B

Backward in Time Problems

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Overview

We define a backward in time problem as a boundary–final value problem associated with the linear theory of thermoelasticity. The final data are given at $t = 0$, and we are interested in extrapolating to previous times. It is like we see a movie from the end to the beginning, but it is a problem; we can imagine different beginnings for the same end. It is well known that this type of problem is ill posed (it fails to have a global solution, or it fails to have a unique solution, or the solution does not depend continuously on the data). In order to stabilize such kind of problems, many different techniques have been developed in literature such of those of "solving" ill–posed problems for equations of evolution. Some of these involve the altering of governing equations in such a way as to make such problems well posed. Others involve changing the initial and/or boundary conditions again in such way as to make the problems well posed. Still others involve constraining solutions to lie in a certain constraint set.

This class of problems was studied by Ames and Payne [1] who derived stabilizing criteria for solutions of the Cauchy problem for the standard equations of dynamical linear thermoelasticity backward in time. They also obtained inequalities establishing continuous dependence on the coupling parameter that links the elastic displacement field and the temperature field.

This type of problems has been initially considered by Serrin [2] who established uniqueness results for the Navier–Stokes equations backward in time. Explicit uniqueness and stability criteria for the classical Navier–Stokes equations backward in time were obtained by Knops and Payne [3] and Galdi and Straughan [4]. For an overview of improperly posed problems, the reader may look in the important study made by Ames and Straughan [5].

A study of uniqueness and continuous dependence upon mild requirements concerning the thermoelastic coefficients for the solution of the boundary–value problems associated with the linear theory of thermoelasticity has been made by Ciarletta [6]. These results have been extended for the theory of thermo-microstretch elastic materials by Bulgariu [7]. The spatial behavior of the thermoelastic process backward in time has been studied by Ciarletta and Chiriţă [8]. A time-weighted volume measure is used for establishing a first-order partial differential inequality which implies the spatial estimate describing the spatial exponential decay of the thermoelastic process backward in time. The asymptotic partition of energy for the thermoelastic process backward in time was obtained by Ciarletta and Chiriţă [9].

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The main purpose of this chapter is to present the backward in time problems associated with the linear theory of thermoelasticity. Firstly, we transform the boundary–final value problem on the time interval $(-t_0, 0]$ into a boundary–initial value problem on $[0, t_0)$, using the change of variable $t \rightsquigarrow -t$. Then we study the uniqueness, continuous dependence of the solutions of the boundary–final value problem with respect to the final data and establish some energy estimates.

The Boundary–Final Value Problem

We suppose that B is a body made by an anisotropic and inhomogeneous thermoelastic material. We consider the boundary–final value problem associated with the linear theory of thermoelasticity on the time interval $(-t_0, 0]$, $t_0 > 0$ may be infinity. The fundamental system of field equations consists of the strain–displacement relation

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

the constitutive equations

$$\sigma_{ij} = C_{ijkl}e_{kl} - M_{ij}\theta \tag{2}$$

$$q_i = -k_{ij}\theta_{,j} \tag{3}$$

in $\overline{B} \times (-t_0, 0]$, the equations of motion

$$\sigma_{ji,j} + \rho b_i = \rho \ddot{u}_i \tag{4}$$

and energy equation

$$-q_{i,i} - T_0 M_{ij}\dot{e}_{ij} + \rho r = c\dot{\theta} \tag{5}$$

in $B \times (-t_0, 0]$, where ρ is the density mass. Constitutive equations satisfy the symmetry relations:

$$C_{ijkl} = C_{klij} = C_{jikl}, \quad M_{ij} = M_{ji}, \quad k_{ij} = k_{ji} \tag{6}$$

The boundary–final value problem $(\mathcal{P})$ is defined by relations (1)–(5) with the final conditions

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = \dot{u}_i^0(\mathbf{x}) \quad \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}), \quad \mathbf{x} \in \overline{B} \tag{7}$$

and the boundary conditions

$$\begin{aligned} &u_i(\mathbf{x}, t) = \hat{u}_i(\mathbf{x}, t), \quad \text{on } \Sigma_1 \times (-t_0, 0] \\ &s_i(\mathbf{x}, t) \equiv \sigma_{ji}(\mathbf{x}, t)n_j(\mathbf{x}) = \hat{s}_i(\mathbf{x}, t), \text{ on } \Sigma_2 \times (-t_0, 0] \\ &\theta(\mathbf{x}, t) = \hat{\theta}(\mathbf{x}, t), \quad \text{on } \Sigma_3 \times (-t_0, 0] \\ &q(\mathbf{x}, t) \equiv q_i(\mathbf{x}, t)n_i(\mathbf{x}) = \hat{q}(\mathbf{x}, t), \text{ on } \Sigma_4 \times (-t_0, 0] \end{aligned} \tag{8}$$

where $u_i^0, \dot{u}_i^0, \theta^0, \hat{u}_i, \hat{s}_i, \hat{\theta}$, and $\hat{q}$ are prescribed functions and $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ are subsurfaces of ∂B, such that $\Sigma_1 \cup \overline{\Sigma}_2 = \Sigma_3 \cup \overline{\Sigma}_4 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$.

We now describe the work of Ciarletta [6] and consider that

$$\rho(\mathbf{x}) \geq \rho_0 > 0, \quad \mathbf{x} \in \overline{B} \tag{9}$$

$$k_{ij}\xi_i\xi_j \geq k_0\xi_i\xi_i > 0, \quad \forall \xi_i \tag{10}$$

where $k_0 > 0$ can be identified with the minimum of the positive eigenvalues of k_{ij} on $\overline{B}$.

Since we will consider the coupled theory of thermoelasticity, we will assume that

$$m = \sup_{\overline{B}} (M_{ij}M_{ij})^{1/2} > 0 \tag{11}$$

We also set

$$m^* = \sup_{\overline{B}} (M_{ir,r}M_{is,s})^{1/2} \geq 0 \tag{12}$$

Thus, we can deduce that

$$M_{ir}M_{is}\theta_{,r}\theta_{,s} \leq m^2\theta_{,i}\theta_{,i} \tag{13}$$

The Transformed Problem

Using the change of variable $t \rightsquigarrow -t$, we transform the boundary–final value problem $(\mathcal{P})$

into a boundary–initial value problem $(\mathcal{P}^*)$ defined by equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{14}$$

$$\sigma_{ij} = C_{ijkl}e_{kl} - M_{ij}\theta \tag{15}$$

$$q_i = -k_{ij}\theta_{,j} \tag{16}$$

in $\overline{B} \times [0, t_0)$ and

$$\sigma_{ji,j} + \rho b_i = \rho \ddot{u}_i \tag{17}$$

$$-q_{i,i} + T_0 M_{ij}\dot{e}_{ij} + \rho r = -c\dot{\theta} \tag{18}$$

in $B \times [0, t_0)$, with the initial conditions

$$\begin{aligned} u_i(\mathbf{x}, 0) &= u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = \dot{u}_i^0(\mathbf{x}) \\ \theta(\mathbf{x}, 0) &= \theta^0(\mathbf{x}), \quad \mathbf{x} \in \overline{B} \end{aligned} \tag{19}$$

and the boundary conditions

$$\begin{aligned} u_i(\mathbf{x}, t) &= \hat{u}_i(\mathbf{x}, t), \quad \text{on } \Sigma_1 \times [0, \mathrm{t}_0) \\ s_i(\mathbf{x}, t) &= \hat{s}_i(\mathbf{x}, t), \quad \text{on } \Sigma_2 \times [0, \mathrm{t}_0) \\ \theta(\mathbf{x}, t) &= \hat{\theta}(\mathbf{x}, t), \quad \text{on } \Sigma_3 \times [0, \mathrm{t}_0) \\ q(\mathbf{x}, t) &= \hat{q}(\mathbf{x}, t), \quad \text{on } \Sigma_4 \times [0, \mathrm{t}_0) \end{aligned} \tag{20}$$

We observe that only the energy equation has a different form in the two considered problems because only in this equation occurs the first-order derivative with respect to time.

By a solution of the boundary–initial value problem $(\mathcal{P}^*)$, Ciarletta [6] understands an ordered array $p = [u_i, e_{ij}, \sigma_{ij}, \theta, \theta_{,i}, q_i]$ with the components sufficient smooth for all the calculation.

Using some Lagrange–Brun identities, Ciarletta [6] established some auxiliary results concerning the solutions of the boundary–initial value problem $(\mathcal{P}^*)$.

Lemma 1. *Let $p = [u_i, e_{ij}, \sigma_{ij}, \theta, \theta_{,i}, q_i]$ be a solution of the boundary–initial value problem $(\mathcal{P}^*)$ corresponding to the external given data $\mathcal{D} = [b_i, r; u_i^0, \dot{u}_i^0, \theta^0; \hat{u}_i, \hat{s}_i, \hat{\theta}, \hat{q}]$. Then, for all $t \in [0, t_0)$, we have*

$$\begin{aligned} &\int_B [\rho \dot{u}_i(t)\dot{u}_i(t) + C_{ijkl}e_{ij}(t)e_{kl}(t)]dv \\ &= \int_B [\rho \dot{u}_i(0)\dot{u}_i(0) + C_{ijkl}e_{ij}(0)e_{kl}(0)]dv \\ &+2\int_0^t \int_B \rho b_i(\eta)\dot{u}_i(\eta)dvd\eta \\ &+2\int_0^t \int_{\partial B} s_i(\eta)\dot{u}_i(\eta)dad\eta \\ &+2\int_0^t \int_B M_{ij}\theta(\eta)\dot{e}_{ij}(\eta)dvd\eta \end{aligned} \tag{21}$$

$$\begin{aligned} &\int_B \frac{1}{T_0}c\theta^2(t)dv - 2\int_0^t \int_B \frac{1}{T_0}k_{ij}\theta_{,i}(\eta)\theta_{,j}(\eta)dvd\eta \\ &= \int_B \frac{1}{T_0}c\theta^2(0)dv - 2\int_0^t \int_B \frac{1}{T_0}\rho r(\eta)\theta(\eta)dvd\eta \\ &+ 2\int_0^t \int_{\partial B} \frac{1}{T_0}q(\eta)\theta(\eta)dad\eta \\ &- 2\int_0^t \int_B M_{ij}\theta(\eta)\dot{e}_{ij}(\eta)dvd\eta \end{aligned} \tag{22}$$

Lemma 2. *Let $p = [u_i, e_{ij}, \sigma_{ij}, \theta, \theta_{,i}, q_i]$ be a solution of the boundary–initial value problem $(\mathcal{P}^*)$ corresponding to the external given data $\mathcal{D} = [b_i, r; u_i^0, \dot{u}_i^0, \theta^0; \hat{u}_i, \hat{s}_i, \hat{\theta}, \hat{q}]$. Then, for all $t \in [0, \frac{t_0}{2})$, we have*

$$\begin{aligned} &\int_B \left\{\rho \dot{u}_i(t)\dot{u}_i(t) - \left[C_{ijkl}e_{ij}(t)e_{kl}(t) + \frac{1}{T_0}c\theta^2(t)\right]\right\}dv \\ &= \int_B \Big\{\rho \dot{u}_i(0)\dot{u}_i(2t) - \big[C_{ijkl}e_{ij}(0)e_{kl}(2t) \\ &+\frac{1}{T_0}c\theta(0)\theta(2t)\big]\Big\}dv + \int_0^t \int_B \Big\{\rho \dot{u}_i(t+\eta)b_i(t-\eta) \\ &-\rho \dot{u}_i(t-\eta)b_i(t+\eta) + \frac{1}{T_0}\rho\big[\theta(t+\eta)r(t-\eta) \\ &-\theta(t-\eta)r(t+\eta)\big]\Big\}dvd\eta + \int_0^t \int_{\partial B} \Big\{\dot{u}_i(t+\eta)s_i(t-\eta) \\ &-\dot{u}_i(t-\eta)s_i(t+\eta) + \frac{1}{T_0}\big[\theta(t-\eta)q(t+\eta) \\ &-\theta(t+\eta)q(t-\eta)\big]\Big\}dad\eta \end{aligned} \tag{23}$$

Proof. Let us consider $\eta \in [0, t], t \in [0, \frac{t_0}{2})$. If we integrate the identity

$$-\frac{\partial}{\partial\eta}[\rho\dot{u}_i(t-\eta)\dot{u}_i(t+\eta)] = \rho\dot{u}_i(t+\eta)\ddot{u}_i(t-\eta) - \rho\dot{u}_i(t-\eta)\ddot{u}_i(t+\eta) \tag{24}$$

we obtain

$$\rho\dot{u}_i(t)\dot{u}_i(t) = \rho\dot{u}_i(0)\dot{u}_i(2t) + \int_0^t \rho[\dot{u}_i(t+\eta)\ddot{u}_i(t-\eta) - \rho\dot{u}_i(t-\eta)\ddot{u}_i(t+\eta)]d\eta \tag{25}$$

Considering the relation (17) at the moment $t+\eta$ multiplied with $\dot{u}_i(t-\eta)$, combined with the same relation at the moment $t-\eta$, and multiplied with $\dot{u}_i(t+\eta)$, we obtain

$$\begin{aligned}\rho[\dot{u}_i(t+\eta)\ddot{u}_i(t-\eta) - \dot{u}_i(t-\eta)\ddot{u}_i(t+\eta)] &= \rho\dot{u}_i(t+\eta)b_i(t-\eta) - \rho\dot{u}_i(t-\eta)b_i(t+\eta) \\ &+ [\dot{u}_i(t+\eta)\sigma_{ji}(t-\eta) - \dot{u}_i(t-\eta)\sigma_{ji}(t+\eta)]_{,j} \\ &+ [\sigma_{ij}(t+\eta)\dot{e}_{ij}(t-\eta) - \sigma_{ij}(t-\eta)\dot{e}_{ij}(t+\eta)]\end{aligned} \tag{26}$$

In the same way like above, combining relations (15), (16), and (18), we get

$$\begin{aligned}\sigma_{ij}(t+\eta)\dot{e}_{ij}(t-\eta) - \sigma_{ij}(t-\eta)\dot{e}_{ij}(t+\eta) &= -\frac{\partial}{\partial\eta}\left[C_{ijkl}e_{ij}(t-\eta)e_{ij}(t+\eta)\right] \\ &- \frac{\partial}{\partial\eta}\left[\frac{1}{T_0}c\theta(t-\eta)\theta(t+\eta)\right] \\ &+ \frac{1}{T_0}\rho[\theta(t+\eta)r(t-\eta) - \theta(t-\eta)r(t+\eta)] \\ &+ \left\{\frac{1}{T_0}[\theta(t-\eta)q_i(t+\eta) - \theta(t+\eta)q_i(t-\eta)]\right\}_{,j}\end{aligned} \tag{27}$$

Finally, combining relations (25)–(27), integrating over B the result and using the divergence theorem, we get relation (23) and the proof is complete. This proof is an example of using some Lagrange–Brun identities.

Uniqueness Results

In this section, we present some results established by Ciarletta [6] regarding the uniqueness of the boundary–initial value problem $(\mathcal{P}^*)$ under various assumptions of the thermoelastic coefficients and the density mass. He assumed that symmetry relations (6) are satisfied and that meas $\Sigma_4 = 0$. He considered the hypotheses:

$(\mathbf{H_1})$ Relations (9) and (10) are true.

$(\mathbf{H_2})$ C_{ijkl} is a positive semidefinite tensor.

$(\mathbf{H_3})$ c is nonpositive.

The hypotheses $(\mathbf{H_1})$ and $(\mathbf{H_2})$ are used extensively in the studies on the thermoelastic problems. Knops and Payne [3] outlined that hypothesis $(\mathbf{H_3})$ is reasonable and can characterize a certain state supported by the thermoelastic body.

Theorem 1. *Assuming that the hypotheses* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold true, then the boundary–initial problem* $(\mathcal{P}^*)$ *has at most one solution.*

Proof. Let $p^{(\alpha)} = [u_i^{(\alpha)}, e_{ij}^{(\alpha)}, \sigma_{ij}^{(\alpha)}, \theta^{(\alpha)}, \theta_{,i}^{(\alpha)}, q_i^{(\alpha)}]$ be two solutions of the boundary–initial problem $(\mathcal{P}^*)$ corresponding to the same external given data $(\mathcal{D})$. Then their difference $p = p^{(1)} - p^{(2)}$ is a solution for $(\mathcal{P}^*)$ corresponding to the null external given data. We will demonstrate that solution p is the null solution.

Using the results from the two lemmas presented in the previous section, we deduce that

$$\int_B \rho\dot{u}_i(t)\dot{u}_i(t)dv = \int_0^t \int_B \frac{1}{T_0}k_{ij}\theta_{,i}(\eta)\theta_{,j}(\eta)dvd\eta \tag{28}$$

for $t \in [0, \frac{t_0}{2})$. We note that meas $\Sigma_4 = 0$ implies that $\theta(\mathbf{x}, t) = 0$ on $\partial B \times [0, t_0)$, and hence, we have

$$-2\int_0^t \int_B M_{ij}\theta(\eta)\dot{e}_{ij}(\eta)dvd\eta = 2\int_0^t \int_B [M_{ij,j}\theta(\eta)\dot{u}_i(\eta) + M_{ij}\theta_{,j}(\eta)\dot{u}_i(\eta)]dvd\eta \tag{29}$$

and also [1]

$$\int_B \theta_{,i}(t)\theta_{,i}(t)dv \geq \lambda \int_B \theta^2(t)dv, \quad \lambda = const. > 0 \tag{30}$$

where λ is the first eigenvalue of clamped membrane problem.

If we use the hypotheses $(\mathbf{H_1})$ and $(\mathbf{H_2})$, the Schwarz's inequality, and relations (28)–(30), we conclude that

$$\begin{aligned} &\int_B \left[\rho\dot{u}_i(t)\dot{u}_i(t) + C_{ijkl}e_{ij}(t)e_{kl}(t)\right]dv \\ &\leq \left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)\int_0^t \int_B \rho\dot{u}_i(\eta)\dot{u}_i(\eta)dvd\eta \\ &+ \frac{T_0}{\rho_0 k_0}\left(\varepsilon_1\lambda^{-1}m^{*2}+\varepsilon_2 m^2\right)\int_B \rho\dot{u}_i(t)\dot{u}_i(t)dv \end{aligned} \tag{31}$$

for all $\varepsilon_1, \varepsilon_2 > 0$ and with $t \in [0, \frac{t_0}{2})$. We further choose the arbitrary parameters $\varepsilon_1, \varepsilon_2$ so small to have

$$\beta \equiv 1 - \frac{T_0}{\rho_0 k_0}\left(\varepsilon_1\lambda^{-1}m^{*2}+\varepsilon_2 m^2\right) > 0 \tag{32}$$

From (31) and (32), we deduce

$$\begin{aligned} &\int_B \rho\dot{u}_i(t)\dot{u}_i(t)dv \leq \frac{1}{\beta}\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right) \\ &\times \int_0^t \int_B \rho\dot{u}_i(\eta)\dot{u}_i(\eta)dvd\eta \end{aligned} \tag{33}$$

for $t \in [0, \frac{t_0}{2})$, and using Gronwall's lemma, we obtain

$$\int_B \rho\dot{u}_i(t)\dot{u}_i(t)dv = 0, \quad t \in \left[0, \frac{t_0}{2}\right) \tag{34}$$

This relation is true only if p is the null solution over $B \times [0, \frac{t_0}{2})$. If $t_0 = \infty$, then we have uniqueness of solutions of $(\mathcal{P}^*)$. Otherwise, for $t_0 < \infty$, we repeat the procedure of proof on the interval $[\frac{t_0}{2}, t_0)$ and get the null solution over $B \times [\frac{t_0}{2}, \frac{3t_0}{4})$. Continuing this procedure, we demonstrate the uniqueness.

Using the same techniques, we can also prove the following results:

Theorem 2. *Assuming that the hypotheses* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$ *hold true, then the boundary–initial problem* $(\mathcal{P}^*)$ *has at most one solution.*

Corollary 1. *Assuming that the hypotheses* $(\mathbf{H_1})$, $(\mathbf{H_2})$, *and* $(\mathbf{H_3})$ *hold true, then the boundary–initial problem* $(\mathcal{P}^*)$ *has at most one solution.*

In the proof of the final corollary, we don't need any procedures of extension of the solution like in the proofs of the theorems.

Continuous Dependence with Respect to the Final Data

In this section, we describe two results of continuous dependence with respect to the final data who becomes initial data for the transformed boundary–initial problem $(\mathcal{P}^*)$. First, we derive the continuous dependence inequality valid on the whole interval $[0, T)$ considering only the hypotheses $(\mathbf{H_1})$, $(\mathbf{H_2})$, and $(\mathbf{H_3})$ (from the previous section) without imposing any constraint restriction upon the solution. This result was obtained by Ciarletta [6].

Theorem 3. *Suppose that the hypotheses* $(\mathbf{H_1})$, $(\mathbf{H_2})$, *and* $(\mathbf{H_3})$ *hold true. Let* $[u_i, e_{ij}, \sigma_{ij}, \theta, \theta_{,i}, q_i]$ *be a solution of the boundary–initial value problem* $(\mathcal{P}^*)$ *corresponding to the external given data* $\mathcal{D}_0 = [0, 0; u_i^0, \dot{u}_i^0, \theta^0; 0, 0, 0, 0]$. *Then we have the estimate*

$$\varepsilon(t) + \int_0^t \int_B \frac{1}{T_0} k_{ij}\theta_{,i}(\eta)\theta_{,j}(\eta)dvd\eta \leq \varepsilon(0)e^{Mt} \tag{35}$$

$$\forall t \in [0, t_0)$$

where

$$\varepsilon(t) = \int_B \left[\rho \dot{u}_i(t)\dot{u}_i(t) + C_{ijkl}e_{ij}(t)e_{kl}(t) - \frac{1}{T_0}c\theta^2(t)\right]dv \tag{36}$$

$$M = \frac{8T_0}{\rho_0 k_0}(\lambda^{-1}m^{*2} + m^2) \tag{37}$$

Proof. We easily note, because of the linearity of the problem $(\mathcal{P}^*)$, that the continuous dependence under perturbations of arbitrary initial data is equivalent to the stability of the null solution. Combining the conclusions of the Lemma 1, written for the external given data $\mathcal{D}_0$, with the relation (36), we deduce

$$\varepsilon(t) + 2\int_0^t \int_B \frac{1}{T_0}k_{ij}\theta_{,i}(\eta)\theta_{,j}(\eta)dvd\eta = \varepsilon(0) - 4\int_0^t \int_B [M_{ij,j}\theta(\eta) + M_{ij}\theta_{,j}(\eta)]\dot{u}_i(\eta)dvd\eta \tag{38}$$

for $t \in [0, t_0)$. Now, using Schwarz's inequality and the arithmetic–geometric mean inequality combined with relations (11)–(13), we get

$$\begin{aligned}
&4\int_0^t \int_B [M_{ij,j}\theta(\eta) + M_{ij}\theta_{,j}(\eta)]\dot{u}_i(\eta)dvd\eta \\
&\le \frac{2}{\varepsilon}\int_0^t \int_B \rho\dot{u}_i(\eta)\dot{u}_i(\eta)dvd\eta \\
&+ \frac{2\varepsilon}{\rho_0}\int_0^t \int_B [M_{ir,r}\theta(\eta) + M_{ir}\theta_{,r}(\eta)] \\
&\times [M_{is,s}\theta(\eta) + M_{is}\theta_{,s}(\eta)]dvd\eta \\
&\le \frac{2}{\varepsilon}\int_0^t \int_B \rho\dot{u}_i(\eta)\dot{u}_i(\eta)dvd\eta + \frac{4\varepsilon T_0}{\rho_0 k_0}(\lambda^{-1}m^{*2} + m^2) \\
&\times \int_0^t \int_B \frac{1}{T_0}k_{ij}\theta_{,i}(\eta)\theta_{,j}(\eta)dvd\eta, \quad \forall\varepsilon > 0
\end{aligned} \tag{39}$$

It is now convenient to take

$$\varepsilon = \frac{\rho_0 k_0}{4T_0(\lambda^{-1}m^{*2} + m^2)} \tag{40}$$

in inequality (39) and use the result in relation (38) to get

$$\begin{aligned}
&\varepsilon(t) + \int_0^t \int_B \frac{1}{T_0}k_{ij}\theta_{,i}(\eta)\theta_{,j}(\eta)dvd\eta \\
&\le \varepsilon(0) + M\int_0^t \int_B \rho\dot{u}_i(\eta)\dot{u}_i(\eta)dvd\eta, \\
&t \in [0, t_0)
\end{aligned} \tag{41}$$

This final relation obtained is a Gronwall-type inequality that produces the relation (35), and the proof is complete.

The second result of continuous dependence with respect to the final data was established by Ames and Payne [1]. The conditions imposed by them are the following: strain energy form is positive definite so that

$$\int_B C_{ijkl}\phi_{ij}(\mathbf{x})\phi_{kl}(\mathbf{x})dv \ge 0, \quad \forall\phi_{ij}(\mathbf{x}) \tag{42}$$

the hypotheses (9)–(12) are again made, and $c > 0$.

Ames and Payne [1] introduce the class of functions ψ defined on $B \times [0, t_0)$ that satisfy

$$\max_{t\in[0,t_0)} \int_B \psi^2(t)dv \le G^2 \tag{43}$$

for a prescribed constant G, and they assume that the temperature θ belongs to this class.

The continuous dependence inequality obtained

$$E(t) \le \{4[G^2 + E(0)]^{1/2}E(0)^{1/2} + 3E_1(0)\}e^{4K^2t/k_0} \tag{44}$$

is valid for $0 \le t \le \frac{t_0}{2}$, where

$$E_1(t) = \frac{1}{2}\int_B [\rho\dot{u}_i(t)\dot{u}_i(t) + C_{ijkl}u_{i,j}(t)u_{k,l}(t)]dv \tag{45}$$

$$E(t) = E_1(t) + \frac{1}{2}\int_B \theta^2(t)dv \tag{46}$$

$$K = \max_B \left\{ \sqrt{\frac{M_{ij}M_{ij}}{\rho}}, \sqrt{\frac{M_{ir,r}M_{is,s}}{\rho\lambda}} \right\} \quad (47)$$

Other Uniqueness Results

In addition to the results presented in the section *Uniqueness Results*, Chiriţă [10] studied the uniqueness of solution for the thermoelastic processes backward in time in an appropriate class of displacement–temperature fields. The density mass and the specific heat are considered strictly positive, and the conductivity tensor is assumed positive definite. In what follows, we present the two main theorems established by Chiriţă [10], just sketching the proofs. From this moment, we will consider $t_0 = \infty$.

Theorem 4. *Suppose that the density mass* ρ *is strictly positive on* B, k_{ij} *is a positive-definite tensor,* C_{ijkl} *is a negative semidefinite tensor, and* $c > 0$ *in* B. *Then there exists a strictly positive constant* α *so that in the class of thermoelastic processes* $[u_i, e_{ij}, \sigma_{ij}, \theta, \theta_{,i}, q_i]$, *defined on* $\overline{B} \times [0, \infty)$, *that satisfy*

$$\int_0^t \int_B \frac{1}{T_0} k_{ij}\theta_{,i}(\eta)\theta_{,j}(\eta)dvd\eta \le M_1^2 e^{\alpha t}, \ \forall t \in [0, \infty) \quad (48)$$

with $M_1 = const.$, *the boundary–initial value problem* $(\mathcal{P}^*)$ *has at most one solution, provided meas* $\Sigma_3 \neq 0$.

Proof. Assuming that k_{ij} is a positive-definite tensor, relation (10) holds true.

Let us consider $[u_i, e_{ij}, \sigma_{ij}, \theta, \theta_{,i}, q_i]$ a solution of the boundary–initial value problem $(\mathcal{P}^*)$ corresponding to zero external given data. For proving the uniqueness of the solution, we will demonstrate that $[u_i, e_{ij}, \sigma_{ij}, \theta, \theta_{,i}, q_i] = \mathbf{0}$ in $\overline{B} \times [0, \infty)$. Combining the results from Lemmas 1 and 2, one can obtain

$$\int_B \left[C_{ijkl}e_{ij}(t)e_{kl}(t) + \frac{1}{T_0}c\theta^2(t) \right] dv = \int_0^t \int_B \frac{1}{T_0} k_{ij}\theta_{,i}(\eta)\theta_{,j}(\eta)dvd\eta, \ t \in [0, \infty) \quad (49)$$

Since the elasticity tensor C_{ijkl} is a negative semidefinite tensor, it follows, from the identities (10), (30), and (49), that

$$\int_0^t \int_B \frac{1}{T_0} k_{ij}\theta_{,i}(\eta)\theta_{,j}(\eta)dvd\eta \le \omega^2 \int_B \frac{1}{T_0} k_{ij}\theta_{,i}(t)\theta_{,j}(t)dv, \ t \in [0, \infty) \quad (50)$$

with

$$\omega^2 = \frac{\sup_{\overline{B}} c}{\lambda k_0} \quad (51)$$

By setting

$$\varphi^2(t) = \int_0^t \int_B \frac{1}{T_0} k_{ij}\theta_{,i}(\eta)\theta_{,j}(\eta)dvd\eta, \ t \in [0, \infty) \quad (52)$$

relation (51) becomes

$$\varphi^2(t) \le 2\omega^2\varphi(t)\dot{\varphi}(t), \ t \in [0, \infty) \quad (53)$$

We can now discuss the differential inequality just obtained. If $\varphi(t) = 0$ for all $t \in [0, \infty)$, then it follows that

$$\theta_{,i}(\mathbf{x}, t) = 0, \text{ in } B \times [0, \infty) \quad (54)$$

and hence, recalling that meas $\Sigma_3 \neq 0$, we deduce that

$$\theta(\mathbf{x}, t) = 0, \text{ in } \overline{B} \times [0, \infty) \quad (55)$$

Further, using relation (53) into identity (28) and using the zero initial conditions, we get

$$u_i(\mathbf{x}, t) = 0, \text{ in } \overline{B} \times [0, \infty) \quad (56)$$

and the uniqueness result is obtained.

Let us suppose now that there exist $\tau \in (0, \infty)$ so that $\varphi(\tau) > 0$, and hence, we have

$$\varphi(t) > 0, \forall t \in [\tau, \infty) \tag{57}$$

Multiplying relation (53) with $e^{-t/(2\omega^2)}$, we obtain

$$\frac{d}{dt}[\varphi(t)e^{-t/(2\omega^2)}] \geq 0, \forall t \in [\tau, \infty) \tag{58}$$

and hence, we deduce

$$\begin{aligned} &\varphi(\tau)e^{-\tau/(2\omega^2)} \leq \varphi(t)e^{-t/(2\omega^2)} \\ &\leq \lim_{t\to\infty} [\varphi(t)e^{-t/(2\omega^2)}], \forall t \in [\tau, \infty) \end{aligned} \tag{59}$$

Choosing that

$$0 \leq \alpha \leq \frac{1}{\omega^2} \tag{60}$$

then from (48), we have

$$\lim_{t\to\infty} [\varphi(t)e^{-t/(2\omega^2)}] = 0 \tag{61}$$

and therefore $\varphi(t) = 0$, for all $t \in [\tau, \infty)$, in contradiction with our assumption expressed by (57). Thus, $\varphi(t) = 0$, and so we have the uniqueness result.

This result of uniqueness for the backward in time processes associated with the linear theory of thermoelasticity completes the study made by Ciarletta [6]. To obtain this result of uniqueness for $c > 0$, it is necessary to assume that the solution belongs to the class of thermoelastic processes $[u_i, e_{ij}, \sigma_{ij}, \theta, \theta_{,i}, q_i]$ that satisfy the constraint (48). The next theorem shows in what conditions the result of uniqueness remains valid if we remove the condition that C_{ijkl} is a negative semidefinite tensor.

Theorem 5. *Suppose that the density mass ρ is strictly positive on B, k_{ij} is a positive-definite tensor, $c > 0$ in B, and meas $\Sigma_4 = 0$. Then, the boundary–initial value problem $(\mathcal{P}^*)$ has at most one solution that lies in the class of thermoelastic processes $[u_i, e_{ij}, \sigma_{ij}, \theta, \theta_{,i}, q_i]$, defined on $\overline{B} \times [0, \infty)$, that satisfy*

$$\lim_{t\to\infty} \int_0^t \int_0^\eta \int_B \frac{1}{T_0} k_{ij}\theta_{,i}(\tau)\theta_{,j}(\tau)dv d\tau d\eta = 0 \tag{62}$$

The assumption embedded in (62) is more restrictive than given in (48), and so the class of thermoelastic processes $[u_i, e_{ij}, \sigma_{ij}, \theta, \theta_{,i}, q_i]$, where the uniqueness result holds true, is essentially reduced.

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Ballistic-Diffusive Approximation: A New Look

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Overview

Boltzmann transport equation (BTE)-based modeling of thermal transport has been a topic of recent interest. Numerous models have emerged that attempt to explain the diffusive, wavelike, and ballistic heat transfer processes that exist in solids due to material heterogeneity, size effects, and the like [1, 2]. The Equation of Phonon Radiative Transport (EPRT) has emerged as a promising model for thermal transport simulations in dielectric media [3]. When modeling materials and devices with EPRT, the primary variable is a function of not only space and time but also velocity. This eliminates the ability to apply standard discretization methods in space and time without first dealing with discretizing this velocity component. The popular discrete ordinates method is accurate but adds numerous degrees of freedom per solution node [4]. The ballistic-diffusive approximation (BDA) has been formulated as an approximate solution method to the EPRT for the purpose of avoiding difficulties associated with velocity discretization [5]. The BDA claims to provide an efficient alternative for applications with complex geometry requiring many degrees of freedom where solving the EPRT would become cumbersome.

In the following exposition, we give an alternative derivation of the ballistic-diffusive equations. We find it to be more simple and transparent than the original derivation. We clearly state all assumptions made in this approach making it more comprehensible and enlightening. We then give a one-dimensional thin solid film problem as an illustrative example. A complete numerical formulation in space and time, that is second-order accurate and has controllable numerical dissipation, is given next. We close with concluding remarks.

Governing Equations

The Boltzmann transport equation can be used to accurately model out-of-equilibrium systems where a particle description of the energy carriers is appropriate [6]. Thermal transport in solids can be described as particle driven when quantization is considered. Phonons are quasiparticles that result from the quantization of vibrational modes in a solid crystal. Thus, the BTE describes thermal transport in a dielectric solid. The BTE states that the distribution of phonons, $f = f(\mathbf{r}, \mathbf{v}, t)$, at any location in phase space, $(\mathbf{r}, \mathbf{v} = ||\mathbf{v}||\hat{\mathbf{s}})$, and point in time, t, is governed by

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{r}}\frac{d\mathbf{r}}{dt} + \frac{\partial f}{\partial \mathbf{v}}\frac{d\mathbf{v}}{dt} = \left(\frac{\partial f}{\partial t}\right)_{\text{collision}} \tag{1}$$

Note that the term on the right-hand side of (1) is not a literal partial derivative. It represents the change in distribution function due to scattering of particles – in our case, phonons. Like the EPRT, the BDA is derived from this equation. The essential assumption of the BDA is that the distribution of thermal carriers can be split into two different types – ballistic and diffusive (denoted by a subscript b and d, respectively), such that the total distribution function is given by

$$f = f_b + f_d \tag{2}$$

It is important to note the definitions of these carriers. The general physical description is as follows: Ballistic transport is a nonlocal phenomenon where phonons travel without scattering. This invalidates the temperature gradient assumption inherent to the Fourier and Cattaneo models of heat flux. Diffusive transport is a local effect where the gradient assumption is appropriate. The convention used in the derivation of the BDA assumes ballistic carriers are those phonons which have left their source and have yet to scatter.

Definitions

We define internal energy, u, and heat flux, $\mathbf{q}$, in the manner standard to solid-state physics [7]

$$u = \frac{1}{4\pi} \int_{4\pi} \int_{\omega} \hbar\omega (f_b + f_d) D d\omega d\Omega \\ = u_b + u_d \tag{3}$$

$$\mathbf{q} = \frac{1}{4\pi} \int_{4\pi} \int_{\omega} \mathbf{v}\hbar\omega (f_b + f_d) D d\omega d\Omega \\ = \mathbf{q}_b + \mathbf{q}_d \tag{4}$$

where $\hbar$ is Planck's constant divided by 2π, ω is the phonon frequency ($\hbar\omega$ being the fundamental unit of vibrational energy in the crystal), $D = D(\omega)$ is the phonon density of states, and Ω is the solid angle.

The definition of temperature in an out-of-equilibrium, nonlocal system is problematic. When using the BDA, there are two conceptual pitfalls. Firstly, the system we are considering is, by design, out of thermodynamic equilibrium. Thus, no real temperature can be assigned to the entire film. We also cannot, as in classical heat transfer which assumes local thermodynamic equilibrium, assign a local temperature. Thus the temperature, T, should be understood only as a measure of the local internal energy of the solid. Temperature is related to the internal energy by the volumetric heat capacity, C. When using the BDA, we consider two simultaneous temperatures (ballistic and diffusive) which make up the total temperature

$$CT = C(T_b + T_d) = u_b + u_d = u \tag{5}$$

with the caveat that this definition, while simple, presents another conceptual problem – temperatures are not additive. Again, the notion of the temperatures, T_b and T_d, simply being a measure of the internal energy provides a reasonable interpretation of this definition. Two different conventions have been used in previous derivations of BDA with respect to temperature. In [5], this two-temperature notion is used. To circumvent the problem of temperature additivity, a second convention was used in [8] which defines $u_b + u_d = CT$. For simplicity in the derivation of the governing equations, we use the temperature additivity convention.

Diffusive Part

To derive a constitutive relationship between heat flux and temperature for diffusive phonons, we begin with the BTE, (1). This equation can be simplified by assuming there are no external forces on the phonons which eliminates the third term. We also replace the right-hand side with the relaxation time approximation resulting in

$$\frac{\partial f_d}{\partial t} + \mathbf{v} \cdot \nabla f_d = \frac{f^0 - f_d}{\tau} \tag{6}$$

where f^0 represents the equilibrium distribution function given for phonons by the Planck distribution. Multiplying (6) by $\frac{1}{4\pi}\mathbf{v}\hbar\omega D\tau$ then integrating over frequency and solid angle and applying (4) gives the desired result. We consider each term individually. For the time derivative term,

$$\int_{4\pi} \int_{\omega} \frac{1}{4\pi} \mathbf{v}\hbar\omega D\tau \frac{\partial f_d}{\partial t} d\omega d\Omega \\ = \tau \frac{\partial}{\partial t} \left[\frac{1}{4\pi} \int_{4\pi} \int_{\omega} \mathbf{v}\hbar\omega f_d D d\omega d\Omega \right] = \tau \frac{\partial \mathbf{q}_d}{\partial t} \tag{7}$$

where we have assumed that $\mathbf{v}, \omega, D$, and τ are all time independent. For the gradient term, we get

$$\int_{4\pi} \int_{\omega} (\mathbf{v} \cdot \mathbf{v}) \frac{1}{4\pi} \hbar\omega D\tau \nabla f_d d\omega d\Omega \\ = \frac{1}{4\pi} \int_{4\pi} \int_{\omega} \|\mathbf{v}\|^2 \tau\hbar\omega D \frac{df^0}{dT} d\omega d\Omega \; \nabla T_d = k \nabla T_d \tag{8}$$

An important assumption has been made here (touched on in section "Definitions") that says that the change in distribution function in space can be approximated by a temperature gradient, $\nabla f_d = \frac{df^0}{dT_d} \nabla T_d$. The introduction of this expression brings with it the assumption of local thermodynamic equilibrium. For carriers that are behaving diffusively – constantly scattering and contributing to the local internal energy of the crystal – this is a reasonable assumption. We have also defined the thermal conductivity following from [9] with an added directional dependence:

$$k = \frac{1}{4\pi}\int_{4\pi}\int_{\omega}||\mathbf{v}||^2\tau\hbar\omega D\frac{df^0}{dT_d}d\omega d\Omega \tag{9}$$

To treat the equilibrium term, we must assume velocity to be an odd function of direction, i.e., $\mathbf{v}(-\hat{\mathbf{s}}) = -\mathbf{v}(\hat{\mathbf{s}})$. Then since f^0 is an even function of direction, the integral of the product over direction must be zero:

$$\frac{1}{4\pi}\int_{4\pi}\int_{\omega}\mathbf{v}\hbar\omega D f^0 d\omega d\Omega = 0 \tag{10}$$

Finally, we have the last term which simply yields the definition of heat flux:

$$\frac{1}{4\pi}\int_{4\pi}\int_{\omega}\mathbf{v}\hbar\omega D f_d d\omega d\Omega = \mathbf{q}_d \tag{11}$$

We are left with what is known as the Cattaneo definition of heat flux:

$$\mathbf{q}_d + \tau\frac{\partial\mathbf{q}_d}{\partial t} = -k\nabla T_d \tag{12}$$

We now have a constitutive relation for the diffusive heat flux which will be used later when considering conservation of energy in a differential volume of dielectric material to obtain a governing equation for the diffusive transport.

Ballistic Part

To handle the ballistic part of the distribution function, we consider a relaxation time BTE with no source term

$$\frac{\partial f_b}{\partial t} + \mathbf{v}\cdot\nabla f_b = \frac{-f_b}{\tau} \tag{13}$$

The absence of the equilibrium distribution function in (13) can be understood by considering that, by our definition of ballistic, no phonons governed by this equation have been scattered. Thus, none of these phonons can be brought to equilibrium and reemitted. While the interpretation of f^0 as an equilibrium distribution function (such as a Planck distribution for phonons) is common, in the sense of EPRT this term represents the contribution to a distribution at some location resulting from scattering in other areas of phase space. By definition, these phonons are not to be included as ballistic carriers, hence the term's absence. Conveniently, (13) has a known solution given by

$$f_b(\mathbf{r},\hat{\mathbf{s}},t) = f_w\left(t - \frac{s-s_0}{||\mathbf{v}||}, \mathbf{r} - (s-s_0)\hat{\mathbf{s}}\right) \exp\left(-\int_{s_0}^{s}\frac{1}{||\mathbf{v}||\tau}ds\right) \tag{14}$$

where f_w is the distribution of thermal carriers at a boundary or source and $s - s_0$ is a distance along the direction, $\hat{\mathbf{s}}$. The time delay in this function is due to the finite speed of the phonons.

For future use we derive another relation from (13) by multiplying it by $\frac{1}{4\pi}\hbar\omega D\tau$ and integrating over frequency and solid angle. Considering term by term,

$$\tau\frac{\partial}{\partial t}\left[\frac{1}{4\pi}\int_{4\pi}\int_{\omega}\hbar\omega f_b D d\omega d\Omega\right] = \tau\frac{\partial u_b}{\partial t} = C\tau\frac{\partial T_b}{\partial t} \tag{15}$$

by definitions (3) and (5).

$$\tau\nabla\cdot\left[\frac{1}{4\pi}\int_{4\pi}\int_{\omega}\mathbf{v}\hbar\omega f_b D d\omega d\Omega\right] = \tau\nabla\cdot\mathbf{q}_b \tag{16}$$

and trivially,

$$\frac{1}{4\pi}\int_{4\pi}\int_{\omega}\hbar\,\omega f_b D d\omega d\Omega = u_b = CT_b \tag{17}$$

resulting in

$$C\tau\frac{\partial T_b}{\partial t} + \tau\nabla\cdot\mathbf{q}_b + CT_b = 0 \tag{18}$$

Although this equation could be solved, it is not necessary since (14) along with (3) and (4) entirely describes the ballistic contribution of the heat carriers. It has been given only for use in the derivation of the governing equation for diffusive carriers given in the next section.

Energy Balance

In order to obtain a governing equation for diffusive thermal transport in space and time, we impose that energy is conserved for any differential volume in space:

$$\frac{\partial u}{\partial t} = C\left(\frac{\partial T_b}{\partial t} + \frac{\partial T_d}{\partial t}\right) = -\nabla \cdot \mathbf{q} + S$$
$$= -\nabla \cdot \mathbf{q}_b - \nabla \cdot \mathbf{q}_d + S \quad (19)$$

where S is a volumetric heat source in the film (note that phonons generated by a source within the film behave ballistically much like phonons generated from the surface. Treatment of this phenomenon is beyond the scope of this work, but can be found in [10]). This statement represents the last relationship needed to establish a governing equation for the diffusive carriers. While we do not write out the steps, the following equation is obtained by substituting first law and the energy conservation statement (19) and its time derivative into the divergence of (12) to eliminate $\mathbf{q}_d$. Three of the four remaining ballistic terms can be removed using (18) resulting in the governing equation for diffusive carriers

$$C\tau\frac{\partial^2 T_d}{\partial t^2} + C\frac{\partial T_d}{\partial t} = k\nabla^2 T_d - \nabla \cdot \mathbf{q}_b + S + \tau\frac{\partial S}{\partial t} \quad (20)$$

This equation is only in terms of T_d since $\mathbf{q}_b$ is given by (14) and (4):

$$\mathbf{q}_b(\mathbf{r}, t) = q_b\,\hat{\mathbf{s}}_q = \frac{1}{4\pi}\int_{4\pi}\int_{\omega} ||\mathbf{v}||\hbar\omega D f_w \left(t - \frac{s - s_0}{||\mathbf{v}||}, \mathbf{r} - (s - s_0)\hat{\mathbf{s}}\right) \exp\left(-\int_{s_0}^{s}\frac{1}{||\mathbf{v}||\tau}ds\right)(\hat{\mathbf{s}}_q \cdot \hat{\mathbf{s}})\,d\omega d\Omega \quad (21)$$

Boundary Conditions

The boundary conditions of the BDA are of utmost importance in its derivation and application. For the ballistic part, a prescribed distribution function is specified at the boundary

$$f_b(\mathbf{r}_w, \hat{\mathbf{s}}, t) = f_w(\hat{\mathbf{s}}, t) \quad (22)$$

where $\mathbf{r}_w$ is a location on the boundary. No phonons emitted from the wall contribute to the diffusive component. Thus, only energy carriers incident upon the boundary contribute to the heat flux so

$$\mathbf{q}_d \cdot \hat{\mathbf{n}} = \frac{1}{4\pi}\int_{4\pi}\int_{\omega}\hbar\omega f D(\mathbf{v}\cdot\hat{\mathbf{n}})\,d\omega d\Omega \quad (23)$$

By approximating the carrier distribution as isotropic with respect to direction and integrating over the resulting half sphere gives, using (3),

$$\mathbf{q}_d \cdot \hat{\mathbf{n}} = -\frac{1}{2}||\mathbf{v}||u_d \quad (24)$$

Further details on this boundary condition can be found in [8] and [11]. By dotting the Cattaneo expression for the heat flux of diffusive carriers, (12), with the wall unit normal, $\hat{\mathbf{n}}$, and substituting in the above expression, we get a boundary condition for the diffusive carriers

$$||\mathbf{v}||CT_d + ||\mathbf{v}||C\tau\frac{\partial T_d}{\partial t} = 2k\nabla T_d \cdot \hat{\mathbf{n}} \quad (25)$$

The right-hand side of this equation is particularly important because it naturally arises in the finite-element implementation of (20).

This completes the derivation of the BDA. Solution to (20) with the ballistic source given by (21), subject to the boundary condition (25), and initial conditions

$$T_d(\mathbf{r}_w, 0) = T_{d0} \quad (26)$$

$$\dot{T}_d(\mathbf{r}_w, 0) = \dot{T}_{d0} \quad (27)$$

gives the diffusive temperature everywhere at any time. The ballistic temperature can be computed from (14) with the prescribed boundary conditions (22).

Example Problem

The problem we consider is a thin, dielectric film of thickness, L, at an initial temperature of $T = T_0 = T_{b0} + T_{d0}$. At time, $t = 0$, the left boundary at $\eta = 0$ is raised to a temperature of $T = T_1 = T_{b1} + T_{d1}$. Transport in this film can be considered one-dimensional. For simplicity, phonon velocity is assumed uniform in every direction, i.e., not a function of direction. As a result, we may consider propagation direction to be only a function of polar angle, θ. We define θ as the angle between the unit vector in the x -direction and the propagation direction. This prompts the introduction of the directional cosine, $\mu = \cos\,\theta$. The integrations over solid angle in our definitions of internal energy (3) and heat flux (4) are now carried out over the domain $\mu = [-1, 1]$ and normalized by a factor of $\frac{1}{2}$. We introduce the following non-dimensionalization:

$$\begin{aligned} &\xi = \frac{t}{\tau} \qquad \eta = \frac{x}{L} \qquad Kn = \frac{\lambda}{L} \\ &\theta = \frac{T - T_0}{T_1 - T_0} = \frac{T - T_0}{\Delta T} \\ &\theta_b = \frac{T_b - T_{b0}}{\Delta T} = \frac{u_b - u_{b0}}{C\Delta T} \qquad \theta_d = \frac{T_d - T_{d0}}{\Delta T} \\ &\mathrm{q}_b = \frac{q_b - q_{b0}}{C\|\mathbf{v}\|\Delta T} \end{aligned} \tag{28}$$

where Kn is the Knudsen number relating the mean free path of the thermal carrier, λ, to the characteristic domain size. We also assume the classical relation, $k = \frac{1}{3}C||\mathbf{v}||\lambda$. Substitution into (20) yields

$$\frac{\partial^2\theta_d}{\partial\xi^2} + C\frac{\partial\theta_d}{\partial\xi} - \frac{Kn^2}{3}\frac{\partial^2\theta_d}{\partial\eta^2} = -Kn\frac{\partial \mathrm{q}_b}{\partial\eta} \tag{29}$$

This equation is subject to the boundary conditions

$$\theta_d + \frac{\partial\theta_d}{\partial\xi} = -\frac{2Kn}{3}\frac{\partial\theta_d}{\partial\eta} \quad \text{at} \quad \eta = 0 \tag{30}$$

$$\theta_d + \frac{\partial\theta_d}{\partial\xi} = \frac{2Kn}{3}\frac{\partial\theta_d}{\partial\eta} \quad \text{at} \quad \eta = L \tag{31}$$

and initial conditions

$$\theta_d(\eta, 0) = 0 \tag{32}$$

$$\dot{\theta}_d(\eta, 0) = 0 \tag{33}$$

Note that the initial values stated above (indicated by a subscript 0) are not trivial to compute in general, but we need not consider them in our example because of the non-dimensionalization. There exists an inconsistency in the initial conditions as a result of the ballistic-diffusive approximation, (20). This raises concerns about the ability of the BDA to make steady-state predictions such as thermal conductivity. Details can be found in [8].

For the ballistic part, we can get an analytical result for both components necessary to the solution of this example, the ballistic temperature profile and the spatial derivative of the ballistic heat flux appearing in (29). To start we non-dimensionalize (13) and get

$$\frac{\partial f_b}{\partial\xi} + \mu Kn\frac{\partial f_b}{\partial\eta} + f_b = 0 \tag{34}$$

To get convenient analytical results, we introduce the new quantity, $u_{b\mu}$, defined by

$$u_{b\mu} = \int_\omega \hbar\omega f_b D d\omega \tag{35}$$

that can be thought of as the directional component of the total internal energy. The nondimensional ballistic temperature is, therefore,

$$\theta_b = \int_{-1}^{1} u_{b\mu} d\mu \tag{36}$$

Integrating (34) over μ and using (35) gives

$$\frac{\partial u_{b\mu}}{\partial \xi} + \mu Kn \frac{\partial u_{b\mu}}{\partial \eta} + u_{b\mu} = 0 \quad (37)$$

which has a known solution of

$$u_{b\mu} = u_{w\mu}\left(\xi - \frac{\eta}{Kn\mu}\right) e^{-\frac{\eta}{Kn\mu}} \quad (38)$$

where $u_{w\mu}$ is the directional wall internal energy defined such that a constant temperature of $\theta_{b1} = 1$ is maintained at the left boundary for $\xi \geq 0$. For $\xi < 0$, $\theta_{b1} = 0$. Thus, $u_{w\mu}$ can be defined as the Heaviside step function, $\mathcal{H}$. We have

$$\theta_b(\eta, \xi) = \frac{1}{2}\int_0^1 \mathcal{H}\left(\xi - \frac{\eta}{Kn\mu}\right) e^{-\frac{\eta}{Kn\mu}} d\mu \quad (39)$$

where the integration is from 0 to 1 since ballistic phonons on the left boundary only travel in the positive direction. Then following from our definition of heat flux, (4), we have

$$\mathrm{q}_b(\eta, \xi) = \frac{1}{2}\int_0^1 \mathcal{H}\left(\xi - \frac{\eta}{Kn\mu}\right) \mu e^{-\frac{\eta}{Kn\mu}} d\mu \quad (40)$$

Thus,

$$\frac{\partial \mathrm{q}_b}{\partial \eta} = -\frac{1}{2Kn}\int_0^1 \left\{ H\left(\xi - \frac{\eta}{Kn\mu}\right) e^{-\frac{\eta}{Kn\mu}} + \delta\left(\xi - \frac{\eta}{Kn\mu}\right) e^{-\frac{\eta}{Kn\mu}} \right\} d\mu \quad (41)$$

where δ is the Dirac delta function.

All of the equations necessary to solve our non-dimensionalized 1D problem are now defined. The diffusive portion of the temperature can be found by solving (34) with a ballistic flux term defined by (41) subject to boundary conditions (30) and (31) along with initial conditions (32) and (33). The ballistic portion of the temperature is given by (39).

Numerical Formulation in Space

To discretize in space we utilize the Galerkin finite-element method with linear shape functions. For (29) this yields the semi-discrete equations as

$$[M]\{\ddot{\theta}\} + [C_1]\{\dot{\theta}\} + [K_1]\{\theta\} = \{q_1\} \quad (42)$$

For a particular element, e,

$$[M] = \int_0^h \lfloor N \rfloor^T \lfloor N \rfloor d\eta \quad (43)$$

$$[C_1] = \int_0^h \lfloor N \rfloor^T \lfloor N \rfloor d\eta \quad (44)$$

$$[K_1] = \frac{K_n^2}{3}\int_0^h \lfloor B \rfloor^T \lfloor B \rfloor d\eta \quad (45)$$

$$\{q_1\} = \int_0^h \lfloor N \rfloor^T d\eta \left\{ \frac{\partial \mathrm{q}_b}{\partial \eta} \right\} \quad (46)$$

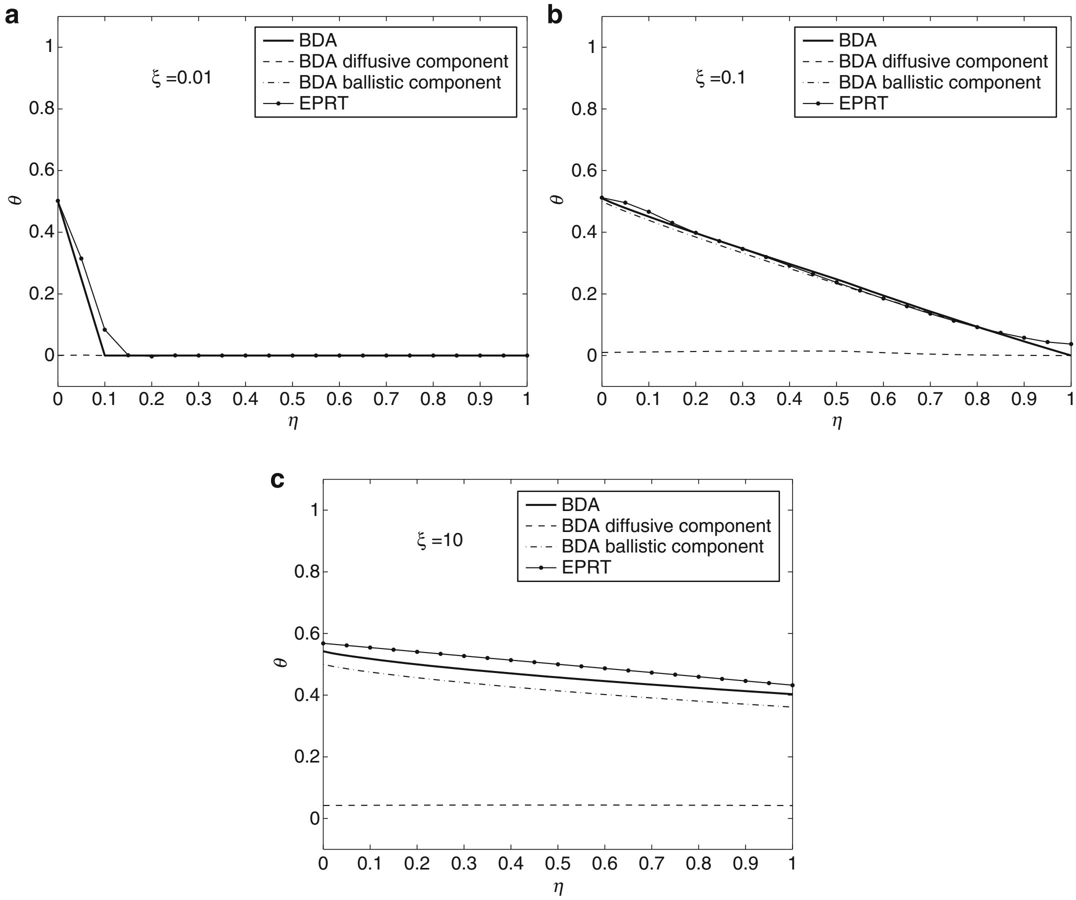
where h is the element length, $\lfloor N \rfloor = \left[1 - \frac{\eta}{h} \; \frac{\eta}{h}\right]$ and $\lfloor B \rfloor = \frac{d}{d\eta}\lfloor N \rfloor$. For clarity, $[\cdot]$ indicates a matrix, $\{\cdot\}$ indicates a column vector, and $\lfloor \cdot \rfloor$ indicates a row vector. The Robin-type boundary conditions given by (30) and (31) should be applied cautiously. Proper application results in additions to the right-hand side of (42) as well as the capacitance matrix, $[C]$, and the stiffness matrix, $[K]$. These contributions are denoted by $\{q_{BC}\}$, $[C_{BC}]$, $[K_{BC}]$, respectfully. The final semi-discretized system takes the form

$$[M]\{\ddot{\theta}\} + [C]\{\dot{\theta}\} + [K]\{\theta\} = \{q\} \quad (47)$$

where $\{q\} = \{q_1\} + \{q_{BC}\}$, $[C] = [C_1] + [C_{BC}]$, and $[K] = [K_1] + [K_{BC}]$. We have now reduced a partial differential equation in space and time, (29), to a set of ordinary differential equations in time, (47).

Discretization in Time

The Generalized Single Step Single Solve (GS4) computational framework has been recently developed to yield a family of second-order accurate, implicit, unconditionally stable algorithms



Ballistic-Diffusive Approximation: A New Look, Fig. 1 Simulation results in the highly ballistic limit, $Kn = 10$ for various snapshots in time. The ballistic portion of the temperature can be seen to be dominant

with controllable numerical dissipation on the zeroth, first, and second-order time derivatives as well as zero-order overshooting behavior [12, 13]. We obtain a fully discretized system by applying the GS4 framework to (47) along with the initial conditions

$$\{\theta\}(0) = 0 \tag{48}$$

$$\{\dot{\theta}\}(0) = 0 \tag{49}$$

The result is the following system:

$$[M]\left\{\widetilde{\ddot{\theta}}\right\} + [C]\left\{\widetilde{\dot{\theta}}\right\} + [K]\{\widetilde{\theta}\} = \{\widetilde{q}\} \tag{50}$$

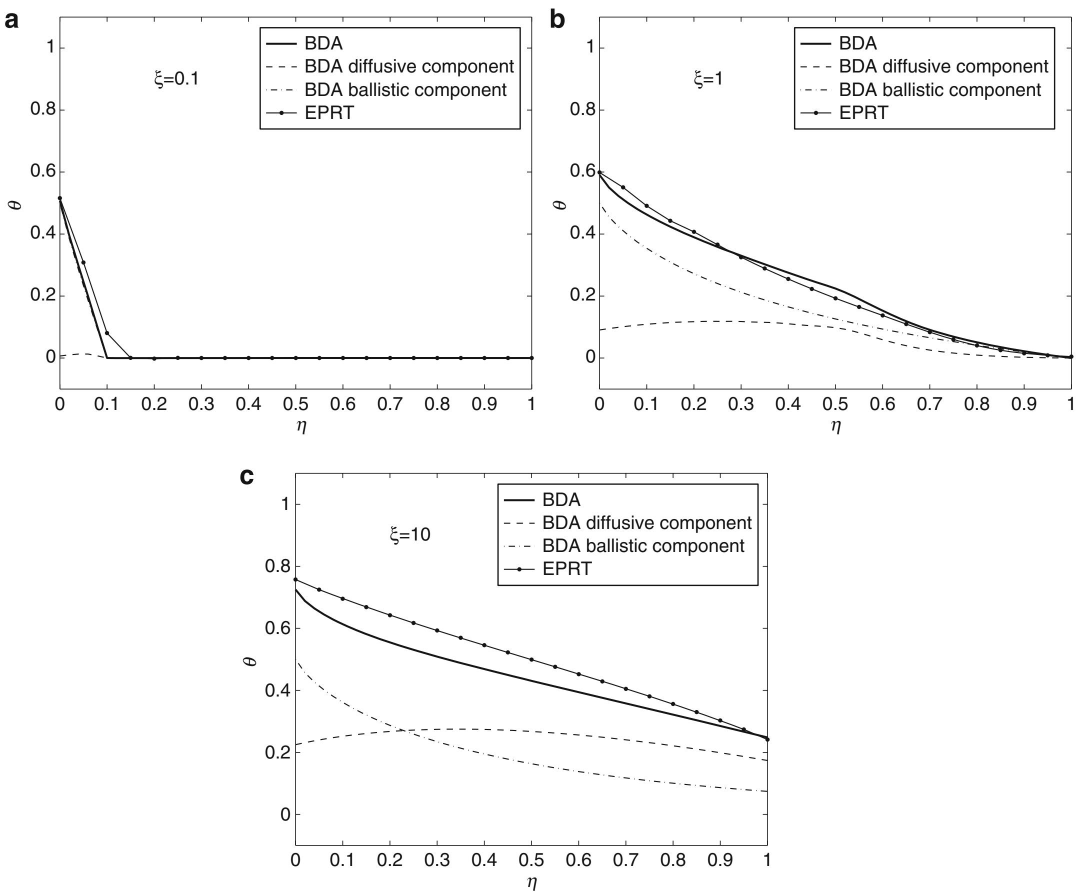
Where

$$\left\{\widetilde{\ddot{\theta}}\right\} = \{\ddot{\theta}\}_n + \Lambda_6 W_1\left(\{\ddot{\theta}\}_{n+1} - \{\ddot{\theta}\}_n\right) \tag{51}$$

$$\begin{aligned}\{\widetilde{\dot{\theta}}\} &= \{\dot{\theta}\}_n + \Lambda_4 W_1 \Delta t\{\ddot{\theta}\}_n \\ &\quad + \Lambda_5 W_2 \Delta t\left(\left\{\ddot{\theta}\right\}_{n+1} - \left\{\ddot{\theta}\right\}_n\right)\end{aligned} \tag{52}$$

$$\begin{aligned}\{\widetilde{\theta}\} &= \{\theta\}_n + \Lambda_1 W_1 \Delta t\{\dot{\theta}\}_n + \Lambda_2 W_2 \Delta t^2\{\ddot{\theta}\}_n \\ &\quad + \Lambda_3 W_3 \Delta t^2(\{\ddot{\theta}\}_{n+1} - \{\ddot{\theta}\}_n)\end{aligned} \tag{53}$$

$$\{\widetilde{q}\} = (1 - W_1)\{q\}_n + W_1\{q\}_{n+1} \tag{54}$$



Ballistic-Diffusive Approximation: A New Look, Fig. 2 Simulation results in the transition regime, $Kn = 1$ for various snapshots in time. The ballistic and diffusive contributions of the temperature are shown to both be important in the total profile result

and the subscript n indicates the timestep. Substituting these into (50), we can solve for $\{\Delta\ddot{\theta}\} = \{\ddot{\theta}\}_{n+1} - \{\ddot{\theta}\}_n$ from

$$\begin{aligned}
&(\Lambda_6 W_1[M] + \Lambda_5 W_2 \Delta t[C] + \Lambda_3 W_3 \Delta t^2[K])\{\Delta\ddot{\theta}\} \\
&= -[M]\{\ddot{\theta}\}_n - [C](\{\dot{\theta}\}_n + \Lambda_4 W_1 \Delta t\{\ddot{\theta}\}_n) \\
&\quad - [K](\{\theta\}_n + \Lambda_1 W_1 \Delta t\{\dot{\theta}\}_n + \Lambda_2 W_2 \Delta t^2\{\ddot{\theta}\}_n) \\
&\quad + (1 - W_1)\{q\}_n + W_1\{q\}_{n+1}
\end{aligned} \tag{55}$$

Once we have $\{\Delta\ddot{\theta}\}$, we can find dimensionless temperature and its first- and second-order derivatives in time, at time $t = n + 1$, using the updates

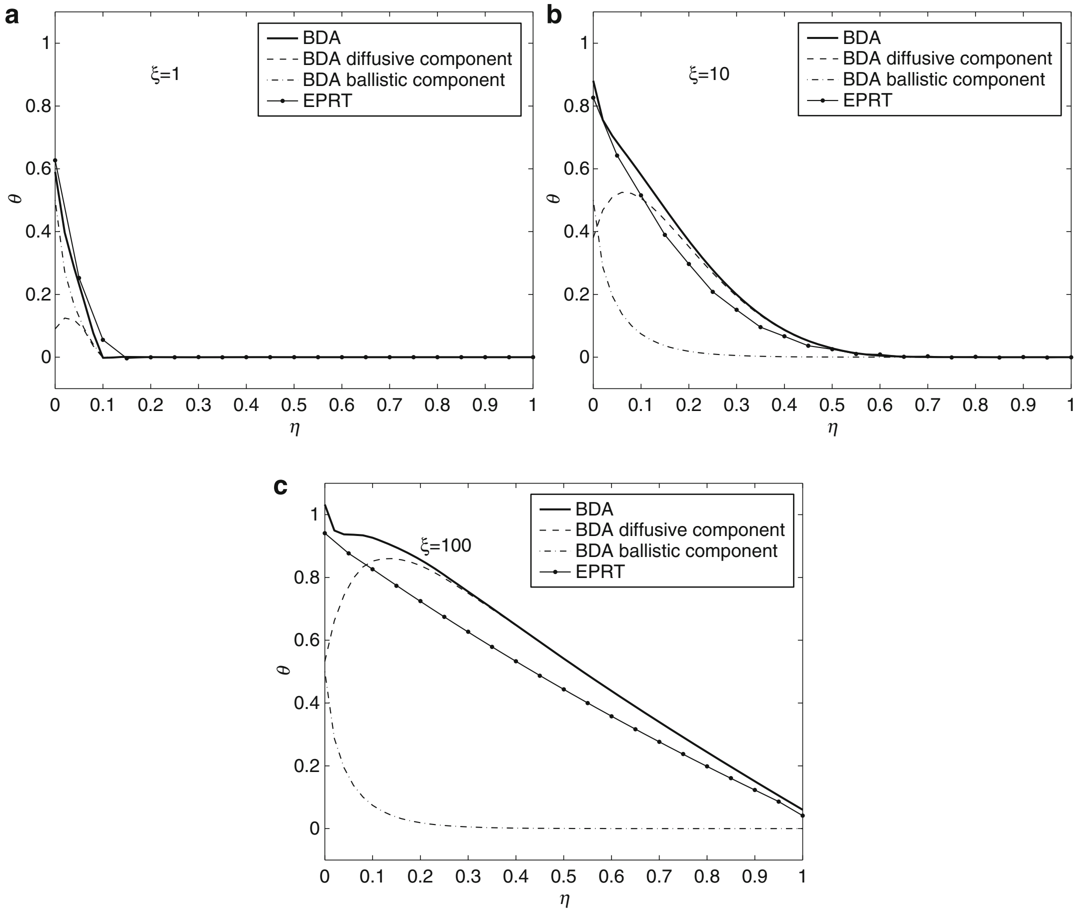
$$\{\ddot{\theta}\}_{n+1} = \{\ddot{\theta}\}_n + \{\Delta\ddot{\theta}\} \tag{56}$$

$$\{\dot{\theta}\}_{n+1} = \{\dot{\theta}\}_n + \lambda_4 \Delta t\{\ddot{\theta}\}_n + \lambda_5 \Delta t\{\Delta\ddot{\theta}\} \tag{57}$$

$$\begin{aligned}
\{\theta\}_{n+1} = &\{\theta\}_n + \lambda_1 \Delta t\{\dot{\theta}\}_n \\
&+ \lambda_2 \Delta t^2\{\ddot{\theta}\}_n + \lambda_3 \Delta t^2\{\Delta\ddot{\theta}\}
\end{aligned} \tag{58}$$

where

$$\Lambda_1 W_1 = \frac{3 + \rho_\infty^{\min} + \rho_\infty^{\max} - \rho_\infty^{\min}\rho_\infty^{\max}}{2(1 + \rho_\infty^{\min})(1 + \rho_\infty^{\max})}$$



Ballistic-Diffusive Approximation: A New Look, Fig. 3 Simulation results in the highly diffusive regime, $Kn = 0.1$ for various snapshots in time. Ballistic effects are only important near the wall where phonons are being emitted

$$\Lambda_2 W_2 = \frac{1}{(1+\rho_\infty^{\min})(1+\rho_\infty^{\max})}$$

$$\Lambda_3 W_3 = \frac{1}{(1+\rho_\infty^{\min})(1+\rho_\infty^{\max})(1+\rho_\infty^{s})}$$

$$\Lambda_4 W_1 = \frac{3+\rho_\infty^{\min}+\rho_\infty^{\max}-\rho_\infty^{\min}\rho_\infty^{\max}}{2(1+\rho_\infty^{\min})(1+\rho_\infty^{\max})}$$

$$\Lambda_5 W_2 = \frac{2}{(1+\rho_\infty^{\min})(1+\rho_\infty^{\max})(1+\rho_\infty^{s})} \qquad (59)$$

$$\Lambda_6 W_1 = \frac{2+\rho_\infty^{\min}+\rho_\infty^{\max}+\rho_\infty^{s}-\rho_\infty^{\min}\rho_\infty^{\max}\rho_\infty^{s}}{(1+\rho_\infty^{\min})(1+\rho_\infty^{\max})(1+\rho_\infty^{s})}$$

$$W_1 = \frac{3+\rho_\infty^{\min}+\rho_\infty^{\max}-\rho_\infty^{\min}\rho_\infty^{\max}}{2(1+\rho_\infty^{\min})(1+\rho_\infty^{\max})}$$

$$\lambda_1 = 1, \quad \lambda_2 = 1/2, \quad \lambda_4 = 1$$

$$\lambda_3 = \frac{1}{2(1+\rho_\infty^{s})}, \quad \lambda_5 = \frac{1}{1+\rho_\infty^{s}}$$

are the algorithmic parameters which can be controlled via a set of user-defined parameters $(\rho_\infty^{\min}, \rho_\infty^{\max}, \rho_\infty^{s})$ associated with the high-frequency damping of the variables $(\{\theta\}, \{\dot{\theta}\}, \{\ddot{\theta}\})$, respectively. Note that the algorithm given by (59) corresponds to the so-called $V0$ family of algorithms of GS4-2. There also exists a family of $U0$ algorithms. Details can be found in [13].

Numerical Results

Temperature profiles have been computed for three different values of the Knudsen number, Kn. These values represent three transport regimes:

- $Kn = 10$, corresponding to highly ballistic transport. Results can be seen in Fig. 1.
- $Kn = 1$, where ballistic and diffusive effects are both important. Results can be seen in Fig. 2.
- $Kn = 0.1$, indicating highly diffusive transport. Results can be seen in Fig. 3.

All results have been plotted against a validated numerical solution of the EPRT using the discrete ordinates method and a weighted residual approach as shown in [4]. The EPRT solutions seen in Figs. 1–3 used 16 ordinates and 20 linear finite elements for a total of 320 degrees of freedom and 20 timesteps. BDA results show good agreement with EPRT solutions. For results given in Figs. 1a–3a, BDA solutions were computed using 50 elements and 20 timesteps. For Figs. 3b–c, 500 elements were used. Fewer elements, in these two cases, resulted in exceedingly poor agreement with the EPRT results. This refinement was still unable to produce a satisfactory agreement with EPRT, as can be seen in Fig. 3c. In addition, the high number of elements led to slow solution times due to numerous integral evaluations of (41).

Concluding Remarks

We have presented a new derivation of the ballistic-diffusive approximation, originally introduced by Chen [5, 8], that requires no complicated approximation techniques. We find this approach more straightforward and elucidating than the original. A simple 1D problem was formulated entirely, and a complete numerical procedure has been described for its solution. The solution included a numerical time integration technique that is second-order accurate in time with controllable numerical dissipation. While the BDA has been developed as a computationally efficient alternative to the Equation of Phonon Radiative Transport, it requires expensive integral evaluations for every point in space and time due to the ballistic distribution (14) and ballistic flux present in the governing equation (20). Particular numerical difficulty has been found in the diffusive limit, $L >> \lambda$, as the system approaches steady state. More work needs to be done to truly evaluate the claim of computational efficiency.

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Basic Theorems in Thermoelastostatics of Bodies with Microtemperatures

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Synonyms

Boundary value problems; Equilibrium theory; Potential method; Thermoelasticity with microtemperatures; Uniqueness and existence theorems

Overview

Grot [5] developed a theory of thermodynamics for elastic materials with microstructure whose microelements, in addition to microdeformations, possess microtemperatures. Iesan and Quintanilla [9] investigated the linear theory of thermoelasticity with microtemperatures and proved an existence result and the continuous dependence of solutions upon initial data and body loads.

Since then many papers have been written on this subject. The fundamental solutions of the theory of thermoelasticity with microtemperatures were constructed by Svanadze [21]. The representations of Galerkin type and general solutions of equations of dynamic and steady vibrations in this theory were obtained by Scalia and Svanadze [16]. Casas and Quintanilla [2] studied the exponential stability of solution in the theory of thermoelasticity with microtemperatures. Using the potential method and the theory of singular integral equations, the basic boundary value problems (BVPs) of steady vibrations were investigated by Svanadze [20] and Scalia and Svanadze [18]. Within the frame of the linear theory, the representation of the general solution and fundamental solution for steady vibrations are established by Scalia and Svanadze [17], and uniqueness and existence theorems are proved.

A theory of micromorphic fluid with microtemperatures has been established by Riha [14, 15]. For an extensive review and the basic results in the microcontinuum field theories, see the books of Eringen [3] and Iesan [8].

The study of BVPs of mathematical physics by the classical potential method has notched up a century. The application of this method to the three-dimensional basic BVPs of the theory of elasticity reduces these problems to two-dimensional singular integral equations [12].

The theory of multidimensional singular integral equations has presently been worked out with sufficient completeness [1, 11–13]. This theory makes it possible to investigate three-dimensional problems not only of classical theory of elasticity but also problems of generalized theories. An extensive review of works on the potential method can be found in Gegelia and Jentsch [4].

Scalia et al. [19] established various analytical results in the linear theory of thermoelastostatics of bodies with microtemperatures, and some basic results of the classical theories of elasticity and thermoelasticity are generalized (see [6, 7, 12]).

This entry is concerned with the equilibrium theory of thermoelastic solids with microtemperatures. First, we present some uniqueness results. Then, we use the fundamental matrix to obtain a representation of Somigliana type. The potentials of single layer and double layer are used to reduce the BVPs to singular integral equations for which Fredholm's basic theorems are valid. Finally, the existence and uniqueness theorems are established.

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Basic Equations. Boundary Value Problems

We consider an isotropic elastic material with microstructure that occupies the region Ω of the Euclidean three-dimensional space E^3. Let $\mathbf{x} = (x_1, x_2, x_3)$ be the point of E^3, $\mathbf{D_x} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$.

The fundamental system of field equations in the linear equilibrium theory of thermoelasticity with microtemperatures consists of the equations of equilibrium [5]:

$$t_{jl,j} + \rho F_l^{(1)} = 0 \qquad (1)$$

The balance of energy

$$q_{l,l} + \rho s = 0 \qquad (2)$$

The first moment of energy

$$q_{jl,j} + q_l - Q_l + \rho F_l^{(2)} = 0 \qquad (3)$$

The constitutive equations

$$\begin{aligned} t_{jl} &= (\lambda\, e_{rr} - \beta\theta)\, \delta_{jl} + 2\mu e_{jl} \\ q_l &= k\, \theta_{,l} + k_1\, w_l \\ q_{jl} &= -k_4\, w_{r,r}\delta_{jl} - k_5\, w_{j,l} - k_6\, w_{l,j} \\ Q_l &= (k_1 - k_2) w_l + (k - k_3)\, \theta_{,l} \end{aligned} \qquad (4)$$

The geometrical equations

$$e_{lj} = \frac{1}{2}(u_{l,j} + u_{j,l}) \qquad (5)$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector, $\mathbf{w} = (w_1, w_2, w_3)$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 ($T_0 > 0$), t_{jl} is the stress tensor, ρ is the reference mass density ($\rho > 0$), $\mathbf{F}^{(1)} = (F_1^{(1)}, F_2^{(1)}, F_3^{(1)})$ is the body force, $\mathbf{q} = (q_1, q_2, q_3)$ is the heat flux vector, s is the heat supply, q_{jl} is the component of first heat flux moment tensor, $\mathbf{Q} = (Q_1, Q_2, Q_3)$ is the mean heat flux vector, $\mathbf{F}^{(2)} = (F_1^{(2)}, F_2^{(2)}, F_3^{(2)})$ is first heat source moment vector, $\lambda, \mu, \beta, k, k_1, k_2, \cdots, k_6$ are constitutive coefficients, δ_{lj} is the Kronecker delta, e_{lj} is the component of strain tensor, the subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and $j, l = 1, 2, 3$, and repeated indices are summed over the range (1, 2, 3).

By virtue of (4) and (5), system (1)–(3) can be expressed in terms of the displacement vector $\mathbf{u}$, the microtemperature vector $\mathbf{w}$, and the temperature θ. We obtain the system of equations of the linear equilibrium theory of thermoelasticity with microtemperatures [5]:

$$\begin{aligned} &\mu\, \Delta \mathbf{u} + (\lambda + \mu)\, \mathrm{grad\, div}\, \mathbf{u} - \beta\, \mathrm{grad}\, \theta = -\rho \mathbf{F}^{(1)} \\ &k_6\, \Delta \mathbf{w} + (k_4 + k_5)\, \mathrm{grad\, div}\, \mathbf{w} - k_3 \\ &\qquad \mathrm{grad}\, \theta - k_2 \mathbf{w} = \rho\, \mathbf{F}^{(2)} \\ &\quad k\, \Delta\, \theta + k_1\, \mathrm{div}\, \mathbf{w} = -\rho\, s \end{aligned} \qquad (6)$$

We introduce the matrix differential operator:

$$\begin{aligned} &\mathbf{A}(\mathbf{D_x}) = \left(A_{pq}(\mathbf{D_x})\right)_{7\times 7} \\ &A_{lj}(\mathbf{D_x}) = \mu \Delta\, \delta_{lj} + (\lambda + \mu)\, \frac{\partial^2}{\partial x_l \partial x_j} \\ &A_{l+3;j+3}(\mathbf{D_x}) = (k_6 \Delta - k_2)\delta_{lj} + (k_4 + k_5)\, \frac{\partial^2}{\partial x_l \partial x_j} \\ &A_{l7}(\mathbf{D_x}) = -\beta \frac{\partial}{\partial x_l} \qquad A_{l+3;7}(\mathbf{D_x}) = -k_3 \frac{\partial}{\partial x_l} \\ &A_{7;l+3}(\mathbf{D_x}) = k_1 \frac{\partial}{\partial x_l} \qquad A_{77}(\mathbf{D_x}) = k\Delta \\ &A_{l;j+3}(\mathbf{D_x}) = A_{l+3;j}(\mathbf{D_x}) = A_{7l}(\mathbf{D_x}) = 0 \\ &\qquad l, j = 1, 2, 3 \end{aligned}$$

The system (6) can be written as

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \qquad (7)$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \theta)$, $\mathbf{F} = (-\rho \mathbf{F}^{(1)}, \rho \mathbf{F}^{(2)}, -\rho s)$ and $\mathbf{x} \in \Omega$.

Let S be the smooth closed surface surrounding the finite domain Ω^+ in E^3, $\bar{\Omega}^+ = \Omega^+ \cup S$, $\Omega^- = E^3 \backslash \bar{\Omega}^+$, $\bar{\Omega}^- = \Omega^- \cup S$.

Definition 1. A vector function $\mathbf{U} = (U_1, U_2, \cdots, U_7)$ is called *regular* in Ω^-(or Ω^+) if

1.
$$U_l \in C^2(\Omega^-) \cap C^1(\bar{\Omega}^-)$$
$$(\text{or } \; U_l \in C^2(\Omega^+) \cap C^1(\bar{\Omega}^+))$$

2.
$$U_l(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \qquad \frac{\partial}{\partial x_j} U_l(\mathbf{x}) = o(|\mathbf{x}|^{-1})$$
$$\text{for} \qquad |\mathbf{x}| \gg 1 \tag{8}$$

where $j = 1, 2, 3, \quad l = 1, 2, \cdots, 7$.

In the sequel, we use the matrix differential operators

$$\mathbf{P}^{(m)}(\mathbf{D_x}, \mathbf{n}) = \left(P_{lj}^{(m)}(\mathbf{D_x}, \mathbf{n})\right)_{3\times 3}$$
$$P_{lj}^{(1)}(\mathbf{D_x}, \mathbf{n}) = \mu \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + \mu n_j \frac{\partial}{\partial x_l} + \lambda n_l \frac{\partial}{\partial x_j}$$
$$P_{lj}^{(2)}(\mathbf{D_x}, \mathbf{n}) = k_6 \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + k_5 n_j \frac{\partial}{\partial x_l} + k_4 n_l \frac{\partial}{\partial x_j}$$

and

$$\mathbf{P}(\mathbf{D_x}, \mathbf{n}) = (P_{lj}(\mathbf{D_x}, \mathbf{n}))_{7\times 7}$$
$$P_{lj}(\mathbf{D_x}, \mathbf{n}) = P_{lj}^{(1)}(\mathbf{D_x}, \mathbf{n}) \quad P_{l+3;j+3}(\mathbf{D_x}, \mathbf{n}) = P_{lj}^{(2)}(\mathbf{D_x}, \mathbf{n})$$
$$P_{l7}(\mathbf{D_x}, \mathbf{n}) = -\beta n_l \qquad P_{7;l+3}(\mathbf{D_x}, \mathbf{n}) = k_1 n_l$$
$$P_{77}(\mathbf{D_x}, \mathbf{n}) = k \frac{\partial}{\partial \mathbf{n}} \qquad P_{l;j+3} = P_{l+3;j} = P_{l+3;7} = P_{7l} = 0$$

where $\mathbf{n} = (n_1, n_2, n_3)$, $\mathbf{n}(\mathbf{z})$ is the external unit normal vector to S at $\mathbf{z}$; $\frac{\partial}{\partial \mathbf{n}}$ is the derivative along the vector $\mathbf{n}$, $m = 1,2$, and l, $j = 1,2,3$; $\mathbf{P}(\mathbf{D_x}, \mathbf{n})\mathbf{U}(\mathbf{x})$ is the stress vector in the theory of thermoelasticity with microtemperatures [9].

The internal and external basic BVPs of the equilibrium theory of thermoelasticity with microtemperatures are formulated as follows. Find a regular (classical) solution to system (7) for $\mathbf{x} \in \Omega^+$ satisfying the boundary condition

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$$

in the internal *Problem* $(I)^+_{\mathbf{F},\mathbf{f}}$ and

$$\{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$$

in the internal *Problem* $(II)^+_{\mathbf{F},\mathbf{f}}$. Find a regular (classical) solution to system (7) for $\mathbf{x} \in \Omega^-$ satisfying the boundary condition

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$$

in the external *Problem* $(I)^-_{\mathbf{F},\mathbf{f}}$ and

$$\{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$$

in the external *Problem* $(II)^-_{\mathbf{F},\mathbf{f}}$. Here $\mathbf{F}$ and $\mathbf{f}$ are the known seven-component vector functions, and supp $\mathbf{F}$ is a finite domain in Ω^-.

Uniqueness Theorems

In this section, first, we present the Green's formulae of the linear equilibrium theory of thermoelasticity with microtemperatures, and then we prove the uniqueness theorems of regular solutions of above formulated BVPs.

We introduce the notation

$$\begin{aligned} W(\mathbf{U}, \mathbf{U}') = & W^{(1)}(\mathbf{u}, \mathbf{u}') \\ & + W^{(2)}(\mathbf{w}, \mathbf{w}') - \beta\, \theta \operatorname{div} \mathbf{u}' \\ & + k_2 \mathbf{w}\, \mathbf{w}' + k_1\, \mathbf{w} \operatorname{grad} \theta' \\ & + k_3 \operatorname{grad} \theta\, \mathbf{w}' \\ & + k \operatorname{grad} \theta \operatorname{grad} \theta' \end{aligned}$$

where $\mathbf{u}' = (u'_1, u'_2, u'_3)$ and $\mathbf{w}' = (w'_1, w'_2, w'_3)$ are three-component vector functions, θ' is scalar function, $\mathbf{U}' = (\mathbf{u}', \mathbf{w}', \theta')$, and

$$W^{(1)}(\mathbf{u},\mathbf{u}') = \frac{1}{3}(3\lambda+2\mu)\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{u}' + \mu\left[\frac{1}{2}\sum_{l,j=1;l\neq j}^{3}\left(\frac{\partial u_j}{\partial x_l}+\frac{\partial u_l}{\partial x_j}\right)\left(\frac{\partial u'_j}{\partial x_l}+\frac{\partial u'_l}{\partial x_j}\right) + \frac{1}{3}\sum_{l,j=1}^{3}\left(\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\right)\left(\frac{\partial u'_l}{\partial x_l}-\frac{\partial u'_j}{\partial x_j}\right)\right]$$

$$W^{(2)}(\mathbf{w},\mathbf{w}') = \frac{1}{3}(3k_4+k_5+k_6)\operatorname{div}\mathbf{w}\operatorname{div}\mathbf{w}' + \frac{1}{2}(k_6-k_5)\operatorname{curl}\mathbf{w}\operatorname{curl}\mathbf{w}' + \frac{1}{2}(k_6+k_5)\left[\frac{1}{2}\sum_{l,j=1;l\neq j}^{3}\left(\frac{\partial w_j}{\partial x_l}+\frac{\partial w_l}{\partial x_j}\right)\left(\frac{\partial w'_j}{\partial x_l}+\frac{\partial w'_l}{\partial x_j}\right) + \frac{1}{3}\sum_{l,j=1}^{3}\left(\frac{\partial w_l}{\partial x_l}-\frac{\partial w_j}{\partial x_j}\right)\left(\frac{\partial w'_l}{\partial x_l}-\frac{\partial w'_j}{\partial x_j}\right)\right]$$

As in classical theory of elasticity (see, for details [12]), we can prove the following results (Green's theorems in the linear equilibrium theory of thermoelasticity with microtemperatures).

Theorem 1. *If* $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \theta)$ *is a regular vector field in* Ω^+ *and* $\mathbf{U}' = (\mathbf{u}', \mathbf{w}', \theta') \in C^1(\Omega^+)$, *then*

$$\int_{\Omega^+} [\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\,\mathbf{U}'(\mathbf{x}) + W(\mathbf{U},\mathbf{U}')]d\mathbf{x} = \int_S \mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\mathbf{U}'(\mathbf{z})d_{\mathbf{z}}S$$

Theorem 2. *If* $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \theta)$ *is a regular vector field in* Ω^-, $\mathbf{U}' = (\mathbf{u}', \mathbf{w}', \theta') \in C^1(\Omega^-)$, *and*

$$U'(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \qquad \frac{\partial}{\partial x_j}U'(\mathbf{x}) = o(|\mathbf{x}|^{-1})$$

$$\text{for} \quad |\mathbf{x}| \gg 1 \qquad j = 1, 2, 3$$

then

$$\int_{\Omega^-} [\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\,\mathbf{U}'(\mathbf{x}) + W(\mathbf{U},\mathbf{U}')]d\mathbf{x} = -\int_S \mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\mathbf{U}'(\mathbf{z})d_{\mathbf{z}}S$$

Theorem 3. *If* $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \theta)$ *is a regular vector field in* Ω^+ *and* $\mathbf{U}' = (\mathbf{u}', \mathbf{w}', \theta') \in C^1(\Omega^+)$, *then*

$$\int_{\Omega^+} \Big[\Big(\mathbf{A}^{(1)}\,\mathbf{u}(\mathbf{x}) - \beta\operatorname{grad}\theta\Big)\,\mathbf{u}'(\mathbf{x}) + W^{(1)}(\mathbf{u},\mathbf{u}') - \beta\,\theta\operatorname{div}\mathbf{u}'(\mathbf{x})\Big]d\mathbf{x} = \int_S \mathbf{P}^{(1)}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{v}(\mathbf{z})\mathbf{u}'(\mathbf{z})d_{\mathbf{z}}S$$

$$\int_{\Omega^+} \Big[\Big(\mathbf{A}^{(2)}(\mathbf{D_x})\,\mathbf{w} - k_2\mathbf{w} - k_3\operatorname{grad}\theta\Big)\,\mathbf{w}'(\mathbf{x}) + W^{(2)}(\mathbf{w},\mathbf{w}') + (k_2\mathbf{w} + k_3\operatorname{grad}\theta)\,\mathbf{w}'(\mathbf{x})\Big]d\mathbf{x} = \int_S \mathbf{P}^{(2)}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{w}(\mathbf{z})\mathbf{w}'(\mathbf{z})d_{\mathbf{z}}S$$

We are now in a position to prove the following uniqueness theorems in the equilibrium theory of thermoelasticity with microtemperatures.

