Chapter 3
The Adventure as Experienced by the Teachers

In Chap. 2 we presented an adventure in the learning of fractions and decimal numbers, with our perspective being that of the students who were doing the learning. In this chapter we step back and have another look at the same adventure. We will first set out the context: when and why the curriculum was created, the research questions underlying it and the school and research environment in which it was embedded. With that in hand, we will look again at the adventure itself, this time from the perspective of the teachers. In Chap. 4, we will take one further step back and examine the mathematical context and the reasoning behind the mathematical choices made in constructing the curriculum from the perspective of the researchers.

Background of the Project

Two elements of the background that were described in Chap. 1 are sufficiently pertinent to this chapter that we will start by reproducing them: The lessons described in Chap. 2 took place at the COREM *(*Center for Observation and Research on Mathematics Teaching), which was a regular public school in a blue collar district on the edge of Bordeaux equipped with a carefully thought out and agreed to set of research arrangements. On the physical side, the arrangements consisted of an observation classroom in which classes would occasionally be held – often enough so that the students found them routine. The classroom was equipped with a multitude of video cameras and enough space for observers to sit unobtrusively. Other arrangements were far more complex, involving an extra teacher at each level and an agreement among the teachers, administrators and researchers setting out the responsibilities and rights of each. Nothing involving that many humans could possibly glide smoothly through the years, but the fundamental idea proved robust, and the École Michelet functioned as a rich resource for researchers for two and a half decades.

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On the theoretical front, the background has roots that can be traced back through the generations, but came to the foreground in the 1960s under the title of constructivism. The title stems from the underlying tenet that knowledge is constructed in the human mind rather than absorbed by it. Applications of that tenet range from the radical constructivist belief that absolutely no information should be conveyed to students directly, to the naïve conviction that having children manipulate some physical objects that an adult can see to represent a mathematical concept will result in the children understanding the concept itself. In the interest of providing some solid research in support of the theory itself, the researchers set themselves the goal of taking some serious piece of mathematics and proving that in certain conditions the children – all the children, together – could create, understand, learn, use and love that mathematics. Accompanying that was the goal of studying the conditions themselves.

Clearly the mathematics to be used for this experiment had to be both significant and challenging. After some consideration they made a choice that will resonate with elementary teachers worldwide: fractions, or more properly, rational and decimal numbers. They had, in fact, some reservations about whether rational numbers should be taught at all, but rational numbers were firmly part of the national requirements, and likewise firmly a heavy-duty challenge for teachers and students, so they met the criteria.

The remaining element of background concerns the format for the learning adventure itself. All of the researchers were strongly committed to the Theory of Situations – in particular to the hypothesis that children could learn mathematical concepts by being put into carefully designed Situations in which they would need to construct them – but had an equally strong commitment to the principle that before people were asked to accept it they should be presented with solid research validating it. This pair of commitments helped fuel the drive to create the COREM. Once it was created, the first goal was to design research to test the theory. At the heart of that research was the curriculum that provided the adventure described in Chap. 2.

We will postpone until the next chapter a discussion of some of the mathematical choices and how they relate to the more common structure for the teaching of this topic. Our next goal is rather to set the stage for the reader to re-examine the adventure from the vantage point of the teacher. To do that, however, requires a deeper understanding of the philosophy behind the Theory of Situations and some of its practical consequences.

Public opinion in the sixties was exerting pressure for the mathematics taught in schools to resemble as much as possible, and as early as possible, the mathematics practiced and produced by mathematicians. Some even felt that from pre-school to university everything could be taught in a unique "definitive" form. However utopian the idea may appear today, at the time it didn't seem impossible to meet that challenge, or at least to study it seriously.

To do so required that the activity of mathematicians be modeled, and then that conditions be imagined that were realizable by the teacher and would lead the students to produce on their own, by a similar activity, some current mathematical knowledge. In point of fact, there is no such thing as a "mathematical activity" that does not depend on its objective, and the historical genesis of any mathematical concept is so complex and so much wrapped up in its history that it defies reproduction by any isolated modern individual. Furthermore, understanding a notion like that of rational or decimal number implies that at the end of the learning process a subject has at her disposal a collection of widely varied, logically interlinked pieces of knowledge. Thinking in terms of this end organization leads to an ordering of teaching based on logical relations, for example a locally or completely axiomatic ordering. This is the thinking that dictated the classical didactical methods.

But mathematical concepts are constructed in the course of a far richer story involving questions, problems and solutions, where a much wider collection of reasons comes into play. The researchers' idea was to realize a process of construction of rational and decimal numbers simulating that sort of genesis. That is, a process making minimal use of pieces of knowledge imported by the teacher for reasons invisible to the students. This type of project was subsequently labeled constructivist.
The initial objective of the experiment was thus an attempt to establish an "existence theorem":

- Would it be possible to produce and discuss such a process?
- Would the students – all of the students – be able to engage in it?
- Could the result of the process be, for each of the students, a state of knowledge *at least equal* to that obtained by current, standard methods?

The realization of the process made no sense unless simultaneously each lesson was conceived, studied, corrected and criticized with the most severe of theoretical, pragmatic and methodological instruments. These instruments were mostly derived from the Theory of Situations, but they were heavily modified in the course of the experiment. Another goal was thus that the instruments should progress. *The second objective was to clarify and complete the Theory of Didactical Situations.*

On the other hand, there was no question of relying on imagination and fantasy and then waiting to see if the results were satisfactory. Children are not laboratory animals. The methodological and deontological principles were very different from those in use today in that domain. In this real experiment, we set both minimal objectives in terms of success rates relative to median results at other schools, and time limits. If the method we used had not made it possible to achieve the results normally attained by classical methods in the specified amount of time, we would have had the teachers follow some alternative activities – if necessary using other methods. The comparison between two methods was thus made *on equal results on curricular objectives*, by comparing

- the time and effort required to achieve this result,
- various differences in results that were not evaluated and were often impossible to propose as objectives, of which we will speak later,
- certain qualitative differences, some of them affective: pleasure and motivation, for the students and the teacher.

The third objective was essentially to know if the use of activities similar to those of mathematicians would give the scholastic knowledge of students different qualities from that obtained by the standard teaching methods of the period.

How then were the activities set up to simulate the ways in which mathematicians generate a concept? One aspect will undoubtedly have struck the reader in the course of Chap. 2: a great deal of mathematical progress is made communally, by mathematicians bouncing ideas around, building on each others' thinking (possibly over the course of decades, but that's another story!) Correspondingly, in this curriculum a lot of class time is spent with students working together towards some mathematical goal. Given the current teaching culture, in which the expectation is that each teacher should be constantly monitoring the state of knowledge of each student and fixing up any individuals who are lagging, this can be disconcerting. A sports metaphor is perhaps the most useful tool for illuminating the situation:

How do children learn to play rugby in England (or America or Aquitaine?) Children watch the game being played and have an idea of what is going on. People run around with a funny shaped ball that if you drop it clearly doesn't do anything you want it to. After watching a bunch of kids playing for a while, a new kid asks to join in. They let him know that if he wants to be accepted he has to run in a particular direction and that he needs to throw the ball to somebody else before he gets trapped with it, that he shouldn't knock down or sock an adversary, nor cry when somebody else gets the ball. The rest of the game he learns as he plays. After a while he will even be dealing with subtleties like playing a particular position, but he doesn't need those subtleties either to enjoy the game or to make a genuine contribution to his new-found team. And if he and his team stick together for a period of time, taking on various other neighborhood teams and profiting from some low-pressure coaching, they will all learn from each other and develop both individual and team strengths. On the other hand, if someone were to break into this process and attempt at regular intervals to measure how "good" each child is at rugby, or just which skills each one has mastered, the effort would be not only futile, but damaging to the whole team's progress both individually and collectively.

