstrates the ability to adapt communication strategies to new environments and audiences. Able to incorporate ethical and critical dimensions to the communication
Level 2 C	Some personal meaning has been extracted and their understanding reflects this internalised view. The concepts of mathematical discourse have become a part of their knowledge. Nevertheless, the concepts remains narrow and shallow and relatively disconnected from other concepts	Demonstrates the ability to apply given communication strategies and procedures in a variety of contexts and to novel problems	Can evaluate a professional situation and identify key communication strategies
Level 1 P	Demonstrates the ability to describe and define the basic concepts of the mathematical discourse, but has not demonstrated an ability to be able to elaborate or to reflect on the meaning of the concepts	Demonstrates knowledge of basic communication strategies and can practice the rules of a given communication strategy	Demonstrates a basic understanding of how to communicate mathematically and a basic understanding of the significance of these in professional practice
Level 0 F	Demonstrates inability to describe and define the basic concepts of mathematical communication	Demonstrates no knowledge of mathematical communication strategies and is not able to practice the rules	Demonstrates no ability to communicate mathematics or the significance of mathematics in professional practice

Australian Professional Development Framework

Table A3.1 Indicators for each level of operation for the core knowledge domain of the professional development framework (From Brown et al. 2010)

Core knowledge	Teaching classes	Coordinating units	Leading programs
1. The subject material	Unit content Prerequisite knowledge Applications of unit content	The purpose of the unit How concepts in the unit relate to other units across and within year levels History of concepts in the unit Applications of the concepts in the unit	External content requirements for professional bodies, accreditation Connections between discipline and other areas in the university Currency with trends in discipline area
2. Appropriate methods for teaching and learning in the subject area and at the level of the academic program	Ways of communicating in the discipline Different approaches to explaining concepts Different approaches and techniques for solving problems	Structuring the unit Designing and developing teaching activities and resources to align with learning outcomes Adapting materials for different learners Strategies for teaching large and small classes	Current teaching practices in the field Teaching approaches being used across the program to support course level outcomes
3. How students learn, both generally and in the subject	Contemporary learning theory: How to differentiate teaching depending on student background and context How to cater to different learning styles Engagement and scaffolding	Providing opportunities for students to engage with the unit content in different ways Sequencing the curriculum content to support development of learning outcomes Implications of students' discipline backgrounds	Contemporary learning theory in the discipline area Methods of structuring and sequencing of content across a program to enable appropriate development of concepts over a degree program Cognitive maturity of students

(continued)

Table A3.1 (continued)

Core knowledge	Teaching classes	Coordinating units	Leading programs
4. The use of appropriate learning technologies	Available teaching technologies and how to use them	LMS unit development and design skills How technologies can be used to represent concepts and facilitate collaboration to effectively achieve learning outcomes Electronic assignment submission and marking approaches Supporting technology use by teaching staff	Available technologies for application in the discipline Emerging technologies for learning and teaching Trends in technology usage in discipline-specific research
5. Methods for evaluating the effectiveness of teaching	Different approaches to collecting and analysing evidence about teaching and student learning	Different approaches to collecting evidence about unit level student outcomes Approaches to analysing effectiveness of the unit	Approaches to program evaluation at institutional and national levels
6. The implications of quality assurance and enhancement for professional practice	University policies relating to teaching Professional development to improve classroom practice	Institutional requirements Professional development to improve unit development and implementation	National trends and policies Communities of practice for benchmarking

Table A3.2 Indicators for each level of operation for the areas of activity domain of the professional development framework (From Brown et al. 2010)