Theorem 4. *If conditions*

$$\mu > 0 \quad 3\lambda + 2\mu > 0 \tag{9}$$

and

$$\begin{aligned} &3k_4 + k_5 + k_6 > 0 \qquad k_6 \pm k_5 > 0 \\ &k > 0 \qquad (k_1 + T_0\,k_3)^2 < 4T_0\,k\,k_2 \end{aligned} \tag{10}$$

are satisfied, then each of the problems $(I)^+_{\mathbf{F},\mathbf{f}}$, $(I)^-_{\mathbf{F},\mathbf{f}}$ *and* $(II)^-_{\mathbf{F},\mathbf{f}}$ *admit at most one regular solution.*

Proof. (a) Suppose that there are two regular solutions of problem $(I)^+_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ corresponds to zero data ($\mathbf{F} = \mathbf{f} = 0$), i.e., $\mathbf{U}$ is a regular solution of problem $(I)^+_{\mathbf{0},\mathbf{0}}$. If $\mathbf{U} = \mathbf{U}'$, then on the basis of Theorem 3, we obtain

$$\int_{\Omega^+} \Big[W^{(1)}(\mathbf{u},\mathbf{u}) - \beta\,\theta\operatorname{div}\mathbf{u}\Big]d\mathbf{x} = 0 \tag{11}$$

$$\int_{\Omega^+} \left[W^{(2)}(\mathbf{w},\mathbf{w}) + (k_2\mathbf{w} + k_3 \operatorname{grad}\theta)\,\mathbf{w}\right] d\mathbf{x} = 0 \tag{12}$$

$$\int_{\Omega^+} [k \operatorname{grad}\theta \operatorname{grad}\theta + k_1 \mathbf{w} \operatorname{grad}\theta]\, d\mathbf{x} = 0 \tag{13}$$

Equations (12) and (13) imply

$$\begin{aligned}\int_{\Omega^+} &\Big[T_0 W^{(2)}(\mathbf{w},\mathbf{w})\\ &+ \Big(T_0 k_2 |\mathbf{w}|^2 + (k_1 + T_0 k_3)\mathbf{w} \operatorname{grad}\theta\\ &+ k\,|\operatorname{grad}\theta|^2\Big)\Big]\, d\mathbf{x} = 0\end{aligned} \tag{14}$$

Keeping in mind (10) from (14), we have

$$\mathbf{w}(\mathbf{x}) = 0 \qquad \theta(\mathbf{x}) = \text{const} \quad \text{for} \quad \mathbf{x} \in \Omega^+$$

In view of homogeneous boundary condition, we have $\theta(\mathbf{x}) = 0$ for $\mathbf{x} \in \Omega^+$, and from (11), we get

$$\int_{\Omega^+} W^{(1)}(\mathbf{u},\mathbf{u}) d\mathbf{x} = 0 \tag{15}$$

It is easy to see that by condition (9), we can write $W^{(1)}(\mathbf{u},\mathbf{u}) \geq 0$ and from (15) we obtain $W^{(1)}(\mathbf{u},\mathbf{u}) = 0$, and hence, $\mathbf{u}$ is the vector of rigid displacement [12]

$$\mathbf{u}(\mathbf{x}) = \mathbf{a} + [\mathbf{b} \times \mathbf{x}] \tag{16}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary real constant three-component vectors and $[\mathbf{b} \times \mathbf{x}]$ is the vector product of vectors $\mathbf{b}$ and $\mathbf{x}$. On the basis of homogeneous boundary condition from (16), we have $\mathbf{u}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in \Omega^+$, and the uniqueness of solution of problem $(I)^+_{\mathbf{F},\mathbf{f}}$ is proved.

(b) Suppose that there are two regular solutions of the external BVP $(K)^-_{\mathbf{F},\mathbf{f}}$, where $K = I, II$. Then their difference $\mathbf{U}$ corresponds to zero data ($\mathbf{F} = \mathbf{f} = 0$), i.e., $\mathbf{U}$ is a regular solution of problem $(K)^-_{\mathbf{0},\mathbf{0}}$. Quite similarly, we obtain

$$\mathbf{w}(\mathbf{x}) = \mathbf{0} \qquad \theta(\mathbf{x}) = \text{const} \qquad \text{for} \qquad \mathbf{x} \in \Omega^-$$

In view of (8), we have $\theta(\mathbf{x}) = 0$ for, $\mathbf{x} \in \Omega^-$ and by theorem 2 we get $W^{(1)}(\mathbf{u},\mathbf{u}) = 0$. Hence, vector $\mathbf{u}$ has the form (16) for $\mathbf{x} \in \Omega^-$. Keeping in mind condition (8), we have $\mathbf{u}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in \Omega^-$, and the uniqueness of solution of problem $(K)^-_{\mathbf{F},\mathbf{f}}$ is proved, where $K = I, II$.

Similarly we can prove the following theorem:

Theorem 5. *If the conditions (9) and (10) are satisfied, then any two regular solutions of the BVP* $(II)^+_{\mathbf{F},\mathbf{f}}$ *may differ only to within additive vector* $\mathbf{U}^{(0)} = (\mathbf{u}^{(0)}, \mathbf{w}^{(0)}, \theta^{(0)})$, *where*

$$\begin{aligned}&\mathbf{u}^{(0)}(\mathbf{x}) = \mathbf{a} + [\mathbf{b} \times \mathbf{x}] + c_2\mathbf{x} \qquad \mathbf{w}^{(0)}(\mathbf{x}) = 0\\ &\theta^{(0)}(\mathbf{x}) = c_1 \qquad \textit{for} \quad \mathbf{x} \in \Omega^+\end{aligned}$$

where $\mathbf{a}$ *and* $\mathbf{b}$ *are arbitrary real constant three-component vectors and* c_1 *is an arbitrary real constant and* $c_2 = \frac{c_1 \beta}{3\lambda + 2\mu}$.

As in classical theory of elasticity [10], we can prove uniqueness of regular solution of the problems $(I)^+_{\mathbf{F},\mathbf{f}}$ and $(I)^-_{\mathbf{F},\mathbf{f}}$ in more weak conditions that (9) and (10). We have the following result [19]:

Theorem 6. *If conditions*

$$\mu > 0 \qquad \lambda + 2\mu > 0 \tag{17}$$

and

$$\begin{aligned}&k > 0 \qquad k_6 > 0 \qquad k_7 > 0\\ &(k_1 + T_0 k_3)^2 < 4T_0 k k_2\end{aligned} \tag{18}$$

are satisfied, then the problems $(I)^+_{\mathbf{F},\mathbf{f}}$ *and* $(I)^-_{\mathbf{F},\mathbf{f}}$ *admit at most one regular solution, where* $k_7 = k_4 + k_5 + k_6$.

Remark 1. Obviously, from (9) and (10) we have (17) and (18), respectively. Indeed, (9) and (10) imply

$$\begin{aligned}\lambda + 2\mu &= \frac{1}{3}[(3\lambda + 2\mu) + 4\mu] > 0\\ k_6 &= \frac{1}{2}[(k_6 + k_5) + (k_6 - k_5)] > 0\\ k_7 &= \frac{1}{3}[(3k_4 + k_5 + k_6) + 2(k_6 + k_5)] > 0\end{aligned}$$

Remark 2. The uniqueness theorem of the first BVP (when on the boundary is prescribed the displacement vector) of the classical theory elasticity is proved in the condition (17) [see, for details [10]].

Representation of Somigliana Type

In this section, the formulae of integral representations of regular vector and regular solution of system (6) in the domains Ω^+ and $\Omega^-$are obtained.

Definition 2. The matrix $\tilde{\mathbf{A}}(\mathbf{D_x}) = (\tilde{A}_{lj}(\mathbf{D_x}))_{7\times 7}$, where $\tilde{A}_{lj}(\mathbf{D_x}) = A_{jl}(-\mathbf{D_x})$, $l,j = 1,2,\cdots,7$, will be called *the associated operator* of differential operator $\mathbf{A}$.

Hence, the associated operator $\tilde{\mathbf{A}}$ satisfies the condition $\tilde{\mathbf{A}}(\mathbf{D_x}) = \mathbf{A}^T(-\mathbf{D_x})$, where superscript "$T$" denotes transposition. The homogeneous associated system of (6) will be the following system:

$$\begin{aligned} &\mu\,\Delta\mathbf{u} + (\lambda+\mu)\text{ grad div }\mathbf{u} = \mathbf{0}\\ &k_6\,\Delta\mathbf{w} + (k_4+k_5)\text{ grad div }\\ &\mathbf{w} - k_1\text{ grad }\theta - k_2\mathbf{w} = \mathbf{0}\\ &k\,\Delta\,\theta + \beta\text{ div }\mathbf{u} + k_3\text{ div }\mathbf{w} = 0 \end{aligned} \tag{19}$$

Let $\tilde{\mathbf{\Gamma}}(\mathbf{x}) = (\tilde{\Gamma}_{lj}(\mathbf{x}))_{7\times 7}$ be a fundamental solution (fundamental matrix) of the system (19) (operator $\tilde{\mathbf{A}}(\mathbf{D_x})$).

By the method, developed in Svanadze [21], we can construct the matrix $\tilde{\mathbf{\Gamma}}(\mathbf{x})$. Obviously, $\tilde{\mathbf{\Gamma}}(\mathbf{x}) = \mathbf{\Gamma}^T(-\mathbf{x})$, where $\mathbf{\Gamma}(\mathbf{x})$ is the fundamental matrix of operator $\mathbf{A}(\mathbf{D_x})$ and is constructed by means of elementary functions in Svanadze [21].

The following basic properties of matrix $\tilde{\mathbf{\Gamma}}(\mathbf{x})$ may be easily verified:

Theorem 7. *Each column of the matrix $\tilde{\mathbf{\Gamma}}(\mathbf{x})$, considered as a vector, satisfies the associated system (19) at every point of E^3 except the origin*, i.e.,

$$\tilde{\mathbf{A}}(\mathbf{D_x})\,\tilde{\mathbf{\Gamma}}(\mathbf{x}) = \mathbf{0} \qquad \text{for} \quad \mathbf{x} \neq \mathbf{0}$$

Theorem 8. *The elements of the matrix $\tilde{\mathbf{\Gamma}}(\mathbf{x}) - \mathbf{\Psi}(\mathbf{x})$ are bounded at $\mathbf{x} = \mathbf{0}$, while the first derivatives have isolated singularities of the kind $|\mathbf{x}|^{-1}$*, i.e.,

$$\begin{aligned} &\tilde{\Gamma}_{lj}(\mathbf{x}) - \Psi_{lj}(\mathbf{x}) = O(1)\\ &\frac{\partial}{\partial x_m}\left(\tilde{\Gamma}_{lj}(\mathbf{x}) - \Psi_{lj}(\mathbf{x})\right) = O(|\mathbf{x}|^{-1})\\ &l,j,m = 1,2,3 \end{aligned}$$

where $\mathbf{\Psi}(\mathbf{x})$ is the fundamental solution of system

$$\begin{aligned} &\mu\,\Delta\mathbf{u} + (\lambda+\mu)\text{ grad div }\mathbf{u} = \mathbf{0}\\ &k_6\,\Delta\mathbf{w} + (k_4+k_5)\text{ grad div }\mathbf{w} = \mathbf{0}\\ &k\,\Delta\,\theta = 0 \end{aligned}$$

and has the following form [21]:

$$\begin{aligned} &\mathbf{\Psi}(\mathbf{x}) = (\tilde{\Psi}_{lj}(\mathbf{x}))_{7\times 7}\\ &\Psi_{lj}(\mathbf{x}) = \left(\frac{1}{\mu}\Delta\,\delta_{lj} - \frac{\lambda+\mu}{\mu\,\mu_0}\frac{\partial^2}{\partial x_l\,\partial x_j}\right)\left(-\frac{|\mathbf{x}|}{8\pi}\right)\\ &\Psi_{l+3;j+3}(\mathbf{x}) = \left(\frac{1}{k_6}\Delta\,\delta_{lj} - \frac{k_4+k_5}{k_6\,k_7}\frac{\partial^2}{\partial x_l\,\partial x_j}\right)\left(-\frac{|\mathbf{x}|}{8\pi}\right)\\ &\Psi_{77}(\mathbf{x}) = -\frac{1}{4\pi k\,|\mathbf{x}|}\\ &\Psi_{l;j+3} = \Psi_{l+3;j} = \Psi_{l7} = \Psi_{l+3;7} = \Psi_{7l}\\ &\quad = \Psi_{7;l+3} \quad l,j = 1,2,3 \end{aligned}$$

Let the vector $\tilde{U}_l$ be the l-th column of the matrix $\tilde{U} = (\tilde{U}_{jl})_{7\times 7}$, $\tilde{u}_l = (\tilde{U}_{1l}, \tilde{U}_{2l}, \tilde{U}_{3l})^T$, $\tilde{w}_l = (\tilde{U}_{4l}, \tilde{U}_{5l}, \tilde{U}_{6l})^T$, $\tilde{\theta}_l = \tilde{U}_{7l}$, $l = 1,2,\cdots,7$. In the sequel, we use the matrix differential operator

$$\begin{aligned} &\tilde{\mathbf{P}}(\mathbf{D_z},\mathbf{n}) = \left(\tilde{P}_{lj}(\mathbf{D_z},\mathbf{n})\right)_{7\times 7} \qquad \tilde{P}_{lj} = P_{lj}^{(1)}\\ &\tilde{P}_{l+3;j+3} = P_{lj}^{(2)}\\ &\tilde{P}_{7;j+3} = k_3\,n_j \qquad \tilde{P}_{77} = k\,\frac{\partial}{\partial\mathbf{n}}\\ &\tilde{P}_{l;j+3} = \tilde{P}_{l7} = \tilde{P}_{l+3;j} = \tilde{P}_{l+3;7} = \tilde{P}_{7j} = 0\\ &l,j = 1,2,3 \end{aligned}$$

As in classical theory of elasticity [12], we can prove the following results (Green's theorems in the linear theory of thermoelasticity with microtemperatures for the domains Ω^+ and Ω^-) [19]:

Theorem 9. *If U and* $\tilde{U}_l\,(l = 1, 2, \cdots, 7)$ *are regular vectors in* Ω^+*, then*

$$\int_{\Omega^+} \Big\{ [\tilde{\mathbf{A}}(\mathbf{D_y})\tilde{\mathbf{U}}(\mathbf{y})]^T \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^T \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y}) \Big\} d\mathbf{y} = \int_S \Big\{ [\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{U}}(\mathbf{z})]^T \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^T \mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \Big\} d_{\mathbf{z}}S$$

Theorem 10. *If U and* $\tilde{U}_l\,(l = 1, 2, \cdots, 7)$ *are regular vectors in* Ω^-*, then*

$$\int_{\Omega^-} \Big\{ [\tilde{\mathbf{A}}(\mathbf{D_y})\tilde{\mathbf{U}}(\mathbf{y})]^T \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^T \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y}) \Big\} d\mathbf{y} = -\int_S \Big\{ [\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{U}}(\mathbf{z})]^T \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^T \mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \Big\} d_{\mathbf{z}}S$$

Theorems 9 and 10 lead the formulae of integral representations of regular vector in the equilibrium theory of thermoelasticity with microtemperatures for the domains Ω^+ and Ω^-.

Theorem 11. *If U is a regular vector in* Ω^+*, then*

$$\delta_1(\mathbf{x})\,\mathbf{U}(\mathbf{x}) = \int_S \Big\{ [\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{\Gamma}}(\mathbf{z} - \mathbf{x})]^T \mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x} - \mathbf{z})\,\mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \Big\} d_{\mathbf{z}}S + \int_{\Omega^+} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y})\,\mathbf{A}(\mathbf{D_y})\,\mathbf{U}(\mathbf{y})d\mathbf{y} \tag{20}$$

where

$$\delta_1(\mathbf{x}) = \begin{cases} 1 & \text{for} \quad \mathbf{x} \in \Omega^+ \\ 0 & \text{for} \quad \mathbf{x} \in \Omega^- \end{cases}$$

Theorem 12. *If U is a regular vector in* Ω^-*, then*

$$\delta_2(\mathbf{x})\,\mathbf{U}(\mathbf{x}) = -\int_S \Big\{ [\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{\Gamma}}(\mathbf{z} - \mathbf{x})]^T \mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x} - \mathbf{z})\,\mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \} d_{\mathbf{z}}S + \int_{\Omega^-} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y})\,\mathbf{A}(\mathbf{D_y})\,\mathbf{U}(\mathbf{y})d\mathbf{y} \tag{21}$$

where

$$\delta_2(\mathbf{x}) = \begin{cases} 1 & \text{for} \quad \mathbf{x} \in \Omega^- \\ 0 & \text{for} \quad \mathbf{x} \in \Omega^+ \end{cases}$$

Equations (20) and (21) will be called the *representation formulae of Somigliana type* in the equilibrium theory of thermoelasticity with microtemperatures.

Theorems 11 and 12 lead to the following results:

Corollary 1. *If U is a regular solution of the homogeneous equation*

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) = \mathbf{0} \tag{22}$$

for $\mathbf{x} \in \Omega^+$*, then*

$$\mathbf{U}(\mathbf{x}) = \int_S \Big\{ [\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{\Gamma}}(\mathbf{z} - \mathbf{x})]^T \mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x} - \mathbf{z})\,\mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \Big\} d_{\mathbf{z}}S$$

Corollary 2. *If U is a regular solution of the homogeneous equation (22) for* $\mathbf{x} \in \Omega^-$*, then*

$$\mathbf{U}(\mathbf{x}) = -\int_S \Big\{ [\tilde{\mathbf{P}}(\mathbf{D_z},\mathbf{n})\tilde{\mathbf{\Gamma}}(\mathbf{z}-\mathbf{x})]^T \mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x}-\mathbf{z})\,\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z}) \Big\} d_{\mathbf{z}}S$$

Corollary 3. *The regular solution of the homogeneous equation (22) has continuous partial derivatives of any order at an arbitrary point not belonging to S.*

Thermoelastopotentials

In this section, we present the basic properties of the thermoelastopotentials and the singular integral operators.

We introduce the single-layer potential

$$\mathbf{Z}^{(1)}(\mathbf{x},\mathbf{g}) = \int_S \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\,\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$$

the double-layer potential

$$\mathbf{Z}^{(2)}(\mathbf{x},\mathbf{g}) = \int_S [\tilde{\mathbf{P}}(\mathbf{D_y},\mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^{\mathrm{T}}(\mathbf{x}-\mathbf{y})]^{\mathrm{T}}\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$$

and the volume potential

$$\mathbf{Z}^{(3)}(\mathbf{x},\phi,\Omega^{\pm}) = \int_{\Omega^{\pm}} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})d\mathbf{y}$$

where $\mathbf{g}$ and ϕ are seven-component vectors.

Remark 3. By Theorems 11 and 12, the regular in Ω^+ (or in Ω^-) solution of (7) is represented by sum of single layer, double layer, and volume of potentials:

$$\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(2)}(\mathbf{x},\mathbf{U}) - \mathbf{Z}^{(1)}(\mathbf{x},\mathbf{PU}) + \mathbf{Z}^{(3)}(\mathbf{x},\mathbf{F},\Omega^+) \quad \text{for } \mathbf{x}\in\Omega^+$$

$$\Big(\text{or } \mathbf{U}(\mathbf{x}) = -\mathbf{Z}^{(2)}(\mathbf{x},\mathbf{U}) + \mathbf{Z}^{(1)}(\mathbf{x},\mathbf{PU}) + \mathbf{Z}^{(3)}(\mathbf{x},\mathbf{F},\Omega^-) \quad \text{for } \mathbf{x}\in\Omega^-\Big)$$

The basic properties of potentials are given in the following theorems:

Theorem 13. *If* $S\in C^{m+1,\alpha_1}$, $\mathbf{g}\in C^{m,\alpha_2}(S)$, $0<\alpha_2<\alpha_1\le 1$, *and m is a nonnegative whole number, then*

(a) $\mathbf{Z}^{(1)}(\cdot,\mathbf{g})\in C^{0,\alpha_2}(E^3)\cap C^{m+1,\alpha_2}(\bar{\Omega}^{\pm})\cap C^{\infty}(\Omega^{\pm})$
(b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Z}^{(1)}(\mathbf{x},\mathbf{g}) = \mathbf{0}$ *for* $\mathbf{x}\in\Omega^{\pm}$
(c) $\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g}) = \int_S \mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{\Gamma}(\mathbf{z}-\mathbf{y})\,\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$ is the singular integral for $\mathbf{z}\in S$.
(d) $\Big\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g})\Big\}^{\pm} = \mp\frac{1}{2}\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g})$ *for* $\mathbf{z}\in S$

Theorem 14. *If* $S\in C^{m+1,\alpha_1}$, $\mathbf{g}\in C^{m,\alpha_2}(S)$, $0<\alpha_2<\alpha_1\le 1$, *then*

(a) $\mathbf{Z}^{(2)}(\cdot,\mathbf{g})\in C^{m,\alpha_2}(\bar{\Omega}^{\pm})\cap C^{\infty}(\Omega^{\pm})$
(b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Z}^{(2)}(\mathbf{x},\mathbf{g}) = \mathbf{0}$ *for* $\mathbf{x}\in\Omega^{\pm}$
(c) $\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g}) = \int_S [\tilde{\mathbf{P}}(\mathbf{D_y},\mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^{\mathrm{T}}(\mathbf{z}-\mathbf{y})]^{\mathrm{T}}\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$ *is the singular integral for* $\mathbf{z}\in S$.
(d) $\{\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})\}^{\pm} = \pm\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})$ *for nonnegative whole number m and* $\mathbf{z}\in S$.
(e) $\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})\}^{+} = \{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})\}^{-}$ *for the natural number m and* $\mathbf{z}\in S$.

Theorem 15. *If* $S\in C^{1,\alpha_1}$, $\phi\in C^{0,\alpha_2}(\Omega^+)$, $0<\alpha_2<\alpha_1\le 1$, then

(a) $\mathbf{Z}^{(3)}(\cdot,\phi,\Omega^+)\in C^{1,\alpha_2}(E^3)\cap C^2(\Omega^+)\cap C^{2,\alpha_2}(\bar{\Omega}_0^+)$
(b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Z}^{(3)}(x,\phi,\Omega^+) = \phi(\mathbf{x})$ *for* $\mathbf{x}\in\Omega^+$ *where* Ω_0^+ *is a domain in* R^3 *and* $\Omega_0^+\subset\Omega^+$.

Theorem 16. *If* $S\in C^{1,\alpha_1}$, $\mathrm{supp}\phi = \Omega\subset\Omega^-$, $\phi\in C^{0,\alpha_2}(\Omega^-)$, $0<\alpha_2<\alpha_1\le 1$, *then*

(a) $\mathbf{Z}^{(3)}(\cdot,\phi,\Omega^-)\in C^{1,\alpha_2}(E^3)\cap C^2(\Omega^-)\cap C^{2,\alpha_2}(\bar{\Omega}_0^-)$
(b) $\mathbf{A}(\mathbf{D_x})\mathbf{Z}^{(3)}(\mathbf{x},\phi,\Omega^-) = \phi(\mathbf{x})$ *for* $\mathbf{x}\in\Omega^-$ *where* Ω *is a finite domain in* E^3 *and* $\bar{\Omega}_0^-\subset\Omega^-$.

Theorems 13–16 can be proved similarly to the corresponding theorems in the classical theory of elasticity and thermoelasticity (for details, see [12], Chapters V and X). The proposition e) of Theorem 14 is generalization of the Lyapunov-Tauber theorem for the double-layer potential of the classical theory of elasticity (see [12], Ch. V).

We introduce the notation

$$\begin{aligned}\mathcal{K}^{(1)}\mathbf{g}(\mathbf{z}) &= \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})\\ \mathcal{K}^{(2)}\mathbf{g}(\mathbf{z}) &= \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D}_\mathbf{z},\mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g})\\ \mathcal{K}^{(3)}\mathbf{g}(\mathbf{z}) &= -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g})\\ \mathcal{K}_{(\eta)}\mathbf{g}(\mathbf{z}) &= \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \eta\,\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g}) \qquad \mathbf{z}\in S\end{aligned} \tag{23}$$

where η is an arbitrary complex number. By virtue of Theorems 13 and 14, $\mathcal{K}^{(1)}$, $\mathcal{K}^{(2)}$, $\mathcal{K}^{(3)}$, and $\mathcal{K}_{(\eta)}$ are the singular integral operators. Obviously, the operators $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$ are adjoint with respect to each.

Remark 4. For the definitions of a singular integral operator, a normal-type singular integral operator, the symbol, and the index of operators, see, e.g., Kupradze et al. [12]. The basic theory of multidimensional singular integral equations is given in Kupradze et al. [12] and Mikhlin [13].

In the sequel we need the following lemma:

Lemma 1. *If $\mathcal{L}$ is a continuous curve on the complex plane connecting the origin with the point η_0 and $\mathcal{K}_{(\eta)}$ is a normal-type operator for any $\eta\in\mathcal{L}$, then the index of the operator $\mathcal{K}_{(\eta_0)}$ vanishes*, i.e.,

$$\operatorname{ind}\mathcal{K}_{(\eta_0)} = 0$$

Lemma 1 is proved in Kupradze et al. [12].

Theorem 17. *If conditions (9) and (10) are satisfied, then the singular integral operator $\mathcal{K}^{(p)}$ is of the normal type with the index equals to zero, where $p = 1,2,3$.*

Proof. Let $\boldsymbol{\sigma}^{(p)} = (\sigma_{lj}^{(p)})_{7\times 7}$ be the symbol of the operator $\mathcal{K}^{(p)}$ $(p = 1,2,3)$. From (23) we have

$$\det\ \boldsymbol{\sigma}^{(1)} = \det\ \boldsymbol{\sigma}^{(2)} = -\det\ \boldsymbol{\sigma}^{(3)} = -\frac{1}{2}\,\sigma_1\,\sigma_2 \tag{24}$$

where

$$\begin{aligned}\sigma_1 &= \frac{(\lambda+\mu)(\lambda+3\mu)}{8(\lambda+2\mu)^2}\\ \sigma_2 &= \frac{1}{32\,k_6^2\,k_7^2}(k_5+k_6)(k_6+k_7)\\ &\quad(2k_6k_7 - k_5k_7 + k_4k_6)\end{aligned} \tag{25}$$

Keeping in mind the relations (9) and (10) from (25), we have $\sigma_1 > 0$ and $\sigma_2 > 0$. Obviously, from (24) we obtain

$$\det\ \boldsymbol{\sigma}^{(m)} < 0 \qquad \det\ \boldsymbol{\sigma}^{(3)} > 0 \qquad \text{for}\quad m = 1,2 \tag{26}$$

Hence, the operators $\mathcal{K}^{(1)}$, $\mathcal{K}^{(2)}$, and $\mathcal{K}^{(3)}$ are of the normal type.

Let $\boldsymbol{\sigma}_{(\eta)}$ and ind $\mathcal{K}_{(\eta)}$ be the symbol and the index of the operator $\mathcal{K}_{(\eta)}$, respectively. It may be easily shown that $\det\ \boldsymbol{\sigma}_{(\eta)}$ vanishes only at four points η_1, η_2, η_3, and η_4 of the complex plane. By virtue of (26) and $\det\ \boldsymbol{\sigma}_1 = \det\ \boldsymbol{\sigma}^{(1)}$, we get $\eta_l \neq 1$ for $l = 1,2,3,4$. By Lemma 1, we obtain

$$\operatorname{ind}\mathcal{K}^{(1)} = \operatorname{ind}\mathcal{K}_{(1)} = 0$$

Equation $\operatorname{ind}\mathcal{K}^{(2)} = 0$ is proved in a quite similar manner. ◇

Theorem 17 leads the following result:

Theorem 18. *If conditions (9) and (10) are satisfied, then Fredholm's theorems are valid for the following singular integral equation:*

$$\mathcal{K}^{(p)}\,\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \qquad \textit{for}\quad \mathbf{z}\in S$$

where **f** *is a seven-component vector function on S and $p = 1,2,3$.*

Existence Theorems

In this section, we establish the existence of regular solutions of the basic BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{-}_{\mathbf{F},\mathbf{f}}$ by means of the potential method and the theory of singular integral equations.

Obviously, by virtue of Theorems 15 and 16, the volume potentials $\mathbf{Z}^{(3)}(\mathbf{x},\mathbf{F},\Omega^+)$ and $\mathbf{Z}^{(3)}(\mathbf{x},\mathbf{F},\Omega^-)$ are the partial regular solutions of (7) in Ω^+ and Ω^-, respectively, where $\mathbf{F}\in C^{0,\alpha_2}(\Omega^\pm)$, $0<\alpha_2\le 1$; $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-. Therefore, further we will consider problems $(I)^{\pm}_{0,\mathbf{f}}$ and $(II)^{-}_{0,\mathbf{f}}$.

Problems $(I)^{+}_{0,\mathbf{f}}$ and $(II)^{-}_{0,\mathbf{f}}$. The regular solution of problem $(I)^{+}_{0,\mathbf{f}}$ is sought in the form of double-layer potential

$$\mathbf{U}(\mathbf{x})=\mathbf{Z}^{(2)}(\mathbf{x},\mathbf{g})\qquad\text{for}\qquad \mathbf{x}\in\Omega^+\tag{27}$$

where $\mathbf{g}$ is the unknown seven-component vector. Taking into account the boundary property of potential of double layer (see Theorem 14) and boundary condition of problem $(I)^{+}_{0,\mathbf{f}}$, we obtain, for determining the unknown vector $\mathbf{g}$, the following singular integral equation:

$$\mathcal{K}^{(1)}\,\mathbf{g}(\mathbf{z})=\mathbf{f}(\mathbf{z})\qquad\text{for}\qquad \mathbf{z}\in S\tag{28}$$

On the other hand, the regular solution of problem $(II)^{-}_{0,\mathbf{f}}$ is sought in the form of single-layer potential

$$\mathbf{U}(\mathbf{x})=\mathbf{Z}^{(1)}(\mathbf{x},\boldsymbol{\varphi})\qquad\text{for}\qquad \mathbf{x}\in\Omega^-\tag{29}$$

where $\boldsymbol{\varphi}$ is the unknown seven-component vector. On the basis of the boundary property of single-layer potential (see Theorem 13) and boundary condition of problem $(II)^{-}_{0,\mathbf{f}}$, we obtain, for determining the unknown vector $\boldsymbol{\varphi}$, the following singular integral equation:

$$\mathcal{K}^{(2)}\,\boldsymbol{\varphi}(\mathbf{z})=\mathbf{f}(\mathbf{z})\qquad\text{for}\qquad \mathbf{z}\in S\tag{30}$$

On the basis of Theorem 18, the Fredholm's theorems are valid for (28) and (30). It is easy to see that the homogeneous equations $\mathcal{K}^{(1)}\,\mathbf{g}(\mathbf{z})=\mathbf{0}$ and $\mathcal{K}^{(2)}\,\boldsymbol{\varphi}(\mathbf{z})=\mathbf{0}$ admit only trivial solutions [19]. On the basis of the Fredholm's theorems, there exist solutions of singular integral equations (28) and (30). We have thereby proved the following existence theorems:

Theorem 19. *If* $S\in C^{2,\alpha_1}$, $\mathbf{f}\in C^{1,\alpha_2}(S)$, $0<\alpha_2<\alpha_1\le 1$, *then a regular solution of problem* $(I)^{+}_{0,\mathbf{f}}$ *exists, is unique, and is represented by double-layer potential (27), where* $\mathbf{g}$ *is a solution of the singular integral equation (28), which is always solvable for an arbitrary vector* $\mathbf{f}$.

Theorem 20. *If* $S\in C^{1,\alpha_1}$, $\mathbf{f}\in C^{0,\alpha_2}(S)$, $0<\alpha_2<\alpha_1\le 1$, *then a regular solution of problem* $(II)^{-}_{0,\mathbf{f}}$ *exists, is unique, and is represented by single-layer potential (29), where* φ *is a solution of the singular integral equation (30), which is always solvable for an arbitrary vector* $\mathbf{f}$.

Problem $(I)^{-}_{0,\mathbf{f}}$. We have the following result:

Theorem 21. *If* $S\in C^{2,\alpha_1}$, $\mathbf{f}\in C^{1,\alpha_2}(S)$, $0<\alpha_2<\alpha_1\le 1$, *then a regular solution of problem* $(I)^{-}_{0,\mathbf{f}}$ *exists, is unique, and is represented by sum of double-layer and single-layer potentials*

$$\mathbf{U}(\mathbf{x})=\mathbf{Z}^{(2)}(\mathbf{x},\mathbf{g})+a\,\mathbf{Z}^{(1)}(\mathbf{x},\mathbf{g})\qquad\text{for}\qquad \mathbf{x}\in\Omega^-$$

where a *is an arbitrary positive number and* $\mathbf{g}$ *is a solution of the singular integral equation*

$$\mathcal{K}^{(3)}\,\mathbf{g}(\mathbf{z})+a\,\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g})=\mathbf{f}(\mathbf{z})\qquad\text{for}\qquad \mathbf{z}\in S$$

which is always solvable for arbitrary vector $\mathbf{f}$.

Theorem 21 is proved quite similarly as Theorem 19.

Cross-References

- ▶ Fundamental Solutions in Thermoelasticity Theory
- ▶ Fundamental Solutions in Thermoelastostatics of Micromorphic Solids
- ▶ Large Existence of Solutions in Thermoelasticity Theory of Steady Vibrations
- ▶ Potentials in Thermoelasticity Theory
- ▶ Representations of Solutions in Thermoelasticity Theory

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Beams

► Beams, Thermal Stresses
► Thermal Post-Buckling Paths of Beams

Beams, Thermal Stresses

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Synonyms

Beams

Overview

A beam is one of the most important members or parts in machine structures. It is a long bar which chiefly causes bending deformation. Bending deformation in beams subjected to mechanical loads is developed for addressing the relationship between strain and curvature and so on. Bending deformation in beams subjected to thermal loads is also developed for addressing the relationship among strain, curvature, thermal expansion, and thermal stress.

Thermal stresses in beams are treated in homogeneous beams, composite beams with two and more homogeneous beams, and inhomogeneous beams whose material properties depend on position. Temperatures which vary in the lateral direction to the beam are treated so that only the symmetry deformations are treated.

Stresses in Beams Subjected to Mechanical Loads

A beam is defined as a long bar which chiefly causes bending by acting of lateral loads such as concentrated load P, distributed load q, and/or external moment M_0 as shown in Fig. 1.

A beam also may be regarded as a mass of longitudinal fibers, which are straight in longitudinal direction. It is assumed that Bernoulli-Euler hypothesis is valid, which means that a plane section perpendicular to the longitudinal fibers before loading remains a plane perpendicular to the deformed longitudinal fibers after loading. Now, we take out an element ABCD with a minute width dx from the beam in Fig. 1 and consider the shapes before and after the deformation as shown in Fig. 2. Although bending moment M at an arbitrary cross section is a function of position x, the moments induced at both cross sections in an element with a minute width dx are almost same as shown in Fig. 2b.

The element ABCD changes to the element A′B′C′D′ after deformation. Longitudinal fibers on the convex side A′B′ are extended, and the fibers on the concave side C′D′ are compressed. There must be a plane in which the elongation of fibers is zero and the length of fibers is dx in the beam. The plane is called the neutral plane (plane m′n′ in Fig. 2b). Any fiber pq at a distance of y from the neutral plane in the element changes to the fiber $p'q'$. The bending strain ε at the distance y from the neutral plane after bending is

$$\begin{aligned}\varepsilon &= \frac{\overset{\frown}{p'q'} - \overline{pq}}{\overline{pq}} = \frac{\overset{\frown}{p'q'} - \overline{mn}}{\overline{mn}} = \frac{\overset{\frown}{p'q'} - \overset{\frown}{m'n'}}{\overset{\frown}{m'n'}} \\ &= \frac{(\rho + y)d\theta - \rho d\theta}{\rho d\theta} = \frac{y}{\rho}\end{aligned} \tag{1}$$

where ρ is the radius of curvature at the neutral plane. Hooke's law gives the normal stress as follows:

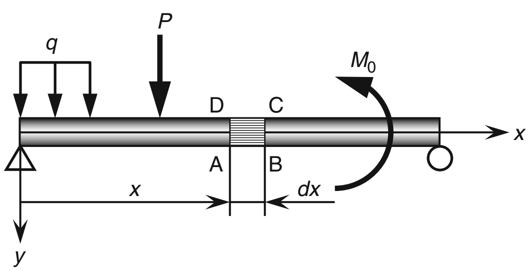
$$\sigma = E\varepsilon = E\frac{y}{\rho} \tag{2}$$

The normal stress σ is called the bending stress. The distribution of the bending stress is proportional to the distance y from the neutral plane. If the axial force is not applied to the beam, the equilibrium of force and the equilibrium of moment on the plane A′D′ give the following relations:

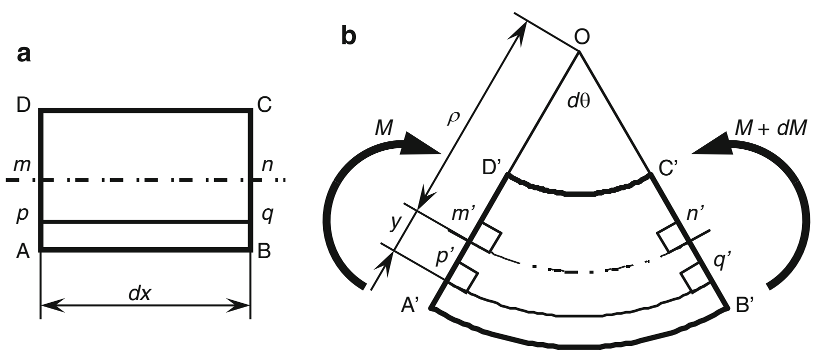
$$\int_A \sigma dA = 0 \tag{3}$$

$$\int_A \sigma y dA = M \tag{4}$$

where dA is a small element area of cross section at the distance y from the neutral plane. When the material is homogeneous in the cross section,



Beams, Thermal Stresses, Fig. 1 Beams subjected to mechanical loads



Beams, Thermal Stresses, Fig. 2 Deformation of beam with a minute width dx. (**a**) before deformation; (**b**) after deformation

Young's modulus is independent of the integral with respect to the area. Therefore, the substitution of (2) into (3) and (4) gives the following relations:

$$\int_A \sigma dA = \int_A E\varepsilon dA = \frac{E}{\rho}\int_A y dA = 0 \qquad (5)$$

$$M = \int_A E\varepsilon y dA = \frac{E}{\rho}\int_A y^2 dA = \frac{E}{\rho} I \qquad (6)$$

where I is the moment of inertia of the cross section with respect to the neutral axis and is defined by

$$I = \int_A y^2 dA \qquad (7)$$

Equation (5) suggests that the neutral axis passes through the centroid of the cross section. Eliminating the radius of curvature ρ from (2) and (6), the bending stress is expressed by

$$\sigma = \frac{My}{I} \qquad (8)$$

Thermal Stresses in Beams

When a beam is subjected to a temperature change T and is unconstrained, it will expand freely. The free thermal strain ε is given as

$$\varepsilon = \alpha T \qquad (9)$$

where α is the coefficient of linear thermal expansion. The bending deformation in beams subjected to the mechanical loads is induced by the strain which is a function of position y as shown in (1). The beam subjected to the temperature change also induces the bending deformation when the strain is a function of position y in (9), i.e., the coefficient of linear thermal expansion α and/or the temperature change T are functions of position y. So the homogeneous materials have the bending deformation only when the temperature change is a function of position y, since the material properties are constant in homogeneous materials. If the temperature change is uniform in homogeneous materials, the long bar just caused only the elongation or the shrinkage. On the other hand, the uniform temperature change can induce the bending deformation in the inhomogeneous material beams or the composite beams, since the linear thermal expansion α is a function of position y. We explain thermal stresses in beams due to two causes: the temperature change and the material property difference.

Thermal Stresses in Beams Due to Temperature Change

Let us consider the strain and stress in beams, which consist of homogeneous materials, subjected to thermal loads. When the beam is subjected to thermal loads, it will be deformed. The deformation will consist of axial elongation and of bending when the thermal loads change along y direction. We assume that the Bernoulli-Euler assumption is valid also in the case of thermal loads as same as mechanical loads. We take the neutral axis to pass through the centroid of the cross section of the beam. A fiber pq at a distance of y from the neutral axis in the element elongates to $p'q'$ as shown in Fig. 2. As a fiber mn at the neutral axis elongates to $m'n'$ due to the temperature change, the strain ε at the distance y from the neutral axis after deformation due to the temperature change is

$$\begin{aligned}
\varepsilon &= \frac{\widehat{p'q'} - \overline{pq}}{\overline{pq}} = \frac{\widehat{p'q'} - \overline{mn}}{\overline{mn}} \\
&= \frac{\left(\widehat{p'q'} - \widehat{m'n'}\right) + \left(\widehat{m'n'} - \overline{mn}\right)}{\overline{mn}} \\
&= \frac{\widehat{m'n'} - \overline{mn}}{\overline{mn}} + \frac{\widehat{p'q'} - \widehat{m'n'}}{\overline{mn}} \\
&= \varepsilon_0 + \frac{(\rho + y)d\theta - \rho d\theta}{\rho d\theta}\frac{\rho d\theta}{\overline{mn}} \\
&= \varepsilon_0 + \frac{y}{\rho}\left(1 + \frac{\widehat{m'n'} - \overline{mn}}{\overline{mn}}\right) \\
&= \varepsilon_0 + \frac{y}{\rho}\left(1 + \frac{\rho d\theta - \overline{mn}}{\overline{mn}}\right) \\
&= \varepsilon_0 + \frac{y}{\rho}(1 + \varepsilon_0) \cong \varepsilon_0 + \frac{y}{\rho}
\end{aligned} \qquad (10)$$

where ρ is the radius of curvature at the neutral plane and ε_0 is the axial strain at the neutral plane.

The strain ε at a distance of y from the neutral axis in the beam consists of the free thermal strain and the strain due to the bending stress σ_x:

$$\varepsilon = \alpha T + \frac{\sigma_x}{E} = \varepsilon_0 + \frac{y}{\rho} \tag{11}$$

Solving this equation for σ_x, we find

$$\sigma_x = -\alpha ET + \varepsilon_0 E + E\frac{y}{\rho} \tag{12}$$

Since the beam is free from external forces, the equilibrium of force and that of moment gives the following relations:

$$\int_A \sigma_x dA = 0 \tag{13}$$

$$\int_A \sigma_x y dA = 0 \tag{14}$$

where dA is a small element area of the cross section at a distance of y from the neutral plane. Substitution of (12) into (13) and (14) gives the axial strain ε_0 and the curvature $1/\rho$ at the neutral plane $y = 0$ as follows:

$$\varepsilon_0 = \frac{1}{EA}\int_A \alpha ET(y)dA \tag{15}$$

$$\frac{1}{\rho} = \frac{1}{EI}\int_A \alpha ET(y)ydA \tag{16}$$

where I is the moment of inertia of the cross section. Then, the substitution of (15) and (16) into (11) gives the thermal stress

$$\sigma_x(y) = -\alpha ET(y) + \frac{1}{A}\int_A \alpha ET(y)dA + \frac{y}{I}\int_A \alpha ET(y)ydA \tag{17}$$

Equation (17) gives the general solution for the thermal stress in the beam under thermal loads. Now, we consider the boundary conditions at the end of the beam on extension and bending: the restrained extension in the axial direction means that the axial strain is zero; the free extension in the axial direction means that the axial strain is not zero and a finite value; the restrained bending means that the curvature is zero; and the free bending means that the curvature is not zero and a finite value. The combination of these conditions gives the four boundary conditions and the thermal stresses:

(a) The thermal stress for the beam with perfectly clamped ends is

$$\sigma(y) = -\alpha ET(y) \tag{18}$$

(b) The thermal stress for the beam with free extension and restrained bending ends is

$$\sigma(y) = -\alpha ET(y) + \frac{1}{A}\int_A \alpha ET(y)dA \tag{19}$$

(c) The thermal stress for the beam with restrained extension and free bending ends is

$$\sigma(y) = -\alpha ET(y) + \frac{y}{I}\int_A \alpha ET(y)ydA \tag{20}$$

(d) The thermal stress for the beam with free extension and free bending is given with (17).

Rectangular cross section is one of the most commonly used cross-sectional shapes of the beams. The thermal stress σ_x for the beam with rectangular cross section of the height h and the width b is

$$\sigma_x(y) = -\alpha ET(y) + \frac{1}{h}\int_{-h/2}^{h/2} \alpha ET(y)dy + \frac{12y}{h^3}\int_{-h/2}^{h/2} \alpha ET(y)ydy \tag{21}$$

since the area dA and A and the moment of inertia I are given as follows:

$$dA = bdy, \quad A = bh, \quad I = \frac{bh^3}{12}$$

We use homogeneous materials in almost cases. Now we consider the case that the temperature change is constant and is not a function of position y. Then the curvature $1/\rho$ is zero from (16). The beam has no bending deformation any more.

Thermal Stresses in Beams Due to Material Properties Difference

Thermal Stresses in Composite Beams

We consider the thermal stresses in the composite beams which consist of multiple homogeneous beams with different material properties. A simple example is two parallel beams clamped at each end to rigid and non-heat-conducting plates as shown in Fig. 3a. Each beam has different cross section area A_i and temperature change T_i $(i = 1, 2)$. The coordinate system is shown in Fig. 3a. The origin of the coordinate y is taken at arbitrary position between two beams, the origin of the local coordinate y_i is taken at the centroid of the cross section of each beam, e denotes the distance between the centroids of cross section of both beams, and e_i denotes the distance from $y = 0$ to the centroid of the cross section of each bar.

The moments of the area of the cross section for each beam are zero:

$$\int_{A_i} y_i dA_i = 0 \quad (i = 1,\ 2) \tag{22}$$

because the origin of the coordinate system passes through the centroid of the section of each beam.

When ε_0 and ρ denote the axial strain at $y = 0$ and the radius of curvature at $y = 0$, respectively, as shown in Fig. 3b, the strains ε_{xi} at a distance of y from the origin are expressed by the following equation as same as (11)

$$\varepsilon_{xi} = \alpha_i T_i(y_i) + \frac{\sigma_{xi}}{E_i} = \varepsilon_0 + \frac{y}{\rho} \qquad (i = 1, 2) \tag{23}$$

Then the stresses σ_{xi} may be expressed by

$$\sigma_{xi} = E_i\left[\varepsilon_0 + \frac{y}{\rho} - \alpha_i T_i(y_i)\right] \qquad (i = 1, 2) \tag{24}$$

The equilibrium of force and that of moment given with (13) and (14) are valid since the beam is free from the external forces. Using the relationship $y_2 = y - e_2$ and $y_1 = y - e_1$, we determine the normal strain ε_0 and the curvature $1/\rho$ at $y = 0$ to satisfy (13) and (14) as follows:

$$\varepsilon_0 = \frac{P_T I_{E2} - M_T I_{E1}}{I_{E0} I_{E2} - I_{E1}^2}, \qquad \frac{1}{\rho} = \frac{M_T I_{E0} - P_T I_{E1}}{I_{E0} I_{E2} - I_{E1}^2} \tag{25}$$

where

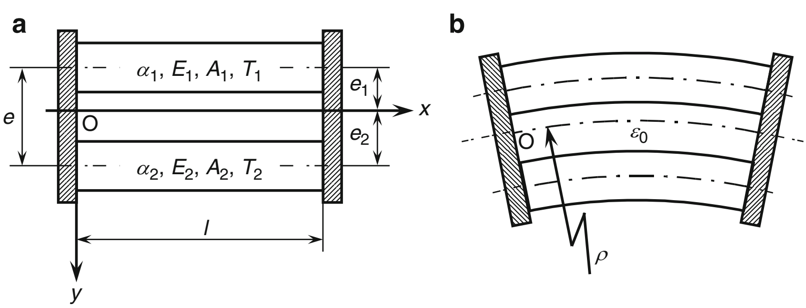
$$I_{E0} = E_1 A_1 + E_2 A_2$$

$$I_{E1} = e_2 E_2 A_2 - e_1 E_1 A_1$$

$$I_{E2} = E_1(I_1 + A_1 e_1^2) + E_2(I_2 + A_2 e_2^2)$$

$$P_T = P_{T1} + P_{T2}$$

$$P_{T1} = \int_{A_1} \alpha_1 E_1 T_1(y_1) dA_1$$



Beams, Thermal Stresses, Fig. 3 Two parallel beams clamped to rigid plates at both ends. (**a**) before deformation; (**b**) after deformation

$$P_{T2} = \int_{A_2} \alpha_2 E_2 T_2(y_2) dA_2$$

$$M_T = M_{T1} + M_{T2} + e_2 P_{T2} - e_1 P_{T1}$$

$$M_{T1} = \int_{A_1} \alpha_1 E_1 T_1(y_1) y_1 dA_1$$

$$M_{T2} = \int_{A_2} \alpha_2 E_2 T_2(y_2) y_2 dA_2$$

Thus, the thermal stresses may be expressed by

$$\begin{aligned} \sigma_{x1}(y_1) &= -\alpha_1 E_1 T_1(y_1) \\ &\quad + E_1 \frac{P_T I_{E2} - M_T I_{E1}}{I_{E0} I_{E2} - I_{E1}^2} \\ &\quad + (y_1 - e_1) E_1 \frac{M_T I_{E0} - P_T I_{E1}}{I_{E0} I_{E2} - I_{E1}^2} \\ \sigma_{x2}(y_2) &= -\alpha_2 E_2 T_2(y_2) \\ &\quad + E_2 \frac{P_T I_{E2} - M_T I_{E1}}{I_{E0} I_{E2} - I_{E1}^2} \\ &\quad + (y_2 + e_2) E_2 \frac{M_T I_{E0} - P_T I_{E1}}{I_{E0} I_{E2} - I_{E1}^2} \end{aligned} \tag{26}$$

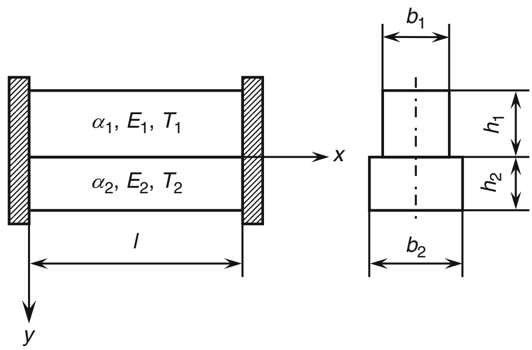
We consider a beam of two layers, which has different rectangular cross section $b_1 \times h_1$ and $b_2 \times h_2$, respectively, bonded together and subjected to temperature change $T_i(y)$ as shown in Fig. 4 as a special case of two parallel beams clamped of to rigid plates at both ends in Fig. 3.

The cross sectional areas dA_i and A_i and the moment of inertia I_i and e_i are given as follows:

$$dA_i = b_i dy_i, \quad A_i = b_i h_i, \quad I_i = \frac{b_i h_i^3}{12}, \quad e_i = \frac{h_i}{2} \tag{27}$$

Substituting (25) into (23), the normal strain and the curvature $1/\rho$ at bonding surface are given as follows:

$$\begin{aligned} \varepsilon_0 &= \frac{2}{D} \Big[2P_T (E_2 b_2 h_2^3 + E_1 b_1 h_1^3) \\ &\quad - 3M_T (E_2 b_2 h_2^2 - E_1 b_1 h_1^2) \Big] \\ \frac{1}{\rho} &= \frac{6}{D} \Big[2M_T (E_2 b_2 h_2 + E_1 b_1 h_1) \\ &\quad - P_T (E_2 b_2 h_2^2 - E_1 b_1 h_1^2) \Big] \end{aligned} \tag{28}$$



Beams, Thermal Stresses, Fig. 4 A beam consisting of two layers bonded together

where

$$P_T = \int_{-h_1}^{0} \alpha_1 E_1 T_1(y) b_1 dy + \int_{0}^{h_2} \alpha_2 E_2 T_2(y) b_2 dy$$

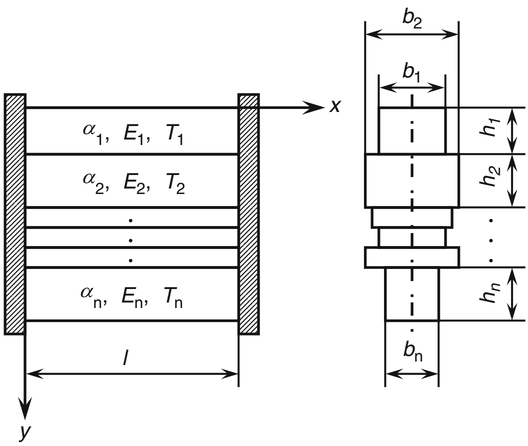
$$M_T = \int_{-h_1}^{0} \alpha_1 E_1 T_1(y) y b_1 dy + \int_{0}^{h_2} \alpha_2 E_2 T_2(y) y b_2 dy$$

$$D = (E_2 b_2 h_2^2 - E_1 b_1 h_1^2)^2 + 4E_1 E_2 b_1 b_2 h_1 h_2 (h_1 + h_2)^2$$

The thermal stresses are given with (24).

Now we consider the case that the temperature changes in each layer are constant, i.e., $T_1(y) = T_2(y) =$ constant. The curvature $1/\rho$ is not zero in this case. The beam yields the bending deformation and the thermal stresses. Bimetal is one of such composite beam used as thermostat. It consists of two different thin metal plates, which are selected to yield large deformation when it is heated by surrounding thermal field. The large bending deformation may switch off when the temperature exceeds the preset temperature.

We consider an n-layered composite beam with rectangular cross sections subjected to temperature change $T_i(y)$ as shown in Fig. 5. We take the origin $y = 0$ of the coordinate system y at the upper surface of the composite beam.



Beams, Thermal Stresses, Fig. 5 Multilayered composite beam

When ε_0 and ρ denote the axial strain at the upper surface $y = 0$ and the radius of curvature at $y = 0$, respectively, the strains at a distance of y from the origin are expressed by

$$\varepsilon_x = \alpha_i T_i(y) + \frac{\sigma_{xi}}{E_i} = \varepsilon_0 + \frac{y}{\rho} \qquad (i = 1, 2, \ldots, n) \tag{29}$$

Then, the stresses σ_{xi} may be expressed by

$$\sigma_{xi} = E_i\left[\varepsilon_0 + \frac{y}{\rho} - \alpha_i T_i(y_i)\right] \qquad (i = 1, 2, \ldots, n) \tag{30}$$

When external forces do not act on the layers, the equilibrium of force and moment are given by

$$\sum_{i=1}^{n} \int_{y_{i-1}}^{y_i} \sigma_{xi} b_i dy = 0 \tag{31}$$

$$\sum_{i=1}^{n} \int_{y_{i-1}}^{y_i} \sigma_{xi} b_i y dy = 0 \tag{32}$$

where b_i and $h_i = y_i - y_{i-1}$ denote the width and the height of each layer, respectively, y_i $(i = 1, 2, \ldots, n)$ means the lower surface of the i-th beam, and $y_0 = 0$. Substitution of (30) into (31) and (32) yields the simultaneous equation on the normal strain ε_0 and the curvature $1/\rho$ at $y = 0$. Solving the simultaneous equation, we obtain the normal strain ε_0 and the curvature $1/\rho$ at $y = 0$ as follows as same as (25). The thermal stresses may be expressed from (30) as follows:

$$\sigma_{xi} = E_i\left[\frac{P_T I_{E2} - M_T I_{E1}}{I_{E0} I_{E2} - I_{E1}^2} + y\frac{M_T I_{E0} - P_T I_{E1}}{I_{E0} I_{E2} - I_{E1}^2} - \alpha_i T_i(y_i)\right] \tag{33}$$

$$(i = 1, 2, \ldots, n)$$

where

$$I_{E0} = \sum_{i=1}^{n} E_i b_i h_i$$

$$I_{E1} = \frac{1}{2}\sum_{i=1}^{n} E_i b_i (y_i^2 - y_{i-1}^2)$$

$$I_{E2} = \frac{1}{3}\sum_{i=1}^{n} E_i b_i (y_i^3 - y_{i-1}^3)$$

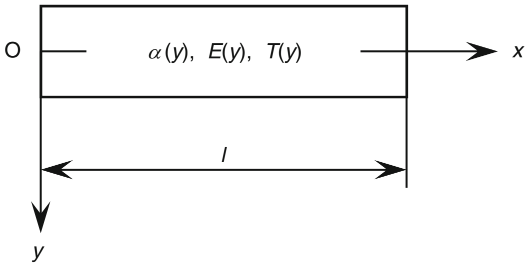
$$P_T = \sum_{i=1}^{n} \int_{y_{i-1}}^{y_i} \alpha_i E_i T_i(y) b_i dy$$

$$M_T = \sum_{i=1}^{n} \int_{y_{i-1}}^{y_i} \alpha_i E_i T_i(y) y b_i dy$$

Thermal Stresses in Inhomogeneous Beams

When the thickness of each layer h_i in the multi-layered composite beam is infinitely small, the beam is regarded as one beam whose material properties such as the coefficient of linear thermal expansion α and Young's modulus E depend on position. The materials are known as inhomogeneous materials such as quenched steel, functionally graded materials, and so on. We consider the thermal stress in an inhomogeneous beam, in which the coefficient of linear thermal expansion and Young's modulus are arbitrary functions of position y as shown in Fig. 6.

We take the origin $y = 0$ of the coordinate system at the centroid of the cross section of the beam. The relationship between the bending strains is given by



Beams, Thermal Stresses, Fig. 6 Inhomogeneous beam

$$\alpha(y)T(y) + \frac{\sigma_x}{E(y)} = \varepsilon_0 + \frac{y}{\rho} \tag{34}$$

where ε_0 and $1/\rho$ denote the normal strain and the curvature at $y = 0$, respectively. Solving (34), the stresses σ_x may be expressed by

$$\sigma_x = E(y)\left[\varepsilon_0 + \frac{y}{\rho} - \alpha(y)T(y)\right] \tag{35}$$

Since the beam is free from external forces and moments, the equilibrium of force and moment in (13) and (14) is also valid in this problem. The normal strain ε_0 and the curvature $1/\rho$ are determined in the same expression in (25) to satisfy (13) and (14):

$$\sigma_x(y) = E(y)\left[\frac{P_T I_{E2} - M_T I_{E1}}{I_{E0} I_{E2} - I_{E1}^2} + y\frac{M_T I_{E0} - P_T I_{E1}}{I_{E0} I_{E2} - I_{E1}^2} - \alpha(y)T(y)\right] \tag{36}$$

where

$$I_{E0} = \int_A E(y)dA$$

$$I_{E1} = \int_A E(y)y dA$$

$$I_{E2} = \int_A E(y)y^2 dA$$

$$P_T = \int_A \alpha(y)E(y)T(y)dA$$

$$M_T = \int_A \alpha(y)E(y)T(y)y dA$$

References

1. Noda N, Hetnarski RB, Tanigawa Y (2003) Thermal stresses, 2nd edn. Taylor & Francis, New York/London

BEM

▶ Boundary Element Method in Generalized Thermoelasticity
▶ Boundary Element Method in Heat Conduction
▶ Boundary Element Method in Inverse Heat Conduction Problem

Bending

▶ Saint-Venant's Problem for Cosserat Elastic Shells

Bifurcation

▶ Higher-Order Beam Theories
▶ Slender Beams, Thermal Buckling

Binary Mixtures

▶ Large Existence of Solutions in Thermoelasticity Theory of Steady Vibrations
▶ Potentials in Thermoelasticity Theory

Biopolymers

▶ Coupled and Generalized Thermoviscoelasticity

Biot Number

▶ Lumped Thermal Capacity Model

Bitumen Materials

▶ Coupled and Generalized Thermoviscoelasticity

Body Force Analogy for Thermoelasticity

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Overview

The classical body force analogy for static problems of thermoelasticity is firstly stated. Then we extended toward dynamic coupled problems. We consider two dynamic problems, namely, a thermal problem without body forces, but with a given distribution of transient sources of heat, and a force problem without sources of heat but with body forces. Both problems are treated within the coupled theory of thermoelasticity such that temperature must also be taken into account in the force problem. We restrict our considerations to the one-dimensional case, and we show that, given suitable boundary and initial conditions, a distribution of body forces can be constructed such that the dynamic displacements in both problems become equal. This analogy is checked by means of illustrative analytical examples. We also discuss the relations between the stresses and the temperature in both problems, and we mention that a similar analogy can be established, requiring the temperatures in both problems to be equal. In the end, extensions of the presented methodology with respect to three-dimensional formulations and with respect to the theory of generalized thermoelasticity are briefly presented.

Introduction

In statics, a well-known body force analogy does exist, which dates back to Duhamel, and which does not refer to stresses, but which refers to thermal displacements. A contemporary discussion of this Duhamel displacement analogy is given in the comprehensive book on the mathematical theory of elasticity (see, e.g., Hetnarski and Ignaczak [1], Noda, Hetnarski, and Tanigawa [2]). In its classical form, it can be stated as follows. Consider the static deformation of an isotropic linear thermoelastic body under the action of a given temperature. Then the thermal stresses can be obtained by addition of an imaginary pressure to the isothermal stresses that follow by solving the isothermal governing equations with certain imaginary body forces and surface tractions (see [2] for a contemporary proof). Moreover, the thermal displacements due to the given temperature are identical to the isothermal displacements due to the imaginary body forces and surface tractions. This follows by comparing the thermal boundary-value problem in hand with the imaginary isothermal boundary-value problem introduced by the body force analogy. We subsequently refer to the latter result when talking about a body force analogy.

The question of a dynamic extension of the aforementioned static body force analogy has gained interest in connection with the compensation of force-induced vibrations in a linear elastic body by smart actuation. This field is also known as shape control in the literature (see, e.g., the review papers by Irschik [3] and Ziegler [4] and the literature cited therein). The actuating strains that correspond to the physical effects are denoted as eigenstrains [5]. In the context of a transient actuation, it has been shown by Irschik and Pichler [6, 7] that the vibrations induced by transient forces in an anisotropic linear elastic body can be exactly compensated when a statically admissible transient stress is used as actuation stress. Temperature required for displacement compensation can easily be computed. Conversely, assuming that the temperature is given and reinserting the corresponding actuation stress in the equilibrium equation and boundary conditions for the statically admissible stress, imaginary body forces and surface tractions can be computed such that the displacements due to the temperature are exactly compensated.

This is not necessarily so, when coupling of deformation and heating should be taken into account in the heat conduction equation, which, however, is often the case under dynamic conditions. The deformations due to the forces then act as a driving source in the coupled heat conduction equation such that an additional temperature arises, which enters the stress–strain–temperature relationship. Nevertheless, the classical body force analogy can be extended to the coupled theory in the following sense. The thermal displacements arising from an initial-boundary-value problem of the coupled theory of thermoelasticity are identical to isothermal dynamic displacements due to suitable imaginary body forces and surface tractions. The value of the latter formulation lies in the fact that results of the coupled theory can be checked by results of the simpler isothermal theory. The drawback, however, of such an imaginary type of analogy is that it cannot be utilized directly in applications because coupling, if it is of any practical importance, should be taken into account in both the thermal problem and the force problem.

This entry is a complete extension of the classical static body force analogy to the coupled theory of dynamic thermoelasticity [8]. We restrict our considerations to the one-dimensional case, and we show that, given suitable boundary and initial conditions, a distribution of body forces can be constructed such that the dynamic displacements in both problems become equal. This physical analogy is checked by means of illustrative analytical examples. We also discuss the relations between the stresses and the temperature in both problems, and we point out that a similar analogy can be established that requires the temperatures in both problems to be equal. Extensions of the presented methodology with respect to three-dimensional formulations and with respect to the theory of generalized thermoelasticity (see, e.g., [9, 10]) are presented shortly. Some technical books related to the generalized thermoelasticity have been published (see, e.g., Ignaczak and Ostoja-Starzewski [11]).

The Classical Body Force Analogy for Static Problems of Thermoelasticity

The basic equations are written as follows.

The strain–displacement equation:

$$\mathbf{E} = \frac{\mathbf{1}}{\mathbf{2}}(\nabla\mathbf{U} + \nabla\mathbf{U}^{\mathbf{T}}) \tag{1}$$

The equilibrium equation:

$$\text{div}\,\mathbf{S} + \mathbf{b} = \mathbf{0} \tag{2}$$

The energy equation:

$$\text{-div}\,\mathbf{q} + r = c\,\dot{\theta} \tag{3}$$

The stress–strain–temperature equation:

$$\mathbf{S} = \mathbf{2}\mu\,\mathbf{E} + \lambda(\text{tr}\,\mathbf{E})\mathbf{1} - \beta\,\theta\,\mathbf{1} \tag{4}$$

The heat equation:

$$\mathbf{q} = -\mathbf{k}\nabla\theta \tag{5}$$

Here, **S** stands for the stress tensor, and **E** is the strain tensor. The ρ, λ, μ, β, c, and k are material parameters, such that density, Lame's constants, thermoelastic constant, specific heat, and thermal conductivity, which we take as independent of temperature θ. Sources of heat are denoted as r and are assumed to be given. Once and for all we assume that the fields introduced in (1)–(5) are sufficiently smooth, in order that our mathematical operations appear to be justified. Hence, we particularly exclude the presence of singular surfaces.

Introduction of (5) into (3) yields the heat conduction equation:

$$k\,\nabla^2\theta = c\dot{\theta} - r \tag{6}$$

Introducing (4) into (2), we obtain Navier's equation:

$$\mu\nabla^2\mathbf{U} + (\lambda + \mu)\nabla(\text{div}\,\mathbf{U}) - \beta\nabla\theta + \mathbf{b} = \mathbf{0} \tag{7}$$

Navier's equation (7) for thermoelasticity become Navier's equation for isothermal elasticity if the gradient of temperature change is taken as a body force. The generalized Hooke's law (4) can be rewritten in the form:

$$\mathbf{S} = \mathbf{S}^* - \beta\,\theta\,\mathbf{1} \tag{8}$$

where

$$\mathbf{S}^* = 2\mu\,\mathbf{E} + \lambda(\mathrm{tr}\,\mathbf{E})\mathbf{1} \tag{9}$$

Equations (9) correspond to Hooke's law for isothermal conditions. Substitution of (8) into (2) leads to:

$$\mathrm{div}\,\mathbf{S}^* + \mathbf{b}^* = \mathbf{0} \tag{10}$$

where

$$\mathbf{b}^* = \mathbf{b} - \beta\nabla\theta$$

This equation may be regarded as the body force which contains the imaginary body forces $-\beta\nabla\theta$ so that equilibrium equations (10) may be considered as the equilibrium equations under isothermal conditions.

This analysis shows that thermal stresses can be obtained by addition of the pressure $-\beta\,\theta$ to the isothermal stresses $\mathbf{S}^*$ which can be obtained by solving the isothermal governing equations with imaginary body forces and imaginary surface tractions $\beta\nabla\theta$.

The Body Force Analogy for Dynamic Coupled Problems of Thermoelasticity

We consider two one-dimensional coupled thermoelastic fields taken the coupling of the strain to conduction of heat into account, where the boundary conditions and initial conditions for the displacement and temperature of these fields are assumed equal. We denote the displacement and temperature in the two problems as U, θ and $\bar{U}$, $\bar{\theta}$, respectively. The basic equations of the first problem are written as follows.

The equation of motion:

$$\frac{\partial\sigma}{\partial X} + b = \rho\ddot{U} \tag{11}$$

The stress–strain–temperature relation:

$$\sigma = Y\varepsilon + \theta M \tag{12}$$

The strain–displacement relation:

$$\varepsilon = \frac{\partial U}{\partial X} \tag{13}$$

The energy equation:

$$-\frac{\partial q}{\partial X} + r = c\dot{\theta} - \theta_0 M\dot{\varepsilon} \tag{14}$$

The heat equation:

$$q = -K\frac{\partial\theta}{\partial X} \tag{15}$$

where σ stands for the stress, and ε is the strain. The axial coordinate is denoted by X, and ρ, Y, M, c, and K are material parameters, which we take as independent of temperature θ. Source of heat is denoted as r and is assumed to be given. Note that the coupling of the strain to conduction of heat is taken into account by the last term in the energy equation, (14), such that we deal with the so-called coupled theory of thermoelasticity.

Once and for all, we assume that the fields introduced in (11)–(15) are sufficiently smooth, so that our mathematical operations appear to be justified. Hence, we particularly exclude the presence of singular surfaces.

Introduction of (15) and (13) into (14) yields:

$$K\frac{\partial^2\theta}{\partial X^2} = c\dot{\theta} - \theta_0 M\frac{\partial\dot{U}}{\partial X} - r \tag{16}$$

By introducing (12) into (11) and using relation (3), we obtain:

$$Y\frac{\partial^2 U}{\partial X^2} + M\frac{\partial\theta}{\partial X} + b = \rho\ddot{U} \tag{17}$$

When we use same procedure and an analogous notation, the basic equations of second thermoelastic problem are:

$$K\frac{\partial^2\bar{\theta}}{\partial X^2} = c\dot{\bar{\theta}} - \theta_0 M\frac{\partial\dot{\bar{U}}}{\partial X} - \bar{r} \tag{18}$$

$$Y\frac{\partial^2\bar{U}}{\partial X^2} + M\frac{\partial\bar{\theta}}{\partial X} + \bar{b} = \rho\ddot{\bar{U}} \tag{19}$$

We now assume that $b = 0$ and $\bar{r} = 0$. The problem according to (16) and (17) will then be denoted as the thermal problem, whereas the problem stated in (18) and (19) will subsequently be called the force problem as same as the case of dynamical thermoelasticity problem. Then the fundamental equations of the thermal problem are:

$$K\frac{\partial^2\theta}{\partial X^2} = c\dot{\theta} - \theta_0 M\frac{\partial\dot{U}}{\partial X} - r \tag{20}$$

$$Y\frac{\partial^2 U}{\partial X^2} + M\frac{\partial\theta}{\partial X} = \rho\ddot{U} \tag{21}$$

while the force problem reads:

$$K\frac{\partial^2\bar{\theta}}{\partial X^2} = c\dot{\bar{\theta}} - \theta_0 M\frac{\partial\dot{\bar{U}}}{\partial X} \tag{22}$$

$$Y\frac{\partial^2\bar{U}}{\partial X^2} + M\frac{\partial\bar{\theta}}{\partial X} + \bar{b} = \rho\ddot{\bar{U}} \tag{23}$$

It is essential to note that the coupled theory of thermoelasticity is taken into account in both the thermal and the force problems.

We now introduce new notations:

$$U^* = U - \bar{U}$$

$$\Theta^* = \theta - \bar{\theta} \tag{24}$$

Subtracting (21) from (23) and (20) from (22) yields the following equations:

$$Y\frac{\partial^2 U^*}{\partial X^2} + M\frac{\partial\Theta^*}{\partial X} - \bar{b} = \rho\ddot{U}^* \tag{25}$$

$$K\frac{\partial^2\Theta^*}{\partial X^2} = c\dot{\Theta}^* - \theta_0 M\frac{\partial\dot{U}^*}{\partial X} - r \tag{26}$$

To extend the classical body force analogy to the coupled theory of dynamic thermoelasticity, we seek the condition $U^* = 0$, that is, $U = \bar{U}$. Hence, we take the sources of heat, r, in (26) to be given, and we seek for body forces $\bar{b}$ in (25) such that the displacements in the thermal and force problems become equal.

The first step in our solution is as follows. We assume that:

$$\bar{b} = M\frac{\partial\Theta^*}{\partial X} \tag{27}$$

Then (25) reduces to:

$$Y\frac{\partial^2 U^*}{\partial X^2} = \rho\ddot{U}^* \tag{28}$$

which indeed has $U^* = 0$ as a solution because the influent terms of temperature and body force are disappear.

In the present context, the body force $\bar{b}$ is sought such that the displacements in the thermal and force problems become equal. Therefore, an equation for computing the body force $\bar{b}$ is needed. We suggest the following construction. The differentiation of (27) yields:

$$\frac{\partial^2\Theta^*}{\partial X^2} = \frac{1}{M}\frac{\partial\bar{b}}{\partial X} \tag{29}$$

By use of (19), (16) becomes:

$$\frac{K}{M}\frac{\partial\bar{b}}{\partial X} = c\dot{\Theta}^* - \theta_0 M\frac{\partial\dot{U}^*}{\partial X} - r \tag{30}$$

Further differentiation of this equation and using (27) causes:

$$\frac{K}{M}\frac{\partial^2\bar{b}}{\partial X^2} = \frac{c}{M}\dot{\bar{b}} - \theta_0 M\frac{\partial^2\dot{U}^*}{\partial X^2} - \frac{\partial r}{\partial X} \tag{31}$$

Hence, when we assume that $U^* = 0$, we obtain the following partial differential equation for the body force:

$$\frac{\partial^2 \bar{b}}{\partial X^2} - \frac{c}{K}\dot{\bar{b}} = -\frac{M}{K}\frac{\partial r}{\partial X} \tag{32}$$

In other words, when we can guarantee that $U^* = 0$ is the solution of (28), then (32) can be used to compute a body force $\bar{b}$ that equalizes the displacements of the thermal problem due to the sources of heat, r. To ensure a trivial solution of (28), initial and boundary conditions must be taken into account.

We state two cases for the mechanical and thermal boundary conditions and initial conditions, which indeed have $U^* = 0$ as their solution.

Case 1.

$$\begin{aligned} &\text{Boundary condition } \mathrm{U}^* = 0, \frac{\partial \Theta^*}{\partial \mathrm{X}} = 0 \\ &\text{Initial condition } \mathrm{U}^* = 0, \Theta^* = 0 \end{aligned} \tag{33}$$

Case 2.

$$\begin{aligned} &\textit{Boundary condition } \frac{\partial U^*}{\partial X} = 0, \Theta^* = 0 \\ &\textit{Initial condition } U^* = 0, \Theta^* = 0 \end{aligned} \tag{34}$$

In both cases, the mechanical and thermal initial conditions of the force and the thermal problem are taken as equal. At the boundaries, either the displacements (case 1) or the stresses (case 2) are equal in the force and the thermal problems. Furthermore, either the temperature (case 2) or the heat flux (case 1) must be equal at the boundary. The two cases stated in (33) and (34) thus cover a fairly wide range of problems with possibly inhomogeneous boundary and initial conditions.

To compute a suitable solution of (32) that is consistent with the preceding derivation, boundary conditions and initial conditions are needed. As will be seen in the subsequent analytical examples, these conditions are provided by the thermal conditions in (23) and (24).

We now turn to the stresses that are produced in the force and thermal problems, assuming the displacements to be equal. Subtracting the stress–strain–temperature relation for the two problems, we find from the stress–strain–temperature relation (12):

$$\sigma^* = \sigma - \bar{\sigma} = \Theta^* M \tag{35}$$

because the difference in strain vanishes. The difference in temperature must then obey (27) with $U^* = 0$.

Of course, one could have posed the analogy in alternative setting. For example, one may ask for a distribution of sources of heat so that the displacements are equal in the force and displacement problems, where this time the body forces are given. The solution for this scenario follows from (32), which can now be integrated directly for r. Moreover, one might ask for body forces so that the temperature in both problems becomes equal. The displacements will then be generally different. The strategy for solving this problem can closely follow the preceding considerations.

One starts from (25) and (26), assuming the temperature difference to be zero. One then eliminates the displacements to obtain a differential relation for the body force. The problem of an equal temperature, however, appears to be of little practical relevance, because the temperature associated with the force problem can be expected to be small. Instead of writing down the corresponding relations, we therefore turn to an exemplary justification of the body force analogy for equal displacements.

First, we consider case 1 for one-dimensional body with the region $0 \le X \le L$, boundary condition $\bar{b} = 0$ and initial condition $\bar{b} = 0$.

As an example, the source of heat r is taken as:

$$r = r_0 \cos\frac{\pi X}{L}\exp(-at) \tag{36}$$

where r_0 and a are arbitrary positive constants. We put:

$$\bar{b} = \bar{b}_0 \sin\frac{\pi X}{L}\exp(-at) \tag{37}$$

which is satisfied the boundary conditions. In this case, it seems that the initial condition is not be satisfied. In this case, Heaviside unit step function that is discontinuous at $t = 0$ is omitted in order to avoid the difficulty in differentiation at $t = 0$. By substitution of

(36) and (37) into differential equation (32), we obtain the body force $\bar{b}$ as:

$$\bar{b} = -\frac{M}{K}\frac{\pi}{L}\frac{1}{(\pi/L)^2 - (c/K)a} r_0 \sin\frac{\pi X}{L} \times \exp(-at) \tag{38}$$

When (36) and (38) are introduced into (25) and (26), the following equations are obtained:

$$\frac{\partial^2 U^*}{\partial X^2} - \frac{\rho}{Y}\ddot{U}^* + \frac{M}{Y}\frac{\partial \Theta^*}{\partial X} = -\frac{1}{Y}\frac{M}{K}\frac{\pi}{L}\frac{1}{(\pi/L)^2 - (c/K)a} r_0 \sin\frac{\pi X}{L}\exp(-at) \tag{39}$$

$$\frac{\theta_0 M}{K}\frac{\partial \dot{U}^*}{\partial X} + \frac{\partial^2 \Theta^*}{\partial X^2} - \frac{c}{K}\dot{\Theta}^* = -\frac{1}{K} r_0 \cos\frac{\pi X}{L}\exp(-at) \tag{40}$$

We take the solutions of differential equations (39) and (40) to be:

$$U^* = A\sin\frac{\pi X}{L}\exp(-at) \tag{41}$$

$$\Theta^* = B\cos\frac{\pi X}{L}\exp(-at) \tag{42}$$

to satisfy the boundary condition $U^* = 0$, $\frac{\partial \Theta^*}{\partial X} = 0$ and initial condition $U^* = 0$, $\Theta^* = 0$.

Then we obtain the displacement and temperature:

$$U^* = 0 \tag{43}$$

$$\Theta^* = \frac{1}{K}\frac{1}{(\pi/L)^2 - (c/K)a} r_0 \cos\frac{\pi X}{L}\exp(-at) \tag{44}$$

From (38) and (44), assumed relation (27) is proved and expected equation (43) is also obtained. Equation (44) shows that the difference of temperature between the thermal and force problems of coupled thermoelasticity is coincident with that of decoupled thermoelasticity.

Furthermore, to have an independent proof, we insert (37) into (22) and (23) and seek the solution, which must then be equal to the solution of (20) and (21) with (36).

Inserting (36) into (22) and (23) and subsequent treatment similar to the aforementioned procedure provides the following particular solutions (not taking into account the initial conditions):

$$\bar{U} = -\frac{1}{K}\frac{M}{Y}\frac{\pi}{L}\frac{1}{\Delta} r_0 \sin\frac{\pi X}{L}\exp(-at) \tag{45}$$

$$\bar{\theta} = \frac{1}{K}\frac{M}{Y}\left(\frac{\pi}{L}\right)^2 \frac{\theta_0 M}{K} a \frac{1}{(\pi/L)^2 - (c/K)a} \frac{1}{\Delta} r_0 \cos\frac{\pi X}{L}\exp(-at) \tag{46}$$

where

$$\Delta = \left[\left(\frac{\pi}{L}\right)^2 - \frac{c}{K}a\right]\left[\left(\frac{\pi}{L}\right)^2 + \frac{\rho}{Y}a^2\right] - \frac{\theta_0 M}{K}\frac{M}{Y}\left(\frac{\pi}{L}\right)^2 a \tag{47}$$

Apply the same procedure to (20) and (21) with (36) produces:

$$U = -\frac{1}{K}\frac{M}{Y}\frac{\pi}{L}\frac{1}{\Delta} r_0 \sin\frac{\pi X}{L}\exp(-at) \tag{48}$$

$$\theta = \frac{1}{K}\left[\left(\frac{\pi}{L}\right)^2 + \frac{\rho}{Y}a^2\right]\frac{1}{\Delta} r_0 \cos\frac{\pi X}{L}\exp(-at) \tag{49}$$

From these equations, we obtain, for the particular solutions:

$$U^* = U - \bar{U} = 0 \tag{50}$$

$$\Theta^* = \theta - \bar{\theta} = \frac{1}{K}\frac{1}{(\pi/L)^2 - (c/K)a} r_0 \cos\frac{\pi X}{L}\exp(-at) \tag{51}$$

Hence, if the initial conditions are the same for U and $\bar{U}$, then the total solutions for displacements are equal. Equations (50) and (51) are coincident with (43) and (44), respectively; therefore, the justifications of (43) and (44) are reconfirmed.

Next we consider case 2, (34). From (30), the boundary condition becomes:

$$\frac{\partial \bar{b}}{\partial X} = -\frac{M}{K} r \tag{52}$$

and the initial condition is:

$$\bar{b} = 0 \tag{53}$$

We take the example that:

$$r = r_0 \sin \frac{\pi X}{L} \exp(-at) \tag{54}$$

Then the boundary conditions reduce to $\frac{\partial \bar{b}}{\partial X} = 0$ at the boundaries $X = 0$ and $X = L$, so the solution of differential equation (32) is:

$$\bar{b} = \frac{M}{K} \frac{\pi}{L} \frac{1}{(\pi/L)^2 - (c/K)a} r_0 \cos \frac{\pi X}{L} \exp(-at) \tag{55}$$

Equation (25) with (55) and (26) with (54) are represented as follows:

$$\frac{\partial^2 U^*}{\partial X^2} - \frac{\rho}{Y} \ddot{U}^* + \frac{M}{Y} \frac{\partial \Theta^*}{\partial X} = \frac{1}{Y} \frac{M}{K} \frac{\pi}{L} \frac{1}{(\pi/L)^2 - (c/K)a} r_0 \cos \frac{\pi X}{L} \exp(-ax) \tag{56}$$

$$\frac{\theta_0 M}{K} \frac{\partial \dot{U}^*}{\partial X} + \frac{\partial^2 \Theta^*}{\partial X^2} - \frac{c}{K} \dot{\Theta}^* = -\frac{1}{K} r_0 \sin \frac{\pi X}{L} \exp(-at) \tag{57}$$

Applying the same procedure to case 1 produces:

$$U^* = 0 \tag{58}$$

$$\Theta^* = \frac{1}{K} \frac{1}{(\pi/L)^2 - (c/K)a} r_0 \sin \frac{\pi X}{L} \exp(-at) \tag{59}$$

In accordance with case 1, the individual solutions are obtained as follows:

$$\bar{U} = \frac{1}{K} \frac{M}{Y} \frac{\pi}{L} \frac{1}{\Delta} r_0 \cos \frac{\pi X}{L} \exp(-at) \tag{60}$$

$$\bar{\theta} = \frac{1}{K} \frac{M}{Y} \left(\frac{\pi}{L}\right)^2 \frac{\theta_0 M}{K} a \frac{1}{(\pi/L)^2 - (c/K)a} \frac{1}{\Delta} r_0 \cos \frac{\pi X}{L} \exp(-at) \tag{61}$$

$$U = \frac{1}{K} \frac{M}{Y} \frac{\pi}{L} \frac{1}{\Delta} r_0 \cos \frac{\pi X}{L} \exp(-at) \tag{62}$$

$$\theta = \frac{1}{K} \left[\left(\frac{\pi}{L}\right)^2 + \frac{\rho}{Y} a^2 \right] \frac{1}{\Delta} r_0 \sin \frac{\pi X}{L} \exp(-at) \tag{63}$$

where Δ is expressed as (47). Then, (58) and (59) are reconfirmed by the use of these equations.

The Classical Body Force Analogy for Generalized Thermoelasticity

The basic equations for generalized theory of the first problem are written as follows:

$$\mathbf{E} = \frac{\mathbf{1}}{\mathbf{2}} (\nabla \mathbf{U} + \nabla \mathbf{U}^{\mathbf{T}}) \tag{64}$$

$$\operatorname{div} \mathbf{S} + \mathbf{b} = \rho \ddot{\mathbf{U}} \tag{65}$$

$$\text{-div}\, \mathbf{q} + m\theta_0 \operatorname{tr} \dot{\mathbf{E}} + r = c\, \dot{\theta} \tag{66}$$

$$\mathbf{S} = \mathbf{2}\mu \mathbf{E} + \lambda (\operatorname{tr} \mathbf{E}) \mathbf{1} + \mathbf{m}\, \theta\, \mathbf{1} \tag{67}$$

$$\mathbf{q} + t_0^* \dot{\mathbf{q}} = -k \nabla \theta \tag{68}$$

Here, **S** stands for the stress tensor, and **E** is the strain tensor. The ρ, λ, μ, m, c and k are material parameters, which we take as independent of θ. Sources of heat are denoted as r and are assumed to be given. Note that the coupling of the strain to conduction of heat is taken into account by the last term in the energy equation, (66) and the relaxation time t_0^* is included in the heat equation, (68), such that we deal with the theory of generalized thermoelasticity called as Lord and Shulman's theory [12]. Once and for all, we assume that the fields introduced in (64)–(68) are sufficiently smooth, in order that our mathematical operations appear to be justified. Hence, we particularly exclude the presence of singular surfaces.

Introduction of (68) and (64) into (66) yields:

$$k\,\nabla^2\theta = c\left(\dot{\theta} + t_0^*\ddot{\theta}\right) - \theta_0 m\left(\nabla\,\dot{\mathbf{U}} + t_0^*\nabla\ddot{\mathbf{U}}\right) - \left(r + t_0^*\dot{r}\right) \tag{69}$$

Introducing (67) into (65) and using the relation (64) and assuming that $\mathbf{b} = \mathbf{0}$, we obtain:

$$\mu\nabla^2\mathbf{U} + (\lambda + \mu)\nabla(\operatorname{div}\mathbf{U}) + m\nabla\theta = \rho\;\ddot{\mathbf{U}} \tag{70}$$

When we use same procedure and an analogous notation, the basic equations of second thermoelastic problem are:

$$k\,\nabla^2\bar{\theta} = c\left(\dot{\bar{\theta}} + t_0^*\ddot{\bar{\theta}}\right) - \theta_0 m\left(\nabla\,\dot{\bar{U}} + t_0^*\nabla\ddot{\bar{U}}\right) \tag{71}$$

$$\mu\nabla^2\bar{\mathbf{U}} + (\lambda + \mu)\nabla(\operatorname{div}\mathbf{U}) + \mathrm{m}\nabla\theta + \mathbf{b} = \rho\ddot{\bar{U}} \tag{72}$$

where we assume that $\bar{r} = 0$. The problem according to (69) and (70) then will be denoted as the thermal problem, while the problem stated in (71) and (72) will be called the force problem subsequently.

We now introduce new notations:

$$\mathbf{U}^* = \mathbf{U} - \mathbf{U}$$

$$\boldsymbol{\Theta}^* = \theta - \bar{\theta} \tag{73}$$

Subtracting (70) from (72) and (69) from (71) yields the following equations:

$$\mu\nabla^2\mathbf{U}^* + (\lambda + \mu)\nabla(\operatorname{div}\mathbf{U}^*) + \mathrm{m}\nabla\boldsymbol{\Theta}^* - \mathbf{b} = \rho\;\mathbf{U}^* \tag{74}$$

$$k\nabla^2\boldsymbol{\Theta}^* = c\left(\dot{\boldsymbol{\Theta}}^* + t_0^*\ddot{\boldsymbol{\Theta}}^*\right) - \theta_0 m\nabla\left(\dot{\mathbf{U}}^* + t_0^*\ddot{\mathbf{U}}^*\right) - \left(r + t_0^*\dot{r}\right) \tag{75}$$

In order to extend the classical body force analogy to the generalized theory of thermoelasticity, we seek the condition $\mathbf{U}^* = \mathbf{0}$. Hence, we take the sources of heat r in (75) to be given, and we seek for body forces $\bar{\mathbf{b}}$ in (74), such that the displacements in the thermal and in the force problem become equal.

The first step in our solution for the body force analogy stated in the previous Section is as follows. We assume that:

$$\bar{\mathbf{b}} = m\nabla\boldsymbol{\Theta}^* \tag{76}$$

Then (74) reduces to:

$$\mu\nabla^2\mathbf{U}^* + (\lambda + \mu)\nabla(\operatorname{div}\mathbf{U}^*) = \rho\;\mathbf{U}^* \tag{77}$$

which indeed has $\mathbf{U}^* = \mathbf{0}$ as a solution. In the present context, the body force $\bar{\mathbf{b}}$ is sought such that the displacements in the thermal and the force problem become equal. An equation for computing the body force $\bar{\mathbf{b}}$ is therefore needed. We suggest the following construction. The differentiation of (76) yields:

$$\nabla^2\boldsymbol{\Theta}^* = \frac{1}{m}\nabla\bar{\mathbf{b}} \tag{78}$$

By use of (78), (22) becomes:

$$\frac{k}{m}\nabla\bar{\mathbf{b}} = c\left(\dot{\boldsymbol{\Theta}}^* + t_0^*\ddot{\boldsymbol{\Theta}}^*\right) - \theta_0 m\nabla\left(\dot{\mathbf{U}}^* + t_0^*\ddot{\mathbf{U}}^*\right) - \left(r + t_0^*\dot{r}\right) \tag{79}$$

A further differentiation of this equation and using (76) causes:

$$\frac{k}{m}\nabla^2\bar{\mathbf{b}} = \frac{c}{m}(\dot{\bar{\mathbf{b}}} + t_0^*\ddot{\bar{\mathbf{b}}}) - \theta_0 m\nabla^2(\dot{\mathbf{U}}^* + t_0^*\ddot{\mathbf{U}}^*) - \nabla(r + t_0^*\dot{r}) \tag{80}$$

Hence, when we assume that $\mathbf{U}^* = \mathbf{0}$, we obtain the following partial differential equation for $\bar{\mathbf{b}}$:

$$\nabla^2\bar{\mathbf{b}} - \frac{c}{k}\left(\dot{\bar{\mathbf{b}}} + t_0^*\ddot{\bar{\mathbf{b}}}\right) = -\frac{m}{k}\nabla\left(r + t_0^*\dot{r}\right) \tag{81}$$

In other words, when we can guarantee that $\mathbf{U}^* = \mathbf{0}$ is the solution of (75), then (81) can be used to compute a body force $\bar{\mathbf{b}}$ which equalizes the displacements of the thermal problem due to the sources of heat r. The remaining procedures are omitted here.