In the same way, the class *does* mathematics, with everything that that necessitates and all the satisfaction it produces. Each student participates and does certain things, personally and according to her lights. What she does visibly makes a contribution to a group task, even if she doesn't fully understand every aspect of it herself. At many stages, individuals would be disconcerted and the collective rhythm would be broken if the teacher were to cut in with a form of assessment that implied that everyone ought to be able to answer some particular collection of questions. Nonetheless, as the process goes along, the whole class is developing both individual and collective understandings that lead ultimately to the knowledge in question, complex though it be.

Looking more deeply into the nature and structure of these activities requires a brief preparatory excursion into what appears at first to be a simple semantic issue, but definitely is not (it took Warfield several years to accept that it was not simple, and she is still grappling with its complexities.) In the place where the English language has a single verb: "to know" and a single noun: "knowledge", the French language has two verbs: "*savoir*" and "*connaître*" and four nouns: *"savoir"*, "*savoirs",* "*connaissance",* and *"connaissances"*. After numerous unsuccessful efforts to bend or dragoon the English language into conveying what we wanted it to we finally agreed that we needed simply to leave the words in French and clarify for our readers what they were saying. *Savoir,* then, deals with the kind of knowledge implied in the statement "I know that for a fact." This is not to say that each *savoir* is actually a fact. It can be a procedure, or a connection, or some other nugget of knowledge. What characterizes the knowing or the knowledge is that it is solid and certain and that it is or can be shared. In more formal terms, a *savoir* is reference knowledge. A *connaissance*, on the other hand, is more landscape than landmark. It is the feeling that the current situation is similar to a previous one whose results might be useful, or the suspicion that that tempting tactic might be a trap. It might be a little vague, or even occasionally incorrect, and furthermore it may be so unarticulated that a person is unaware of having it, but it is what gives meaning to the *savoirs*. Without a landscape, landmarks do not have much of a function.[1]

For the many occasions when this distinction is needed for understanding the issues under discussion we will use the appropriate French term. When it is not, and especially when the distinction is a distraction, we will stick with the English.

The Relationship with the Theory of Situations

With these distinctions in hand, we are equipped to take a closer look at Situations and how Guy Brousseau's theories play out in this particular curriculum. Structurally, it is easiest to think in terms of the slightly oversimplified model of a small number of general Situations in which more limited Situations are embedded (we use capital letters to distinguish these from the everyday situations that are part of the happenstance of normal life.) A general Situation would be, for instance, the exploration of commensuration that results from measuring the thickness of sheets of paper, or the exploration of the ordering of rational numbers that results from bracketing them with intervals. Such Situations are not teaching objectives, nor even problems that students must learn to recognize in order to answer them by repeating some algorithm. They are many-faceted adventures that pull together a whole conglomeration of *connaissances* that will be provoked, activated, invented, used, modified, and verified, around a project of a mathematical nature dealing with an essential mathematical notion. Within these general Situations are sequences of more limited Situations, again not focused on some specific learning objective but rather on the progression of the general adventure. Nonetheless, they are reproduced with a high enough density to be recognizable and to provoke, justify and accompany the learning, at least implicitly, of answers that suit the particular need (not necessarily immediately correct and appropriate ones.) Before long the students' answers arrive at a level of maturity such that they can be identified (recognized as stable, identical and useful), named, and sometimes made explicit by the teacher and/or the students themselves. This begins the production of *savoirs*, though at this point most are of only temporary use and value.

[1] This distinction is discussed further in Chap. 5.

Future learning depends strongly on the *set* of these results of an activity: the knowledge of the general Situation (the adventure) and of simpler and more identifiable Situations, *connaissances* that can be produced, improvised or manifested but only within the Situation, formulations that may be provisional and opportunistic, and of course *savoirs* that are recognized, verified, practiced, certified, detachable and exportable by analogy.

These *savoirs* constitute the only part that can be more or less formally evaluated, and as a result they tend to be regarded by some people as the only objective of teaching. Evaluation of *savoirs* alone, however, is totally inappropriate as a global instrument of evaluation and especially disastrous for making decisions about teaching (thus in particular for decisions by the teacher). To take the manifestation of these *savoirs* as the daily indicator, unique objective and unique criterion for success engages the teacher in a paradigm of extremely closed and not very productive didactical choices. Essentially it results in reproduction of the conditions of the evaluation, with a few variations and explanations to attempt to extend the useful domain of the required answer.

In reality, the teacher needs to take into account and manage the evolution of all the forms of knowledge constituting a given *connaissance*. She can only do it with powerful, attractive Situations where many different pieces of knowledge are at work at the same time, in a learning process with many repercussions, like the ones that result from the real mathematical Situations proposed here. This does not mean that learning flows "naturally" from the students' encounter with a few assignments. No Situation could possibly lead the students to the institutionalized knowledge that remains the essential, effective and contractual objective of teaching. The teacher has an on-going responsibility to keep up the level of interest of the students and the production of *connaissances* and *savoirs* of all sorts that the students themselves perceive as the results of their efforts.

What we are talking about here is a collective adventure that produces many bits of spontaneous learning that would swiftly evaporate if the process did not give the teacher and the students the possibility of unceasingly realizing the steps of a recognized didactical process. Situations do not relieve the teacher of professional responsibilities and obligations. What they provide is an opportunity for the teacher to give a meaning, a context and an objective for the knowledge the Situation gives rise to. They also allow the teacher to escape the pressures and paradoxes created by the pedagogical stance of teacher as authority and student as obedient absorber.

We have just distinguished several forms of "a" piece of knowledge. The Theory of Situations analyzes the conditions of evolution of the sets of these forms of knowledge that are at the disposition of teachers. We need to say a word about how these different sets of knowledge are determined by the position of those who are using them. The organization of *connaissances* and *savoirs* by the scientific community serves as a reference, but knowledge that the teacher wants and needs to teach is necessarily a transposed version. And cognitive psychology shows us unambiguously that student knowledge differs considerably from student to student and consequently also differs from what the teacher wants or believes himself to be teaching. Does that mean that the teacher ought to adapt himself to all those individual differences and make them the object of his work?

The videotapes of these lessons show us that the explicit object of knowledge, the one the teacher and students are working on, is the one defined by the Situation. The propositions or responses of the students are taken up only insofar as they are intelligible and useful for the advancement of the adventure. This position completely changes the relationship of the students to the knowledge that is in the course of being collectively constructed, and extends that change to all aspects of the process.

The students do not lack occasions for individually exercising their capacities. They have, in fact, more such occasions than in many traditionally taught courses. These occasions give the teacher a chance to follow the progress of the students' work without making each exercise into a blunt and decisive test calling for an immediate didactical response from the teacher. The pressure on the ones who are falling behind to catch up with the group is collective, and it is all the stronger for that.

Before we progress to the teacher's perspective on this learning adventure, let us take another, deeper look at the knowledge that the teacher is managing. In the Theory of Situations, and indeed for any thoughtful teaching, every lesson is built on various types of prior knowledge. An effective lesson modifies the knowledge, transforms it, completes it. But only a small portion of the knowledge at work in the course of a lesson attains, by the end of that particular lesson, a state that permits the students to formulate it and fully understand it (and thus to be able to write it down as a response to a standardized (decontextualized) question).