Areas of activity	Teaching classes	Coordinating units	Leading programs
1. Design and planning of learning activities and/or programmes of study	Designing and planning classes Structuring the class and the teaching activities	Constructive alignment of unit Liaising with relevant stakeholders Writing unit outlines Selecting texts and resources Including a range of learning activities and resources	Determining course level learning outcomes Mapping program curricula Mapping graduate attributes Differentiating curriculum depending on stage in program Determining appropriate division of content and student workload between units
2. Teaching and/or supporting student learning	Effective communication Encouraging participation and interaction Considering student diversity Creating a culture of inquiry Incorporating a range of strategies for teaching in small groups Giving students opportunities to engage with feedback and reflect on their work	Leading and managing small teaching teams Ensuring there are appropriate channels of feedback and support for students Incorporating a range of strategies for teaching large groups	Providing support and guidance for teaching teams to support effective practice Providing appropriate structures to support students Encouraging and providing for professional development of staff to improve teaching and learning Encouraging and modelling appropriate use of learning technologies
3. Assessment and giving feedback to learners	Using different types of assessment Providing effective and timely feedback to individuals	Designing effective and aligned assessment tasks Employing a range of assessment tasks Providing clear instructions and marking criteria Moderating between markers	Ensuring appropriate variety and balance of assessment tasks across the program Calibrating levels of difficulty between units Leading the implementation of current assessment practices

(continued)

Table A3.2 (continued)

Areas of activity	Teaching classes	Coordinating units	Leading programs
4. Developing effective environments and student support and guidance	Creating a positive culture in the classroom Engaging students Encouraging student interaction	Providing support and guidance to tutors Facilitating student interaction in the unit Implementing avenues of unit-wide student support	Providing program level support for students Employing, training and supporting teaching staff (including sessionals)
5. Integration of scholarship, research and professional activities with teaching and supporting learning	Using tertiary teaching literature to inform classroom practice Integrating own and others research into teaching	Using current learning and teaching research to inform curriculum design	Conducting and encouraging research-based approaches to learning and teaching
6. Evaluation of practice and continuing professional development	Collecting evidence to evaluate teaching Analysing and reflecting on collected data Undertaking relevant professional development	Collecting data from a variety of sources to enable critical reflection upon unit Reflecting upon performance based upon analysis of unit data Engaging relevant professional development to improve unit	Managing and monitoring programs based on feedback Supporting professional development of departmental staff Contributing to professional bodies and communities of practice

Table A3.3 Indicators for each level of operation for the core values domain of the professional development framework (From Brown et al. 2010)

Core values	Teaching classes	Coordinating units	Leading programs
1. Respect for individual learners	Developing a supportive and inclusive learning environment Demonstrating inter-cultural competence	Planning for students with differing backgrounds and future pathways Providing accessible resources Choosing inclusive texts and learning examples Identifying and nurturing high achieving students	Leading by example – demonstrating inclusive practice with students and staff Providing opportunities and pathways for high achieving students
2. Advancement of the discipline	Instilling enthusiasm for the discipline amongst students	Raising awareness of opportunities for students to participate in discipline activities	Providing opportunities for students to participate in discipline activities such as seminars and summer programs
3. Commitment to incorporating the process and outcomes of relevant research, scholarship and/or professional practice	Valuing the use of discipline-specific education theory in teaching practices	Sourcing and sharing relevant discipline-specific knowledge and, teaching and learning research with members of the teaching team	Leading the integration of educational research into learning and teaching across the department Setting up conditions for learning and teaching research that could contribute to the literature
4. Commitment to development of learning communities	Creating and participating in learning and teaching communities Collaborating with unit coordinator and colleagues	Facilitating collaboration between teaching staff Facilitating and participating in peer observation and review	Leading the implementation of policies that support learning and teaching collaborations Leading and encouraging participation in learning and teaching communities
5. Commitment to encouraging participation in higher education, acknowledging diversity and promoting equality of opportunity	Directing students to support and resources	Monitoring student progress Negotiating support or alternative pathways for students at risk	Planning, implementing and raising awareness of study pathways and resources for students from a diversity of backgrounds
6. Commitment to continuing professional development and evaluation of practice	Engaging in reflective practice Seeking opportunities for professional development	Seeking opportunities for professional development for self and members of team	Creating opportunities for professional development across the department Promoting a scholarly approach to learning and teaching

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