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Body Force Method

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Synonyms

Boundary element method

Definitions

Body force method (BFM) is a numerical stress analysis method based on the superposition of a series of closed-form fundamental solutions, which is usually a stress field due to an isolated point force acting in an infinite elastic body. Essentially, BFM is similar to the boundary element method (BEM), but BFM has clearer physical meaning than the latter. In BFM, the solution process of an elastic problem is as follows: (a) starting from solution of point force, so the equilibrium condition and compatibility condition are satisfied; (b) after superposing point forces, determine the distribution of point force so as to satisfy boundary condition. The superposed elastic field is the solution of the given problem, because the solution of elastic problem is unique.

In the early stage of the progress of the method, it was mainly applied to elastostatic problems for calculating the stress concentration factors of notches and the stress intensity factors of cracks. In recent decades, the method was extended so as to be applicable to the elastic–plastic problems, the elastodynamic problems, the thermoelastic problems, and the inverse problems of cracks. Also, the method was used to study the theoretical problems of stress analysis such as the linear notch mechanics and the effect of elastic constants on stresses.

Overview

Nisitani and Chen have made great contributions to the development of body force method. The body force method was originally proposed by Nisitani [1] in 1967 as a versatile method of numerical stress analysis, and Chen expanded the research fields of BFM. In the early stage of the progress of the method, it was mainly applied to elastostatic problems for calculating the stress concentration factors of notches and the stress intensity factors of cracks [2–6]. In the 1980s and 1990s, the method was extended so as to be applicable to the elastic–plastic problems [7, 8], the elastodynamic problems [9–11], the thermoelastic problems [12], and the inverse problems of cracks [13, 14]. Also, the method was used to study the theoretical problems of stress analysis such as the linear notch mechanics [15, 16] and the effect of elastic constants on stresses [17, 18].

Comparing to the more popular numerical methods, such as the finite element method (FEM) and the finite-difference method (FDM), which can be classified as the domain methods, the BFM distinguishes itself as a boundary method (similar to the boundary element method, BEM), meaning that the numerical discretization is conducted at reduced spatial dimension. For example, for problems in three spatial dimensions, the discretization is performed on the bounding surface only, and in two spatial dimensions, the discretization is on the boundary contour only. This reduced dimension leads to smaller linear systems, less computer memory requirements, and more efficient computation.

Basic Methodology

The framework and basic methodology of the BFM are well summarized in [19, 20], and some of the following contents are recited from them. BFM is a method based on the principle of superposition. An unknown elastic field can be expressed by superposing some specific known elastic fields with some unknown weight magnitudes because the elastic fields are superposable. That is as follows [20],

$$\boxed{\begin{array}{r}\textit{elastic field}\\ \textit{under inverstigation}\end{array}} = \sum \boxed{\begin{array}{r}\textit{specific known}\\ \textit{solution}\\ \textit{of elastic field}\end{array}} \times \boxed{\begin{array}{r}\textit{unknown weight}\\ \textit{magnitudes}\end{array}} \tag{1}$$

So, the solution of the given problem could be transformed to solve the unknown weight magnitudes. In BFM, the fundamental solutions are the elastic fields due to a point force and/or a discrepancy. By using these fundamental solutions, the given problem is treated as if it exists in an infinite body, where the point forces and/or discrepancies are continuously distributed along an imaginary boundary.

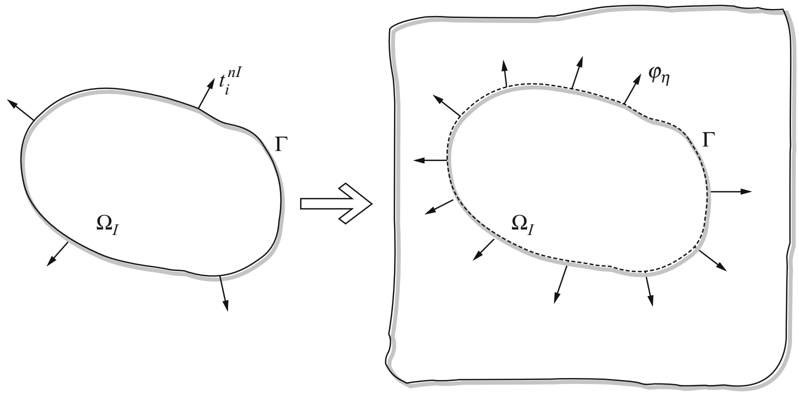
As shown in Fig. 1, an arbitrary elastic body is subjected to a given surface force $t_i^{nI}(Q),\ \ Q \in \Gamma$, and could be simulated by the elastic field caused by the point forces continuously distributed along the imaginary boundary Γ, with a density of $\varphi_\eta(Q)$, in an infinite body. The displacement and stress in the domain Ω_I are obtained by superposition of the displacement and stress due to all the point forces acting in the infinite body as follows [20]:

$$u_i(P) = \int_\Gamma u_i(P, Q_\eta)\varphi_\eta(Q)d\Gamma(Q)\ \ (P \in \Omega_I) \tag{2}$$

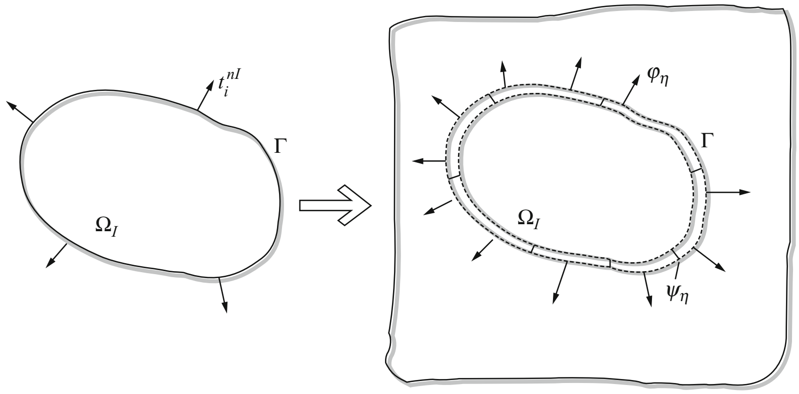
$$\sigma_{ij}(P) = \int_\Gamma \sigma_{ij}(P, Q_\eta)\varphi_\eta(Q)d\Gamma(Q)\ \ (P \in \Omega_I) \tag{3}$$

where $u_i(P, Q_\eta)$ and $\sigma_{ij}(P, Q_\eta)$ are the i-direction displacement and the stress σ_{ij} at a point P due to the η- direction unit point force acting at a point Q respectively.

The unknown density $\varphi_\eta(Q)$ of point forces in (2) and (3) can be determined from the displacement or stress boundary condition equation. When we get the value of $\varphi_\eta(Q)$, we could calculate the stress at an arbitrary point $P, P \in \Omega_I$ from (2) and (3).



Body Force Method, Fig. 1 Elastic field simulated by one in an infinite body, in which point forces are distributed along an imaginary boundary



Body Force Method, Fig. 2 Elastic field simulated by one in an infinite body, in which discrepancies are distributed along an imaginary boundary

Similarly, if the fundamental solution is the elastic field due to a discrepancy, the elastic field under investigation could be transformed to solve the unknown density, $\psi_\eta(Q)$, of the discrepancies distributed along the imaginary boundary in an infinite body, as shown in Fig. 2.

Thus, the displacement and stress in the domain Ω_I are expressed as follows [20]:

$$u_i(P) = \int_\Gamma u_i^*(P, Q_\eta^n)\psi_\eta(Q)d\Gamma(Q) \ \ (P \in \Omega_I) \tag{4}$$

$$\sigma_{ij}(P) = \int_\Gamma \sigma_{ij}^*(P, Q_\eta^n)\psi_\eta(Q)d\Gamma(Q) \ \ (P \in \Omega_I) \tag{5}$$

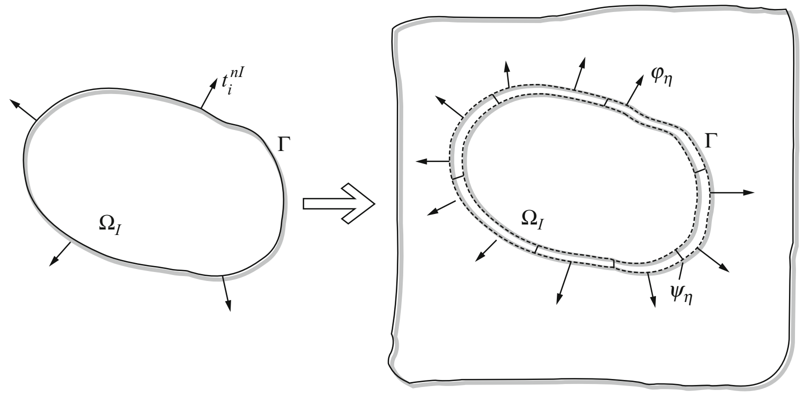
where $u_i^*(P, Q_\eta^n)$ and $\sigma_{ij}^*(P, Q_\eta^n)$ are the i-direction displacement and the stress σ_{ij} at a point P due to the η -direction unit concentrated discrepancy of facets, whose normal vector is of the n-direction, acting at a point Q, respectively.

In (4) and (5), the unknown is the density of discrepancies $\psi_\eta(Q)$, which can be solved from the boundary condition equation for displacement or traction, as in the case of $\varphi_\eta(Q)$ being the unknown in (2) and (3).

In a more general case, as shown in Fig. 3, both of the elastic fields due to point forces and that due to discrepancies can be used as the fundamental solutions at the same time; the elastic field in the domain Ω_I could be expressed by a sum of those due to the point forces and the discrepancies. The displacement and stress in the domain Ω_I are written in the following form [20]:

$$u_i(P) = \int_\Gamma u_i(P, Q_\eta)\varphi_\eta(Q)d\Gamma(Q) + \int_\Gamma u_i^*(P, Q_\eta^n)\psi_\eta(Q)d\Gamma(Q) \ \ (P \in \Omega_I) \tag{6}$$

Body Force Method, Fig. 3 Elastic field simulated by one in an infinite body, in which point forces and discrepancies are distributed along an imaginary boundary



$$\sigma_{ij}(P) = \int_\Gamma \sigma_{ij}(P, Q_\eta)\varphi_\eta(Q)d\Gamma(Q) + \int_\Gamma \sigma^*_{ij}(P, Q^n_\eta)\psi_\eta(Q)d\Gamma(Q) \quad (P \in \Omega_I) \tag{7}$$

When solving the two unknowns, $\varphi_\eta(Q)$ and $\psi_\eta(Q)$, in (6) and (7), one of them could be assumed arbitrarily, and the other one is determined from the boundary condition equations, which are obtained from (6) or (7) by moving the observation point P to the boundary Γ.

In BFM, the stress field in the domain Ω_I is simulated by the stress field in the subdomain Ω_I in an infinite body, so that the fundamental solutions, which are the elastic fields due to point forces and/or discrepancy in an infinite body, can be used. It could be proved that the densities of point forces and/or discrepancies in (2)–(7) are existent and the stress field under investigation is fully equivalent to that caused by the point forces and/or discrepancies with the solved densities [20]. In the process of proving, an auxiliary domain Ω_E is needed. Its shape is defined as an infinite body with a cavity, which has the same shape as the boundary of the domain under investigation. The auxiliary domain Ω_E is assumed to subject to an arbitrary surface force $t_i^{nE}(Q)$, $Q \in \Gamma$ on the boundary and an arbitrary body force $f_i^E(Q)$ in the domain, which result in a displacement $u_i^E(Q)$ on the boundary. There will be a gap between the domain Ω_I and the auxiliary domain Ω_E because the boundary displacement of these two domains is inconsistent. In order to prevent the disturbing of the stress field in domain Ω_I, the gap could be simply treated as discrepancies distributed along the boundary Γ of the auxiliary domain Ω_E. After this inserting, we could obtain a perfect infinite body.

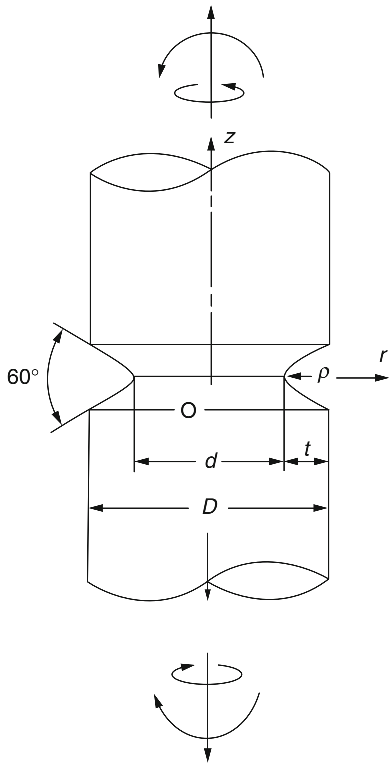
The whole process implies that the elastic field in the domain under investigation can be obtained simply by a superposition of all the actions applied in the infinite body, which includes the body force acting on both domains, the point force acting along the imaginary boundary, and the discrepancies distributed along the boundary. More detailed process of the proving could be seen in [20].

Key Research Findings

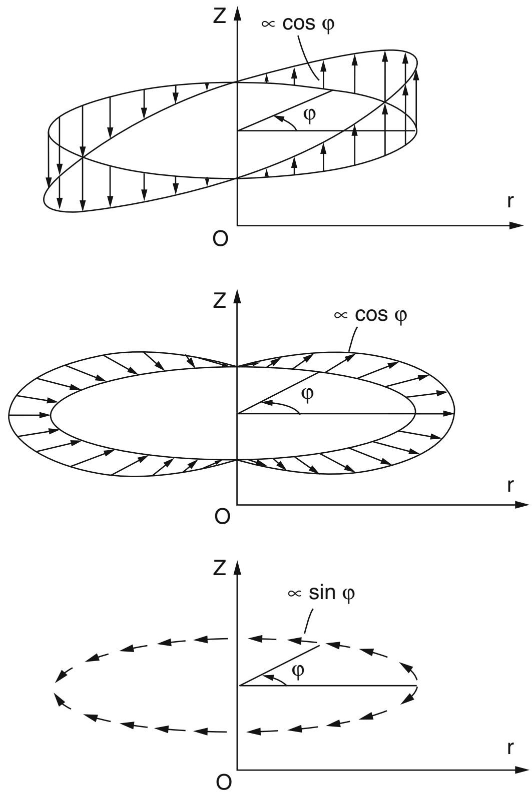
Stress Analysis

Stress Concentration Factor Calculation

The BFM was used to calculate stress concentration factors by Nisitani and Noda [21]. The stress concentration problem of a cylindrical bar with a circumferential groove (Fig. 4) is mainly concerned in designing of shafts, and many fatigue test specimens are also fabricated with V-shaped circumferential groove. In solving this problem, the stress fields due to ring forces in an infinite body, as shown in Fig. 5,



Body Force Method, Fig. 4 A cylindrical bar with a V-shaped circumferential groove [21]



Body Force Method, Fig. 5 Fundamental solutions for bending problems in BFM [21]

are used as the fundamental solutions [21]. Figure 6 shows the stress concentration factors of a 60° V-shaped notch under bending. As a result of the systematic calculations, it has been found that the stress concentration factors obtained by Neuber's trigonometric rule used currently have nonconservative errors of about 10 % for a wide range of notch depths.

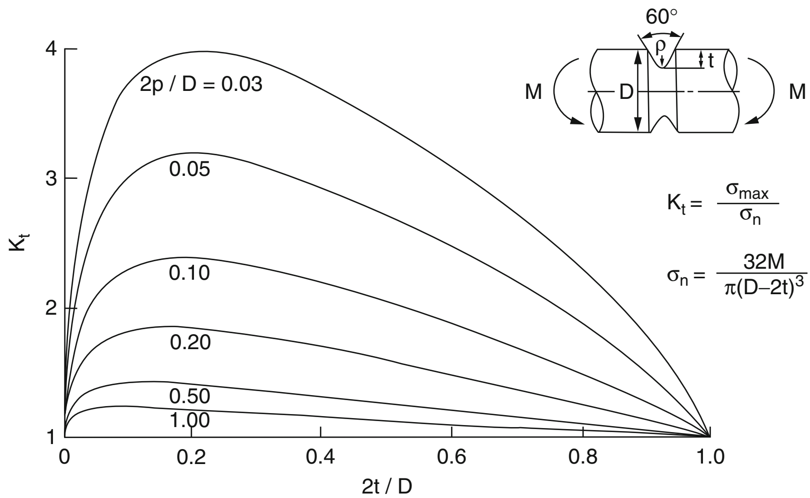
Singular Thermal Stress Calculation

Nisitani and Chen studied the singular thermal stress problems using BFM [19]. This problem is shown in Fig. 7; an infinite plate contains an inclusion and the temperature changes from T to $T + \Delta T$, where $G_1,\ \nu_1,\ \alpha_1$ and $G_2,\ \nu_2,\ \alpha_2$ denote shear modulus, Poisson's ratio, coefficient of thermal expansion for the infinite plate, and the inclusion, respectively.

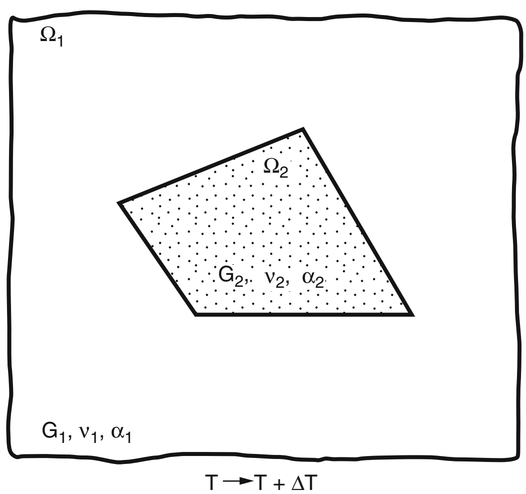
Based on BFM, Nisitani and Chen found that the singular thermal stress field near a corner tip caused by the thermal mismatch of different moduli (Fig. 8d) is equivalent to the singular elastic stress field caused by body forces, applied at points on the interface with a density of $\phi_T = \frac{4G_2(\alpha_2 - \alpha_1)}{k_2 - 1}\Delta T$, where $k_2 = (3 - \nu_2)/(1 + \nu_2)$ for plane stress and $k_2 = 3 - \nu_2$ for plane strain. This finding enables us to obtain the stress intensity factors of a singular thermal stress field from the available data on the stress intensity factors of a singular elastic stress field caused by point force [19].

Crack Problems

The BFM has an obvious advantage in dealing with crack problems. The singular stress field



Body Force Method, Fig. 6 Stress concentration factors of a 60° V-shaped notch under bending [21]



Body Force Method, Fig. 7 Thermoelastic problem of an infinite plate containing an inclusion for a temperature change from T to $T + \Delta T$ [19]

near the crack tip could be well simulated by using BFM. In the body force method, force doublets are used in place of the discrepancies because the discrepancy can be caused by force doublets. In two-dimensional crack problems, the definition of force doublets used in the body force method is given in Fig. 9 [20], from which it is seen that a unit tension-type force doublet acting in $x_i x_i$ -direction would cause an i-direction discrepancy acting on an i-direction facet and having a magnitude of C_Δ, which could be expressed as [20],

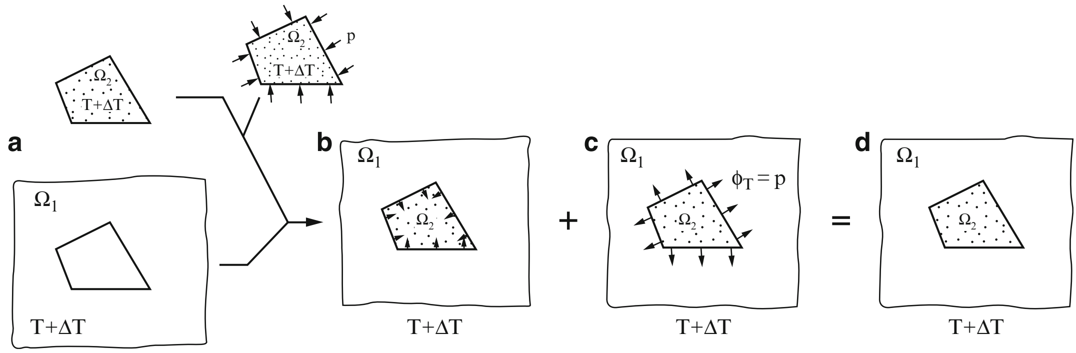
$$C_\Delta = \frac{(k-1)}{G(|k+1)} \tag{8}$$

and a unit shear-type force doublet acting in $x_i x_j$ -direction would cause an i-direction discrepancy acting on an j-direction facet and having a magnitude of [20]

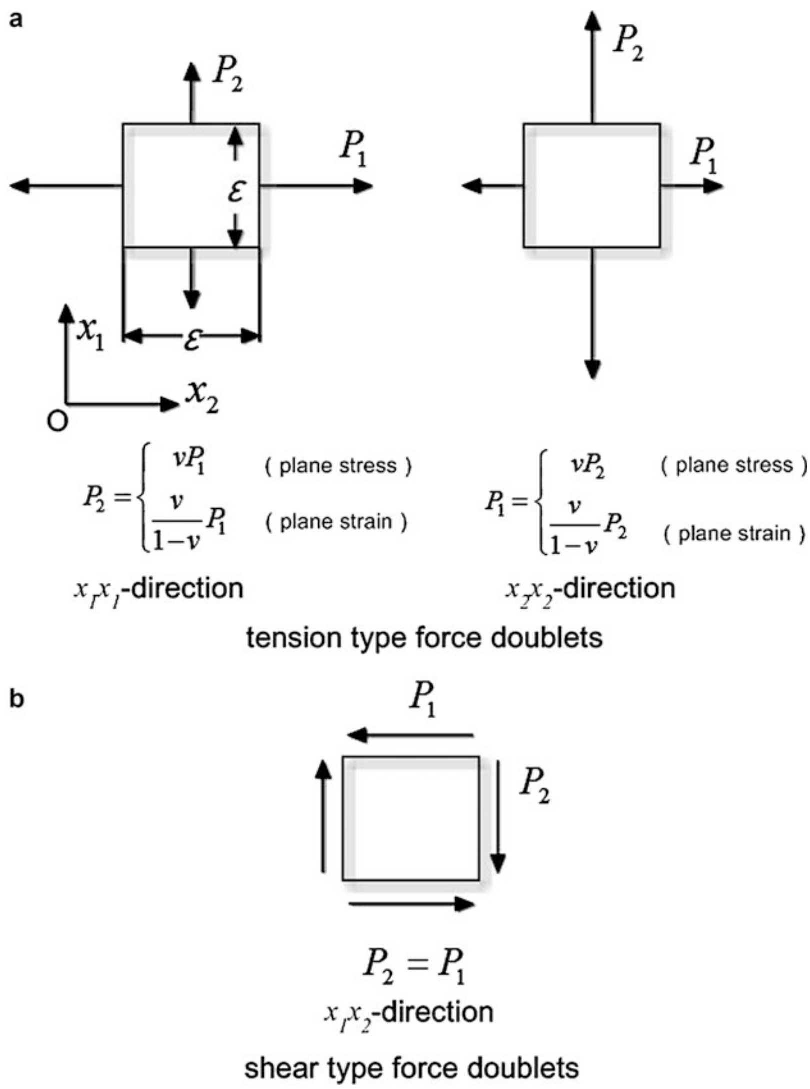
$$C_\Delta = \frac{1}{G} \tag{9}$$

where G and v denote the shear modulus and Poisson's ratio, respectively, and $k = (3-v)/(1+v)$ for plane stress and $k = 3 - 4v$ for plane strain.

In BFM, the crack problem could be transformed to solve the unknown densities $x_{\eta\xi}(Q)$ of force doublets, which are acting on the imaginary crack surface Γ_{crack} in a body without crack. Concerning the force balance condition on the imaginary crack surface, we can get an integral equation which could be solved by using

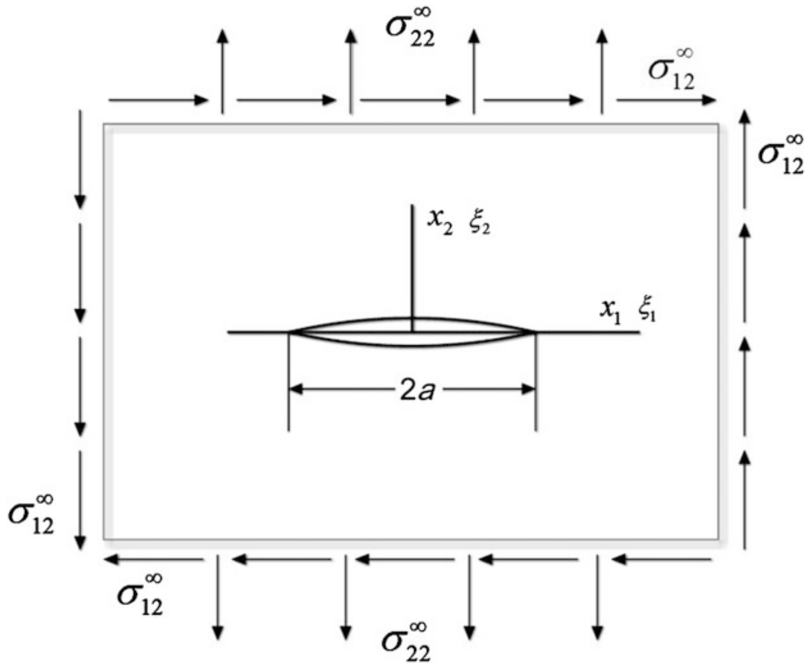


Body Force Method, Fig. 8 Equivalence between the singular thermal stress field and that caused by body force [19]



Body Force Method, Fig. 9 Force doublets for two-dimensional crack problems. (**a**) Tension-type force doublets. (**b**) Shear-type force doublets [20]

Body Force Method, Fig. 10 A crack in an infinite plate subjected to a uniform tension and shear



numerical method. The imaginary crack surface is divided into several elements, and in each element, the density of the force doublets is assumed to be constant. After this approximation, the integral equation is transformed into a system of linear algebraic equations and could be solved easily. A quite accurate solution could be obtained even when the division number of the crack surface is not large (e.g., the division number is 50) [22].

Stress Intensity Factors for 2D Crack Problem

As the simplest case, Fig. 10 shows a two-dimensional crack in an infinite plate under remote mechanical load. When this problem is analyzed by BFM, a series of tension-type force doublets and shear-type force doublets are assumed to distribute along the imaginary crack line in an infinite plate. The tension-type force doublets are acting in x_2x_2 directions, and the shear-type force doublets are acting in x_1x_2 directions. For the problem of Fig. 10, the densities of these force doublets can be obtained in a closed form as follows [20]:

$$\begin{cases} x_{22}(\xi_1) = \dfrac{(k+1)^2\sigma_{22}^{\infty}}{2(k-1)}\sqrt{a^2-\xi_1^2} \\ x_{12}(\xi_1) = \dfrac{(k+1)\sigma_{12}^{\infty}}{2}\sqrt{a^2-\xi_1^2} \end{cases} \tag{10}$$

where a is the half length of the crack and ξ_1 is the distance measured from the crack center point.

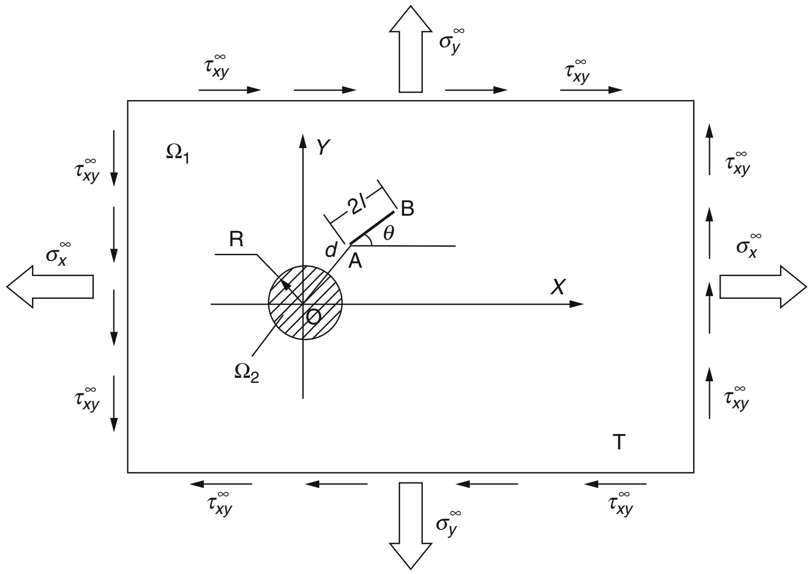
Note that both $x_{22}(\xi_1)$ and $x_{12}(\xi_1)$ could be expressed in the form of (constant) $\times\sqrt{a^2-\xi_1^2}$. We can define a basic density function of the force doublets $\varphi(\xi_1)$, expressed as [20]

$$\varphi(\xi_1) = \sqrt{a^2-\xi_1^2} \tag{11}$$

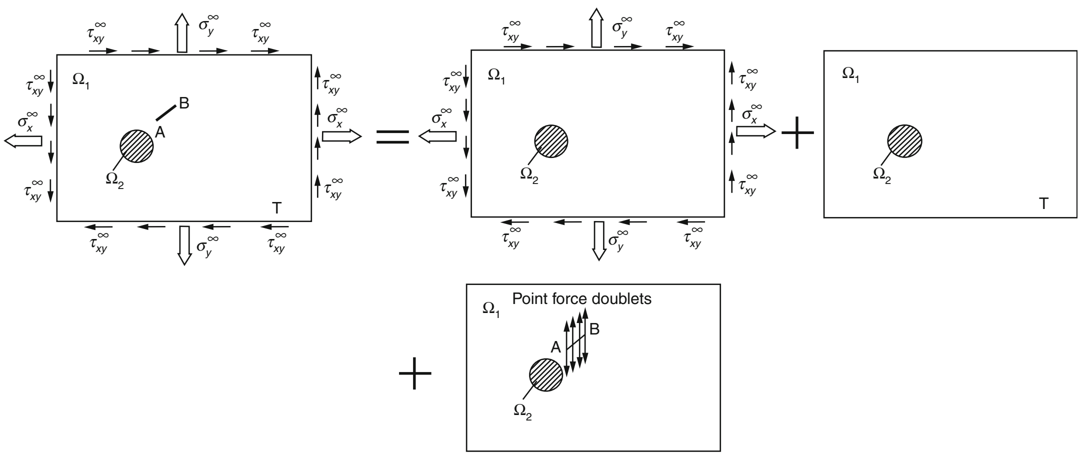
And (10) could be changed to the form [20],

$$\begin{cases} x_{22}(\xi_1) = W_2(\xi_1)\phi(\xi_1) \\ x_{12}(\xi_1) = W_1(\xi_1)\phi(\xi_1) \end{cases} \tag{12}$$

Another advantage of using basic density functions is that the intensities of the singular



Body Force Method, Fig. 11 A crack in the neighborhood of an inclusion under thermomechanical loads [22]



Body Force Method, Fig. 12 Interaction of crack and inclusion under thermomechanical loads is transformed to three simple problems [22]

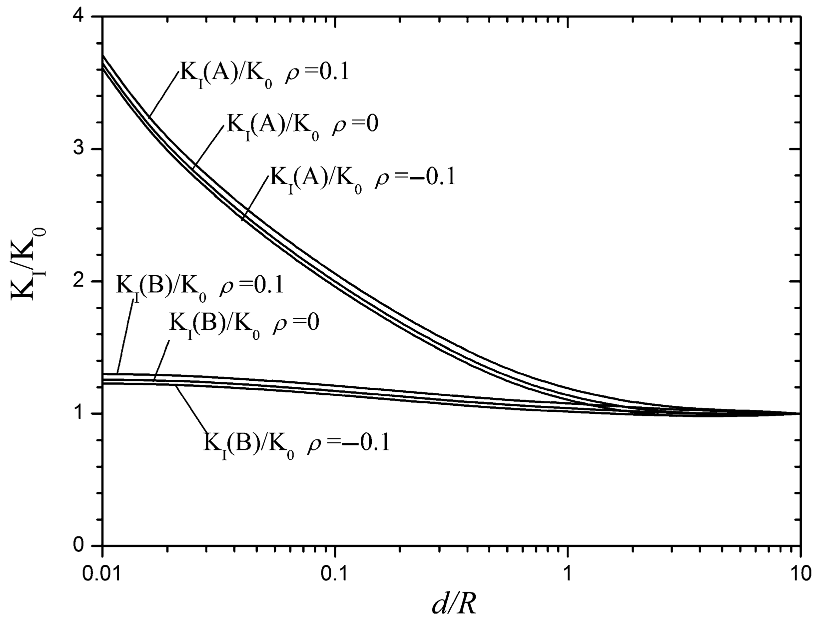
stress field can be directly obtained from the values of the weight functions at the crack tip, $W_1(a)$ and $W_2(a)$, by the following expressions [20]:

$$\begin{cases} K_I = \dfrac{2(k-1)}{(k+1)^2}\sqrt{\pi a}W_2(a) \\ K_{II} = \dfrac{2}{(k+1)}\sqrt{\pi a}W_1(a) \end{cases} \tag{13}$$

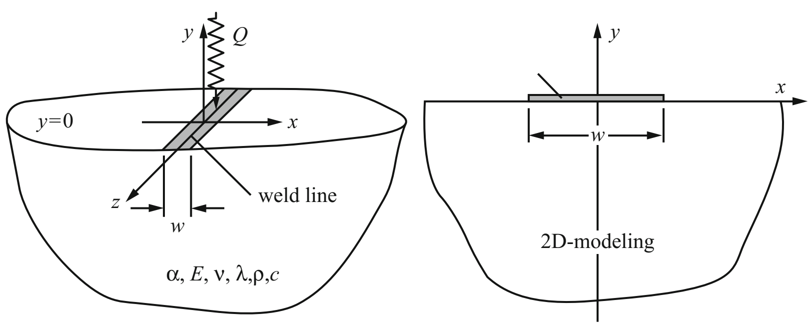
Interaction of Crack and Inclusion Under Thermomechanical Loads

Chen Yong studied the effects of nonmetallic inclusion of Al_2O_3 on the SIFs in FGH 95 PM superalloys (similar to Rene95 PM superalloy) under the coupling of remote mechanical loads and thermal loads by using BFM [22]. The problem is described in Fig. 11.

As shown in Fig. 12, to use the principle of superposition, the inclusion and crack interaction



Body Force Method, Fig. 13 Normalized stress intensity factors for Al2O3/FGH95 under combined thermal loads and mechanical loads at different distances d/R from the crack tip A to the interface [22]



Body Force Method, Fig. 14 Schematic illustration of a welding model [23]

problem under thermomechanical loads is transformed to three simple problems, which are as follows: (a) a plate with an inclusion under remote mechanical loads, (b) a plate with an inclusion in a uniform temperature field, and (c) a plate with an inclusion and a series of force doublets acting on the imaginary crack line.

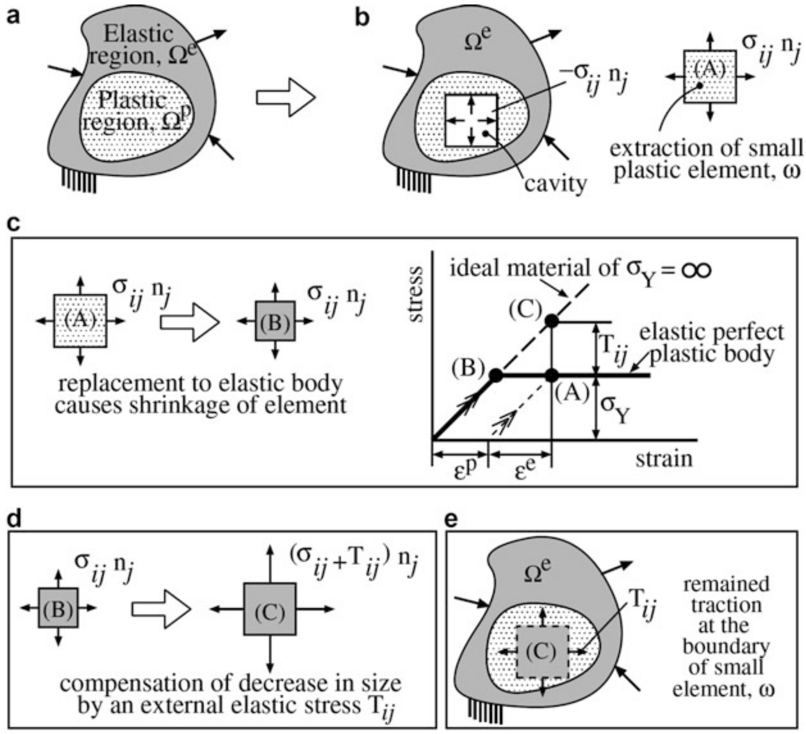
Figure 13 shows normalized SIFs for Al_2O_3/FGH95 under different combined thermal and mechanical loads (different values of ρ and the definition of ρ can be seen in [22]) at different distances d/l from the crack tip A to the interface. Compression thermal stresses ($\rho = -0.1$) lead to slightly decreasing SIFs, which means the crack was slightly stabilized when it is propagating toward the inclusion. But the reduction amount of SIFs is very small. Thermal stress has little effect on the SIFs, so the SIFs under combined loads are dominated by the mechanical load.

Elastic–Plastic Problems

Calculation of Welding Residual Stress

Body force method can be extended so as to be applicable to elastic–plastic problems. The basic

Body Force Method, Fig. 15 Procedure for the replacement of the plastic strain ε_{ij}^{p} by a force doublet of magnitude Ti j. (**a**) Elastic–plastic body. (**b**) Extraction of yielded element without changing stresses. (**c**) Replacement to elastic element. (**d**) Compensation of element shrinkage. (**e**) Partially replaced body [23]



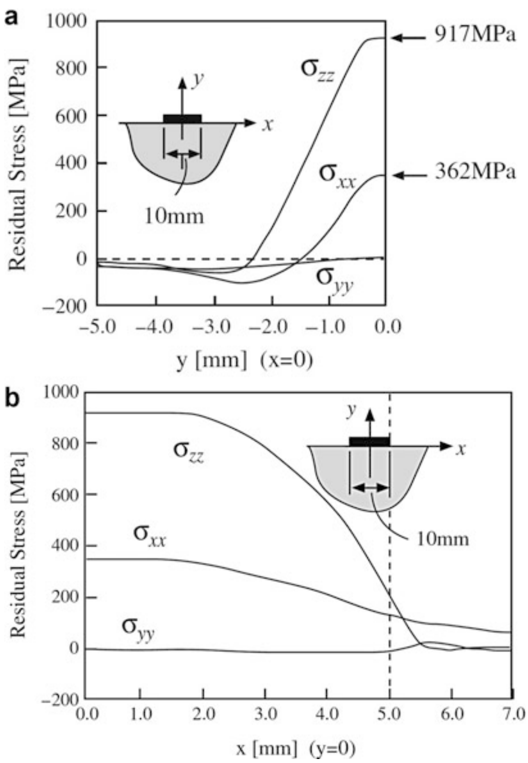
principle of analysis is the same as in the case of elastic problems. That is, the problem is replaced by an infinite body without any plastic zone and the solution of the problem is obtained by superposition of the fundamental elastic fields. We can transform the problem to a problem of an infinite body by inserting the domain under investigation into an auxiliary domain. Therefore, the key point for analyzing the elastic–plastic problems is how to replace the plastic zone by an elastic element. Saimoto and Imai's research result could be shown here as an example [23].

Saimoto and Imai studied the problem of thermally induced residual stress under plane-strain constraint based on BFM [23]. As a numerical example, a simple problem of limited plasticity due to a uniform strength of transient line heat source of finite width, which is applied to a surface of a semi-infinite solid for a short duration of time, was considered.

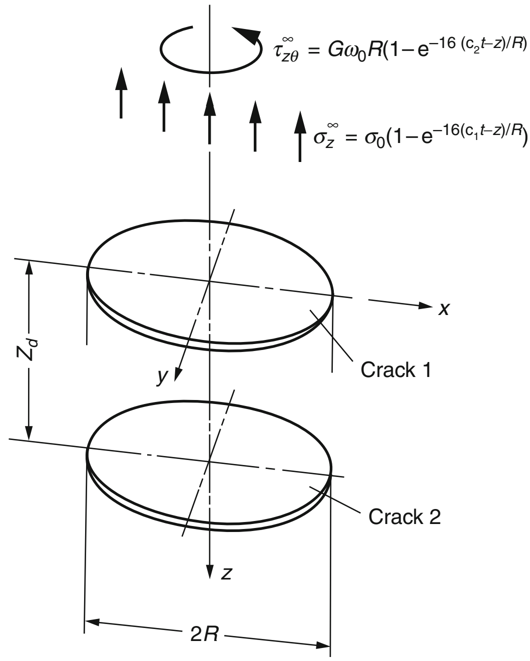
Figure 14 shows a simplified welding model. A transient line heater of uniform strength and of width "*w*" is applied to a surface of a semi-infinite medium for a short duration of time with a chosen strength so that the total heating energy delivered from the heater resembles the one expected under actual welding of stainless steel. Residual stress in the out-of-plane (z) direction becomes significant.

Figure 15 shows the process of replacement of the plastic strain by a force doublet. For more information of the transforming procedure, please see ref. [23]. And Fig. 16 shows the Residual stress after cooling.

According to ref. [23], it was found that the residual stress in the out-of-plane direction σ_{zz} can be estimated independently of the in-plane residual stress components σ_{xx} and σ_{yy}. It was also found that the proposed method provides an effective and efficient scheme for treating a problem of limited plasticity.



Body Force Method, Fig. 16 Residual stress after cooling. (**a**) Residual stress along y axis. (**b**) Residual stress along x axis [23]



Body Force Method, Fig. 17 Two penny-shaped cracks in an infinite body subjected to torsional and tensile impact [20]

Elastodynamic Problem

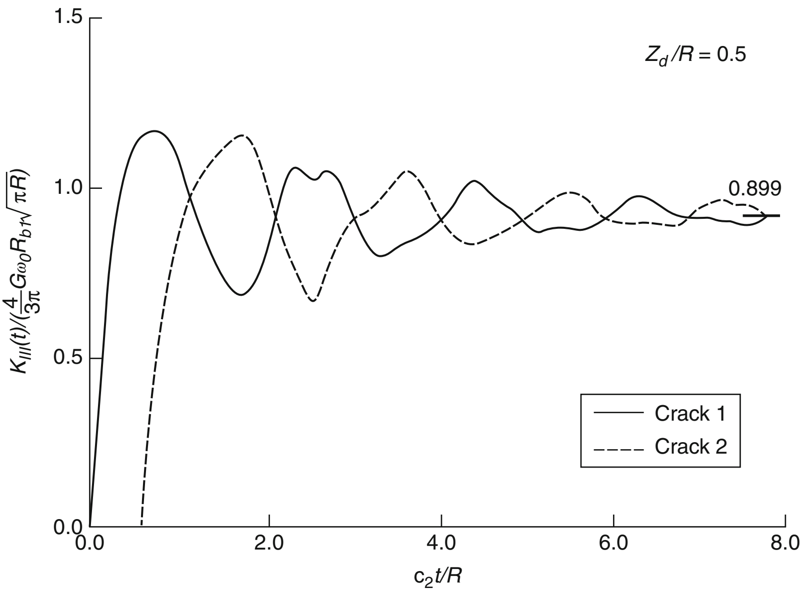
The body force method can also be extended to elastodynamic problems. The procedure of using BFM dealing with elastodynamic problems is similar to that of dealing with elastic–plastic problems.

Consider an elastodynamic problem in a domain Ω_I which is subjected to a given body and surface force; both of them vary with the time. In order to construct an integral equation expressing the elastodynamic field under investigation, we insert the domain under investigation Ω_I into an auxiliary domain Ω_E as in the case of the elastostatic problem. In this case, however, the gaps between domains Ω_I and Ω_E change with the time. Therefore, the density of distributed discrepancies to be embedded along the imaginary boundary is a function of time. For more information, please see ref. [20].

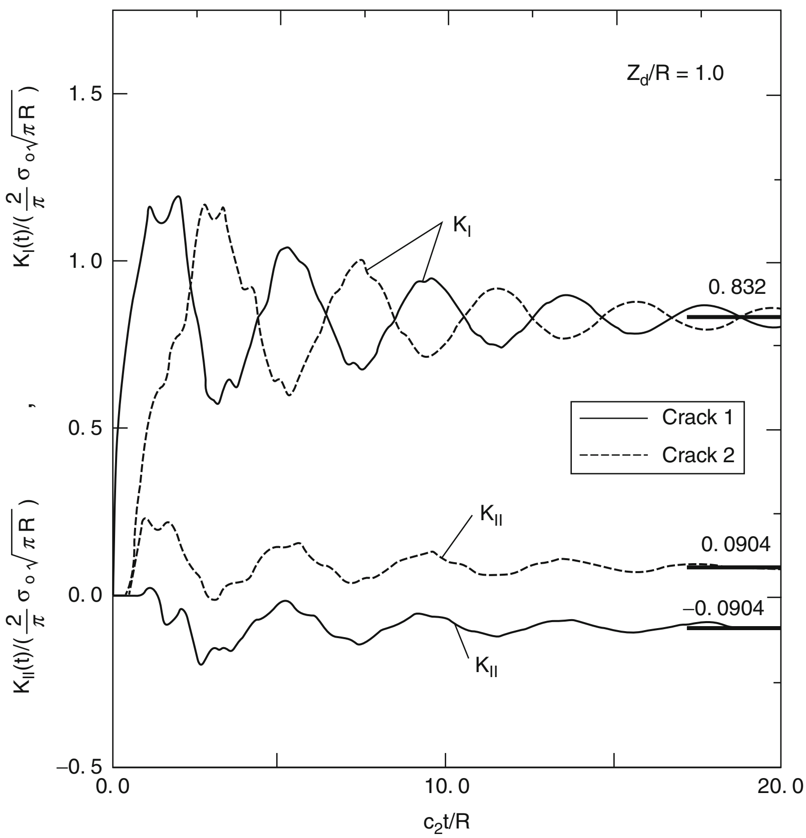
Figure 17 shows two penny-shaped cracks in an infinite body subjected to torsional and tensile impact. And Figs. 18 and 19 show the variation of dynamic stress intensity factors with time for torsional and tensile impact, respectively.

Future Directions for Research

BFM has the advantage of reduced dimension that leads to smaller linear systems, less computer memory requirements, and more efficient computation. But the applications of BFM to solve theoretical and engineering problems are very limited. Partly because the fundamental solutions of point force acting in an infinite body are not easy to obtain. This situation leads to the lack of software platform support. So, one direction of BFM research is the collection and integration of all kinds of basic solutions to build a database and support numerical calculation software



Body Force Method, Fig. 18 Variation of dynamic stress intensity factors with time for torsional impact [20]



Body Force Method, Fig. 19 Variation of dynamic stress intensity factors with time for tensile impact [20] platform. On the other hand, the utilization of BFM in nonlinear and dynamic problems should be strengthened.

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Body Force Method for Thermoelasticity

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Overview

The body force method (BFM) is a numerical technique for stress analysis, originally proposed by Professor Hironobu Nisitani in 1967 [1]. The essence of BFM is to transform a given elastic problem to an equivalent problem of infinite domain in which body forces are embedded. That is, a solution of given elastic problem is expressed by a superposition of elementary solutions called fundamental solution. As a fundamental solution to be superposed, stress fields due to an isolated point force acting in an infinite plate (for 2D problems) [2] or an infinite solid (for 3D problems) [3] are preferably used. According to the ordinal classification, BFM is considered as an indirect boundary element method as the stress and displacement at a reference point is expressed in terms of densities of body force. In BFM, density of body force is an unknown function to be determined through boundary conditions. If boundary condition is satisfied exactly, it means that an exact solution is derived by BFM. In thermoelastic analysis, not only the elastic field due to point force but also a thermoelastic field due to point heat acting in an infinite domain is used as a fundamental solution.

Basic Concept

Based on the principle of BFM, any elastic problem is transformed into a problem of a complete infinite domain without any hole or crack. That is, a boundary of given problem is replaced by an equivalent imaginary boundary along which body force or body force doublets are embedded. Let us consider a problem shown in Fig. 1. The given problem (left of Fig. 1) is replaced by a problem of an infinite domain (right of Fig. 1) without an oval notch but with distributed point forces. In this figure, Ω is a region corresponding to the given problem. Ω_c is an auxiliary region introduced to produce a perfect infinite domain. The body force is embedded along the periphery of Ω_c, which is referred as an imaginary boundary Γ. The reference point P is an arbitrary point in a region Ω, and the source point Q lies on Γ. In BFM, stresses and displacements at a point P can be calculated by

$$\sigma_{ij}(P) = \sigma_{ij}^{\infty}(P) + \int_{\Gamma} \sigma_{ij}^{k}(P,Q)\phi_k(Q)d\Gamma \quad (P \in \Omega, Q \in \Gamma) \tag{1}$$

for stress components and

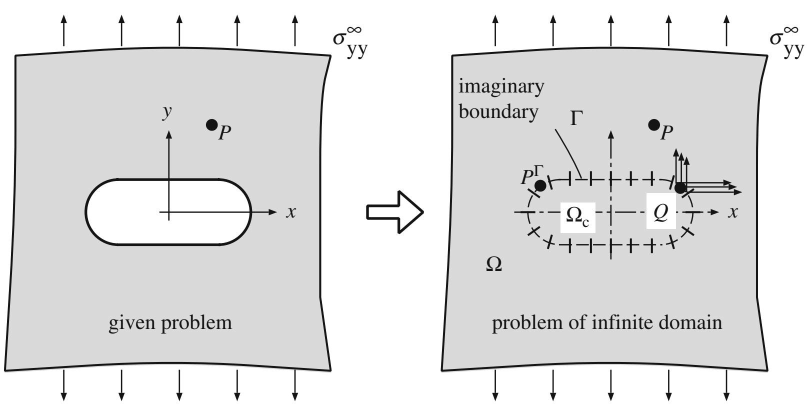
$$u_i(P) = u_i^{\infty}(P) + \int_{\Gamma} u_i^{k}(P,Q)\phi_k(Q)d\Gamma \quad (P \in \Omega, Q \in \Gamma) \tag{2}$$

for displacement components. In (1) and (2), a summation convention is taken account. That is, i and j are free indices, and k is a dummy index which takes either of (x, y) for 2D and (x, y, z) for 3D problems.

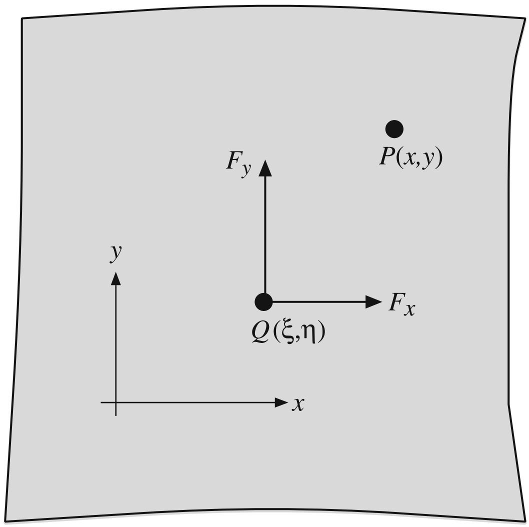
In these equations, $\sigma_{ij}^{k}(P,Q)$ and $u_i^{k}(P,Q)$ are the fundamental solutions. For instance, $\sigma_{xx}^{y}(P,Q)$ is a σ_{xx} component of stress at a reference point P due to a unit magnitude of point force of y direction acting at a source point Q in an infinite medium, and $u_y^{x}(P,Q)$ is a y component of displacement at a reference point P due to a unit magnitude of point force of x direction acting at a source point Q in an infinite medium. $\phi_x(Q)$ and $\phi_y(Q)$ are the unknown density functions of body force to be determined so that the boundary conditions are satisfied.

Concrete Forms of Fundamental Solutions for 2D Elasticity

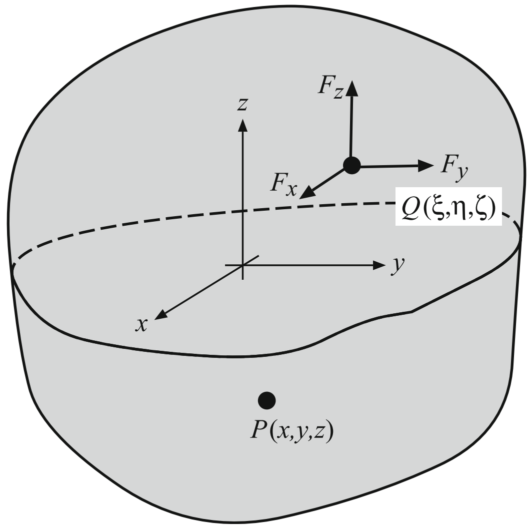
The simplest and the most frequently used fundamental solution in 2D analysis is a stress field due to a unit magnitude of an isolated point force acting in an infinite plate (Fig. 2). The complete



Body Force Method for Thermoelasticity, Fig. 1 Basic concept of body force method (tension of infinite plate with oval notch)



Body Force Method for Thermoelasticity, Fig. 2 The fundamental solutions for 2D problems



Body Force Method for Thermoelasticity, Fig. 3 The fundamental solutions for 3D problems

expressions of set of fundamental solutions for 2D problems [2] are as follows:

$$\left.\begin{aligned}
\sigma_{xx}^{x}(P,Q) &= -\frac{x-\xi}{2\pi(\kappa+1)}\left[\frac{\kappa}{r^2}+\frac{3(x-\xi)^2-(y-\eta)^2}{r^4}\right]\\
\sigma_{yy}^{x}(P,Q) &= \frac{x-\xi}{2\pi(\kappa+1)}\left[\frac{\kappa}{r^2}-\frac{(x-\xi)^2+5(y-\eta)^2}{r^4}\right]\\
\tau_{xy}^{x}(P,Q) &= -\frac{y-\eta}{2\pi(\kappa+1)}\left[\frac{\kappa}{r^2}+\frac{3(x-\xi)^2-(y-\eta)^2}{r^4}\right]
\end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned}
\sigma_{xx}^{y}(P,Q) &= \frac{y-\eta}{2\pi(\kappa+1)}\left[\frac{\kappa}{r^2}-\frac{5(x-\xi)^2+(y-\eta)^2}{r^4}\right]\\
\sigma_{yy}^{y}(P,Q) &= -\frac{y-\eta}{2\pi(\kappa+1)}\left[\frac{\kappa}{r^2}-\frac{(x-\xi)^2-3(y-\eta)^2}{r^4}\right]\\
\tau_{xy}^{y}(P,Q) &= -\frac{x-\xi}{2\pi(\kappa+1)}\left[\frac{\kappa}{r^2}-\frac{(x-\xi)^2-3(y-\eta)^2}{r^4}\right]
\end{aligned}\right\} \quad (4)$$

and

$$\left.\begin{aligned}
u_x^x(P,Q) &= \frac{1}{2\pi G(\kappa+1)}\left[\kappa\log\frac{1}{r}+\frac{(x-\xi)^2}{r^2}\right]\\
u_x^y(P,Q) = u_y^x(P,Q) &= \frac{1}{2\pi G(\kappa+1)}\frac{(x-\xi)(y-\eta)}{r^2}\\
u_y^y(P,Q) &= \frac{1}{2\pi G(\kappa+1)}\left[\kappa\log\frac{1}{r}+\frac{(y-\eta)^2}{r^2}\right]
\end{aligned}\right\} \quad (5)$$

where (x,y) are the coordinates of reference point P and (ξ,η) are the coordinates of source point Q. κ is Kolosov's index defined as $\kappa=\frac{3-\nu}{1+\nu}$ for plane stress and $\kappa=3-4\nu$ for plane strain, in which ν is Poisson's ratio. G is a shear modulus of the material, and r is a distance between P and Q so that $r^2=(x-\xi)^2+(y-\eta)^2$.

Concrete Forms of Fundamental Solutions for 3D Elasticity

The simplest and the most frequently used fundamental solution in 3D analysis is a stress field due to a unit magnitude of an isolated point force acting in an infinite solid (Fig. 3). The complete expressions of fundamental solution for 3D problems [3] are as follows:

$$\left.\begin{aligned}
\sigma_{xx}^{x}(P,Q) &= -\frac{x-\xi}{8\pi(1-\nu)}\left[\frac{1-2\nu}{r^3}+3\frac{(x-\xi)^2}{r^5}\right]\\
\sigma_{yy}^{x}(P,Q) &= \frac{x-\xi}{8\pi(1-\nu)}\left[\frac{1-2\nu}{r^3}-3\frac{(y-\eta)^2}{r^5}\right]\\
\sigma_{zz}^{x}(P,Q) &= \frac{x-\xi}{8\pi(1-\nu)}\left[\frac{1-2\nu}{r^3}-3\frac{(z-\zeta)^2}{r^5}\right]\\
\tau_{yz}^{x}(P,Q) &= -\frac{3}{8\pi(1-\nu)}\frac{(x-\xi)(y-\eta)(z-\zeta)}{r^5}\\
\tau_{zx}^{x}(P,Q) &= -\frac{z-\zeta}{8\pi(1-\nu)}\left[\frac{1-2\nu}{r^3}+3\frac{(x-\xi)^2}{r^5}\right]\\
\tau_{xy}^{x}(P,Q) &= -\frac{y-\eta}{8\pi(1-\nu)}\left[\frac{1-2\nu}{r^3}+3\frac{(x-\xi)^2}{r^5}\right]
\end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned}
\sigma_{xx}^{y}(P,Q) &= \frac{y-\eta}{8\pi(1-v)}\left[\frac{1-2v}{r^3}-3\frac{(x-\xi)^2}{r^5}\right]\\
\sigma_{yy}^{y}(P,Q) &= -\frac{y-\eta}{8\pi(1-v)}\left[\frac{1-2v}{r^3}+3\frac{(y-\eta)^2}{r^5}\right]\\
\sigma_{zz}^{y}(P,Q) &= \frac{y-\eta}{8\pi(1-v)}\left[\frac{1-2v}{r^3}-3\frac{(z-\zeta)^2}{r^5}\right]\\
\tau_{yz}^{y}(P,Q) &= -\frac{z-\zeta}{8\pi(1-v)}\left[\frac{1-2v}{r^3}+3\frac{(y-\eta)^2}{r^5}\right]\\
\tau_{zx}^{y}(P,Q) &= -\frac{3}{8\pi(1-v)}\frac{(x-\xi)(y-\eta)(z-\zeta)}{r^5}\\
\tau_{xy}^{y}(P,Q) &= -\frac{x-\xi}{8\pi(1-v)}\left[\frac{1-2v}{r^3}+3\frac{(y-\eta)^2}{r^5}\right]
\end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned}
\sigma_{xx}^{z}(P,Q) &= \frac{z-\zeta}{8\pi(1-v)}\left[\frac{1-2v}{r^3}-3\frac{(x-\xi)^2}{r^5}\right]\\
\sigma_{yy}^{z}(P,Q) &= \frac{z-\zeta}{8\pi(1-v)}\left[\frac{1-2v}{r^3}-3\frac{(y-\eta)^2}{r^5}\right]\\
\sigma_{zz}^{z}(P,Q) &= -\frac{z-\zeta}{8\pi(1-v)}\left[\frac{1-2v}{r^3}+3\frac{(z-\zeta)^2}{r^5}\right]\\
\tau_{yz}^{z}(P,Q) &= -\frac{y-\eta}{8\pi(1-v)}\left[\frac{1-2v}{r^3}+3\frac{(z-\zeta)^2}{r^5}\right]\\
\tau_{zx}^{z}(P,Q) &= -\frac{x-\xi}{8\pi(1-v)}\left[\frac{1-2v}{r^3}+3\frac{(z-\zeta)^2}{r^5}\right]\\
\tau_{xy}^{z}(P,Q) &= -\frac{3}{8\pi(1-v)}\frac{(x-\xi)(y-\eta)(z-\zeta)}{r^5}
\end{aligned}\right\} \quad (8)$$

and

$$\left.\begin{aligned}
u_x^x(P,Q) &= \frac{1}{16\pi G(1-v)}\left[\frac{3-4v}{r}+\frac{(x-\xi)^2}{r^3}\right]\\
u_y^x(P,Q) &= u_x^y(P,Q) = \frac{1}{16\pi G(1-v)}\frac{(x-\xi)(y-\eta)}{r^3}\\
u_z^x(P,Q) &= u_x^z(P,Q) = \frac{1}{16\pi G(1-v)}\frac{(z-\zeta)(x-\xi)}{r^3}
\end{aligned}\right\} \quad (9)$$

$$\left.\begin{aligned}
u_x^y(P,Q) &= u_y^x(P,Q) = \frac{1}{16\pi G(1-v)}\frac{(x-\xi)(y-\eta)}{r^3}\\
u_y^y(P,Q) &= \frac{1}{16\pi G(1-v)}\left[\frac{3-4v}{r}+\frac{(y-\eta)^2}{r^3}\right]\\
u_z^y(P,Q) &= u_y^z(P,Q) = \frac{1}{16\pi G(1-v)}\frac{(y-\eta)(z-\zeta)}{r^3}
\end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned}
u_x^z(P,Q) &= u_z^x(P,Q) = \frac{1}{16\pi G(1-v)}\frac{(z-\zeta)(x-\xi)}{r^3}\\
u_y^z(P,Q) &= u_z^y(P,Q) = \frac{1}{16\pi G(1-v)}\frac{(y-\eta)(z-\zeta)}{r^3}\\
u_z^z(P,Q) &= \frac{1}{16\pi G(1-v)}\left[\frac{3-4v}{r}+\frac{(z-\zeta)^2}{r^3}\right]
\end{aligned}\right\} \quad (11)$$

where (x, y, z) are the coordinates of reference point P and (ξ, η, ζ) are the coordinates of source point Q. r is a distance between P and Q so that $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$. When the reference point P approaches to the source point Q, stress solutions designate a singularity of order $O(r^{-1})$ for 2D and $O(r^{-2})$ for 3D problems. Therefore, in case if P approaches to the imaginary boundary Γ, a special consideration for a boundary integral is required.

An Example: A Circular Hole Problem

Solution of Stresses in Terms of Boundary Integrals

Consider an infinite plate with a circular hole of radius a and is subjected to a uniform tensile stress σ_{yy}^{∞} at infinity. Let the origin of coordinate system coincide with a center of circular hole and express the coordinate of the source point $Q(\xi, \eta)$ on Γ by means of circumferential angle θ such that $\xi = a\cos\theta$ and $\eta = a\sin\theta$. Then, the components of stresses and displacements at a reference point $P(x, y)$ can be written as follows:

$$\begin{aligned}
\sigma_{xx}(x,y) &= \oint\{\sigma_{xx}^{x}(x,y,\xi,\eta)\phi_x(\theta)\\
&\quad + \sigma_{xx}^{y}(x,y,\xi,\eta)\phi_y(\theta)\}ad\theta\\
\sigma_{yy}(x,y) &= \sigma_{yy}^{\infty} + \oint\{\sigma_{yy}^{x}(x,y,\xi,\eta)\phi_x(\theta)\\
&\quad + \sigma_{yy}^{y}(x,y,\xi,\eta)\phi_y(\theta)\}ad\theta\\
\tau_{xy}(x,y) &= \oint\{\tau_{xy}^{x}(x,y,\xi,\eta)\phi_x(\theta)\\
&\quad + \tau_{xy}^{y}(x,y,\xi,\eta)\phi_y(\theta)\}ad\theta
\end{aligned} \quad (12)$$

In these expressions, $d\Gamma$ (an infinitesimal length measured along the imaginary boundary) is replaced by $ad\theta$, and the integral is evaluated as a contour integral.

The Boundary Condition

In the limit of P approaching to Γ from interior of Ω, since the circular hole is considered as free of traction, the following boundary conditions are imposed:

$$\lim_{P \to P^\Gamma} t_i(P) = \lim_{P \to P^\Gamma} \sigma_{ij}(P) n_j(P^\Gamma) = 0 \quad (P \in \Omega, P^\Gamma \in \Gamma) \tag{13}$$

where P^Γ is an arbitrary point on Γ and $n_j(P^\Gamma)$ is a j-th component of unit outward normal at P^Γ. In more concrete expression, the following two conditions are held irrespective to the angular coordinate of reference point φ with which the coordinates of reference point are assumed as $x = (a+\varepsilon)\cos\varphi$ and $y = (a+\varepsilon)\sin\varphi$.

$$t_x(P^\Gamma) = \lim_{\varepsilon \to 0, \varepsilon > 0} \{\sigma_{xx}(x,y)\cos\varphi + \tau_{xy}(x,y)\sin\varphi\} = 0 \tag{14}$$

$$t_y(P^\Gamma) = \lim_{\varepsilon \to 0, \varepsilon > 0} \{\tau_{xy}(x,y)\cos\varphi + \sigma_{yy}(x,y)\sin\varphi\} = 0 \tag{15}$$

Substituting (12) into (14) and (15), a set of boundary integral equations for the determination of unknown density functions $\phi_x(\theta)$ and $\phi_y(\theta)$ are obtained. It should be noted that integral included in the right-hand side of (14) and (15) have to be examined before taking a limit $\varepsilon \to 0$, otherwise the boundary integral may yield an improper result. This fact is caused by a presence of the singularity in the point force solution.

Introduction of the Basic Density Function

In order to compute unknown density functions $\phi_x(\theta)$ and $\phi_y(\theta)$ numerically, Γ is divided into N segments $\Gamma_1, \Gamma_2, \cdots, \Gamma_N$, and value of density function on each segment is assumed to take some designated variation. If Γ (an imaginary circle) is equally divided into N segments, then n-th segment Γ_n has a shape of circular arc and is described by an angular extent $2\pi\frac{n-1}{N} < \theta < 2\pi\frac{n}{N}$. In this discretization scheme, it is expected that the approximation is improved with increase of division number N; however, increase in N also causes increase of memory consumption and calculation time. In order to overcome this problem, the concept of basic density function is introduced [4]. Introduction of the basic density function is a unique strategy which greatly improves the efficiency of numerical analysis in BFM. In a circular hole problem, the basic density function is chosen as a component of unit outward normal at Q on Γ. Then density function to be determined is replaced by a product of basic density function and unknown weight value such that

$$\phi_x(\theta) = \cos\theta \times W_x^n,\ \phi_y(\theta) = \sin\theta \times W_y^n \quad (n = 1, 2, \cdots, N) \tag{16}$$

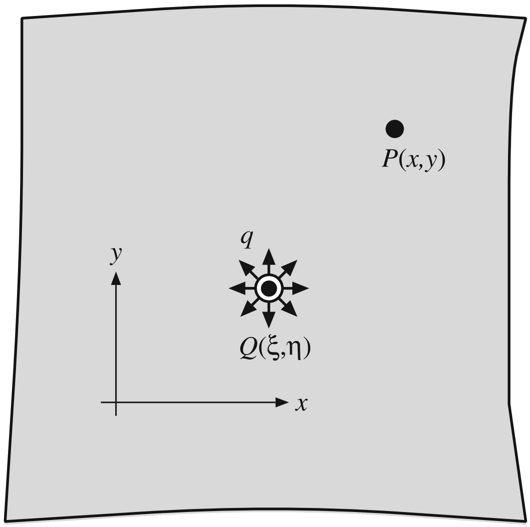
As a result, (12) can be rewritten as

$$\sigma_{ij}(x,y) = \sigma_{ij}{}^\infty(x,y) + \sum_{n=1}^{N} W_x^n \int_{2\pi\frac{n-1}{N}}^{2\pi\frac{n}{N}} \sigma_{ij}^x(x,y,a\cos\theta,a\sin\theta)\cos\theta \times a d\theta + \sum_{n=1}^{N} W_y^n \int_{2\pi\frac{n-1}{N}}^{2\pi\frac{n}{N}} \sigma_{ij}^y(x,y,a\cos\theta,a\sin\theta)\sin\theta \times a d\theta \tag{17}$$

Therefore, there are $2N$ unknown weights $W_x^1, W_x^2, \cdots, W_y^N$ are to be determined. These unknowns uniquely determined from condition of free of traction at N of representative reference points. In usual analysis, these representative points are set at a middle point of each divided segment. In this manner, the integral equation is transformed into a form of simultaneous equations for the determination of unknown weights. It should be noted again when the reference point exists very near to the segment where body forces are acting, a special care is required to ensure the accuracy of boundary integral.

Extension to Thermoelasticity

In thermoelastic analysis, the given temperature conditions are satisfied by superposing a temperature field due to an isolated heat source acting in



Body Force Method for Thermoelasticity, Fig. 4 Thermoelastic field due to point heat acting in an infinite plate

an infinite domain first [5]. Then, the isothermal mechanical problem is solved considering the influence of distributed heat sources. In case of a transient thermal stress analysis, fundamental solution depending on time variable is employed. For instance, thermoelastic fields due to an isolated point heat acting in an infinite plate (Fig. 4), which can be used for stationary thermoelastic problems, are as follows:

$$T(x,y) = -\frac{q}{2\pi\lambda}\ln r \tag{18}$$

$$\begin{aligned}
\sigma_{xx}(x,y) &= \frac{\beta E q}{8\pi\lambda}\left\{2\ \ln r - \frac{(x-\xi)^2-(y-\eta)^2}{r^2}\right\}\\
\sigma_{yy}(x,y) &= \frac{\beta E q}{8\pi\lambda}\left\{2\ \ln r + \frac{(x-\xi)^2-(y-\eta)^2}{r^2}\right\}\\
\tau_{xy}(x,y) &= -\frac{\beta E q}{4\pi\lambda}\frac{(x-\xi)(y-\eta)}{r^2}
\end{aligned} \tag{19}$$

where q is a magnitude of point heat per thickness, λ is thermal conductivity, E is Young's modulus, and β is thermal stress constant defined by $\beta = \alpha$ for plane stress and $\beta = \frac{\alpha}{1-\nu}$ for plane strain in which α is a coefficient of linear expansion. (ξ, η) is a coordinate at which point heat acts, and (x, y) is a coordinate of reference point. r is a distance between source and reference points $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$ as defined previously.

The fundamental thermoelastic fields shown in (18) and (19) are brought as follows. Consider an infinite plate of homogeneous thermoelastic property whose thermal conductivity is λ. When point heat source of magnitude q per thickness B is applied at the origin, corresponding temperature field is symmetric so that the temperature is a function of distance from the origin r alone. In such situation, the equation of energy balance ignoring the heat dissipation from the surface of plate can be finally deduced in polar coordinate system as [2]

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \tag{20}$$

It should be noted that this equation is valid except the origin where heat source is applied. The general solution of (20) is

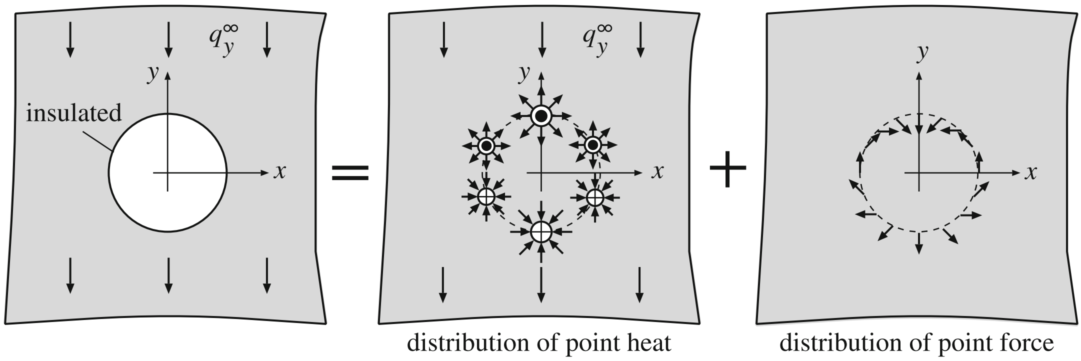
$$T(r) = C_1 \ln r + C_2 \tag{21}$$

where C_1 and C_2 are arbitrary constants. Apparently, C_2 corresponds to a uniform temperature rise. On the other hand, C_1 is determined from the condition that the total heat passing through a cylindrical area of radius r and a thickness B is constant at $-\lambda\frac{dT}{dr} \times 2\pi r B = qB$. Therefore, $C_1 = -\frac{q}{2\pi\lambda}$ is concluded. Under the presence of thermal strain, the Hooke's law for plane theory of thermoelasticity in the polar coordinates system becomes

$$\varepsilon_r = \frac{\sigma_r - \nu\sigma_\theta}{E} + \alpha T,\ \varepsilon_\theta = \frac{\sigma_\theta - \nu\sigma_r}{E} + \alpha T,\ \gamma_{r\theta} = \frac{\tau_{r\theta}}{G} \tag{22}$$

Due to the symmetry of the problem, the strain components ε_r and ε_θ are the function of radial displacement u alone, and $\gamma_{r\theta}$ is simply zero so that

$$\varepsilon_r = \frac{du}{dr},\ \varepsilon_\theta = \frac{u}{r}, \gamma_{r\theta} = 0 \tag{23}$$



Body Force Method for Thermoelasticity, Fig. 5 Analysis of thermoelastic field of an infinite plate with uniform heat flow at infinity disturbed by an insulated circular hole of free of traction

Therefore, stress components are expressed using displacement function $u(r)$ and temperature function $T(r)$ as

$$\begin{aligned}\sigma_r &= \frac{E}{1-\nu^2}\left[\frac{du}{dr}+\nu\frac{u}{r}-(1+\nu)\alpha T\right]\\ \sigma_\theta &= \frac{E}{1-\nu^2}\left[\frac{u}{r}+\nu\frac{du}{dr}-(1+\nu)\alpha T\right]\end{aligned} \tag{24}$$

Then, the equation of equilibrium, $\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}=0$, becomes

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=(1+\nu)\alpha\frac{dT}{dr} \tag{25}$$

Substituting $\frac{dT}{dr}=-\frac{q}{2\pi\lambda}\frac{1}{r}$, particular solution of (25) is

$$\begin{aligned}u &= (1+\nu)\alpha\frac{1}{r}\int T(r)rdr\\ &= -\frac{(1+\nu)\alpha q}{4\pi\lambda}r\left(\ln r-\frac{1}{2}\right)\end{aligned} \tag{26}$$

Substitution (26) into (24), the corresponding thermal stresses are

$$\begin{aligned}\sigma_r &= \frac{\alpha E q}{8\pi\lambda}(2\ln r-1)\\ \sigma_\theta &= \frac{\alpha E q}{8\pi\lambda}(2\ln r+1)\\ \tau_{r\theta} &= 0\end{aligned} \tag{27}$$

It should be noted that (27) is a thermoelastic solution for plane stress problems. By examining a stress transformation from polar coordinate system to Cartesian coordinate system, (19) is obtained from (27).

Based on the principle of BFM, problem shown in Fig. 5 is treated in the following manner. In a first step, a point heat source is distributed continuously along an imaginary boundary Γ with unknown density $\rho(\theta)$. Then, the components of heat flux $q_x=-\lambda\frac{\partial T}{\partial x}$ and $q_y=-\lambda\frac{\partial T}{\partial y}$ at arbitrary point (x,y), $(x^2+y^2>a)$ are expressed as

$$q_x(x,y)=\frac{1}{2\pi}\oint\frac{\rho(\theta)(x-a\cos\theta)ad\theta}{(x-a\cos\theta)^2+(y-a\sin\theta)^2} \tag{28}$$

$$q_y(x,y)=q_y^\infty+\frac{1}{2\pi}\oint\frac{\rho(\theta)(y-a\sin\theta)ad\theta}{(x-a\cos\theta)^2+(y-a\sin\theta)^2} \tag{29}$$

where a is a radius of an imaginary circular boundary along which thermally insulated condition is imposed. The unknown density of distributed heat source $\rho(\theta)$ is determined from the condition that the heat flux that flows perpendicular to the boundary of an imaginary circular boundary is zero; therefore,

$$\lim_{\varepsilon\to 0^+}\{q_x(x,y)\cos\varphi+q_y(x,y)\sin\varphi\}=0 \tag{30}$$

(30) should be held for arbitrary angular coordinate φ $(0 \leq \varphi \leq 2\pi)$ when $x = (a + \varepsilon)\cos\varphi$ and $y = (a + \varepsilon)\sin\varphi$. That is, (30) is a boundary integral equation for the determination of unknown density of heat source $\rho(\theta)$. After the determination of density of heat source, the corresponding thermal stress field is calculated from (19). In general, the obtained thermal stress field corresponding to the temperature rise does not satisfy the given mechanical boundary conditions. Therefore, the appropriate body force solution (the isothermal elasticity solution) is superposed so that the given mechanical boundary condition is totally satisfied.

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Bonded Circular Inclusions in Plane Thermoelasticity

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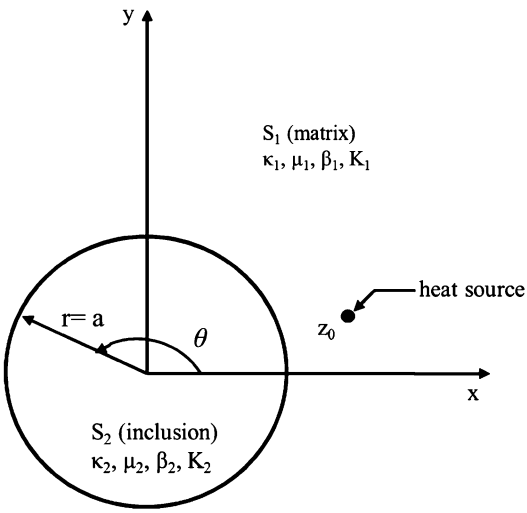
Overview

The thermal stresses induced by an insulated hole in an isotropic medium was first studied by Florence and Goodier [1, 2]. The same problem was solved by Chen [3] for an orthotropic medium containing a circular or elliptic hole. Hwu [4] studied the thermal stresses in an anisotropic body under uniform heat flow disturbed by an insulated elliptic hole using the Stroh formalism [5]. Following the Lekhnitskii complex potential approach, Tarn and Wang [6] found the thermal stresses in anisotropic bodies with a hole or a rigid inclusion. Recently, Kattis and Meguid [7] gave a solution of thermoelastic problems of an elastic curvilinear inclusion embedded in an elastic matrix where all singularities are located in the matrix. In this study, we aim to provide a general solution to the elastic inclusion problem subjected to a point heat source or a uniform heat flow. A point heat source considered in this note resides either outside or inside the circular inclusion. The analysis is based upon the complex variable theory and the method of analytical continuation which allows us to express the general solutions of the temperature and stress functions in a compact form. Some special examples are solved in closed form and are compared with existing analytical solutions, such as a point heat source in the circular disk and an infinite matrix with a circular elastic inclusion under a remote uniform heat flow.

Problem Formulation

Consider a circular elastic inclusion perfectly bonded to an infinite matrix subjected to a point heat source located either in the matrix (including infinity) or in the inclusion and a uniform heat flow applied at infinity. The regions occupied by the elastic matrix ($|z| > a$) and the inclusion ($|z| < a$) will be referred to as regions S_1 and S_2, respectively, and the quantities associated with these regions will be denoted by the corresponding subscripts (see Fig. 1). A point heat source in the system or a uniform heat flow at infinity causes a thermal stress distribution as a result of the different thermoelastic properties of the two phases. For a two-dimensional heat conduction problem, the resultant heat flow Q_j and the temperature T_j can be expressed in terms of a single complex potential $g_j'(z)$ as



Bonded Circular Inclusions in Plane Thermoelasticity, Fig. 1 A bonded circular inclusion subjected to a point heat source outside the inclusion.

$$Q_j = \int (q_{xj}dy - q_{yj}dx) = -k_j \operatorname{Im}[g_j{}'(z)] \quad (1)$$

$$T_j = \operatorname{Re}[g_j{}'(z)] \quad (2)$$

where Re and Im denote the real part and imaginary part of the bracketed expression, respectively. The quantities q_{xj}, q_{yj} in (1) are the components of heat flux in the x and y-direction, respectively, and k_j stands for the heat conductivity with $j = 1$ for S_1 and $j = 2$ for S_2. Once the heat conduction problem is solved, the temperature function $g_j{}'(z)$ is determined. For a two-dimensional theory of thermoelasticity, the components of the displacement and traction force can be expressed in terms of two stress functions $\phi_j(z)$, $\psi_j(z)$ and a temperature function $g_j{}'(z)$ as

$$2\mu_j(u_j + iv_j) = \kappa_j\phi_j(z) - \overline{z\phi_j{}'(z)} - \overline{\psi_j(z)} + 2\mu_j\beta_j \int g_j{}'(z)dz \quad (3)$$

$$-Y_j + iX_j = \phi_j(z) + \overline{z\phi_j{}'(z)} + \overline{\psi_j(z)} \quad (4)$$

where μ_j is the shear modulus, and $\kappa_j = (3 - v_j)/(1 + v_j)$, $\beta_j = \alpha_j$ for plane stress and $\kappa_j = 3 - 4v_j$,

$\beta_j = (1 + v_j)\,\alpha_j$ for plane strain with v_j being the Poisson's ratio and α_j the thermal expansion coefficients. Primes denote differentiation with respect to z and a superimposed bar denotes the complex conjugate. For the condition that both the stresses and displacements are single-valued either in the matrix or in the inclusion, the stress functions $\phi_j(z)$, $\psi_j(z)$ must take the form

$$\phi_j(z) = A_j z \ln z + B_j \ln z + \phi_j{}^*(z) \quad (5)$$

$$\psi_j(z) = C_j \ln z + \psi_j{}^*(z) \quad (6)$$

where A_j is a real constant and B_j, C_j are complex constants which are related by the following equations

$$(\kappa_j + 1)A_j z + \kappa_j B_j + \overline{C_j} = \frac{-2\mu_j\beta_j}{2\pi i} \oint_{c_j} g_j{}'(t)dt \quad (7)$$

$$B_j - \overline{C_j} = 0 \quad (8)$$

where c_j is any surrounding contour within the region S_j ($j = 1, 2$). Note that the singularity of the term $z \ln z$ appeared in the stress functions, (5), results from the logarithmic singularity of the temperature function induced by a point heat source. The two holomorphic functions $\phi_j{}^*(z)$ and $\psi_j{}^*(z)$ in (5) and (6), respectively, can be expressed in a series form as

$$\phi_1{}^* = \sum_{m=1}^{\infty} L_m z^{-m}, \quad \psi_1{}^* = \sum_{m=1}^{\infty} M_m z^{-m} \quad (9)$$

$$\phi_2{}^* = \sum_{m=1}^{\infty} N_m z^{m}, \quad \psi_2{}^* = \sum_{m=1}^{\infty} P_m z^{m} \quad (10)$$

where the constant coefficients L_m, M_m N_m and P_m may be determined from the interface continuity conditions.

Temperature Field

Consider a point heat source is located outside the inclusion (see Fig. 1), the temperature functions in the matrix and in the inclusion, respectively, can be written as

$$g_1'(z) = g_0'(z) + g_1'(z) \quad (11)$$

$$g_2'(z) = g_2'(z) \quad (12)$$

where $g'_0(z)$ represents the function associated with the unperturbed field which is related to the solutions of homogeneous media and is holomorphic in the entire domain except a singular point under a point heat source, and the points at zero or infinity. $g'_1(z)$ (or $g'_2(z)$) is the function corresponding to the perturbed field of matrix (or inclusion) and is holomorphic in region S_1, (or S_2.) except some singular points. In the present study, the temperature function $g'_0(z)$ is given as [8]

$$g_0'(z) = -\frac{q}{2\pi k_1} \ln(z - z_0) \quad (13)$$

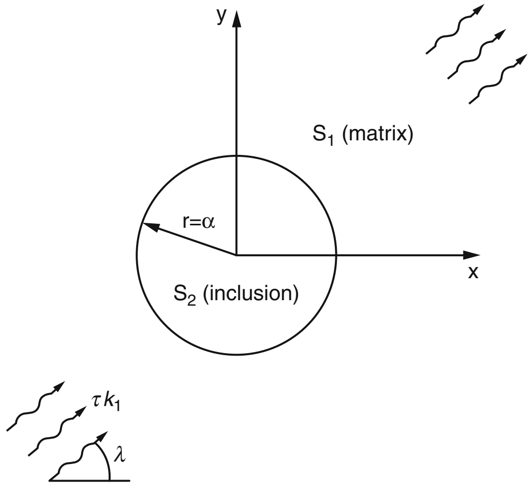
for a point heat source with the strength q located at the point $z = z_0$ in the matrix, and

$$g_0'(z) = \tau e^{-i\lambda z} z \quad (14)$$

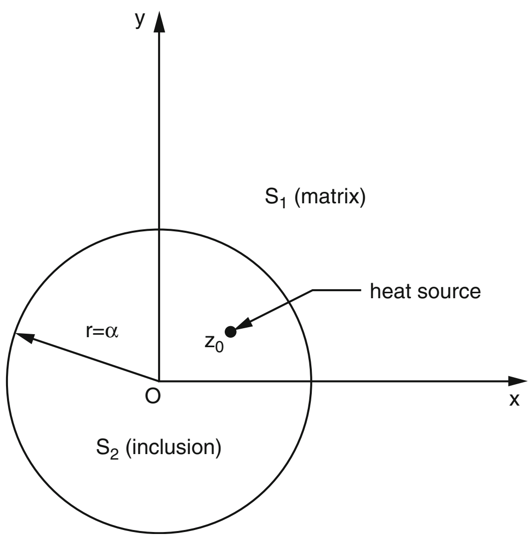
for a remote uniform heat flow with the constant temperature gradient τ directed at an angle λ with respect to the positive x-axis (see Fig. 2). $g'_1(z)$ and $g'_2(z)$ in (11) and (12), respectively, will be determined from the interface continuity conditions, i.e., $T_1 = T_2$ and $Q_1 = Q_2$ along the interface $z = \sigma = ae^{i\theta}$. Using the above boundary conditions and applying the method of analytical continuation, we obtain the final results as

$$g_1'(z) = g_0'(z) + \frac{k_1 - k_2}{k_1 + k_2} \overline{g_0'}\left(\frac{a^2}{z}\right) \quad (15)$$

$$g_2'(z) = \frac{2k_1}{k_1 + k_2} g_0'(z) \quad (16)$$



Bonded Circular Inclusions in Plane Thermoelasticity, Fig. 2 A bonded circular inclusion subjected to a remote uniform heat flow.



Bonded Circular Inclusions in Plane Thermoelasticity, Fig. 3 A bonded circular inclusion subjected to a point heat source inside the inclusion.

If a point heat source is located inside the inclusion (see Fig. 3), the temperature functions can be written as

$$g_1'(z) = \frac{-q}{2\pi k_1} \ln\frac{z}{a} + g_1'(z) \quad (17)$$

$$g_2{}'(z) = \frac{-q}{2\pi k_2} \ln \frac{z}{a} + g_0{}'(z) + g_2{}'(z) \quad (18)$$

where $g'_0(z)$ is given by

$$g_0{}'(z) = \frac{-q}{2\pi k_2} \ln(a - \frac{az_0}{z}) \quad (19)$$

Using the interface continuity conditions as mentioned above and the method of analytical continuation, the final expression for the temperature functions becomes

$$g_1{}'(z) = \frac{-q}{2\pi k_1} \ln \frac{z}{a} - \frac{q}{\pi(k_1 + k_2)} \ln(a - \frac{az_0}{z}) \quad (20)$$

$$g_2{}'(z) = \frac{-q}{2\pi k_2} \ln(z - z_0) - \frac{(k_2 - k_1)q}{2\pi(k_1 + k_2)k_2} \ln(a - \frac{a\overline{z_0}}{z}) \quad (21)$$

Thermal Stress Field

Having the temperature functions as derived previously, the general solutions for the stress and displacement fields can be obtained in terms of the complex potentials given in (5) and (6) in which the constant coefficients A_j, B_j, and C_j may be determined from (7) and (8) while the two holomorphic functions $\phi_i{}^*(z)$ and $\psi_i{}^*(z)$ will be obtained from the interface continuity conditions.