In general, before it can emerge as a *savoir* and be exported out of the situations in which it has made its original appearance, knowledge must progress as a *connaissance* in hidden forms through different lessons, often numerous and widely dispersed. A *connaissance* is initially tightly attached to specific situations and limited by the role it plays in those situations. To be detached from them and take its place as a *savoir* it must be recognized, formulated and analyzed. That can be a long process, one that constitutes a genesis of that *savoir*. In every lesson several notions are under construction, often in different stages. Thus the teacher manages (teaches, provokes, sustains, rectifies, etc.) a whole bundle of different *connaissances* and *savoirs* in varying stages of development. The means of managing each one is a Situation – or rather the role that a Situation makes that knowledge play by provoking or justifying its use, its transformation or its replacement. The teacher must thus add or deepen Situations and the means of resolving them and also find within them the questions that keep the process unfolding.

On the other hand, at a given moment, even if the Situation being worked on as well as the knowledge needed to resolve it are common to all the students, the relationships that individual students have with the Situation and the knowledge are all different. The maturing of a piece of knowledge is frequently spread over several lessons. The behaviors of the other students form part of the didactical Situation, and consequently it is not possible to synchronize all of the didactical events among all of the students. At any given moment the teacher must be able to deal with leftover, undigested bits and forms of knowledge as well as newly arising ones. That does not mean that she needs to prolong the process in order to keep addressing the old forms, but that she must not make it impossible to progress if the knowledge is still a bit imperfect. In order to do that, she must constantly assess both the state of knowledge of the class as a whole and that of each individual student. This provides a totally different kind of information from that provided by an examination using a pre-set collection of questions. With this information she is able to make and carry out a continuous sequence of didactical decisions.

The teacher is dealing at the same time, without confusing them, with class knowledge and each student's knowledge. These are different forms of knowledge, and are differently manifested. Hence the knowledge that reaches maturity in the course of a given lesson does not show up in the same way for all of the students. The process must make it possible for the knowledge that is indispensable for community use to be shared as swiftly as possible by the whole class, while leaving some leeway for less immediately crucial knowledge to be developed at different paces by different students.

To consider the objectives and results of a lesson exclusively from the point of view of certain *savoirs*, focusing especially on which ones have not been acquired (which in effect is the normal tactic) is insufficient for managing and conducting a learning process and in the long run dangerous. The minimal objective of a lesson should be to make it possible to approach the next lesson in good condition. The results of a lesson are represented by the number of lessons that can be taken up after doing it that couldn't have been taken up if it had not been done.

A particularly clear illustration of a Situation where class knowledge and individual knowledge tend to diverge and require a lot of managing is the sequence in which the decimal numbers are motivated and introduced by using intervals to bracket a fraction [Modules 4 and 5], about which there will be further discussion later in the chapter. These lessons make unusually heavy use of class knowledge as distinct from individual knowledge. Certainly by the end of the sequence, the individual knowledge of all (or essentially all) of the class includes the forms and uses and management of decimal numbers, and furthermore a well internalized notion that they resolve some messy problems with rational numbers. On the other hand, at many of the intermediate stages the process depends almost exclusively on a more general form of shared knowledge, where everyone is engaged, and everyone has enough partial knowledge to play a genuine part in developing the Situation, but very few if any have the whole picture in their heads. The results in terms of depth of conceptual understanding are well worth the effort, but there is no denying that the process is extremely challenging for the teacher!

The Perspective of the Teacher

Let us move on, then, to the perspective of the teacher. The adventure of these students was also – and above all – that of the teacher. What decisions did he need to make, based on what indications? Our look at the adventure from the student perspective does not tell in what ways the teacher was free to adapt his lessons to the results of the students. There seems to be a great discrepancy between the complexity of the lessons and knowledge that the teacher was responsible for and the apparent simplicity of the knowledge – that of an ordinary class – ultimately provided and formally verified. What is the meaning of phrases like “All the students took part in the activity and finished it”, “The students understood that …” or, “After that the students knew that …”? What was the final result? Why do there seem to be so few formal “learning exercises”? It is the adventure of the teacher that we will try to describe here to respond to these questions.

The teachers who had to manage this curriculum had a high density of aid from a team of researchers and advisors who explained the design, tried to understand the difficulties encountered, and attempted to respond to them. The teachers took part in figuring out the concepts their advisors were using and understood them very well. We will be speaking of the teachers here in their role as instruments of the work. But these teachers were solely responsible for what the students did. They had not only the right but the duty to refuse any suggestion that seemed to them not to be good for the students, and to put an immediate stop to any activity that got out of their control. Very swiftly, by reproducing the same curriculum each year, they familiarized themselves with the profound modifications required in the ways of managing class, and adapted themselves marvelously to it. This is why, in this chapter where we want to look at the adventure from the spontaneous point of view of the teachers, we must anticipate the following chapter and mention some theoretical concepts.

In circumstances where testing plays a heavy role in the evaluation of teachers, schools and even the whole system, teachers are under pressure to focus on results that can be observed by means of standardized tests. Most of their decisions then depend on this ultimate step of the teaching process, and most of the techniques that are considered acceptable are based on the corresponding type of formalized reference knowledge, or *savoir*. The present curriculum offers an alternative by working with all of the *connaissances* – general knowledge in all its forms and stages of development – that preceed and accompany *savoirs* without themselves being either *savoirs* or scholastic objectives. These *connaissances* are picked up in encounters and dealings with appropriate situations. They play the same role that the family environment plays in the learning of native language.

In the course of the process of teaching that we are presenting, a *connaissance* evolves and changes form, use and meaning. In this way it becomes more precise and complete and ends up being known in the canonical form that the culture assigns it, as a *savoir*. This *savoir* results primarily from living with these *connaissances* in many forms. All of them contribute more or less to the moment when it is suddenly obvious that “Everybody knows that…”. Knowing how to recite the rules of the road requires much more effort and is less effective than knowing them because one has practised them assiduously and knows the reasons for them.

The success of each step depends on the previous ones and more or less conditions the possibilities for the ones after it. The collection of these steps constitutes the process of teaching and learning of a *savoir*. In the course of each step, a number of *connaissances* are engaged, each at a different stage of development and evolving towards a different *savoir*. The same *connaissance* presents itself in diverse forms: decision, formulation, explanation, which appear and evolve in appropriate situations.

The teacher does not evaluate *connaissances* like *savoirs*: it is how the activity itself works out that indicates how the project is advancing. The importance of having each student participate and do what she has to do is not the result of an abstract educational intention, it is a necessity – like talking, sharing a culture or a project – a piece of evidence for the students and the teacher. That way the students participate in the development of the curriculum. That is what they call "doing mathematics" as opposed to "learning mathematics" (which they also need to do at times).

Carrying out such a process is at once more complex, more demanding of the teacher in terms of engagement, and less alarming for all parties.

A metaphor might help: the teacher braids a rope whose strands are evolving *connaissances*. A particular *connaissance* may appear and develop and wind itself in with some other *connaissances,* then disappear from sight, only to reappear further along the rope as a new strand that develops in perhaps a different direction and winds itself in with yet another set of *connaissances.* The thicker the resulting rope, the stronger the knowledge that it represents.

The art of the teacher resides in the possibility of observing each stage of the progression of the curriculum and associating with it the decisions most favorable to the stages that follow. Sensitive observations and reliable models for decisions are essential conditions for obtaining chains of decisions – though not the only conditions.