We now consider a heat source located in the matrix for which the temperature functions $g'_1(z)$, $g'_2(z)$ have been given in (15) and (16), respectively. Substituting (15) and (16) into (7) and using (8), one obtains

$$A_1 = \frac{\mu_1 \beta_1 q}{\pi k_1 (1 + \kappa_1)}, \quad B_1 = \frac{-\mu_1 \beta_1 q z_0}{\pi k_1 (1 + \kappa_1)} \quad for \quad |z| > |z_0| \quad (22)$$

$$A_1 = B_1 = 0, \quad for \quad a < |z| < |z_0| \quad (23)$$

And

$$A_2 = B_2 = 0, \quad for \quad |z| < a \quad (24)$$

Since the inclusion and the matrix are assumed to be perfectly bonded along the interface, the displacements and surface tractions at the interface must be continuous, i.e., $u_1 + iv_1 = u_2 + iv_2$ and $-Y_1 + iX_1 = -Y_2 + iX_2$ along the interface $z = \sigma = ae^{i\theta}$. Using the above boundary conditions and applying the method of analytical continuations, the constant coefficients appeared in (9) and (10) are obtained as

$$L_m = -\frac{2\mu_1 \mu_2 \beta_1}{\mu_1 + \mu_2 \kappa_1} b_m \quad (25)$$

$$N_1 = \frac{-2\mu_1 \mu_2 (\beta_2 c_1 - \beta_1 a_1)}{(\kappa_1 \mu_1 + 2\mu_2 - \mu_1)} \quad (26)$$

$$N_m = -\frac{2\mu_1 \mu_2}{\mu_2 + \mu_1 \kappa_2} (\beta_2 c_m - \beta_1 a_m), for \quad m \geq 2 \quad (27)$$

$$M_1 = [2N_1 - A_1(1 + \ln a^2)]a^2 \quad (28)$$

$$M_2 = \overline{N_2} a^4 - B_1 a^2 \quad (29)$$

$$M_m = (m-2)a^2 L_{m-2} + a^{2m} \overline{N_m}, for \quad m \geq 3 \quad (30)$$

$$P_m = a^{-2m} \overline{L_m} - (m+2)a^2 N_{m+2} \quad (31)$$

Having the solutions in (25)–(31), the final expression of the stress functions can then be determined by substituting (9)–(10) and (22)–(24) into (5) and (6).

If a heat source is located in the inclusion, the constant coefficients in (5) and (6) can be determined by substituting (20) and (21) into (7) and (8) as

$$A_1 = \frac{\mu_1 \beta_1 q}{\pi k_1 (1 + \kappa_1)}, \quad B_1 = 0 \, for \quad |z| > a \quad (32)$$

$$A_2 = \frac{\mu_2 \beta_2 q}{\pi k_2 (1 + \kappa_2)}, B_2 = \frac{-\mu_2 \beta_2 q z_0}{\pi k_2 (1 + \kappa_2)} \quad for \quad |z_0| < |z| < a \tag{33}$$

And

$$A_2 = B_2 = 0, \quad for \quad |z| < |z_0| \tag{34}$$

By using the interface continuity conditions and the method of analytical continuation, we finally obtain

$$L_m = -\frac{2\mu_1\mu_2}{\mu_1 + \mu_2\kappa_1}(\beta_1 e_m - \beta_2 h_m) \tag{35}$$

$$N_1 = \frac{-(\mu_2 A_2 - \mu_1 A_2)(1 + \ln a^2) - 2\mu_1\mu_2(\beta_2 f_1 - \beta_1 d_1)}{(\kappa_2\mu_1 + 2\mu_2 - \mu_1)} \tag{36}$$

$$N_2 = -\frac{2\mu_1\mu_2}{\mu_2 + \mu_1\kappa_2}\beta_2 f_2 - \frac{\mu_2\overline{B_2} - \mu_1\overline{B_2}}{(\mu_2 + \mu_1\kappa_2)a^2} \tag{37}$$

$$N_m = -\frac{2\mu_1\mu_2}{\mu_2 + \mu_1\kappa_2}\beta_2 f_m, for \quad m \geq 3 \tag{38}$$

$$M_1 = [2N_1 + (A_2 - A_1)(1 + \ln a^2)]a^2 \tag{39}$$

$$M_2 = \overline{N_2}a^4 + (B_2 - B_1)a^2 \tag{40}$$

$$M_m = (m-2)a^2 L_{m-2} + a^{2m}\overline{N_m}, for \quad m \geq 3 \tag{41}$$

$$P_m = a^{-2}\overline{L_m} - (m+2)a^2 N_{m+2} \tag{42}$$

With the results in (35)–(42) the general solutions for the stress functions can then be obtained by substituting (32)–(34) and (9)–(10) into (5) and (6).

Examples

As our first example, we consider a circular elastic inclusion perfectly bonded to a matrix which is subjected to a uniform heat flux with the temperature gradient T directed at an angle k with respect to the positive x-axis (see Fig. 2). The solution of temperature functions can be easily obtained by substituting (14) into (15) and (16) as

$$g_1{}'(z) = \tau e^{-i\lambda} z + \frac{\tau(k_1 - k_2)}{(k_1 + k_2)} e^{i\lambda} \frac{a^2}{z} \tag{43}$$

$$g_2{}'(z) = \frac{2k_1\tau}{(k_1 + k_2)} e^{-i\lambda} z \tag{44}$$

Applying the formulae given in Section 4.1, the final solutions for the stress field can be given by

$$\phi_1(z) = \frac{-2\mu_1\beta_1\tau}{(1+\kappa_1)} \frac{(k_1 - k_2)}{(k_1 + k_2)} a^2 e^{-i\lambda} \ln z \tag{45}$$

$$\psi_1(z) = \frac{-2\mu_1\beta_1\tau}{(1+\kappa_1)} \frac{(k_1 - k_2)}{(k_1 + k_2)} a^2 e^{-i\lambda} \ln z - \frac{\mu_1\mu_2\tau}{(\mu_1\kappa_2 + \mu_2)} \left(\frac{2k_1\beta_2}{(k_1 + k_2)} - \beta_1 \right) a^4 e^{i\lambda} \frac{1}{z^2} + \frac{-2\mu_1\beta_1\tau}{(1+\kappa_1)} \frac{(k_1 - k_2)}{(k_1 + k_2)} a^4 e^{i\lambda} \frac{1}{z^2} \tag{46}$$

$$\phi_2(z) = -\frac{2\mu_1\mu_2\tau}{\mu_1\kappa_2 + \mu_2} \left(\frac{2k_1\beta_2}{(k_1 + k_2)} - \beta_1 \right) e^{-i\lambda} z^2 \tag{47}$$

$$\psi_2(z) = 0 \tag{48}$$

The interfacial stresses along the inclusion boundary can be performed by using field solutions of the matrix or inclusion as

$$\sigma_{rr} = -\frac{2\mu_1\mu_2\tau}{(\mu_1\kappa_2 + \mu_2)} \left(\frac{2k_1\beta_2}{(k_1 + k_2)} - \beta_1 \right) a\cos(\theta - \lambda)$$

$$\sigma_{r\theta} = -\frac{\mu_1\mu_2\tau}{(\mu_1\kappa_2 + \mu_2)} \left(\frac{2k_1\beta_2}{(k_1 + k_2)} - \beta_1 \right) a\sin(\theta - \lambda)$$

$$(\sigma_{\theta\theta})_1 = \left[\frac{-8\mu_1\beta_1\tau(k_1-k_2)}{(1+\kappa_1)(k_1+k_2)} + \frac{2\mu_1\mu_2\tau}{(\mu_1\kappa_2+\mu_2)} \times \left(\frac{2k_1\beta_2}{(k_1+k_2)} - \beta_1\right)\right] a\cos(\theta-\lambda)$$

$$(\sigma_{\theta\theta})_2 = -\frac{12\mu_1\mu_2\tau}{(\mu_1\kappa_2+\mu_2)}\left(\frac{2k_1\beta_2}{(k_1+k_2)} - \beta_1\right) a\cos(\theta-\lambda)$$

When the inclusion is assumed to be an insulated and traction free hole, the hoop stress along the hole boundary can be obtained by letting $k_2 = 0$ and $\mu_2 = 0$ as

$$\sigma_{\theta\theta} = -\frac{8\mu\beta\tau}{(1+\kappa)} a\cos(\theta-\lambda)$$

which is in agreement with the result of Florence and Goodier [1, 2]. For a special case of $k_1 = k_2$, $\mu_1 = \mu_2$ and $\beta_1 = \beta_2$, the solutions of the corresponding homogeneous problem is trivially given as

$$\phi(z) = \psi(z) = 0 \qquad (49)$$

This is expected that there is no thermal stresses induced by a homogeneous body under the condition of free expansion.

As a second example we consider the inclusion subjected to a point beat source acting at the origin. The temperature functions can be obtained by putting $z_0 = 0$ into (20) and (21) as

$$g_1'(z) = -\frac{q}{2\pi k_1}\ln\frac{z}{a} - \frac{a}{\pi(k_1+k_2)}\ln a \qquad (50)$$

$$g_2'(z) = -\frac{q}{2\pi k_2}\ln z - \frac{(k_2-k_1)q}{2\pi(k_1+k_2)k_2}\ln a \qquad (51)$$

A direct application of the formulae given in Section 4.2, the stress functions can be obtained as

$$\phi_1(z) = \frac{\mu_1\beta_1 q}{\pi k_1(1+\kappa_1)} z\ln z \qquad (52)$$

$$\psi_1(z) = M_1/z \qquad (53)$$

$$\phi_1(z) = \frac{\mu_1\beta_1 q}{\pi k_1(1+\kappa_1)} z\ln z \qquad (54)$$

$$\psi_1(z) = M_1/z \qquad (55)$$

where N_1 and M_1 are

$$N_1 = \frac{q}{\kappa_2\mu_1 + 2\mu_2 - \mu_1}\left\{\frac{\mu_2\beta_2(\mu_1-\mu_2)(1+\ln a^2)}{\pi k_2(1+\kappa_2)} - \frac{\mu_1\mu_2}{\pi}\left[\frac{\beta_2}{k_2} + \frac{(k_1-k_2)\beta_2\ln a}{(k_1+k_2)k_2} - \frac{\beta_1}{k_1}(1+\ln a) + \frac{2\beta_1\ln a}{(k_1+k_2)}\right]\right\}$$

$$M_1 = 2N_1a^2 + \frac{q}{\pi}\left[\frac{\mu_2\beta_2}{k_2(1+\kappa_2)} - \frac{\mu_1\beta_1}{k_1(1+\kappa_1)}\right](1+\ln a^2)a^2$$

Note that, the stresses would not be bounded either at zero or at infinity due to the presence of the singular term $z\ln z$ appeared in the stress functions induced by a point heat source. Nevertheless, the solutions are useful as the outer boundary of a body remains finite.

Consider a circular disk where the boundary surface is assumed to be free of traction and remain zero temperature. The corresponding temperature function and stress functions, respectively, can be obtained by letting $k_1 = \infty$ in (51) and $\mu_1 = 0$ in (54) as

$$g'(z) = -\frac{q}{2\pi k}\ln\frac{z}{a} \qquad (56)$$

$$\phi(z) = \frac{\mu\beta q}{\pi k(1+\kappa)} z\ln\frac{z}{a} - \frac{\mu\beta q}{2\pi k(1+\kappa)} z \qquad (57)$$

$$\psi(z) = 0 \qquad (58)$$

Accordingly, the stress components are given by

$$\sigma_{rr} = \frac{2\mu\beta q}{\pi k(1+\kappa)}\ln\frac{r}{a}$$

$$\sigma_{\theta\theta} = \frac{2\mu\beta q}{\pi k(1+\kappa)} \ln\frac{r}{a} + \frac{2\mu\beta q}{\pi k(1+\kappa)}$$

$$\sigma_{r\theta} = 0$$

which is the same as the results obtained by Parkus [9] essentially by guessing.

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Bonded Dissimilar Materials

▶ Edge Crack, Bimaterial Systems

Bonded Joints

▶ Thermal Stresses in Hybrid Composite Joints

Boundary Element Method

▶ Body Force Method
▶ Boundary Element Method in Generalized Thermoelasticity
▶ Boundary Element Method in Heat Conduction
▶ Boundary Element Method in Inverse Heat Conduction Problem

B

Boundary Element Method in Generalized Thermoelasticity

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Synonyms

BEM; Boundary element method; Boundary integral equation method; Boundary integral method

Overview

Many generalizations to the equations of the coupled theory of thermoelasticity due to Biot [1] are presently available. The first to appear is the theory of generalized thermoelasticity with one relaxation time introduced by Lord and Shulman [2]. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and, hence, automatically eliminates the paradox of infinite speeds of propagation inherent in the coupled theory of thermoelasticity. This theory was extended [3] by Dhaliwal and Sherief to include the effects of anisotropy. The uniqueness of solution of these equations was proved by Ignaczak [4] and by Sherief [5]. The boundary element formulation was conducted by Anwar and Sherief in [6]. The fundamental solution was obtained by Sherief [7] for the spherical case and by Sherief and Anwar [8] for the cylindrical case.

The second generalization to the coupled theory, known as the theory of thermoelasticity with two relaxation times, was obtained by Green and Lindsay [9]. In this theory, the classical Fourier law of heat conduction is not violated when the body under consideration has a center of symmetry. The uniqueness of solution for this theory was proved by Green [10]. The fundamental solution was obtained by Sherief [11]. The boundary element formulation was conducted by Anwar and Sherief in [12].

The boundary integral equation method (BEM) has been applied successfully in many branches of applied mathematics. The reason for this is its simplicity, efficiency, and ease of implementation compared to other numerical methods based on domain discretization such as the finite element method. Sladek and Sladek [13] set up the BEM formulation for coupled thermoelasticity.

In this work, a formulation of the boundary integral equation method for thermoelasticity with both one and two relaxation times is given. Fundamental solutions of the corresponding differential equations are obtained. A reciprocity theorem is derived. An outline of the implementation of the boundary element method is discussed for the solution of the above boundary equations.

The Mathematical Model

We shall consider a homogeneous isotropic thermoelastic solid occupying the region V and bounded by a smooth surface S. We shall also assume that the initial state of the medium is quiescent. Throughout this work, a comma denotes material derivatives and the summation convention is used. The governing equations for thermoelasticity with one–two relaxation times consist of [9]:

1. The equations of motion:

$$\mu u_{i,kk} + (\lambda+\mu)u_{k,ki} + \rho F_i - \gamma\left(T_{,i} + \upsilon\frac{\partial T_{,i}}{\partial t}\right) = \rho\frac{\partial^2 u_i}{\partial t^2} \quad (1)$$

where λ, μ are Lamé constants; ρ is the density; t is the time variable; T is the absolute temperature of the medium; u_i is the displacement component in the x_i-direction; F_i is the component of the body force per unit mass in the x_i-direction and $\gamma = (3\lambda + 2\mu)\alpha_t$, α_t being the coefficient of linear thermal expansion; and v is a constant with the dimensions of time that acts as a relaxation time.

2. The equation of heat conduction:

$$\chi T_{,ii} = \rho c_E\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)T + \left(1 + \alpha\tau_0\frac{\partial}{\partial t}\right)(\gamma T_0\dot{e} - \rho Q) \quad (2)$$

where χ is the thermal conductivity of the medium, e is the cubical dilatation, c_E is the specific heat at constant strain, T_0 is a reference temperature assumed to be such that $\left|\frac{T-T_0}{T_0}\right| << 1$, τ_0 is another relaxation time, and Q is the strength of the applied heat source per unit mass. For Lord-Shulman theory $\upsilon = 0$, $\alpha = 1$ and for Green-Lindsay theory, $\alpha = 0$.

3. The constitutive relations:

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e\delta_{ij} - \gamma\left(T - T_0 + v\frac{\partial T}{\partial t}\right)\delta_{ij} \quad (3)$$

where e_{ij} are the strain components given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ and } e = e_{ii} = u_{i,i} \quad (4)$$

Let us introduce the nondimensional variables

$$x_i^* = c\eta x_i,\ u_i^* = c\eta u_i, t^* = c^2\eta t,$$
$$\tau_0^* = c^2\eta\tau_0,\ \upsilon^* = c^2\eta\upsilon$$
$$\sigma_{ij}^* = \frac{\sigma_{ij}}{\lambda+2\mu}, \theta = \frac{\gamma(T-T_0)}{\lambda+2\mu},$$
$$Q^* = \frac{\rho Q}{\chi T_0\eta^2 c^2},\ F_i^* = \frac{\rho F_i}{\eta c(\lambda+2\mu)}$$

where $c^2 = \frac{\lambda+2\mu}{\rho}$ is the square of the velocity of longitudinal waves and $\eta = \frac{\rho c_E}{\chi}$. In terms of these variables Equations (1–3), respectively, take the following forms (dropping the asterisks for convenience):

$$u_{i,kk} + (\beta^2 - 1)u_{k,ki} + F_i - \beta^2\left(\theta_{,i} + \upsilon\frac{\partial\theta_{,i}}{\partial t}\right) = \beta^2\frac{\partial^2 u_i}{\partial t^2} \quad (5)$$

$$\theta_{,ii} = \frac{\partial \theta}{\partial t} + \tau_0 \frac{\partial^2 \theta}{\partial t^2} + \epsilon \left(\frac{\partial}{\partial t} + \alpha \tau_0 \frac{\partial^2}{\partial t^2} \right) e - \left(1 + \alpha \tau_0 \frac{\partial}{\partial t} \right) Q \quad (6)$$

$$\sigma_{ij} = (\beta^2 - 2)\, e\, \delta_{ij} + 2e_{ij} - \beta^2 \left(\theta + \upsilon \frac{\partial \theta}{\partial t} \right) \delta_{ij} \quad (7)$$

where $\beta^2 = \frac{\lambda + 2\mu}{\mu}$ and $\epsilon = T_0 \gamma^2 / \rho\, C_E (\lambda + 2\mu)$.

The boundary conditions of the problem will be taken as follows:

1. Mechanical condition: The traction $p_i = \sigma_{ij} n_j$ is specified on a part $S_1 \subset S$ while the displacement u, is specified on $S - S_1$
2. Thermal conditions: The temperature increment θ is specified on a part $S_2 \subset S$ while the normal derivative $\frac{\partial \theta}{\partial n}$ is specified on $S_2 \subset S$, where $\mathbf{n}(\mathrm{r})$ is the outward normal to the surface S. The above conditions can be stated as

$$\sigma_{ij} n_j = p_{i0}(r, t) \;\; on\; S_1 \quad (8)$$

$$u_i = u_{i0}(r, t) \;\; on\; \overline{S - S_1} \quad (9)$$

$$\theta = \theta_0(r, t) \;\; on\; S_2 \quad (10)$$

$$\theta_{,n} = \theta_{0,n}(r, t) \;\; on\; \overline{S - S_2} \quad (11)$$

Using (4) and (7), the traction $p_i(r, t)$ can be written as

$$p_i(r, t) = \left[(\beta^2 - 2) u_{j,j} - \beta^2 \left\{ \theta + v \frac{\partial \theta}{\partial t} \right\} \right] n_i + (u_{i,j} + u_{j,i}) n_j \quad (12)$$

Formulation in the Laplace-Transform Domain

The Laplace transform of a function $f\ (t)$ is defined by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Applying Laplace transform to both sides of (5)–(7) and using the homogeneous initial conditions, we obtain

$$\overline{u}_{i,kk} + (\beta^2 - 1)\overline{u}_{k,ki} + \overline{F}_i - \beta^2 (1 + v\, s) \overline{\theta}_{,i} = \beta^2 s^2 \overline{u}_i \quad (13)$$

$$\overline{\theta}_{,ii} = s\,(1 + \tau_0\, s)\, \overline{\theta} + (1 + \alpha \tau_0 s)\, (\varepsilon\, s\, \overline{u}_{k,k} - \overline{Q}) \quad (14)$$

$$\overline{\sigma}_{ij} = (\beta^2 - 2)\, \overline{u}_{k,k}\, \delta_{ij} + 2e_{ij} - \beta^2 (1 + v\, s)\, \overline{\theta}\, \delta_{ij} \quad (15)$$

Using the following Helmholtz decomposition for the displacement and body forces

$$u_i = \varphi_{,i} + \varepsilon_{ijk}\, \psi_{j,k} \quad (16)$$

$$F_i = X_{,i} + \varepsilon_{ijk}\, Y_{j,k} \quad (17)$$

Equations (13)–(15) yield

$$\beta^2 \left(\nabla^2 - s^2 \right) \overline{\varphi} + \overline{X} = \beta^2 (1 + v\, s) \overline{\theta} \quad (18)$$

$$(\nabla^2 - \beta^2 s^2)\, \overline{\psi_i} = -\overline{Y_i} \quad (19)$$

$$(\nabla^2 - s\,(1 + \tau_0\, s))\, \overline{\theta} - \varepsilon\, s\, (1 + \alpha \tau_0 s)\, \nabla^2 \overline{\varphi} = -(1 + \alpha \tau_0 s)\, \overline{Q} \quad (20)$$

Fundamental Solutions in the Laplace-Transform Domain

We shall consider two cases. The first is that of an instantaneous concentrated heat source acting at the point x = y in the absence of externally applied body forces. The second case is that when there is no heat source acting inside the medium with an instantaneous concentrated body force acting in the direction of one of the coordinate axes.

Case 1

We take

$$Q = \delta(x-y)\,\delta(t), \quad F_i = 0$$

The fundamental solutions corresponding to this case will be denoted by primes. Substituting the above values in Equations (18–20), we get

$$\left(\nabla^2 - s^2\right)\overline{\varphi'} = (1+v\,s)\,\overline{\theta'} \tag{21}$$

$$\left(\nabla^2 - \beta^2 s^2\right)\overline{\psi_i'} = 0 \tag{22}$$

$$\left(\nabla^2 - s\,(1+\tau_0\,s)\right)\overline{\theta'} - \varepsilon\, s\,(1+\alpha\tau_0 s)\,\nabla^2\overline{\varphi'} = -(1+\alpha\tau_0 s)\,\delta\,(x-y) \tag{23}$$

Equation (22) immediately yields

$$\overline{\psi_i'} = 0 \tag{24}$$

while (21) and (23) give, upon elimination of $\overline{\theta'}$, the following equation satisfied by $\overline{\varphi'}$
$\left\{\nabla^4 - [s^2 + s(1+\tau_0 s) + \epsilon s(1+v s)]\nabla^2 + s^3(1+\tau_0 s)\right\}\overline{\varphi'} = -(1+v s)\,(1+\alpha\tau_0 s)\,\delta(x-y)$ This equation can be factorized as

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)\,\overline{\varphi'} = -(1+v\,s)\,(1+\alpha\tau_0 s)\,\delta(x-y) \tag{25}$$

where k_1^2 and k_2^2 are the roots of the characteristic equation

$$k^4 - k^2\left[(s^2 + \varepsilon(s + \tau_0\,\alpha\,s^2)(1+v\,s) + (s+\tau_0\,s^2)\right] + s^3(1+\tau_0\,s) = 0$$

The sum and product of the roots of this equation satisfy the relations

$$k_1^2 + k_2^2 = (s^2 + \varepsilon(s+\tau_0\,\alpha\,s^2)(1+v\,s) + (s+\tau_0\,s^2) \tag{26a}$$

$$k_1^2 . k_2^2 = s^3(1+\tau_0 s) \tag{26b}$$

Using Helmholtz equations in space [14], namely,

$$\frac{1}{\nabla^2 - k^2}[\delta(r)] = \frac{-1}{4\pi\,r}e^{-k\,r}$$

the solution of (25) takes the form

$$\overline{\varphi'} = \frac{(1+v\,s)(1+\alpha\tau_0 s)}{4\pi\,r(k_1^2 - k_2^2)}\left[e^{-k_1\,r} - e^{-k_2\,r}\right] \tag{27}$$

where $r = \sqrt{(x_i - y_i)(x_i - y_i)}$

From (16), (24), and (27), it follows that

$$\overline{u}_i'(r,s) = \frac{-(1+v\,s)\,(1+\alpha\tau_0 s)}{4\pi\,r^2(k_1^2 - k_2^2)}r_{,i}\left[(k_1 r + 1)e^{-k_1 r} - (k_2 r + 1)e^{-k_2 r}\right] \tag{28}$$

where $r_{,i} = (x_i - y_i)/r$

From Equations (21) and (27), we obtain

$$\overline{\theta}'(r,s) = \frac{1}{4\pi\,r\,(k_1^2 - k_2^2)}\left[(k_1^2 - s^2)e^{-k_1 r} - (k_2^2 - s^2)e^{-k_2 r}\right] \tag{29}$$

The Laplace transform of the traction vector can be obtained from (15) as

$$\overline{p}_l'(r,s) = \left[(\beta^2 - 2)\overline{u}_{k,k}' - \beta^2(1+v\,s)\overline{\theta}'\right]n_l + (\overline{u}_{l,k}' + \overline{u}_{k,l}')n_k$$

It can easily be seen from (28) that

$$\overline{u}_{i,j}' = \frac{(1+v\,s)\,(1+\alpha\tau_0 s)}{4\pi\,r^3(k_1^2 - k_2^2)} \times \left\{r_{,i}r_{,j}f_1 + (2r_{,i}r_{,j} - \delta_{ij})f_2\right\} \tag{30}$$

where

$$f_1 = (k_1^2 r^2 + k_1 r + 1)e^{-k_1 r} - (k_2^2 r^2 + k_2 r + 1)e^{-k_2 r}$$

$$f_2 = (k_1 r + 1)e^{-k_1 r} - (k_2 r + 1)e^{-k_2 r}$$

Using the above expressions, the traction vector takes the form

$$\overline{p}_l'(r,s) = \frac{(1+\nu s)(1+\alpha\tau_0 s)}{4\pi r^3 (k_1^2 - k_2^2)} \{[(\beta^2 - 2)(f_1 - f_2) - \beta^2 r^2 f_3] n_l + 2[r_{,l} r_{,k}(f_1 + 2f_2) - \delta_{lk} f_2] n_k\} \quad (31)$$

where

$$f_3 = (k_1^2 - s^2)e^{-k_1 r} - (k_2^2 - s^2)e^{-k_2 r}$$

The expression for $\frac{\partial \overline{\theta}'}{\partial n}$ will be used later on in this entry. This is obtained from (29) as follows:

$$\frac{\partial \overline{\theta}'}{\partial n} = -\overline{\theta}'_{,i}\, n_i = \frac{r_{,i}\, n_i}{4\pi(k_1^2 - k_2^2)\, r^2} f_4 \quad (32)$$

where

$$f_4 = (k_1^2 - s^2)(k_1 r + 1)e^{-k_1 r} - (k_2^2 - s^2)(k_2 r + 1)e^{-k_2 r}$$

Case 2

We take

$$Q = 0, \qquad F_i = F_i^{(j)} = \delta_{ij}\,\delta(t)\ \delta(x - y)$$

The fundamental solutions in this case will be denoted by a superscript *(j)*.

The governing equations for this case take the form

$$\beta^2(\nabla^2 - s^2)\ \overline{\varphi}^{(j)} + \overline{X}^{(j)} = \beta^2(1 + \nu s)\overline{\theta}^{(j)} \quad (33)$$

$$(\nabla^2 - \beta^2 s^2)\,\overline{\psi_i}^{(j)} = -\overline{Y_i}^{(j)} \quad (34)$$

$$(\nabla^2 - s(1 + \tau_0 s))\,\overline{\theta}^{(j)} - \varepsilon s(1 + \alpha\tau_0 s)\,\nabla^2 \overline{\varphi}^{(j)} = 0 \quad (35)$$

The scalar and vector potentials in (17) have the forms

$$X = X^{(j)} = \frac{-1}{4\pi}\left(\frac{1}{r}\right)_{,j} \quad Y_i = Y_i^{(j)} = \frac{-1}{4\pi}\varepsilon_{ilj}\left(\frac{1}{r}\right)_{,l} \quad (36)$$

Eliminating $\overline{\theta}^{(j)}$ from (33) and (35) and substituting in the resulting equationsfor $X^{(j)}$ given in (36), we get

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)\overline{\varphi}^{(j)} = \frac{1}{4\pi\beta^2}[\nabla^2 - s(1 + \tau_0 s)]\left(\frac{1}{r}\right)_{,j} \quad (37)$$

To solve this equation, we make the change of variable $\overline{\varphi}^{(j)} = g_{,j}$ to arrive at

$$g = \frac{1}{4\pi\beta^2(k_1^2 - k_2^2)}[\nabla^2 - s(1 + \tau_0 s)](F_1 - F_2)$$

where F_i is the solution of the equation

$$(\nabla^2 - k_i^2)F_i = \frac{1}{r}, \ i = 1, 2$$

The solution of these equations bounded both at the origin and at infinity is givenby

$$F_i = \frac{1}{k_i^2 r}[e^{-k_i r} - 1], \ i = 1, 2$$

Collecting the previous results and using (26a, b), we arrive at

$$\overline{\varphi}^{(j)} = \frac{r_{,j}}{4\pi\beta^2 s^2 r^2} \times \left[1 + \frac{1}{k_1^2 - k_2^2}\{(1 + k_1 r)(k_2^2 - s^2)e^{-k_1 r} - (1 + k_2 r)(k_1^2 - s^2)e^{-k_2 r}\}\right] \quad (38)$$

Substituting from the second of (36) into (34) and using the substitution

$$\overline{\psi}_i^{(j)} = \epsilon_{ilj} G_{,l}$$

we get as before

$$\overline{\psi}_l^{(j)} = \frac{1}{4\pi\beta^2 s^2 r^2}\varepsilon_{lmj} r_{,m}\left[(1+\beta sr)e^{-\beta sr} - 1\right] \quad (39)$$

Substituting from (38) and (39) into (16) and using the relation

$$r_{,ij} = \frac{\delta_{ij} - r_{,i} r_{,j}}{r}$$

we get

$$\overline{u}_i^{(j)} = \frac{1}{r}\left[U_1 \delta_{ij} + U_2\, r_{,i} r_{,j}\right] \quad (40)$$

where

$$U_1 = \frac{1}{4\pi\beta^2 s^2 r^2}\left\{(1+\beta sr)e^{-\beta sr} + s^2 \sum_{n=1}^{2} (-1)^{n-1} A_n (1+k_n r) e^{-k_n r}\right\}$$

$$U_2 = \frac{-1}{4\pi\beta^2 s^2 r^2} \times \left\{(3+3\beta sr + \beta^2 s^2 r^2)e^{-\beta sr} + s^2 \sum_{n=1}^{2} (-1)^{n-1} A_n (3+3k_n r + k_n^2 r^2) e^{-k_n r}\right\}$$

where

$$A_n = \frac{k_n^2 - s(1+\tau_0 s)}{k_n^2 (k_2^2 - k_1^2)}, n = 1,\ 2$$

Finally, to obtain the temperature function in this case, it can easily be checked that

$$\overline{\theta}^{(j)} = \frac{-\varepsilon\, s}{\beta^2 (1+\nu\, s)} \overline{u}_j' \quad (41a)$$

This equation together with (28) give

$$\overline{\theta}^{(j)} = \frac{\varepsilon\, s\,(1+\alpha\tau_0 s)}{4\pi\,\beta^2 r^2 (k_1^2 - k_2^2)} r_{,j} f_2 \quad (41b)$$

Equation (41b) leads to

$$\frac{\partial\overline{\theta}^{(j)}}{\partial\, n} = \frac{\varepsilon\, s\ (1+\alpha\tau_0 s)\, n_i}{4\pi\,\beta^2 r^3\,(k_1^2 - k_2^2)}\left[\delta_{ij} f_2 - r_{,i} r_{,j} f_5\right] \quad (42)$$

where

$$f_5 = (k_1^2 r^2 + 3k_1 r + 3)e^{-k_1 r} - (k_2^2 r^2 + 3k_2 r + 3)e^{-k_2 r}$$

The Laplace transform of the traction vector for this case can also be obtained from (15) as

$$\overline{p}_l^{(j)}(r,s) = \left[(\beta^2 - 2)\,\overline{u}_{k,k}^{(j)} - \beta^2(1+\nu\, s)\overline{\theta}^{(j)}\right] n_l + 2\left[\overline{u}_{l,k}^{(j)} + \overline{u}_{k,l}^{(j)}\right] n_k$$

From (40), we obtain

$$\overline{u}_{k,l}^{(j)} = \frac{1}{r^2}\left[U_3 \delta_{kj} r_{,l} + U_2(\delta_{kl} r_{,j} + \delta_{jl} r_{,k}) + U_4 r_{,j} r_{,k} r_{,l}\right] \quad (43)$$

and

$$\overline{u}_{k,k}^{(j)} = \frac{1}{r^2}[U_3 + 4U_2 + U_4] r_{,j} \quad (44)$$

where

$U_3 = r\frac{\partial U_1}{\partial r} - U_1$ and $U_4 = r\frac{\partial U_2}{\partial r} - U_2$

Substituting from (43) and (44) into the expression for $\overline{p}_l^{(j)}(r,s)$, we get

$$\overline{p}_l^{(j)}(r,s) = \frac{1}{r^2}\left[(\beta^2 - 2)[U_3 + 4U_2 + U_4] + 2U_2 + (1+\nu\, s) f_2\right] r_{,j} n_l + (U_3 + U_2) r_{,l} n_j + \left[(U_3 + U_2) + 2U_4 r_{,j} r_{,l}\right] r_{,k} n_k \quad (45)$$

Reciprocity Theorem

This section is devoted to the derivation of a reciprocity theorem that shall be used later in this entry to obtain integral representations of the displacement and temperature distributions in terms of the boundary values of the problem.

Assume we have two systems of causes (heat sources and body forces) (Q, F_i) and (Q', F_i). Due to the application of these causes, we get the effects (displacements and temperature) (u_i, θ) and (u_i', θ'), respectively. Multiplying both sides of (15) by $\bar{e}'_{ij}$ and integrating over the volume V, we obtain

$$\int_V \bar{\sigma}_{ij}\bar{e}'_{ij}dV = \int_V \left[2\bar{e}_{ij}\bar{e}'_{ij} + (\beta^2 - 2)\bar{e}\,\bar{e}' - \beta^2(1 + v\,s)\,\bar{\theta}\,\bar{e}'\right] dV$$

Subtracting this equation from its counterpart obtained by interchanging the two systems, we get upon using (4) and integration by parts

$$\int_S \left[\bar{\sigma}_{ij}\,\bar{u}'_i - \bar{\sigma}'_{ij}\,\bar{u}_i\right] n_j dS - \int_V \left[\bar{\sigma}_{ij,j}\,\bar{u}'_i - \bar{\sigma}'_{ij,j}\,\bar{u}_i\right] dV = \beta^2(1 + v\,s)\int_V \left[\bar{\theta}'\,\bar{e} - \bar{\theta}\,\bar{e}'\right] dV$$

Using the transformed equations of motion in nondimensional form $\bar{\sigma}_{ij,j} = \beta^2 s^2 \bar{u}_i - \bar{F}_i$ and $\bar{\sigma}'_{ij,j} = \beta^2 s^2 \bar{u}'_i - \bar{F}'_i$we obtain

$$\int_S \left[\bar{\sigma}_{ij}\,\bar{u}'_i - \bar{\sigma}'_{ij}\,\bar{u}_i\right] n_j dS + \int_V \left[\bar{F}_i\,\bar{u}'_i - \bar{F}'_i\,\bar{u}_i\right] dV = \beta^2(1 + v\,s)\int_V \left[\bar{\theta}'\,\bar{e} - \bar{\theta}\,\bar{e}'\right] dV \tag{46}$$

Multiplying both sides of (14) by $\bar{\theta}'$ and following the same procedure as above, we get

$$\int_S \left[\bar{\theta}'\,\bar{\theta}_{,n} - \bar{\theta}\,\bar{\theta}'_{,n}\right] dS = (1 + \alpha\tau_0 s)\left\{\varepsilon\,s\int_V \left[\bar{\theta}'\,\bar{e} - \bar{\theta}\,\bar{e}'\right] dV - \int_V \left[\bar{\theta}'\,\bar{Q} - \bar{\theta}\,\bar{Q}'\right] dV\right\} \tag{47}$$

Eliminating $\int_V \left[\bar{\theta}'\,\bar{e} - \bar{\theta}\,\bar{e}'\right] dV$ between (46) and (47), we get

$$\begin{aligned}
&\varepsilon\,s\,(1 + \alpha\tau_0 s)\int_V \left[\bar{F}_i\,\bar{u}'_i - \bar{F}'_i\,\bar{u}_i\right] dV \\
&+ \beta^2(1 + v\,s)\int_V \left[\bar{Q}'\,\bar{\theta} - \bar{Q}\,\bar{\theta}'\right] dV \\
&= \beta^2(1 + v\,s)\int_S \left[\bar{\theta}'\,\bar{\theta}_{,n} - \bar{\theta}\,\bar{\theta}'_{,n}\right] dS \\
&- \varepsilon\,s\,(1 + \alpha\tau_0 s)\int_S \left[\bar{\sigma}_{ij}\,\bar{u}'_i - \bar{\sigma}'_{ij}\,\bar{u}_i\right] n_j dS
\end{aligned} \tag{48}$$

Boundary Integral Equations

In order to obtain integral representations for the transformed displacement and temperature distributions, we shall take the causes and effects (Q, F_i) and (u_i, θ) of the previous section to denote those of a given distribution under consideration.

The second system of causes and effects will be first taken as those obtained above in Case 1, i.e., we take

$Q' = \delta(x - y)\delta(t),\;\; F'_i = 0$ where $y \in V \cup S$

The corresponding transformed displacement $\bar{u}_i$ and temperature $\bar{\theta}'$ are given by

Equations (28) and (29). It follows from (48) that

$$\begin{aligned}
&(1 + v\,s)\Delta(x)\,\bar{\theta}(x,s) = (1 + v\,s) \\
&\int_S \left[\bar{\theta}'\,\bar{\theta}_{,n} - \bar{\theta}\,\bar{\theta}'_{,n}\right] dS + (1 + v\,s) \\
&\int_V \bar{\theta}'\bar{Q}\,dV - \frac{\varepsilon\,s\,(1 + \alpha\tau_0 s)}{\beta^2} \\
&\left\{\int_S \left[\bar{\sigma}_{ij}\,\bar{u}'_i - \bar{\sigma}'_{ij}\,\bar{u}_i\right] n_j dS + \int_V \bar{u}'_i\bar{F}_i dV\right\}
\end{aligned} \tag{49}$$

where $\Delta(x) = \begin{cases} 1 & if\ x \in V \\ 0 & if\ x \notin V \cup S \\ \frac{1}{2} & if\ x \in S \end{cases}$

Next, we take second system of causes and effects to denote those of Case 2, i.e., we take

$Q^{(j)} = 0,\;\; F_i^{(j)} = \delta_{ij}\delta(x - y)\delta(t)$ where $y \in V \cup S$***

The corresponding transformed displacement $\bar{u}_i^{(j)}$ and temperature $\bar{\theta}'^{(j)}$ are given by (41) and (40). It follows from (48) that

$$\varepsilon s(1+\alpha\tau_0 s)\Delta(x)\,\overline{u}_j(x,s)=\varepsilon s(1+\alpha\tau_0 s)$$
$$\left\{\int_V \overline{u}_i^{(j)}\overline{F}_i dV+\int_S\left[\overline{\sigma}_{ij}\,\overline{u}_i^{(j)}-\overline{\sigma}_{ij}^{(j)}\,\overline{u}_i\right]n_j dS\right\}$$
$$-\beta^2(1+v\,s)\left\{\int_V \overline{\theta}^{(j)}\overline{Q}\,dV+\int_S \overline{\theta}^{(j)}\overline{\theta}_{,n}-\overline{\theta}\,\overline{\theta}_{,n}^{(j)}dS\right\}$$

Substituting from (41a) into the above equation, we get

$$(1+\alpha\tau_0 s)\Delta(x)\,\overline{u}_j(x,s)=(1+\alpha\tau_0 s)\left\{\int_V \overline{u}_i^{(j)}\overline{F}_i dV+\int_S\left[\overline{\sigma}_{ij}\,\overline{u}_i^{(j)}-\overline{\sigma}_{ij}^{(j)}\,\overline{u}_i\right]n_j dS\right\}+\int_V \overline{u}_j'\,\overline{Q}\,dV+\int_S(\overline{u}_j'\overline{\theta}_{,n}-\overline{\theta}\,\overline{u}_{j,n}')\,dS \tag{50}$$

Using the convolution theorem of the Laplace transforms [15], namely,

$$L^{-1}\left[\overline{f}_1(s)\overline{f}_2(s)\right]=\int_0^t f_1(z)f_2(t-z)dz$$

together with (49) yields

$$\Delta(x)\left(1+v\frac{\partial}{\partial t}\right)\theta(x,t)=\int_0^t\int_S\left[\frac{\partial\theta(\eta,z)}{\partial n(\eta)}-\theta(\eta,z)\frac{\partial}{\partial n(\eta)}\right]\left(1-v\frac{\partial}{\partial z}\right)\theta'(x-\eta,t-z)dS_n dz$$
$$+\int_0^t\int_V Q(y,z)\left(1-v\frac{\partial}{\partial z}\right)\theta'(x-\eta,t-z)dV_y dz$$
$$-\frac{\varepsilon}{\beta^2}\int_0^t\int_V F_i(y,z)\left(\frac{\partial}{\partial z}-\alpha\tau_0\frac{\partial^2}{\partial z^2}\right)u_i'(x-y,t-z)dV_y dz$$
$$-\frac{\varepsilon}{\beta^2}\int_0^t\int_S\left[\sigma_{ij}(\eta,z)\left(\frac{\partial}{\partial z}-\alpha\tau_0\frac{\partial^2}{\partial z^2}\right)u_i'(x-\eta,t-z)-u_i(\eta,z)\left(\frac{\partial}{\partial z}-\alpha\tau_0\frac{\partial^2}{\partial z^2}\right)\sigma_{ij}'(x-\eta,t-z)\right]n_j dS_n dz \tag{51}$$

Similarly, inverting the Laplace transforms in (50) gives***

$$(1+\alpha\tau_0 s)\Delta(x)\,\overline{u}_j(x,s)=(1+\alpha\tau_0 s)\left\{\int_V \overline{u}_i^{(j)}\overline{F}_i dV+\int_S\left[\overline{\sigma}_{ij}\,\overline{u}_i^{(j)}-\overline{\sigma}_{ij}^{(j)}\,\overline{u}_i\right]n_j dS\right\}+\int_V \overline{u}_j'\,\overline{Q}\,dV+\int_S(\overline{u}_j'\overline{\theta}_{,n}-\overline{\theta}\,\overline{u}_{j,n}')\,dS$$

$$\Delta(x)\left(1+\alpha\tau_0\frac{\partial}{\partial t}\right)u_j(x,t)=\int_0^t\int_S\left[\frac{\partial\theta(\eta,z)}{\partial n(\eta)}-\theta(\eta,z)\frac{\partial}{\partial n(\eta)}\right]\overline{u}_j'(x-\eta,t-z)dS_n dz+\int_0^t\int_V Q(y,z)\,\overline{u}_j'(x-y,t-z)dV_y dz$$
$$+\int_0^t\int_V F_i(y,z)\left(1-\alpha\tau_0\frac{\partial}{\partial z}\right)u_i^{(j)}(x-y,t-z)\,dV_y dz$$
$$+\int_0^t\int_S\left[\sigma_{ij}(\eta,z)\left(1-\alpha\tau_0\frac{\partial}{\partial z}\right)u_i^{(j)}(x-\eta,t-z)\,n_j dS_n dz\right.$$
$$\left.-\int_0^t\int_S\left[-u_i(\eta,z)\left(1-\alpha\tau_0\frac{\partial}{\partial z}\right)\sigma_{ij}^{(i)}(x-\eta,t-z)\,n_j dS_n dz\right]\right. \tag{52}$$

Equations (51) and (52) can be written more concisely as

$$\Delta(x)\left(1+v\frac{\partial}{\partial t}\right)\theta(x,t)=W_1(x,t) \tag{53}$$

$$\Delta(x)\left(1+\alpha\tau_0\frac{\partial}{\partial t}\right)u_j(x,t)=W_2^{(j)}(x,t) \tag{54}$$

where $W_1(x, t)$ and $W_2^{(j)}(x,t)$ are the right-hand sides of (51) and (52), respectively.

The solutions of these equations have the forms:

$$\Delta(x)\theta(x,t)=\frac{1}{v}e^{-t/v}\int_0^t e^{z/v}W_1(x,z)dz \tag{55}$$

$$\Delta(x)u_j(x,t) = \frac{1}{\alpha\tau_0} e^{-t/\alpha\tau_0} \int_0^t e^{z/\alpha\tau_0} W_2^{(j)}(x,z)dz \tag{56}$$

Letting $x \to \xi \in S$ in (55) and (56), we obtain

$$\theta(\xi,t) = \frac{2}{v} e^{-t/v} \int_0^t e^{z/v} W_1(\xi,z)dz \tag{57}$$

$$u_j(\xi,t) = \frac{2}{\alpha\tau_0} e^{-t/\alpha\tau_0} \int_0^t e^{z/\alpha\tau_0} W_2^{(j)}(\xi,z)\,dz \tag{58}$$

Equations (57) and (58) together with the boundary conditions (8)–(11) and (31), (32), (42), and (45) can be used to set up the system of linear equations of the boundary integral equation method.

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Boundary Element Method in Heat Conduction

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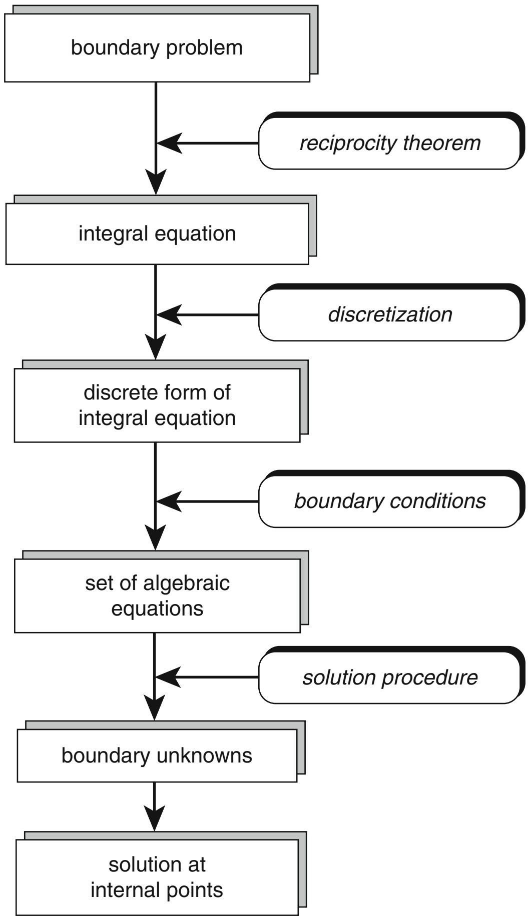
Synonyms

BEM; Boundary element method

Overview

The boundary element method (BEM) can be considered as an interesting alternative to other general numerical techniques like finite volume method, finite difference method, or finite element method. All these techniques are capable of solving boundary value problems, including heat conduction ones.

The boundary element method can be seen as a particular version of the method of weighted residuals [1]. BEM equations can be obtained by weighting the residual of the governing equation with a so-called *fundamental solution* and integrating twice by parts over the domain. As a result, an integral equation is obtained, which in many cases contains boundary integrals only. This can be interpreted as “reducing” dimensionality of the problem by one. The word reducing has been put into apostrophes since the physical field (e.g., temperature) still varies with the original independent variables but the object which needs to be discretized, i.e., contour line (in 2–D computations) or surface (in 3–D computations), has one dimensionless than the physical problem itself.



Boundary Element Method in Heat Conduction, Fig. 1 Successive stages of solving boundary problems by BEM

In the next sections, the fundamentals of the boundary element method are briefly described for both steady state as well as for transient heat conduction problems. Presented material generally consists of several major steps which are schematically summarized in Fig. 1. The first step is to convert the primary boundary problem into an integral equation. This equation is then discretized, and after substitution of boundary conditions, a system of algebraic equations is obtained. Solution of this system offers all unknown quantities along the boundary. Solution at internal points (if needed) is calculated from the boundary values. This procedure involves only simple matrix operations. More details can easily be found in main BEM textbooks, e.g., [2–10].

Formulations discussed within next two subsections refer generally to linear heat conduction. Nonlinear problems of all types (material, boundary, geometry, and/or body force nonlinearities) can also be tackled with BEM, although resulting equations require additional operations/transformations to possess the boundary-only character. General information on the application of BEM to nonlinear heat conduction can be found in many textbooks, e.g., [4–8].

Integral Equations of BEM for Stationary Heat Conduction

Steady-state temperature field $T(\mathbf{r})$ in a domain with heat sources $\dot{q}_v(\mathbf{r})$ is for linear heat conduction governed by Poisson equation

$$k\,\nabla^2 T(\mathbf{r}) + \dot{q}_v(\mathbf{r}) = 0 \tag{1}$$

where k stands for the heat conductivity.

According to weighted residuals method [1], the residual of this equation weighted with fundamental solution T^* and integrated over the domain produces zero:

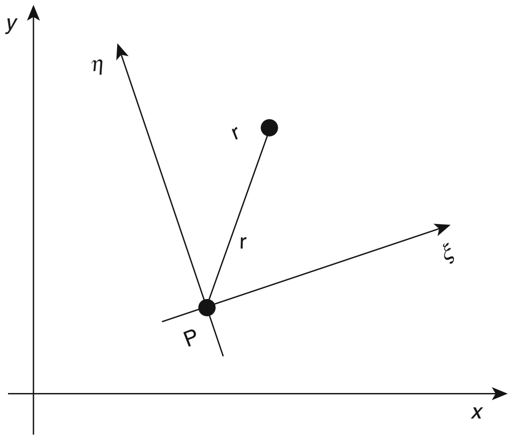
$$\int_V T^*\left(k\,\nabla^2 T + \dot{q}_v\right) \mathrm{d}V = 0 \tag{2}$$

Making use of the reciprocity theorem, which holds for any two temperature fields acting within any domain V with boundary S, i.e.,

$$\int_V \left(T\,\nabla^2 T^* - T^*\,\nabla^2 T\right) \mathrm{d}V = \int_S \left(T\,\frac{\partial T^*}{\partial n} - T^*\,\frac{\partial T}{\partial n}\right) \mathrm{d}S \tag{3}$$

Equation (2) is transformed into following form:

$$\int_V T\,\nabla^2 T^*\,\mathrm{d}V + \int_V T^*\,\frac{1}{k}\,\dot{q}_v\,\mathrm{d}V = \int_S \left(T\,\frac{\partial T^*}{\partial n} - T^*\,\frac{\partial T}{\partial n}\right) \mathrm{d}S \tag{4}$$



Boundary Element Method in Heat Conduction, Fig. 2 Global (x, y) and local (ξ, η) coordinate systems for calculating fundamental solution T^*

Since there is no limitation regarding temperature field T^*, the most convenient choice is to demand that it satisfies the following differential Equation [2, 11]:

$$\nabla^2 T^*(\mathbf{p}, \mathbf{r}) = \delta(\mathbf{p}, \mathbf{r}) \tag{5}$$

with the right-hand side being the Dirac's delta function δ depending on source point $\mathbf{p}$ and field point $\mathbf{r}$.

The function T^* varies with two parameters: coordinates of source point $\mathbf{p}$ and coordinates of field point $\mathbf{r}$, at which one observes the effects of the source. Positioning the local coordinate system at point $\mathbf{p}$ (cf Fig. 2) makes the problem symmetric. As a result, the fundamental solution T^* can be expressed in a local coordinate system only in terms of a distance r between points $\mathbf{p}$ and $\mathbf{r}$:

$$T^* = \begin{cases} \dfrac{1}{2\pi} \ln r & \text{2–D problems} \\ \dfrac{1}{4\pi} \dfrac{1}{r} & \text{3–D problems} \end{cases} \tag{6}$$

Differentiating temperature fields T and T^* along the outward normal, the normal components of the boundary heat fluxes q and q^* are obtained:

$$q = -k \frac{\partial T}{\partial n} \tag{7}$$

$$q^* = -k \frac{\partial T^*}{\partial n} \tag{8}$$

Substituting (5), (8), and (8) into (2), one arrives at the following expression:

$$\int_V \left[T^*(\mathbf{p}, \mathbf{r}) \frac{1}{k} \dot{q}_v(\mathbf{r}) + T(\mathbf{r}) \delta(\mathbf{p}, \mathbf{r}) \right] \mathrm{d}V(\mathbf{r}) = \int_S \left[T(\mathbf{r}) \frac{\partial T^*(\mathbf{p}, \mathbf{r})}{\partial n(\mathbf{r})} - T^*(\mathbf{p}, \mathbf{r}) \frac{\partial T(\mathbf{r})}{\partial n(\mathbf{r})} \right] \mathrm{d}S(\mathbf{r}) \tag{9}$$

The integration as well as the differentiation in (9) is performed with respect to the coordinates of field point $\mathbf{r}$ which is emphasized by the appropriate argument of those operations.

Taking into account the filtering property of Dirac's delta function and after performing simple algebra manipulation, the following equation is obtained:

$$\begin{aligned} k\, c(\mathbf{p})\, T(\mathbf{p}) &+ \int_S q^*(\mathbf{p}, \mathbf{r})\, T(\mathbf{r})\, \mathrm{d}S(\mathbf{r}) \\ &= \int_S T^*(\mathbf{p}, \mathbf{r})\, q(\mathbf{r})\, \mathrm{d}S(\mathbf{r}) \\ &\quad - \int_V T^*(\mathbf{p}, \mathbf{r})\, \dot{q}_v(\mathbf{r})\, \mathrm{d}V(\mathbf{r}) \end{aligned} \tag{10}$$

where the free coefficient $c(\mathbf{p})$ is dependent on the location of source point $\mathbf{p}$. If this point lies outside the domain V, the coefficient $c(\mathbf{p}) = 0$. For all points located inside the domain V, the coefficient $c(\mathbf{p}) = 1$. If point $\mathbf{p}$ belongs to the boundary S, this coefficient is equal to the internal angle the boundary makes at point $\mathbf{p}$ [2–10]. Thus, for a smooth boundary, the value $c(\mathbf{p}) = 0.5$ is obtained.

For the sake of notation simplicity, the temperature at point $\mathbf{p}$ will be denoted as T_i whereas coefficient $c(\mathbf{p})$ will be replaced by c_i from now on. Simultaneously, the arguments of the integration and differentiation operations will be dropped. Thus, (10) takes the following form:

$$k\, c_i\, T_i + \int_S q^*\, T\, \mathrm{d}S = \int_S T^*\, q\, \mathrm{d}S - \int_V T^*\, \dot{q}_v\, \mathrm{d}V \tag{11}$$

It should be noticed that (11) is an integral equation with respect to temperature T and heat flux q. It is an integral representation of the solution of the primary boundary value problem (1).

If the domain under consideration does not contain any internal heat sources, the above equation is simplified further to the following form:

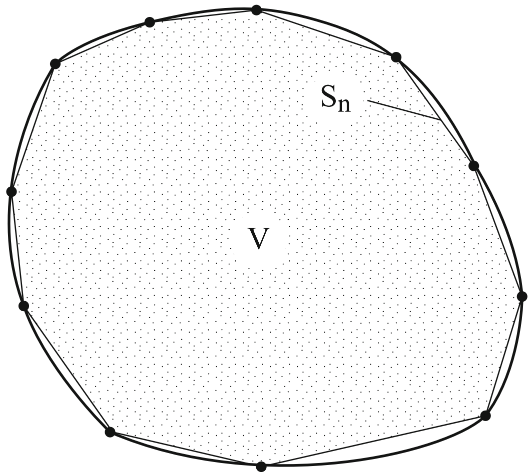
$$k\, c_i\, T_i + \int_S q^*\, T\, \mathrm{d}S = \int_S T^*\, q\, \mathrm{d}S \tag{12}$$

Discretization of the BEM Integral Equation

Analytical solutions of (11) and/or (12) are restricted to very simple geometries only. Therefore, for a general case, the integral equation is discretized and solved numerically.

Because (12) does not contain any domain integrals, only the boundary needs to be discretized. The first step is to divide it into N segments. These segments (appropriately small) are then replaced by so-called *boundary elements* S_n. In the simplest situation, boundary elements are straight line segments. This is schematically shown for a 2–D case in Fig. 3. However, higher order elements are also available like quadratic, cubic, and splines. Special elements like segments of circle, ellipse, and segments of sphere offer high accuracy while modeling objects built of those geometrical primitives.

Once the boundary elements are generated, changes of temperature and heat flux within each boundary element are related to their nodal values through the shape functions. Depending on the approximation made constant, linear, quadratic, etc., elements can be utilized. These three main possibilities are schematically shown in Fig. 4. In matrix notation, approximations of temperature and heat flux for each boundary element can be expressed by the following general system:



Boundary Element Method in Heat Conduction, Fig. 3 Geometrical model of the region using boundary elements

$$T = \Phi^T\, \mathbf{T}_n \tag{13}$$

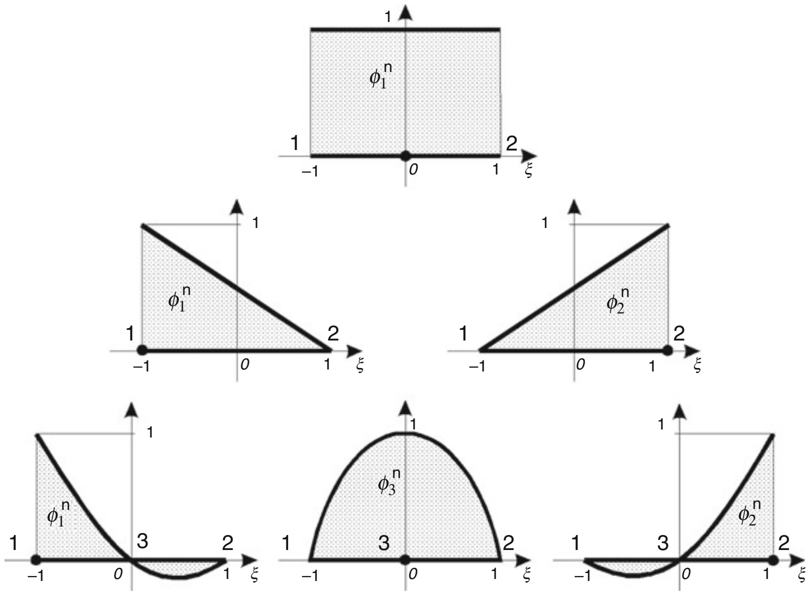
$$q = \Phi^T\, \mathbf{q}_n \tag{14}$$

with $\mathbf{T}_n$ and $\mathbf{q}_n$ being the vectors referring to nth boundary element and containing nodal values of temperature and heat flux, respectively. The row matrix Φ^T contains the local shape functions. The dimension of this matrix depends on the type of boundary element: equal to 1 for constant elements and equal to 2 for linear ones, while it takes the value 3 for quadratic elements.

It should be noted that a temperature field modeled using constant elements cannot be continuous. Only higher order elements are capable of representing continuous (but not smooth) variations of temperature. However, even these elements generally do not guarantee continuity in heat fluxes. It results from the fact that the derivative of temperature along an outward normal is not continuous at any corner. Thus, flux continuity can only be forced along smooth faces; otherwise, a jump of flux occurs (points A and B in Fig. 5). Because of this, also B-splines have been used to account for higher order interelement continuity.

Influence Matrices

Discretization of the boundary S into boundary elements allows one to replace the integrals



Boundary Element Method in Heat Conduction, Fig. 4 Constant, linear, and quadratic shape functions for a line element

in (12) by the summation of integrals, each one along a particular boundary element S_n:

$$k\,c_i\,T_i + \sum_{n=1}^{N} \int_{S_n} q^*\,T\,\mathrm{d}S_n = \sum_{n=1}^{N} \int_{S_n} T^*\,q\,\mathrm{d}S_n \tag{15}$$

Considering now approximations (13) and (14), one can express the (15) in the following form:

$$k\,c_i\,T_i + \sum_{n=1}^{N} \left\{ \int_{S_n} \Phi^T\ q^*\,\mathrm{d}S_n \right\} \mathbf{T}_n = \sum_{n=1}^{N} \left\{ \int_{S_n} \Phi^T\ T^*\,\mathrm{d}S_n \right\} \mathbf{q}_n \tag{16}$$

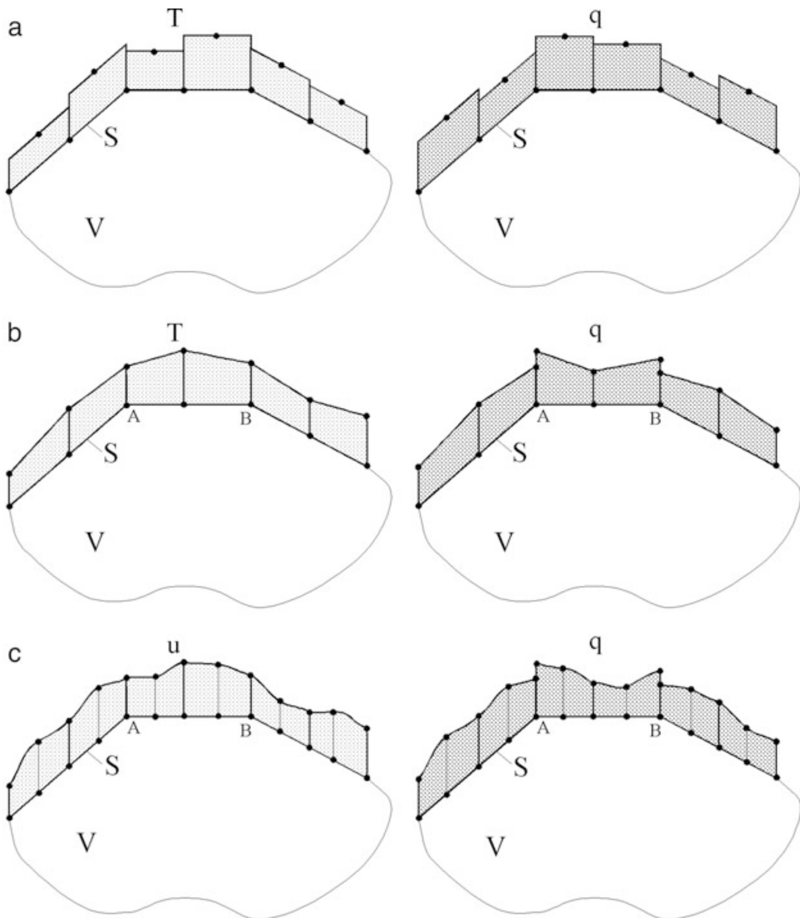
Integrals over boundary elements S_n are calculated numerically in the local coordinate system. This involves the Jacobian of the transformation between global and local coordinate systems. For each boundary element, row matrices $\mathbf{h}_{in}$ and $\mathbf{g}_{in}$ are produced with the number of elements matching the dimension of matrix Φ^T:

$$\mathbf{h}_{in} = \int_{S_n} \Phi^T\ q^*\,\mathrm{d}S_n \tag{17}$$

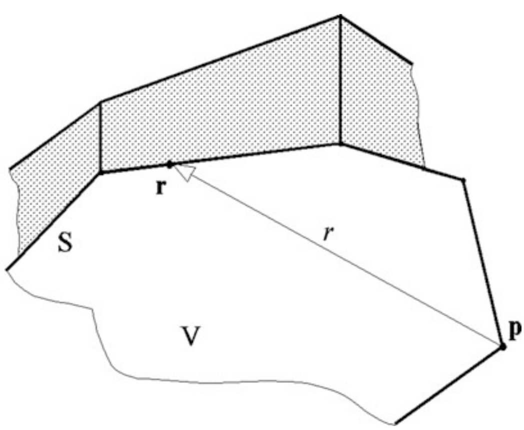
$$\mathbf{g}_{in} = \int_{S_n} \Phi^T\ T^*\,\mathrm{d}S_n \tag{18}$$

Because the fundamental solution depends on the distance r between the source point $\mathbf{p}$ (called also observation point) and the field point $\mathbf{r}$, some of the integrals become singular. It happens when the source point and field point both belong to the same boundary element, cf Figs. 6 and 7. While regular boundary integrals are computed accurately enough using standard Gauss quadratures [12], singular integrals require special treatment. Among others, transformations like Telles's transformation [13] are applicable. They generally cancel the singularity by making the Jacobian tend to zero at the same time that the distance r approaches zero, and simultaneously, they group the integration points in the vicinity of singularity. Thus, high accuracy of integration can still be achieved. Alternatively, singular integrals can be determined from the rigid body condition. For more details, the reader is referred to BEM monographs [2–10].

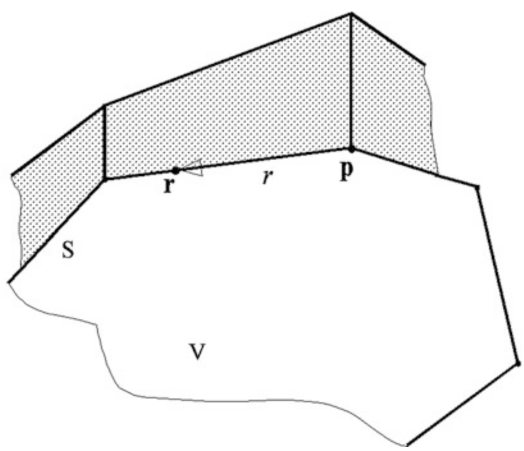
Introducing (17) and (18) into (16), the following relationship is obtained for each source point:



Boundary Element Method in Heat Conduction, Fig. 5 Changes of temperature and heat flux along the boundary for different types of boundary elements: (**a**) constant elements, (**b**) linear elements, (**c**) quadratic elements



Boundary Element Method in Heat Conduction, Fig. 6 Integration along regular element



Boundary Element Method in Heat Conduction, Fig. 7 Integration along singular element

$$k\, c_i\, T_i + \sum_{n=1}^{N} \mathbf{h}_{in}\, \mathbf{T}_n = \sum_{n=1}^{N} \mathbf{g}_{in}\, \mathbf{q}_n \qquad (19)$$

There are no limitations regarding the location of source point i. However, when generating the BEM equations, it is of common practice to locate it subsequently at all boundary nodes. This means that the number of generated equations of type (19) matches the number boundary nodes.

Since the sought temperature field is usually continuous, the next step is to assemble this system of equations with respect to nodal temperatures. This step is required only for elements other than constant and involves the collection of appropriate entries of matrix $\mathbf{h}_{in}$ to form so-called *temperature influence matrix* $\mathbf{H}$. The reason is that all extreme temperature nodes belong at the same time to all adjacent elements. Thus, all neighboring elements contribute during integration to these entries of $\mathbf{H}$ matrix which are associated with the common node. The assembling procedure also forms the global vector of nodal temperatures $\mathbf{T}$.

According to Fig. 5, heat fluxes are generally not continuous. So, they are not being assembled at this stage. They only form a global $\mathbf{Q}$ vector, but still, discontinuity at extreme points is preserved. As a consequence, so-called *flux influence matrix* $\mathbf{G}$ also remains not assembled. The number of its columns matches exactly the size of vector $\mathbf{Q}$ and is equal to $2N$ for linear elements and to $3N$ for quadratic ones.

Finally, the following discrete form of the integral equation is obtained:

$$\mathbf{H}\,\mathbf{T} = \mathbf{G}\,\mathbf{Q} \qquad (20)$$

This equation is a direct relationship between temperatures and heat fluxes along the boundary. Because of that, the boundary element method is very convenient when solving inverse thermal problems.

It is clear that in order to solve the system, additional information has to be provided from boundary conditions.

Handling Boundary Conditions

It is important to notice that boundary conditions are usually formulated for the particular faces. For constant elements, the nodes are located at the center of each element. As a consequence, prescribed values directly refer to nodal values.

If higher order elements are used, the situation is more complicated since they share the extreme nodes. If the boundary is smooth at a considered node, fluxes "before" and "after" the node are the same, unless they are prescribed as different. Thus, independently on boundary condition, only one unknown exists at such a point, either unique flux or temperature.

Corners need special treatment. This procedure is out of the scope of this book and will not be discussed here. The interested reader should refer to BEM textbooks. Nevertheless, system (20) can easily be rearranged to form a set of linear equations:

$$\mathbf{A}\,\mathbf{x} = \mathbf{f} \qquad (21)$$

Its right-hand side is obtained by multiplication of the appropriate columns of the influence matrices by values known from boundary conditions.

Components of the vector of unknowns $\mathbf{x}$ as well as components of the main matrix $\mathbf{A}$ depend on the boundary condition at the considered node as well as on the type of boundary element. For example, for constant elements, one obtains

$$A_{in} = \begin{cases} -G_{in} & \text{for Dirichlet's boundary condition} \\ H_{in} & \text{for Neumann's boundary condition} \\ H_{in} - h\,G_{in} & \text{for Robin's boundary condition} \end{cases} \qquad (22)$$

$$x_n = \begin{cases} q_n & \text{for Dirichlet's boundary condition} \\ T_n & \text{for Neumann's and Robin's boundary conditions} \end{cases} \qquad (23)$$

where h stands for the heat transfer coefficient. For linear and quadratic elements, the formulae have a similar structure, although they are more complicated.

It is worth stressing that the main matrix **A** is nonsymmetric and fully populated. Solution of the system (21) is obtained employing classical methods, e.g., Gauss elimination. Problems with a high number of degrees of freedom may require the use of a block solver which operates only on part of matrix **A**. The size of the block depends on the memory available.

Solution at Internal Points

Once the system (21) is solved and the distribution of temperature and heat flux along the boundary becomes available, one can solve for the temperature at any internal point. This procedure consists of considering a new source point i in (16), but no domain discretization is required. It involves one integration along the boundary, this time to form the integrals for the internal point. It should be noticed that all such integrals are regular ones, although some of them might be "nearly singular," i.e., when the internal point is near the boundary. In that case, it is recommended to treat them in a similar way as purely singular integrals.

By generating the *internal influence matrices* $\mathbf{H}_{int}$ and $\mathbf{G}_{int}$ and keeping in mind that for each internal point coefficient c_i is equal to 1, a vector of internal temperatures $\mathbf{T}_{int}$ can be obtained by simple matrix multiplications, i.e.,

$$\mathbf{T}_{int} = -\mathbf{H}_{int}\,\mathbf{T} + \mathbf{G}_{int}\,\mathbf{Q} \tag{24}$$

It is easy to recognize that matrix $\mathbf{H}_{int}$ is a rectangular one. The number of columns is equal to the number of boundary nodes, but the number of rows is equal to the number of considered internal points. Columns of matrix $\mathbf{G}_{int}$ remain unassembled like in the case of matrix **G** for boundary nodes.

Problems with Internal Heat Sources

For boundary problems with internal heat sources (either real or fictitious), the (11) instead of (12) must be considered, and because of the domain integral, its discretization is no longer restricted to the boundary S only. In early BEM works, integrals of this type have been calculated by domain discretization. If heat sources are a known function of space only, the domain integrals do not introduce any new unknowns. Subdivision of the domain into cells is, however, cumbersome and time-consuming, and particularly, in three dimensions, this is a difficult task even when automatic mesh generators are available. Moreover, the integration over the whole domain has to be performed as many times as the total number of nodes. This noticeably affects the efficiency of the method and causes the BEM to lose its main advantage which is the boundary – only formulation of the problem.

In many practical situations, the domain integral occurring in (11) can be transformed into its equivalent boundary form using the *dual reciprocity method* (DRM) [14] or *multiple reciprocity method* (MRM) [15]. Full details of these techniques can also be found in monographs [16] and [17].

BEM for Transient Heat Transfer Problems

BEM equations for transient heat conduction problems can generally be derived similarly to the procedure presented in previous section. The essential differences consist of one more integration required due to dependence of temperature field on time and application of different fundamental solutions. While the fundamental solution for steady-state problems is defined by (5), for transient heat conduction, it results from

$$\nabla^2 T^* + \frac{1}{\kappa}\frac{\partial T^*}{\partial t} = -\frac{1}{\kappa}\,\delta_i\,\delta(t-\tau) \tag{25}$$

where κ stand for thermal diffusivity. The time-dependent fundamental solution, being a temperature field at any point under consideration at time t due to spontaneous heat source acting at point i and at time τ, is of the form [11]

$$T^* = \frac{1}{[4\,\pi\,\kappa\,(t-\tau)]^{(d/2)}}\exp\left[-\frac{r^2}{4\,\kappa\,(t-\tau)}\right]H(t-\tau) \tag{26}$$

where r stands for the distance between source and field points, d is the number of spatial dimensions of the problem, and $H(t-\tau)$ denotes Heaviside function.

The time-dependent fundamental solution defined by (25) and (26) leads to the following integral equation describing the solution of transient heat conduction:

$$c_{i,t}\, T_{i,t} = \frac{\kappa}{k}\int_0^t \int_S q^*\, T\, \mathrm{d}S\, \mathrm{d}\tau - \frac{\kappa}{k}\int_0^t \int_S T^*\, q\, \mathrm{d}S\, \mathrm{d}\tau + \int_V T_0\, T^*|_{\tau=0}\, \mathrm{d}V \tag{27}$$

where T_0 is the initial condition, i.e., temperature field within the domain V at time $t = 0$.

It should be noted that in integral (27), both boundary as well as domain integrals occur. Though, discretization of this equation involves subdivision of the boundary S into boundary elements, subdivision of the volume V into cells, and finally subdivision of time variable into M time intervals. The latter discretization can employ constant, linear, etc. variation of temperature with time [2–10]. Discretization with respect to space results in the system of linear equations:

$$\sum_{k=1}^{M} \mathbf{H}_k^M\, \mathbf{T}_k = \sum_{k=1}^{M} \mathbf{G}_k^M\, \mathbf{Q}_k + \mathbf{B}_0\, \mathbf{T}_0 \tag{28}$$

For time constant elements, integration with respect to time can be performed analytically producing the following formulae:

$$\{H_k^M\}_{ij} = \frac{1}{2}\,\delta_{ij}\,\delta_{kM} - \frac{1}{2\pi}\int_{S_j} \frac{1}{r}\exp\left(-\frac{r^2}{4\,\kappa[t_M - t_{k-1}]}\right)\frac{\partial r}{\partial n}\,\mathrm{d}S + \frac{1}{2\pi}\int_{S_j} \frac{1}{r}\exp\left(-\frac{r^2}{4\,\kappa[t_M - t_k]}\right)\frac{\partial r}{\partial n}\,\mathrm{d}S \tag{29}$$

$$\{G_k^M\}_{ij} = -\frac{1}{4\pi k}\int_{S_j} \mathrm{Ei}\left(\frac{r^2}{4\,\kappa[t_M - t_{k-1}]}\right)\mathrm{d}S + \frac{1}{4\pi k}\int_{S_j} \mathrm{Ei}\left(\frac{r^2}{4\,\kappa[t_M - t_k]}\right)\mathrm{d}S \tag{30}$$

where Ei stands for exponential integral and $\delta_{i,j}$ is the *Kronecker* symbol.

Once the boundary conditions are applied, one can solve system (28) for unknown boundary quantities.

When the initial condition satisfies the Laplace equation, domain integral in (27) can be converted to the boundary by introducing a new variable which is a difference between temperature T and initial temperature T_0. Then, integrating with respect to time always from initial time equal to zero, domain integration can be omitted. However, because formulation (28) consists of summation relevant to the number of time steps, it becomes less and less efficient when the process proceeds in time, i.e., more terms in (28) are required to represent solution. Therefore, more recently, alternative approaches of solving transient processes have been proposed. The dual reciprocity method and the multiple reciprocity method both apply the time-independent fundamental solution and treat the temporal derivative of temperature as fictitious heat sources, i.e.,

$$k\, c_i\, T_i + \int_S q^*\, T\, \mathrm{d}S = \int_S T^*\, q\, \mathrm{d}S + \frac{k}{\kappa}\int_V T^*\, \dot{T}\, \mathrm{d}V \tag{31}$$

Domain integral occurring in the above equation can be transformed into equivalent boundary integrals either by DRM [16] or by MRM [17]. Then, the final integral equation is discretized and converted into the set of algebraic equations. Coefficients in this set are expressed in terms of boundary influence matrices **H** and **G**.

Formulation (31) offers a considerable reduction in computing effort. It is also interesting to point out that such an approach is equivalent to an assumption that the transient problem has already reached a regular regime in which temperature field is practically controlled by boundary conditions only. The influence of the initial condition has become so weak that it can simply be neglected. In practice, the combination of both formulations seems to be the most advisable.

Final Remarks

The previous sections contain only fundamental and rather classical BEM formulations for the linear heat conduction problems. It should be stressed however that two already mentioned techniques, i.e., dual reciprocity method and multiply reciprocity method can be applied not only to problems with internal heat sources but also to the heat conduction problems with all types of nonlinearities. Details can be found in monographs [16] and [17].

More recent ideas, including the fast multipole BEM, are also worth studying, e.g., [18]. Reader interested in open-source BEM software can also visit some WEB pages, e.g., [19] or [20].

Acknowledgement Permission from Computational Mechanics Publication to use figures included in this chapter is gratefully acknowledged.

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Boundary Element Method in Inverse Heat Conduction Problem

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Synonyms

BEM; Boundary element method

Overview

The essence of the boundary element method consists in transformation of a partial differential equation with given conditions into the form of an integral equation. This transformation may be achieved, for example, by using the weighted residual method followed by integration by parts method or by using the Green's second identity [1–3]. A result of the transformation mentioned above is an integral connected with differential operator which by using a fundamental solution is reduced to an unknown function. If the differential equation has non-zero right hand side then as a result of the transformation of the differential equation into the integral equation appears a domain integral which may be transformed into a boundary integral [4, 5]. In order to determine the value of the boundary integrals a boundary discretization and an interpolation of integrand are made which in the end leads to the problem of solving a system of linear algebraic equations.

The first advantage of the boundary element method is carrying out the boundary discretization which simplifies considerably a generation of the grid on the boundary and setting boundary conditions. The dimension of a matrix connected with a vector of unknowns is significantly smaller in the boundary element method when compared with the finite element method (in the first case this matrix is a dense one, and in the second – a sparse one).

The second advantage of the boundary element method is connected with the use of the fundamental solution which improves the accuracy of the solution and enables considering the boundary conditions in infinity. To the contrary, in the finite element method, considering the boundary conditions in infinity requires a huge number of elements.

The third advantage results from the appearance of the unknown function and its derivative in boundary integrals of the domain. Both the functions are interpolated with the same accuracy which allows to obtain better approximation of temperature and its gradient.

The fourth advantage of the boundary element method is the possibility of using discontinuous elements (taking into account discontinuity of heat flux while passing from one element to another).

The boundary element method is more difficult to implement than the finite element method.

Boundary Element Method for Helmholtz's Equation

Since many heat conduction problems may be reduced to the problem of solving the Helmholtz's equation, thus we will discuss the boundary element method on the example of this typical equation.

Let us consider a heat conduction equation in the form of

$$\rho c \frac{\partial T(x,t)}{\partial t} = \operatorname{div}(k \nabla T(x,t)) + p(x,t) \qquad x = (x_1, x_2, x_3), \quad t > 0, \quad x \in \Omega \tag{1}$$

Since the domain Ω is very often irregular in order to find a solution under given conditions we use the numerical methods to both the spatial variables and time variable. Applying backward difference quotient formula to approximate partial time derivative we have

$$\frac{\partial T(x,t)}{\partial t} \approx \frac{T(x,t) - T(x,t-\Delta t)}{\Delta t}$$

and the (1) will take the approximate form

$$\operatorname{div}(k \nabla T(x,t)) - \frac{\rho \cdot c}{\Delta t} \cdot T(x,t) = -\frac{\rho \cdot c}{\Delta t} \cdot T(x,t-\Delta t) - p(x,t) \tag{2}$$

Now, assuming that the parameter k, ρ, c are constant, the (2) can be written as follows

$$\Delta T(x,t) - \beta^2 T(x,t) = -\beta^2 Q(x,t) \tag{3}$$

$$\beta^2 = -\frac{\rho \cdot c}{k \cdot \Delta t}, \quad Q = T(x, t - \Delta t) + \frac{\Delta t}{\rho \cdot c} \cdot p(x, t)$$

Here, in the (3) time t plays the role of parameter as well. Let us notice that for $\beta = 0$ the (3) describes a stationary distribution of temperature modelled by the equation

$$\Delta T(x) = -\frac{1}{k} p(x), \quad x = (x_1, x_2, x_3) \in \Omega \quad (4)$$

We can transform the (3) to the boundary integral equation with the boundary conditions of the three possible forms, Fig. 1

1. Dirichlet's type boundary condition

$$T(x) = f(x), \; x \in \Gamma_A \quad (5)$$

2. Neumann's type boundary condition

$$q(x) = -k \frac{\partial T}{\partial n} = g(x), \; x \in \Gamma_B \quad (6)$$

3. Mixed type boundary condition

$$-k \frac{\partial T}{\partial n} = h(T(x) - T_{\text{fluid}}), \; x \in \Gamma_C, \; h > 0 \quad (7)$$

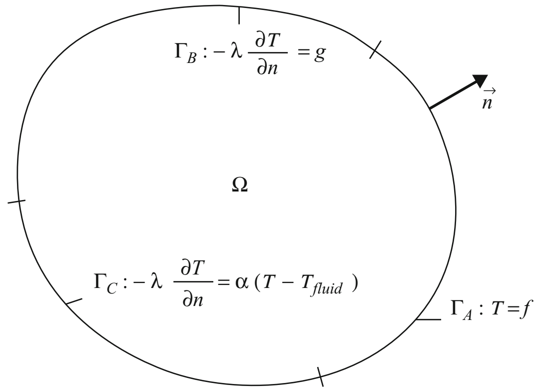
The first step to formulate the boundary element method is the transformation of the (3) to an integral equation. Multiplying the (3) by the test function T^* and integrating over the domain Ω we have

$$\int_\Omega \left(\Delta T - \beta^2 T + \beta^2 Q\right) T^* d\Omega = 0 \quad (8)$$

Since

$$\begin{aligned} \operatorname{div}(T^* \nabla T) &= \nabla T^* \nabla T + T^* \cdot \Delta T \\ \operatorname{div}(T \cdot \nabla T^*) &= \nabla T \cdot \nabla T^* + T \cdot \Delta T^* \end{aligned} \quad (9)$$

thus for the first term in (8) on the basis of (9) and Gauss' theorem we have



Boundary Element Method in Inverse Heat Conduction Problem, Fig. 1 Calculation domain

$$\begin{aligned} \int_\Omega \Delta T \cdot T^* d\Omega &= \int_\Omega \operatorname{div}(T^* \cdot \nabla T) d\Omega \\ &- \int_\Omega \nabla T^* \nabla T d\Omega = \int_\Omega \operatorname{div}(T^* \cdot \nabla T) d\Omega \\ &- \int_\Omega \operatorname{div}(T \cdot \nabla T^*) d\Omega + \int_\Omega T \cdot T^* d\Omega \\ &= \int_\Gamma T^* \cdot \nabla T \cdot \vec{n} d\Omega - \int_\Gamma T \cdot \nabla T^* \cdot \vec{n} d\Omega \\ &+ \int_\Omega T \cdot \Delta T^* \cdot d\Omega \end{aligned} \quad (10)$$

Therefore the (8) may be transformed into the form

$$\begin{aligned} \int_\Omega T(\Delta T^* - \beta^2 T^*) d\Omega &= \int_\Gamma \left(T \frac{\partial T^*}{\partial n} - u^* \frac{\partial T}{\partial n} \right) d\Gamma \\ &- \beta^2 \int_\Omega Q \cdot T^* \cdot d\Omega \end{aligned} \quad (11)$$

The integral on the left hand side (11) is reduced to the function T if the test function T^* satisfies the fundamental equation

$$\Delta T^* - \beta^2 T^* = -\delta(x, \xi) \quad (12)$$

If it is so, we have

$$\int_{\Omega} T(x)\cdot\delta(x,\xi)d\Omega = T(x),\quad x,\xi\in\Omega \qquad (13)$$

The possible explicite forms of the function T^* and its derivative $q^* = T_i^*\cdot n_i$ are as follows [3, 5], with the distance $r = \sqrt{(\xi_i - x_i)(\xi_i - x_i)}$

– For Laplace's equation for the 2D case

$$T^*(x,\xi) = -\frac{1}{2}\cdot\ln r,\ q^*(x,\xi) = -\frac{(x_i-\xi_i)\cdot n_i}{2\pi r}$$

– For Laplace's equation for the 3D case

$$T^*(x,\xi) = \frac{1}{4\pi r},\ q^*(x,\xi) = -\frac{(x_i-\xi_i)\cdot n_i}{4\pi\cdot r^{3/2}}$$

– For Helmholtz's equation for the 2D case

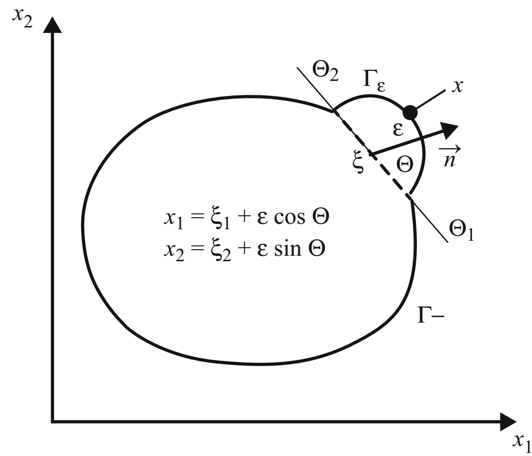
$$T^*(r,\beta) = \frac{1}{2\pi}\cdot K_0(\beta r),$$
$$q^*(r,\beta) = -\frac{\beta}{2\pi}\cdot K_1(\beta r)$$

– For Helmholtz's equation for the 3D case

$$T^*(r,\beta) = \frac{\exp(\beta r)}{4\pi r},$$
$$q^*(r,\beta) = -(1-\beta\cdot r)\cdot\frac{\exp(\beta r)}{4\pi r^2}\cdot\frac{\partial r}{\partial n}$$

Taking into account the property (13) in (11) we obtain

$$T(\xi) = \int_{\Gamma}(T^*\cdot q - T\cdot q^*)d\Gamma + \beta^2\int_{\Omega} Q\cdot T^* d\Omega,\ q = \frac{\partial T}{\partial n},\ \ \xi\in\Omega \qquad (14)$$



Boundary Element Method in Inverse Heat Conduction Problem, Fig. 2 Boundary Γ augmented by Γ_ε

If temperature T distribution and heat flux q on the boundary Γ are known then the temperature distribution within the domain Ω may be determined from the (14). To obtain an equation including only data on the boundary Γ in (14) one should approach with the coordinate ξ to the boundary Γ. The equation obtained in such a way is called a boundary integral equation (BIE). Because of the singularity of the fundamental solution T^* and the derivate q^*, if $\xi \to x \in \Gamma$ one should make a boundary transition. In order to do that we divide the integration boundary into two $\Gamma - \Gamma_\varepsilon$ and Γ_ε, Fig. 2, and split the integral disjoint over Γ into two boundary integrals

$$\int_{\Gamma} q^* T d\Gamma = \int_{\Gamma-\Gamma_\varepsilon} q^* T d\Gamma + \int_{\Gamma_\varepsilon} q^* T d\Gamma = I_1 + I_2 \qquad (15)$$

Introducing the polar coordinates in the domain bounded by the arc Γ_ε we have

$$q^* = -\frac{1}{2\pi r}\cdot\frac{\partial r}{\partial n} = -\frac{1}{2\pi\varepsilon},\quad x\in\Gamma_e;$$

therefore

$$I_2 = \int_{\Gamma_\varepsilon} q^* u \cdot d\Gamma = -\frac{u(\xi)}{2\pi} \int_{\Gamma_\varepsilon} \frac{1}{\varepsilon}\varepsilon \cdot d\Theta = -\frac{u(\xi)}{2\pi} \int_{\Theta_1}^{\Theta_2} d\Theta = -\frac{u(\xi)}{2\pi}(\Theta_2 - \Theta_1) \quad (16)$$

The integral $\int_\Gamma T^* \cdot q \cdot d\Gamma$ is a weakly singular one and is equal to zero over the arc Γ_ε for $\varepsilon \to 0$.

Thus, for $\xi \to x \in \Gamma$ the boundary integral equation has the form

$$c(\xi) \cdot T(\xi) = \int_\Gamma (T^* q - T \cdot q^*) d\Gamma + \beta^2 \int_\Omega Q \cdot T^* d\Omega, \quad \xi \in \Gamma \quad (17)$$

$$c = \begin{cases} 1 & x \in \Omega \\ 0 & x \notin \Omega \\ 1 + \lim\limits_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} q^* d\Gamma, & \xi \in \Gamma \end{cases} \quad (18)$$

and for the 2D case

$$\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} q^* d\Gamma = 1 - \frac{\Theta_2 - \Theta_1}{2\pi} \quad (19)$$

For the boundary $\Gamma \in C^1$, in the 2D case $\Theta_2 - \Theta_1 = \pi$, then c =1/2, and by analogy for the 3D case, c = ¼.

Discretization of Boundary

Approximation of the boundary Γ is done by dividing it to elements $\Gamma^{(1)}$, $\Gamma^{(2)}$,. . ., $\Gamma^{(NE)}$ with a disjoint interiors, *NE* is the number of boundary elements. Each of elements may possesses one or more nodes, Fig. 3.