We will first turn our attention to the basic question:

How does the teacher manage the progression of the Situations and the learning of the whole class? How does she continually assess each student's behavior along the way towards appropriating some mathematical concept, and how does she deal with possible divergences from the intentions of the curriculum?

The accuracy of the curriculum and the intimate knowledge of it that the teacher acquires in successive reproductions of it are helpful and reassuring. But a closer look reveals a wide array of possible accidents, detours and divergences. The success of the curriculum and of the students is a result of constant vigilance over certain variables, of constant exercise of subtle choices of judicious decisions, and of clever corrections to prevent the students from losing interest, scattering and giving up.

For simplification, the teacher distinguishes four major types of lessons:

1. Lessons introducing a concept

   These are lessons that introduce the students to an important new mathematical notion: the Thickness of a Sheet of Paper (Module 1, Lesson 1), the Puzzle (Module 8, Lesson 1), the Enlargement of the Optimist (Module 9, Lesson 1), the Pantograph (Module 14, Lesson 1). These lessons are fundamental ones, which we were able to conceive in such a way that they almost invariably produce the desired behavior from the students. The role of the teacher is far from negligible, but it consists entirely of predicting and preventing any accident from messing up or slowing down the dynamic of the game, of directing the didactical phases with spirit and conviction, of discretely encouraging perseverance on the part of some whose energy is flagging, of welcoming student involvement with interest even when it is slightly off track and leaving the Situation to make any necessary corrections to these indispensable contributions.

2. Intermediate Lessons

The students invest their fresh, new knowledge in intermediate Situations to solve known problems in slightly unfamiliar conditions. The teachers found these lessons to be the easiest to manage. It is always a matter of resolving a mathematical question. The students discover new knowledge in problems that make them use and use again the reasoning and calculations that are becoming familiar, but are not yet frozen into scholarly conventions.

For example, Module 8, Lesson 4, the Image of a Decimal, is a typical transition lesson. The introduction of the Situation appears to be the same as that of the Puzzle which immediately precedes it, but the measurements are in decimal numbers, not integers. The children have just finished constructing decimal numbers as a means of comparing and ordering fractions, but these decimal numbers are not yet objects of *savoir*, directly usable in a canonical manner. Sometimes they function almost like whole numbers (for ordering and operations), sometimes the students have to go back to their fractional form to figure out their still somewhat astonishing behavior. The teacher has not yet established one of the different modes of calculating fractions as a canonical method, which would have transformed the whole Situation into an exercise.

The proposed Situation, like that of the Puzzle, has the capacity to reject a fair number of the incorrect answers without the teacher having to intervene. On the other hand, its mathematical objective is considerably more modest than that of the Puzzle, which is designed to produce the discovery of a whole new property.

The students work in groups of three, but each student has the responsibility of producing a piece identical to that of his neighbors, which must fit with theirs to produce a tessellation. This task gives rise to observations that are not an objective of the sequence, but do serve to maintain the interest of the students. For example, some of the groups set about to calculate the eight segments of the perimeter independently, but observation of symmetries enables others to see that they can get by with just three calculations. They point it out to their teammates, which brings out some questions and explanations.

The teacher circulates among the desks and observes the progress of the operations. She might intervene if something of no specific importance interferes with the work of the students, but not to suggest or correct the reasoning or realizations of the students. Only if an error is manifestly sterile, blocking, and incapable of fulfilling its role of pointing students in the right direction does she step in.

Decimal measurement to the nearest millimeter is one of the results of a preceding phase that is built on 3 years of familiarity with the ruler. But it is not an objective of this lesson, especially since an error in the measurement of the model would only surface very late in the process. The teacher has two students measure the sides and write their measurements on the board for everyone to use. The work of the students deals with the method of calculation and the calculation itself. The teacher insists on having students carry out the calculations individually before comparing the results with others in the group. But it is acceptable to help a comrade with one or two of the calculations, and to discuss which method to choose. There is absolutely no obligation for all the members of the group to use the same method. However, a member of a group is allowed to insist on understanding what another member did.

Students are thus motivated to use methods that others can understand or at least reproduce so as to make their own calculations. But they are also motivated to have several methods available if possible, so as to present them for their classmates and the teacher to admire in the presentation phase of the lesson. The teacher permits this goal but doesn't encourage it much, in order to avoid the proliferation of equivocal propositions that could soak up a lot of everybody's time and energy.

No explicit reference is made to the procedures used in the previous lesson. The students do not have to reproduce what they did the day before, just to use it for inspiration without losing sight of what they are now doing. That makes an implicit rule for the teacher, who must avoid saying, for instance, "Just do what we did yesterday!" That would be a purely didactical argument. This Situation is different and should officially be examined independently. The similarities are the student's responsibility. To be sure, the expectation is that the student will use or try to use what he did the day before, but of his own volition.

Since the numbers are the same for everybody, it is hard to maintain the uncertainty. It is absolutely necessary that the individual part be respected and be required for the making of the pieces. The teacher needs to verify that each student has had to carry out by herself some calculations similar to the ones from the lesson before. If it is needed, the teacher gives different groups projects with different dimensions.

This lesson is close to being a classical exercise. The children do carry out similar calculations over and over, but here it is in a completely different spirit. These calculations are justified by a collective task, not by a personal project of perfectionism required by a monitor. Knowledge is made evident by its use in a new "adventure"; it is going to become familiar, with or without the aid of formal description, which will not turn up until it is needed for the development of further knowledge. In this process, the pressure to turn the scholarly activity into an individual formal learning project is minimal. The engine is the participation in the construction of a collective and individual culture.

3. Terminal lessons

The following lesson (Module 8, Lesson 5) proceeds just the same way, but it is a different type: it provides a conclusion and an institutionalization. It looks like a continuation of the preceding one – it takes up the same *milieu* and it is still about a fixed enlargement: $1 \rightarrow 3.5$. But the questions are very different and not "motivated". The teacher asks for the images of a bunch of numbers that are clearly of a particular kind: 1/10, 1/100, 1/1,000, etc. In the process of carrying out the calculations, the students come to the realization that they can now deduce from what they already know a new (for them) rule for division of a decimal number by 10, 100, 1,000, and that they can say it, prove it, practice it on demand, and require other people to understand them without having to repeat their demonstrations. It is just a question of recognizing what they already know how to do and nailing it down with rules and words that express what they already think and know. The numerous calculations that they have to make are justified by a goal held by the community of students, but it is a knowledge goal that they can see approaching and that they achieve.

This looks exactly like a classical lesson, except that it is the students who are supposed to guess and establish the *savoir* that is to be learned in order to resolve the situation proposed. It would be only an exercise if the method had been laid out in advance. It is completely simple to solve, and the students work individually. The question is different and arbitrary, but the answer is known (not for sure, but it can be guessed.)

The numerous individual attempts are not repeated exercises. They are attempts, more or less successful and more or less appreciated by the others. The goal is to be able to continue taking part in the common adventure with the other students, to be able to present one's ideas and bring in one's work. It is not the pursuit of a personal egocentric goal supported only by the undependable satisfaction of adults.

The formulation of the rule for dividing decimal numbers by 10, 100 and 1,000 is stated by the students at the request of the teacher, accepted (i.e., institutionalized) as a *savoir* and immediately applied in exercises that are promptly corrected. This is the normal method, and it has the usual results. Many of the students understand, all of them make some correct calculations and many make mistakes. The teacher is not expected to hold out for an immediate, definitive, and general success on this important question. Because it is used frequently, they will be reminded of it often and the teacher can follow the individual progress of the students until they get it. The goal of Situations of institutionalization is for the students to know that they have a common repertoire of objects, terms and *savoirs*, which can be best understood in exchanges with others if they use the conventional solutions, terms and explanations.

4. The process of generating a concept

The most complex lessons for the teacher are those where for an extended period she must manage provisional, uncertain knowledge in order to bring out different aspects of a concept. Ambiguities are only gradually resolved, nothing is formalized but nothing should be forgotten.

The best example of this type of process is the sequence of Situations leading up to the construction of the decimal numbers (Module 4, from Lesson 1 to Lesson 4). In this type of sequence the teacher and the children use and evolve *connaissances* that cannot be set up as *savoirs*. Every Situation prepares for the one that follows as much by the questions it raises as by the answers it provides. The most important thing for the children is remembering not the specific outcomes of the adventure but the things they have encountered along the way – intervals, end-points, interval lengths, the search for a strategy for reducing the interval of uncertainty, etc. Nothing is to be learned in final form, but all the calculations they have made contribute to an incomparable familiarity with the rational and decimal number line, and with the calculation and location of those numbers.

These lessons have to do with the order and topology of the rational and decimal numbers. They come close to reproducing an almost historic and scientific development, but the objectives and real significance of the sequence remain obscure to the students until quite late. It's a matter of comparing the size of fractions, finding an interval around them, estimating them, ordering them, improving on the intervals, and the like. At the end of the route, after Module 8, Lesson 5, the students invent a method that could be called a division, but that for them is just the method of finding a decimal expression for a fraction by locating it in successively smaller decimal number intervals.

In the opinion of the teachers, the first lessons of this sequence were the most difficult ones to manage in the whole curriculum. Nonetheless, they were successfully reproduced every year for 25 years with the same results.

Within the lessons, whatever the type, the teacher must make choices based on the state of knowledge of the students, which brings us to our next question:

What are the manifestations of student mathematical activity with respect to a connaissance?

In the course of carrying out a Situation, the teacher must keep track of the functioning and evolution of many forms of *connaissances* related to the *savoirs* that she wants her students to acquire.

A major mathematical *connaissance* makes its appearance in the curriculum as an initiative of the student in different roles and conditions, and generally roughly in the following forms and order:

Observable Aspects of Connaissances

- Student decisions. For these the *connaissances* need only be adequate for decision making, whatever the form in which they are conceived (Action Situations)
- Formulations that may be improvised but must be intelligible (Formulation or Communication Situations)
- Proofs that it are valid, and consistent with what is already known. The proofs must be recognized as valid by peers (Validation Situations)
- *Savoirs* extracted from their context and offered in a situation where there may be doubts about their pertinence or utility, but not about their validity.

Savoirs follow a different route, since their status as reference knowledge needs direct action from the teacher. They nonetheless need to be kept track of.

Manifestation of Savoirs

- As a reference: its definition or certain of its properties, expressed in a canonical fashion, are declared or recognized by the teacher as personal, interpersonal or cultural references (Institutionalization Situations)
- Explicit investment of these references by the students in problems or exercises and in proofs.
- Casual use as references or implicit knowledge in new uncertain situations (Action Situations)

Although the succession is not arbitrary, it is also not a formal necessity. It can be adapted and be responsive to the possibilities and necessities of the curriculum, which itself is subject to many other constraints. In this curriculum, all unnecessary steps and digressions have been eliminated. Some Situations produce a rapid evolution while in other cases several Situations are necessary to achieve a single step. Different *connaissances* are involved in the same Situation, in different forms and roles. They may advance all together or separate themselves in conjectures or reasoning. This process simulates as authentically as possible a genuine mathematical activity.

We will not detail here the tangle of *connaissances* and *savoirs* that turn up together in the course of each lesson, each evolving in a specific way under the influence of successive Situations in the course of the curriculum. The teacher must stay conscious of the dependences that come into play among these *connaissances* in the course of the different steps. The reader can follow the twists and turns of the adventure in Chap. 2. Here the issue is to understand the action of the teacher while the adventure is in progress.

Teaching a mathematical subject presents a teacher with two essential and distinct types of difficulties: those connected to carrying out each episode (a whole lesson or a particular phase: assignment, exercise, correction, assessment, etc.) and those connected to the total trajectory: choice of successive episodes and the passage from one episode to another (or from one lesson to another.) The former have to do with the actual activities of the students moment by moment, and the latter with the possibility that these activities can succeed in producing a coherent culture, and a capacity among the students for undertaking new activities. Concretely, in the second case, for the teacher it is a matter of evaluating the possibility of undertaking the next phases of the curriculum based on the earlier ones.

In terms of Situations, the result of a particular episode consists of all the Situations that can be taken up thereafter with a good chance of success but could not have been before it, and of all the ones that will not have to be revisited at the end of the teaching sequence thanks to having done it.

In the curriculum that we are presenting, the principal instrument of regulation at the disposition of the teacher is the choice of the moment of institutionalization. In supporting autonomous activities of the students, the Situations bring out questions, convictions, declarations, arguments, *connaissances* that are justified only by their temporary use in the students' thought process and in this particular Situation. The cultural value of these *connaissances* – their actual validity, their canonical formulation, their place among *savoirs* – is not something the students can deduce from their role in the Situation. Furthermore in the course of the Situation events turn up that are known to only one student or group of students or even to the whole class, and the students don't know their value and may suspect that they are temporary, since the Situation itself may modify them.

Institutionalization is an act or process that causes a fact or *connaissance* to pass from one sphere to a larger one. For example, the teacher tells the whole class about something done by a student or group of students, or summarizes the session from the day before and the state of the question being studied, or describes a result that everyone can now count on, or confirms that a conclusion conforms to the truth and is recognized by science or society, indicates that a result was the objective of the lesson, etc.

Institutionalization has a slightly ambiguous status. The *connaissance* or convention in question is certainly precise and well determined, but the affirmation that everyone will henceforth know it, practice it without opposition, and use it as means and reference is a convention and above all a gamble. The fact that no one is supposed to be ignorant of a law does not turn that law into a sure practice. The fact that not everyone respects or is able to respect the law is not a reason to give it up.

Institutionalization of *connaissances* is a Situation in the course of which the teacher recognizes as valid and accepted by society the *connaissances* that the students have come up with in the course or conclusion of a Situation or series of Situations and that they propose as a reference. This event concludes the phase of quest for *connaissances* on the part of the students and determines the *savoirs* that they can take as certain.

Normally institutionalization signifies that thereafter each student will be authorized to refer to this *savoir* to support an opinion, and is assumed to be capable of producing it with precision and confidence and using it correctly. The teacher guarantees that this *savoir* is exportable and recognized outside of the classroom by society as a whole. Clearly nobody in the class but the teacher can give this guarantee.

So the question is how to determine the moment at which institutionalization can be made to have the best chance of succeeding with all of the students. Done prematurely and suddenly it would isolate the few students who were the first to be able to understand it and submit to it and would tend to make the rest appear to be rebelling against a communal law. Not only that, it would make the latter submit to a servile relationship with *savoir*, to learn and apply a rule that they do not understand and that they can only acquire by procedures foreign to their understanding. At the other end of the scale, an excessively scrupulous institutionalization would wait until each and every student understood and could put the rule into practice. Waiting that long would cause an excessive delay in pursuit of acculturation to other *savoirs*.