The boundary Γ can be approximated by straight segments, Figs. 4 and 5 or curved ones for the 2D case and plain or curved surfaces for the 3D case. Figure 4 shows the subdivision of the boundary into the straight segments which imposes the interpolation of temperature T and heat flux q by the piecewise linear function. Namely

$$\left.\begin{array}{l} T(\eta) = T_i \cdot \varphi_1(\eta) + T_{i+1} \cdot \varphi_2(\eta) = \{\varphi\}^T \{T\} \\ q(\eta) = q_i \cdot \varphi_1(\eta) + q_{i+1} \cdot \varphi_2(\eta) = \{\varphi\}^T \{q\} \\ \varphi_1(\eta) = 1 - \eta, \quad \varphi_2(\eta) = \eta, \quad \eta \in \langle 0, 1 \rangle \end{array}\right\} \quad (20)$$

$$d\Gamma = \sqrt{(x_{1i} - x_{1,i+1})^2 + (x_{2i} - x_{2,i+1})^2} \cdot d\eta = \Delta s_i \cdot d\eta.$$

We can use the discontinuous approximation of temperature and heat flux to the boundaries with corners or in polygonal domains.

Introducing a discrete form of the solution (20) to the boundary (18) we obtain

$$c(\xi) \cdot T(\xi) + \sum_{i=1}^{NE} \Delta s_i \int_{\Gamma^{(i)}} q^*(\xi, \eta) \cdot \{\varphi(\xi)\}^T \cdot d\eta \cdot \{T\} = \sum_{i=1}^{NE} \Delta s_i \cdot \int_{\Gamma^{(i)}} T^*(\xi, \eta) \cdot \{\varphi(\xi)\}^T d\eta \cdot \{q\} + \beta^2 \int_\Omega T^*(\xi, x) \cdot Q(x) \cdot d\Omega \quad (21)$$

For subsequent collocation points ξ_k, $k = 1, \ldots,$ N equal to the number of nodes (piecewise continuous linear approximation) after discretization the solution (21) takes the matrix form

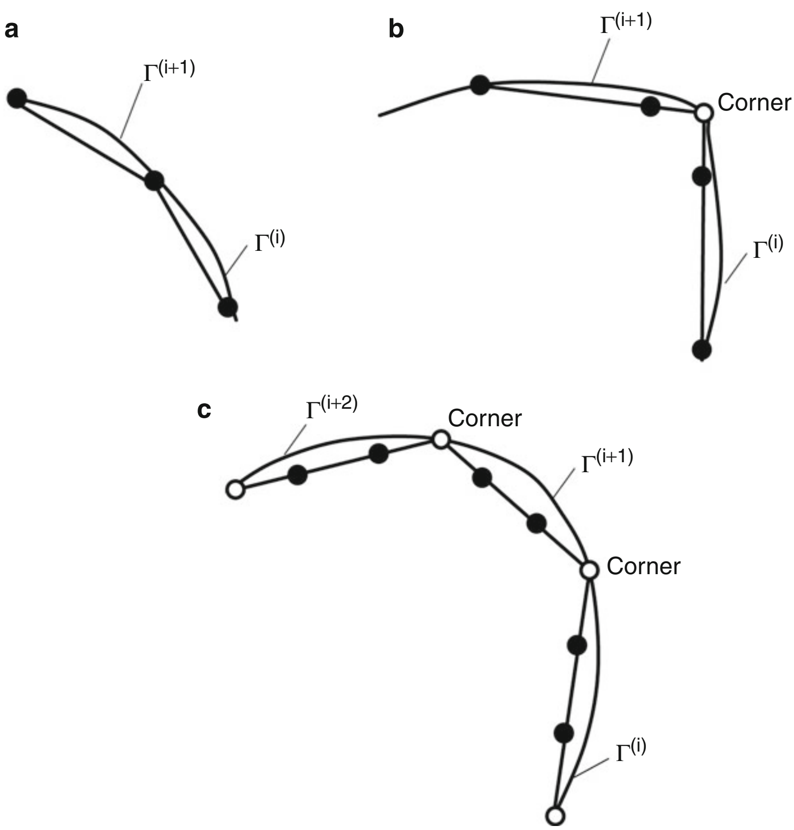
$$[H]\{T\} = [G]\{q\} + [P]\{Q\} \quad (22)$$

$$\dim[H] = \dim[G] = N \times N, \ \dim[P] = N \times NQ$$

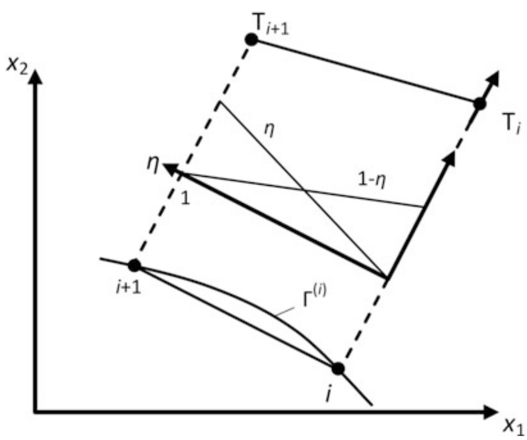
In case of discontinuous approximation (Fig. 6) the number of unknowns and collocation points is equal to 2 N.

Boundary Conditions

Separating from the dependence (22) matrices connected with the boundaries Γ_A, Γ_B and Γ_C, we may write this dependence in the matrix form

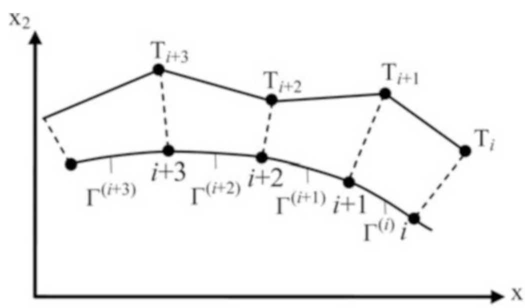


Boundary Element Method in Inverse Heat Conduction Problem, Fig. 3 Division of the boundary into elements: (**a**) collocation points coincide with nodes, (**b**) the collocation point is removed from the node (a single corner), (**c**) collocation points are inside the element



Boundary Element Method in Inverse Heat Conduction Problem, Fig. 4 Linear approximation of temperature and heat flux in element $\Gamma^{(i)}$

$$[H_A, H_B, H_C]\begin{Bmatrix} \{T_A\} \\ \{T_B\} \\ \{T_C\} \end{Bmatrix} = [G_A, G_B, G_C]\begin{Bmatrix} q_A \\ q_B \\ q_C \end{Bmatrix} + [P]\{Q\}$$

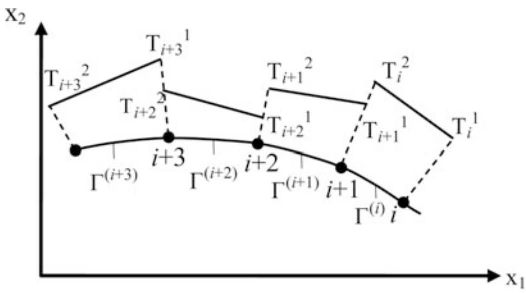


Boundary Element Method in Inverse Heat Conduction Problem, Fig. 5 Linear continuous approximation of temperature

or equivalently

$$[H_A]\{T_A\} + [H_B]\{T_B\} + [H_C]\{T_C\} = [G_A]\{q_A\} + [G_B]\{q_B\} + [G_C]\{q_C\} + [P\{Q\}] \tag{23}$$

Now taking into account the conditions (5)–(7) we obtain



Boundary Element Method in Inverse Heat Conduction Problem, Fig. 6 Segmental discontinuous approximation of temperature

$$\left[-G_A, H_B, H_C + \frac{h}{k} G_C\right] \begin{Bmatrix} \{q_A\} \\ \{T_B\} \\ \{T_C\} \end{Bmatrix} = \left[-H_A, -\frac{1}{\lambda} G_B, \frac{h}{k} G_C\right] \begin{Bmatrix} \{f\} \\ \{g\} \\ T_{\text{fluid}} \end{Bmatrix} + [P]\{Q\} \tag{24}$$

or, in a compact form

$$[HS] \begin{Bmatrix} \{q_A\} \\ \{T_B\} \\ \{T_C\} \end{Bmatrix} = [GS] \begin{Bmatrix} \{f\} \\ \{g\} \\ \{T_{\text{fluid}}\} \end{Bmatrix} + [P]\{Q\}$$

Premultiplying both sides by the matrix $[HS]^{-1}$

$$\begin{Bmatrix} \{q_A\} \\ \{T_B\} \\ \{T_C\} \end{Bmatrix} = [HG] \begin{Bmatrix} \{f\} \\ \{g\} \\ \{T_{\text{fluid}}\} \end{Bmatrix} + [HP]\{Q\} \tag{25}$$

For determined temperature distribution $\{T_C\}$ (25) we have $q_C = -\frac{\partial T}{\partial n} = h(T_C - T_{\text{fluid}})/k$, thus the vector of temperature $\{T\}$ and heat flux $\{q\}$ on the entire boundary Γ is known. Therefore the temperature distribution at a given point $\xi \in \Omega$ is represented by the dependence (17), which may be written in the following form $c = 1$ ($\Gamma \in C^1$)

$$T(\xi) = \{HU(\xi)\}^T\{q\} + \{GU(\xi)\}^T\{T\} + \{PU(\xi)\}^T\{Q\}, \ \xi \in \Omega \tag{26}$$

Perturbed temperature $T(\xi)$ for perturbed data $T + \delta T$, $q + \delta q$, $Q + \delta Q$ on the basis of the dependence (26) is represented as follows

$$T(\xi) + \delta T(\xi) = \{HU(\xi)\}^T\{q + \delta q\} + \{GU(\xi)\}^T\{T + \delta T\} + \{PU(\xi)\}^T\{Q + \delta Q\} \tag{27}$$

Subtracting the dependence (26) from the (27) one, we obtain a temperature error at the point $\xi \in \Omega$ as the function of a perturbed value (data error).

$$\delta T(\xi) = \{HU(\xi)\}^T\{\delta q\} + \{GU(\xi)\}^T\{\delta T\} + \{PU(\xi)\}^T\{\delta Q\} \tag{28}$$

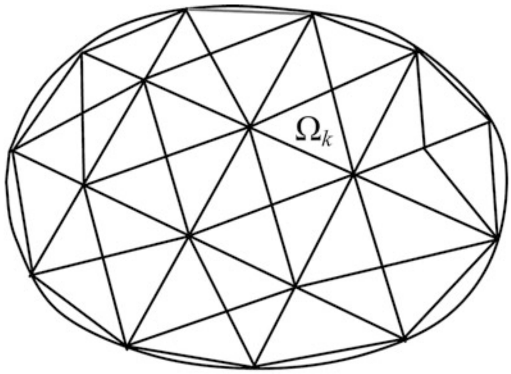
Having considered the boundary conditions represented by the vector (25) we may write the dependence (26) in the form

$$\begin{aligned} T(\xi) &= \left\{\{HU_A\}^T, \{HU_B\}^T, \{HU_C\}^T\right\} \begin{Bmatrix} \{q_A\} \\ \{q_B\} \\ \{q_C\} \end{Bmatrix} + \left\{\{GU_A\}^T, \{GU_B\}^T, \{GU_C\}^T\right\} \begin{Bmatrix} \{T_A\} \\ \{T_B\} \\ \{T_C\} \end{Bmatrix} \\ &= \{SH\}^T \begin{Bmatrix} \{q_A\} \\ \{q_B\} \\ \{q_C\} \end{Bmatrix} + \{SG\}^T \begin{Bmatrix} \{f\} \\ \{g\} \\ \{T_{\text{fluid}}\} \end{Bmatrix} \\ &= \{SH\}^T[HG] \begin{Bmatrix} \{f\} \\ \{g\} \\ \{T_{\text{fluid}}\} \end{Bmatrix} + \{SG\}^T \begin{Bmatrix} \{f\} \\ \{g\} \\ \{T_{\text{fluid}}\} \end{Bmatrix} \\ &\quad + \{SH\}^T[HP]\{Q\} = \{S_A\}^T\{f\} + \{S_B\}^T\{g\} \\ &\quad + \{S_C\}^T\{T_{\text{fluid}}\} + \{SP\}^T\{Q\} \end{aligned} \tag{29}$$

For each ξ value the vectors $\{HU\}$, $\{GU\}$ and $\{PU\}$ have to be determined from scratch.

Calculation of Domain Integral

An appearance of the integral $I_\Omega = \int_\Omega Q \cdot T^* d\Omega$ is a certain inconvenience in the boundary element method. A domain integral can be determined by



$$\int_{\Omega} T^* Q d\Omega = \sum_{k=1}^{Na} \int_{\Omega_k} T^* Q d\Omega \qquad \Omega = \bigcup_{k=1}^{Na} \Omega_k$$

Boundary Element Method in Inverse Heat Conduction Problem, Fig. 7 Division of the domain Ω into finite elements

using the finite element method or by reducing it to a sequence of boundary integrals (it is particularly useful if the domain Ω is unbounded) [5].

Figure 7 shows the discretization of the domain Ω with the use of the simplest possible finite elements [3]. For better approximation of the boundary (with the same number of boundary nodes) we can use curvilinear elements.

The second approach consists on transformation of the integral I_Ω into the boundary integrals [6].

To determine the integral I_Ω we will use the fundamental solution u^* of n-th order for which the following equality occurs

$$\begin{aligned} \Delta^n u^* = T^* \quad &\text{hence} \quad \Delta^{n+1} u^* = \Delta T^* = \delta \\ &\text{and} \quad \Delta^{n-j} u^* = \Delta^{-(j+1)} \delta, \quad n > 1 \end{aligned} \tag{30}$$

Therefore n + 1 times using the integration by parts we have

$$\begin{aligned} I_\Omega = \int_\Omega Q \cdot T^* d\Omega &= \int_\Omega \Delta^{n+1} Q \cdot u^* d\Omega \\ &+ \sum_{j=0}^{n} \left[\frac{\partial}{\partial n} \left(\Delta^j Q \right) \cdot \Delta^{n-j} u^* - \Delta^j Q \right. \\ &\left. \times \cdot \frac{\partial}{\partial n} \left(\Delta^{n-j} u^* \right) \right] \cdot d\Gamma \end{aligned}$$

where $\Delta^{n-j} u^* = \Delta^{-(j+1)} \delta$ and

$$\Delta^{-(j+1)} \delta = \begin{cases} \dfrac{1}{2} \cdot \dfrac{r^{2j+1}}{2(2j+1)!}, \; r = |x - \xi|, \; \text{1D case} \\ \dfrac{1}{2\pi} \cdot \dfrac{r^{2j}}{(2^j \cdot j!)^2} \cdot \left(\sum_{k=1}^{j} \dfrac{1}{k} - \ln r \right), \; \text{2D case} \\ \dfrac{1}{4\pi} \cdot \dfrac{r^{2j+1}}{(2(j+1))!}, \; \text{3D case} \end{cases} \tag{31}$$

Similarity, for the 2D case [4]

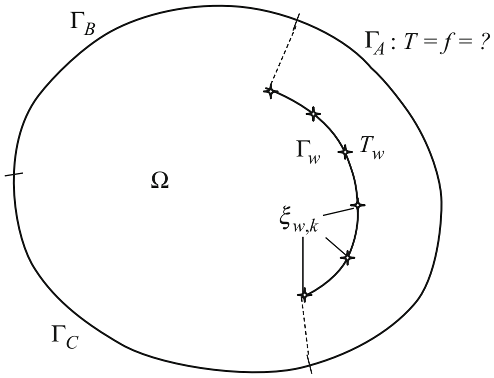
$$\left| \int_\Omega \Delta^{n+1} Q \cdot u^* \cdot d\Omega \right| \le \frac{\max\left(\left| \Delta^{n+1} Q \right| \right)}{\pi \cdot 2^{2(n+1)} (n+1) \cdot (n!)^2} \cdot \left(\frac{1}{2(n+1)} + \sum_{k=1}^{n} \frac{1}{k} \right) \tag{32}$$

Inverse Problem

The well posed problems for the heat conduction (1) require: knowledge of temperature coefficients λ, ρ, c and the source function, the determination of the domain Ω, the initial condition and the boundary conditions. If one of these data is missing then from the physical point of view it can be completed by the temperature measurement in the interior points of the domain Ω [7]. The problem stated in such a way and called an inverse problem is ill posed in Hadamard's sense. It means that a small perturbation of data may lead to huge perturbation in solving the (1). To minimize this negative phenomenon a regularization of the inverse problem is used.

The inverse problems mentioned above may be one of the following types: initial (unknown initial temperature distribution), boundary (unknown boundary condition on a part of the boundary), coefficient (unknown coefficient λ, ρ and c), geometric (the shape of a part of the domain's boundary is unknown).

One of the problems which most often appears in practice is an inverse boundary problem for which a distribution of temperature $T = f$ on the boundary Γ_A, Fig. 8, is sought after. Let the number of measuring points on the interior boundary Γ_w be denoted by N_w while the number N_T of nodes on the boundary Γ_T be $N_T \le N_w$.



Boundary Element Method in Inverse Heat Conduction Problem, Fig. 8 Calculation domain with interior temperature measurement points

In order to find the unknown distribution of the function *f* in the nodes of the boundary Γ_A we use the dependence (29), that is

$$\begin{aligned} T(\xi_{w,k}) &= \{S_A(\xi_{w,k})\}^T\{f\} + \{S_B(\xi_{w,k})\}^T\{g\} \\ &\quad + \{S_C(\xi_{w,k})\}^T\{T_{\text{fluid}}\} + \{SP(\xi_{w,k})\}^T\{Q\} \\ &= \{S_A(\xi_{w,k})\}^T\{f\} + R_k,\ k = 1,2,\ldots,N_w \end{aligned} \quad (33)$$

We obtain the discrete distribution of the function *f* from the minimization of the distance in the least squares sense between the measured temperature $T_{w,k}$, $k = 1,2,\ldots, N_w$ and the one determined from the solution (33). Namely the error-functional, is represented as follows:

$$\begin{aligned} I(\{f\}) &= \sum_{k=1}^{N_w}\left(T(\xi_{w,k}) - T_{w,k}\right)^2 \\ &= \sum_{k=1}^{N_w}\left(\{S_A(\xi_{w,k})\}^T\{f\} + R_k - T_{w,k}\right)^2 \\ &= \|[AA]\{f\} - \{TR\}\|^2,\ \{AA_i\}^T = \{S_A(\xi_{w,i})\}^T \\ TR_i &= T_{w,i} - R_i \end{aligned} \quad (34)$$

The minimization of the functional (34) is equivalent to solving the system of linear equation

$$[AA]\{f\} = \{TR\} \text{ hence } \{f\} = [AA]^+\{TR\} \quad (35)$$

where $[AA]^+$ is a pseudoinverse matrix.

The influence of the data error δTR on the solution *f* results from the dependence (35), that is

$$\{\delta f\} = [AA]^+\{\delta TR\} \quad (36)$$

If small changes in $\|\delta TR\|$ produce huge changes in $\|\delta f\|$ then the inverse problem requires regularization [7]. After making regularization in the Tichonov's sense the functional (34) takes the form

$$\begin{aligned} I_\alpha(\{f\}) &= \|[AA]\{f\} - \{TR\}\|^2 \\ &\quad + \alpha^2\|[B]\{f - f_0\}\|^2, \quad \alpha > 0 \end{aligned} \quad (37)$$

where *B* is called the regularization matrix (in the special case $B = I$) while $\{f_0\}$ is the approximation of $\{f\}$ (in many cases $\{f_0\} = \{0\}$).

The minimization of the functional (37) is equivalent to solving the overdetermined system of linear algebraic equations in the least squares sense

$$\begin{bmatrix} [AA] \\ \alpha[B] \end{bmatrix} = \begin{bmatrix} \{TR\} \\ \alpha[f_0] \end{bmatrix} \text{ or } [AA_\alpha]\{f\} = \{TR_\alpha\} \quad (38)$$

Thus

$$\{f\} = [AA_\alpha]^+\{TR_\alpha\} \quad (39)$$

A number of methods to determine the optimum value of the parameter α is shown in the paper [8]. A solution of a non-stationary inverse problem with the use of the fundamental solution dependent on space and time variables is presented in the paper [9].

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Boundary Integral Equation Method

▶ Application of Boundary Integral Equation (BIE) Method in Thermoelastodynamic Problem
▶ Boundary Element Method in Generalized Thermoelasticity

Boundary Integral Equations for Notch Problems

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Overview

The magnification of stresses at geometric discontinuities is of great importance in engineering design. In particular, local stresses may be highly enhanced in notched materials arising from abrupt changes of shape. This would result in a substantial decrease of the load-bearing capacity of structural members. Various methods of calculating stress concentration factors have been developed for two-dimensional elasticity problems. Using a series expansion method, Ling [1] solved elastic problems with different types of notches. Bowie and Freese [2] analyzed the notch problem by using complex variable theory in conjunction with the conformal mapping method. Nisitani [3] used the method of body force (or Green's function) to solve the notch problem in a semi-infinite plate or in a strip.

An alternative method for solving notch problems may be formulated in terms of a system of boundary integral equations. This method has clear advantages in solving the problem by applying a numerical treatment. In the derivation of boundary integral equations, the selection of the auxiliary function determines whether the kernels have weak or strong singularities. The kernel with Cauchy-type singularity has been widely used to solve many crack problems [4]. On the other hand, the integral equation with a logarithmic kernel has been proved to easily perform the numerical computation by Cheung and Chen [5]. This method with weak singularity has been used to solve some crack problems associated with an elastic half-plane medium [6], two bonded half-plane media [7], and a circular inclusion perfectly bonded in an infinite matrix [8]. Based on the earlier derivation, Chen and Cheung [9] reformulated a new boundary integral equation to deal with the notch problem in plane elasticity. We aim to further extend the aforementioned method to solve notch problems in plane thermoelasticity. In the derivation of singular integral equations, instead of using the components of heat flux and the components of stress, the resultant heat flow $\boldsymbol{Q}$ and the resultant force $-Y+iX$ are used to formulate the boundary conditions along the notch surface. This would result in singular integral equations with a logarithmic kernel instead of a Cauchy-type kernel. Three different types of notches in an infinite medium under a remote uniform heat flow are considered as our examples to illustrate the use of the approach. Some available exact solutions are provided to compare with the calculated numerical results to demonstrate the accuracy of the study.

Formulation of Integral Equation: Thermal Field

For the two-dimensional steady-state heat conduction problem, the temperature function, which satisfies the Laplace equation, can be expressed in terms of a single analytic function $\theta(z)$. With this function, both the temperature T and the resultant heat flow Q are written as

$$T = \mathrm{Re}[\theta(z)] \tag{1}$$

$$Q = \int \left(q_x dy - q_y dx\right) = -k\mathrm{Im}[\theta(z)] \tag{2}$$

where Re and Im denote the real and imaginary parts of the bracketed expression, respectively. The quantities q_x and q_y in (2) are the components of heat flux in the x and y directions, respectively, and k is the heat conductivity. Consider a remote uniform heat flux approached from the negative x-axis obstructed by the presence of an insulated notch or hole in an infinite medium. The current problem can be treated as a sum of the corresponding infinite medium problem without notches and a corrective term. The solution associated with the former problem can be easily expressed as

$$Q_0(z) = q\mathrm{Im}[z] \tag{3}$$

with q being the strength of heat flux applied at infinity. On the other hand, a corrective solution associated with an infinite medium with a single notch can be obtained by assuming a continuous distribution of dislocations with the density $b_0(s)$ placed along a given contour L as

$$\theta(z) = -\frac{i}{2\pi}\int_L \log(z-t)b_0(s)ds \tag{4}$$

The resultant heat flow across the notch surface can be obtained by substituting (4) into (2) as

$$Q(z) = \frac{k}{4\pi}\int_L \left[\log(z-t) + \log(\bar{z}-\bar{t})\right] ds + c_0,\ z \in L \tag{5}$$

where a bar will be used to indicate a conjugate complex quantity and c_0 is a constant to be determined. Based on the superposition principle, the boundary integral equation for an infinite medium containing an insulated notch is then established as follows:

$$\frac{k}{2\pi}\int_L \log(|z-t|)b_0(s)ds + c_0 = -q\mathrm{Im}[z], z \in L \tag{6}$$

In addition, the single-valued condition of the temperature must be satisfied, i.e.,

$$\int_L b_0(s)ds = 0 \tag{7}$$

Equation 6 together with (7) constitutes a boundary integral equation for solving the unknown function $b_0(s)$. Once the function $b_0(s)$ is determined, the temperature function $\theta(z)$ in (4) will be obtained accordingly.

Formulation of Integral Equation

For a two-dimensional thermoelastic problem, the components of the displacement and the traction force can be expressed in terms of two analytic functions, $\phi(z)$ and $\psi(z)$, and a temperature function $\theta(z)$ as [10]

$$2\mu(u+iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\Psi(z)} + 2\mu\beta\int \theta(z)dz \tag{8}$$

$$-Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\Psi(z)} \tag{9}$$

where μ is the shear modulus, $\kappa = (3-v)/(1+v)$ and $\beta = \alpha$ for plane stress, and $\kappa = 3-4v$ and $\beta = (1+v)\alpha$ for plane strain, with v Poisson's ratio and α the thermal expansion coefficient. The stress functions associated with an infinite medium containing a single notch can be obtained by assuming a continuous distribution of edge dislocations with densities $b_1(s)$ and $b_2(s)$ placed along a given contour L as

$$\phi(z)\frac{i\mu}{\pi(1+\kappa)} \times \int_L [b_1(s) + ib_2(s)]\log(z-t)ds \tag{10}$$

$$\psi(z) = \frac{-i\mu}{\pi(1+\kappa)} \int_L [b_1(s) - ib_2(s)] \log(z-t)ds - \frac{i\mu}{\pi(1+\kappa)} \int_L \frac{[b_1(s) + ib_2(s)]\bar{t}}{z-t} ds \tag{11}$$

Substituting (10) and (11) into (9) yields

$$-Y + iX = \int_L K_1(t, \bar{t}, z, \bar{z})[b_1(s) + ib_2(s)]ds + \int_L K_2(t, \bar{t}, z, \bar{z})[b_1(s) - ib_2(s)]ds + c_1 + ic_2, \quad z \in L \tag{12}$$

where

$$K_1(t, \bar{t}, z, \bar{z}) = \frac{2i\mu}{\pi(1+\kappa)} \log|z-t| \tag{13}$$

$$K_2(t, \bar{t}, z, \bar{z}) = \frac{i\mu}{\pi(1+\kappa)} \frac{t-z}{\overline{z-\bar{t}}} \tag{14}$$

For the traction-free condition along the notch surface, we now have the boundary integral equation

$$\int_L K_1(t, \bar{t}, z, \bar{z})[b_1(s) + ib_2(s)]ds + \int_L K_2(t, \bar{t}, z, \bar{z})[b_1(s) - ib_2(s)] ds + c_1 + ic_2 = 0 \tag{15}$$

Furthermore, the single-valued condition of the displacement must be satisfied, i.e.,

$$\int_L [b_1(s) + ib_2(s)]ds - \int_L \beta\left[\int b_0(\xi)d\xi\right]ds = 0 \tag{16}$$

Equation 15 together with (16) constitutes a boundary integral equation for solving the unknown functions $b_1(s)$ and $b_2(s)$. Once these two functions are determined, the stress functions $\phi(z)$ and $\psi(z)$ in Eqs. (10) and (11), respectively, will be obtained accordingly.

Numerical Results and Discussion

The dislocation functions $b_0(s)$ in (6) and $b_1(s)$ and $b_2(s)$ in (15) together with the subsidiary conditions (7) and (16) will be solved numerically using the appropriate interpolation formulas. For performing the numerical calculation, the contour L is replaced by a polygon of N line segments. The interpolation formulas for line segments in local coordinates $s_j (1 \le j \le N)$ are taken as [9]

$$b_i(s_j) = b_{i,j} \frac{d_j - s_j}{2d_j} + b_{i,j+1} \frac{d_j - s_j}{2d_j} \times (i = 0, 1, 2) \tag{17}$$

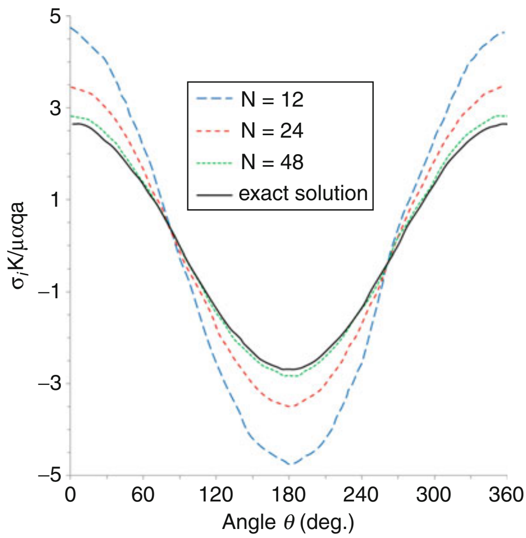
where $d_j (1 \le j \le N)$ are the half-length for each line segment and $b_{i,j} (1 \le j \le N)$ are the unknown coefficients to be determined. If the preceding formulas are used, the boundary integral equation (6) together with the subsidiary condition (7) can be carried out to yield $N+2$ algebraic equations for solving $N+2$ unknown coefficients $(b_{0,0}, b_{0,1}, b_{0,2}, \ldots, b_{0,N}, c_0)$. Similarly, the boundary integral equation [Eq. (15)] together with the subsidiary condition (16) can be arranged to yield $2N+4$ algebraic equations for solving $2N+4$ unknown constants $(b_{1,0}, b_{1,1}, \ldots, b_{1,N}, b_{2,0}, b_{2,1}, \ldots, b_{1,0}, c_1, c_2)$. Once the stress functions are determined, the tangential stresses or hoop stresses along the notch surface may be evaluated by

$$\sigma_i = 4\mathrm{Re}\{\phi'(z)\}, \quad z \in L \tag{18}$$

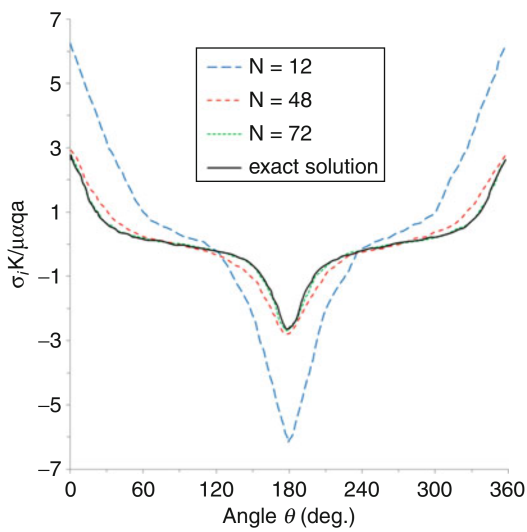
Circular Hole

As our first example, an insulated circular hole in an infinite medium under a remote uniform heat flow is considered. For performing the numerical technique, the contour of the circular hole is replaced by a polygon of N line elements discreted with a number of N points expressed by

$$x_i = a\cos\left[\frac{2(i-1)\pi}{N}\right], \quad y_i = b\sin\left[\frac{2(i-1)\pi}{N}\right] \quad (i = 1, 2, \ldots, N) \tag{19}$$

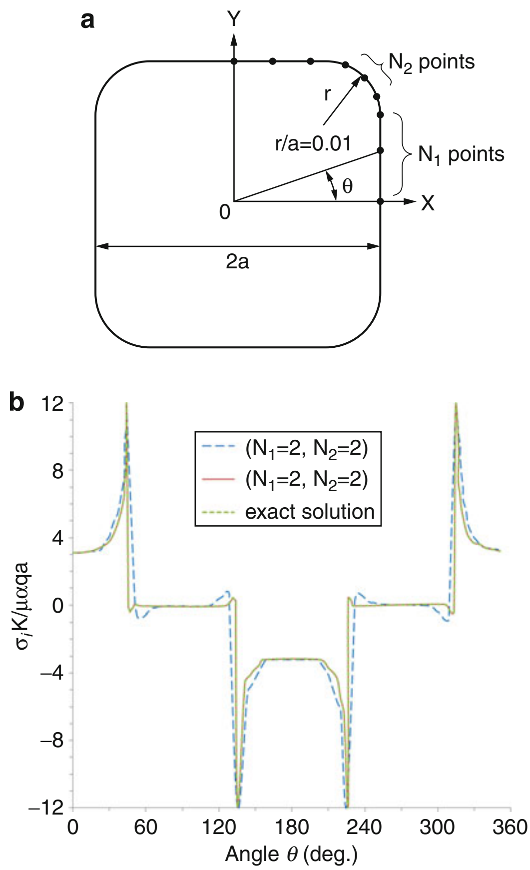


Boundary Integral Equations for Notch Problems, Fig. 1 Comparisons between the calculated and exact values of hoop stresses of the circular hole



Boundary Integral Equations for Notch Problems, Fig. 2 Comparisons between the calculated and exact values of hoop stresses of the elliptic hole

The calculated hoop stresses and the corresponding exact results are displayed in Fig. 1. It can be seen that the calculated numerical results agree very well with the corresponding exact solutions with the number of line segments $N = 48$.



Boundary Integral Equations for Notch Problems, Fig. 3 (**a**) Insulated square hole embedded into an infinite matrix. (**b**) Comparisons between the calculated and exact values of hoop stresses of the square hole

Elliptic Hole

As our second example, we consider an insulated elliptic hole with the semiaxes a and $b = a/2$ under a remote uniform heat flow. Similar to the preceding approach, some discreted points along the elliptic hole are expressed by

$$x_i = a\cos\left[\frac{2(i-1)\pi}{N}\right], y_i = b\sin\left[\frac{2(i-1)\pi}{N}\right] \quad (i = 1, 2, \ldots, N) \tag{20}$$

The calculated hoop stresses and the corresponding exact results are displayed in

Fig. 2. It shows that the error between the numerical results and exact solutions is within 2 % with the number of line segments $N = 72$.

Square Hole

As our third example, the notch problem of square hole with round corners is considered and shown in Fig. 3a. In the following numerical analysis, the choice of N_1 points is selected along the straight portions, whereas the choice of N_2 is selected along the round corners. The exact results and numerical hoop stresses of two cases with $N_1 = N_2 = 2$ and $N_1 = N_2 = 4$ are displayed in Fig. 3b. Good accuracy is also observed for the square hole problem with $N_1 = N_2 = 4$.

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Boundary Integral Formulation of the Plane Problems of Thermoelastostatics

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Overview

The theory of thermoelasticity has many applications in various problems of engineering, environment, and biology, for example, problems of aircraft, chemical and mechanical engineering, and geothermal effects, and in the different situations where the heat effects cannot be neglected. This is true, in particular, in conductors of electricity, where the contribution of Joule heat cannot, in general, be disregarded.

A model that lends itself easily to treatment within the boundary integral methods is that of static, uncoupled linear theory of thermoelasticity for isotropic media. This is due to the fact that the differential operators in the governing equations can be dealt with through the Laplacian operator, and the solutions can be expressed in terms of harmonic functions.

This model excludes any temperature dependence of the material constants of the medium. Moreover, the heat problem is totally independent of the mechanical parameters and may be solved separately, as a first step towards the complete solution of the thermoelastic problem. In what follows, we shall present a boundary integral formulation of this model.

An extensive literature exists on the subject of uncoupled, linear thermoelasticity and on the various analytical and numerical techniques adopted for the solution of different problems [1]. Classical analytical methods of solution have been widely used to tackle problems with relatively simple geometries to get exact or approximate analytical solutions. Alternatively, the numerical methods based on finite difference or on finite element techniques have become more popular in the recent years.

Although successful in dealing with problems with complex geometry, the numerical methods necessitate laborious calculations and high-capacity computing machines in order to discretize the field equations in the bulk and the boundary conditions on the boundary and to deal with the problems of convergence, stability, and computing time saving. In comparison, the boundary methods have some advantages: First, they rely on well-established theoretical results of existence and uniqueness of solutions of integral equations. Second, the tackling of the problem is carried out in two consecutive stages, during which the unknown functions are first computed on the boundary of the domain of solution, then in the bulk by quadrature, which means saving of computing time. Third, the involved computational aspects are simpler than for numerical methods and usually produce good approximate solutions, with no drastic limitations on the shape or connectivity of the boundary. A thorough exposition of boundary integral methods and their applications in potential theory and in elastostatics may be found in [2].

For the case under consideration, the Theory of Potential provides us with the necessary tool, in the form of a boundary integral representation of harmonic functions. This is recalled in Appendix A in different forms convenient for our purposes. Applications of the Theory of Potential to problems of elasticity may be found in [3]. For conciseness, we shall restrict further considerations to plane problems of thermoelasticity. The notations and analysis follow those of [4] and [5]. The basic equations and boundary conditions are quoted without proof. Details may be found in relevant textbooks.

Problem Formulation

Consider an infinite cylinder of a thermoelastic, isotropic medium acted upon by thermal sources in the bulk and surface forces on the lateral boundary. The setting is such that there is no dependence of the solution on the z-coordinate measured along the axis of the cylinder, with respect to a system of orthogonal Cartesian coordinates (x, y, z) with center O in the body. The unit vectors along the axes are denoted $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

It is required to find the distribution of temperature in space and the deformations and stresses occurring in the body.

Let D be the normal cross section of the cylinder containing the origin O. Region D is assumed simply connected and bounded by a sufficiently smooth contour C with parametric representation

$$x = x(s), \quad y = y(s), \quad 0 \le s \le s_1 \tag{1}$$

at each point of which the unit outwards normal $\mathbf{n}$ is uniquely defined. Here, s denotes the arc length as measured on C in the usual positive sense associated with D, from a fixed point Q_0 to a general boundary point Q. Let $\mathbf{t}$ be the unit vector tangent to C at Q in the sense of increase of s. One has

$$\mathbf{t} = (\dot{x}(s), \dot{y}(s)), \qquad \mathbf{n} = (\dot{y}(s), -\dot{x}(s)) \tag{2}$$

and the “dot” means derivative with respect to s.

Thermal Problem

The general equations of linear thermoelasticity are well established and may be found in [1]. In what follows, we shall quote these equations without proof, to be used throughout the text.

In the steady state, the temperature θ as measured from a reference temperature θ_0 satisfies Poisson’s equation

$$\Delta\theta = -\frac{q}{k} \tag{3}$$

where $q(x, y)$ is a given function representing the rate of heat supply per unit volume arising from heat sources present in the region occupied by the medium and k is the constant coefficient of heat conduction.

The temperature function θ solving equation (3) is represented as

$$\theta = \theta_H + \theta_p \tag{4}$$

where θ_H is a harmonic function and θ_p is any particular solution of (3). This latter function can be taken in the form of Newton's potential

$$\theta_p(x, y) = \frac{1}{2\pi} \int\int_D \frac{q(x', y')}{k} \ln \frac{1}{r}\, dx' dy' \tag{5}$$

where r is the distance between the two points with coordinates (x, y) and (x', y') in D. In the case of a constant heat supply q_0,

$$\theta_p = -\frac{q_0}{4k}\left(x^2 + y^2\right) \tag{6}$$

Three cases are considered for the thermal boundary conditions. For each of them, we shall show how to obtain the functions $\theta_H(x, y)$ and $\theta_H^c(x, y)$ in $\bar{D}$, as this information is needed for the solution of the thermoelastic problem.

Dirichlet Problem

The temperature on the boundary C of the domain D is a given continuous function $\theta^*(s)$. The Dirichlet problem can be formulated as follows: to find a solution to (3) which belongs to the class $C^2(D) \cap C(\bar{D})$ and which is equal to the prescribed function θ^* on the boundary:

$$\theta(s) = \theta^*(s), \quad 0 \le s \le s_1 \tag{7}$$

The harmonic function $\theta_H(x, y)$ is expressed on the boundary in terms of $\theta^*(s)$ as

$$\theta_H(s) = \theta^*(s) - \theta_p(s) \tag{8}$$

where $\theta_p(s)$ denotes the restriction of the function $\theta_p(x, y)$ to the boundary C. Substituting (8) into (54) for $f(s)$, the boundary function $\theta_H^c(s)$ is found to satisfy the following Fredholm integral equation of the first kind:

$$\oint_C \theta_H^c\, g_{t'}\, ds' = \pi[\theta^*(s) - \theta_p(s)] - \oint_C [\theta^*(s) - \theta_p(s)] g_{n'}\, ds' \tag{9}$$

From the solution of (9), using (8), (49) and (52), one then obtains the functions $\theta_H(x, y)$ and $\theta_H^c(x, y)$ in D. This determines completely the temperature function $\theta(x, y)$.

The Dirichlet problem for Laplace's equation has not more than one solution. For definiteness, a specific value can be assigned to the function θ_H^c at any chosen point in $\bar{D}$ without affecting the solution. In case this point is in D, the condition may be transformed into a boundary integral relation by means of (52).

Neumann Problem

The normal derivative of the temperature on the boundary, which represents the heat flux on the boundary C of the domain D, is a prescribed continuous function $\theta_n^*(s)$. The Neumann problem can be formulated as follows: to find a solution to (3) which belongs to the class $C^2(D) \cap C^1(\bar{D})$ and which has a normal derivative equal to the prescribed function θ_n^* on the boundary:

$$\frac{\partial \theta}{\partial n}(s) = \theta_n^*(s), \quad 0 \le s \le s_1 \tag{10}$$

Equation (4) yields

$$\frac{\partial}{\partial n}\theta_H(s) = \theta_n^*(s) - \frac{\partial}{\partial n}\theta_p(s) \tag{11}$$

Substituting this into (54) yields the following Fredholm integral equation of the second kind for the unknown function $\theta_H(s)$:

$$\theta_H(s) - \frac{1}{\pi}\oint_C \theta_H\, g_{n'}\, ds' = -\frac{1}{\pi}\oint_C [\theta_n^*(s') - \frac{\partial}{\partial n'}\theta_p(s')] g'\, ds' \tag{12}$$

The solution of this equation produces the boundary function $\theta_H(s)$. Now substitute this solution and (11) into (56) to get $\theta_H^c(s)$. As for Dirichlet problem, functions $\theta_H(x, y)$ and $\theta_H^c(x, y)$ are obtained in D from (49) and (52). This determines the temperature function $\theta(x, y)$ in $\bar{D}$.

The solution of Neumann problem for Laplace's equation is uniquely determined up to an additive arbitrary constant. Thus, one can specify the values of θ_H and θ_H^c, each at an arbitrarily chosen point in $\overline{D}$. If the point is in D, then the corresponding condition may be transformed into a boundary integral relation using (49) and (52).

Robin Problem

The normal derivative of the temperature is related to the temperature on the boundary C by a relation of the form

$$\frac{\partial\theta}{\partial n}(s) = -\frac{Bi}{k}[\theta(s) - \theta_e(s)] \qquad (13)$$

where Bi is Biot constant and $\theta_e(s)$ denotes the external (ambient) temperature on the boundary C. In other words, the heat flux is proportional to the difference of temperatures on both sides of the boundary C. The ambient temperature θ_e is assumed continuous on the boundary C. The Robin problem can be formulated as follows: to find a solution to equation (3) which belongs to the class $C^2(D) \cap C^1(\bar{D})$ and which has a normal derivative satisfying condition (13) on the boundary.

The radiation condition may be reformulated in terms of $\theta_H(s)$ with the help of (4) to read

$$\frac{\partial\theta_H}{\partial n}(s) = -\frac{Bi}{k}\left[\theta_H(s) + \theta_p(s) - \theta_e(s)\right] - \frac{\partial\theta_p}{\partial n}(s) \qquad (14)$$

Substituting this expression into (54) yields the following Fredholm integral equation of the second kind for the function $\theta_H(s)$:

$$\theta_H - \frac{1}{\pi}\oint_C \theta_H\left(g_{n'} + \frac{Bi}{k}g'\right)ds' = \theta^{**}(s) \qquad (15)$$

where

$$\theta^{**}(s) = \frac{1}{\pi}\oint_C\left[\frac{Bi}{k}\left(\theta_p - \theta_e\right) + \frac{\partial\theta_p}{\partial n}\right]g'\,ds'$$

Substituting (14) and the solution of (15) into (56) yields $\theta_H^c(s)$ at once. Finally, functions $\theta_H(x,y)$ and $\theta_H^c(x,y)$ are obtained in D from (49) and (52) as before. This determines the temperature function $\theta(x,y)$ in $\bar{D}$. As for Dirichlet problem, one can specify the value of $\theta_H^c(x,y)$ at a chosen point in $\bar{D}$.

Mechanical Problem

In the absence of body forces, the equations of equilibrium are automatically satisfied if the identically nonvanishing stress components are defined through the stress function Ψ by the relations

$$\sigma_{xx} = \frac{\partial^2\Psi}{\partial y^2},\ \sigma_{yy} = \frac{\partial^2\Psi}{\partial x^2},\ \sigma_{xy} = -\frac{\partial^2\Psi}{\partial x\partial y} \qquad (16)$$

The generalized Hooke's law reads

$$\sigma_{xx} = \frac{\nu E}{(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{E}{(1+\nu)}\frac{\partial u}{\partial x} - \frac{\alpha E}{(1-2\nu)}\theta \qquad (17)$$

$$\sigma_{yy} = \frac{\nu E}{(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{E}{(1+\nu)}\frac{\partial v}{\partial y} - \frac{\alpha E}{(1-2\nu)}\theta \qquad (18)$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (19)$$

where u and v are the Cartesian components of displacement and E, ν, and α are Young's modulus, Poisson's ratio, and the coefficient of linear thermal expansion, respectively, for the considered elastic medium.

Besides the local equilibrium conditions, the global equilibrium conditions of the body must also be satisfied. This requires that the resultant force and the resultant couple applied to the boundary of the body must vanish. This, evidently, will put some restrictions on the external agents, whether mechanical or thermal or of any other origin, affecting the body.

The compatibility condition for the solution of (17)–(18), viewed as first-order partial differential equations in the displacement components, leads to the following nonhomogeneous biharmonic equation for the stress function Ψ:

$$\Delta^2\Psi = \frac{\alpha E q}{k(1-v)} \tag{20}$$

The stress function Ψ solving (20) may be represented as

$$\Psi = x\phi + y\phi^c + \psi + \Psi_p \tag{21}$$

where ϕ and ψ are harmonic functions belonging to the class of functions $C^2(\bar{D})$ and Ψ_p is any particular solution of (20). Without loss of generality, this function may be taken as any particular solution of Poisson's equation

$$\Delta\Psi_p = -\frac{\alpha E}{1-v}\theta_p \tag{22}$$

since such a solution is also a particular solution of (20) and will differ from any other particular solution of this equation by a harmonic function that could be incorporated into the harmonic function ψ in (21).

The particular solution Ψ_p may be expressed in the form of Newton's potential

$$\Psi_p(x,y) = \frac{1}{2\pi}\frac{\alpha E}{1-v}\int\int_D \theta_p(x',y')\ln\frac{1}{r}dx'dy' \tag{23}$$

For constant heat supply,

$$\Psi_p = \frac{\alpha E q_0}{64\,k\,(1-v)}\left(x^2+y^2\right)^2 \tag{24}$$

The components of the mechanical displacement are easily obtained [4] as

$$\frac{E}{1+v}\,u = -\frac{\partial\Psi}{\partial x} + 4(1-v)\phi + \frac{E}{1+v}\,u_\theta \tag{25}$$

and

$$\frac{E}{1+v}\,v = -\frac{\partial\Psi}{\partial y} + 4(1-v)\phi^c + \frac{E}{1+v}\,v_\theta \tag{26}$$

with

$$u_\theta = \alpha(1+v)\int_{M_0}^{M}\left(\theta_H\,dx - \theta_H^c\,dy\right) \tag{27}$$

and

$$v_\theta = \alpha(1+v)\int_{M_0}^{M}\left(\theta_H^c\,dx + \theta_H\,dy\right) \tag{28}$$

The line integrations are taken along any path inside the region $\bar{D}$ joining an arbitrarily chosen, fixed point $M_0 \in \bar{D}$ to the general field point $M \in \bar{D}$ where the functions u_θ and v_θ are evaluated. The Cauchy-Riemann conditions for θ_H ensure that both line integrals are path independent. Consequently, if point M_0 lies on the boundary C, the integrals can also be taken on boundary segments. In any case, the given representation provides single-valued displacements.

The problem now reduces to the determination of four harmonic functions ϕ, ϕ^c, ψ, and ψ^c on the boundary C as a first step, then in the bulk through the use of the boundary representation of harmonic functions. Although function ψ^c does not appear in the expressions given above for the stress and the displacement components, it will be involved in the formulation of the boundary conditions as explained below.

The available relations to achieve our goal are thus the boundary representations of the four harmonic functions, together with two boundary conditions, applied either on stress or on displacement components. This requires the expression of the stresses and displacements in terms of the harmonic functions ϕ, ϕ^c and ψ. One has

$$\sigma_{xx} = x\frac{\partial^2\phi}{\partial y^2} + 2\frac{\partial\phi^c}{\partial y} + y\frac{\partial^2\phi^c}{\partial y^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\Psi_p}{\partial y^2} \tag{29}$$

$$\sigma_{yy} = x\frac{\partial^2\phi}{\partial x^2} + 2\frac{\partial\phi}{\partial x} + y\frac{\partial^2\phi^c}{\partial x^2} + \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\Psi_p}{\partial x^2} \tag{30}$$

$$\sigma_{xy} = -x\frac{\partial^2\phi}{\partial x\partial y} - y\frac{\partial^2\phi^c}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\Psi_p}{\partial x\partial y} \tag{31}$$

from which one obtains

$$\sigma_{xx} + \sigma_{yy} = 4\frac{\partial\phi}{\partial x} - \frac{\alpha E}{1-v}\theta_p = 4\frac{\partial\phi^c}{\partial y} - \frac{\alpha E}{1-v}\theta_p \tag{32}$$

Thus, the derivatives $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi^c}{\partial y}$ must be univalued functions in $\bar{D}$. As for the function ψ, it has univalued second derivatives as appears from relations (29)–(31).

The mechanical displacement components are rewritten as

$$\frac{E}{1+v}u = (3-4v)\phi - x\frac{\partial\phi}{\partial x} - y\frac{\partial\phi^c}{\partial x} - \frac{\partial\psi}{\partial x} - \frac{\partial\Psi_p}{\partial x} + \frac{E}{1+v}u_\theta \tag{33}$$

and

$$\frac{E}{1+v}v = (3-4v)\phi^c - x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi^c}{\partial y} - \frac{\partial\psi}{\partial y} - \frac{\partial\Psi_p}{\partial y} + \frac{E}{1+v}v_\theta \tag{34}$$

In fact, other conditions are still required to eliminate the possible rigid body motion and the arbitrariness of solutions inherent to the solution of plane problems of elasticity in stresses. Such additional conditions depend on the type of the mechanical boundary conditions and have been investigated in [4]–[6].

For the above formulae for the stresses and the displacements to be useful in completing the boundary formulation of the problem, the first and the second derivatives appearing in them must be expressed through derivatives taken along the boundary. The way to achieve this is explained in Appendix B.

In what follows, we consider two main boundary-value problems of elasticity.

First Fundamental Problem of Elasticity

The external force distribution **f** per unit length of the boundary is specified.

Let

$$\mathbf{f} = f_x\mathbf{i} + f_y\mathbf{j} = f_s\mathbf{t} + f_n\mathbf{n} \tag{35}$$

Then, at a general boundary point Q, the stress vector satisfies

$$\sigma_n = \mathbf{f}$$

or in components

$$\sigma_{xx}n_x + \sigma_{xy}n_y = f_x, \quad \sigma_{xy}n_x + \sigma_{yy}n_y = f_y$$

Substituting for σ_{xx}, σ_{xy}, and σ_{yy} in terms of the stress function Ψ and for n_x and n_y, the last two relations yield

$$f_x(s) = \left(\frac{\partial\Psi}{\partial y}\right)^\bullet(s), f_y(s) = -\left(\frac{\partial\Psi}{\partial x}\right)^\bullet(s) \tag{36}$$

The upper "dot" to the right of the bracket means derivative of the quantity between brackets along the direction of the tangent to the boundary. This derivative may be calculated with the help of (21) and the results of Appendix B. One can also find expressions for the normal and the tangential components of the applied force on the boundary.

It is important to notice that the solution of the first fundamental problem may include a rigid body motion, and one may want to eliminate this by imposing conditions to prohibit the motion of an arbitrarily chosen point in $\bar{D}$, as well as the rigid body rotation around this point. Such conditions may be transformed into boundary integral relations and added to the previous set of boundary relations. For example, the rigid body motion can be prevented if the following three conditions are enforced at the origin of coordinates $O \in D$:

$$u = v = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

Second Fundamental Problem of Elasticity

The mechanical displacement **d** on the boundary C is specified. In this case, relations (33) and (34) directly provide the required boundary relations. Written in terms of the tangential and the normal displacements d_s and d_n, these relations read

$$\begin{aligned}\frac{E}{1+v}d_s = &- x\dot{\phi} - y\dot{\phi}^c - \dot{\psi} \\ &+ (3-4v)(\dot{x}\phi + \dot{y}\,\phi^c) \\ &- \dot{\Psi}_p + \frac{E}{1+v}(u_\theta)_s\end{aligned} \quad (37)$$

and

$$\begin{aligned}\frac{E}{1+v}d_n = &y\dot{\phi} - x\dot{\phi}^c - \psi^c \\ &+ (3-4v)(\dot{y}\,\phi - \dot{x}\,\phi^c) - \frac{\partial \Psi_p}{\partial n} + \frac{E}{1+v}(u_\theta)_n\end{aligned} \quad (38)$$

where $(u_\theta)_s$ and $(u_\theta)_n$ may be calculated from the relation

$$(u_\theta)_s\,\mathbf{t} + (u_\theta)_n\,\mathbf{n} = u_\theta\,\mathbf{i} + v_\theta\,\mathbf{j} \quad (39)$$

In the second fundamental problem of elasticity, the rigid body motion is automatically prohibited.

Uniqueness of Solution of the Mechanical Problem

Even if the rigid body motion is eliminated from the elastic solution, some indeterminacy will still persist in the solution. This is due to the fact that one can still force the stress function Ψ and its first derivatives to have predetermined values at an arbitrary point in $\bar{D}$ without affecting the values of stresses and displacements in the body. Also, function ψ^c may be given a predetermined value at any boundary point. Thus, a total of four conditions can still be imposed on the solution. For points in D, these conditions can be transformed into boundary integral relations using (49) and (52), to be added to the set of integral relations.

Example

As an illustration of the proposed scheme and to put in evidence the necessity of recurring to numerical treatments, we present here below the solution of one of the few problems of thermoelasticity that can be handled analytically, namely, the problem of a circular elastic cylinder subjected to a uniform heating in the bulk and to a thermal radiation condition, together with a uniform external pressure on the lateral boundary. Let the normal cross section be a circle of radius a centered at the origin of coordinates, and let q_0 be the constant heat supply in the bulk and p_0 the uniform pressure on the boundary. The usual polar coordinates (r, φ) defined in the plane of the cross section will be used. The parametric equations of the contour are

$$x(\varphi) = a\cos\varphi,\ \ y(\varphi) = a\sin\varphi,\ \ 0 \le \varphi < 2\pi$$

and the arc length on this curve is given by $s = a\varphi$.

For simplicity, the ambient temperature is taken as

$$\theta_e(\varphi) = T_2\,\cos 2\varphi \quad (40)$$

The particular solution θ_p in (5) is

$$\theta_p = -\frac{q_0}{4k}r^2 \quad (41)$$

In view of the above choice for the ambient temperature θ_e, a solution for the harmonic part of the temperature on the boundary is assumed in the form

$$\theta_H(\theta) = K_1 + K_2\cos 2\varphi \quad (42)$$

To calculate the constants K_1 and K_2, it is imperative to evaluate the kernels appearing in the integral equation (15) for θ_H. The distance R between two points (r, φ) and (r', φ') on the boundary is

$$R = \left[r^2 + r'^2 - 2rr'\cos(\varphi' - \varphi)\right]^{1/2} \tag{43}$$

so that

$$\begin{aligned} g &= \ln R \\ &= \ln\left[r^2 + r'^2 - 2rr'\cos(\varphi' - \varphi)\right]^{1/2} \end{aligned} \tag{44}$$

The normal and the tangential derivatives of g on the boundary mean the derivatives with respect to r' and to φ', respectively, taken at the boundary point (r', φ'). Inside the integrals, one further substitutes $r = r' = a$. The following formulae are finally obtained:

$$g' = \ln\left(2\sin\frac{\varphi' - \varphi}{2}\right) \tag{45}$$

$$g_{n'} = \frac{1}{2a}, \quad g_{t'} = \frac{1}{2a}\frac{\sin(\varphi' - \varphi)}{1 - \cos(\varphi' - \varphi)} \tag{46}$$

This information is now substituted into the integral equation (15) for θ_H. Equating the singular parts on both sides of the equation and using the well-known integral [7]

$$\int_0^\pi \ln\sin(x)\cos[2m(x-p)]dx = -\frac{\pi\cos 2mp}{2m}$$

one finally obtains

$$K_1 = \frac{q_0 a^2}{4k}\left(1 + \frac{2}{B}\right), K_2 = \frac{B}{2+B}T_2,$$

with $B = \frac{aB_i}{k}$. Inside the domain,

$$\theta_H(r,\varphi) = K_1 + K_2\left(\frac{r}{a}\right)^2 \cos 2\varphi$$

$$\theta_H^c(r,\varphi) = K_2\left(\frac{r}{a}\right)^2 \sin 2\varphi$$

$$\theta(r,\varphi) = K_1 + K_2\left(\frac{r}{a}\right)^2 \cos 2\varphi - \frac{q_0}{4k}r^2$$

Next, one proceeds to the determination of the functions u_θ and v_θ in (27) and (28). The integration path may be taken along the radius starting at the origin O and ending at the general field point $M(r,\varphi)$. One obtains

$$\frac{u_\theta}{a(1+v)\alpha} = K_1\left(\frac{r}{a}\right)\cos\varphi + = \frac{1}{3}K_2\left(\frac{r}{a}\right)^3 \cos 3\varphi$$

$$\frac{v_\theta}{a(1+v)\alpha} = K_1\left(\frac{r}{a}\right)\sin\varphi + \frac{1}{3}K_2\left(\frac{r}{a}\right)^3 \sin 3\varphi$$

Turning now to the mechanical problem, the particular solution Ψ_p to (22) is

$$\Psi_p = \frac{\alpha E q_0}{64k(1-v)}r^4$$

The restrictions of the harmonic functions ϕ and ψ on the boundary are sought for in the form:

$$\phi(\varphi) = F_0 + F_1\cos\varphi$$

$$\phi^c(\varphi) = F_1\sin\varphi$$

$$\psi(\varphi) = E_0 + E_1\cos\varphi$$

$$\psi^c(\varphi) = E_1\sin\varphi$$

hence inside the domain,

$$\phi(r,\varphi) = F_0 + F_1\left(\frac{r}{a}\right)\cos\varphi = F_0 + F_1\frac{x}{a}$$

$$\phi^c(r,\varphi) = F_1\left(\frac{r}{a}\right)\sin\varphi = F_1\frac{y}{a}$$

$$\psi(r,\varphi) = E_0 + E_1\left(\frac{r}{a}\right)\cos\varphi = E_0 + E_1\frac{x}{a}$$

$$\psi^c(r,\varphi) = E_1\left(\frac{r}{a}\right)\sin\varphi = E_1\frac{y}{a}$$

and

$$\Psi(r,\varphi) = E_0 + (aF_0 + E_1)\tfrac{r}{a}\cos\varphi + aF_1\left(\tfrac{r}{a}\right)^2 + \tfrac{\alpha E q_0}{64k(1-v)}r^4$$

These relations clearly satisfy the boundary integral equations representing the harmonic functions ϕ and ψ. To satisfy the boundary conditions (36) for the first fundamental problem of elasticity, substitute the above solution into these two boundary conditions to get, after lengthy manipulations,

$$F_1 = -\frac{a}{2}\left(p_0 + \frac{\beta E}{4(1-v)}\right)$$

where

$$\beta = \frac{\alpha q_0 a^2}{4k}$$

For convenience of calculations, we have invariably used both types of coordinates (x, y) and (r, φ).

We are now in a position to write down the displacement components everywhere inside the circle and on the boundary. The remaining constants F_0, E_0, and E_1 may be determined from the conditions of elimination of the rigid body motion and the additional conditions stated above. It is easy to show that the relation

$$(3 - 4v)F_0 - \frac{E_1}{a} = 0$$

guarantees the vanishing of the displacement components u and v at the center of the circle, thus preventing any rigid body displacement, in which case one gets

$$\frac{1}{1+v}\frac{u}{a} = \left[2(1-2v)\frac{F_1}{aE} - \frac{\beta}{4(1-v)}\left(\frac{r}{a}\right)^2\right]\frac{x}{a} + \frac{1}{1+v}\frac{u_\theta}{a},$$
$$\frac{1}{1+v}\frac{v}{a} = \left[2(1-2v)\frac{F_1}{aE} - \frac{\beta}{4(1-v)}\left(\frac{r}{a}\right)^2\right]\frac{y}{a} + \frac{1}{1+v}\frac{v_\theta}{a}$$

One also verifies that there is no rigid body rotation around the center of the circle. Finally, we impose the conditions of vanishing of the stress function and its first derivatives at the point $(r = a, \varphi = 0)$ on the boundary:

$$\Psi(a,0) = \frac{\partial\Psi}{\partial r}(a,0) = \frac{\partial\Psi}{\partial\varphi}(a,0) = 0$$

This completes the determination of the constants:

$$E_0 = -\frac{a^2}{2}\left(p_0 - \frac{\beta E}{8(1-v)}\right),\ F_0 = \frac{ap_0}{4(1-v)},$$
$$E_1 = \frac{3-4v}{4(1-v)}a^2 p_0$$

Taking in consideration a sign error in the expression for Ψ_p in [5], the present results are identical to those given therein.

Conclusions

1. The plane problem of static, linear uncoupled thermoelasticity for isotropic media has been uniformly formulated in stresses within a boundary integral procedure in terms of three harmonic functions and their harmonic conjugates, subject to thermal and mechanical boundary conditions.
2. No limitations were imposed on the shape of the boundary or on the given boundary conditions, except smoothness. When the boundary is not smooth enough, for example, if it includes corner points, the contour must be rounded off properly at those points.
3. The present approach can be integrated within the extensive literature on harmonic function representations in different coordinate systems to provide practical series solutions of the boundary integral equations.
4. The arising boundary integral equations have kernels with removable singularities. A way to treat the singularities for numerical solution may be found in [4].
5. There is well-established Theory of Integral Equations to deal with the arising boundary integral equations, in what concerns existence, uniqueness, and stability of solutions.
6. When the shape of the contour is simple enough (e.g., circle, ellipse), one may try to find an analytical solution to the problem. From a computational point of view, when no analytic solutions are possible, the contours are partitioned in the usual way, and the boundary integral equations lead to rectangular systems of linear algebraic equations in the values of the harmonic functions at the nodal points.
7. For numerical calculations, it is important to carry out careful evaluation of the matrix entries generated by the removable singularities.
8. Numerous experiments have indicated that accurate calculation of the first and the

second boundary derivatives of functions and of the possibly existing Newton's potentials is necessary, in order for the method to perform efficiently.

9. The induced errors caused by the partitioning procedure can usually be kept low, and the obtained solutions are acceptable approximate solutions to difficult boundary-value problems of plane, linear thermoelasticity of isotropic media.
10. When the parameter in the equations of the contour is different from the arc length, the above formulation needs to be changed. This may be noticed from the outset as (2) will assume different forms.
11. More general boundary formulations can be used to treat other models of continuous media.

A. Boundary Integral Representation of Harmonic Functions

We list here below some useful variations of the boundary integral representation of a harmonic function. We denote by R the distance between a general point $P(x, y) \in D$ and the current integration point $Q(s')$ on the boundary C of D. For the sake of conciseness, we introduce the following notations for the function $\ln R$ and its derivatives along the normal and the tangent directions on C at the point $Q(s')$:

$$g' = \ln R, \quad g_{n'} = \frac{\partial}{\partial n'} \ln R, \quad g_{t'} = \frac{\partial}{\partial s'} \ln R.$$

It is important to note that the following relations remain invariant if the variable R in the logarithm is replaced by $\frac{R}{\ell}$, where ℓ is a characteristic length of the problem. This remark is based on an important property of harmonic functions, according to which the contour integral of the normal derivative of a harmonic function on the boundary vanishes.

Let $f \in C^2(\bar{D})$ be harmonic in D, and let f^c denote its harmonic conjugate. The following Cauchy-Riemann relations take place:

$$\frac{\partial f}{\partial n'} = \frac{\partial f^c}{\partial s'}, \quad \frac{\partial f}{\partial s'} = -\frac{\partial f^c}{\partial n'} \tag{47}$$

In these notations, the boundary integral representation of function f reads

$$f(x, y) = \frac{1}{2\pi} \oint_C \left(f g_{n'} - \frac{\partial f}{\partial n'} g' \right) ds' \tag{48}$$

or using integration by parts together with the Cauchy-Riemann conditions:

$$f(x, y) = \frac{1}{2\pi} \oint_C (f g_{n'} + f^c g_{t'}) ds' \tag{49}$$

The harmonic conjugate of (48) is

$$f^c(x, y) = \frac{1}{2\pi} \oint_C \left(f \frac{\partial \Theta}{\partial n'} - \frac{\partial f}{\partial n'} \Theta \right) ds' \tag{50}$$

where

$$\Theta = \tan^{-1} \frac{y - y(s')}{x - x(s')} \tag{51}$$

Replacing f by f^c in (48), one obtains

$$f^c(x, y) = \frac{1}{2\pi} \oint_C \left(f^c g_{n'} - \frac{\partial f^c}{\partial n'} g' \right) ds' \tag{52}$$

or in the equivalent form,

$$f^c(x, y) = \frac{1}{2\pi} \oint_C (f^c g_{n'} - f g_{t'}) ds' \tag{53}$$

When point (x, y) tends to a boundary point $Q(s)$, relations (48)–(50) and (52)–(53) yield, respectively,

$$f(s) = \frac{1}{\pi} \oint_C \left(f g_{n'} - \frac{\partial f}{\partial n'} g' \right) ds' \tag{54}$$

$$f(s) = \frac{1}{\pi} \oint_C (f g_{n'} + f^c g_{t'}) ds' \tag{55}$$

$$f^c(s) = \frac{1}{\pi} \oint_C \left(f \frac{\partial \Theta}{\partial n'} - \frac{\partial f}{\partial n'} \Theta \right) ds' \tag{56}$$

$$f^c(s) = \frac{1}{\pi} \oint_C \left(f^c g_{n'} - \frac{\partial f^c}{\partial n'} g' \right) ds' \quad (57)$$

and

$$f^c(s) = \frac{1}{\pi} \oint_C (f^c g_{n'} - f g_{t'}) ds' \quad (58)$$

B. Derivatives of Boundary Functions

Let $f \in C^2(\bar{D})$ be harmonic in D. It is required to find the first-order derivatives of f with respect to (x, y) in terms of the derivative of f taken along the boundary C of D. One has

$$\dot{f} = \nabla f \cdot \mathbf{t} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} \quad (59)$$

and

$$\frac{\partial f}{\partial n} = \nabla f \cdot \mathbf{n} = \frac{\partial f}{\partial x}\dot{y} - \frac{\partial f}{\partial y}\dot{x} \quad (60)$$

Therefore,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial n}\dot{y} + \dot{f}\,\dot{x} = \dot{f}^c\,\dot{y} + \dot{f}\,\dot{x} \quad (61)$$

and

$$\frac{\partial f}{\partial y} = \dot{f}\,\dot{y} - \frac{\partial f}{\partial n}\dot{x} = \dot{f}\,\dot{y} - \dot{f}^c\,\dot{x} \quad (62)$$

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Boundary Integral Method

▶ Boundary Element Method in Generalized Thermoelasticity

Boundary Value Problems

▶ Basic Theorems in Thermoelastostatics of Bodies with Microtemperatures
▶ Boundary Value Problems in Two-Dimensional Elastostatics of Anisotropic Solids
▶ Boundary Value Problems of Elastostatics of Hemitropic Solids
▶ Boundary-Value Problems Resulting in Thermoelastic Shock Wave Propagation

Boundary Value Problems in Two-Dimensional Elastostatics of Anisotropic Solids

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Synonyms

Boundary value problems

Overview

The method of boundary integral equations [1] is a powerful tool for solving boundary problems in the theory of elasticity. Compared with domain methods like the finite element method and the finite difference method in which the entire domain needs to be discretized, the boundary integral method (BIE) requires only the discretization of the boundary alone. Here and below are considered two classical problems of the two-dimensional theory of elasticity: the first, when tractions are prescribed at the boundary, and the second, when displacements are prescribed at the boundary. These problems are reduced to regular integral equations in regions with a smooth boundary.

Introduction

Recall that there are different ways for reducing boundary value problems to integral equations. One of them, usually used in practice, is based on the direct formulation, that is, the formulation directly dealing with the primitive variables (displacement or traction) prescribed at the boundary. For basic elastic problems, this method leads to singular integral equations of the first kind. Of course, if the posed problem has the zero index [2], it can be regularized and reduced to a system of Fredholm second-kind equations, but its discretization leads to an ill-conditioned system of linear algebraic equations. Therefore, it is desirable to use second-kind equations for solving basic elastic problems, since the resulting system of linear algebraic equations is well conditioned. We use here an indirect method, based on introduction of a fictitious and nonphysical double-layer density, fitting to solve a boundary value problem. It is natural to ask yourself whether it is possible to write the system of regular equations for the given problem without using the procedure of regularization. In the three-dimensional isotropic elasticity, it is known as the so-called antennae, H. Weyl's potential [1], whose application immediately leads to regular integral equations for basic three-dimensional problems of the isotropic theory of elasticity. It was noted in [3] that it corresponds to a solution of an elastic problem, obtained by superposition of solutions, loaded at the surface by a concentrated force (the Boussinesq solution). In the two-dimensional elasticity, it is known as the similar solution [4] of the static isotropic traction problem. The complex system of second-kind regular integral equations, advanced by D. I. Sherman [5] to solve the traction anisotropic elastic problem, is well known in the two-dimensional elasticity. In reality that author studied its modification consisting in assigning at a boundary first-order derivatives of the stress function. This modification has been considered recently by this author by means of a new approach [6]. The author's effort to perform the limiting transition to an isotropic material using his equations was useless. Therefore, this author has undertaken an effort to produce the correct derivation of required potentials. It was successful; it turned out that the difference of complex parameters of an elastic material was absent in Sherman's equations in the denominator, forbidding the limiting transition to an isotropic case. His equations are simply a result of a guess. The system of equations of the two-dimensional theory of elasticity has two pairs of distinct complex characteristics; this author have shown [4, 6, 7], and [8] that simplicity of characteristic roots essentially simplifies the procedure of reducing an anisotropic elastic boundary value problem to the system of integral equations. It is adequate to the considered boundary value problem. Moreover, this approach lets us use minimum assumptions on smoothness of a boundary and boundary data. In distinction to the approach based on the knowledge of the fundamental solution, which reduces the considered problem to the system of singular integral equations, this approach immediately reduces the problem to the system of regular equations for regions with a smooth boundary. This author also wants to add that the explicit construction of the required potentials is a nontrivial computational problem, even in the simplest case of an orthotropic material, when reducing a considered boundary value problem to singular integral equations. The limiting transition to an isotropic material can be performed without any difficulty. A large number of works are devoted to the deformation of anisotropic cylinders with various symmetry properties of the material. In this entry we study the plain strain problem when the material is homogeneous and has a plane of elastic symmetry, normal to the axis of cylinder. Let Q be the cross section of the cylinder. Throughout this entry a rectangular Cartesian coordinate system $Ox_k(k = 1, 2, 3)$ is used. The coordinate frame is chosen such that the x_3-axis is parallel to the generators of the cylinder. We shall employ the usual summation and differential conventions. Latin subscripts (unless otherwise stated) are understood to range over integers (1, 2). In this entry we consider the linear theory of elastostatics for anisotropic bodies [13]. It was S. G. Mikhlin [13] in 1936, who initiated the study of plane anisotropic elasticity in Russia. Up to now there are no correct set-ups of boundary value problems for anisotropic elastic solids. Some work for isotropic solids was done by authors of [12] and [14].

Preliminaries

The constitutive relations are

$$\begin{aligned} \varepsilon_{11} &= a_{11}\sigma_{11} + a_{12}\sigma_{22} + a_{16}\sigma_{12} \\ \varepsilon_{22} &= a_{12}\sigma_{11} + a_{22}\sigma_{22} + a_{26}\sigma_{12} \\ 2\varepsilon_{12} &= a_{16}\sigma_{11} + a_{26}\sigma_{22} + a_{66}\sigma_{12} \end{aligned}$$

where

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

are strains, $\sigma_{i,j}$ are stresses, and u_1,u_2 are displacements. The positive definiteness of the compliances' matrix $(a_{ij}), i, j = 1, 2, 6$ is assumed. Here and below, all considered functions are defined in Q. Excluding displacements u_1,u_2 by differentiation of strains, we derive the equation of the strain's compatibility:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0$$

Now, substitute into this equation strains from the constitutive relations and represent stresses as second order derivatives of a function $u(x_1,x_2)$:

$$\sigma_{11} = \frac{\partial^2 u}{\partial x_2^2}, \quad \sigma_{22} = \frac{\partial^2 u}{\partial x_1^2}, \quad \sigma_{12} = -\frac{\partial^2 u}{\partial x_1 \partial x_2}$$

In the absence of body forces, the equilibrium equations are identically satisfied, and the stress function $u(x_1,x_2)$ satisfies the elliptic fourth-order equation:

$$\begin{aligned} L(u) = L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)u(x_1, x_2) = a_{22}\frac{\partial^4 u}{\partial x_1^4} \\ - 2a_{26}\frac{\partial^4 u}{\partial x_1^3 \partial x_2} + (2a_{12} + a_{66})\frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} \\ - 2_{26}\frac{\partial^4 u}{\partial x_1 \partial x_2^3} + a_{11}\frac{\partial^4 u}{\partial x_2^4} = 0 \end{aligned} \tag{1}$$

Following Lekhnitskii [9], associate with equation (1) the (characteristic) equation

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0.$$

It has two pairs of complex conjugate roots (in terms of Lekhnitskii, complex parameters of an elastic material) $\mu_k = \alpha_k + i\beta_k, \beta_k > 0$, $\bar{\mu}_k = \alpha_k - i\beta_k, k = 1, 2,$ since (1) is an elliptic equation with real coefficients. Write $u(x_1, x_2)$ as the sum $u(x_1, x_2) = w_1(x_1, x_2) + w_2(x_1, x_2)$. Here $w_k(x_1,x_2)$ is a quasi-harmonic function, that is, a solution of a "quasi-harmonic" equation

$$(\beta_k^2 + \alpha_k^2)\frac{\partial^2 w_k}{\partial x_1^2} - 2\alpha_k\beta_k\frac{\partial^2 w_k}{\partial x_1 \partial x_2} + \frac{\partial^2 w_k}{\partial x_2^2} = 0$$

as the change of independent variables $y_1 = x_1 + \alpha_k x_2, y_2 = \beta_k x_2$ reduces it to Laplace's equation. The quotation marks at the term "quasi-harmonic" below are omitted. In a simply connected region, any quasi-harmonic function $w_k(x_1, x_2), k = 1, 2$ can be represented as the real part of a holomorphic function of the (complex) argument $z_k = x_1 + \mu_k x_2$, $w_k(x_1, x_2) = Re\varphi_k(z_k)$. As result, the general solution of (1) is

$$u(x_1, x_2) = Re\{\varphi_1(x_1 + \mu_1 x_2) + \varphi_2(x_1 + \mu_2 x_2)\}$$

Here and below, prime denotes differentiation with respect to the argument in parentheses. Let Q be a simply connected, bounded, counter-clockwise-oriented plane region with a Lyapunov boundary ∂Q, that is, it is assumed that ∂Q has a uniformly Hölder continuous inward normal field $v(z) = (n_1, n_2)$. It means that the functions $x_k(s)$ specifying the shape of a region are continuous and continuously differentiable, and, moreover, there is a positive α, $0 < \alpha < 1$, such that

$$|x'_k(s) - x'_k(s_0)| < c|s - s_0|^\alpha$$

Here s is the arc-length parameter, and $C^k(Q)$ and $C^k(\bar{Q})$ are, respectively, the spaces of real functions that are continuously differentiable up to the order k in Q and $\bar{Q}$. $C^{0,\alpha}(\bar{Q})$ and $C^{0,\alpha}(\partial Q)$ are, respectively, the spaces of real continuous functions satisfying in $\bar{Q}$ and ∂Q a uniform Hölder condition with some exponent $\alpha, 0 < \alpha < 1$. $C^{k,\alpha}(\bar{Q})$ is the subclass of $C^k(\bar{Q})$ consisting from functions u such that $D^v u \in C^{0,\alpha}(\bar{Q}), |v| = k$. Here v is the multi-index $v = (k_1, k_2)$:

$$D^v u = \frac{\partial^v u}{\partial x_1^{k_1} \partial x_2^{k_2}}, \quad k_1 + k_2 = v$$

The spaces $C^{l,\alpha}(\partial Q)$ are defined in a similar way. It is also assumed that the origin of coordinates lies inside the region. Let $u_1(x_1, x_2), u_2(x_1, x_2)$ be the displacements in the anisotropic elasticity; they can be written as

$$\begin{aligned} u_1 &= Re[b_{11}\Phi_1(z_1) + b_{12}\Phi_2(z_2)] + \alpha x_2 + \delta_1 \\ u_2 &= Re[b_{21}\Phi_1(z_1) + b_{22}\Phi_2(z_2)] - \alpha x_1 + \delta_2 \end{aligned} \quad (2)$$

Here $\Phi_k(z_k) = \varphi'_k(z_k)$, terms $\alpha x_2 + \delta_1$, and $-\alpha x_1 + \delta_2$ answer for a rigid displacement of an elastic body:

$$\begin{aligned} b_{1k} &= a_{11}\mu_k^2 + a_{12} - a_{16}\mu_k \\ b_{2k} &= a_{12}\mu_k + a_{22}\mu_k^{-1} - a_{26} \end{aligned} \quad (3)$$

Then stresses $\sigma_{ij}, i, j = 1, 2$ are then rewritten as derivatives of the analytical functions $\Phi_k(z_k), z_k x_1 + \mu_k x_2, k = 1, 2$ as

$$\begin{aligned} \sigma_{11} &= Re[\mu_1^2\Phi'_1(z_1) + \mu_2^2\Phi'_2(z_2)] \\ \sigma_{22} &= Re[\Phi'_1(z_1) + \Phi'_2(z_2)] \\ \sigma_{12} &= -Re[\mu_1\Phi'_1(z_1) + \mu_2\Phi'_2(z_2)] \end{aligned} \quad (4)$$

Consider the integral equation

$$f(s) + \lambda \int_a^b K(s, s_0) f(s_0)\, d s_0 = g(s)$$

where λ, a, and b are real parameters and $f(s), g(s), K(s, s_0)$ are real functions. The function $K(s, s_0)$ is defined in the plane (x, s) in the square $a < s, s_0 < b$. According to the definition of S. G. Mikhlin [10], the equation above is a Fredholm equation of the second kind for the function $f(s)$, if $g(s)$ and $K(s, s_0)$ are square integrable in the square $a < s, s_0 < b$. Recall that Fredholm assumed continuity of the kernel $K(s, s_0)$ in the same square. If a boundary of a plane region is a Lyapunov curve, then the kernel of the integral operator

$$\frac{1}{\pi i} \int_{\partial Q} \frac{f(s)\, d t}{t - z}$$

where $f(s)$ is a real function,

$$\begin{aligned} t &= t(s) = x_1(s) + i x_2(s), \\ d t &= (x'_1(s) + i x'_2(s))\, d s, \quad z = x_1 + i x_2 \end{aligned}$$

is a kernel with a weak singularity, that is, it can be written as a fraction

$$K(s - s_0) = \frac{a(s, s_0)}{|s - s_0|^\alpha}, 0 < \alpha < 1$$

where $a(s, s_0)$ is a bounded function. It can be proved (Mikhlin [11]) that if a kernel of an integral operator has a weak singularity, all iterated kernels, beginning from some, are bounded. Hence, the equations with weakly singular kernels are Fredholm.

The Traction Boundary Value Problem

Let $\mathbf{n} = (n_1, n_2), n_1 = -x_2'(s), n_2 = x'_1(s)$ be the internal normal vector to a rectifiable Jordan boundary ∂Q of length $L > 0$ of a simply connected region Q in the plane R^2 with a boundary of the class $C^{1,\lambda}(0,L), 0<\lambda<1$. In another words, it is assumed that the functions $x_1(s), x_2(s)$, defining a boundary, are continuous and continuously differentiable, their derivatives satisfy the Hölder condition. Assume (without loss of generality) that the origin of coordinates belongs to a region Q. Tractions are written at a boundary as

$$\begin{aligned} \sigma_{11}n_1 + \sigma_{12}n_2|_{\partial Q} &= g_1(s) \\ \sigma_{12}n_1 + \sigma_{22}n_2|_{\partial Q} &= g_2(s) \end{aligned} \tag{5}$$

where s is the arc-length parameter. Put $t_k(s) = x_1(s) + \mu_k x_2(s), k = 1,2$; prime sign below denotes the derivative with respect to s. If stresses $\sigma_{ij}, i,j = 1,2$ are square integrable in Q, the solution of the traction problems (1) and (5) is unique up to a rigid displacement. Indeed, the volume strain energy

$$2a(u,u) = \int_Q \{a_{11}\sigma_{11}^2 + 2a_{12}\sigma_{11}a_{22} + 2a_{16}\sigma_{11}\sigma_{12} + a_{22}\sigma_{22}^2 + 2a_{26}\sigma_{22}\sigma_{12} + a_{66}\sigma_{12}^2\}dx$$

is positive defined and as a consequence there is a positive constant $C > 0$, such that

$$a(u,u) \geq C \int_Q \left\{ \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2 \right\} dx_1\, dx_2$$

If $u(x_1, x_2)$ is a solution of the homogeneous problem, stresses and strains vanish inside a region.

On the other hand, the proof of the uniqueness of a solution of problems (1) and (5), given above, uses only a piecewise smoothness of a boundary; it means that the Fredholm alternative is true for regions more general than bounded regions with a Lyapunov boundary. Boundary conditions (5) can be rewritten as

$$\begin{aligned} &Re\{-\mu_1 t_1'(s_0)\Phi_1'(t_1(s_0)) - \mu_2 t_2'(s_0)\Phi_2'(t_2(s_0))\} \\ &= g_1(s_0), \\ &Re\{t_1'(s_0)\Phi_1'(t_1(s_0)) + t_2'(s_0)\Phi_2'(t_2(s_0))\} = g_2(s_0) \end{aligned} \tag{6}$$

Here $t_k(s_0) = x_1(s_0) + \mu_k x_2(s_0), k = 1,2$. Write the functions $\Phi'_1(z_1), \Phi'_2(z_2)$ as Cauchy-type integrals with unknown densities $b_k(s), k = 1,2$:

$$\Phi'_k(z_k) = \frac{1}{\pi i} \int_{\partial Q} \frac{b_k(s)(t_k(s))^{-1} ds}{t_k - z_k}, k = 1,2$$

Densities $b_k(s), k = 1,2$ are determined from the simple system of equations

$$-\mu_1 b_1 - \mu_2 b_2 = f_1(s), \quad b_1 + b_2 = f_2(s) \tag{7}$$

Here the functions $f_k(s), k = 1,2$ are real. As result, the first-order derivatives are rewritten in terms of variables z_1, z_2 in the required form. These computations can be done only if the inequality $\mu_1 - \mu_2 \neq 0$ is satisfied. The latter inequality is equivalent to the condition of Ya. B. Lopatinskii. Having solved it, we shall get that

$$\begin{aligned} \Phi'_1(z_1) &= -\frac{1}{\pi i(\mu_1 - \mu_2)} \int_{\partial Q} \frac{(f_1 + \mu_2 f_2)[t_1'(s)]^{-1} dt_1}{t_1 - z_1} \\ \Phi'_2(z_2) &= \frac{1}{\pi i(\mu_1 - \mu_2)} \int_{\partial Q} \frac{(f_1 + \mu_1 f_2)[t_2'(s)]^{-1} dt_2}{t_2 - z_2} \end{aligned} \tag{8}$$

Recall the formulae of Sokhotski-Plemelj [2]

$$\lim_{z_j \to t_j} \frac{1}{\pi i} \int_{\partial Q} \varphi(s) \frac{dt_j}{t_j - z_j} = \varphi(s_0) + \frac{1}{\pi i} \int_{\partial Q} \varphi(s) \frac{dt_j}{t_j - t_{j0}} \tag{9}$$

if $z \in Q_i$, and

$$\lim_{z_j \to t_j} \frac{1}{\pi i} \int_{\partial Q} \varphi(s) \frac{d\,t_j}{t_j - z_j} = -\varphi(s) + \frac{1}{\pi i} \int_{\partial Q} \varphi(s) \times \frac{d\,t_j}{t_j - t_{j0}}$$

if $z \in Q_e$. Here $Q_i = Q$, and Q_e is the region, external to Q, $t_{j0} = t_j(s_0)$. Of course, it is assumed that $\varphi(s) \in C^{0,\lambda}(\partial Q)$. It follows that

$$\sigma_{kj}(\mathbf{u}(x,\mathbf{f})n_j)_i(s_0) - \sigma_{kj}(\mathbf{u}(x,\mathbf{f})n_j)_e(s_0) = 2f_k(s_0), k = 1,2$$

Here $\sigma_{kj}(\mathbf{u}(x,\mathbf{f})n_j)_i(s_0)$ is the limiting value of the traction vector inside a region, and, correspondingly, $\sigma_{kj}(\mathbf{u}(x,\mathbf{f})n_j)_e(s_0)$ is its limiting value outside a region, and $\mathbf{u}(x,\mathbf{f})$ is the vectorial simple-layer potential defined below by formulas (12) and (13).

$$f_1(s_0) + Re\frac{\mu_1 t_1'(s_0)}{\pi i(\mu_1 - \mu_2)} \int_{\partial Q} \frac{(f_1 + \mu_2 f_2)[t_1'(s)]^{-1} dt_1}{t_1 - t_{10}} + - Re\frac{\mu_2 t_2'(s_0)}{\pi i(\mu_1 - \mu_2)} \int_{\partial Q} \frac{(f_1 + \mu_1 f_2)[t_2'(s)]^{-1} dt_2}{t_2 - t_{20}} = g_1(s_0) \quad (10)$$

$$f_2(s_0) - Re\frac{t_1'(s_0)}{\pi i(\mu_1 - \mu_2)} \int_{\partial Q} \frac{(f_1 + \mu_2 f_2)[t_1'(s)]^{-1} dt_1}{t_1 - t_{10}} + Re\frac{t_2'(s_0)}{\pi i(\mu_1 - \mu_2)} \int_{\partial Q} \frac{(f_1 + \mu_1 f_2)[t_2'(s)]^{-1} dt_2}{t_2 - t_{20}} = g_2(s_0) \quad (11)$$

Here $t_{k0} = x_1(s_0) + \mu_k x_2(s_0), k = 1,2$

Existence Results

Therefore, these mutually adjoint systems are Fredholm solvable. Now, for solubility of systems (10) and (11), it is necessary and sufficient that right-hand sides of equations (10) and (11) be orthogonal to all solutions of (12) and (13), and vice versa. Therefore, it is necessary to require vanishing of the principal vector and the principal vector of acting forces. In other words,

$$\int_{\partial Q} g_k(s)\,d\,s = 0, k = 1,2$$

$$\int_{\partial Q} (x_1(s)g_2(s) - x_2(s)g_1(s))\,d\,s = 0$$

It is easy to verify its fulfillment. Indeed, multiply equations (10) and (11) on ds_0 and integrate the result with respect to s_0, bearing in mind that

$$\int_{\partial Q} \frac{t'_{k0}\,d\,s_0}{t_k - t_{k0}} = -\pi i$$

Then left-hand sides of (10) and (11) become zero and we obtain two first equalities. In a similar way, multiply the first equation on $x_1(s_0)$, the second onto $-x_2(s_0)$, change the order of integration, and integrate with respect to s_0. Then you shall get the third equality in the previous formula. Hence, these equalities are necessary for solubility of these equations. The solution of the traction problem is given by the simple-layer potential, which is written as

$$u_1(x_1, x_2, f) = Re\frac{b_{11}}{\pi i(\mu_1 - \mu_2)} \int_{\partial Q} (f_1 + \mu_2 f_2)\ln(z_1 - t_1)ds - Re\frac{b_{12}}{\pi i(\mu_1 - \mu_2)} \int_{\partial Q} (f_1 + \mu_1 f_2)\ln(z_2 - t_2)ds \quad (12)$$

$$u_2(x_1, x_2, f) = Re\frac{b_{21}}{\pi i(\mu_1 - \mu_2)} \int_{\partial Q} (f_1 + \mu_2 f_2)\ln(z_1 - t_1)ds - Re\frac{b_{22}}{\pi i(\mu_1 - \mu_2)} \int_{\partial Q} (f_1 + \mu_1 f_2)\ln(z_2 - t_2)ds \quad (13)$$

Fix, for example, the principal branch of logarithm. Consider in more detail properties of the simple-layer potential. For existence of integrals (14) and (15) is sufficient continuity of f_1, f_2; then u_1, u_2 are continuous in the whole plane but have the logarithmic growth at infinity. Now,

Lemma 1. *The simple-layer potential with a continuous density* $\mathbf{f} = (f_1, f_2)$*, satisfying the equation*

$$\int_{\partial Q} f_i\, ds = 0, i = 1, 2$$

satisfies also the estimates

$$|u_i(x,f)| < \frac{c}{|x|}, \quad \left|\frac{\partial u_i}{\partial x_j}\right| < \frac{c_1}{|x|^2}, i,j = 1,2, |x| \sqrt{x_1^2 + x_2^2}.$$

Indeed, consider the typical integral, entering into (14) and (15):

$$\int_{\partial Q} f(s) \ln(z_k - t_k)\, ds$$

Here $f(s)$ is any density. Write it as a derivative:

$$f(s) = \frac{d}{ds}\left(\int_0^s f(s)\, ds\right) = \frac{d}{ds}\tau(s)$$

After integration by parts we shall get that

$$\int_{\partial Q} \frac{d\tau}{ds} \ln(z_k - t_k)\, ds = \tau(s) \ln(z_1 - t_1)|_{\partial Q} - \int_{\partial Q} \tau(s) \frac{d\, t_k}{t_k - z_k}$$

The first summand in the right-hand side vanishes if $\tau(L) = 0$, and the second summand is a single-valued Cauchy-type integral. The equality $\tau(L) = 0$ means that conditions of the lemma are satisfied. Other assertions are obvious. Therefore, functions $v_1(x_1, x_2), v_2(x_1, x_2)$ are also single valued. Put $\mathbf{f} = (f_1, f_2)$. Introduce now a short notation for a simple-layer potential: put

$$\mathbf{u}(x, \mathbf{f}) = (u_1(x, \mathbf{f}), u_2(x, \mathbf{f})),$$

$$u_i(x, \mathbf{f}) = \sum_{j=1}^{2} \int_{\partial Q} G_{ij} f_j(s)\, ds, i = 1, 2$$

understanding by $G_{ij}, i, j = 1, 2$ kernels of integral operators in (10) and (11). Or, quite shortly, put

$$\mathbf{u}(x, \mathbf{f}) = \int_{\partial Q} G(x - y)\mathbf{f}\, ds$$

where $G(x)$ is the matrix $(G_{ij}(x)), i, j = 1, 2$

Lemma 2. *The system of integral equations* (10) *and* (11) *has exactly three linearly independent solutions:*

$$\mathbf{f}_1 = (1, 0), \quad \mathbf{f}_2 = (0, 1), \mathbf{f}_3 = (-x_2(s), x_1(s))$$

Indeed, if it is not so, then the system of equations (12) and (13), adjoint to (10) and (11), would have more than three linearly independent solutions $\varphi_k(s), 1 \le k, k > 3$. To any of them answer a simple-layer potential $\mathbf{v}(x, f_k)$, satisfying conditions $\sigma_{ij}(\mathbf{v}(x, f_k) n_j = 0$ at the boundary. Let there be one more solution $f_4(s)$, linearly independent from others; then

$$f(s) = f_4(s) - \sum_{j=1}^{3} c_j f_j(s)$$

where $c_j, j = 1, 2, 3$ are arbitrary real constants, is also a solution of the homogeneous system of equations (10) and (11). Write up simple-layer potentials:

$$\mathbf{u}(x, f_4), \mathbf{u}(x, f_j), j = 1, 2, 3$$

As solutions of homogeneous boundary value problems, they are rigid displacement vectors and so

$$\mathbf{u}(x) = \mathbf{u}(x, f_4) - \sum_{j=1}^{3} \mathbf{u}(x, f_j)$$

is a rigid displacement vector. Now choose constants c_1, c_2, c_3 to satisfy the conditions

$$\mathbf{u}(0) = 0, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0$$

But then v^k satisfies the system

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = 0, i, j = 1, 2$$

Rewrite these conditions as

$$\sum_{j=1}^{3} c_j \int_{\partial Q} G(x_2)\varphi_j(s)\, d\, s = \int_{\partial Q} G(x_2)\, d\, s$$

$$\sum_{j=1}^{3} c_j \int_{\partial Q} G_0(x_2)\varphi_j(s)\, d\, s = \int_{\partial Q} G_0(x_2)\, d\, s$$

where

$$G_0(x_2) = \frac{\partial G_2}{\partial x_2} - \frac{\partial G_1}{\partial x_1}$$

The determinant of this system is different from zero by linear independence of $\varphi_j, j = 1, 2, 3$. Solve this system with respect to $c_j, j = 1, 2, 3$. Then $u(x_1, x_2) = 0, (x_1, x_2) \in Q_i$. Continuity of a potential in the whole plane implies that $u(x) = 0$ at ∂Q. As $u(x,)$ satisfies conditions of the lemma 1, then at infinity

$$|u(x_1, x_2)| < \frac{c}{\sqrt{x_1^2 + x_2^2}}, |\sigma_{ij}(u(x))| < \frac{c_1}{x_1^2 + x_2^2}$$

and so $u(x_1, x_2) = 0, (x_1, x_2) \in Q_e$. But then according to the Sokhotski-Plemelj formula, the jump of the traction vector is equal zero at the boundary and then $f_4 = 0$. This is a contradiction.

Denote a solution of the internal boundary value problem by (I, Q_i) and the solution of the external by (I, Q_e). Then

Lemma 3. *The problem (I, Q_i) has unique solution (up to a linear combination) $c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3$, where*

$$\varphi_1 = (0, 1), \varphi_2 = (1, 0), \varphi_3 = (-x_2, x_1)$$

Functions $\varphi_k, k = 1, 2, 3$ are a complete set of solutions of the system

$$\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} = 0, i, k = 1, 2$$

Its proof follows from the integral identity

$$\int_{Q_i} \sigma_{ij}(\mathbf{v})\mathbf{e_{ij}}(\mathbf{v})\, \mathbf{d\, x} = \int_{\partial \mathbf{Q}} \sigma_{\mathbf{ij}}(\mathbf{v})\mathbf{v_i}\nu_{\mathbf{j}}\, \mathbf{d\, s}$$

Here ν is the external normal vector of Q_i. For functions $\mathbf{v}$, with properties

$$|\mathbf{v}| \le c, |\frac{\partial v_k}{\partial x_i}| \le \frac{c}{|x|^2}, i, k = 1, 2 \qquad (14)$$

for large |x| and satisfying the homogeneous system of elastic equations in the unbounded region, Q_e holds the integral identity

$$\int_{Q_e} \sigma_{ij}(\mathbf{v})e_{ij}(\mathbf{v})\, d\, x = -\int_{\partial Q} \sigma_{ij}(\mathbf{v})v_i\nu_j\, d\, s \qquad (15)$$

The last equality is obtained from the previous integral identity for a bounded region. Indeed, consider the region lying between the boundary ∂Q and the circumference of sufficiently large radius R, with a center lying inside Q_i. The previous equality holds for this region. The integral, taken along the external circumference, tends to zero, when $R \to \infty$. Hence, equality (15) follows from the given above assertion and the absolute convergence of the integral with respect to Q_e. This implies the following lemma.

Lemma 4. *The solution of the problem (I, Q_e) in the class of functions, subject to (1.13), is unique up to a constant vector, and if $\mathbf{u} \to \mathbf{0}$, when $|x| \to \infty$, then it is unique.* Notice that the system of equations (10) and (11) can be modified to make it uniquely solvable. Indeed, introduce to equation (10) summands

$$-Re\frac{\mu_1}{2\pi i(\mu_1-\mu_2)}\frac{t_1'(s_0)}{t_1(s_0)}\int_{\partial Q}(f_1+\mu_2 f_2)\,ds$$

$$-Re\frac{\mu_2}{2\pi i(\mu_1-\mu_2)}\frac{t_2'(s_0)}{t_2(s_0)}\int_{\partial Q}(f_1+\mu_1 f_2)\,ds$$

$$-Re\frac{1}{4\pi i}\left\{-\mu_1\frac{\partial}{\partial s_0}\frac{1}{t_1(s_0)}-\mu_2\frac{\partial}{\partial s_0}\frac{1}{t_2(s_0)}\right\}M$$

and equation (11) by summands

$$-Re\frac{1}{2\pi i(\mu_1-\mu_2)}\frac{t_1'(s_0)}{t_1(s_0)}\int_{\partial Q}(f_1+\mu_2 f_2)\,ds$$

$$+Re\frac{1}{2\pi i(\mu_1-\mu_2)}\frac{t_2'(s_0)}{t_2(s_0)}\int_{\partial Q}(f_1+\mu_1 f_2)\,ds$$

$$-Re\frac{1}{4\pi i}\left\{\frac{\partial}{\partial s_0}\frac{1}{t_1(s_0)}+\frac{\partial}{\partial s_0}\frac{1}{t_2(s_0)}\right\}M$$

Here M is a real constant. As

$$\int_{\partial Q}\frac{dt_k(s_0)}{t_k(s_0)}=2\pi i, k=1,2$$

(the origin of coordinates is inside a region), it follows that

$$\int_{\partial Q}f_k(s)\,ds=\int_{\partial Q}g_k(s)\,ds, k=1,2$$

$$M=\int_{\partial Q}(-x_2(s)g_1(s)+x_1(s)g_2(s))\,ds$$

It is clear that if the principal vector and the principal moment of applied forces are equal to zero, the system of equations with these applied summands is equivalent to the system of equations (10) and (11). It is necessary to assume that $g_k(s)\in C^{0,\lambda}(\partial Q), k=1,2.$ Then $f_k(s)\in C^{0,\lambda}(\partial Q), k=1,2.$ Moreover, it is not quite obvious that it is the system of regular equations. Rewrite equations (10) and (11) as

$$f_1(s_0)+Re\frac{t_1'(s_0)}{\pi i}\int_{\partial Q}f_1(s)\frac{[t_1'(s)]^{-1}dt_1}{t_1-t_{10}}$$
$$+Re\frac{\mu_2}{\pi i(\mu_1-\mu_2)}\int_{\partial Q}(f_1+\mu_1 f_2)\left\{\frac{t_2'(s_0)}{t_2-t_{20}}\right.$$
$$\left.-\frac{t_1'(s_0)}{t_1-t_{10}}\right\}ds=g_1(s_0) \tag{16}$$

$$f_2(s_0)+Re\frac{t_2'(s_0)}{\pi i}\int_{\partial Q}f_2(s)\frac{(t_2'(s))^{-1}dt_1}{t_2-t_{20}}$$
$$+Re\frac{1}{\pi i(\mu_1-\mu_2)}\int_{\partial Q}(f_1+\mu_2 f_2)\left\{\frac{t_1'(s_0)}{t_1-t_{10}}\right.$$
$$\left.-\frac{t_2'(s_0)}{t_2-t_{20}}\right\}ds=g_2(s_0) \tag{17}$$

Therefore, it is the system of regular equations. Formulate now the principal result of the previous investigation.

Theorem 1. *Assume that*

$$g_k(s)\in C^{0,\lambda}(\partial Q), k=1,2, \partial Q$$
$$\in C^{1,\lambda}(0,L), 0<\lambda<1.$$

Under the given above assumptions, equality to zero of the principal vector and the principal moment of applied forces is the necessary and sufficient condition for existence of the solution of the system (10) and (11) belonging to $C^{0,\lambda}(0,L)$. Then $u_k(x)\in C^{1,\lambda}(\overline{Q}), k=1,2.$

Indeed, let us seek the solution as a single-layer potential $\mathbf{u}=(u_1(x,\mathbf{f}),u_2(x,\mathbf{f}))$. Then density $\mathbf{f}=(f_1,f_2)$ satisfies systems (10) and (11). As two pairs of adjoint integral equations (10) and (11) and (12) and (13) are Fredholm solvable, the system of equations (10) and (11) is solvable if and only if the vector function $\mathbf{g}=(g_1,g_2)$ is orthogonal to all solutions of the homogeneous system

(12) and (13). By lemma 2 the general solution of the homogeneous system (12) and (13) is a linear combination of the three vector functions $\varphi(x) = c_1(1,0) + c_2(0,1) + c_3(-x_2, x_1)$. Conditions (1.10) are conditions of orthogonality of a solution of a homogeneous system (10) and (11) to the vector function $\mathbf{g} = (g_1, g_2)$, and so they guaranty the existence of the system of (10) and (11). Necessity follows from Betti's formula. Indeed, we have the conditions

$$0 = \int_{\partial Q} \sigma_{ij} n_j \varphi_i \, d\, s = \int_{\partial Q} g_i \varphi_i \, d\, s$$

which coinciding with the equilibrium conditions. It is not hard to see that smoothness of solution grows up according to the growth of a boundary and boundary data. Now,

Theorem 2. *Assume that*

$$g_k(s) \in C^{l,\lambda}(\partial Q), k = 1, 2, \partial Q \in C^{l+1,\lambda}(0, L), 0 < \lambda < 1, l \geq 1.$$

Then $C^{l+1,\lambda}(\bar{Q})$. It follows immediately from the properties of the Cauchy-type integral (see theorem 1.10 from [11]).

The Case of an Isotropic Material

For an isotropic material,

$$a_{11} = a_{22} = \frac{1}{E}, a_{12} = \frac{-v}{E}, a_{66} = \frac{1}{G}, a_{16} = a_{26} = 0$$

Here E is the Young's modulus, G is the shear modulus, and v is the Poisson's ratio. The limiting transition, when $\mu_1, \mu_2 \to i$, is easily performed. As a result, we get the system of equations for an isotropic material:

$$f_1(s_0) + Re\frac{t'(s_0)}{\pi i} \int_{\partial Q} \frac{f_1(s)[t'(s)]^{-1} dt}{t - t_0} + Re\frac{1}{2\pi i} \int_{\partial Q} (f_1 + if_2) \frac{(\bar{t} - \bar{t}_0)dt - (t - t_0)d\bar{t}}{(t - t_0)^2} = g_1(s_0) \quad (18)$$

$$f_2(s_0) + Re\frac{t'(s_0)}{\pi i} \int_{\partial Q} \frac{f_2(s)[t'(s)]^{-1} dt}{t - t_0} + Re\frac{i}{2\pi i} \int_{\partial Q} (f_1 + if_2) \frac{(\bar{t} - \bar{t}_0)dt - (t - t_0)d\bar{t}}{(t - t_0)^2} = g_2(s_0) \quad (19)$$

Here $z = x_1 + ix_2, t_0 = x_1(s_0) + ix_2(s_0)$. It is clear that this system can be reduced to one (complex) equation with the respect to the complex density $\omega(s) = f_1(s) + if_2(s)$. In the complex notation it is equivalent to the system of equations from [5]. Denote by $u_1^1(x), u_2^1(x)$ displacements for an isotropic material. Then

$$u_1^1(x) = Re\left\{\frac{2}{E}\frac{1}{\pi}\int_{\partial Q} f_1(s)\ln(z - t)ds + \frac{1+v}{E}\frac{1}{\pi}\int_{\partial Q} f_1(s)\frac{i(\eta - x_2)}{t - z}ds\right] + Re\left[\frac{1-v}{E}\frac{1}{\pi}\int_{\partial Q} if_2(s)\ln(z - t)ds + \frac{1+v}{E}\frac{1}{\pi}\int_{\partial Q} f_2(s)\frac{i(\eta - x_2)}{t - z}ds\right\} \quad (20)$$

$$u_2^1(x) = Re\left\{\frac{1-v}{E}\frac{1}{\pi_j}\int_{\partial Q} f_1(s)\ln(z-t)ds - \frac{1+v}{E}\frac{1}{\pi i}\int_{\partial Q} f_1(s)\frac{i(\eta - x_2)}{t-z}ds\right] + Re\left[\frac{2}{E}\frac{1}{\pi i}\int_{\partial Q} if_2(s)\ln(z-t)ds - \frac{1+v}{E}\frac{1}{\pi_j}\int_{\partial Q} if_2(s)\frac{i(\eta - x_2)}{t-z}ds\right\} \quad (21)$$

Here $\xi = x_1(s), \eta = x_2(s)$

The Displacement Boundary Value Problem

Here we discuss (in short) the displacement boundary value problem. As in the previous section, consider a bounded simply connected region Q with the boundary ∂Q of the Lyapunov class and prescribe displacements at the boundary. Consider the boundary value problem of determination of the stress-strain state when displacements

$$Re\{b_{11}\Phi_1(z_1)+b_{12}\Phi_2(z_2)\}=g_1(s_0)$$
$$Re\{b_{21}\Phi_1(z_1)+b_{22}\Phi_2(z_2)\}=g_2(s_0)$$
$$g_k(s_0)\in C^{0,\alpha}(\partial Q),\ k=1,2$$

are prescribed at the boundary. First-order derivatives of displacements should be square integrable in the closed region $\bar{Q}$, as otherwise a solution of this problem is non-unique. Introduce the functions $\Phi_k(z_k)(k=1,2)$ as Cauchy-type integrals

$$\Phi_k(z_k)=\frac{1}{\pi i}\int\limits_{\partial Q}\frac{\omega_k(s)dt_k}{t_k-z_k}k=1,2$$

where $t_k=x_1(s)+\mu_k x_2(s)k=1,2.$

Let $f_k(s)(k=1,2)$ be two real functions. Solve the system of equations:

$$b_{11}\omega_1+b_{12}\omega_2=f_1(s),\quad b_{21}\omega_1+b_{22}\omega_2=f_2(s)$$

Put

$$u_1|_{\partial Q}=g_1(s),\quad u_2|_{\partial Q}=g_2(s),g_k(s)\in C^{0,\alpha}(\partial Q) \tag{22}$$

$$\Phi_k(z_k)=\frac{1}{\pi i}\int\limits_{\partial Q}\frac{\omega_k(s)\,d\,t_k}{t_k-z_k}\quad k=1,2$$

Write $\varphi_k(z_k),k=1,2$ as Cauchy-type integrals with complex densities $m_k(s)$:

$$\varphi_k(z_k)=\frac{1}{\pi i}\int\limits_{\partial Q}\frac{\omega_k(s)\,d\,t_k}{t_k-z_k},k=1,2$$

Just as for the traction problem, by means of the Plemelj formula, we obtain the system of linear equations $m_k(s),k=1,2,$

$$b_{11}\omega_1+b_{12}\omega_2=g_1,b_{21}\omega_1+b_{22}\omega_2=g_2 \tag{23}$$

and solve it by the Cramer's rule. Its determinant is

$$\delta=\frac{\mu_1-\mu_2}{\mu_1\mu_2}[\mu_1\mu_2(a_{11}a_{22}-a_{12}^2)-a_{22}a_{66}]$$

and it is distinct from zero if $\mu_1\neq\mu_2$. Put $\delta=\xi(\mu_1-\mu_2)$. Here ξ depends only on the symmetric functions of roots of the characteristic equation and, therefore, on elastic coefficients. It is assumed further that $\xi\neq 0$. It is always different from zero in elastic problems by positivity of the stress energy. For example, for an orthotropic material ($a_{16}=0,a_{26}=0$, it is equal to

$$\xi=(\gamma_1\gamma_2)^{-1}\left\{(a_{11}a_{22}-a_{12}^2)\sqrt{a_{22}a_{11}^{-1}}+a_{22}a_{66}\right\}$$

and it is always positive, as

$$a_{11}a_{22}-a_{12}^2>0,a_{11}>0,a_{22}>0,a_{66}>0$$

For an isotropic material $\xi=(1+v)(3-v)E^{-2}$, where E is the Young's modulus and v is the Poisson's ratio.

Hence, $u_k(x_2,x_2),k=1,2$ can be written as

$$u_1(x_1,x_2)=Re\frac{1}{\pi i}\int\limits_{\partial Q}\frac{f_1(s)\,dt_1}{t_1-z_1}$$
$$+Re\frac{b_{12}}{\pi i\delta}\int\limits_{\partial Q}(-b_{21}f_1(s)+b_{11}f_2(s))\left(\frac{dt_2}{t_2-z_2}-\frac{dt_1}{t_1-z_1}\right)$$

$$u_2(x_1,x_2)=Re\frac{1}{\pi i}\int\limits_{\partial Q}\frac{f_2(s)\,dt_2}{t_2-z_2}$$
$$+Re\frac{b_{21}}{\pi i\delta}\int\limits_{\partial Q}(b_{22}f_1(s)-b_{12}f_2(s))\left(\frac{dt_1}{t_1-z_1}-\frac{dt_2}{t_2-z_2}\right)$$

As a result, we arrive to the system of regular integral equations:

$$f_1(s_0)+Re\frac{1}{\pi i}\int\limits_{\partial Q}\frac{f_1(s)dt_1}{t_1-t_{10}}+Re\frac{b_{12}}{\pi i\delta}\int\limits_{\partial Q}(-b_{21}f_1(s)$$
$$+b_{11}f_2(s))\left(\frac{dt_2}{t_2-t_{20}}-\frac{dt_1}{t_1-t_{10}}\right)=g_1(s_0) \tag{24}$$

$$f_2(s_0) + Re\frac{1}{\pi i}\int\limits_{\partial Q}\frac{f_2(s)dt_2}{t_2 - t_{20}} + Re\frac{b_{21}}{\pi i\delta}\int\limits_{\partial Q}(b_{22}f_1(s) - b_{12}f_2(s))\left(\frac{dt_1}{t_1 - t_{10}} - \frac{dt_2}{t_2 - t_{20}}\right) = g_2(s_0) \quad (25)$$

The Case of an Isotropic Solid

Now, in the limiting transition to an isotropic material due to the presence of the factor $\mu_1 - \mu_2$ in the denominator δ, the difference

$$\frac{1}{(\mu_1 - \mu_2)}\left(\frac{d\,t_2}{t_2 - z_2} - \frac{d\,t_1}{t_1 - z_1}\right)$$

tends in the limit to the difference

$$\frac{x'_1(s)(x_2(s) - x_2) - x'_2(s)(x_1(s) - x_1)}{(t - z)^2}$$

and the limiting values of displacements are expressed as

$$u_1^0 = Re\frac{1}{\pi i}\int\limits_{\partial Q}\frac{f_1(s)\,dt}{t - z} - Re\frac{i}{\pi i}\frac{1+v}{3-v}\int\limits_{\partial Q}(f_1 + if_2)\frac{x'_1(s)(x_2(s) - x_2) - x'_2(s)(x_1(s) - x_1)}{(t-z)^2}ds$$

$$u_2^0 = Re\frac{1}{\pi i}\int\limits_{\partial Q}\frac{f_2(s)\,dt}{t - z} - Re\frac{1}{\pi i}\frac{1+v}{3-v}\int\limits_{\partial Q}(f_1 + if_2)\frac{x'_1(s)(x_2(s) - x_2) - x'_2(s)(x_1(s) - x_1)}{(t-z)^2}ds$$

where the superscript 0 refers to the field of displacements in an isotropic material. We get the formulae and the integral equation of D. I. Sherman for an isotropic material. The author sincerely thanks the editor, Professor Dorin Iesan, for careful editing of this entry.

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Boundary Value Problems of Elastostatics of Hemitropic Solids

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Synonyms

Boundary value problems; Chiral; Hemitropic; Noncentrosymmetric

Overview

Classical elasticity associates only the three translational degrees of freedom to material points of the body, and all the mechanical characteristics are expressed by the corresponding displacement vector. On the contrary, micropolar theory, by including intrinsic rotation of a material particle (a structural unit of the medium), provides a rather complex model of an elastic body that can support body forces and body couple vectors as well as force stress vectors and couple stress vectors at the surface. Consequently, in micropolar theory, in the course of deformation, not only a displacement but also a rotation takes place, and all the mechanical quantities are written in terms of the displacement and microrotation vectors.

The origin of the rational theories of micropolar continua goes back to the outstanding French scholars, the brothers Eugène Maurice Pierre Cosserat and François Cosserat [3, 4], who gave a development of the mechanics of continuous media in which material points are considered as oriented particles and have the six degrees of freedom defined by 3 displacement components and 3 microrotation components (for the history of the theory of micropolar elasticity, see [5, 8, 13], and the references therein).

A micropolar solid which is not isotropic with respect to mirror reflections (i.e., inversion) is called *hemitropic, noncentrosymmetric*, or *chiral*. Materials may exhibit chirality on the atomic scale, as in quartz and in biological molecules – DNA – as well as on a large scale, as in composites with helical or screw-shaped inclusions, certain types of nanotubes, bone, fabricated structures such as foams, chiral sculptured thin films, and twisted fibers.

Refined mathematical models describing the hemitropic properties of elastic materials have been proposed by Aero and Kuvshinski [1, 2] (for historical notes, see also [5, 6], and the references therein). In the mathematical theory of elasticity for hemitropic continua, there are introduced the *asymmetric force stress tensor* and *couple stress tensor*, which are kinematically related with the *asymmetric strain tensor* and *asymmetric torsion tensor*.

The governing equations in this model become very involved and generate 6×6 matrix partial differential operator of second order.

Field Equations

Denote by $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ the displacement vector and the microrotation vector, respectively. Note that the microrotation vector ω in the micropolar elasticity theory is kinematically distinct from the macrorotation vector $\frac{1}{2}\,\mathrm{curl}\, u$.

In the linear theory of micropolar elasticity, the asymmetric strain tensor u_{pq} and asymmetric micro-strain (torsion-flexure) tensor ω_{pq} are defined as

$$u_{pq} = \partial_p u_q - \varepsilon_{pqk}\omega_k, \quad \omega_{pq} = \partial_p \omega_q, \quad p, q = 1, 2, 3 \tag{1}$$

while the force stress tensor σ_{pq} and the couple stress tensor μ_{pq} read as follows (the constitutive equations):

$$\begin{aligned}\sigma_{pq} &= (\mu+\alpha)u_{pq} + (\mu-\alpha)u_{qp} + \lambda u_{kk}\delta_{pq} \\ &\quad + (\kappa+\nu)\omega_{pq} + (\kappa-\nu)\omega_{qp} + \delta\omega_{kk}\delta_{pq} \\ &= (\mu+\alpha)\partial_p u_q + (\mu-\alpha)\partial_q u_p + \lambda\delta_{pq}\mathrm{div}\, u \\ &\quad + (\kappa+\nu)\partial_p\omega_q + (\kappa-\nu)\partial_q\omega_p \\ &\quad + \delta\,\delta_{pq}\mathrm{div}\omega - 2\alpha\varepsilon_{pqk}\omega_k\end{aligned} \tag{2}$$

$$\begin{aligned}\mu_{pq} &= (\kappa+\nu)u_{pq} + (\kappa-\nu)u_{qp} + \delta\, u_{kk}\delta_{pq} \\ &\quad + (\gamma+\varepsilon)\omega_{pq} + (\gamma-\varepsilon)\omega_{qp} + \beta\,\omega_{kk}\delta_{pq} \\ &= (\kappa+\nu)\partial_p u_q + (\kappa-\nu)\partial_q u_p + \delta\,\delta_{pq}\mathrm{div}\, u \\ &\quad + (\gamma+\varepsilon)\partial_p\omega_q + (\gamma-\varepsilon)\partial_q\omega_p + \beta\,\delta_{pq}\mathrm{div}\,\omega \\ &\quad - 2\nu\varepsilon_{qpk}\omega_k\end{aligned} \tag{3}$$

where summation over repeated indices is meant form one to three, $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, δ_{pq} is the Kronecker delta, ε_{pqk} is the permutation (Levi–Civitá) symbol, and $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \kappa$, and ε are the material constants [1, 2, 5].

The components of the force stress vector $\sigma^{(n)}$ and the couple stress vector $\mu^{(n)}$, acting on a surface element with a normal vector $n = (n_1, n_2, n_3)$, read as

$$\sigma_q^{(n)} = \sum_{p=1}^{3} \sigma_{pq} n_p,$$
$$\mu_q^{(n)} = \sum_{p=1}^{3} \mu_{pq} n_p, \quad q = 1,2,3$$

Let us introduce the generalized stress operator (6×6 matrix differential operator)

$$T(\partial, n) = \begin{bmatrix} T^{(1)}(\partial, n) & T^{(2)}(\partial, n) \\ T^{(3)}(\partial, n) & T^{(4)}(\partial, n) \end{bmatrix}_{6\times 6}, \qquad (4)$$
$$T^{(j)} = \left[T_{pq}^{(j)}\right]_{3\times 3}, \quad j = \overline{1,4}$$

where

$$T_{pq}^{(1)}(\partial, n) = (\mu+\alpha)\delta_{pq}\partial_n + (\mu-\alpha)n_q\partial_p + \lambda n_p \partial_q$$
$$T_{pq}^{(2)}(\partial, n) = (\kappa+\nu)\delta_{pq}\partial_n + (\kappa-\nu)n_q\partial_p + \delta n_p \partial_q - 2\alpha\, \varepsilon_{pqk} n_k$$
$$T_{pq}^{(3)}(\partial, n) = (\kappa+\nu)\delta_{pq}\partial_n + (\kappa-\nu)n_q\partial_p + \delta n_p \partial_q$$
$$T_{pq}^{(4)}(\partial, n) = (\gamma+\varepsilon)\delta_{pq}\partial_n + (\gamma-\varepsilon)n_q\partial_p + \beta n_p \partial_q - 2\nu\, \varepsilon_{pqk} n_k$$

Here ∂_n stands for the normal derivative. Then

$$\left(\sigma^{(n)}, \mu^{(n)}\right)^\top = T(\partial, n)\, U \quad \text{for} \quad U = (u, \omega)^\top$$

Basic equations of dynamics in the linear micropolar theory of elasticity have the form

$$\partial_p \sigma_{pq}(x,t) + \varrho F_q(x,t) = \varrho\, \ddot{u}_q(x,t), \; q = 1,2,3$$
$$\partial_p \mu_{pq}(x,t) + \varepsilon_{qlr}\sigma_{lr}(x,t) + \varrho G_q(x,t) = J\, \ddot{\omega}_q(x,t), \quad q = 1,2,3$$

where t is the time variable, $F = (F_1, F_2, F_3)^\top$ and $G = (G_1, G_2, G_3)^\top$ are the body force and body couple vectors, ϱ is the mass density of the elastic material, and J is a constant characterizing the so called spin torque corresponding to the microrotations (i.e., the moment of inertia per unit volume).

Using relations (2)–(3), the above dynamic equations can be rewritten in terms of the displacement and microrotation vectors,

$$(\mu+\alpha)\Delta u(x,t) + (\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div} u(x,t) + (\kappa+\nu)\Delta\omega(x,t) + (\delta+\kappa-\nu)\operatorname{grad}\operatorname{div}\omega(x,t) + 2\alpha\operatorname{curl}\omega(x,t) + \varrho F(x,t) = \varrho\, \ddot{u}(x,t)$$
$$(\kappa+\nu)\Delta u(x,t) + (\delta+\kappa-\nu)\operatorname{grad}\operatorname{div} u(x,t) + 2\alpha\operatorname{curl} u(x,t) + (\gamma+\varepsilon)\Delta\omega(x,t) + (\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\omega(x,t) + 4\nu\operatorname{curl}\omega(x,t) - 4\alpha\,\omega(x,t) + \varrho G(x,t) = J\,\ddot{\omega}(x,t)$$

where $\Delta \equiv \Delta(\partial) = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator.

If all the quantities involved in these equations are harmonic time dependent, i.e., $u(x,t) = u(x)\exp\{-it\tau\}$, $\omega(x,t) = \omega(x)\exp\{-it\tau\}$, $F(x,t) = F(x)\exp\{-it\tau\}$, $G(x,t) = G(x)\exp\{-it\tau\}$, with $\tau \in E^1$ and $i = \sqrt{-1}$, we obtain the *steady state oscillation equations*:

$$(\mu+\alpha)\Delta u(x) + (\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div} u(x) + (\kappa+\nu)\Delta\omega(x) + (\delta+\kappa-\nu)\operatorname{grad}\operatorname{div}\omega(x) + 2\alpha\operatorname{curl}\omega(x) + \varrho\,\tau^2 u(x) = -\varrho F(x)$$
$$(\kappa+\nu)\Delta u(x) + (\delta+\kappa-\nu)\operatorname{grad}\operatorname{div} u(x) + 2\alpha\operatorname{curl} u(x) + (\gamma+\varepsilon)\Delta\omega(x) + (\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\omega(x) + 4\nu\operatorname{curl}\omega(x) + (J\,\tau^2 - 4\alpha)\,\omega(x) = -\varrho G(x)$$

Here u, ω, F, and G are complex-valued vector functions, and τ is a frequency parameter.

If $\tau = \tau_1 + i\,\tau_2$ is a complex parameter with $\tau_2 \neq 0$, then the above equations are called the *pseudo-oscillation equations*, while for $\tau = 0$ they represent the *equilibrium equations of statics*:

$$
\begin{aligned}
&(\mu+\alpha)\Delta u(x)+(\lambda+\mu-\alpha)\text{grad}\,\text{div}\,u(x)\\
&\quad+(\kappa+\nu)\Delta\omega(x)+(\delta+\kappa-\nu)\text{grad}\,\text{div}\,\omega(x)\\
&\quad+2\alpha\ \text{curl}\ \omega(x)=-\varrho F(x)\\
&(\kappa+\nu)\Delta u(x)+(\delta+\kappa-\nu)\text{grad}\,\text{div}\,u(x)\\
&\quad+2\alpha\,\text{curl}\,u(x)+(\gamma+\varepsilon)\Delta\omega(x)\\
&\quad+(\beta+\gamma-\varepsilon)\text{grad}\,\text{div}\,\omega(x)+4\nu\ \text{curl}\,\omega(x)\\
&\quad-4\alpha\,\omega(x)=-\varrho G(x)
\end{aligned}
\tag{5}
$$

Introduce the matrix differential operator corresponding to the equilibrium system (5):

$$
A(\partial):=\begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial)\\ L^{(3)}(\partial) & L^{(4)}(\partial)\end{bmatrix}_{6\times 6}
$$

where

$$
\begin{aligned}
L^{(1)}(\partial)&=(\mu+\alpha)I_3\Delta+(\lambda+\mu-\alpha)Q(\partial)\\
L^{(2)}(\partial)&=L^{(3)}(\partial)=(\kappa+\nu)\Delta I_3+(\delta+\kappa-\nu)Q(\partial)\\
&\qquad+2\alpha R(\partial)\\
L^{(4)}(\partial)&=[(\gamma+\varepsilon)\Delta-4\alpha]I_3+(\beta+\gamma-\varepsilon)Q(\partial)\\
&\qquad+4\nu R(\partial)
\end{aligned}
$$

Here $R(\partial)=[-\varepsilon_{kjl}\partial_l]_{3\times3}$, $Q(\partial)=[\,\partial_k\partial_j\,]_{3\times3}$, and I_k stands for the $k\times k$ unit matrix.

The operator $A(\partial)$ is formally self-adjoint $A(\partial)=[A(-\partial)]^\top$ and strongly elliptic.

Due to the above notation, (5) can be rewritten in matrix form as

$$
\begin{aligned}
&A(\partial)U(x)=\Phi(x)\ U=(u,\omega)^\top,\\
&\Phi=(\Phi^{(1)},\Phi^{(2)})^\top=(-\varrho F(x),-\varrho G(x))^\top
\end{aligned}
$$

Green's Formulae

For real-valued vectors $U:=(u,\omega)^\top$, $U':=(u',\omega')^\top\in[C^2(\overline{\Omega})]^6$, the following Green formulae hold

$$
\begin{aligned}
&\int_\Omega [U'\cdot A(\partial)U+E(U',U)]\,dx\\
&\quad=\int_{\partial\Omega} U'\cdot T(\partial,n)U\,dS\\
&\int_\Omega [U'\cdot A(\partial)U-A(\partial)U'\cdot U]\,dx\\
&\quad=\int_{\partial\Omega}[U'\cdot T(\partial,n)U-T(\partial,n)U'\cdot U]\,dS
\end{aligned}
\tag{6}
$$

where $\Omega\subset E^3$ is a bounded domain with a smooth boundary manifold $\partial\Omega$, $\overline{\Omega}=\Omega\cup\partial\Omega$, n is the outward unit normal vector to $\partial\Omega$, $a\cdot b$ denotes the usual scalar product of two vectors $a,b\in E^m$: $a\cdot b=\sum_{j=1}^{m}a_j b_j$ and $E(\cdot,\cdot)$ is the so called *energy bilinear form*

$$
\begin{aligned}
E(U',U)=E(U,U')=\sum_{p,q=1}^{3}\{&(\mu+\alpha)\,u'_{pq}\,u_{pq}\\
&+(\mu-\alpha)\,u'_{pq}\,u_{qp}+(\kappa+\nu)(u'_{pq}\,\omega_{pq}+\omega'_{pq}\,u_{pq})\\
&+(\kappa-\nu)(u'_{pq}\,\omega_{qp}+\omega'_{pq}\,u_{qp})\\
&+(\gamma+\varepsilon)\,\omega'_{pq}\,\omega_{pq}+(\gamma-\varepsilon)\,\omega'_{pq}\,\omega_{qp}\\
&+\delta(u'_{pp}\,\omega_{qq}+\omega'_{qq}\,u_{pp})+\lambda\,u'_{pp}\,u_{qq}\\
&+\beta\,\omega'_{pp}\,\omega_{qq}\}
\end{aligned}
\tag{7}
$$

Sylvester's theorem gives the necessary and sufficient conditions for positive definiteness of the quadratic form $E(U,U)$ with respect to the variables (1) [7]

$$
\begin{aligned}
&\mu>0,\ \ \alpha>0,\ \ \gamma>0,\ \ \varepsilon>0,\ \ \lambda+2\mu>0,\\
&\mu\gamma-\varkappa^2>0,\ \ \alpha\varepsilon-\nu^2>0,\ (\lambda+\mu)(\beta+\gamma)\\
&-(\delta+)\varkappa^2>0,\ \ (3\lambda+2\mu)(3\beta+2\gamma)\\
&-(3\delta+2)\varkappa^2>0,\ \mu[(\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2]\\
&+(\lambda+\mu)(\mu\gamma-\varkappa^2)>0,\ \mu[(3\lambda+2\mu)(3\beta+2\gamma)\\
&-(3\delta+2)\varkappa^2]+(3\lambda+2\mu)(\mu\gamma-\varkappa^2)>0
\end{aligned}
$$

If in addition the inequality $3\lambda+2\mu>0$ is fulfilled, which is very natural in the classical elasticity, then the above conditions are equivalent to the following simultaneous inequalities:

$$\mu > 0,\ \ \alpha > 0,\ \ \gamma > 0,\ \ \varepsilon > 0,\ \ 3\lambda + 2\mu > 0,$$
$$\mu\gamma - \varkappa^2 > 0,\ \ \alpha\varepsilon - \nu^2 > 0$$
$$(\mu + \alpha)(\gamma + \varepsilon) - (\varkappa + \nu)^2 > 0,$$
$$(3\lambda + 2\mu)(3\beta + 2\gamma) - (3\delta + 2)\varkappa^2 > 0$$

From (7) and (1) it follows that

$$\begin{aligned}E(U,U') =& \frac{3\lambda+2\mu}{3}\left(\operatorname{div} u + \frac{3\delta+2\kappa}{3\lambda+2\mu}\operatorname{div}\omega\right) \times \left(\operatorname{div} u' + \frac{3\delta+2\kappa}{3\lambda+2\mu}\operatorname{div}\omega'\right)\\ &+\frac{1}{3}\left(3\beta+2\gamma-\frac{(3\delta+2\kappa)^2}{3\lambda+2\mu}\right)(\operatorname{div}\omega)(\operatorname{div}\omega')\\ &+\frac{\mu}{2}\sum_{k,j=1,k\neq j}^{3}\left[\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\kappa}{\mu}\left(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right)\right] \times \left[\frac{\partial u'_k}{\partial x_j}+\frac{\partial u'_j}{\partial x_k}+\frac{\kappa}{\mu}\left(\frac{\partial \omega'_k}{\partial x_j}+\frac{\partial \omega'_j}{\partial x_k}\right)\right]\\ &+\frac{\mu}{3}\sum_{k,j=1}^{3}\left[\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\kappa}{\mu}\left(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\right)\right] \times \left[\frac{\partial u'_k}{\partial x_k}-\frac{\partial u'_j}{\partial x_j}+\frac{\kappa}{\mu}\left(\frac{\partial \omega'_k}{\partial x_k}-\frac{\partial \omega'_j}{\partial x_j}\right)\right]\\ &+\left(\gamma-\frac{\kappa^2}{\mu}\right)\sum_{k,j=1,k\neq j}^{3}\left[\frac{1}{2}\left(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right) \times \left(\frac{\partial {\omega'}_k}{\partial x_j}+\frac{\partial {\omega'}_j}{\partial x_k}\right)+\frac{1}{3}\left(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\right) \times \left(\frac{\partial {\omega'}_k}{\partial x_k}-\frac{\partial {\omega'}_j}{\partial x_j}\right)\right]\\ &+\alpha\left(\operatorname{curl} u +\frac{\nu}{\alpha}\operatorname{curl}\omega - 2\omega\right)\cdot \times \left(\operatorname{curl} u' +\frac{\nu}{\alpha}\operatorname{curl}\omega' - 2\omega'\right)\\ &+\left(\varepsilon-\frac{\nu^2}{\alpha}\right)\operatorname{curl}\omega\cdot\operatorname{curl}\omega'\end{aligned}$$

In particular,

$$\begin{aligned}E(U,U) =& \frac{3\lambda+2\mu}{3}\left(\operatorname{div} u + \frac{3\delta+2\kappa}{3\lambda+2\mu}\operatorname{div}\omega\right)^2\\ &+\frac{1}{3}\left(3\beta+2\gamma-\frac{(3\delta+2\kappa)^2}{3\lambda+2\mu}\right)(\operatorname{div}\omega)^2\\ &+\frac{\mu}{2}\sum_{k,j=1,k\neq j}^{3}\left[\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\kappa}{\mu}\left(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right)\right]^2\\ &+\frac{\mu}{3}\sum_{k,j=1}^{3}\left[\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\kappa}{\mu}\left(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x}\right)\right]^2\\ &+\left(\gamma-\frac{\kappa^2}{\mu}\right)\sum_{k,j=1,k\neq j}^{3}\left[\frac{1}{2}\left(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right)^2 +\frac{1}{3}\left(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\right)^2\right]+\left(\varepsilon-\frac{\nu^2}{\alpha}\right)(\operatorname{curl}\omega)^2\\ &+\alpha\left(\operatorname{curl} u+\frac{\nu}{\alpha}\operatorname{curl}\omega-2\omega\right)^2\end{aligned}$$

Theorem 1. *Let* $U = (u,\omega)^\top \in [C^1(\Omega)]^6$ *be a real-valued vector and* $E(U,U) = 0$ *in* Ω. *Then* $U = (u,\omega)^\top$ *is a generalized rigid displacement vector with*

$$u(x) = [a \times x] + b, \qquad \omega(x) = a, \quad x \in \Omega \tag{8}$$

where a *and* b *are arbitrary three-dimensional constant vectors.*

If $U = U^{(1)} + i\,U^{(2)}$ is a complex-valued vector, where $U^{(j)} = (u^{(j)}, \omega^{(j)})^\top$, $j = 1,2$, are real-valued vectors, then due to the positive definiteness of the energy form for real-valued vector functions, we have

$$E(U,\overline{U}) \geq c_0 \sum_{p,q=1}^{3}\left[(u_{pq}^{(1)})^2 + (u_{pq}^{(2)})^2 + (\omega_{pq}^{(1)})^2 + (\omega_{pq}^{(2)})^2\right]$$

where c_0 is a positive constant depending only on the material parameters, and $u_{pq}^{(j)}$ and $\omega_{pq}^{(j)}$ are defined by formulae (1) with $u^{(j)}$ and $\omega^{(j)}$ for u and ω.

From the positive definiteness of the energy form $E(\cdot,\cdot)$ with respect to the variables (1), it easily follows that there exist positive constants c_1 and c_2 such that for an arbitrary real-valued vector $U \in [C^1(\overline{\Omega})]^6$,

$$\mathcal{B}(U,U) = \int_\Omega E(U,U)dx$$
$$\geq c_1 \int_\Omega \left\{ \sum_{p,q=1}^{3} [(\partial_p u_q)^2 + (\partial_p \omega_q)^2] + \sum_{p=1}^{3} (u_p^2 + \omega_p^2) \right\} dx$$
$$- c_2 \int_\Omega \sum_{p=1}^{3} (u_p^2 + \omega_p^2)\, dx$$

i.e., the following Korn's type inequality holds

$$\mathcal{B}(U,U) \geq c_1 \, ||U||^2_{[H_2^1(\Omega)]^6} - c_2 \, ||U||^2_{[H_2^0(\Omega)]^6}$$

where $||\cdot||_{[H_2^s(\Omega)]^6}$ denotes the norm in the Bessel potential space $[H_2^s(\Omega)]^6$.

By standard approach, Green's formula (6) can be extended to Lipschitz domains and to the case of vector functions $U \in [W_2^1(\Omega)]^6$, $A(\partial)U \in [L_2(\Omega)]^6$, and $U' \in [W_2^1(\Omega)]^6$

$$\int_\Omega [U' \cdot A(\partial)U + E(U',U)]\, dx = \langle U', T(\partial,n)U \rangle_{\partial\Omega} \tag{9}$$

where $\langle \cdot,\cdot \rangle_{\partial\Omega}$ denotes the duality between the Bessel potential spaces $[H_2^{\frac{1}{2}}(\partial\Omega)]^6$ and $[H_2^{-\frac{1}{2}}(\partial\Omega)]^6$, which extends the usual L_2-inner product. The functional $T(\partial,n)U \in [H_2^{-\frac{1}{2}}(\partial\Omega)]^6$ is determined by the relation (9).

Boundary Value Problems and Uniqueness Theorems

Let $\Omega^+ \subset E^3$ be a bounded domain with a smooth boundary manifold $\partial\Omega^+$ and $\Omega^- = E^3 \setminus \overline{\Omega^+}$. Denote by n the outward unit normal vector to $\partial\Omega^+$. The symbols $[\cdot]^\pm$ denote the interior and exterior limits on $\partial\Omega^\pm$ from $\Omega^\pm$.

In the hemitropic elasticity, the basic boundary value problems of statics are formulated as follows. Find a solution $U \in (u,\omega)^\top$ to the differential equation

$$A(\partial)U(x) = \Phi(x) \;\; \text{in} \;\; \Omega^\pm \tag{10}$$

satisfying one of the following boundary conditions on $S = \partial\Omega^\pm$:

Problem $(I)^\pm$ (the Dirichlet-type problem)

$$[U(x)]^\pm = f(x), \quad x \in S \tag{11}$$

Problem $(II)^\pm$ (the Neumann-type problem)

$$[T(\partial,n)U(x)]^\pm = F(x), \quad x \in S \tag{12}$$

Problem $(III)^\pm$ (a mixed-type problem)

$$[U(x)]^\pm = f_D(x), \quad x \in S_D \tag{13}$$

$$[T(\partial,n)U(x)]^\pm = F_N(x), \quad x \in S \tag{14}$$

Problem $(IV)^\pm$ (the Robin-type problem)

$$[T(\partial,n)U(x)]^\pm \pm \chi(x)\, [U(x)]^\pm = F(x), \quad x \in S \tag{15}$$

where S_D and S_N are two open disjoint parts of S and $\overline{S_D} \cup \overline{S_N} = S$, and $\chi(x)$ is a given smooth nonnegative function on S that does not vanish identically.

In addition, in the case of exterior problems, the following decay conditions at infinity should be satisfied

$$u(x) = \mathcal{O}(|x|^{-1}), \;\; \partial_j u(x) = \mathcal{O}(|x|^{-2}), \;\; j = 1,2,3$$
$$\omega(x) = \mathcal{O}(|x|^{-2}), \;\; \partial_j \omega(x) = \mathcal{O}(|x|^{-3}), \;\; j = 1,2,3 \tag{16}$$

as $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2} \to +\infty$.

In the case of regular setting, $U \in [C^1(\overline{\Omega^\pm})]^6 \cap [C^2(\Omega^\pm)]^6$ and the vector function Φ and the boundary data f and F belong to some Hölder spaces

$$\Phi \in [C^{0,a}(\overline{\Omega^{\pm}})]^6,\ f \in [C^{1,a}(S)]^6,\ F \in [C^{0,a}(S)]^6,\quad 0 < a \le 1$$

while in the case of weak formulation, $U \in [W_2^1(\Omega^+)]^6$ or $U \in [W_{2,loc}^1(\Omega^-)]^6$ and the data belong to the corresponding Sobolev–Slobodetskii or Bessel potential spaces

$$f \in [H_2^{\frac{1}{2}}(S)]^6,\ \ F \in [H_2^{-\frac{1}{2}}(S)]^6,\ \ f_D \in [H_2^{\frac{1}{2}}(S_D)]^6$$
$$F_N \in [H_2^{-\frac{1}{2}}(S_N)]^6,\ \ \Phi \in [L_2(E^3)]^6$$

where supp Φ is compact.

Note that in the case of weak setting, the differential equation (10) is understood in the distributional sense, and the Dirichlet-type conditions (11) and (13) are understood in the usual trace sense, while the Neumann-type conditions (12) and (14) and the Robin-type condition (15) are understood in the functional sense with $[T(\partial, n)U]^{\pm} \in [H_2^{-\frac{1}{2}}(S)]^6$ defined in (9).

Theorem 2. *The homogeneous versions of the BVPs $(I)^{\pm}$, $(II)^{-}$, $(III)^{\pm}$, and $(IV)^{\pm}$ have only the trivial solution in the space of regular vector functions satisfying the decay conditions (16), while the homogeneous problem $(II)^{+}$ has the vector (8) as a general solution.*

The same uniqueness results hold also true for Lipschitz domains and for weak solutions $U \in [W_2^1(\Omega^+)]^6$ or $U \in [W_{2,loc}^1(\Omega^-)]^6$ satisfying the decay conditions (16).

Fundamental Matrix and Potentials

The fundamental matrix $\Gamma(x)$ of the differential operator $A(\partial)$ reads as follows [7]:

$$\Gamma(x) = \left\| \begin{matrix} \Gamma^{(1)}(x) & \Gamma^{(2)}(x) \\ \Gamma^{(3)}(x) & \Gamma^{(4)}(x) \end{matrix} \right\|_{6\times 6},$$
$$\Gamma^{(m)}(x) = \| \Gamma_{kj}^{(m)}(x) \|_{3\times 3},\quad m = \overline{1,4}$$

where

$$\begin{aligned}
\Gamma_{kj}^{(1)}(x) = -\frac{1}{4\pi}\Bigg\{ & \left[\frac{\gamma+\varepsilon}{d_1}\frac{1}{|x|} + \sum_{l=1}^{2} k_l^2 c_{1l}\frac{e^{-k_l|x|}-1}{|x|}\right]\delta_{kj} \\
& -\frac{\partial^2}{\partial x_k \partial x_j}\left[\frac{\lambda+\mu}{\mu(\lambda+2\mu)}\frac{|x|}{2} + \sum_{l=1}^{3} c_{1l}\frac{e^{-k_l|x|}-1}{|x|}\right] \\
& + \sum_{l,p=1}^{3} c_{2l}\,\varepsilon_{kjp}\frac{\partial}{\partial x_p}\frac{e^{-k_l|x|}-1}{|x|}\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{kj}^{(2)}(x) = \Gamma_{kj}^{(3)}(x) = -\frac{1}{4\pi}\Bigg\{ & -\left[\frac{\kappa+\nu}{d_1}\frac{1}{|x|} + \sum_{l=1}^{2} k_l^2 c_{3l}\frac{e^{-k_l|x|}-1}{|x|}\right]\delta_{kj} \\
& + \frac{\partial^2}{\partial x_k \partial x_j}\sum_{l=1}^{3} c_{3l}\frac{e^{-k_l|x|}-1}{|x|} \\
& + \sum_{l,p=1}^{3} c_{4l}\,\varepsilon_{kjp}\frac{\partial}{\partial x_p}\frac{e^{-k_l|x|}-1}{|x|}\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{kj}^{(4)}(x) = -\frac{1}{4\pi}\Bigg\{ & \left[\frac{\mu+\alpha}{d_1}\frac{1}{|x|} + \sum_{l=1}^{2} k_l^2 c_{5l}\frac{e^{-k_l|x|}-1}{|x|}\right]\delta_{kj} \\
& - \frac{\partial^2}{\partial x_k \partial x_j}\sum_{l=1}^{3} c_{5l}\frac{e^{-k_l|x|}-1}{|x|} \\
& + \sum_{l,p=1}^{3} c_{6l}\,\varepsilon_{kjp}\frac{\partial}{\partial x_p}\frac{e^{-k_l|x|}-1}{|x|}\Bigg\}
\end{aligned}$$

Here

$$\begin{aligned}
c_{1l} &= \frac{c_l}{d_1}(k_l^2 - k_3^2)\{[(\gamma+\varepsilon)k_l^2 - 4\alpha](k_l^2 - \lambda_1^2) + 4\nu d_3 k_l^2\} \\
c_{2l} &= \frac{c_l k_l^2}{d_1}(k_l^2 - k_3^2)\left\{[d_3(\gamma+\varepsilon) - 4\nu]k_l^2 + \frac{16\alpha^2\kappa}{d_1}\right\} \\
c_{3l} &= \frac{c_l k_l^2}{d_1}(k_l^2 - k_3^2)[(k_l^2 - \lambda_1^2)(\kappa+\nu) - 2\alpha d_3] \\
c_{4l} &= \frac{c_l k_l^2}{d_1}(k_l^2 - k_3^2)[2\alpha(k_l^2 - \lambda_1^2) - d_3 k_l^2(\kappa+\nu)] \\
c_{5l} &= \frac{c_l k_l^2}{d_1}(\mu+\alpha)(k_l^2 - k_3^2)(k_l^2 - \lambda_1^2),
\end{aligned}$$

$$c_{6l} = \frac{c_l\, d_3\, k_l^4}{d_1}\,(\mu+\alpha)\,(k_l^2 - k_3^2),\ l = 1,2$$
$$c_{13} = -\frac{(\delta+2\,\kappa)^2}{4\,\alpha\,(\lambda+2\,\mu)},\ c_{33} = -\frac{\delta+2\,\kappa}{4\,\alpha\,(\lambda+2\,\mu)^2},$$
$$c_{53} = -\frac{1}{4\,\alpha},\ c_{23} = c_{43} = c_{63} = 0$$
$$c_q = \frac{1}{k_q^4\,(k_{q+1}^2 - k_q^2)\,(k_{q+2}^2 - k_q^2)},\ q = 1,2,3$$
$$k_1 = \overline{k}_2,\quad \mathrm{Re}\,k_1 > 0,\quad k_1^2 + k_2^2 = 2\,\lambda_1^2 - d_3^2,$$
$$k_1^2\,k_2^2 = \lambda_1^4,\ \ k_4 = k_1,\ \ k_5 = k_2$$
$$d_1 = (\mu+\alpha)(\gamma+\varepsilon) - (\kappa+v)^2 > 0,$$
$$d_2 = (\lambda+2\mu)(\beta+2\gamma) - (\delta+2\kappa)^2 > 0$$
$$d_3 = \frac{4\,(\mu\,v - \alpha\,\kappa)}{d_1},\quad \lambda_1^2 = \frac{4\,\alpha\,\mu}{d_1},\quad k_3^2 = \frac{4\,\alpha\,(\lambda+2\,\mu)}{d_2}$$

The matrix $\Gamma(x-y)$ solves the distributional equation $A(\partial)\Gamma(x-y) = I_6\,\delta(x-y)$, where $\delta(\cdot)$ is Dirac's delta distribution. Note that $\Gamma(y-x) = [\,\Gamma(x-y)\,]^\top$.

With the help of the relations

$$\sum_{l=1}^{3} c_l\,k_l^2 = \lambda^{-4}\,k_3^{-2},\quad \sum_{l=1}^{3} c_l\,k_l^{2m} = 0$$
$$\text{for}\quad m = 2,3,\quad \sum_{l=1}^{3} c_l\,k_l^8 = 1$$
$$\sum_{l=1}^{3} c_{1l}\,k_l^2 = -\frac{\beta+2\,\gamma}{d_2} + \frac{\gamma+\varepsilon}{d_1} + \frac{\lambda+\mu}{\mu\,(\lambda+2\,\mu)}$$
$$\sum_{l=1}^{3} c_{3l}\,k_l^2 = -\frac{\delta+2\,\kappa}{d_2} + \frac{v+\kappa}{d_1},$$
$$\sum_{l=1}^{3} c_{5l}\,k_l^2 = -\frac{\lambda+2\,\mu}{d_2} + \frac{\mu+\alpha}{d_1}$$

it can be checked that for sufficiently large $|x|$ and for arbitrary multi-index $s = (s_1, s_2, s_3)$, the following asymptotic relations hold

$$\partial^s\,\Gamma_{kj}(x) = \begin{cases} \mathcal{O}(|x|^{-1-|s|}) & \text{for}\quad k,j = 1,2,3, \\ \mathcal{O}(|x|^{-2-|s|}) & \text{for either}\quad k \ge 4 \\ & \text{or}\quad j \ge 4, \\ & |s| = s_1 + s_2 + s_3 \end{cases}$$

The corresponding single-layer and double-layer potentials and the Newton-type volume potential read as

$$V(\varphi)(x) = \int_S \Gamma(x-y)\,\varphi(y)\,dS_y,\quad x \in E^3 \setminus S$$
$$W(\varphi)(x) = \int_S [T(\partial_y, n(y))\,\Gamma(y-x)]^\top\,\varphi(y)\,dS_y,$$
$$x \in E^3 \setminus S$$
$$N_\Omega(\psi)(x) = \int_\Omega \Gamma(x-y)\,\psi(y)\,dy,\quad x \in E^3 \qquad (17)$$

where $T(\partial, n)$ is the generalized stress operator given by (4), $\varphi = (\varphi_1, \cdots, \varphi_6)^\top$ is a density vector function defined on $S = \partial\Omega$, while a density vector function $\psi = (\psi_1, \cdots, \psi_6)^\top$ is defined on $\Omega \in \{\Omega^+, \Omega^-\}$.

The layer potentials introduced above, and generated by them, boundary integral operators have the following jump and mapping properties (for details see [10] and [12]).

Theorem 3. *Let* $U \in [W_2^1(\Omega^+)]^6$ *with* $A(\partial)U \in [L_2(\Omega^+)]^6$. *Then the following integral representation formula holds*

$$W([U]^+)(x) - V([TU]^+)(x) + N_{\Omega^+}(A(\partial)U)(x) = \begin{cases} U(x) & \text{for } x \in \Omega^+ \\ 0 & \text{for } x \in \Omega^- \end{cases}$$

Theorem 4. *Let* $S \in C^{k+1,a}$ *where* $k \ge 0$ *is an integer*, $0 < a \le 1$, *and let* $0 < b < a$. *Then the operators*

$$V\ :\ [C^{k,b}(S)]^6 \to [C^{k+1,b}(\overline{\Omega^\pm})]^6,$$
$$W\ :\ [C^{k,b}(S)]^6 \to [C^{k,b}(\overline{\Omega^\pm})]^6 \qquad (18)$$

are continuous.

For any $g \in C^{k,b}(S)$ *and any* $x \in S$

$$[V(g)(x)]^\pm = V(g)(x) = \mathcal{H}\,g(x) \qquad (19)$$

$$[T(\partial_x, n(x))V(g)(x)]^{\pm} = [\mp 2^{-1} I_6 + \mathcal{K}]\, g(x) \tag{20}$$

$$[W(g)(x)]^{\pm} = [\pm 2^{-1} I_6 + \mathcal{K}^*]\, g(x) \tag{21}$$

$$[T(\partial_x, n(x))W(g)(x)]^{+} = [T(\partial_x, n(x))W(g)(x)]^{-} = \mathcal{L}\, g(x) \tag{22}$$

where

$$\mathcal{H}\, g(x) := \int_S \Gamma(x-y)\, g(y)\, dS_y$$
$$\mathcal{K}\, g(x) := \int_S T(\partial_x, n(x))\Gamma(x-y)\, g(y)\, dS_y$$
$$\mathcal{K}^*\, g(x) := \int_S [T(\partial_y, n(y))\Gamma(y-x)]^{\top}\, g(y)\, dS_y$$
$$\mathcal{L}\, g(x) := \lim_{\Omega^{\pm}\ni z\to x\in S} T(\partial_z, n(x))\, W(g)(z)$$

The operators V and W in (18) can be extended to the continuous mappings

$$V : [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^1(\Omega^+)]^6 \quad \left[[H_2^{-\frac{1}{2}}(S)]^6 \to [H^1_{2,\,loc}(\Omega^-)]^6 \right]$$
$$W : [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^1(\Omega^+)]^6 \quad \left[[H_2^{\frac{1}{2}}(S)]^6 \to [H^1_{2,\,loc}(\Omega^-)]^6 \right]$$

The jump relations (19)–(22) on S remain valid for the extended operators in the corresponding functional spaces.

Denote by $X_\Omega\{\Lambda^{(1)}, \Lambda^{(2)}, \cdots, \Lambda^{(6)}\}$ the linear span of vectors of generalized rigid displacements in a region Ω, where, for definiteness, we assume that

$$\begin{aligned}
\Lambda^{(1)}(x) &= (0, -x_3, x_2, 1, 0, 0)^{\top}\\
\Lambda^{(2)}(x) &= (x_3, 0, -x_1, 0, 1, 0)^{\top}\\
\Lambda^{(3)}(x) &= (-x_2, x_1, 0, 0, 0, 1)^{\top}\\
\Lambda^{(4)}(x) &= (1, 0, 0, 0, 0, 0)^{\top}\\
\Lambda^{(5)}(x) &= (0, 1, 0, 0, 0, 0)^{\top}\\
\Lambda^{(6)}(x) &= (0, 0, 1, 0, 0, 0)^{\top}
\end{aligned} \tag{23}$$

The restriction of the space $X_\Omega\{\Lambda^{(1)}, \Lambda^{(2)}, \cdots, \Lambda^{(6)}\}$ onto the boundary $S = \partial\Omega$ we denote by $X_S\{\Lambda^{(1)}, \Lambda^{(2)}, \cdots, \Lambda^{(6)}\}$. The system $\{\Lambda^{(j)}(x)\}_{j=1}^6$ is basis in the space of generalized rigid displacement vectors defined by (8).

Theorem 5. *Let S, k, a, and b be as in Theorem 4. Then the operators*

$$\mathcal{H} : [C^{k,b}(S)]^6 \to [C^{k+1,b}(S)]^6 \quad \left[[H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6 \right]$$
$$\mathcal{K} : [C^{k,b}(S)]^6 \to [C^{k,b}(S)]^6 \quad \left[[H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6 \right]$$
$$\mathcal{K}^* : [C^{k,b}(S)]^6 \to [C^{k,b}(S)]^6 \quad \left[[H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6 \right]$$
$$\mathcal{L} : [C^{k+1,b}(S)]^6 \to [C^{k,b}(S)]^6 \quad \left[[H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6 \right]$$

are bounded. Moreover,

(a) $\mathcal{H}$, $\pm\frac{1}{2} I_6 + \mathcal{K}$, $\pm\frac{1}{2} I_6 + \mathcal{K}^*$, *and* $\mathcal{L}$ *are elliptic pseudodifferential operators of order* -1, 0, 0, *and* 1, *respectively;*

(b) $\pm\frac{1}{2} I_6 + \mathcal{K}$ *and* $\pm\frac{1}{2} I_6 + \mathcal{K}^*$ *are mutually adjoint singular integral operators of normal type with index equal to zero. The operators* $\mathcal{H}$, $\frac{1}{2} I_6 + \mathcal{K}$, *and* $\frac{1}{2} I_6 + \mathcal{K}^*$ *are invertible. The inverse of* $\mathcal{H}$

$$\mathcal{H}^{-1} : [C^{k+1,b}(S)]^6 \to [C^{k,b}(S)]^6 \quad \left[[H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6 \right]$$

is a singular integro-differential operator.

The null space of the operator $-\frac{1}{2} I_6 + \mathcal{K}^*$ *is* $X_S\{\Lambda^{(1)}, \Lambda^{(2)}, \cdots, \Lambda^{(6)}\}$;

(c) $\mathcal{L}$ is a singular integro-differential operator and the following equalities hold:

$$\mathcal{K}^*\mathcal{H} = \mathcal{H}\mathcal{K}, \quad \mathcal{L}\mathcal{K}^* = \mathcal{K}\mathcal{L}$$
$$\mathcal{H}\mathcal{L} = -4^{-1} I_6 + (\mathcal{K}^*)^2$$
$$\mathcal{L}\mathcal{H} = -4^{-1} I_6 + \mathcal{K}^2$$

(d) *The operators* $-\mathcal{H}$ *and* $\mathcal{L}$ *are self-adjoint and nonnegative elliptic pseudodifferential operators with positive definite principal symbol matrices and with index equal to zero. Moreover,* $\langle -\mathcal{H}h,\ h\rangle_s \ge c_0 \,\|\, h \,\|_{[H_2^{-\frac{1}{2}}(S)]^6}$ *for all* $h \in [H_2^{-\frac{1}{2}}(S)]^6$ *and* $\langle g, \mathcal{L}\, g\rangle_s \ge 0$ *for all* $g \in [H_2^{\frac{1}{2}}(S)]^6$ *with equality only for*

$$g = ([a \times x] + b,\ a)^{\top} \tag{24}$$

where $a, b \in E^3$ *are arbitrary constant vectors; here* $< \cdot, \cdot >_s$ *denotes the duality between the spaces* $[H_2^{\frac{1}{2}}(S)]^6$ *and* $[H_2^{-\frac{1}{2}}(S)]^6$*;*

(e) *a general solution of the homogeneous equations* $[-\frac{1}{2}I_6 + \mathcal{K}^*]\, g = 0$ *and* $\mathcal{L}\, g = 0$ *is given by* (24), *implying that the operators* $\mathcal{L}$, $-\frac{1}{2}I_6 + \mathcal{K}^*$, *and* $-\frac{1}{2}I_6 + \mathcal{K}$ *have six-dimensional null spaces.*

Existence Results

From the mathematical point of view, without loss of generality, it can be assumed that $\Phi = 0$ in (10), since corresponding particular solutions $U_{\pm}^{(0)}$ can be written explicitly as volume potentials (see (17))

$$U_{\pm}^{(0)}(x) := N_{\Omega^{\pm}}(\Phi)(x) = \int_{\Omega^{\pm}} \Gamma(x-y)\,\Phi(y)\,dy, \quad x \in \Omega^{\pm}$$

Let

$$S = \partial\Omega^{\pm} \in C^{k,a}, \quad f \in C^{1,b}(S), \quad F \in C^{0,b}(S), \quad 0 < b < a \le 1, \quad k \ge 2 \tag{25}$$

Theorem 6. *Let S and f be as in* (25) *with* $k = 2$. *Then Problem* $(I)^+$ *with* $\Phi = 0$ *is uniquely solvable in the space of regular vector functions. Moreover, the solution belongs to the space* $[C^{1,b}(\overline{\Omega^+})]^6 \cap [C^{\infty}(\Omega^+)]^6$, *and it can be represented by the double-layer potential*

$$U(x) = W(g)(x), \quad x \in \Omega^+$$

where the density vector $g \in [C^{1,b}(S)]^6$ is defined by the uniquely solvable singular integral equation

$$\left[2^{-1}I_6 + \mathcal{K}^*\right] g(x) = f(x), \quad x \in S$$

Theorem 7. *Let S and f be as in* (25) *with* $k = 2$. *Then Problem* $(I)^-$ *with* $\Phi = 0$ *is uniquely solvable in the space of regular vector functions satisfying the decay conditions* (16). *Moreover, the solution belongs to the space* $[C^{1,b}(\overline{\Omega^-})]^6 \cap [C^{\infty}(\Omega^-)]^6$, *and it can be represented by a linear combination of the single and double-layer potentials*

$$U(x) = W(g)(x) + V(g)(x), \quad x \in \Omega^-$$

where the density vector $g \in [C^{1,b}(S)]^6$ *is defined by the uniquely solvable integral equation*

$$\left[-2^{-1}I_6 + \mathcal{K}^* + \mathcal{H}\right] g(x) = f(x), \quad x \in S$$

Theorem 8. *Let S and F be as in* (25) *with* $k = 1$. *Then Problem* $(II)^-$ *is uniquely solvable in the space of regular vector functions. Moreover, the solution belongs to the space* $[C^{1,b}(\overline{\Omega^-})]^6 \cap [C^{\infty}(\Omega^-)]^6$, *and it can be represented by the single-layer potential*

$$U(x) = V(h)(x), \quad x \in \Omega^-$$

where the density vector $h \in C^{0,b}(S)$ *is defined by the uniquely solvable integral equation*

$$\left[2^{-1}I_6 + \mathcal{K}\right] h(x) = F(x), \quad x \in S$$

Theorem 9. *The interior Neumann problem* $(II)^+$ *is solvable if and only if*

$$\int_S F(x) \cdot \Lambda^{(j)}(x)\,dS = 0, \quad j = \overline{1,6}$$

where $\Lambda^{(j)}$ are defined in (23) and solutions can be represented by the single-layer potential

$$U(x) = V(h)(x), \quad x \in \Omega^+$$

where the density vector $h \in C^{1,b}(S)$ solves the integral equation

$$\left[-2^{-1}I_6 + \mathcal{K}\right] h(x) = F(x), \qquad x \in S$$

A solution vector U is defined modulo a rigid displacement, while the generalized stress vector TU is determined uniquely.

Similar existence results hold also true for weak solutions in smooth and Lipschitz domains (see [9–11], and [12]).

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Boundary Variational Functional

► Variational Formulation and Nonsmooth Optimization Algorithms in Elastostatic Contact Problems for Cracked Body

Boundary–Initial Value Problems

► Boundary–Initial Value Problems of Thermoelastodynamics

Boundary–Initial Value Problems of Thermoelastodynamics

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Synonyms

Boundary–initial value problems; Linear theory; Thermoelasticity

Overview

In this work, we formulate boundary–initial value problems of the linear thermoelasticity. We formulate the forward in time-coupled problem and the backward in time-coupled problem.

Moreover, we consider the uncoupled problem of thermoelasticity, the problem of the quasi-static theory, and the problem of the equilibrium theory.

Our analysis is based on the works by Truesdell and Noll [9], Truesdell [10], Nowacki [8], Carlson [4], Eringen [5], and Hetnarski and Eslami [7]. The history of the thermoelasticity is fully discussed by Truesdell [10] and Hetnarski and Eslami [7] (see also the reference list of the work by Carlson [4]).

Boundary–Initial Value Problems

In this work, we consider a thermoelastic material which at time $t_0 = 0$ occupies the region B of the three-dimensional Euclidian space E^3, whose boundary is the smooth surface ∂B. In the following, the configuration of the body at the initial time $t_0 = 0$ is considered as the reference configuration.

Throughout this chapter, Latin subscripts take the values $1, 2, 3$, and summation is carried out over repeated indices. Typical conventions for differential operations are implied such as a superposed dot or comma, followed by a subscript to denote the partial derivative with respect to time or to the corresponding Cartesian coordinate, respectively.

We refer the configurations of the body to a fixed system of rectangular axes. In the rest of this chapter, $\mathbf{x}$ denotes the position vector with the components (x_1, x_2, x_3) of a generic point P of the domain B.

Let us consider a fixed time interval $[0, t_1)$, where $t_1 > 0$ can be infinite. Considered two positive integers M and N, we say that a function f defined on $B \times (0, t_1)$ is of class $C^{M,N}$ if the functions

$$\partial^m f^{(n)} \equiv \frac{\partial^m}{\partial x_p \partial x_q \ldots \partial x_k}\left(\frac{\partial^n f}{\partial t^n}\right)$$
$$m \in \{0, 1, \ldots M\}, n \in \{0, 1, \ldots, N\}$$
$$m + n \leq \max\{M, N\}$$

exist and are continuous on $B \times (0, t_1)$.

We denote by T_0 the absolute temperature in the reference configuration, and we suppose that T_0 is a prescribed positive constant. We also suppose that in the natural state, the body is free of initial stresses and entropy.

We assume that the components u_i of the displacement vector are of class C^2 on $B \times (0, t_1)$, while we assume that the variation of temperature θ is of class $C^{2,1}$ on $B \times (0, t_1)$ and continuous together with $\dot{\theta}$ and $\theta_{,i}$ on $\overline{B} \times [0, t_1)$.

In the linear theory, we suppose that $\mathbf{u} = \varepsilon \mathbf{u}'$ and $\theta = \varepsilon \theta'$ where ε is a constant small enough to have $\varepsilon^n \simeq 0$, for $n \geq 2$, and $\mathbf{u}'$ and θ' are independent of ε. We consider the components of the infinitesimal strain tensor e_{ij} given by

$$2e_{ij} = u_{i,j} + u_{j,i} \tag{1}$$

In the following, we denote the stress tensor and the heat flux by σ_{ij} and q_i, respectively. Moreover, we will use the following notations:

i) ρ_0 is the mass density of the continuum at the initial time.
ii) S is the entropy per unit mass.
iii) $\mathbf{b}$ is the body force per unit mass.
iv) r is the heat supply per unit mass.

The equations of the linear theory of thermoelasticity consist of (see [4])

– The equations of motion

$$\sigma_{ji,j} + \rho_0 b_i = \rho_0 \ddot{u}_i \tag{2}$$

– The energy equation

$$\rho_0 T_0 \dot{S} + q_{i,i} = \rho_0 r \tag{3}$$

– The constitutive equations

$$\begin{aligned} \sigma_{ij} &= C_{ijkl} e_{kl} - M_{ij} \theta \\ \rho_0 S &= M_{ij} e_{ij} + a\theta \\ q_i &= -k_{ij} \theta_{,j} \end{aligned} \tag{4}$$

– The geometrical equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{5}$$

The constitutive coefficients $C_{ijkl}, M_{ij}, k_{ij}, \rho_0$ and a are depending on the spatial variables x_1, x_2, x_3, and they have the following properties of symmetry:

$$M_{ij} = M_{ji} \quad C_{ijkl} = C_{klij} = C_{jikl} \tag{6}$$

and

$$k_{ij}\theta_{,i}\theta_{,j} \geq 0 \tag{7}$$

To these equations, we must adjoin boundary conditions and initial conditions. The boundary conditions can be of Dirichlet type or of Neumann type, or we can have mixed boundary conditions.

The initial conditions have the following form:

$$\begin{aligned} u_i(\mathbf{x},0) &= u_{0i}(\mathbf{x}) \\ \dot{u}_i(\mathbf{x},0) &= \dot{u}_{0i}(\mathbf{x}) \\ S(\mathbf{x},0) &= S_0(\mathbf{x}), \quad \mathbf{x} \in \overline{B} \end{aligned} \tag{8}$$

and, in the case of the mixed problem, the boundary conditions are

$$u_i(\mathbf{x},t) = \hat{u}_i(\mathbf{x},t), \quad \text{on} \quad \Sigma_1 \times [0,t_1) \tag{9}$$

$$\begin{aligned} s_i(\mathbf{x},t) &= \sigma_{ji}(\mathbf{x},t)n_j \\ &= \hat{s}_i(\mathbf{x},t), \quad \text{on} \quad \Sigma_2 \times [0,t_1) \end{aligned} \tag{10}$$

$$\theta(\mathbf{x},t) = \hat{\theta}(\mathbf{x},t), \quad \text{on} \quad \Sigma_3 \times [0,t_1) \tag{11}$$

$$\begin{aligned} q(\mathbf{x},t) &= q_i(\mathbf{x},t)n_i \\ &= \hat{q}(\mathbf{x},t), \quad \text{on} \quad \Sigma_4 \times [0,t_1) \end{aligned} \tag{12}$$

where Σ_s $(s = 1,\ldots,4)$ are subsets of the boundary ∂B so that $\Sigma_1 \cup \overline{\Sigma}_2 = \Sigma_3 \cup \overline{\Sigma}_4 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$, $\mathbf{n}$ is the unit outward normal to the boundary, and $u_{0i}, \dot{u}_{0i}, S_0, \hat{u}_i, \hat{s}_i, \hat{\theta}$, and $\hat{q}$ are prescribed fields.

In the linear theory of isotropic materials, we have only five constitutive coefficients λ, μ, m, k, and a so that the constitutive equations (4) become

$$q_i = -k\theta_{,i} \tag{13}$$

$$\begin{aligned} \sigma_{ij} &= \lambda e_{rr}\delta_{ij} + \mu e_{ij} - m\theta\delta_{ij} \\ \rho_0 S &= m e_{rr} + a\theta \end{aligned} \tag{14}$$

Here, the scalars λ and μ are called Lamé moduli, μ is the shear modulus, m is the stress–temperature modulus, and k is the conductivity coefficient assumed to be positive.

We assume that the constitutive coefficients are of class C^1 on $\overline{B}$ while ρ_0 and a are assumed to be continuous on $\overline{B}$ and $\rho_0 > 0$.

We also suppose that the prescribed data are given so that [4]:

i) b_i and r are continuous on $\overline{B} \times [0,t_1)$.
ii) u_{0i}, $\dot{u}_{0i}$, and S_0 are continuous on $\overline{B}$.
iii) $\hat{u}_i$ are continuous on $\Sigma_1 \times [0,t_1)$.
iv) $\hat{s}_i$ are smooth functions on $\Sigma_2 \times [0,t_1)$ and continuous as functions of time.
v) $\hat{\theta}$ is continuous on $\Sigma_3 \times [0,t_1)$.
vi) $\hat{q}$ is smooth on $\Sigma_4 \times [0,t_1)$ and continuous as function of time.

By an admissible thermoelastic process, in the linear theory of thermoelasticity, we mean an ordered array $[u_i, e_{ij}, \sigma_{ij}, \theta, S, q_i]$ with the properties [4]:

i) u_i are of class C^2 on $B \times (0,t_1)$.
ii) $u_i, \dot{u}_i, \ddot{u}_i, u_{i,j}, \dot{u}_{i,j}$ are continuous on $\overline{B} \times [0,t_1)$.
iii) e_{ij} are the components of a symmetric tensor, continuous on $\overline{B} \times [0,t_1)$.
iv) σ_{ij} are the components of a symmetric tensor, of class $C^{1,0}$ on $B \times (0,t_1)$.
v) σ_{ij} and $\sigma_{ji,j}$ are continuous on $\overline{B} \times [0,t_1)$.
vi) θ is of class $C^{2,1}$ on $B \times (0,t_1)$.
vii) $\theta, \theta_{,i}, \dot{\theta}$ are continuous on $\overline{B} \times [0,t_1)$.
viii) S is of class $C^{0,1}$ on $B \times (0,t_1)$.
ix) $S, \dot{S}$ are continuous on $\overline{B} \times [0,t_1)$.
x) q_i are of class $C^{1,0}$ on $B \times (0,t_1)$.
xi) q_i and $q_{i,i}$ are continuous on $\overline{B} \times [0,t_1)$.

The following remark is an immediate consequence of the above definitions and linearity of field equations.

Remark 1. Carlson [4] Let $[u_i, e_{ij}, \sigma_{ij}, \theta, S, q_i]$ and $[\tilde{u}_i, \tilde{e}_{ij}, \tilde{\sigma}_{ij}, \tilde{\theta}, \tilde{S}, \tilde{q}_i]$ be thermoelastic processes corresponding to the external force systems $[\mathbf{s}, \mathbf{f}]$ and $[\tilde{\mathbf{s}}, \tilde{\mathbf{f}}]$, respectively, and to the external thermal systems $[q, r]$ and $[\tilde{q}, \tilde{r}]$, respectively.

If α and $\tilde{\alpha}$ are scalars, then $[\alpha u_i + \tilde{\alpha}\tilde{u}_i, \alpha e_{ij} + \tilde{\alpha}\tilde{e}_{ij}, \alpha\sigma_{ij} + \tilde{\alpha}\tilde{\sigma}_{ij}, \alpha\theta + \tilde{\alpha}\tilde{\theta}, \alpha S + \tilde{\alpha}\tilde{S}, \alpha q_i + \tilde{\alpha}\tilde{q}_i]$ is a thermoelastic process corresponding to the external force system $[\alpha\mathbf{s} + \tilde{\alpha}\tilde{\mathbf{s}}, \alpha\mathbf{f} + \tilde{\alpha}\tilde{\mathbf{f}}]$ and to the external thermal system $[\alpha q + \tilde{\alpha}\tilde{q}, \alpha r + \tilde{\alpha}\tilde{r}]$.

The above remark proves that the set of all admissible thermoelastic processes may be organized as a linear space endowed with natural addition and scalar multiplication.

By a solution of the mixed boundary–initial value problem, we mean an admissible thermoelastic process which satisfies the (2)–(5) and the conditions (8)–(12).

Let us introduce the relations (4) and (5) into (2) and (3). Then, we can formulate the boundary–initial value problem in terms of displacement u_i and temperature variation θ only. Thus, we have the differential system

$$\begin{aligned} (C_{ijkl}u_{k,l})_{,j} - (M_{ij}\theta)_{,j} - \rho_0\ddot{u}_i &= -\rho_0 b_i \\ (k_{ij}\theta_{,j})_{,i} - T_0 M_{ij}\dot{u}_{i,j} - c\dot{\theta} &= -\rho_0 r \end{aligned} \tag{15}$$

with the initial conditions

$$\begin{aligned} u_i(\mathbf{x},0) &= u_{0i}(\mathbf{x}) \\ \dot{u}_i(\mathbf{x},0) &= \dot{u}_{0i}(\mathbf{x}) \\ \theta(\mathbf{x},0) &= \theta_0(\mathbf{x}), \quad \mathbf{x} \in \overline{B} \end{aligned} \tag{16}$$

and the boundary conditions

$$u_i(\mathbf{x},t) = \hat{u}_i(\mathbf{x},t), \quad \text{on} \quad \Sigma_1 \times [0,t_1) \tag{17}$$

$$\begin{aligned} &(C_{ijkl}u_{k,l} - M_{ij}\theta)(\mathbf{x},t)n_j \\ &= \hat{s}_i(\mathbf{x},t), \quad \text{on} \quad \Sigma_2 \times [0,t_1) \end{aligned} \tag{18}$$

$$\theta(\mathbf{x},t) = \hat{\theta}(\mathbf{x},t), \quad \text{on} \quad \Sigma_3 \times [0,t_1) \tag{19}$$

$$k_{ij}\theta_{,j}(\mathbf{x},t)n_i = \hat{q}(\mathbf{x},t), \quad \text{on} \quad \Sigma_4 \times [0,t_1) \tag{20}$$

In the above relations, we have used the specific heat $c = T_0 a$, and we have introduced the notation:

$$a\theta_0 = \rho_0 S_0 - M_{ij}u_{0i,j} \tag{21}$$

If the body is homogeneous, then the system of partial differential equations (15) reduces to

$$\begin{aligned} C_{ijkl}u_{k,lj} - M_{ij}\theta_{,j} - \rho_0\ddot{u}_i &= -\rho_0 b_i \\ k_{ij}\theta_{,ji} - T_0 M_{ij}\dot{u}_{i,j} - c\dot{\theta} &= -\rho_0 r \end{aligned} \tag{22}$$

Moreover, if the body is homogeneous and isotropic, then the equations are

$$\begin{aligned} \mu u_{i,jj} + (\lambda+\mu)u_{j,ji} - m\theta_{,i} - \rho_0\ddot{u}_i &= -\rho_0 b_i \\ k\theta_{,ii} - T_0 m\dot{u}_{j,j} - c\dot{\theta} &= -\rho_0 r \end{aligned} \tag{23}$$

Concerning the solution of the mixed boundary value problem, we have the following uniqueness result [4]:

Theorem 1. *Suppose the elasticity tensor C_{ijkl} is positive semi-definite and the specific heat c is strictly positive. Then the mixed problem has at most one solution.*

Sometimes in applications, the boundary condition for the heat flux is considered in the form of the following convection condition:

$$\mathbf{q}\cdot\mathbf{n} = h(T - T_e) \quad \text{on} \quad \partial B \tag{24}$$

where T is the temperature of the solid's boundary, T_e is the ambient temperature and h is the convection coefficient. The last two quantities, h and T_e, are determined by experiments. Other types of boundary conditions for the heat flux can be found in the books [3, 7, 8].

Up to now, we have formulated forward in time problems. In the last part of this section, we formulate the boundary–final value problem known as the backward in time problem. We consider the boundary–final value problem of the linear theory of thermoelasticity on the interval $(-t_1, 0]$, where $t_1 > 0$ may be infinite. All the quantities have the same significations as in the formulation of the forward in time problem defined above.

In terms of displacement u_i and temperature variation θ, the boundary–final value problem is defined by the equations

$$\begin{aligned}&(C_{ijkl}u_{k,l})_{,j} - (M_{ij}\theta)_{,j} - \rho_0 \ddot{u}_i = -\rho_0 b_i \\ &(k_{ij}\theta_{,j})_{,i} - T_0 M_{ij}\dot{u}_{i,j} - c\dot{\theta} = -\rho_0 r, \quad \text{in} \quad B \times (-t_1, 0)\end{aligned} \tag{25}$$

the final conditions

$$\begin{aligned}u_i(\mathbf{x}, 0) &= u_{0i}(\mathbf{x}) \\ \dot{u}_i(\mathbf{x}, 0) &= \dot{u}_{0i}(\mathbf{x}) \\ \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), &\quad \mathbf{x} \in \overline{B}\end{aligned} \tag{26}$$

and the boundary conditions

$$u_i(\mathbf{x}, t) = \hat{u}_i(\mathbf{x}, t), \quad \text{on} \quad \Sigma_1 \times (-t_1, 0] \tag{27}$$

$$\begin{aligned}&(C_{ijkl}u_{k,l} - M_{ij}\theta)(\mathbf{x}, t)n_j \\ &\quad = \hat{s}_i(\mathbf{x}, t), \quad \text{on} \quad \Sigma_2 \times (-t_1, 0]\end{aligned} \tag{28}$$

$$\theta(\mathbf{x}, t) = \hat{\theta}(\mathbf{x}, t), \quad \text{on} \quad \Sigma_3 \times (-t_1, 0] \tag{29}$$

$$k_{ij}\theta_{,j}(\mathbf{x}, t)n_i = \hat{q}(\mathbf{x}, t), \quad \text{on} \quad \Sigma_4 \times (-t_1, 0] \tag{30}$$

where Σ_s $(s = 1, \ldots, 4)$ are subsets of the boundary ∂B so that $\Sigma_1 \cup \overline{\Sigma}_2 = \Sigma_3 \cup \overline{\Sigma}_4 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$, $\mathbf{n}$ is the unit outward normal to the boundary, and $u_{0i}, \dot{u}_{0i}$, S_0, $\hat{u}_i$, $\hat{s}_i$, $\hat{\theta}$, and $\hat{q}$ are prescribed fields.

The backward in time problems lead to ill-posed problems. By means of the change $t \rightsquigarrow -t$ we can transform the above boundary–final value problem into a boundary–initial problem defined by the equations

$$\begin{aligned}&(C_{ijkl}u_{k,l})_{,j} - (M_{ij}\theta)_{,j} - \rho_0 \ddot{u}_i = -\rho_0 b_i \\ &(k_{ij}\theta_{,j})_{,i} + T_0 M_{ij}\dot{u}_{i,j} + c\dot{\theta} = -\rho_0 r, \quad \text{in} \quad B \times (0, t_1)\end{aligned} \tag{31}$$

the initial conditions

$$\begin{aligned}u_i(\mathbf{x}, 0) &= u_{0i}(\mathbf{x}) \\ \dot{u}_i(\mathbf{x}, 0) &= \dot{u}_{0i}(\mathbf{x}) \\ \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), &\quad \mathbf{x} \in \overline{B}\end{aligned} \tag{32}$$

and the boundary conditions

$$u_i(\mathbf{x}, t) = \hat{u}_i(\mathbf{x}, t), \quad \text{on} \quad \Sigma_1 \times [0, t_1) \tag{33}$$

$$\begin{aligned}&(C_{ijkl}u_{k,l} - M_{ij}\theta)(\mathbf{x}, t)n_j \\ &\quad = \hat{s}_i(\mathbf{x}, t), \quad \text{on} \quad \Sigma_2 \times [0, t_1)\end{aligned} \tag{34}$$

$$\theta = \hat{\theta}, \quad \text{on} \quad \Sigma_3 \times [0, t_1) \tag{35}$$

$$k_{ij}\theta_{,j}(\mathbf{x}, t)n_i = \hat{q}(\mathbf{x}, t), \quad \text{on} \quad \Sigma_4 \times [0, t_1) \tag{36}$$

We remark the energy equation is changed by this transformation, while the first three equations have the same form as in the case of final value problem. This class of problems was first considered by Ames and Payne [1] (see also the book by Ames and Straughan [2]).

Uncoupled Problems

In the study of certain materials, it has been observed that in the energy equation $(15)_2$, the term $T_0 M_{ij}\dot{u}_{i,j}$ can be neglected, and the corresponding predicted results are in concordance with the experiments. In such a case, the energy equation $(15)_2$ is replaced by

$$\left(k_{ij}\theta_{,j}\right)_{,i} - c\dot{\theta} = -\rho_0 r \tag{37}$$

and the mixed problem becomes high simplified. In fact, the mixed problem leads to the separate study of two boundary–initial value problems. The first is concerned with the above equation together with the initial and boundary conditions for the temperature variation θ, a problem relating only the temperature variation. Assuming solved this problem for the temperature variation θ, the second boundary–initial problem consists of the differential system $(15)_1$ with the initial and boundary conditions in terms of the displacement u_i in which the temperature variation is assumed prescribed. This last boundary–initial value problem represents a boundary–initial value problem of the linear elastodynamics in which the components of the body force vector are

$$b_i - \frac{1}{\rho_0}\left(M_{ij}\theta\right)_{,j} \tag{38}$$

and the stress boundary condition is

$$s_i(\mathbf{x},t) = \tilde{s}_i(\mathbf{x},t) + M_{ij}\theta(\mathbf{x},t)n_j \quad (39)$$

This last case is called the uncoupled theory of the thermoelasticity. The coupled theory concerns the studies of the interaction between the deformation of elastic materials and the thermal field. Thermoelasticity gives the tools to investigate the stresses produced by the temperature field and to calculate the distribution of temperature due to the action of internal forces.