Institutionalization can apply to *connaissances*, but also to Situations. When the development of the Situation becomes confused, the reactions and the various more or less true or false "*connaissances*" diverge. Nothing more can be understood the same way by the whole class in the natural course of the actual Situation. These differences make the pursuit of the proposed communal activity impossible. The teacher must then pull everyone together with "What has happened so far? What was the Situation we started with? What did some of you do? What did others of you do? Where are we now with the problem?" This re-framing of the Situation informs all of the students what is in question, what deserves to be noticed and what remains the object of the action, which can then resume its course (unless the essential part of what was of interest in the Situation has had to be revealed).

This approach to institutionalization contrasts sharply with the curricula (such as the daisy-chain programs) that are reduced to a sequence of institutionalizations. Each lesson, each exercise and each *savoir* presented is considered to be both necessary and sufficient for proceeding to the following step. Every question is considered to be equally key and definitive and the only *connaissances* considered are *savoirs* and errors. At every step the student is supposed to make an effort sufficient to succeed in completely acquiring a given, indispensable *savoir*.

Institutionalization marks the separation between things that are of the order of *connaissances* – temporary, personal, in question – and things that are accepted as definitive, agreed to, common and sure.

In making decisions, teachers must be conscious of the whole structure of the curriculum, which brings up the question: What are the dependencies between lessons and between things learned?

How does the progression of one lesson depend on that of the preceding lessons? What are the indicators of good progress in the process? What are the possibilities for intervention by the teacher if something goes off track or fails? What constitutes a failure and what is just an episode? All these questions are tightly linked.

How a lesson develops can depend on how the previous one developed. The second one can depend on the *savoirs* learned in the course of the first. Sometimes the students cannot do, say, understand or learn what they are supposed to because they cannot use the necessary *savoir* because they did not learn it beforehand.

The precaution of never using a single word or property that has not previously been defined or demonstrated is the basis of the general, deductive organization of mathematics. This organization is often used as a model for the teaching of science and even for the acquisition of all scholarly knowledge. The teacher wants to be able to report that he has made available to the students all the necessary elements and the only possible cause for failure of his lesson would be failure of previous teaching or inability of the students to understand the construction under way. But before *connaissances* can be definitively cast in the bronze of organized *savoir* they must be established by complex processes very different from this fiction. In this curriculum we try to have the students reproduce or simulate such a process.

To this end, we have installed at the heart of the lessons Situations that are steps in an adventure. A Situation exists independently of the actions and modifications of the protagonists.

Two successive Situations are linked if what is produced in the first conditions what can be produced in the second. We distinguish at least two types of dependence:

1. Two lessons may be linked because the second (in time) uses or resumes use of *connaissances* that have been *established* in the first. They are connected by a structural relationship of *savoir*: for example the second lesson studies the corollaries of a statement established in the first.

   The reality of the learning sequence does not necessarily follow the order of an exposition of *savoirs*. It is not indispensible for the students to have fully understood and learned everything that has been defined or demonstrated for them. The study of the consequences, extensions and "uses" of taught *savoir* is indispensible to explore, know and understand a definition (which furthermore is often the result of a concentrated result of a complex process.) Otherwise stated, the appropriation of a *connaissance*, even presented in a strict axiomatic order, depends as much on the lessons that follow it as those that preceded it. The process must be considered as a whole.

2. Two lessons can also be linked not by the relationship of the *savoirs* that they offer for learning, but by the questions that these *savoirs* are supposed to answer. The result of an unresolved Situation can be a new question that gives rise to a new Situation which itself may make it possible to resolve the initial problem.[2]
3. A lesson can obviously be linked to a previous one both by questions arising from the previous one and by the consequences of the *connaissances* established in it.

For example, when the teacher asks, "Are the measurements of thickness numbers?" he introduces a natural sequence of lessons (i.e., one that the students could practically run themselves) sparked by the questions "What do we do with the natural numbers that we may or may not be able to do with the measurements of thickness?" And certainly also new *connaissances* are established using the preceding ones.

What Then Are the Causes of Learning and the Reasons for Knowing?

Having an individual reproduce the same task is the antique means of having him learn it and execute it more easily in all circumstances. The learning can be observed through the progress of the student in the perfection of execution (reliability, speed, precision). The link between the successive steps is essentially the state of the student. One cannot pass from one task to a more complex one unless the student has satisfied certain required conditions. If there is a connection between the things learned, a progression in the complexity of the tasks, only the teacher responds to it. Thus it is the state of the student that is the link between two steps of the learning process. The student reproduces calculations in order to know how to do them. And if learning makes no progress he has only himself to blame, his characteristics, qualities or faults. The teacher and society reinforce the blame and question the properties and virtues of the individuals who are recipients of the keys to perfect learning.

The learning process with which we were experimenting here is completely different. This formal (and universal) learning process has only a marginal place in it. The repetitions of "exercises" are not motivated by a direct desire to enrich oneself by knowing how to do them. They are steps in the realization of a task that has its own significance and interest and that is a goal shared with others. The Situations proposed are not solely destined to be the causes for learning for individuals, they are first of all destined to determine the reason for some *savoir* to exist, the role that it plays in people's relationship with each other and the world, and the role that humans play in society thanks to that *savoir.*

[2] An experiment that we carried out demonstrated how a sequence of Situations each issuing from the previous ones by questions produced by the students was able to generate the discovery of limits of frequencies and of measures of events without the teacher's ever proposing a Situation beyond the initial one or supplying information or a personal solution – and without the notion of chance ever being mentioned! (Brousseau, Brousseau, & Warfield, 2002)

There are large moral, cultural and epistemological differences between reproducing a calculation in order to advance a common task and reproducing a calculation simply to know how to do it oneself.

Finally, the conduct of the lessons and of the sequences they formed depended heavily on the observation by the teachers of a certain collection of indicators, on the verification that a certain number of appropriate and expected corrections were resulting from the combined impact of the Situations and the teacher's interventions in these Situations.

How Does the Teacher Use Assessment of and Within the Curriculum?

The teacher assesses the Situation under way, the *savoirs* in action and the students. These assessments are subordinate to the possibilities for action that are available to the teacher and depend on the assessment.

The purpose of assessing the Situation is to determine moment by moment whether it is best to let the Situation proceed or whether it is time to intervene and either to redirect its course or to interrupt it. For example: Is some additional commentary on the assignment needed so that all of the students have some project for action (whatever it may be) that will let them get into the problem? On the other hand, at what point would supplementary information make the necessary efforts useless?

The decision depends on the expected profit from the amount of additional time accorded to the intervention. It is difficult to describe in a few lines all of the factors that need taking into account: the fatigue or loss of interest of the students, the amount of useful information that can be harvested (not just plain success.) Sometimes teachers content themselves with one correct proposition (the success of one student or group). Sometimes it is important that each participant obtain a proposition to present to the other students.

This assessment applies simultaneously to reality – to facts – and to their meaning, that is, the possibilities for interpreting them offered by the Situation. Sometimes making each and every student experience all of the difficulties and their solutions is completely superfluous. Students may be able to benefit by proxy from the experience of others. Sometime simulations are sufficient, while other events need to be really experienced. Deciding to organize a Situation in such a way as to produce the discovery of the properties of a mathematical notion is a non-trivial decision. It takes what might be a considerable amount of time for the sake of what might be a trivial significance. Any time that that is possible the Situation needs to be reduced. Often a short definition followed by an illustration of examples and counterexamples is the best solution if that definition is useful in the project in progress. Often a simulation can be worth more than an actual heavy realization.