In the uncoupled theory, the function θ vanishes when $r, \theta_0, \hat{\theta}$, and $\hat{q}$ are zero. In the coupled theory, this is not true: there is a variation of the temperature due to the mechanical deformation. This variation also produces a mechanical deformation.

Sometimes the inertial terms are not taken into account. Then, the (2) is replaced by

$$\sigma_{ji,j} + \rho_0 b_i = 0 \quad (40)$$

In such a case, we obtain the so-called quasi-static theory of thermoelasticity. Then the basic equations of the quasi-static theory are (40) and (3)–(5).

Let us consider now the equilibrium theory. Then the fundamental system of field equations consist of

– The equations of equilibrium

$$\sigma_{ji,j} + \rho_0 b_i = 0 \quad (41)$$

– The energy equation

$$q_{i,i} = \rho_0 r \quad (42)$$

– The constitutive equations

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}e_{kl} - M_{ij}\theta \\ \rho_0 S &= M_{ij}e_{ij} + a\theta \\ q_i &= -k_{ij}\theta_{,j} \end{aligned} \quad (43)$$

– The geometrical equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (44)$$

To these equations, we must adjoin boundary conditions which can be of Dirichlet type or of Neumann type, or we can have mixed boundary conditions.

In the case of the mixed problem, the boundary conditions are

$$u_i(\mathbf{x}) = \hat{u}_i(\mathbf{x}), \quad \text{on} \quad \Sigma_1 \quad (45)$$

$$s_i(\mathbf{x}) = \sigma_{ji}(\mathbf{x})n_j = \hat{s}_i(\mathbf{x}), \quad \text{on} \quad \Sigma_2 \quad (46)$$

$$\theta(\mathbf{x}) = \hat{\theta}(\mathbf{x}), \quad \text{on} \quad \Sigma_3 \quad (47)$$

$$q(\mathbf{x}) = q_i(\mathbf{x})n_i = \hat{q}(\mathbf{x}), \quad \text{on} \quad \Sigma_4 \quad (48)$$

where Σ_s $(s = 1, \ldots, 4)$ are subsets of the boundary ∂B such that $\Sigma_1 \cup \overline{\Sigma}_2 = \Sigma_3 \cup \overline{\Sigma}_4 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$, and $\hat{u}_i$, $\hat{s}_i$, $\hat{\theta}$, and $\hat{q}$ are prescribed fields.

We suppose that the prescribed data are given so that [4]:

i) b_i and r are continuous on $\overline{B}$.
ii) $\hat{u}_i$ are continuous on Σ_1.
iii) $\hat{s}_i$ are smooth functions on Σ_2.
iv) $\hat{\theta}$ is continuous on Σ_3.
v) $\hat{q}$ is smooth on Σ_4.

Let us remark that the above system is uncoupled in the sense that the temperature can be found by solving the heat flow problem given by

$$\begin{aligned} (k_{ij}\theta_{,j})_{,i} &= -\rho r, \quad \text{in} \quad B \\ \theta(\mathbf{x}) &= \hat{\theta}(\mathbf{x}), \quad \text{on} \quad \Sigma_3 \\ q_i(\mathbf{x})n_i &= \hat{q}(\mathbf{x}), \quad \text{on} \quad \Sigma_4 \end{aligned} \quad (49)$$

From the above problem, we can remark that the mechanical deformation does not influence the variation of the temperature. In the following, we can suppose that the temperature field is already determined.

By an admissible state, in the linear equilibrium theory of thermoelasticity, we mean an ordered array $[u_i, e_{ij}, \sigma_{ij}]$ with the properties [4]:

i) u_i are of class C^2 on B.
ii) u_i and $u_{i,j}$ are continuous on $\overline{B}$.
iii) e_{ij} are the components of a symmetric tensor, continuous on $\overline{B}$.

iv) σ_{ij} are the components of a symmetric tensor, of class C^1 on B.
v) σ_{ij} and $\sigma_{ji,j}$ are continuous on $\overline{B}$.

Thus, the equilibrium problem consists in finding an admissible state which satisfies the problem defined by the (41), (43), and (44) and the boundary conditions (45) and (46), where θ is a known function. We also have that the set of all admissible states may be organized as a linear space endowed with natural addition and scalar multiplication.

Regarding the equilibrium problem, we have the following uniqueness theorem [4]:

Theorem 2. *Let the elasticity tensor C_{ijkl} be positive definite. Then any two solutions of the mixed equilibrium problem are equal modulo a rigid displacement. Moreover, if Σ_1 is nonempty, the mixed problem has at most one solution.*

In the equilibrium problem, let us consider that $\Sigma_2 = \partial B$. So, we know the surface traction on entire boundary of the body. For this problem, we intend to give a formulation of the problem only in terms of the stress tensor.

We assume that the tensor C_{ijkl} is invertible. From the constitutive equations $(43)_1$, we have that there exists a tensor A_{ijkl} so that

$$e_{ij} = A_{ijkl}\sigma_{kl} + \alpha_{ij}\theta \tag{50}$$

where

$$\alpha_{ij} = A_{ijkl}M_{kl} \tag{51}$$

We assume that A_{ijkl} and α_{ij} are of class C^2 on $\overline{B}$ and that the domain B is simply connected. In view of the compatibility conditions (see [6]), it follows that the stress tensor σ_{ij}, of class C^2 on $\overline{B}$, corresponds to a solution of the equilibrium problem if and only if it is solution of the problem defined by the equations

$$\begin{aligned} \sigma_{ji,j} + \rho_0 b_i &= 0 \\ \varepsilon_{pim}\varepsilon_{qjn}(A_{ijkl}\sigma_{kl} + \alpha_{ij}\theta)_{,mn} &= 0, \quad \text{in} \quad B \end{aligned} \tag{52}$$

and the boundary condition

$$\sigma_{ij}n_j = \hat{s}_i, \quad \text{on} \quad \partial B \tag{53}$$

where ε_{ijk} is the alternating symbol.

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Boundary-Value Problems Resulting in Thermoelastic Shock Wave Propagation

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Synonyms

Boundary value problems; Conditions of compatibility; Ray series; Shock waves

Overview

Thermoelastic shock waves are generated and then propagate only in *nonlinear media*. Discontinuities in such main fields as stresses, strains, velocities, and temperature take place along the shock wave front, while displacements remain to be continuous. But distinct to the surfaces of strong discontinuities propagating in linear media, normal velocities of the shock waves depend on the discontinuities in the stress and temperature fields. Moreover, in the general case of the shock wave propagation, for the values experiencing discontinuities, it is impossible to obtain transport equations which could describe the evolution of these discontinuities. The reason is that the velocity and the strong discontinuity surface of the shock wave are a priori unknown, and they should be determined during solving the problem.

However, for acceleration waves propagating in nonlinear media, the evolution equations exist but, contrary to linear media, in the form of the nonlinear partial differential equations. A solution of these equations subjected to certain conditions for the discontinuities at the initial instant of time allows one to describe such a phenomenon as the transition of a weak wave into a shock wave, which is called as the *wave breakdown*.

The transport equation could be obtained also for weak shock waves assuming that the magnitudes of the relevant discontinuities across the shock wave front are small. It has been shown [1] that ignoring the products of jumps across the shock wave front, the propagation condition of “linear weak shock waves” could be obtained (▶ Propagation of Shock Waves in Thermoelastic Solids in View of Singular Surfaces).

For solving the boundary-value problems of the linear isotropic or anisotropic thermoelasticity with due regard for a finite speed of heat propagation, the ray method (▶ Ray Expansion Theory, ▶ Ray Method for Solving Boundary-Value Problems of Anisotropic Thermoelasticity with Thermal Relaxation) was developed by Rossikhin [2, 3], Gonsovskii et al. [4], and Rossikhin and Shitikova [5, 6]. It has been shown that the jumps of strong and weak discontinuities propagating with constant velocities arise in a thermoelastic medium as a result of instantaneous variations in temperature or heat flow on the boundary surface or as a result of combined subjection (thermal and mechanical).

The ray method for nonlinear elastic and nonlinear viscoelastic media without regard for thermal effects has been devised, respectively, by Rossikhin and Shitikova [6, 7] and Burenin and Rossikhin [8]. One-dimensional boundary-value problems on shock subjection upon the medium boundary surface have been solved, resulting in the propagation of one or three shock waves (one longitudinal wave and two quasi-transverse waves) in the medium depending on whether the nonlinear medium under consideration was unstressed or prestressed at the moment of shock subjection. Independently of Rossikhin and his collaborators, Prasad [9] has suggested a similar ray method for solving certain nonlinear partial differential equations, considering the problem of a piston moving with a constant initial speed through a polytropic gas as an example.

The ray method developed for nonlinear elastic media has been generalized in [10] to unidimensional, plane, and spatial boundary-value problems on instantaneous thermal and mechanical subjection upon the boundary plane of a nonlinear thermoelastic half-space, wherein heat propagates with a finite speed. By the action of initial and boundary conditions, two types of finite amplitude shock wave propagate in such media: quasi-thermal wave (fast wave) and quasi-longitudinal wave (slow wave). Behind the wave fronts, the solution for the desired functions is constructed along the rays in terms of power series (ray series), the coefficients of which are the discontinuities in various orders partial derivatives of the functions to be found with respect to time, but a variable value is the time needed for a disturbance to propagate along the ray from the point under consideration up to the wave front; in so doing the power of the variable value corresponds to the order of partial time derivative of the desired function (▶ Ray Expansion Theory).

Governing Equations

The nonlinear isotropic thermoelastic half-space $X_3 > 0$ is considered, wherein heat propagates with a finite speed. The motion of such a medium in Lagrangian variables in the rectangular Cartesian coordinate system is described by the following set of equations:

$$L_{ij,j} = \rho_0 \dot{v}_i, \qquad v_i = \dot{u}_i \tag{1}$$

$$L_{ij} = \rho_0 \frac{\partial U}{\partial u_{i,j}}, \qquad U = U(K_1, K_2, K_3, S) \tag{2}$$

$$\begin{aligned} &K_1 = \varepsilon_{ii}, \quad K_2 = \varepsilon_{ij}\varepsilon_{ji}, \quad K_3 = \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki}, \\ &2\varepsilon_{ij} = u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} \end{aligned} \tag{3}$$

$$-\rho_0 T\left(\frac{\partial S}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial S}{\partial T}\dot{T}\right) = Q_{k,k}, \qquad T = \frac{\partial U}{\partial S} \tag{4}$$

$$\tau_0 \dot{Q}_i + Q_i = A_{ij}T_{,j}, \qquad A_{ij} = A_{ij}(\varepsilon_{mn}, T) \tag{5}$$

where ρ_0 is the density in the initial state; u_i and v_i are the components of the displacement and velocity vectors, respectively; $u_i = x_i - X_i$; x_i and X_i are the Eulerian and Lagrangian coordinates of medium particles, respectively; $L_{ij} \neq L_{ji}$ are the Lagrangian stress tensor components; ε_{ij} are the Green finite strain tensor components; K_1, K_2, and K_3 are the strain tensor invariants; U is the internal energy; S is the specific entropy; T is the body's absolute temperature; Q_i are the components of the Lagrangian vector of heat flow per unit square of the surface; τ_0 is the thermal relaxation time; A_{ij} are the thermal conductivity coefficients; an overdot denotes a time derivative; a Latin index after a point indicates a derivative with respect to the corresponding spatial coordinate; and the Latin indices take on the values 1, 2, and 3.

The set of (1–5) involving the equations of motion, the generalized Hooke's law, relations between displacements and strains, the energy conservation law, and the generalized Fourier law is the closed system of twenty six equations in twenty six unknown values: $u_i, v_i, \varepsilon_{ij}, L_{ij}, T, S, Q_i$.

The thermodynamic inequality

$$-x_{i,k}Q_k T_{,i} \geq 0 \tag{6}$$

what results from the second law of thermodynamics, should be added to (1–5).

In further consideration we assume that the values $u_{i,j}$, $\theta = T - T_0$, where T_0 is the medium temperature in the initial state, and S are small values. Let us represent the internal energy $U(K_1, K_2, K_3, S)$, Helmholtz free energy $F(K_1, K_2, K_3, T) = U - TS$, as well as the tensor components A_{ij} by its Taylor series expansions in the vicinity of the natural state:

$$\begin{aligned} \rho_0 U = &\frac{1}{2}\lambda^S K_1^2 + \mu^S K_2 - \kappa^S K_1 S - \kappa_1^S S^2 \\ &+ l_1^S K_1 + l_2^S K_1 K_2 + l_3^S K_3 + \ldots \end{aligned} \tag{7}$$

$$\begin{aligned} \rho_0 F = &\frac{1}{2}\lambda^T K_1^2 + \mu^T K_2 - \kappa^T K_1 \theta - \kappa_1^T \theta^2 \\ &+ l_1^T K_1 + l_2^T K_1 K_2 + l_3^T K_3 + \ldots \end{aligned} \tag{8}$$

$$A_{ij} = k_0\delta_{ij} + \frac{1}{2}k_{ijpq}\left(u_{p,q} + u_{q,p}\right) + k_{ij}\theta + \ldots \tag{9}$$

where the upper indices S and T denote that the corresponding coefficients are referred to the adiabatic and isothermic state, respectively; λ and μ are Lamé's constants; l_1, l_2, and l_3 are Murnaghan coefficients; κ and κ_1 are certain constants defining the thermoelastic process; k_0 is the thermal conductivity; δ_{ij} is Kronecker's symbol; $k_{ij} = k_1\delta_{ij}$; $k_{ijpq} = k_2\delta_{ij}\delta_{pq} + k_3(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})$; and k_1, k_2, and k_3 are some constants of the medium.

Look up the relation between the coefficients with the S and T indices. For this purpose, we use the formulas

$$\sigma_{ij} = \rho_0\left(\frac{\partial U}{\partial \varepsilon_{ij}}\right)_S, \qquad \sigma_{ij} = \rho_0\left(\frac{\partial F}{\partial \varepsilon_{ij}}\right)_T \tag{10}$$

where σ_{ij} are the Kirchhoff stress symmetric tensor components associated with L_{ij} by the relation $L_{ij} = x_{i,k}\sigma_{jk}$, as well as the expression $S = -\partial F/\partial T$ that gives us with (8) taken into account:

$$\rho_0 S = \kappa^T K_1 + 2\kappa_1^T \theta \tag{11}$$

Substituting (7), (8), and (11) into (10) yields two expressions for σ_{ij}:

$$\begin{aligned}\sigma_{ij} &= \lambda^T K_1 \delta_{ij} + 2\mu^T \varepsilon_{ij} - \kappa^T(\rho_0 S - \kappa^T K_1)\\ &\quad \times (2\kappa_1^T)^{-1}\delta_{ij} + 3l_1^T K_1^2 \delta_{ij}\\ &\quad + l_2^T(2K_1\varepsilon_{ij} + K_2\delta_{ij})\\ &\quad + l_3^T(\varepsilon_{kj}\varepsilon_{ik} + \varepsilon_{jk}\varepsilon_{ik} + \varepsilon_{kj}\varepsilon_{ki}) + \ldots\\ \sigma_{ij} &= \lambda^S K_1 \delta_{ij} + 2\mu^S \varepsilon_{ij} - \kappa^S S \delta_{ij} + 3l_1^S K_1^2 \delta_{ij}\\ &\quad + l_2^S(2K_1\varepsilon_{ij} + K_2\delta_{ij})\\ &\quad + l_3^S(\varepsilon_{kj}\varepsilon_{ik} + \varepsilon_{jk}\varepsilon_{ik} + \varepsilon_{kj}\varepsilon_{ki}) + \ldots\end{aligned} \tag{12}$$

the comparison of which results in the required relationships between the coefficients:

$$\begin{aligned}&\lambda^S = \lambda^T + (\kappa^T)^2(2\kappa_1^T)^{-1},\\ &\mu^S = \mu^T l_i^S = l_i^T \quad (i = 1, 2, 3),\\ &\kappa^S = \rho_0\kappa^T(2\kappa_1^T)^{-1}, \quad \kappa_1^S = \rho_0^2(4\kappa_1^T)^{-1}\end{aligned} \tag{13}$$

Using the formulas (13) and considering (11), we obtain the following formula to define the internal energy (7):

$$\begin{aligned}\rho_0 U = &\frac{1}{2}\lambda K_1^2 + \mu K_2 - \kappa K_1\theta - \kappa_1\theta^2 + l_1 K_1^3\\ &+ l_2 K_1 K_2 + l_3 K_3 + \ldots\end{aligned} \tag{14}$$

where it is designated

$$\lambda = \lambda^T - (\kappa^T)^2(\kappa_1^T)^{-1}, \quad \mu = \mu^T, \quad \kappa = 2\kappa^T,$$
$$\kappa_1 = \kappa_1^T, \quad l_i = l_i^T, \quad (i = 1, 2, 3)$$

Shock Waves

Let the shock wave $\Sigma(t)$ be generated in a nonlinear thermoelastic isotropic half-space, the dynamic behavior of which is described by the set of (1–5), as a result of external subjection on its boundary $X_3 = 0$ and then propagate with the normal velocity G. Hereafter the shock wave $\Sigma(t)$ will be interpreted as a limiting layer of the width h at $h \to 0$, within which the desired values L_{ij}, $u_{i,j}$, v_i, Q_i, and T change monotonically and continuously from the magnitudes L_{ij}^+, $u_{i,j}^+$, v_i^+, Q_i^+, and T^+ on the layer's forward boundary to the magnitudes L_{ij}^-, $u_{i,j}^-$, v_i^-, Q_i^-, and T^- on the layer's reverse boundary.

To derive the relations connecting the values to be found on the shock wave, consider that on the wave surface

$$\frac{\partial}{\partial X_i} = \frac{d}{dn}n_i + g^{\alpha\beta}\frac{\partial}{\partial y^\alpha}X_{i,\beta}, \quad \frac{\partial}{\partial t} = -G\frac{d}{dn} + \frac{\delta}{\delta t} \tag{15}$$

where n_i are the components of the normal vector to the wave surface, y^α are the curvilinear coordinates on the surface $\Sigma(t)$, $g^{\alpha\beta}$ is the contravariant metric tensor of the wave surface, d/dn is the derivative with respect to the normal to $\Sigma(t)$, $X_i(y_\alpha, t)$ are the Cartesian coordinates of the surface, $X_{i,\beta} = \partial X_i/\partial y^\beta$, $\delta/\delta t$ is the time derivative along the normal to the wave surface [11], and Greek indices take on the values 1 and 2.

Note that at $h \to 0$, the second terms in (15) can be disregarded in comparison with the first ones. Then substituting formulas (11) and (14) into (1–6) and replacing the partial derivatives with respect to coordinates and time by their expressions (15), upon integrating of the resulted relations with respect to the normal to the surface from $-h/2$ to $h/2$ and the transition to the limit at $h \to 0$, we obtain

$$[\![L_{ij}]\!]n_j + \rho_0 G[\![v_i]\!] = 0 \tag{16}$$

$$\begin{aligned}[\![L_{ij}]\!] = &(\lambda\omega G^{-1} + \kappa[\![T]\!])\left(G^{-1}[\![v_i]\!]n_j - \delta_{ij} - u_{i,j}^+\right)\\ &+ \kappa G^{-1}\{T\}[\![v_i]\!]n_j + \lambda G^{-1}\left(\frac{1}{2}G^{-1}[\![v_k]\!][\![v_k]\!]\delta_{ij}\right.\\ &\left.-[\![v_k]\!]n_l u_{k,l}^+\delta_{ij} - [\![v_i]\!]n_j u_{k,k}^+\right)\\ &+ \mu[\![A_{ij}]\!]\left(u_{i,l}^+ - G^{-1}[\![v_i]\!]n_l\right)\\ &- \mu G^{-1}A_{lj}^+[\![v_i]\!]n_l + \mu[\![A_{ij}]\!]\\ &- \mu G^{-1}\left(u_{k,j}^+[\![v_k]\!]n_i + u_{k,i}^+[\![v_k]\!]n_j\right.\\ &\left.-G^{-1}[\![v_k]\!][\![v_k]\!]n_i n_j\right) - 3\omega G^{-1}l_1\left(2u_{l,l}^+ - \omega G^{-1}\right)\delta_{ij}\\ &+ \frac{1}{4}l_2[\![A_{km}]\!](2A_{km}^+ + [\![A_{km}]\!])\delta_{ij}\\ &- l_2 G^{-1}\left(A_{ij}^+ + [\![A_{ij}]\!]\right)\omega + l_2 u_{k,k}^+[\![A_{ij}]\!]\\ &+ \frac{3}{4}l_3\left([\![A_{ik}]\!]A_{jk}^+ + [\![A_{ik}]\!][\![A_{jk}]\!] + [\![A_{jk}]\!]A_{ik}^+\right)\end{aligned} \tag{17}$$

$$- \kappa_1 G[\![T]\!](2T^+ + [\![T]\!]) + \frac{1}{2}\,\kappa\langle T\rangle \times [\![v_k]\!]\left(n_k - G^{-1}v_k^+ - \frac{1}{2}\,G^{-1}[\![v_k]\!]\right) - [\![Q_k]\!]n_k = 0 \qquad (18)$$

$$\tau_0 G[\![Q_i]\!] = [\![T]\!]\Big(k_0 + k_1\{T\} + \frac{1}{2}\,k_1[\![T]\!] - k_2 G^{-1}\langle v_k\rangle n_k)n_i - k_3 G^{-1}(\langle v_i\rangle n_k + \langle v_k\rangle n_i)n_k[\![T]\!] \qquad (19)$$

where $[\![f]\!] = f^- - f^+$, $\langle f\rangle = f^+ + \frac{1}{2}[\![f]\!]$, $\{T\} = T^+ - T_0$, $\omega = [\![v_i]\!]n_i$, $A_{ij}^+ = u_{i,j}^+ + u_{j,i}^+$

$$[\![A_{ij}]\!] = -G^{-1}([\![v_i]\!]n_j + [\![v_j]\!]n_i) = [\![u_{i,j}]\!] + [\![u_{j,i}]\!] = u_{i,j}^- + u_{j,i}^- - \left(u_{i,j}^+ + u_{j,i}^+\right) = A_{ij}^- - A_{ij}^+$$

To find the equations defining the shock wave velocities, we substitute the values $[\![L_{ij}]\!]$ from (17) into (16) and multiply sequentially the resulted equations by n_i and $X_{i,\alpha}$. To these three equations, one more equation is added which results after eliminating the values $[\![Q_i]\!]$ from (18) and (19). Thus, we obtain

$$\{\rho_0 G^2 - (\lambda + 2\mu - \kappa[\![T]\!] - \kappa\{T\} + a^+)\}\omega G = \frac{1}{2}l\,w^2 + \frac{1}{2}(m + 3l)\omega^2 + b_{ij}^+ n_i X_{j,\delta} w^\delta G + \kappa[\![T]\!]G^2\left(1 + u_{i,j}^+ n_i n_j\right) \qquad (20)$$

$$\{\rho_0 G^2 - (\mu - \kappa[\![T]\!] - \kappa\{T\} + l\omega G^{-1} - l u_{k,k})\}w_\gamma = G(\kappa[\![T]\!] - l\omega G^{-1})u_{i,j}^+ n_j X_{i,\gamma} + B\omega u_{j,i}^+ n_j X_{i,\gamma} \qquad (21)$$

$$\left(\frac{1}{2}\,k_1\tau_0^{-1}G^{-1} + \kappa_1 G\right)[\![T]\!]^2 + \Big\{k_4 G^{-2}\tau_0^{-1}\left(v_n^+ + \frac{1}{2}\,\omega\right) - k_T\tau_0^{-1}G^{-1} + 2\kappa_1 G T^+ - \frac{1}{4}\,\kappa\omega(1 - v_n^+ G^{-1}) + \frac{1}{8}\,\kappa G^{-1}(\omega^2 + w^2)\Big\}[\![T]\!] - \frac{1}{2}\,\kappa T^+\{\omega(1 - v_n^+ G^{-1}) - \frac{1}{2}\,G^{-1}(\omega^2 + w^2)\} = 0 \qquad (22)$$

where $a^+ = p u_{i,i}^+ + 2p_1 u_{i,j}^+ n_i n_j$, $b_{ij}^+ = (l_2 + \frac{3}{2}l_3)A_{ij}^+ + (\lambda + \mu)u_{j,i}^+ + \mu u_{i,j}^+$, $p = \lambda + 6l_1 + 2l_2$, $p_1 = \lambda + 3\mu + 2l_2 + 3l_3$, $B = \mu + l_2 + \frac{3}{2}l_3$, $m = -3(2l_1 + l_2 + \frac{1}{2}l_3)$, $l = -(\lambda + 2\mu + l_2 + \frac{3}{2}l_3)$, $v_n^+ = v_i^+ n_i$, $w_\alpha = [\![v_i]\!]X_{i,\alpha}$, $k_4 = k_2 + 2k_3$, $k_T = k_0 + k_1\{T\}$, $w^2 = w^\alpha w_\alpha$

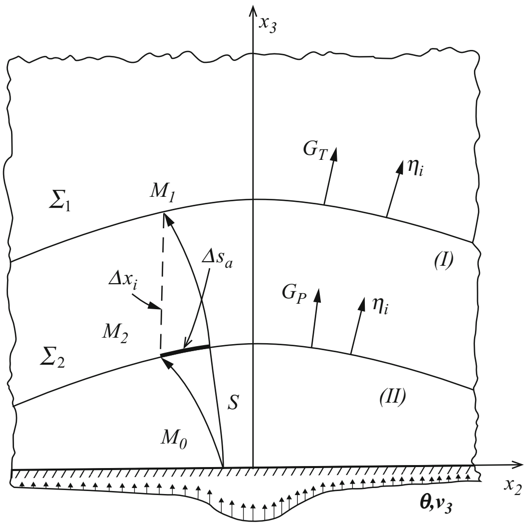
If we restrict ourselves only by linear terms in (20–22) and put the coupling coefficient κ equal to zero, then we are led to the known velocities for three linear waves: thermal wave $c_T = k_0^{1/2}(2\kappa_1 T_0\tau_0)^{-1/2}$, longitudinal wave $c_l = (\lambda + 2\mu)^{1/2}\rho_0^{-1/2}$, and transverse wave $c_t = \mu^{1/2}\rho_0^{-1/2}$.

In the subsequent discussion, we assume that external actions give rise exclusively to the two types of the shock waves: quasi-thermal wave Σ_1 and quasi-longitudinal wave Σ_2 (in the linear case, at $\kappa = 0$ these waves go over into the thermal and longitudinal waves, respectively), in so doing the quasi-transverse wave is weak ($w_\gamma = 0$) and its contribution to the intensity of the shock waves can be disregarded.

Moreover, for definiteness sake, we assume that the quasi-thermal wave is the faster of the two waves and propagates in the constrained medium, i.e., ahead of its front $T^+ = T_0$, $u_{i,j}^+ = v_i^+ = 0$. Then in the zone I between the wave surfaces Σ_1 and Σ_2 (Fig. 1), the half-space material will be in the stressed-strained state which is defined by the intensity of the quasi-thermal wave $\omega_T = \omega|_{\Sigma_1}$, i.e., all the values to be found in the zone I up to the wave surface Σ_2 are expressed in terms of ω_T.

Once the desired functions are calculated within the accuracy of the magnitudes of the second order of infinitesimal when solving boundary-value problems, then in the zone II immediately ahead of the quasi-longitudinal wave front, the functions T^+, $u_{i,j}^+$, and v_i^+ may be represented as the linear expansions in terms of ω_T, since these functions have been multiplied by the magnitudes of the first order of infinitesimal in (20) and (22) (from the foregoing no consideration has been given to (21)) that is

$$T^+ = T_0 + T_{10}\omega_T, \quad u_{i,j}^+ = -\omega_T G_1^{-1} n_i n_j, \quad v_i^+ = \omega_T n_i \qquad (23)$$



Boundary-Value Problems Resulting in Thermoelastic Shock Wave Propagation, Fig. 1 Scheme of the wave front location (this figure is taken from [10] with the permission from Taylor & Francis)

where T_{10} is the constant coefficient to be determined, the lower index T denotes that the given value corresponds to the quasi-thermal wave, and G_1 is the velocity of the quasi-thermal wave in the linear thermoelastic medium ($\kappa \neq 0$) [12]. The second relation in (23) is obtained from the Hadamard condition of compatibility:

$$[\![u_{i,j}]\!] = -G^{-1}[\![v_i]\!]n_j \tag{24}$$

with due regard for $[\![v_i]\!] = \omega n_i$, since the values $w_\gamma = 0$ on the quasi-thermal Σ_1 and quasi-longitudinal Σ_2 waves.

We seek the solution to (20) and (22) on the quasi-thermal wave in the form

$$G_T = G_1 + \alpha_1\omega_T, \quad [\![T]\!]_T = T_{10}\omega_T + T_{11}\omega_T^2 \tag{25}$$

but on the quasi-longitudinal wave, in view of expansion (23), as

$$\begin{aligned} &G_P = G_2 + \alpha_0\omega_T + \alpha_2\omega_P, \\ &[\![T]\!]_P = T_{20}\omega_P + T_{21}\omega_T\omega_P + T_{22}\omega_P^2 \end{aligned} \tag{26}$$

where G_2 is the velocity of the quasi-longitudinal wave in the linear thermoelastic medium [12], α_0, α_1, α_2, T_{10}, T_{20}, T_{11}, T_{22}, and T_{21} are constants to be found, and the lower index P indicates that the given values are taken on the quasi-longitudinal wave.

Substituting sequentially the assumed solutions (25) and (26) into (20) and (22) and equating terms at equal powers of ω_T and ω_P, we determine all constants involved:

$$\begin{aligned}
&G_{1,2} = (r_1 + r_2 \pm r)^{1/2}, \quad r = \left((r_1 - r_2)^2 + r_3^2\right)^{1/2} \\
&\quad r_1 = k_0(4\kappa_1\tau_0T_0)^{-1} + \kappa^2(8\rho_0\kappa_1)^{-1}, \\
&\quad r_2 = (\lambda + 2\mu)(2\rho_0)^{-1}, \quad r_3^2 = \kappa^2(\lambda + 2\mu)(4\rho_0^2\kappa_1)^{-1} \\
&\quad \alpha_0 = \Big\{a - b - [2(r_1 - r_2) \\
&\qquad \times (a + b) - c](2r)^{-1}\Big\}T_{10}(2G_2)^{-1}, \\
&\quad a = (k_1T_0 - k_0)(4\kappa_1\tau_0T_0^2)^{-1} \\
&\quad b = \kappa(2\rho_0)^{-1}, \quad c = \kappa^3(4\kappa_1\rho_0^2)^{-1}, \\
&\alpha_{1,2} = (G_{1,2}C_{1,2}R_{1,2} + D_{1,2}R_{13,23})a_{1,2}^{-1} \\
&C_{1,2} = \frac{1}{2}m + \frac{3}{2}l - \kappa T_{10,20}G_{1,2}, \\
&R_{1,2} = 2\kappa_1T_0G_{1,2}^2 - k_0\tau_0^{-1}, \; D_{1,2} = \kappa G_{1,2}^2, \\
&a_{1,2} = 2(G_{1,2}B_{1,2}R_{1,2} + R_{12,22}D_{1,2})\rho_0G_{1,2} \\
&R_{11,21} = R_{1,2} + 4\kappa_1T_0G_{1,2}^2, \\
&R_{12,22} = R_{11,21}T_{10,20}(2\rho_0G_{1,2})^{-1} - \kappa T_0(2\rho_0)^{-1} \\
&R_{13,23} = -\frac{1}{2}\Big\{(k_1\tau_0^{-1} + 2\kappa_1G_{1,2}^2)G_{1,2}T_{10,20}^2 \\
&\qquad + (k_4\tau_0^{-1} - \frac{1}{2}\kappa G_{1,2}^2)T_{10,20} + \frac{1}{2}\kappa T_0G_{1,2}\Big\} \\
&\quad B_1 = (\rho_0G_1^2 + \lambda + 2\mu)(2\rho_0G_1)^{-1}, \\
&\quad B_2 = (\rho_0G_2^2 - \lambda - 2\mu)(2\rho_0G_2)^{-1} + G_2 - \kappa\rho_0^{-1}T_{20} \\
&T_{10,20} = (\rho_0G_{1,2}^2 - \lambda - 2\mu)(\kappa G_{1,2})^{-1}, \\
&T_{11,22} = 2(B_{1,2}R_{13,23} - C_{1,2}R_{12,22})\rho_0G_{1,2}a_{1,2}^{-1} \\
&\quad T_{21} = \left\{\alpha_0\left[\rho_0 + (\lambda + 2\mu)G_2^{-2}\right]\right. \\
&\qquad \left. + G_1G_2^{-1}\left[\rho_0 - (\lambda + 2\mu)G_1^{-2}\right]\right\}\kappa^{-1}
\end{aligned} \tag{27}$$

When solving the boundary-value problems of nonlinear thermodynamics connected with the shock wave propagation, checking for the fulfillment of the thermodynamic inequality (6), which on the shock wave front takes the form [13]

$$\rho_0 G[\![S]\!] - [\![Q_i/T]\!] n_i \geq 0 \tag{28}$$

is the mandatory condition for the validity of the results obtained.

The inequality (28) may be rewritten, in view of (18), as

$$\begin{aligned}&\frac{1}{2}\kappa\omega\left(\frac{1}{2}\,\omega G^{-1} - u^+_{i,j}n_i n_j - 1\right) + 2\kappa_1 G[\![T]\!]\\&+\left\{\left[k_0 + k_1\left(\{T\} + \frac{1}{2}\,[\![T]\!]\right)\right.\right.\\&\left.\left.-k_4 G^{-1}\left(v^+_n + \frac{1}{2}\,\omega\right)\right] T^+(\tau_0 G)^{-1} + Q^+_n\right\}[\![T]\!]\\&\times\left(T^+ + [\![T]\!]\right)^{-1}\left(T^+\right)^{-1} \geq 0\end{aligned} \tag{29}$$

where $Q^+_n = Q^+_i n_i$

Assume that the compression wave propagates in the medium. In this case, the values ω and v^+_n are negative, but the values $u^+_{i,j}$, Q^+_n, T^+, $\{T\}$, $[\![T]\!]$, and G are positive. Considering that such material's constants as κ, κ_1, τ_0, k_0, k_1, and k_4 are positive as well, we are led to the conclusion that on the compression wave, the inequality (29) is fulfilled automatically.

Problem Formulation and Method of Solution

Let beginning from the moment of time $t = 0$ the values $v_3(0,t)$ and $\theta(0,t)$ be given at the thermoelastic half-space boundary $X_3 = 0$, which can be expanded into Maclaurin series with respect to time t:

$$\begin{aligned}v_3(0,t) &= \sum_{k=0}^{\infty}\frac{1}{k!}\,g_k t^k,\\ \theta = T - T_0 &= \sum_{k=0}^{\infty}\frac{1}{k!}\,m_k t^k\end{aligned} \tag{30}$$

where $g_k = g_k(X_1,X_2)$ and $m_k = m_k(X_1,X_2)$ are known functions of the X_1 and X_2 coordinates.

Hereafter we anticipate that the duration of the dynamic subjection on the half-space boundary is reasonably small, so we can restrict ourselves by two first terms in the expansions (30).

Assume that as a result of the objection of the boundary conditions (30), the two shock waves, quasi-thermal and quasi-longitudinal waves described in detail above, propagate in the half-space. The problem is reduced to the construction of the solution behind the shock wave fronts (the zones I and II in Fig. 1) up to the boundary plane, such that the solution should fulfill both the boundary conditions (30) and relations (20) and (22) on the shock wave fronts.

As the investigative technique, we use the ray method [6, 7] (▶ Ray Expansion Theory) to the effect that the solution behind the wave front Σ_1 in the zone I is constructed in terms of the ray series:

$$\begin{aligned}v_3^{(I)} &= \sum_{k=0}^{\infty}\frac{1}{k!}\,[\![v_{3,(k)}]\!]_T\big|_{t=Z_T}\,(t - Z_T)^k H(t - Z_T)\\ \theta^{(I)} &= \sum_{k=0}^{\infty}\frac{1}{k!}\,[\![\theta_{,(k)}]\!]_T\big|_{t=Z_T}\,(t - Z_T)^k H(t - Z_T)\end{aligned} \tag{31}$$

but the solution in the zone II, i.e., behind the wave front Σ_2 up to the boundary plane, is written as

$$\begin{aligned}v_3^{(II)} &= v_3^{(I)} + \sum_{k=0}^{\infty}\frac{1}{k!}[\![v_{3,(k)}]\!]_P\big|_{t=Z_P}(t-Z_P)^k H(t-Z_P)\\ \theta^{(II)} &= \theta^{(I)} + \sum_{k=0}^{\infty}\frac{1}{k!}[\![\theta_{,(k)}]\!]_P\big|_{t=Z_P}(t-Z_P)^k H(t-Z_P)\end{aligned} \tag{32}$$

where $v_{3,(k)} = \partial^k v_3/\partial t^k$, $\theta_{,(k)} = \partial^k\theta/\partial t^k$, $Z_{T,P} = \int_0^s G^{-1}_{T,P}\,ds$, s is the arc length along the ray, and $H\left(t - Z_{T,P}\right)$ is the unit Heaviside function.

In formulas (31) and (32), a distinction is not made between the rays issued out of one point of the boundary plane and directed perpendicular to the corresponding wave surfaces Σ_1 and Σ_2 at every instant of time, since, as it has shown in [10], the divergence of these rays may be neglected as the magnitudes of the third order of infinitesimal. However, the curvature of rays, which is characterized by the value $\delta n_i/\delta t = -g^{\alpha\beta}X_{i,\beta}G_{,\alpha}$, has, as exemplified by formulas (25) and (26), the first order of smallness and it should be taken into consideration.

Defining the Divergence of the Rays

Let us show following [10] that the divergence of the rays ΔS_α has third order of infinitesimal and it may be neglected. To do this, we use the relation [11]

$$\delta X_i/\delta t = Gn_i \tag{33}$$

and expand the values entering in it into a power series of t restricting by the linear terms

$$\begin{aligned} n_i &= n_i^0 + \left(\frac{\delta n_i}{\delta t}\right)^0 t \\ G_T &= G_1 + \alpha_1\omega_T^0 + \alpha_1\left(\frac{\delta\omega_T}{\delta t}\right)^0 t \\ G_P &= G_2 + \alpha_0\omega_T^0 + \alpha_2\omega_P^0 \\ &\quad + \left\{\alpha_0\left(\frac{\delta\omega_T}{\delta t}\right)^0 + \alpha_2\left(\frac{\delta\omega_P}{\delta t}\right)^0\right\}t \end{aligned} \tag{34}$$

Substituting (34) into (33) and considering that

$$\left(\frac{\delta n_i}{\delta t}\right)^0 = -\left(g^{\alpha\beta}\right)^0(G_{,\alpha})^0(X_{i,\beta})^0$$

after integrating (33), we obtain for the coordinates of two rays the following expressions accurate up to the magnitudes of third-order infinitesimal:

$$\begin{aligned} (X_i)_T &= \left(G_1 + \alpha_1\omega_T^0\right)tn_i^0 + G_1(\delta n_i/\delta t)^0\frac{1}{2}t^2 + (X_i)^0 \\ (X_i)_P &= \left(G_2 + \alpha_0\omega_T^0 + \alpha_2\omega_P^0\right)tn_i^0 \\ &\quad + G_2(\delta n_i/\delta t)^0\frac{1}{2}t^2 + (X_i)^0 \end{aligned} \tag{35}$$

where $(X_i)^0$ are the coordinates of the point M_0 on the boundary plane from which the rays (35) emerge (Fig. 1).

The divergence ΔS_α is calculated as

$$\Delta S_\alpha = \Delta X_i X_{i,\alpha} = \Delta X_i\left\{(X_{i,\alpha})^0 + (\delta X_{i,\alpha}/\delta t)^0 t\right\} \tag{36}$$

where $\Delta X_i = (X_i)_T - (X_i)_P$

Substituting (35) into (36) and considering that

$$(\delta X_{i,\alpha}/\delta t)^0 \approx (G_{,\alpha})^0 n_i^0$$

yield

$$\Delta S_\alpha \approx (G_1 - G_2)t^2(G_{,\alpha})^0 \tag{37}$$

Reference to (37) shows that the value ΔS_α actually has the third order of infinitesimal.

Thus, considering the above reasoning on the half-space boundary, i.e., at $s = 0$, there are two series in (32) take the form

$$\begin{aligned} v_3(0,t) &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\left[\!\left[v_{3,(k)}\right]\!\right]_T + \left[\!\left[v_{3,(k)}\right]\!\right]_P\right)\Big|_{t=0}t^k \\ \theta(0,t) &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\left[\!\left[\theta_{,(k)}\right]\!\right]_T + \left[\!\left[\theta_{,(k)}\right]\!\right]_P\right)\Big|_{t=0}t^k \end{aligned} \tag{38}$$

A comparison of the two terms of the series (30) and (38) yields

$$\begin{aligned} \left(\left[\!\left[v_3\right]\!\right]_T + \left[\!\left[v_3\right]\!\right]_P\right)\big|_{t=0} &= g_0, \\ \left(\left[\!\left[\theta\right]\!\right]_T + \left[\!\left[\theta\right]\!\right]_P\right)\big|_{t=0} &= m_0 \end{aligned} \tag{39}$$

$$\begin{aligned} \left(\left[\!\left[a_3\right]\!\right]_T + \left[\!\left[a_3\right]\!\right]_P\right)\big|_{t=0} &= g_1, \\ \left(\left[\!\left[\dot\theta\right]\!\right]_T + \left[\!\left[\dot\theta\right]\!\right]_P\right)\Big|_{t=0} &= m_1 \end{aligned} \tag{40}$$

where $a_i = \dot v_i$ is the acceleration of medium particles, $[\![\theta]\!] = [\![T]\!]$, and $[\![\dot\theta]\!] = [\![\dot T]\!]$.

The system of (39) after substituting the values $[\![T]\!]_T$ and $[\![T]\!]_P$ from (25) and (26) allows one to find the initial intensities of the quasi-thermal and quasi-longitudinal waves $\omega_T^0 = \omega_T|_{t=0}$ and $\omega_P^0 = \omega_P|_{t=0}$ in terms of the known functions $g_0(X_1, X_2)$ and $m_0(X_1, X_2)$.

Let us show now that the set of (40) permits to find the initial values $(\delta\omega_T/\delta t)^0 = \delta\omega_T/\delta t|_{t=0}$ and $(\delta\omega_P/\delta t)^0 = \delta\omega_P/\delta t|_{t=0}$ in terms of the given functions $g_1(X_1, X_2)$ and $m_1(X_1, X_2)$ and in terms of the initial intensities ω_T^0 and ω_P^0 found on the preceding step. For this purpose, above all it is to be shown that the discontinuities $[\![a_3]\!]_{T,P}$ and $[\![\dot\theta]\!]_{T,P}$ are expressed in terms of the values $\omega_{T,P}$ and $(\delta\omega/\delta t)_{T,P}$. To do this, we write (1), (4), and (5) in discontinuities and apply the condition of compatibility [6] (▶ Ray Expansion Theory) to the resulted relations

$$[\![f_{,i(k)}]\!] = -G^{-1}[\![f_{,(k+1)}]\!]n_i + G^{-1}\frac{\delta[\![f_{,(k)}]\!]}{\delta t}n_i + g^{\alpha\beta}[\![f_{,(k)}]\!]_{,\alpha}X_{i,\beta} \tag{41}$$

where f is a certain function, a Greek index after a point denotes a covariant derivative with respect to the corresponding curvilinear coordinate on the wave surface, and a Latin index in brackets after a point defines a time derivative of the corresponding order.

As this takes place, formulas (16–19) are taken into account, as well as the relation

$$[\![a_i]\!] = \Omega n_i + W^\gamma X_{i,\gamma} \tag{42}$$

which is the decomposition of the acceleration vector discontinuity in terms of three orthogonal vectors: normal vector and two tangential vectors to the wave surface.

As a result, on each shock wave, we obtain the system of three equations of three unknown values: Ω, $W_\delta = W^\gamma g_{\gamma\delta}$, and $[\![\dot T]\!]$, the solution of which has the form

$$\Omega = -\frac{(\Phi_i + \Phi_i^+)Gn_i + F_1}{M + F_2} \tag{43}$$

$$W_\delta = \Delta_\delta \Delta^{-1} \tag{44}$$

$$[\![\dot T]\!] = \eta\xi^{-1} \tag{45}$$

where

$$\begin{aligned}
\Phi_i n_i G &= B[\![v_k]\!][\![v_{jk}]\!]n_j - (\lambda G - p\omega)[\![v_{kk}]\!] \\
&\quad + G[\![L_i]\!]n_i - \rho_0 G\omega\frac{\delta G}{\delta t} + \kappa G[\![\dot T]\!](1 - \omega G^{-1}) \\
&\quad - \frac{\delta[\![v_k]\!]}{\delta t}\{(\lambda + 2\mu + (2l + m)\omega G^{-1} \\
&\quad + \rho_0 G^2 - \kappa[\![T]\!])n_k + lG^{-1}[\![v_k]\!]\}
\end{aligned}$$

$$\begin{aligned}
\Phi_i^+ n_i G &= \left((\lambda + 2l_2)v_{i,j}^+ n_i n_j + (p - l_2)v_{k,k}^+\right)\omega \\
&\quad - \left(\frac{\delta[\![v_k]\!]}{\delta t}n_k + G[\![v_{kk}]\!]\right)(\lambda + 2l_2)u_{i,j}^+ n_i n_j \\
&\quad - p\frac{\delta[\![v_k]\!]}{\delta t}n_k u_{k,k}^+ - 6l_1 G[\![v_{kk}]\!]u_{k,k}^+ \\
&\quad + [\![v_k]\!]\left(\lambda v_{k,j}^+ + l_2\dot A_{k,j}^+\right)n_j \\
&\quad - u_{k,j}^+\left\{(\lambda + l_2)\left(\frac{\delta[\![v_k]\!]}{\delta t}n_j + G[\![v_{kj}]\!]\right)\right. \\
&\quad \left. + l_2\left(\frac{\delta[\![v_j]\!]}{\delta t}n_k + G[\![v_{jk}]\!]\right)\right\} \\
&\quad + [\![v_k]\!]\left(\mu v_{k,j}^+ + \frac{3}{4}l_3\dot A_{j,k}^+\right)n_j \\
&\quad - \left(\mu + \frac{3}{4}l_3\right)\left\{\left(\frac{\delta[\![v_k]\!]}{\delta t}A_{k,i}^+ - [\![v_k]\!]\dot A_{k,i}^+\right)n_i\right. \\
&\quad + u_{k,j}^+ n_j\left(\frac{\delta[\![v_k]\!]}{\delta t} + \frac{\delta[\![v_i]\!]}{\delta t}n_i n_k\right. \\
&\quad \left.\left. - G[\![v_{ik}]\!]n_i\right) - \omega\dot A_{kj}^+ n_j n_k\right\} \\
&\quad - n_i\left(\frac{\delta[\![v_j]\!]}{\delta t}n_k + G[\![v_{jk}]\!]\right)\left(\mu u_{i,k}^+ + \frac{3}{4}l_3 A_{i,k}^+\right)n_j \\
&\quad - \omega\left(\mu v_{i,k}^+ + \frac{3}{4}l_3\dot A_{i,k}^+\right)n_k n_j \\
&\quad - u_{j,k}^+ n_j\left\{\left(\mu + \frac{3}{4}l_3\right)\frac{\delta[\![v_i]\!]}{\delta t}n_i n_k\right. \\
&\quad \left. + \frac{3}{4}l_3\frac{\delta[\![v_k]\!]}{\delta t} + G\left(\mu + \frac{3}{4}l_3\right)[\![v_{ik}]\!]n_i\right\} \\
&\quad + \kappa G\left([\![\dot T]\!]u_{i,j}^+ + [\![T]\!]v_{i,j}^+\right)n_i n_j \\
&\quad - \kappa\left(\omega\dot T^+ - \{T\}\frac{\delta[\![v_i]\!]}{\delta t}n_i\right)
\end{aligned}$$

$$\begin{aligned}
M &= \lambda + 2\mu + (3l + m)\omega G^{-1} \\
&\quad - \rho_0 G^2 - 2\kappa\{T\} + a^+
\end{aligned}$$

$$\begin{aligned}
F_\alpha &= \left\{\left(f_1^{(\alpha)}Y_2^2 - f_2^{(\alpha)}Y_1^2\right)g^{\gamma 1}\right. \\
&\quad \left. + \left(f_2^{(\alpha)}Y_1^1 - f_1^{(\alpha)}Y_2^1\right)g^{\gamma 2}\right\}M_\gamma\Delta^{-1}
\end{aligned}$$

$$M_\gamma = \left[\left(\mu + \frac{3}{4}l_3\right)u_{j,k}^+ - lu_{k,j}^+\right]n_j X_{k,\gamma}$$

$$
\begin{aligned}
f_\delta^{(1)} =& \left\{ \left(l_2 + \frac{3}{4} l_3 \right) \omega - \mu G \right\} [\![v_k]\!]_{,\delta} n_k \\
&+ G[\![L_i]\!] X_{i,\delta} + \left(\mu + \frac{3}{4} l_3 \right) [\![v_k]\!] [\![v_k]\!]_{,\delta} \\
&- \left(\mu + \rho_0 G^2 + l\omega G^{-1} - \kappa [\![T]\!] \right) \frac{\delta [\![v_i]\!]}{\delta t} X_{i,\delta} \\
&+ \omega \left\{ \left(\lambda v_{i,j}^+ + l_2 \dot{A}_{ij}^+ \right) n_j + 6 l_1 v_{k,k}^+ n_i \right\} X_{i,\delta} \\
&- \left(\frac{\delta [\![v_k]\!]}{\delta t} n_k + G [\![v_{kk}]\!] \right) \left\{ \left(\lambda u_{i,j}^+ + l_2 A_{ij}^+ \right) n_j \right. \\
&\left. + 6 l_1 u_{k,k}^+ n_i \right\} X_{i,\delta} - (\lambda + l_2) \frac{\delta [\![v_i]\!]}{\delta t} u_{k,k}^+ X_{i,\delta} \\
&- l_2 G [\![v_j]\!]_{,\delta} n_j u_{k,k}^+ - \left(\mu + \frac{3}{4} l_3 \right) \\
&\times \left\{ \left(\frac{\delta [\![v_k]\!]}{\delta t} A_{ik}^+ - [\![v_k]\!] \dot{A}_{ik}^+ + \frac{\delta [\![v_i]\!]}{\delta t} u_{k,j}^+ n_k n_j \right) X_{i,\delta} \right. \\
&\left. + G u_{k,j}^+ n_j \left([\![v_k]\!]_{,\delta} + g^{\alpha\beta} [\![v_i]\!]_{,\alpha} X_{k,\beta} X_{i,\delta} \right) \right\} \\
&- \left\{ \left(\frac{\delta [\![v_j]\!]}{\delta t} n_k + G [\![v_{jk}]\!] \right) \left(\mu u_{i,k}^+ + \frac{3}{4} l_3 A_{ik}^+ \right) n_j \right. \\
&- \omega n_k \left(\mu v_{i,k}^+ + \frac{3}{4} l_3 \dot{A}_{ik}^+ \right) \\
&\left. + \left(\mu + \frac{3}{4} l_3 \right) u_{j,k}^+ n_j n_k \frac{\delta [\![v_i]\!]}{\delta t} \right\} X_{i,\delta} \\
&- G u_{j,k}^+ n_j \left\{ \left(\mu + \frac{3}{4} l_3 \right) g^{\alpha\beta} [\![v_i]\!]_{,\alpha} X_{k,\beta} X_{i,\delta} \right. \\
&\left. + \frac{3}{4} l_3 [\![v_k]\!]_{,\delta} \right\} + \kappa G X_{i,\delta} n_j \left([\![\dot{T}]\!] u_{i,j}^+ + [\![T]\!] v_{i,j}^+ \right) \\
&+ \kappa \{T\} \frac{\delta [\![v_i]\!]}{\delta t} X_{i,\delta}
\end{aligned}
$$

$$
\begin{aligned}
\eta =& G^{-2} \tau_0^{-1} [\![T]\!] \left\{ k_0 + k_1 \left(\{T\} + \frac{1}{2} [\![T]\!] \right) \right. \\
&\left. - k_4 G^{-1} \langle v_k \rangle \right\} \\
&+ G^{-1} \left\{ k_* G^{-1} \frac{\delta [\![T]\!]}{\delta t} + k_3 \left(A_{ik}^+ + [\![A_{ik}]\!] \right) n_i \right. \\
&\times \left([\![T]\!]_{,\alpha} g^{\alpha\beta} X_{k,\beta} + G^{-1} \frac{\delta [\![T]\!]}{\delta t} n_k \right) \\
&\left. + \left(k_0 [\![T]\!] - k_2 \omega G^{-1} \right) n_k T_{,k}^+ + k_3 [\![A_{ik}]\!] n_i T_{,k}^+ \right\} \\
&- \tau_0 \left\{ G^{-1} \frac{\delta [\![Q_i]\!]}{\delta t} n_j + g^{\alpha\beta} [\![Q_i]\!]_{,\alpha} X_{j,\beta} \right. \\
&- \left(2\kappa_1 \dot{T}^+ + \frac{1}{2} \kappa v_{j,j}^+ \right) [\![T]\!] \\
&- \frac{1}{2} \kappa G^{-1} \left(T^+ + [\![T]\!] \right) \left(\frac{\delta [\![v_i]\!]}{\delta t} n_i - [\![a_i]\!] n_i + G [\![v_{ii}]\!] \right) \\
&- \frac{1}{2} \kappa T^+ G^{-1} \left[\left(\frac{\delta [\![v_i]\!]}{\delta t} n_k - [a_i] n_k + G [\![v_{ik}]\!] \right) \right. \\
&\left. \left. \times \left(u_{i,k}^+ - G^{-1} [\![v_i]\!] n_k \right) - [\![v_i]\!] v_{i,k}^+ n_k \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
k_* =& k_0 + k_1 \left(\{T\} + [\![T]\!] \right) \\
&+ k_2 \left(u_{i,j}^+ n_i n_j - \omega G^{-1} \right)
\end{aligned}
$$

$$
\begin{aligned}
[\![Q_j]\!] =& -(\tau_0 G)^{-1} [\![T]\!] \left\{ \left[k_0 + k_1 \left(\{T\} + \frac{1}{2} [\![T]\!] \right) \right. \right. \\
&\left. - k_2 G^{-1} \langle v_i \rangle n_i \right] n_j \\
&\left. - k_3 G^{-1} \left(\langle v_j \rangle n_k + \langle v_k \rangle n_j \right) n_k \right\}
\end{aligned}
$$

$$
[\![v_{ij}]\!] = g^{\alpha\beta} [\![v_i]\!]_{,\alpha} X_{j,\beta}
$$

$$
[\![L_i]\!] = g^{\alpha\beta} \left(G [\![L_{ij}]\!] \right)_{,\alpha} X_{j,\beta}, \quad \dot{A}_{ij}^+ = v_{i,j}^+ + v_{j,i}^+
$$

$$
f_\delta^{(2)} = \left\{ (\lambda + \mu) u_{i,k}^+ n_k + B A_{ik}^+ n_k + 6 l_1 u_{k,k}^+ n_i \right\} X_{i,\delta}
$$

$$
\Delta = Y_1^1 Y_2^2 - Y_1^2 Y_2^1,
$$

$$
\Delta_1 = f_1 Y_2^2 - f_2 Y_1^2, \quad \Delta_2 = f_2 Y_1^1 - f_1 Y_2^1
$$

$$
f_\delta = f_\delta^{(1)} + \Omega f_\delta^{(2)}
$$

$$
\begin{aligned}
Y_\delta^\alpha =& - \left\{ \mu + l\omega G^{-1} - \kappa [\![T]\!] - \rho_0 G^2 + (\lambda + l_2) u_{k,k}^+ \right. \\
&\left. + \left(2\mu + \frac{3}{2} l_3 \right) u_{k,j}^+ n_k n_j - \kappa \{T\} \right\} g_\delta^\alpha \\
&- g^{\gamma\alpha} X_{k,\gamma} X_{i,\delta} A_{ki}^+ \left(\mu + \frac{3}{4} l_3 \right)
\end{aligned}
$$

$$
\begin{aligned}
\xi =& G^{-2} \left\{ k_* + k_3 \left(A_{ik}^+ + [\![A_{ik}]\!] \right) n_k \right\} \\
&- 2\kappa_1 \tau_0 \left(T^+ + [\![T]\!] \right)
\end{aligned}
$$

When employing formulas (42–45), it is necessary to take into account that immediately ahead of the quasi-thermal wave front the conditions $T^+ = T_0,\ u_{i,j}^+ = v_i^+ = v_{i,j}^+ = \dot{T}^+ = 0$ are fulfilled, but immediately ahead of the quasi-longitudinal wave front relations (23) are valid. As to the values $\dot{v}_{i,j}$ and $\dot{T}^+$, then they can be disregarded, since these values are present only in terms of the second order of infinitesimal in formulas (42–45). From formulas (42–45), as well as from relations (25) and (26), it is seen that the values $[\![a_i]\!]$ and $[\![\theta]\!] = [\![\dot{T}]\!]$ are expressed ultimately in terms of ω and $\delta\omega/\delta t$.

Let us put $t = 0$ in (42–45) and introduce an upper zero index to denote the initial magnitudes of the values. As a result we obtain on the quasi-thermal wave

$$\begin{aligned} [\![a_3]\!]_T|_{t=0} &= a_{11}\omega_T^0 + a_{12}(\delta\omega_T/\delta t)^0 \\ [\![\dot\theta]\!]_T|_{t=0} &= b_{11}\omega_T^0 + b_{12}(\delta\omega_T/\delta t)^0 \end{aligned} \quad (46)$$

where

$$a_{11} = -\kappa k_0 T_{10}\tau_0^{-1}\left\{(\lambda+2\mu-\rho_0 G_1^2)\varphi_1 - \kappa^2 T_0\tau_0\right\}^{-1}$$

$$\begin{aligned} a_{12} = &-\{\kappa(2k_0T_{10}+\tau_0\kappa G_1 T_0) \\ &-\varphi_1 G_1(\lambda+2\mu-\rho_0 G_1^2)\} \\ &\times\left\{G_1\left[(\lambda+2\mu-\rho_0G_1^2)\varphi_1-\kappa^2T_0\tau_0\right]\right\}^{-1}, \end{aligned}$$

$$\varphi_1 = k_0 G_1^{-2} - 2\kappa_1 T_0\tau_0$$

$$b_{11} = (k_0T_{10} - \tau_0^2\kappa G_1 T_0 a_{11})(\tau_0 G_1\varphi_1)^{-1}$$

$$b_{12} = \{2k_0T_{10}+\tau_0\kappa T_0 G_1(1-a_{12})\}(G_1\varphi_1)^{-1}$$

and on the quasi-longitudinal wave

$$\begin{aligned} [\![a_3]\!]_P|_{t=0} &= a_{21}\omega_P^0 + a_{22}(\delta\omega_P/\delta t)^0 \\ [\![\dot\theta]\!]_P|_{t=0} &= b_{21}\omega_P^0 + b_{22}(\delta\omega_P/\delta t)^0 \end{aligned} \quad (47)$$

where

$$\begin{aligned} a_{21} = &-\kappa\delta G_2T_{20}\tau_0^{-1}(\varphi_2+2\kappa_1\tau_0T_0^+) \\ &\times\{(\lambda+2\mu-\rho_0G_2^2-\chi\omega_T^0G_1^{-1})\varphi_2-\kappa^2\tau_0T_0^+\delta\}^{-1} \end{aligned}$$

$$\begin{aligned} a_{22} = &-\{\kappa\delta[2G_2T_{20}(\varphi_2+2\kappa_1\tau_0T_0^+) \\ &+\kappa\tau_0T_0^+\delta]+[(-m-3l+\kappa T_{10}G_1)\omega_T^0G_1^{-1} \\ &-\lambda-2\mu-\rho_0G_2^2]\varphi_2\} \\ &\times\{(\lambda+2\mu-\rho_0G_2^2-\chi\omega_T^0G_1^{-1})\varphi_2-\kappa^2\tau_0T_0^+\delta\}^{-1} \end{aligned}$$

$$\begin{aligned} b_{21} = &\{(\varphi_2+2\kappa_1\tau_0T_0^+)G_2T_{20}\tau_0^{-1} \\ &-\kappa\tau_0T_0^+a_{21}\delta\}\varphi_2^{-1} \end{aligned}$$

$$b_{22} = \left\{2G_2^{-1}T_{20}\left[k_0+k_1T_{10}\omega_T^0 - \frac{1}{2}k_4\omega_T^0(G_1^{-1}+G_2^{-1})\right]+\kappa\tau_0T_0^+(1-a_{22})\delta\right\}\varphi_2^{-1}$$

$$\delta = 1-\omega_T^0G_1^{-1},$$

$$T_0^+ = T_0+T_{10}\omega_T^0,$$

$$\chi = 2s+p+\kappa T_{10}G_1$$

$$\varphi_2 = G_2^{-2}(k_0+k_1T_{10}\omega_T^0-k_4\omega_T^0G_1^{-1})-2\kappa_1\tau_0T_0^+$$

Substituting expressions (25) and (26) into the set of (39) and relations (46) and (47) into the set of (40), and dropping nonlinear terms, we arrive at two systems of linear equations in ω_T^0, ω_P^0, $(\delta\omega_T/\delta t)^0$, and $(\delta\omega_P/\delta t)^0$:

$$\omega_T^0+\omega_P^0 = g_0, \quad T_{10}\omega_T^0+T_{20}\omega_P^0 = m_0 \quad (48)$$

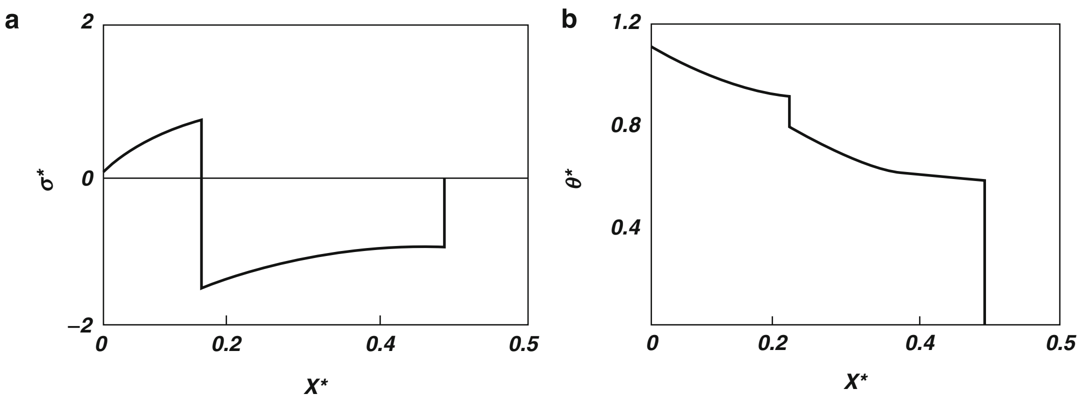
$$\begin{aligned} a_{12}(\delta\omega_T/\delta t)^0 + a_{22}(\delta\omega_P/\delta t)^0 &= g_1 - a_{11}\omega_T^0 - a_{21}\omega_P^0 \\ b_{12}(\delta\omega_T/\delta t)^0 + b_{22}(\delta\omega_P/\delta t)^0 &= m_1 - b_{11}\omega_T^0 - b_{21}\omega_P^0 \end{aligned} \quad (49)$$

Knowing the values $\omega_{T,P}^0$ and $(\delta\omega_{T,P}/\delta t)^0$, the approximate solution of the problem can be constructed in the zones *I* and *II* in terms of the two-term ray expansions (31) and (32), respectively. Really, the discontinuities $[\![v_3]\!]$, $[\![\theta]\!]$, $[\![a_3]\!]$, $[\![\dot\theta]\!]$, the velocities $G_{T,P}$, and eikonals $Z_{T,P}$ entering into the two-term ray expansions (31) and (32) are expressed, as it has been shown above, in terms of the functions $\omega_{T,P}$ and $\delta\omega_{T,P}/\delta t$, but these functions, due to smallness of the time, can be expanded in a power series of t, in so doing restricting by the magnitudes of the first order of infinitesimal, i.e.,

$$\begin{aligned} \omega_{T,P} &= \omega_{T,P}^0 + \left(\delta\omega_{T,P}/\delta t\right)^0 t, \\ \delta\omega_{T,P}/\delta t &= \left(\delta\omega_{T,P}/\delta t\right)^0 \end{aligned} \quad (50)$$

Substituting the values $\omega_{T,P}^0$ and $(\delta\omega_{T,P}/\delta t)^0$ found from (48) and (49) into formulas (50), and then substituting the resulted relations into the discontinuities $[\![v_3]\!]$, $[\![\theta]\!]$, $[\![a_3]\!]$, $[\![\dot\theta]\!]$, the velocities $G_{T,P}$, and eikonals $Z_{T,P}$, we determine all enumerated functions at an arbitrary instant of the time. Thus, the approximate solution is written as in the zone *I*

$$\begin{aligned} v_3^{(I)} = &\left\{\omega_T^0+(\delta\omega_T/\delta t)^0 Z_T \right. \\ &\left. +\Omega_T|_{t=Z_T}(t-Z_T)\right\}H(t-Z_T) \\ \theta^{(I)} = &\left\{T_{10}\left(\omega_T^0+(\delta\omega_T/\delta t)^0 Z_T\right)\right. \\ &\left.+T_{11}(\omega_T^0)^2+[\![\dot\theta]\!]_T|_{t=Z_T}(t-Z_T)\right\}H(t-Z_T) \end{aligned} \quad (51)$$



Boundary-Value Problems Resulting in Thermoelastic Shock Wave Propagation, Fig. 2 The dimensionless coordinate X^* dependence of the dimensionless (**a**) stress σ^* and (**b**) temperature θ^* (this figure is taken from [10] with the permission from Taylor & Francis)

in the zone *II*

$$\begin{aligned} v_3^{(II)} =& v_3^{(I)} + \left\{\omega_P^0 + (\delta\omega_P/\delta t)^0 Z_P \right. \\ & \left. + \Omega_P|_{t=Z_P}(t - Z_P)\right\} H(t - Z_P) \\ \theta^{(II)} =& \theta^{(I)} + \left\{T_{20}\left(\omega_P^0 + (\delta\omega_P/\delta t)^0 Z_P\right)\right. \\ & + T_{21}\omega_T^0\omega_P^0 + T_{22}(\omega_P^0)^2 \\ & \left. + [\![\dot\theta]\!]_P|_{t=Z_P}(t - Z_P)\right\} H(t - Z_P) \end{aligned} \quad (52)$$

where

$$\begin{aligned} Z_T &= \lambda_1^{-1}\, ln\left(1 + \frac{\lambda_1 s}{G_1 + \alpha_1\omega_T^0}\right), \\ Z_P &= \lambda_2^{-1}\, ln\left(1 + \frac{\lambda_2 s}{G_2 + \alpha_0\omega_T^0 + \alpha_2\omega_P^0}\right) \\ \lambda_1 &= \alpha_1(\delta\omega_T/\delta t)^0 (G_1 + \alpha_1\omega_T^0)^{-1}, \\ \lambda_2 &= \alpha_2(\delta\omega_P/\delta t)^0 (G_2 + \alpha_0\omega_T^0 + \alpha_2\omega_P^0)^{-1} \end{aligned}$$

The arc lengths $S_{T,P}$ of two rays perpendicular to the corresponding wave surfaces Σ_1 and Σ_2 are defined by the formulas

$$\begin{aligned} S_T &\approx (G_1 + \alpha_1\omega_T^0)t, \\ S_P &\approx (G_2 + \alpha_0\omega_T^0 + \alpha_2\omega_P^0)t \end{aligned} \quad (53)$$

in so doing the magnitude s changes from S_P to S_T and from 0 to S_P in the zones *I* and *II*, respectively.

Formulas (53) have been derived in [7] with the help of relations (35) considering that

$$S_{T,P} = \int_o^t \left((\dot X_i)_{T,P}(\dot X_i)_{T,P}\right)^{1/2} dt$$

Note that carrying out similar reasoning another boundary-value problem can be solved, where instead of the boundary conditions (30), the following conditions are assigned:

$$\sigma_{33}(0,t) = \sum_{k=0}^{\infty}\frac{1}{k!}d_k t^k, \quad \theta = T - T_0 = \sum_{k=0}^{\infty}\frac{1}{k!}m_k t^k \quad (54)$$

and the temperature and stress can be obtained as the functions of X_1, X_2, X_3, and t using the ray expansions (31) and (32).

As an example, we present in Fig. 2 the solution of the following boundary-value problem: $\theta(0,t) = m_0 = \text{const}$, and $\sigma_{33}(0,t) = 0$. In this case, the rays issued out from one point do not diverge and remain straight, and the wave front is plane one. Reference to Figs. 2a, b shows the dimensionless coordinate $X^* = X_3 c_l 2\kappa_1 T_0 k_0^{-1}$ dependence of the dimensionless stress $\sigma^* = \sigma_{33}(\kappa^T m_0)^{-1}$ and the dimensionless temperature $\theta^* = \theta m_0^{-1}$ at $t^* = tc_l^2 2\kappa_1 T_0 k_0 = 0.15$. During calculations the elastic and thermoelastic constants for iron were taken as in [10]. It can be seen from Fig. 2 that the stress and temperature have discontinuities at two points what corresponds to the location of the quasi-thermal and quasi-longitudinal wave fronts.

Hence, in the case when heat propagates with a finite speed, the solution has pure wave character, and the nonlinear ray method proposed in [10] allows one to solve the boundary-value problem under consideration. But in the case when heat propagates with an infinite speed, the solution contains both wave and diffusive terms, so the ray method is unsuitable.

Cross-References

▶ Propagation of Shock Waves in Thermoelastic Solids in View of Singular Surfaces
▶ Ray Expansion Theory

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Brakes

▶ Brakes, Thermal and Thermoelastic Analysis
▶ Perturbation Methods in Thermoelastic Instability (TEI) with Finite Element Implementation

Brakes, Thermal and Thermoelastic Analysis

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Synonyms

Brakes

Definition

A mechanical friction brake is a technical device that serves to slow down or stop a moving body, or for keeping it at rest. When a brake is acting, the kinetic energy is converted into heat energy via friction.

The main components of the friction brakes are friction elements. In braking, a rotating body or system of bodies (e.g., disks) are in contact with fixed friction elements.

Overview

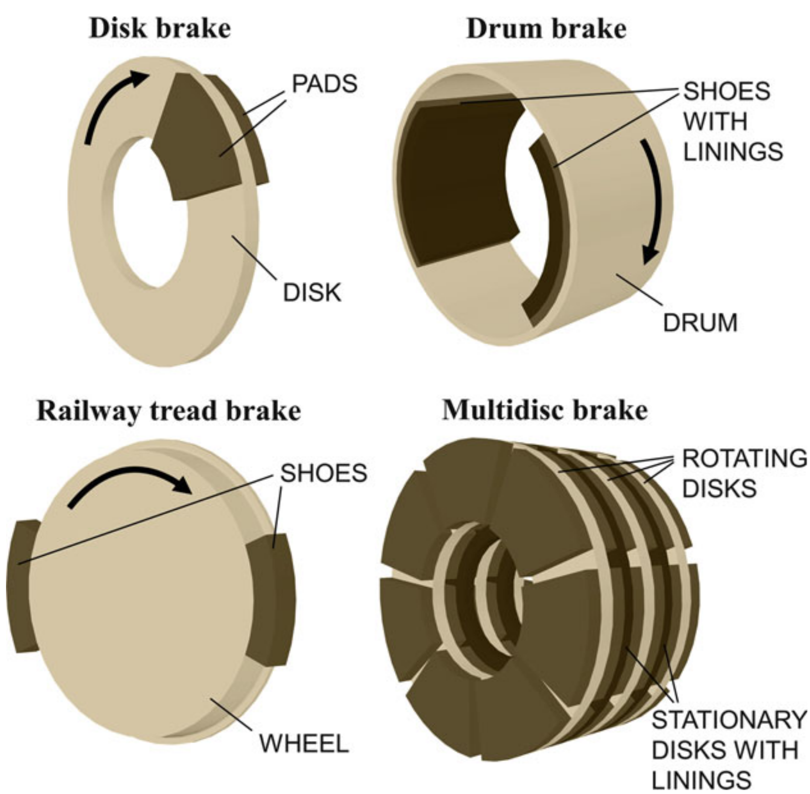
Thermal and thermoelastic analyses of the friction brakes are performed with the aim of understanding a number of associated phenomena, which are listed in Introduction. Typical are different length and time measures, so that the analyses show a different degree of complexity and require different approaches and computational efforts.

The thermal bond among the friction elements is described by contact conditions (1)–(3). The analyses become clearly simple if the heat field of friction elements is solved separately, assuming that the distribution of the heat removal from the contact surface follows (4). It is appropriate to use an analytical approach (see (5)–(17)) for determining the peak surface (flash) temperature, or the mean surface temperature, as it eliminates the necessity to consider 3D geometry of the whole elements. If the problem is to fix the bulk temperature (see Figs. 8–10), the finite element method is used, which also enables consideration of the dependence of the material parameters on temperature, to solve nonlinear problems and to cover the heat removal both via convection and radiation.

A more complex description of the braking action assumes a mutual relationship between the mating parts of the braking system. This allows, for example, the determination of the heat removal from the contact surface more accurately. Thermal deformation of the friction elements can cause nonuniform distribution of contact pressure and consequently also nonuniform distribution of frictional heat generation. This mutual feedback at high sliding velocity can imply ▶ frictionally excited thermoelastic instability.

Introduction

The brakes are classified with respect to the arrangement of the brake elements. The charts of the basic types of brakes are given in Fig. 1. The first three types of brakes of the figure are characterized by intermittent contact. That is, each point of the contact surface of the rotating



Brakes, Thermal and Thermoelastic Analysis, Fig. 1 Charts of friction brakes basic types. Brakes classified with respect to the structural arrangement of the brake elements body is in contact with fixed friction segments only for a limited part of the revolution period.

If the temperature passes beyond a certain critical limit in breaking, undesired effects such as brake fading, hot judder, premature wear, material degradation, and thermal cracks [1, 2] can appear. Apart from macroscopic cracks that usually arise as a consequence of repeated thermomechanical loading, the origin of the cracking network on the friction surface can be observed. Yet the origin is not satisfactorily explained, due to the competition of fretting, thermomechanical, and also material effects [3].

Let us mention, by way of example, some very different brakes. The energy dissipated by the disk brake when stopping a passenger car weighting 1,500 kg from a speed of 100 km/h makes 0.15 MJ. But each 10-disk brake of a Boeing 777 passenger aircraft must be capable of absorbing up to 144 MJ [4]. The most commonly used material for producing current automobile brake disks is cast iron, while in the other case, carbon-based composites are used.

Brake Thermal Analysis

Modeling of Friction Element Contacts

From a microscopic point of view, contact with friction between the two bodies 1 and 2 is a very complex effect, which is affected by the surface roughness, composition of the materials used, their wear, and tear, and so on. We further idealize the reality, and only macroscopic physical entities will be taken into account. We assume that both bodies are homogenous and isotropic and that their contact surfaces are smooth with perfect contact. That is why we, in addition to this, assume equal surface temperatures of both bodies on their contact area (see Fig. 2), that is,

$$T_1(x, y, 0, t) = T_2(x, y, 0, t), t > 0 \qquad (1)$$

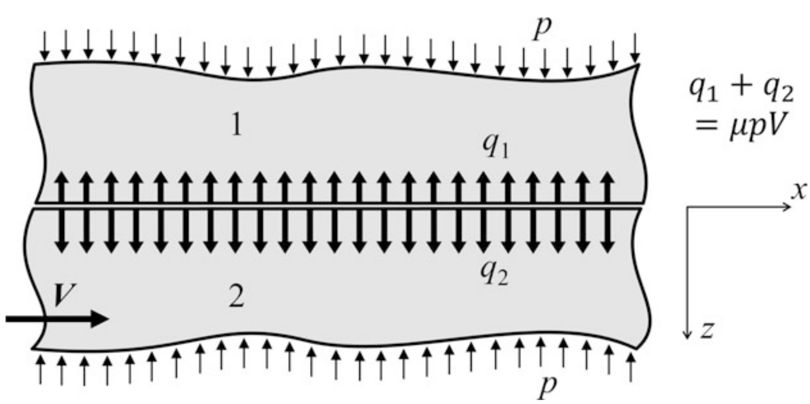
If we denote the sliding velocity of these bodies by V, normal contact pressure by p, and the friction coefficient by μ, then the heat produced by their friction per time unit applied to a unit area will be

$$q = \mu p V \qquad (2)$$

(neglecting the heat due to surface wear). The quantities q, p, and V may depend on time t. Let us further denote the heat flux removed from the contact surface to the body i by q_i, $i = 1{,}2$, and the associated temperature fields $T_i(x,y,z,t)$ according to Fig. 2. It holds that

$$q = q_1 + q_2, \quad q_1 = k_1 \frac{\partial T_1}{\partial z}, \quad q_2 = -k_2 \frac{\partial T_2}{\partial z} \qquad (3)$$

where k_i stands for the thermal conductivity coefficient of the material of body i. The problem of determining the temperature fields T_1 and T_2 will be substantially simplified if we solve the problem of heat conduction through bodies 1 and 2 separately. In such a case we assume the following distribution of respective flows from the contact area to be



Brakes, Thermal and Thermoelastic Analysis, Fig. 2 Charts of two bodies with a slipping contact

$$q_1 = q\left(1 + \frac{A_2}{A_1}\frac{k_2}{k_1}\sqrt{\frac{\kappa_1}{\kappa_2}}\right)^{-1}$$
$$q_2 = q\left(1 + \frac{A_1}{A_2}\frac{k_1}{k_2}\sqrt{\frac{\kappa_2}{\kappa_1}}\right)^{-1} \quad (4)$$

where κ is thermal diffusivity. It holds that $A_1 = A_2$ for full contact and $A_1 \neq A_2$ for intermittent contact, where A_i denotes the size of the corresponding contact area of body i. The relation (4) holds rigorously in case of a slip of two half-spaces with equal temperatures to infinity. It has been derived using the formula for the transient temperature of the semi-infinite body with constant unit heat flux $q(t) \equiv 1$ from contact plane $z = 0$ (see, e.g., [5])

$$\Theta_I(z,t) = \Theta_0 + \frac{2\sqrt{\kappa\, t}}{k}\,\mathrm{ierfc}\left(\frac{z}{2\sqrt{\kappa\, t}}\right) \quad (5)$$
$$z \geq 0, \quad t > 0$$

where Θ_0 is a constant initial temperature. Further we have $\mathrm{ierfc}(\eta) = \frac{1}{\sqrt{\pi}}\exp(-\eta^2) - \eta(1 - \mathrm{erf}(\eta))$, for $\eta > 0$ with the error function $\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta \exp(-\zeta^2)\mathrm{d}\zeta$.

For a contemporary automobile brake with disks made from cast iron with the friction material A of pads from Table 1 and with the value $A_2/A_1 = 7$, approximately 98 % of the produced heat goes into the disk.

Contact Surface Temperature: 1D Approximation

The Péclet number is defined as

$$Pe = \frac{Va}{2\kappa} \quad (6)$$

where $2a$ is the full length of the contact (of the pad). The one-dimensional approximation is useful for Pe > 10 [6, 7]. Let us consider the following 1D problem:

$$\frac{\partial T(z,t)}{\partial t} = \kappa\frac{\partial^2 T(z,t)}{\partial z^2}, \quad 0 \leq z \leq H, \; t > 0,$$
$$-k\frac{\partial T(0,t)}{\partial z} = Q(t), \quad k\frac{\partial T(H,t)}{\partial z} = 0, \quad T(z,0) = \Theta_0 \quad (7)$$

Brakes, Thermal and Thermoelastic Analysis, Table 1 Orientation values of parameters of some materials in use. k – thermal conductivity, κ – thermal diffusivity, E – elastic modulus, α – coefficient of thermal expansion, ρ – density

	k W/(mK)	$\kappa.10^6$ m^2/s	E GPa	$\alpha.10^6$ 1/K	ρ kg/m^3
Cast iron	54	12.98	125	12	7,100
Friction material A	5	3.57	1	10	4,000
Friction material B	1.2	0.52	8	15	3,000
Composite C/SiC	40	22.2	30	0.5	2,300

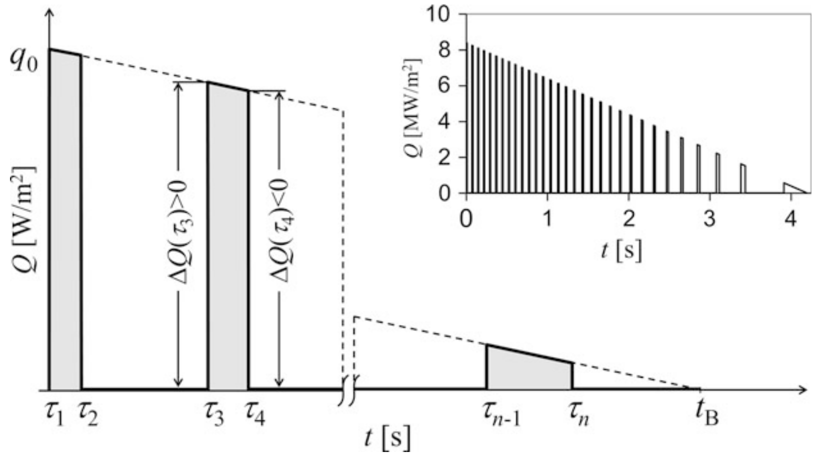
which describes the problem of heat removal from a given point of the body contact surface. For a disk, the symbol H denotes a half of its thickness, while for a drum, the symbol H means its full thickness, as is the case with the pads. In other words, under intensive braking, the heat removal from the surface due to convection and radiation is small in comparison with the friction heating. If the unit flux $Q(t) \equiv 1$ is the case, we receive, using Laplace transform, solution of the problem (7) following [5] in the form

$$\Theta_{II}(z,t) = \Theta_0 + \frac{2\sqrt{\kappa t}}{k}\sum_{n=0}^{\infty}\left\{\mathrm{ierfc}\left(\frac{2H(n+1)-z}{2\sqrt{\kappa t}}\right) + \mathrm{ierfc}\left(\frac{2Hn+z}{2\sqrt{\kappa t}}\right)\right\} \quad (8)$$

Since the flux $Q(t)$ may be time dependent, we receive the general solution of the problem (7) using Duhamel's theorem [5, 8]. After Özisik [8], it holds that

$$T(z,t) = \int_0^t \Theta_{II}(z,t-\tau)\frac{dQ(\tau)}{d\tau}d\tau + \sum_{j,\tau_j \leq t_B}\Theta_{II}(z,t-\tau_j)\,\Delta Q(\tau_j) \quad (9)$$

if the function Q is smooth in the intervals (τ_j, τ_{j+1}) and has a jump point $\Delta Q(\tau_j) = Q^+(\tau_j) - Q^-(\tau_j)$ to the magnitude at the time points τ_j. Let us further



Brakes, Thermal and Thermoelastic Analysis, Fig. 3 Time behavior of heating at a fixed point of the contact surface of a rotating body (brake disk), situation for intermittent contact, and deceleration braking

consider two model cases of braking. In the first case, when $V(t) = V_0$, $0 \leq t \leq t_B$, the subject is continuously braking, that is, preventing unwanted acceleration, for example, while driving downhill. The braking time is t_B. The other case is deceleration braking or emergency braking, and the important matter is the fast slowing down of the vehicle or setting it to the rest, that is, $V(t) = V_0.(1-t/t_B)$. At full contact, by analogy, $Q(t) = q_0$, or $Q(t) = q_0.(1-t/t_B)$ if both the contact pressure p and friction coefficient μ are constant. In the first case, (9) the solution yielded is $T(z,t) = \Theta_{II}(z,t)\, q_0$ for $0 < t \leq t_B$, and $T(z,t) = \{\Theta_{II}(z,t) - \Theta_{II}(z,t-t_B)\}\, q_0$ for $t > t_B$. The first approximation of the surface temperature for emergency braking is the Fazekas known formula [7, 9]

$$T(0,t) \approx \Theta_0 + \frac{2q_0\sqrt{\kappa}}{k\sqrt{\pi}}\sqrt{t}\left(1 - \frac{2t}{3t_B}\right), \quad 0 \leq t \leq t_B \tag{10}$$

(we take $\Theta_{II}(0,t) \approx \Theta_0 + \frac{2\sqrt{\kappa}}{k\sqrt{\pi}}\sqrt{t}$ in (9)). Full contact can be assumed in the problem of pad heating.

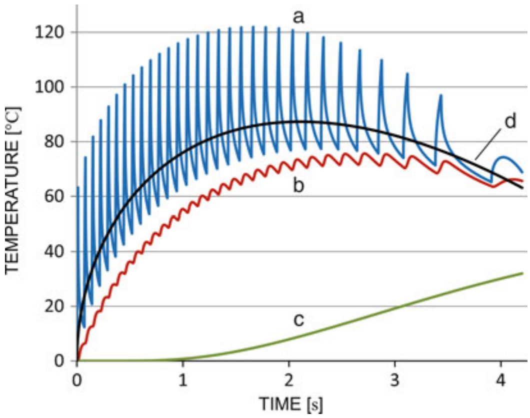
For intermittent contact, the flux $Q(t)$ is positive only if we consider passing a given contact point under the friction pad having $2a$ in width size. In Fig. 3, we have the function $Q(t)$ for emergency braking with a contemporary disk brake. Potential cooling of the surface has not been considered.

Let us consider, by way of example, heating of the brake disk when braking from the initial sliding velocity $V_0 = 11.2$ m/s (which corresponds to the automobile velocity of 100 km/h), braking time $t_B = 4.2$ s, and initial heat flux $q_0 = 8.4$ W/mm^2. The length of trajectory of a given point under the friction pad is $2a = 112$ mm, while its trajectory in full revolution is $L = 780$ mm. The half thickness of the disk is $H = 13.2$ mm. The disk is considered to be made from cast iron and the pads from the friction material A as presented in Table 1. We have high $Pe_0 = 24{,}160$ at the beginning of the braking. The trends in the behavior of the temperatures at the points differently distant from the surface are depicted in Fig. 4.

If the Péclet number is great, the rise ΔT in temperature can be evaluated directly using (8) when the point is passing under the friction pad, which takes the time $t = 2a/V$. For $z = 0$, only the member with $n = 0$ dominantly contributes to the sum for this very short time interval in the corresponding series since $\mathrm{ierfc}(\infty) = 0$. Since $\mathrm{ierfc}(0) = \frac{1}{\sqrt{\pi}}$, we also receive (cf. [5, 6])

$$\Delta T \approx q\frac{2\sqrt{\kappa}}{k\sqrt{\pi}}\sqrt{t} = \frac{2qa}{k\sqrt{\pi}\sqrt{Pe}} \tag{11}$$

The rise in temperature depends on the distance r of a given point from the rotation axis in a brake disk. If the pad holds a central angle φ, then $2a = r\varphi$. Since $V = r\omega$, $q = \mu p r\omega$, a straightforward conclusion follows from (11)



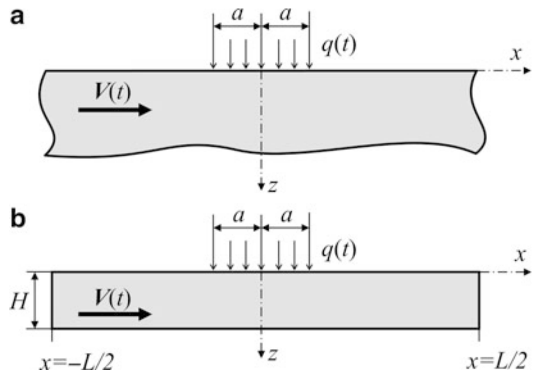
Brakes, Thermal and Thermoelastic Analysis, Fig. 4 Time variation of temperature at selected points of a rotating body in intermittent contact. (**a**) Point at contact surface, (**b**) point 1.5 mm under surface, (**c**) point in the half thickness position of the body (disk), (**d**) the mean surface temperature

that the dependence $\Delta T \approx r \mu p \frac{2\sqrt{\kappa}}{k}\sqrt{\omega \varphi}$ on radius r is linear.

Vernersson [10] puts a relation more general than (11) if railway tread braking is the case. Moreover, it takes into consideration both the influence of tread cooling and heat transfer from the wheel to the rail during their mutual contact.

Yevtushenko and Kuciej [7] have found an analytical solution of a problem more general than (7) using the Laplace transform method. It presents a solution for frictional heating during braking in a three-element tribosystem in full contact. An analytical solution has also been found, for example, for the influence of convective cooling at the outer surface of the friction pad.

As a matter of course, the problem set in (7) can be tackled using numerical procedures such as the finite element method. What is more, it is possible to take into consideration temperature-dependent material parameters, it is possible to differentiate the vented part of the disk, and it is possible to consider a surface of a two-layer cover strap and to cover cooling of the contact surface. Achieving satisfactory accuracy of the numerical solution calls for a sufficiently dense mesh in the neighborhood of the point $z = 0$ (i.e., near the contact).



Brakes, Thermal and Thermoelastic Analysis, Fig. 5 Moving region (body) under immovable thermal source. (**a**) Half-space, (**b**) unfolded drum or half of a disk

2D Unfolded Model

If a quantity of heat W per unit length is instantaneously liberated at time $t = 0$ along the y-axis on the surface of a half-space $z \geq 0$, then the temperature distribution at t is (see [5, 11])

$$\Theta(x,z,t) = \frac{W}{2\pi\, k\, t}\ \exp\left(-\frac{x^2+z^2}{4\kappa\, t}\right) \qquad (12)$$

if the initial temperature $\Theta_0 = 0$. Let us further imagine that the coordinate system (x, z, y) is fixed and that this half-space moves with a velocity $V(t)$ in the direction of the x-axis. In addition, let us consider an immovable heat source $q(t)$ as Fig. 5a shows and which now acts within the band $-a \leq x \leq a$ parallel to axis y. Using (12) and the procedure outlined by Carlslaw and Jaeger [5] or Johnson [11], we get the relation

$$\Theta_{III}(x,z,t) = \frac{1}{2\pi k}\int_0^t \frac{q(\tau)}{t-\tau} \times \int_{-a}^{a} \exp\left(-\frac{(x-s-d(\tau,t))^2+z^2}{4\kappa(t-\tau)}\right) ds\, d\tau \qquad (13)$$

for the unknown temperature field after the time interval t measured from the beginning of the braking. Here, $d(\tau,t) = \int_\tau^t V(\eta)\, d\eta$ is the distance which the body travels between the time points

τ and t. We perform the integration indicated in expression (13) with respect to the spatial variable s, and we write down the result in the dimensionless form

$$\Theta^*_{III}(x^*,z^*,t^*) = \frac{\sqrt{Pe_0}}{4}\int_0^{t^*}\frac{q^*(t^*-\tau^*)}{\sqrt{\tau^*}}\exp\left(-\frac{z^{*2}}{4\tau^*}\right)\cdot \left\{\mathrm{erf}\left(\frac{x^*+1-2\tau^*Pe_0D(\tau^*)}{2\sqrt{\tau^*}}\right) -\mathrm{erf}\left(\frac{x^*-1-2\tau^*Pe_0D(\tau^*)}{2\sqrt{\tau^*}}\right)\right\}d\tau^* \quad (14)$$

$\Theta^*_{III} = \frac{\Theta_{III}}{Tr}$, $Tr = \frac{2q_0a}{k\sqrt{\pi}\sqrt{Pe_0}}$, $x^* = \frac{x}{a}$, $z^* = \frac{z}{a}$, $t^* = \frac{\kappa t}{a^2}$. Here $D \equiv 1$, $q^* \equiv 1$, if $V(t) \equiv V_0$, and $D(\tau^*) = 1 - t^*/t^*_B + \tau^*/t^*_B$, $q^*(\eta) = 1 - \eta^*/t^*_B$ if $V(t) = V_0(1 - t/t_B)$, $q(t) = q_0(1 - t/t_B)$. The reference temperature Tr has been selected with respect to relation (11) comparably with the rise in temperature ΔT.

In the case of an unfolded model of a friction rotating body, it is necessary to determine the temperature field T for a finite region, which is shown in Fig. 5b. Let us introduce two other dimensionless numbers $L^* = \frac{L}{a}$ and $H^* = \frac{H}{a}$. First of all, it is necessary to satisfy the periodicity condition

$$T^*\left(\frac{L^*}{2},z^*,t^*\right) = T^*\left(-\frac{L^*}{2},z^*,t^*\right) \quad (15)$$

$$\frac{\partial T^*}{\partial x^*}\left(\frac{L^*}{2},z^*,t^*\right) = \frac{\partial T^*}{\partial x^*}\left(-\frac{L^*}{2},z^*,t^*\right) \quad \text{for} \quad 0 \le z^* \le H^*,\ 0 < t^*$$

and the conditions for the zero heat flux through the plane $z = H$

$$\frac{\partial T^*}{\partial z^*}(x^*,H^*,t^*) = 0 \quad \text{for} -\frac{L^*}{2} \le x^* \le \frac{L^*}{2},\ 0 < t^* \quad (16)$$

The dimensionless solution has the form

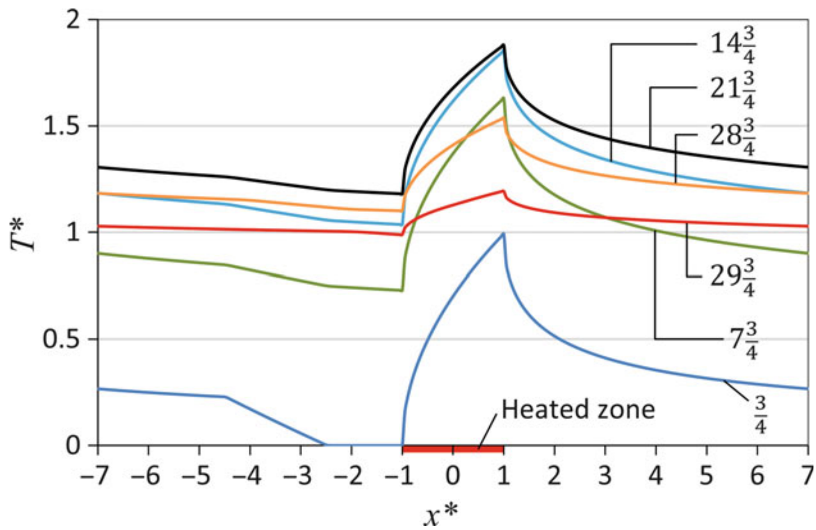
$$T^*(x^*,z^*,t^*) = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\{\Theta^*_{III}(x^*+mL^*,2(n+1)H^* - z^*,t^*) + \Theta^*_{III}(x^*+mL^*,2nH^*+z^*,t^*)\} \quad (17)$$

over $T(x,z,t) = Tr\ T^*(x^*,z^*,t^*) + \Theta_0$. The sum of the series involving the subscript n guarantees that condition (16) has been met, and it represents the changeover from the solution in a half-space to the solution for a band. The sum of the series involving the subscript m leads to fulfilling periodicity condition (15). The dimensionless temperature of the contact surface for the above mentioned example of braking is presented in Fig. 6 for some selected time points.

Finite Element Models

The geometry of the rotating body of a brake can usually be expressed by an axisymmetric model. The vented part of the disk can be regarded as anisotropic material with recalculated values of material parameters. If we want to calculate the flash temperature at least approximately (the accuracy depends on the elements mesh size), it is necessary to specify the heating curve $Q(t,r)$ similar to that shown in Fig. 3 at the contact area. Now, it also depends on the node position, that is, on its radius r. If only establishing the mean surface and body temperatures will suffice, we specify the rating curve in the form $Q(t,r) = q(t,r)\,A_2/A_1$. Figure 9 presents an example of the temperature field for disk Ø312 mm.

In the case of a 3D problem, it is usually possible to make use of the cyclic symmetry of the rotor and to perform the finite element discretization just for the necessary reference segment, in as far as we do not calculate the flash temperature (see Fig. 10). We then specify the condition of periodicity in the planar sections defining the segment – the equality of temperatures and normal heat flows in the pairs of boundary nodes which have equal radial and axial coordinates. If the half of the reference segment can be limited by the planes of symmetry, then we specify the



Brakes, Thermal and Thermoelastic Analysis, Fig. 6 Dimensionless temperature $T^*(x^*, 0, t^*)$ of contact surface with the parameters $Pe_0 = 24{,}160$, $t_B^* = 1.738 \times 10^{-2}$, and $L^* = 13.93$, $H^* = 0.2357$. Temperature variation for ¾, 7¾, 14¾, 21¾, 28¾, and 29¾ of disk rotation measured from the beginning of the braking

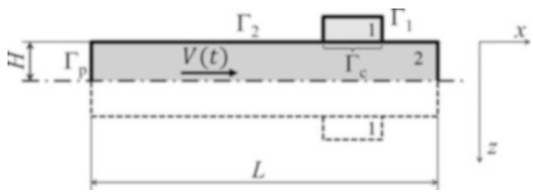
adiabatic conditions as boundary conditions of this semi-model.

In the case of long-standing or repeated braking, it is necessary for the FEM model to also cover the cooling of the brake surfaces by convection and radiation. The rate of heat dissipation into the environment is related to the nature of the coolant (air) flow and is influenced by the arrangement of the space around the brake. The computer-aided simulation of heating and cooling of the brake consequently constitutes an element of the demanding CFD calculations as well.

The temperature analysis of a brake becomes a great deal more strenuous if we require accurate fulfilling of the interface conditions (1)–(3). Let us mention by way of example an "unfolded" 2D model of a disk brake from Fig. 7 (see similarly [12] for a multidisk clutch). Temperature fields $T_i(x,z,t)$ within the regions i meet the equations

$$c_i\rho_i\frac{\partial T_i}{\partial t}+V_i\frac{\partial T_i}{\partial x}=\frac{\partial}{\partial x}\left(k_i\frac{\partial T_i}{\partial x}\right)+\frac{\partial}{\partial z}\left(k_i\frac{\partial T_i}{\partial z}\right),\ i=1,2 \tag{18}$$

where $V_2 = V(t)$, $V_1 = 0$ and next, they fulfill boundary conditions (1)–(3) on Γ_c, $k_2\frac{\partial T_2}{\partial z} = 0$ for $z = H$, $-k_i\frac{\partial T_i}{\partial n_i} = \alpha_i(T_i - T_\infty)$ on Γ_i, $i = 1,2$. Here n_i denotes a vector normal to the boundary of the



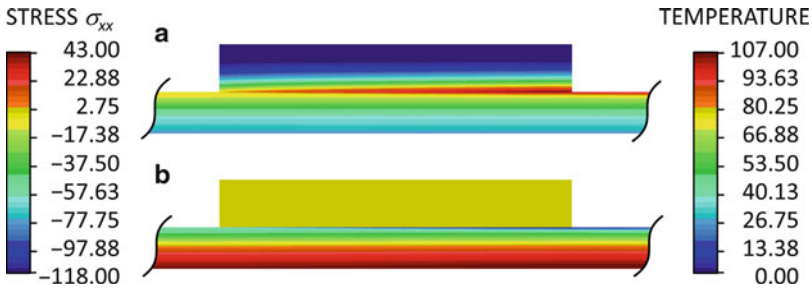
Brakes, Thermal and Thermoelastic Analysis, Fig. 7 Scheme of disk brake 2D model. 1 – pads; 2 – disk; Γ_c – contact surface; Γ_1, Γ_2 – cooled surfaces; Γ_p – section with periodicity condition

region i, T_∞ environment temperature, and α_i the corresponding heat transfer coefficient. Furthermore, the condition of periodicity

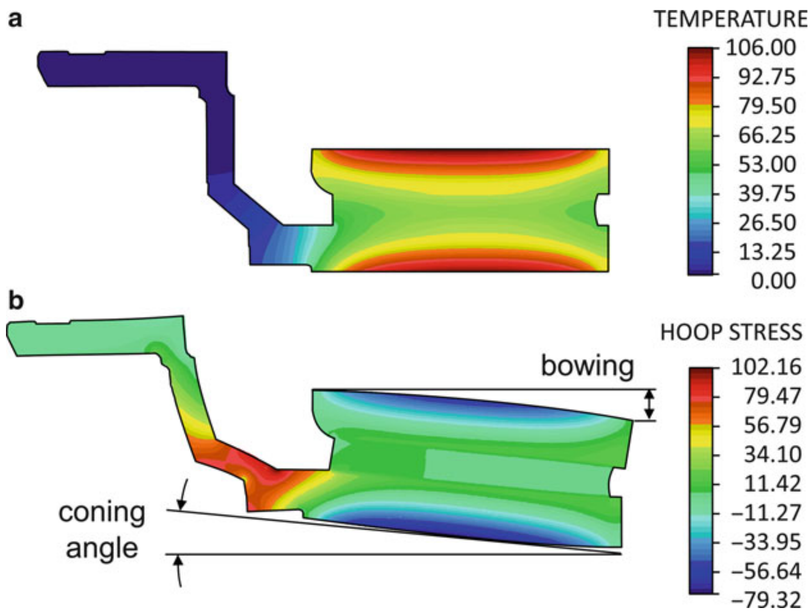
$$T_2(x,z,t) = T_2(x+L,z,t),$$
$$\frac{\partial T_2}{\partial n_2}(x,z,t) = -\frac{\partial T_2}{\partial n_2}(x+L,z,t)$$

must be satisfied for all $(x,z) \in \Gamma_p$ and $t > 0$. The initial condition is $T_i(x,z,0) = T_{i0}(x,z)$ for (x,z) from the region i. The Galerkin finite element discretization method is unstable with this problem, owing to the Péclet number value. This difficulty can be removed using the Petrov-Galerkin method (see, e.g., [13]). A calculated temperature field corresponding to the example of Fig. 4 can be seen in Fig. 8.

The solution of the equations (18) in complete 3D geometry is extremely calculation tough for



Brakes, Thermal and Thermoelastic Analysis, Fig. 8 Heating and stressing of brake 2D model. FEM results for the example of Fig. 4. (**a**) Temperature field at time point 3 s; (**b**) stress σ_{xx} at time point 3 s



Brakes, Thermal and Thermoelastic Analysis, Fig. 9 Heating and stressing of a brake disk. (**a**) Example of the bulk temperature field, (**b**) the corresponding circumferential stress field, enlarged distortion

large Péclet numbers. The effect of the surface heating is restricted to a thin skin adjacent to the surface (e.g., see Fig. 8). Consequently, it is necessary to use very fine finite element mesh near the surface. That is why Floquet and Duborg [14] proposed a new hybrid FFT-FEM method that combines Fourier transformation techniques and a finite element method for the axisymmetric problem of a rotating body. Additional references present Yevtushenko and Grzes [9].

Uncoupled Thermal Stress and Distortion Analysis

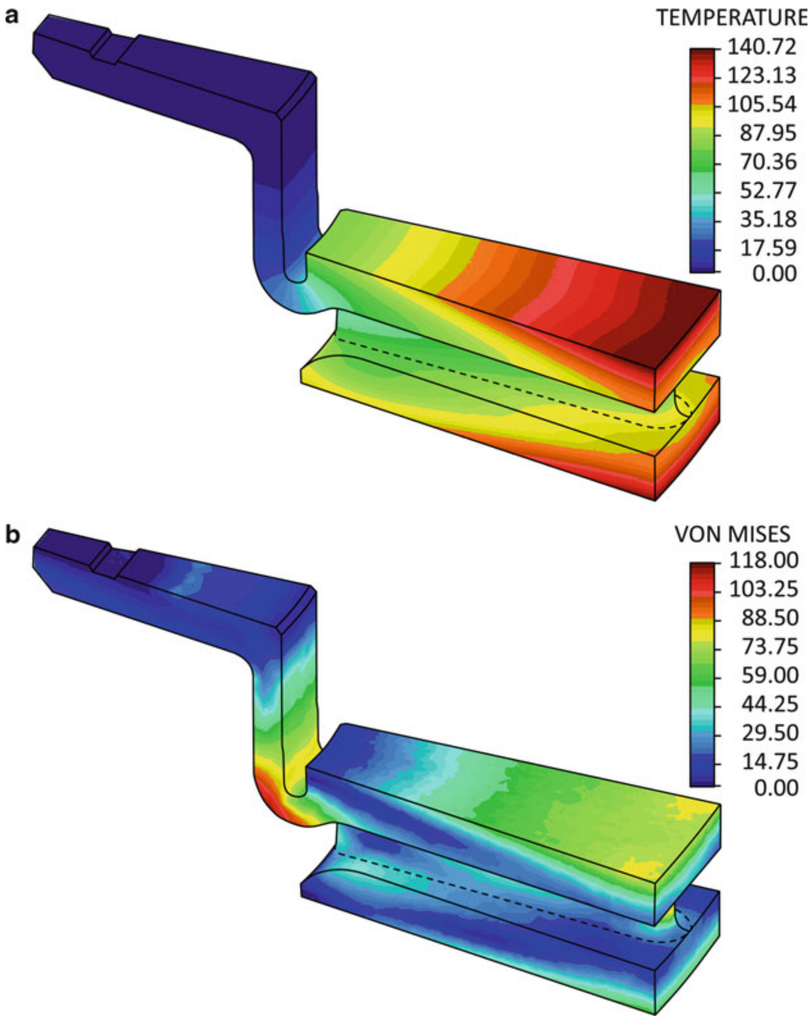
A particular solution to the thermoelastic equation can be obtained in the form of a strain potential (see, e.g., [15, 16])

$$2G\,\mathbf{u} = \nabla\Phi \tag{19}$$

where the scalar potential function Φ satisfies the equation

$$\nabla^2\Phi = \frac{2G(1+\nu)\alpha}{1-\nu}T \tag{20}$$

Let us here denote the displacement vector as $\mathbf{u} = (u_x,u_y,u_z)$, T a temperature field previously calculated, α coefficient of thermal expansion, ν Poisson number, and $G = \frac{E}{2(1+\nu)}$ is the shear modulus with Young's modulus E. As first we want to evaluate the stress acting on the surface and on the thin surface layer due to heating that just takes a short time (the time interval needed for a surface point to pass under the friction pad or duration of one rotation under full contact with a large Péclet number). As we have already seen,



Brakes, Thermal and Thermoelastic Analysis, Fig. 10 Heating and stressing of a brake disk, results of FEM calculation for a 3D model. (**a**) Example of the bulk temperature field, (**b**) the corresponding von Mises stress field

it is then possible to judge the temperature field as a function of only one space variable z (formulae (5), (8), (9)). Then it holds that $\sigma_{xx} = \sigma_{yy} = \frac{\partial^2 \Phi}{\partial z^2}$, $\sigma_{zz} = 0$, and $\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$ for the three normal components of the stress tensor (Barber [15]). Brief heating of an intensity q over a time interval Δt, which causes a rise to the temperature ΔT from the relation (11), brings about a compressive stress of the surface $z = 0$

$$\bar{\sigma}_{xx} = \bar{\sigma}_{yy} \approx -\frac{2E\alpha\sqrt{\kappa}}{(1-\nu)\,k\,\sqrt{\pi}}\,q\,\sqrt{\Delta t} \qquad (21)$$

(we obtain a value of this stress caused by heating with $\Delta T = 63.2$ °C to be -126.4 MPa at the first passage under the friction pad in the example presented in Fig. 4).

Johnson [11] suggests how to find the thermal distortion of the surface due to the moving heat source (a situation corresponding to Fig. 5a). He discloses that the difference of normal displacements of the surface at the trailing edge ($x = a$) and at the leading edge ($x = -a$) is

$$\Delta\,\bar{u}_z \approx -2c\,q\,a^2/Pe \qquad (22)$$

for a large Péclet number where $c = (1+\nu)\,\alpha/k$ is the distortivity of the material. (If we consider the example from Fig. 4, we have $\Delta\,\bar{u}_z \approx -0.6\,\mu\text{m}$ in the first revolution).

The method of evaluating stress and deformation of the surface due to purely mechanical loading, that is, due to normal pressure p and shear forces, suggests, for example, Barber [15] and Johnson [11]. The loading by shear forces is associated with the potential danger of causing the phenomenon known as fretting fatigue.

It is insufficient for putting a value on the influence of bulk temperature on stressing of the friction segment to use only the mentioned particular solution by way of the potential function Φ from (19), (20). That is to say, the displacements u_x and u_y from (19) are zero in case of $\Phi = \Phi(z)$, although we may regard them as free at the end of the friction element. It is possible to exercise the procedure presented in the book from Timoshenko and Goodier [16] applied to the free plate of a thickness H for the initial evaluation. We obtain

$$\begin{aligned} \sigma_{xx}(z) = \sigma_{yy}(z) \approx & -\frac{\alpha E}{1-v}\theta(z) + \frac{1}{H(1-v)}\int_0^H \alpha E\theta(s)\,ds \\ & + \left\{\frac{12(z-H/2)}{H^3(1-v)}\int_0^H \alpha E\theta(s)\,(s-H/2)\,ds\right\}, \\ \theta(z) = & \ T(z) - \Theta_0 \end{aligned} \tag{23}$$

where, in case that $z = H$ is a plane of geometrical and physical symmetry (e.g., brake disk), we put to zero the last addendum, removing the total bending moment, that is, $\{\ldots\} = 0$.

Finally, the thermoelastic stress states and deformation of 2D and 3D models of friction elements can be determined in a satisfactory manner using the finite element method if the corresponding temperature fields are known. The examples of such results are presented in Figs. 8–10. A great deal of attention has been paid both to undesired deformations as well as, for example, the rotor coning of the disk plate due to thermal loading and structural loading events (see, e.g., [17]).

Coupled Thermal and Distortion Analysis

The analysis, which aims to cover the temperature and mechanical contact of mating frictional elements of a brake as accurately as possible, must take into consideration their mutual coupling. That is to say, the temperature deformations of these elements cause nonuniform distribution of the contact pressure which, according to (2), leads subsequently to nonuniform heating, which is the agent of these thermal deformations. Moreover, the contact conditions may change during the braking. It was, for example, Dufrénoy [18] and Day et al. [19] who came along with the analysis of such complex behavior of brakes. They also considered the tribological actions such as wear and the coefficient of friction variations. Computer-aided modeling of thermoelastic wear problems is even more developed (see e.g., Mróz and Páczelt [20]).

Frictionally Excited Thermoelastic Instability
The mutual coupling *thermal deformation – contact pressure – frictional heat generation* leads to the rise of macroscopic hot spots and to the associated hot judder at a high sliding velocity. Barber [21] described the frictionally excited thermoelastic instability as the cause of the phenomenon that is of critical importance in the design of brakes and clutches. The stability analysis showed that the amplitudes of natural modes of perturbation of temperature and contact pressure grow exponentially as $\exp(bt)$. The growth parameter b depends on sliding velocity V, perturbance wavelength, friction coefficient, and also on material parameters of the friction elements (see Table 1) and their geometry. Barber [22] and Al-Shabibi [23] are important reviews in this field.

Some circumstances, more or less unsatisfactorily quantified up to now, contribute to stopping this exponential growth in real braking. First of all, it is a value of friction coefficient decreasing with the temperature growth and a change to the values of material parameters, localization of the contact, and possible origin of plastic deformations or major wear.

Cross-References

▶ Frictionally Excited Thermoelastic Instability (TEI)

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Brittleness Measure of Ceramics

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Synonyms

Perovskite is a calcium titanium oxide mineral species composed of calcium titanate, with the chemical formula CaTiO3.It lends its name to the class of compounds which have the same type of crystal structure as (XIIA2+VIIB4+X2−3)

Overview

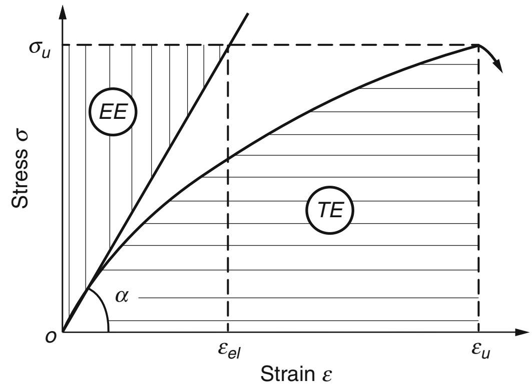
The use of modern ceramics, such as perovskites or transformation-toughened zirconia, calls for more detailed data than those provided by the conventional strength or crack-resistance characteristics. After studying the deformation behavior of brittle materials, a new parameter called brittleness measure (χ) was proposed. If $\chi = 1$, ceramics obey Hooke's law, and their mechanical behavior is similar to that of glass. At the same time, the behavior of these materials under loading may be similar to that of concrete, which cannot be described by the elastic-solid model ($\chi < 1$). It has been found that aluminum oxide and silicon nitride ceramics are characterized by $\chi = 1$, while silicon nitride with silicon carbide and corundum refractory with zirconium dioxide are characterized by $\chi < 1$. Being inelastic, they have an rising R-curve and show insignificant differences in the critical stress-intensity factors for specimens with a notch and sharp crack. The evaluations of such ceramics should be based on the proportionality limit rather than on the ultimate-strength values. Thermal-shock-resistance criteria should take into account the true values of ultimate strain. The experimental data reveal that the evaluation of the performance and mechanical behavior of brittle materials should not disregard their deformation behavior that may affect the reliability of the results. It is shown the brittleness measure χ and ultimate strains may be regarded as important mechanical characteristics of ceramics.

Introduction

The selection of structural ceramics for specific applications is usually based on their strength and crack-resistance characteristics. Less attention is usually given to their deformation capabilities. As a result, these materials are assumed to be linearly elastic and obeying Hooke's law [1], that is, similar to glass. This assumption is fair for certain kinds of ceramics, for example, single-phase dense alumina or silicon nitride ceramics. In many cases, however, the ceramics exhibit inelastic behavior. For instance, the transformation-toughened zirconia-based ceramics happen to be in many respects similar to metals and are sometimes, not without a reason, called "ceramic steel" [2]. In other words, their mechanical behavior is somewhat similar to that of concrete, that is, these materials are inelastic. Thus, some ceramics cannot be described as linearly elastic [3], and the linear elastic fracture mechanics [3], which underlies the evaluation of strength and crack resistance of the materials, may not be directly applicable to those ceramics. Therefore, possible errors in the engineering evaluation of such materials can be excluded by using additional data on their mechanical behavior, such as deformation characteristics.

Brittleness Measure

Studies of ceramics and other brittle materials show [4, 5] that they should be classified according to their deformation performance: (*a*) linearly deformative up to fracture (called "brittle") and



Brittleness Measure of Ceramics, Fig. 1 Stress–strain diagram: σ_u – ultimate stress; ε_{el} – elastic ultimate strain; ε_u – actual ultimate strain; EE and TE – elastic and total fracture energies, respectively

(*b*) nonlinearly deformative after reaching the elasticity limit (called "relatively brittle" in contrast to ductile metals). For this purpose, the brittleness measure χ has to be introduced [6]. It equals to the ratio of the elastic energy (EE in Fig. 1) stored in ceramics to fracture to the total strain energy (TE) spent to fracture. It represents energy characterizing the deviation of the actual deformation of ceramics from the Hooke's law.

The parameter χ can be determined in four-point flexure tests by measuring the load on and deflection of the rectangular-beam specimen and calculating the stress and strain [7]:

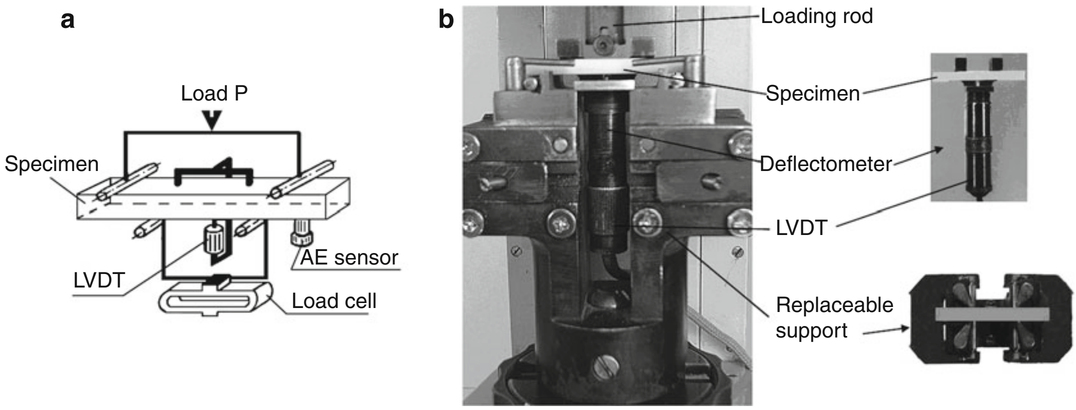
$$\chi = \frac{\sigma_u^2}{2E \int\limits_0^{\varepsilon_u} \sigma d\varepsilon}, \tag{1}$$

where σ_u is the ultimate strength, E is the elastic modulus, ε_u is the ultimate strain, and σ is the stress at the current strain value ε.

The material is linear elastic if $\chi = 1$ or inelastic if $\chi < 1$. To understand what has been stated above, it is expedient to discuss the experimental tests data for elastic and inelastic, single-phase and composite, and transformation-toughened ceramic materials.

Methods and Materials

All the experiments were conducted using a homemade universal loading fixture "Ceramtest"



Brittleness Measure of Ceramics, Fig. 2 Schematic of four-point flexure test (**a**) and the three-point test jig with the deflectometer (**b**)

Brittleness Measure of Ceramics, Table 1 Properties of ceramics

Ceramic	Brittleness measure χ	Density ρ, g/sm^3	Strength σ_u, MPa	Strain ε_u, %	Elastic modulus E, GPa	Stress intensity K_I^R, MPa·m$^{1/2}$	References
ZTA	1.00	3.70	314	0.097	336	3.50	8
Y-PSZ	1.00	6.02	713	0.385	185	4.80	8
Y-TZP	1.00	6.02	1021	0.494	205	5.60	7
Mg-Al-ZrO_2	0.72	4.82	170	0.160	134	3.00	8
Syalon + 30%BN	0.70	2.62	160	0.160	137	3.80	8
Cordierite	0.60	2.26	40	0.096	76	1.93	8
Mg-PSZ	0.44	0.41	506	0.460	204	12.0	8
Al_2O_3 + 6%ZrO_2	0.36	0.36	23	0.720	88	1.17	8
$La_{0.2}Ca_{0.8}CoO_3$	0.25	0.25	140	0.240	148	2.25	12

with a rigid load cell and off-line measuring systems (Fig. 2), which was installed on a conventional universal testing machine. Its jig (Fig. 2b) has rollers that are free to roll in order to eliminate frictional constraints when the specimen surface expands or contracts during loading [8]. The high accuracy of measurements provided by this jig is due to not only the free movement of the specimen surfaces along its axis during the loading but also the symmetry of the force-application points about the loading axis and the uniformity of the bending moment in the pure-bending zone. For four-point flexure, the loading rollers are located with an inner span of 20 mm and an outer span of 40 mm. For three-point flexure, the distance between the support rollers is set at 20 mm. The deflection of the specimen was measured with a special precision LVDT with a resolution of approximately 1 μm per mm of deflection. A semiconductor transducer was used to analyze acoustic emission (AE) signals.

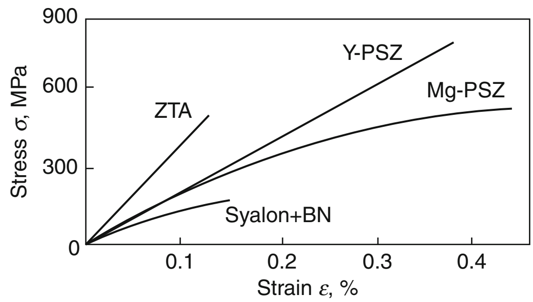
Each material was tested at ambient conditions. The load on the specimen was recorded until fracture as a function of deflection. The resulting curve was used to calculate the brittleness measure with formula (1) and to determine the ultimate strain and static modulus of elasticity. Such tests were also conducted at high and low temperatures using a different testing fixture [9]. The specimens were rectangular beams with dimensions 3 × 5 × 50 mm^3, which were ground longitudinally with the edges rounded (they were placed on the supporting rollers with a side of 5 × 50 mm^2). The crack resistance of ceramics [10] was evaluated on the same specimens in three-point flexure, but the distance between the load-application points was 20 mm. In this case, the specimen was placed on the support rollers with a side of 3 × 50 mm^2, and the critical stress-intensity factors (fracture toughness) K_{Ic} were calculated in accordance with [11].

The following materials were tested: zirconia-toughened alumina, ZTA, with 16 mol % ZrO_2 (Central Institute of Physics of Solids and Materials Science, Germany), zirconia ceramics partially stabilized by magnesia Mg-PSZ (NILCRA Incorp., Australia), $La_{0.8}Ca_{0.2}CoO_3$ perovskite (Norwegian University of Science and Technology, Norway), as well as other ceramics produced in the USSR. The characteristics of the materials are summarized in Table 1.

Results and Discussion

The test data, for example, for ZTA and Y-PSZ (Fig. 3) suggest that these are linearly elastic materials with the brittleness measure $\chi = 1$. The same has been observed for dense single-phase silicon nitride and carbide, yttria, alumina, and many other ceramics [12]. Unlike the above, the multiphase SiAlON + 30% BN, refractory ceramics $Al_2O_3 + ZrO_2$ as well as transformation-toughened Mg-PSZ and Mg-Al-PSZ ceramics exhibit nonlinear deformation behavior ($\chi < 1$). In the latter case, the stress–strain curves also displayed residual deformation after unloading (Fig. 4a) and a continuous acoustic emission, which accompanied the loading of the specimen (Fig. 4b).

In heating [13], brittle ceramics ($\chi = 1$), such Y-TZP, became inelastic, like glass at a certain temperature (Fig. 5a). By contrast, with increase in the test temperature, the ceramics with $\chi < 1$



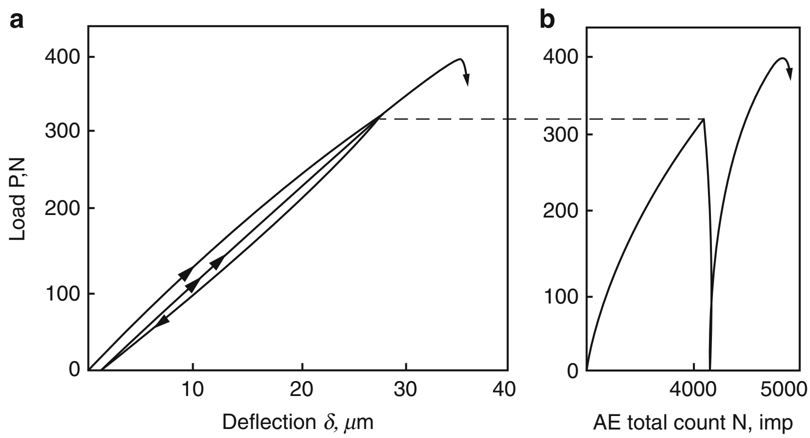
Brittleness Measure of Ceramics, Fig. 3 Stress–strain diagrams of elastic and inelastic ceramics

first became more elastic and then exhibit decreasing inelasticity (Fig. 5b). The temperature dependence of the deformation behavior of ceramics can conveniently be represented by the brittleness measure versus temperature curve (Fig. 6). It is also important to know the temperature dependence of the ultimate strain, ε_u, shown in Fig. 6b. These data are interesting because they indicate the presence of temperature ranges where the ceramics may be more sensitive to cyclic and long-term loading. Normalized brittleness measure ($\bar{\chi}$) may appear useful for evaluation of the change in the deformation behavior of brittle materials with change in test temperature [7] (Fig. 7):

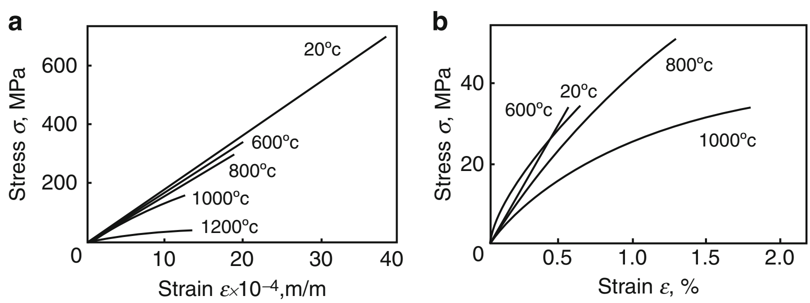
$$\bar{\chi} = \frac{\chi_T - \chi}{\chi} 100\% \qquad (2)$$

where χ_T is the brittleness measure at set temperature.

The fracture of ceramics with $\chi < 1$ differs substantially from ceramics with $\chi = 1$. These materials exhibit different sensitivity to the stress concentration, which is demonstrated by the SENB and SEVNB test data (two substantially different U- and V-shaped stress concentrators in fracture-toughness specimens) collected in Table 2. It can be observed that, when $\chi = 1$, the values of K_{Ic} obtained with the SENB method are much greater than those obtained with the SEVNB method (the lesser the value of χ, the lesser the difference between the values of K_{Ic}). Therefore, the ratio $\varphi = K_{Ic}^{SEVNB}/K_{Ic}^{SEPB}$ was introduced in [10] to identify the deformation

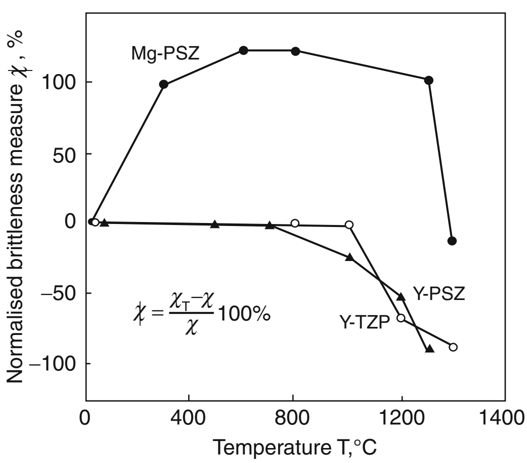


Brittleness Measure of Ceramics, Fig. 4 Cycling loading of Mg-PSZ ceramics: load-deflection diagram (**a**) and an acoustogram (**b**)



Brittleness Measure of Ceramics, Fig. 5 Stress–strain diagram for ceramics at different temperatures: Y-PSZ (**a**) and cordierite (**b**)

B



Brittleness Measure of Ceramics, Fig. 7 Temperature dependence of normalized brittleness measure $\not\chi$

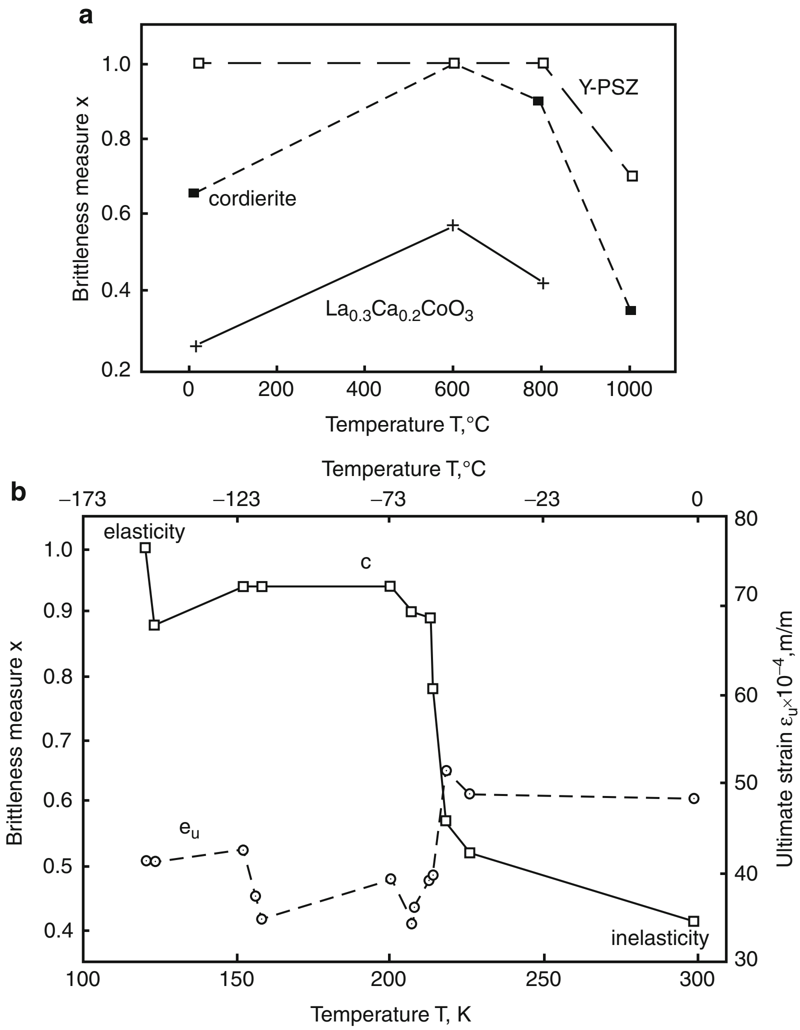
behavior of materials (the greater this ratio, the more inelastic the material is). This ratio is interesting because it is calculated from tests performed on a simpler test fixture than that used for brittleness measure tests (there is no need to measure the deflection of the specimen).

The brittleness measure values can be used for rough estimation of the crack-growth resistance of ceramics (R-curve effect [3]). Such curves are usually linear for elastic ceramics ($\chi = 1$) such as Y-PSZ and nonlinearly increasing for relatively brittle ceramics ($\chi < 1$) such as Mg-PSZ (Fig. 8). This is of great practical importance because it is very difficult to plot R-curves with LEFM methods in a standard testing laboratory.

The foregoing concerns structural ceramics, that is, ceramics bearing mechanical loads during their use. However, evaluation of the deformational behavior of these materials is also important for ceramic products subjected to thermal loading [13]. Considering the refractory ceramics resisting tensile thermal stresses, we may presume that the greater their ultimate strain ε_u (all other conditions being the same), the higher their resistance to thermal deformation. This statement presents the basis for the first criterion of thermal-shock resistance [14]: $R' = \sigma_u / E\alpha$, which involves the ultimate elastic

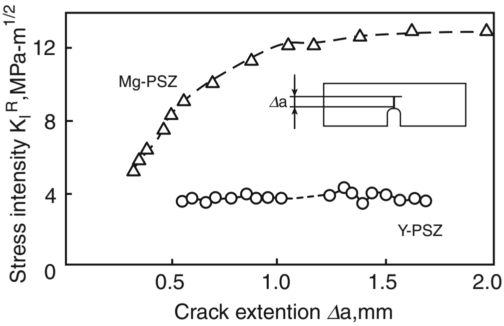
Brittleness Measure of Ceramics, Table 2 Brittleness measure and fracture toughness of glass and ceramics

Materials	Brittleness measure χ	K_{Ic}, MPa·m$^{1/2}$ SEVNB	SENB	Coefficient φ
Soda lime glass	1.00	0.66	1.18	0.562
Y-PSZ	1.00	5.14	9.54	0.538
Si_3N_4	1.00	5.17	9.12	0.567
TS-grade	0.49	9.04	10.33	0.880
$La_{0.2}Ca_{0.8}CoO_3$	0.25	2.20	2.20	1.000



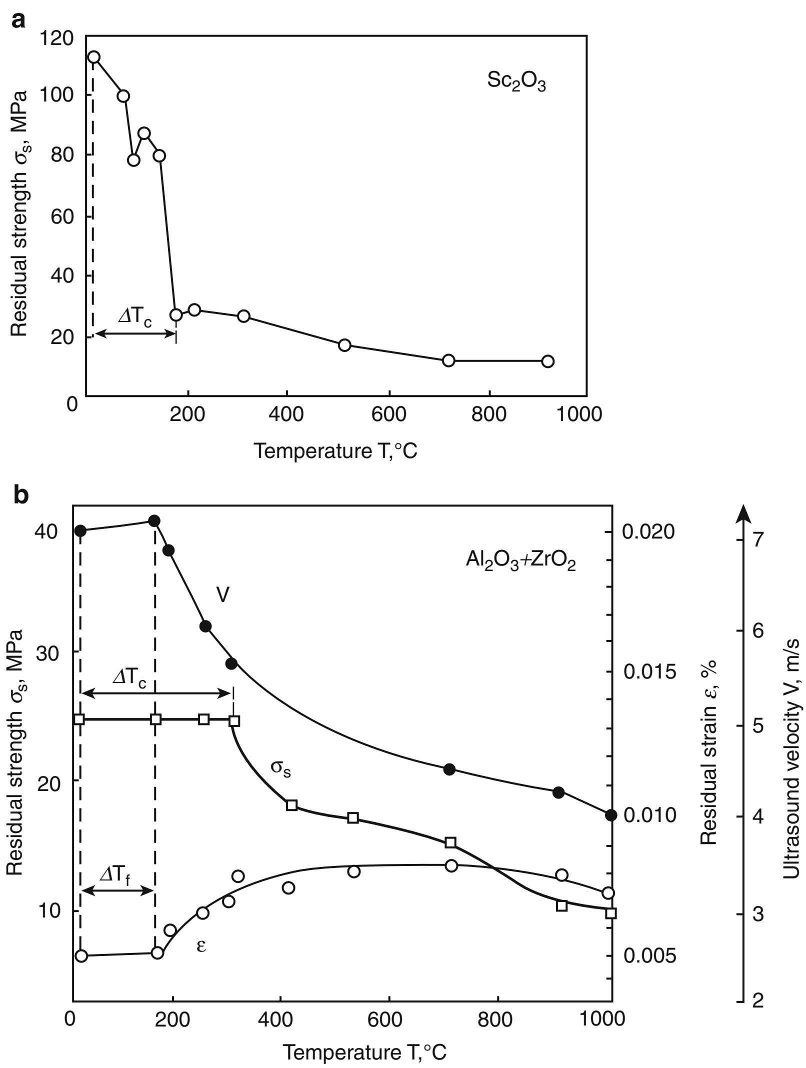
Brittleness Measure of Ceramics, Fig. 6 Temperature dependence of brittleness measure for elastic and inelastic ceramics at high temperatures (**a**) and inelastic Mg-PSZ ceramics at low temperatures (**b**)

strain ε_{el}, calculated in accordance with Hooke's law ($\varepsilon_{el} = \sigma_u/E$) and the linear thermal expansion coefficient α. However, if a ceramic material is inelastic, then the smaller its brittleness measure χ, the greater the difference between ε_{el} and the actual ultimate strain ε_u (Fig. 1). For reliable evaluation of the fracture resistance of brittle materials under thermal loading, this



Brittleness Measure of Ceramics, Fig. 8 R-curves of zirconia ceramics

thermal-shock-resistance criterion should be formulated as $R^1{}_r = \varepsilon_u/\alpha$ [8]. The second (energy) criterion [15] is expressed as $R^{lv} = \gamma E/\sigma_u{}^2$, where γ is the effective surface energy [3]. With the deformation behavior of the material in mind, it becomes clear that the smaller its brittleness measure, the larger the error of the stored energy $(\sigma_u{}^2/E)$. Hence, for reliable evaluation of the fracture resistance of materials under thermal loading, this criterion is given by $R_r = \gamma/\varepsilon_u\sigma_u$ [8]. These new thermal-shock-resistance criteria are also useful for the selection of various industrial



Brittleness Measure of Ceramics, Fig. 9 Thermal-shock damage diagram for Sc_2O_3 ceramics (**a**) and Al_2O_3 + ZrO_2 refractory (**b**) refractories for specific thermal conditions [16]. The above information made it possible to conclude that the brittleness measure can be used for thermal-shock-resistance evaluation: the less the brittleness measure (under otherwise equal conditions), the better the ceramics resist thermal loads.

The deformation behavior of ceramics and refractories should be taken into account when determining the thermal-shock resistance by water quenching using the residual bending strength [16, 17]. According to this method, the test bars are suspended in a vertical tube furnace, heated to the target temperature, and dropped into a water bath [18]. After quenching, the strength of the samples is measured in four-point flexure. The resistance of a material to thermal shock is estimated by maximum thermal-shock temperature difference ΔT_c, which it can withstand without its residual flexure strength being significantly affected. If the material is elastic, it is sufficient to determine its residual strength after quenching (Fig. 9a), which is usually done [19] and substantiated in [20, 21]. If the material is inelastic ($\chi < 1$), the situation is essentially different (Fig. 9b) [8] because ΔT_{c} cannot be a reliable characteristic of the material. In these cases, as the quenching intensity increases, the structural damage of the material increases continuously, and, hence, the residual strength decreases. For this type of ceramics, the value ΔT_f is of importance because it presents a difference between the furnace and quenching-bath temperatures at which the material structure begins to fail. This is manifested as decreasing ultrasonic velocity in the material and increasing deformability of the specimen [8, 22].

Conclusion

It has been shown that, for reliable evaluation of the mechanical behavior of ceramics, it is necessary to take into account their deformation behavior. The brittleness measure and ultimate strain should be considered as the important mechanical characteristics of ceramic materials.

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Buckling

► Buckling and Post-Buckling of Composite Plates Under Thermal Loads
► Ferromagnetic Plates and Shells
► Thermal Post-Buckling Paths of Beams

Buckling and Post-Buckling of Composite Plates Under Thermal Loads

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Synonyms

Buckling

Overview

Plates/panels and shells are the major load bearing structural elements in advanced aerospace, naval, advanced tactical fighters, automobiles, and civil engineering structures. These structural elements are often subjected to hostile environmental conditions. In addition to mechanical loading (in-plane and transverse), these structures in service are often subjected to hygroscopic as well as destabilizing thermal loadings also. The structural components of high-speed aircrafts and spacecrafts are subjected to thermal loading due to aerodynamic pressure and solar radiation heating. Aerodynamic heating causes high temperature, which results in thermal gradient across the thickness and causes very high thermal stresses. A reentry space vehicle encounters hygrothermal loading conditions. Plate structures have reserve strength even after the buckling. The structural elements such as highway pavements, runways, navigation tanks, structural foundations (mat type), machine foundation, launching pads, and supporting system of offshore structures are some of the examples of structures supported on elastic foundation. Examples of footings, mat foundations, ship and bridge structures, etc. constitute an adequate idealization for plates/panels resting on an elastic medium. In order to predict some realistic and accurate response of the laminated composite plate structures subjected to different loading conditions, the nonlinear behavior of plates cannot be ignored, especially for structures where the deformations are significantly large. The large deformations of the plate elements in comparison with the plate thickness produce significant supplementary strains and stresses in the middle plane of the plate. These additional in-plane stresses in turn cause resistance to bending of the plate, and consequently deflection is less than estimated by linear theory; hence, geometric nonlinearity should be incorporated in the mathematical model for better estimation of the strength of the structural member. It is observed that thermal loading greatly affects the transverse normal deformation because at elevated temperature, in addition to reduction of stiffness, bending-stretching coupling is enhanced due to the highly anisotropic thermal expansion behavior, which results in nonlinear response of the laminated composites. It resulted in the investigation of new structural configuration concepts and new fabrication concepts. Hence, the nonlinear analysis of laminated composite plates has become the topic of interest in structural mechanics not only due to their widespread applications but also as the challenging problem with interesting behavior. Nonlinear post-buckling analysis is required to fully exploit the load carrying capacity of the laminated composite plate for optimal and economical design.

Introduction

Buckling of laminated composite plates subjected to in-plane mechanical and thermal loadings has been the interest of many researchers in the past, and even today the elastic and thermoelastic stability analysis is of importance to the researchers due to the fact that the interest they stimulate as classical problems in instability. However, unlike columns, the plate structures resist the in-plane loading with increased deformation beyond initial buckling and in most of the cases represent hardening type of nonlinear behavior in post-buckling range. Hence, it is quite advantageous to study the post-buckling characteristics of the laminated composite plates in order to estimate their reserve strength for economical design. Accurate predictions of the post-buckling response of the laminated composite plate structures are required for efficient and optimal use of the materials in all practical applications. Therefore, the post-buckling response study of the laminated composite rectangular plates has invited the attention of the researchers and scientists.

The light weight feature of the composite materials is one of the most important reason for its wide spread use in almost all high-performance engineering structures mainly in aerospace structures. The structural elements made of laminated composite plates/panels in aerospace structures are usually thin to moderately thick and are subjected to in-plane mechanical loading which consist of uniaxial, biaxial, shear, and a combination of these loadings. In addition to mechanical loading, these are often subjected to thermal and hygroscopic loadings and their combinations. The induced stresses and deformations due to the aforementioned loadings play a significant role in their design. Moreover, plates/panels become unstable and may buckle in elastic region even at a relatively smaller in-plane compressive loading. Thus, buckling is one of the major criteria for design of the plate structures subjected to in-plane compressive loading. The accurate prediction of the post-buckling response of the composite plates is a complex task as compared to conventional single-layered metallic plates because of coupling effects such as bending-stretching, bending-twisting, and weak transverse shear rigidities. Bending-stretching coupling in unsymmetrically laminated plates results in transverse deformation even under the action of pure in-plane loading. Consequently, the existence of bifurcation buckling becomes questionable, and the problem has to be treated as that of a plate with initial imperfections, which may be due to erroneous fabrication process or eccentric in-plane loading. The increased applications of advanced composite and other high-performance engineering materials in structural components, especially in aerospace industry, have stimulated interest of engineering community in the accurate prediction of the buckling and post-buckling response characteristics of fiber-reinforced laminated composite plates/panels.

Employing a perturbation technique Shen [1] presented the thermal post-buckling analysis for a simply supported shear deformable laminated composite plate and resting on elastic foundation subjected to a uniform temperature rise, incorporating temperature-dependent material properties. Nath and Shukla [2, 3] presented the post-buckling of angle-ply and cross-ply laminated plates subjected to in-plane uniform and parabolic temperature. Singh et al. [4] and Thankam et al. [5] obtained the thermal post-buckling response of laminated composite plates, using the finite element method. A nonlinear finite element method to study the post-buckling response and first-ply failure of thin laminated composite plates under uniform temperature was used by Srikanth and Kumar [6]. Shen [7] investigated the post-buckling response of shear deformable laminated composite plates subjected to mechanical or thermal loading using higher-order shear deformation theory and two-step perturbation technique. Based on Reddy's higher-order shear deformation theory and von Karman's nonlinearity, thermomechanical post-buckling analysis of antisymmetric angle-ply and symmetric cross-ply laminated plates under uniaxial compression combined with a uniform temperature rise has been carried out by Shen [8], using a perturbation technique. Shukla and Nath [9, 10] presented analytical solution for buckling and post-buckling of laminated composite plates subjected to thermomechanical loading employing Chebyshev polynomials. Adopting Galerkin procedure and using Newton–Raphson iterative procedure, Girish and Ramachandra [11] presented the thermomechanical post-buckling analysis of composite plates with imperfections. In recent years, considerable amount of work on the post-buckling behavior of laminated composite plates subjected to mechanical, thermal, or thermomechanical loadings has been carried out by the researchers, but relatively lesser attempts are made to study the effect of hygrothermal environment on buckling and post-buckling response of laminated composite plates incorporating moisture and temperature-dependent properties. Utilizing classical laminated plate theory, Sai Ram and Sinha [12] presented the buckling of laminated composites plates under hygrothermal loading using first-order shear deformation theory (FSDT) and linear kinematics, employing finite element method. The effects of moisture and temperature on critical load are presented for simply supported and clamped antisymmetric cross-ply and angle-ply laminates using reduced lamina properties at elevated moisture concentration and temperature incorporating macro-mechanical model. Shen [13] studied the influence of hygrothermal effects on the post-buckling response of simply supported shear deformable laminated plates utilizing Reddy's higher-order shear deformation plate theory and employing perturbation technique. In most of the works, results are confined to uniaxial compression with hygrothermal loading and estimation of buckling load employing linear kinematics and macro-mechanical model for evaluating lamina properties at increased moisture concentration and temperature.

Problem Definition

The mathematical formulation of actual physical problem of laminated composite rectangular plates subjected to hygrothermomechanical loading and supported on elastic subgrades, e.g., Winkler- and Pasternak-type elastic foundations including foundation nonlinearity is presented. The governing differential equations of motion along with boundary conditions and initial conditions are presented for ready reference to the subsequent chapters of nonlinear analysis of moderately thick laminated composite rectangular plates. Based on macro- and micro-mechanics model, the expressions for evaluating temperature- and moisture-concentration-dependent elastic and hygrothermal properties are also presented.

The governing equations of motion of laminated composite rectangular plate resting on nonlinear elastic foundation undergoing moderately large deformation under the influence of hygrothermomechanical loadings are derived. It is assumed that perfect bonding exists between the layers of the laminated composite plate.

Kirchhoff's classical plate theory neglects transverse shear strain, underpredicts deflections, and overpredicts natural frequencies and buckling loads. The errors in deflections, natural frequencies, and buckling loads are even higher for plates made up of advanced composites like graphite-epoxy and boron-epoxy because of very high modulus ratio (the ratio of elastic modulus to shear modulus of these advance composites is of the order of 25–40, instead of 2.6 for conventional isotropic materials), and hence, the effect of transverse shear stresses cannot be ignored. Even thin plate behaves as a thick plate in higher modes of vibration and buckling; hence, an adequate theory is required, which must take into account the effect of transverse shear strains.

Noor and Burton [14] presented an assessment of the different shear deformation theories for multilayered composite plates. Varadan and Bhaskar [15] presented an overview of the different laminate theories and critically examined their relative merits vis-à-vis the three-dimensional elasticity approaches. The necessity of obtaining the response of laminated composite plates with reasonable accuracy has resulted in the development of a variety of two-dimensional shear deformation theories, which are either on stress-based approach or displacement-based approach.

These theories can be grouped in two general categories: (1) theories based on replacing the laminate by an equivalent single-layer anisotropic plate and introducing global displacement and strain and/or stress approximation in the thickness direction, and (2) discrete layer theories based on layer-wise approximations in the thickness direction. The first-order shear deformation theory based on the assumed stress and linear distribution of the in-plane displacement in the thickness direction is due to Reissner [16] and Mindlin [17]. The first-order shear deformation theory yields a constant value of the transverse shear strain through the thickness of the plate and thus requires shear correction, which is incorporated by a factor known as shear correction factor. The shear correction factors are dimensionless quantities introduced to account for the discrepancy between the assumed constant state of shear strain and actual parabolic variation of transverse shear strain across the thickness of the plate. The problem of ad hoc estimation of shear correction factor in first-order shear deformation theory was eliminated in higher-order shear deformation theories. The higher-order shear deformation theories assume higher-order polynomials for displacement through the thickness and, in general, lead to better results. The results obtained by three-dimensional theory of elasticity are recognized as the most accurate results. In the present work, higher-order shear deformation theory (HSDT) with cubic variation of in-plane displacements through the thickness and constant transverse displacement is used, which takes into account the effect of transverse shear. It does not require separate ad hoc estimation of shear correction factors.

The displacement field at a point in the laminated composite plate is expressed as [18]:

$$\begin{Bmatrix} U(x,y,z,t) \\ V(x,y,z,t) \\ W(x,y,z,t) \end{Bmatrix} = \begin{Bmatrix} u_0(x,y,t) \\ v_0(x,y,t) \\ w_0(x,y,t) \end{Bmatrix} + z \begin{Bmatrix} \psi_x(x,y,t) \\ \psi_y(x,y,t) \\ 0 \end{Bmatrix} + z^2 \begin{Bmatrix} u_1(x,y,t) \\ v_1(x,y,t) \\ 0 \end{Bmatrix} + z^3 \begin{Bmatrix} \phi_x(x,y,t) \\ \phi_y(x,y,t) \\ 0 \end{Bmatrix} \quad (1)$$

In order to predict realistic and accurate response of laminated composites, the nonlinear behavior of composites cannot be ignored, especially for thin to moderately thick structures where the deformations are significantly large under severe loading conditions. Also to predict the post-buckling response of laminated composite plates, geometric nonlinearity is incorporated in the mathematical model. Hence, the nonlinear analysis of laminated composite plates has become the topic of research in structural mechanics not only due to their widespread applications but also as the challenging problem with interesting behavior. It has been brought out in the literature that there is no significant difference in the response of moderately thick laminates obtained utilizing von-Karman's nonlinearity and generalized nonlinearity [19, 20]. Shukla [21] also studied the effect of generalized nonlinearity on nonlinear dynamic response of moderately thick laminated composite rectangular plates employing first-order shear deformation theory (FSDT) and concluded that both the formulations, i.e., one based on generalized Green's strain tensor and other based on von Karman's nonlinear kinematics, produce nearly same results with insignificant difference. However, due to the complexities involved in the formulation and large computational efforts involved in the solution, the generalized nonlinearity becomes unattractive to use. The governing differential equations become highly nonlinear in case of generalized nonlinearity. Therefore, it precludes any effort to obtain the solution using conventional analytical technique. Hence, in the present work von Karman's nonlinear kinematics is used for the study of nonlinear response of moderately thick laminated composite plates undergoing moderately large deformations.

The elastic and thermal properties of the composite material are dependent on the temperature. With increase in the temperature, these properties changes considerably reducing the stiffness of the material. It becomes important to consider the temperature-dependent properties of the composite material to study the response of the laminated composite plate in thermal environment. For the analysis with temperature-dependent elastic and thermal properties, the following mathematical model is used in the present work, according to which the properties are assumed as linear functions of temperature change ΔT [1]:

$$E_p(T) = E_p^0(1 + E_p^1 \Delta T) \quad (2)$$

$$\alpha_p(T) = \alpha_p^0(1 + \alpha_p^1 \Delta T) \quad (3)$$

Here E_P refers to E_{11}, E_{22}, G_{12}, G_{23}, G_{13}, and α_P refers to α_{11}, α_{22}. Superscripts "0" refer to the values at reference temperature and "1" refer to the constant coefficients which are as follows:

$$E_p^1 = -0.5 \times 10^{-3}; \quad \alpha_p^1 = 0.2 \times 10^{-3}$$

For hygrothermal analysis, the material properties are evaluated utilizing micro-mechanics model. Since the effect of temperature and moisture is dominant in polymer-based matrix material, the degradation of the composite material properties is estimated by degrading the matrix property only. The matrix mechanical property retention ratio is expressed as [22]:

$$F_m = \left(\frac{T_{gw} - T}{T_{go} - T_o}\right)^{\frac{1}{2}} \quad (4)$$

Where $T = T_o + \Delta T$ and T is the temperature at which material property is to be predicted, T_o is the reference temperature, ΔT is the increase in temperature from reference temperature, and T_{gw} and T_{go} are the glass transition temperatures for wet and reference dry conditions, respectively.

The glass transition temperature for wet material is determined as [23]

$$T_{gw} = \left(0.005C^2 - 0.10C + 1.0\right)T_{go} \quad (5)$$

where $C = C_o + \Delta C$ is the weight percent of moisture in the matrix material, $C_o = 0$ wt.% and ΔC is the increase in moisture concentration. The elastic constants are evaluated from the following equations [24]:

$$E_{11} = E_{f1}V_f + F_mE_mV_m \quad (6)$$

$$E_{22} = (1.0 - \sqrt{V_f})F_mE_m + \frac{F_mE_m\sqrt{V_f}}{1.0 - \sqrt{V_f}\left(1.0 - \frac{F_mE_m}{E_{f2}}\right)} \quad (7)$$

$$G_{12} = (1.0 - \sqrt{V_f})F_mG_m + \frac{F_mG_m\sqrt{V_f}}{1.0 - \sqrt{V_f}\left(1.0 - \frac{F_mG_m}{G_{f12}}\right)} \quad (8)$$

$$\nu_{12} = \nu_{f12}V_f + \nu_mV_m \quad (9)$$

where "V" is volume fraction and subscripts "f" and "m" are used for fiber and matrix, respectively.

The effect of increased temperature and moisture concentration on the coefficients of thermal expansion (α) and hygroscopic expansion (β) is opposite from the corresponding effect on strength and stiffness. Hygroscopic expansion coefficients for fibers are taken as zero ignoring the effect of moisture on the fiber. The matrix hygrothermal property retention ratio is approximated as

$$F_h = 1/\ F_m \quad (10)$$

Coefficients of thermal expansion are expressed as [25]

$$\alpha_{11} = \frac{E_{f1}V_f\alpha_{f1} + F_mE_mV_mF_h\alpha_m}{E_{f1}V_f + F_mE_mV_m} \quad (11)$$

$$\alpha_{22} = \alpha_{f2}V_f + V_mF_h\alpha_m + \frac{V_fV_m\zeta_1\zeta_2}{E_{11}} \quad (12)$$

where $\zeta_1 = (\nu_{f12}F_mE_m - \nu_mE_{f1})$, $\zeta_2 = (\alpha_{f1} - F_h\alpha_m)$

Similarly, coefficients of hygroscopic expansion are expressed as [26]

$$\beta_{11} = \frac{E_{f1}V_f\beta_{f1} + F_mE_mV_mF_h\beta_m}{E_{11}} \quad (13)$$

$$\beta_{22} = \frac{V_mF_h\beta_m[(1 + \nu_m)E_{11} - \nu_{12}E_mF_m]}{E_{11}} \quad (14)$$

The nonlinear elastic foundation is considered as Pasternak type with foundation nonlinearity. It can be modeled as a nonlinear spring and a shear layer. The upthrust due to nonlinear elastic foundation (Pasternak type) can be expressed as [27]

$$R = K_1 W + K_2 W^3 - K_3 \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right) \quad (15)$$

where K_1, K_2, and K_3 are Winkler, nonlinear, and shear foundation parameters, respectively.

The governing equations of motion and proper boundary conditions are derived using Hamilton's principle:

$$(L_a + L_b + L_c)\, d + Q - R = L_e\, d \quad (16)$$

Where

$$\mathbf{L_a} = L_{a1} \frac{\partial^2}{\partial x^2} + L_{a2} \frac{\partial^2}{\partial y^2} + L_{a3} \frac{\partial^2}{\partial x \partial y} + L_{a4} \frac{\partial}{\partial x} + L_{a5} \frac{\partial}{\partial y} + L_{a6}$$

$$\mathbf{L_b} = L_{b1} \frac{\partial^2}{\partial x^2} + L_{b2} \frac{\partial^2}{\partial y^2} + L_{b3} \frac{\partial^2}{\partial x \partial y}$$

$$\mathbf{L_c} = L_{c1} \frac{\partial^2}{\partial x^2} + L_{c2} \frac{\partial^2}{\partial y^2} + L_{c3} \frac{\partial^2}{\partial x \partial y}$$

$$\mathbf{L_e} = L_{e1} \frac{\partial^2}{\partial \tau^2}$$

$$d = \begin{bmatrix} u & v & w & \overline{\psi}_x & \overline{\psi}_y & \overline{u}_1 & \overline{v}_1 & \overline{\phi}_x & \overline{\phi}_y \end{bmatrix}^T$$

$$Q = [0\,0\,Q\,0\,0\,0\,0\,0\,0]^T$$

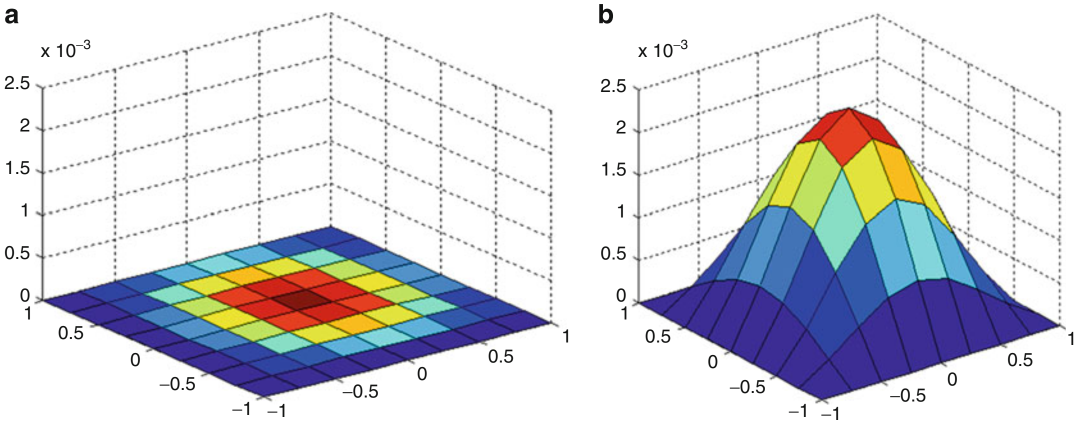
$$R = [0\,0\,R\,0\,0\,0\,0\,0\,0]^T$$

where Q represents nondimensional transverse pressure and R is the nondimensional elastic foundation parameter (Pasternak type with foundation nonlinearity).

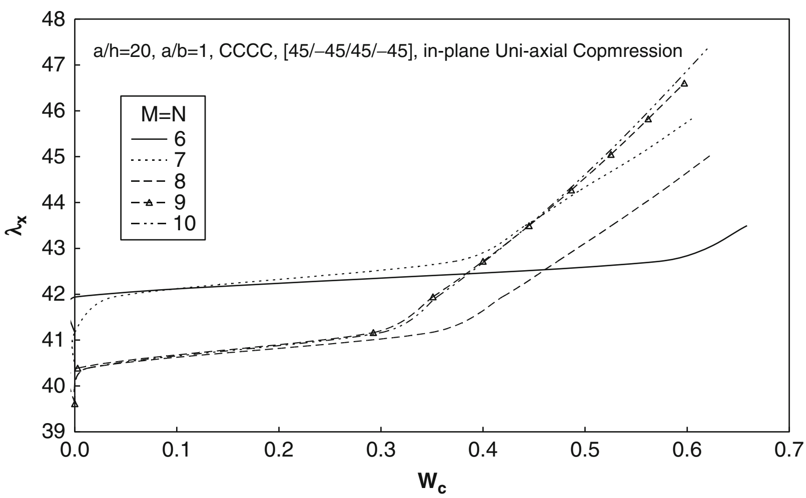
The choice of an accurate and cost efficient computational model depends upon the geometry and loading conditions of the structural elements. The computational power of numerical methods, especially finite element method, is well established for complicated loading and geometry. FEM is employed as a necessity because it is extremely difficult to solve analytically, highly coupled nonlinear differential equations of laminated composite plates subjected to different loading conditions. On the other hand, the results obtained by numerical techniques are also required to be validated before being used for the design purposes, and hence, there is a need to develop and explore the possibility of using some analytical or semi-analytical techniques for the solution of nonlinear problems of laminated composite plates, particularly in the absence of experimental data. The wide spread availability of powerful digital computers suggests examination, evaluation, and improvement on the existing methods and development of the new approaches to solve the complicated problems. The most commonly employed analytical solutions in rectangular domain are based on Fourier series and power series approximations of the spatial functions. In spite of their poor convergence properties, these methods are extensively used for the analysis of plates even today. Fast converging finite double Chebyshev polynomials is used to find out the analytical solutions of the laminated composite plate under different loading conditions and having different boundary conditions that can serve as benchmark solutions.

Results and Discussions

The plate is assumed to be subjected to in-plane mechanical, thermal, thermomechanical, and hygrothermomechanical loadings. In-plane mechanical loading consists of uniaxial, biaxial (compression-compression, compression-tension), shear loadings, and their combinations. The temperature-induced loading is due to either uniform temperature or a linearly varying temperature across the thickness. The moisture concentration is assumed to be uniformly constant throughout the plate. The degradation in material properties due to moisture and temperature is taken into account using micro-mechanics model, and the importance of temperature- and moisture-dependent thermal and mechanical properties on the stability of the composite plates is well elaborated. The elastic foundation is modeled as shear deformable with cubic nonlinearity.



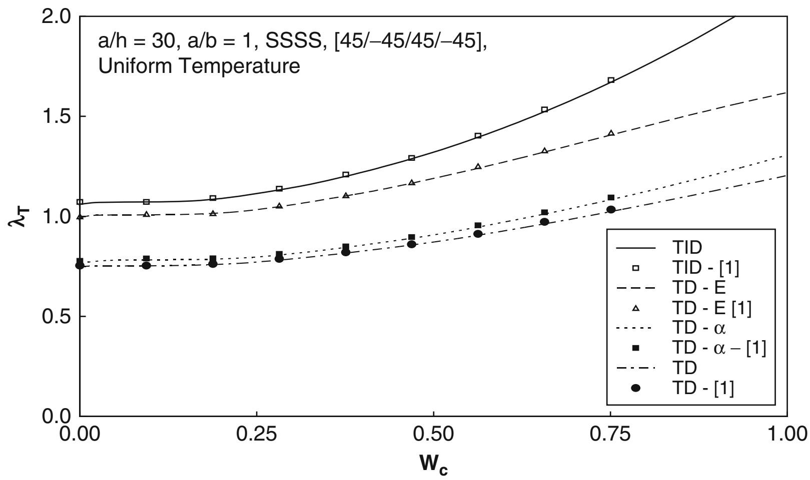
Buckling and Post-Buckling of Composite Plates Under Thermal Loads, Fig. 1 Transverse deflection of the midplane of the simply supported laminated composite [0/90] plate subjected to uniaxial compression (**a**) just before buckling and (**b**) just after buckling



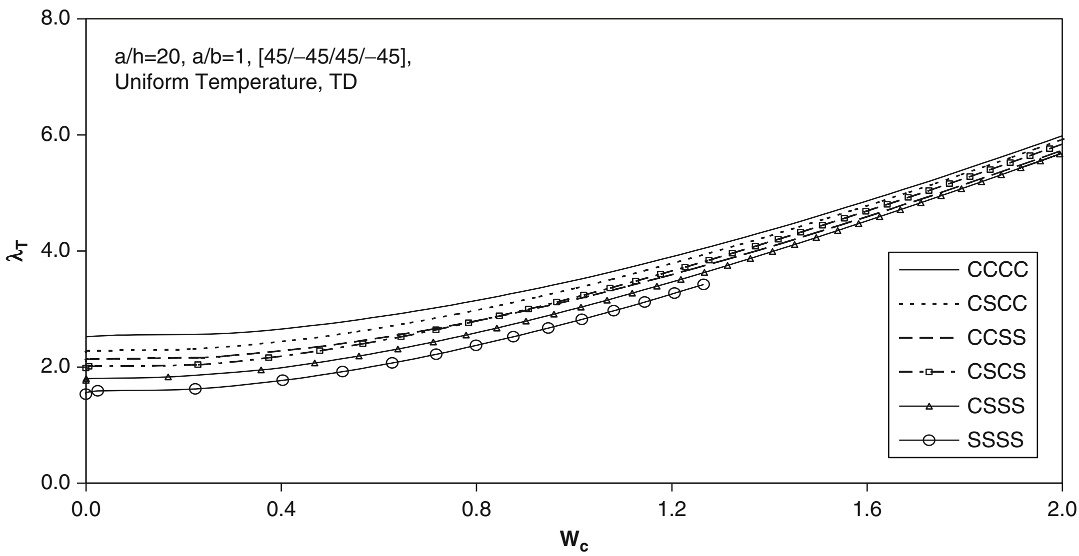
Buckling and Post-Buckling of Composite Plates Under Thermal Loads, Fig. 2 Convergence study of the buckling load and post-buckling response of clamped, square, antisymmetric angle-ply [45/-45/45/-45], laminated composite plate subjected to in-plane uniaxial compression

An incremental iterative approach is adopted, and load/temperature is incremented in small steps. The criterion of sudden jump in characteristic parameter "central deflection" (W_c) due to small increment in the marching variable (loading/temperature) or convergence failure in the iterative procedure at a step is adopted as estimation of the buckling/critical load/temperature. This is very well demonstrated in Figs. 1, 2. Transverse deflection at various points in midplane of the simply supported, laminated composite [0/90] square plate ($a/h = 10$) subjected to in-plane uniaxial compression is obtained and shown in Fig. 1, just before and after the buckling. A clear jump in the deflection is noticed indicating the onset of instability/buckling. This gives fair idea of the buckling load/temperature. In order to facilitate the buckling of plates, a small perturbation in form of transverse pressure ($Q = 10^{-5}$) is applied in all the cases.

In order to show the stability and accuracy of the present solution methodology, the convergence study for the post-buckling response of a four-layered antisymmetric,



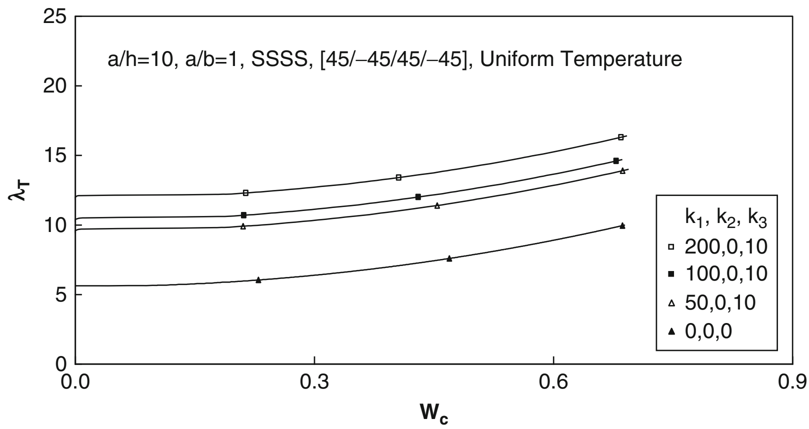
Buckling and Post-Buckling of Composite Plates Under Thermal Loads, Fig. 3 Effect of temperature-dependent thermal and mechanical properties on post-buckling response of simply supported, antisymmetric, angle-ply, laminated plate subjected to in-plane uniform thermal loading



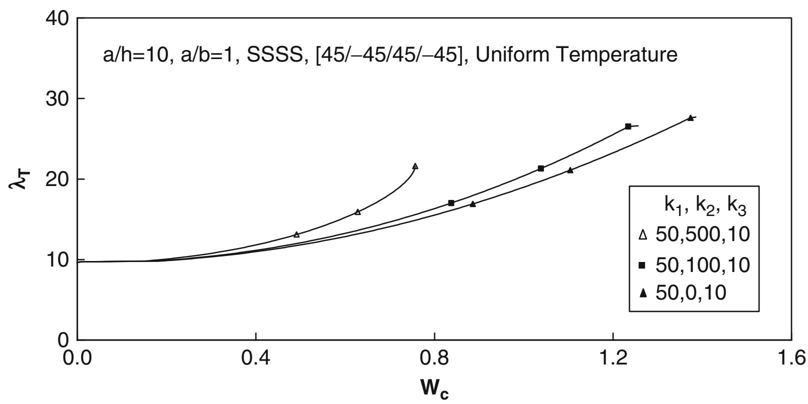
Buckling and Post-Buckling of Composite Plates Under Thermal Loads, Fig. 4 Effect of boundary conditions on thermal post-buckling response of antisymmetric angle-ply laminated composite plate (a/h = 20) subjected to temperature-dependent in-plane uniform thermal loading

angle-ply, clamped (CCCC), square, moderately thick (a/h = 20), laminated plate subjected to in-plane uniaxial compressive loading is carried out and depicted in Fig. 2. It is clear that 9–10-term expansion of each variable in finite double Chebyshev series gives quite good convergence of the post-buckling response. Hence, in the present work 9-term expansion of the variables in Chebyshev series is taken in the analysis.

Figure 3 shows thermal post-buckling load-deflection curve of $[(\pm 45)_2]$ laminated square plate for four different cases of thermoelastic properties along with the results due to Shen [1]. The results obtained are well compared with the results due to Shen [1]. TD-E represents that elastic constants are temperature dependent, but thermal coefficients are temperature independent, whereas TD-α represents that thermal coefficients are temperature dependent, but elastic



Buckling and Post-Buckling of Composite Plates Under Thermal Loads, Fig. 5 Effect of Winkler foundation parameter (k_1) on thermal post-buckling response of simply supported, square, antisymmetric angle-ply plate (a/h = 10) subjected to in-plane uniform thermal loading



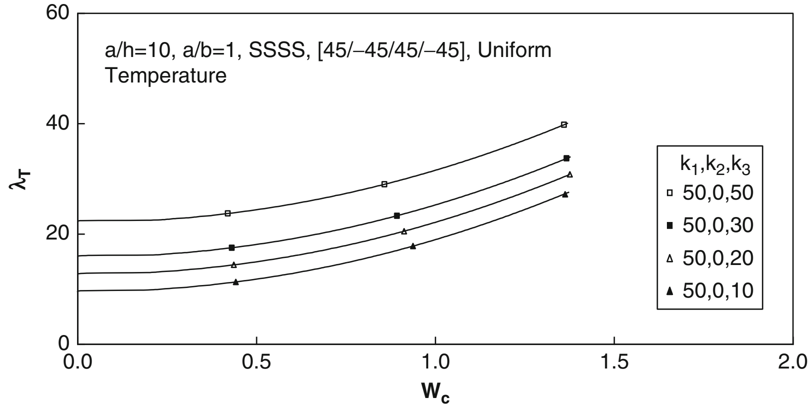
Buckling and Post-Buckling of Composite Plates Under Thermal Loads, Fig. 6 Effect of foundation nonlinearity parameter (k_2) on post-buckling response of simply supported, square, antisymmetric angle-ply plate (a/h = 10) subjected to uniform in-plane thermal loading

constants are temperature independent. It is noted from the figure that buckling/limiting temperature decreases and reserve strength reduces significantly when temperature-dependent properties are taken into consideration.

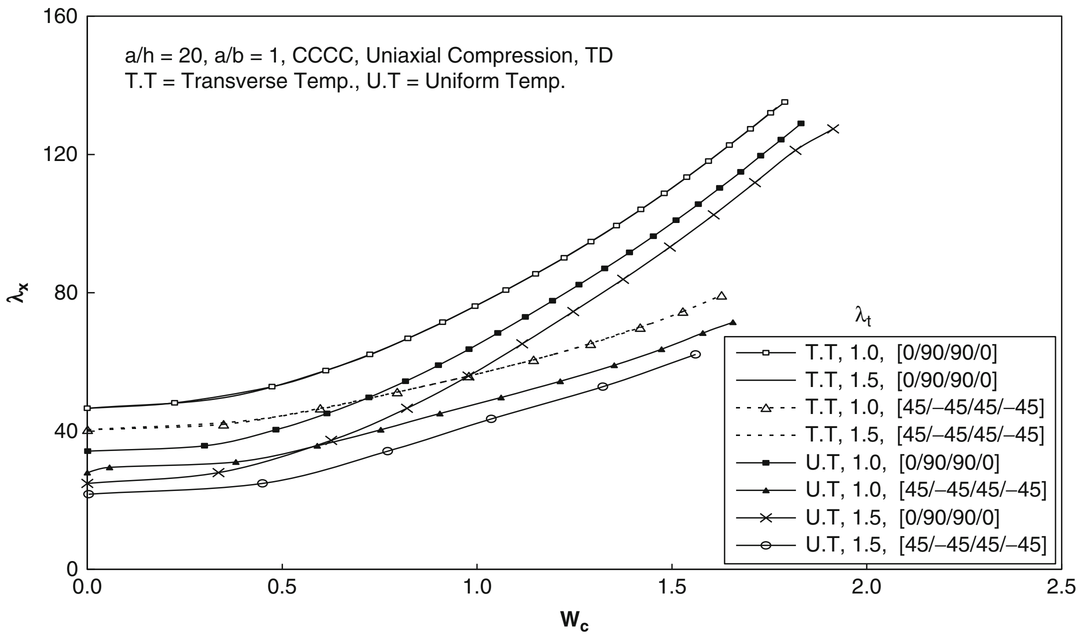
It is also observed that effect of thermal expansion coefficient when considered as varying with temperature on stability of the plate is more pronounced as compared to that of elastic moduli. The effect of temperature-dependent properties on buckling and post-buckling response is mainly due to thermal expansion coefficient.

Figure 4 represents effect of classical and nonclassical boundary conditions on the post-buckling response of 4-layered, antisymmetric, angle-ply, moderately thick (a/h = 20), laminated composite, square plate subjected to uniform temperature. It is noted that decrease in degree of fixity reduces the buckling/limiting temperature; i.e., buckling strength of all edges clamped (CCCC) plate is greater than buckling strength of three-edge clamped and one simply supported (CSCC) followed by two-edge clamped and two simply supported (CCSS and CSCS) and one-edge clamped and three simply supported (CSSS). The buckling/limiting temperature of all edges simply supported (SSSS) plate is lowest.

Figures 5–7 present the influence of Winkler foundation parameters (k_1), foundation nonlinearity parameter (k_2), and shear layer (Pasternak) foundation parameter (k_3), respectively, on the post-buckling response of 4-layered, antisymmetric, angle-ply, simply supported, square, laminated composite plate



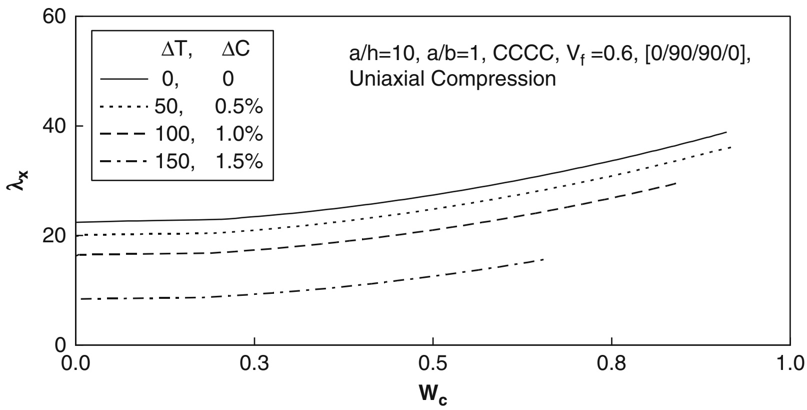
Buckling and Post-Buckling of Composite Plates Under Thermal Loads, Fig. 7 Effect of shear layer foundation parameter (k_3) on thermal post-buckling response of simply supported, square, antisymmetric angle-ply plate (a/h = 10) subjected to in-plane uniform thermal loading



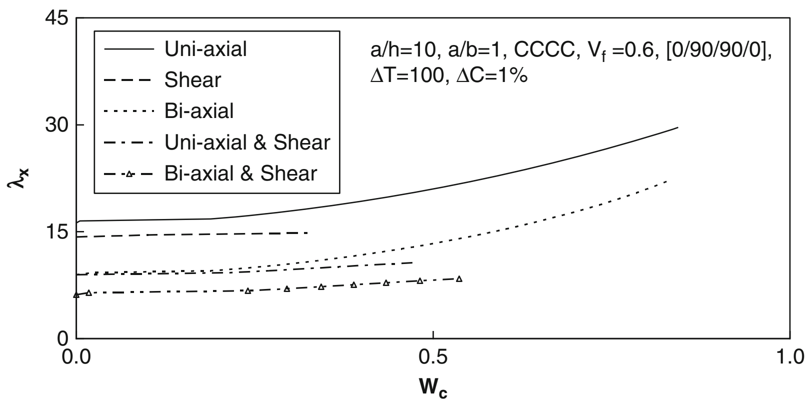
Buckling and Post-Buckling of Composite Plates Under Thermal Loads, Fig. 8 Effect of in-plane uniform temperature and transverse temperature gradient on post-buckling response of clamped, square, laminated composite plate (a/h = 20) subjected to in-plane uniaxial compression

(a/h = 10) subjected to in-plane uniform temperature. It is observed that increase in foundation parameters (k_1 and k_3) increases the buckling load as well as post-buckling strength of the plate, whereas increase in foundation nonlinearity parameter (k_2) increases the post-buckling strength of the plate only without affecting the buckling strength indicating herein that for the accurate evaluation of the reserve strength of the elastically supported plate, the foundation nonlinearity should be considered. It is also noticed that the effect of shear layer (Pasternak foundation) on the stability of the plate is relatively more than that of Winkler foundation.

The thermoelastic post-buckling response of laminated composite plates subjected to in-plane uniaxial compression and uniform temperature or transverse temperature gradient (linearly varying



Buckling and Post-Buckling of Composite Plates Under Thermal Loads, Fig. 9 Effect of moisture and temperature on critical load and post-buckling response of clamped, square, symmetric, cross-ply, laminated composite plate subjected to uniaxial compression



Buckling and Post-Buckling of Composite Plates Under Thermal Loads, Fig. 10 Post-buckling response of clamped, square, symmetric, cross-ply, laminated composite plate subjected to in-plane uniaxial, biaxial, shear loadings, and their combinations

temperature across the thickness of the plate) is studied. Temperature-dependent material properties are considered. Figure 8 depicts the effects of uniform temperature and transverse temperature gradient on post-buckling response of clamped, square, moderately thick (a/h = 20), symmetric cross-ply [0/90/90/0] and antisymmetric angle-ply [45/−45/45/−45], laminated plates subjected to in-plane uniaxial compression. There is decrease in buckling strength of the plates when subjected to uniform in-plane temperature-induced loading along with in-plane edge compressive loading, but in case of transverse temperature, the effect is insignificant. It can also be observed that the post-buckling strength of symmetric cross-ply laminate is more than antisymmetric angle-ply laminate under in-plane uniform temperature as well as transverse temperature gradient.

The effect of moisture and temperature on the buckling load and post-buckling equilibrium path of a clamped, square, symmetric, cross-ply, laminated plate under uniaxial compression is shown in Fig. 9. It is observed that there is decrease in nondimensional buckling load and post-buckling strength with increase in moisture and temperature. The decrease in the buckling load and post-buckling strength is more pronounced at higher moisture concentration and temperature. The buckling load and post-buckling strength is considerably reduced at T = 150 and C = 1.5 % as the working temperature (171 °C) is closer to the lowered glass transition temperature (183.563 °C) at increased moisture concentration (C = 1.5 %). The buckling load is reduced by approximately 26.1 % when the plate is subjected to moisture concentration of 1 % and temperature increase of 100 °C in comparison to the buckling load of the plate without hygrothermal loading. The reduction in the buckling load is 62.2 % at increased moisture concentration of 1.5 % and temperature of 150 °C. It clearly demonstrates the detrimental effect of the increased moisture concentration and temperature on the stability of the plate. It is also seen that the effect of hygroscopic condition on the stability of the plate becomes more significant in presence of the thermal loading.

The hygrothermal effects on the stability of the symmetric, cross-ply, laminated, square plate subjected to combinations of in-plane uniaxial and biaxial edge compressive loading along with in-plane edge shear loading are shown in Fig. 10. It can be observed that the buckling load is lowest when edges have biaxial loading combined with shear loading and highest for in plane uniaxial loading. It is also noted that the buckling load is almost same for biaxial loading and uniaxial loading combined with shear loading, but the post-buckling strength under biaxial loading is higher.

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