In general, Situations define and at the same time more or less dissemble certain *connaissances* that the student is supposed to make use of to accomplish a proposed goal. Certain Situations have the objective of determining whether the student has available, directly, the *connaissance* for the solution. By definition they do not offer the students who do not already have this *connaissance* the possibility of answering. They are assessments for the student, but they are thus not in principle didactical. They answer a question, but say nothing about others that might connect to it.

Others, on the other hand, have the (didactical) property of inducing the production by the student of a *connaissance* that he did not previously have available (in the form of a *savoir*), but that he can conceive (guess, construct, comprehend, etc.), formulate, prove valid and finally "learn" at his own pace.

When a teacher must intervene in order to promote the evolution of a didactical Situation, one of the principal difficulties consists of monitoring the informative value of his interventions. In an effort to stimulate or re-launch the activity of the students he might bring in information that reduces the Situation to the obvious, or on the other hand he might complicate the work of the students by throwing in superfluous intentions or requirements.

Didactique is, for the teacher, the art of showing and hiding his intentions in such a way as to permit the student to discover as a personal response to objective conditions the thing that the teacher wants to teach but cannot reveal without depriving the student of the possibility of doing it himself.

Making *connaissances* contribute to the learning of *savoirs* so as to approximate the real cultural, social and psychological functioning of mathematical thought presents some very real risks: first, the risk of wasting time and energy, next the risk of accidently producing the learning of *connaissances* that are false, or badly established, or badly formulated, inappropriate or culturally unacceptable.

It is very important to know how to interrupt a Situation that is becoming ambiguous, or that doesn't guarantee that the emerging *connaissances* will have a reasonably strong and simple impact. One must not hesitate then simply to state the canonical solution being sought. This danger is eliminated in the curricula that only consider established *savoirs*, as visible objectives and/or as means.

The Assessment of Students and Groups of Students

The goal of assessing students is to predict whether they are going to be able, together or individually, to take on the rest of the curriculum. This is of interest exclusively in the case where it is possible to choose and manage the curriculum on the basis of the results of the assessment.

Naturally the progression of the Situations permits the teacher to adapt a Situation to the possibilities and varied talents of the students. This continuous adaptation is easier than the choice of appropriate exercises and problems. But once a Situation has come and gone, once an adventure has been lived, for better or for worse it can't begin again. Moreover, the construction of *connaissances* and their meanings is common to all, and there is no royal road.

It is at the moment of institutionalization that it seems one must discriminate between those who understand and those who have not gained from the curriculum the resources needed for the proper acquisition of *savoirs.* Then all that is left is classical resources, explanation and repetition. These possibilities must not be neglected. Institutionalization does not put an end to the process of learning. Really useful *connaissances* should be revisited often enough to permit a party of students to rejoin the troop. What a Situation has made the students live, what has been perceived, communicated and explained is not a required object of *savoir*. People from the same society live and communicate with highly varied repertoires.

The Types of Situations That Appear in the Lessons

We made a distinction above among types of lessons, distinguishing them by their function in different stages of the learning process. Another perspective on the teacher's role comes from a dichotomy that is deeply rooted in the Theory of Situations: *didactical Situations* and *a-didactical Situations.*

- In *didactical Situations*, the teacher maintains direct responsibility for all stages of the lesson. She tells the students her intentions, what they will have to do, and what the results should be. She intervenes freely to keep the class traveling on the desired route. In our curriculum the reader can spot these completely classical phases. They were carried out in the classical manner.
- In *a-didactical Situations* it is the students who have the initiative and the responsibility for what comes of the Situation. The teacher thus delegates part of the care for justifying, channeling and correcting the students' decisions to a *milieu* (a problem statement, a physical set-up, a game, an experiment).

The former tend to produce the learning of reference knowledge, either permanent (*savoirs)* or temporary (assignments, rules, etc.). The latter tend to bring into play *connaissances* corresponding to the *savoirs* being taught. *Connaissances* manifest themselves in responses (actions, choices, expressions, trying things out) in circumstances where they seem necessary and adequate.

In didactical Situations, the students' *connaissances* do not develop and are only manifested in the course of applications, and thus after the presentation and acquisition of the necessary *savoirs*. The teacher demonstrates that the expected answer has been given in the preliminaries, or convinces the student that it is his responsibility to deduce it from what he has been given. But in fact *connaissances* can appear before the student has the corresponding *savoir* available in appropriate circumstances. Thus it is possible not just for *connaissances* to follow from the acquisition of *savoirs* but for them to precede and justify that acquisition.

These *connaissances* correspond to a *savoir*, but they may well differ from it (for example they may be true or false, or consist of beliefs, or be questions.) They may also differ from student to student, because they are often individual. They are similar in the sense that they tend to be opportune and adequate in the same circumstances. They are fleeting and cannot be directly evaluated by classical tests. But they are the only means by which students can participate in the adventure of their own learning.

Classical curricula also combine didactical and a-didactical phases, but in this curriculum their roles and relationships and the proportion of time allocated to one or the other are profoundly modified. Our aim was that the *connaissances* be also the means by which the students participated in the epistemological adventure of *savoir* and of their own personal *savoir*. The curriculum presented in Chap. 2 develops *connaissances* that precede, accompany and follow *savoirs*, as happens in the natural exercise of mathematics.

But this ambition complicates the work of the teachers a lot. What is that work, then?

The Types of Didactical Situation and How They Are Conducted

Many didactical Situations were classical and were (and are) in use in all schools. But the reader may also note some didactical Situations of a new type:

Situations of Institutionalization

These were discussed above, but we will expand slightly on them[3]: The teacher directs a session that consists of observing that almost all the students understand *this* and know how to do *that*. He has the students put this *savoir* in order by presenting it himself. He makes the definitions, algorithms and theorems precise and declares that henceforth he is counting on the few students who are still hesitant to look into these questions in order to be able to continue to work with the others. These are the didactical Situations in which the students learn that certain of the *connaissances* that they formed in the course of preceding a-didactical Situations can be organized, formulated and thus proved. They learn that henceforth they need to know them for communication and for reference. These are lessons of institutionalization. They take on a particular importance because of the importance given to the a-didactical Situations for developing *connaissances* before putting them in definitive form.

These lessons are delicate. Only the teacher can judge the best moment to activate this phase of learning. If it is done too late the children will have developed and become entrenched in ill-conceived ways of doing things, inappropriate ways of saying things and fallacious reasons for knowing things – *connaissances* ensconced as *savoirs*, but badly built and difficult to abandon. If it is done too early the *connaissances* will not be sufficiently familiar to support a precise and solid formalization. A large majority of the students must be able to make the change without effort in order for the

[3] See also Chap. 5.

challenge to be met by those who need to make use of the following encounters with this *savoir* to finish learning it and need to do some exercises to make it familiar.

Institutionalization in this case deals with *connaissances* and produces *savoirs* that are durable and if possible definitive. It can also deal with provisional conditions such as the rules of a game. It is often easier to learn the rules of a game while playing it than it is to learn them in advance. The operation is only interesting if the rules are simple, if the game is reasonably easy to repeat, if the student can notice for herself the causes of any difficulties and correct them and if the *connaissances* thus produced are both interesting and useful for the learning being aimed at. In this case, the rules are part of the solution *savoir*.

Situations of Devolution of an A-didactical Situation

Students are only willing to enter into an a-didactical Situation in the hope of finding pleasure and profit. They must have the hope that they will be able to find on their own the essential parts of the solution, and that the search itself will be exciting and intriguing, that it will be reproducible (though an occasional serendipitous victory produces a kind of satisfaction and should be accepted.) Otherwise stated, the "games" chosen must present specific real qualities and notably feed-back that permits the student to check the value of her actions and understand the reasons for it.

This does involve a didactical Situation because the teacher must teach the rules, but his role consists principally of indicating to the student that he has no obligation to tell her what he wants to teach her. The teacher must let the student know that he ardently hopes she will play, but he cannot force her to do so, and that he hopes not so much that she will win as that she will understand and learn something that will enable her to win.

Conducting such a lesson is a difficult art. The teacher must show a great interest in the game itself and give encouragement to all the players, but he must respond indistinguishably to successes and to errors or stupidities, and initially treat discoveries as difficulties. It is the students who must judge what is good to know, true and useful. The teacher must be able to encourage the students and help the weak or the suspicious – but not too much.

These Situations are not made for judging the students but for developing and judging *connaissances*. For that the teacher must supervise numerous parameters: what it costs the students to participate, the speed of their progress, how ideas spread through the class. He must calm fears and also excessive enthusiasm. If the Situation is not well calibrated he will have to make concessions, but he will have to hide them as much as he can.

The a-didactical Situations that permit this devolution cannot be improvisations.

Situations of Evaluation

In the traditional system what is evaluated is essentially the students, indirectly the knowledge acquired, and secondarily the teachers. But this type of summative evaluation gives only partial information, insufficient for making decisions in the course of learning. This information is subject to superficial interpretations produced by reductionist pedagogical ideologies that use them for inappropriate decisions. To be able to negotiate more effective teaching the teachers and students must develop a culture of evaluation by a communal practice of *Situations of evaluation* in the course of teaching. They are the times for the teacher and the students to take stock, to look together over what has been done, what that means and what it would be best to do next. They are the instrument for transmitting a very necessary epistemology and scholarly *didactique*. We cannot describe how the teacher conducts this type of Situation until after we examine the conduct of a-didactical Situations.

The Types of A-didactical Situation and How They Are Conducted

The objective of a-didactical Situations is to induce manifestations of *connaissances* such as decisions (if possible adequate ones), formulations (effective whether or not correct), and/or convincing proofs characteristic of the notion to be taught. They take place before the phases of exposition of *savoir*. That way the *savoirs* become a conclusion that the students can draw, after some preliminary work that bears more resemblance to motivated research than to free exploration of a theme. This approach thus precedes (but note that it does not exclude) the classical presentation that proceeds from the study of a text to be learned and known (definitions, fundamental theorems…) to formal teaching, then to its applications.

The goal of these a-didactical Situations is to facilitate the learning of the corresponding *savoirs* by first making familiar and intelligible what it is that they mean, which is what the students ultimately need to acquire as canonical knowledge. The formal classical learning comes in as a supplement, after phases of intense use of the *connaissances*, motivated by other projects. This type of Situation must be distinguished from classical "discovery situations" in which the teacher has the students visit various aspects of a notion borrowed from a text that is already there.

The teacher must concentrate her efforts and those of her students on the questions posed and the tasks to be carried out and thus avoid creating a direct didactical tension about the *means* of accomplishing these tasks (the *savoir*). Learning is a spontaneous consequence of the activity. The *connaissances* are thus the means and not the official goal of the Situations. At the same time, they are also one of their consequences.

Once a *connaissance* becomes sufficiently familiar it is time to recognize its importance and its place. The students then may well be willing to make an extra effort in the form of exercises "to make it stick" in order to make their use of this recognized *savoir* easier and more fluid. Learning snippets of knowledge under construction head-on and without relevance in order to apply them in conditions as yet undiscovered – the classical method – has its points, but it requires of the students a great deal more confidence, attention and good will.

The choice to teach the use of the *connaissances* before making them the objects of *savoir* to be learned voluntarily is a positive one. It relieves the students of the tension created by the obligation to regard everything presented at any moment in class as equally indispensible and decisive and hence in need of instant learning. Each lesson is thus the occasion to make progress with some *connaissances* and among them to recognize some and institutionalize them, and to exercise others that have already been institutionalized.

This choice requires of the teacher both sophisticated methods of evaluation and complex decision strategies.

Situations of Validation (or Proof)

These are the ones that make the mathematical reasoning of the students most visible, as they produce arguments addressed to their peers with the goal of convincing them. It's a matter of inciting the students to become skeptical about some precise mathematical notion and of giving them a motive and the means not just to check the validity but beyond that to convince the other students.

These Situations develop the capacity to produce, appreciate and judge arguments and in the end to distinguish and reject incorrect rhetorical methods and practices. In organizing debates, the teacher also teaches progressively more formal rules. On the other hand, it is essential not to lose sight of the fact that the important thing is the declaration and its proof. This type of initiation rests principally on the cleverness of the teacher, whose interventions must be attuned to a variety of indices in order to optimize the interest and participation of all the students. This cannot be judged solely on the number of participants, nor on the speed with which the solution is given and established.

She can for instance, organize debates first in very small groups and then in larger groups to bring up alternatives. Whatever the format, the game must be worth the time and effort required for it. The interval between too obvious and too complicated can be a very narrow one. If the whole argument depends on an abstract demonstration, the discussion may degenerate into a debate among two or three "champions" without benefit to the most of the class. Speeding up or slowing down the process, maintaining the engagement and pleasure of each student, avoiding traps posed by individuals, cutting it short or waiting patiently – only a report of the discussions of the debriefings of the teachers and the researchers could do justice to the subtlety of conducting this kind of lesson, often halfway between reality and simulation, and to their influence on the enthusiasm of the students when they were successful.

Situations of Formulation

In order for a Situation of validation to function, the students need to have understood the object of the debate and thus to be capable of formulating the elements of it. Some specific Situations lead to this result by challenging the students to communicate some real information to a partner, either using a known vocabulary or by creating a repertoire or even a provisional specific language like the one for designating the thickness of the sheets of paper or of the enlargements. This type of Situation where actual communication is organized requires particular set-ups and materials for the class and thus must not be trivialized. But conducting them is much less of a burden on the teacher. The times and results are easier for the teacher to regulate and the students to evaluate. Furthermore the students rapidly acquire sufficient knowledge about communication for the teacher to be able simply mention it without actually realizing it, and save actual communications for the cases that merit it.

Situations of Action

Formulations only make sense through their meaning in terms of decisions in a specific conditions. Thus the whole construction is founded on the possibility of giving each mathematical notion to be taught a meaning that is simultaneously significant, correct and fruitful. This meaning is traditionally given by the linguistic means offered by the culture: verbal definitions, explanations and proofs – essentially by texts. The teaching of mathematics is thus reduced to the study of a text with the aid of texts that may be illustrated by a discourse. These means appear economical because they facilitate the communication of the text of the *savoir*. But in reality they are not economical for the students, who grasp concepts better by their function in the course of an action in a situation and by the decisions that it calls for than by descriptions and intellectual proofs. Action Situations, in the large sense, are thus the foundation of the whole edifice for all of the students.

Carrying out an action Situation was fairly easy because it had been well conceived. The teacher had to restrict herself to being satisfied with the first successes. She was not supposed to approve them or spread them around. With the complicity of the students who had found an answer (which they thought was right, or knew it was because it obtained the desired result) she encouraged each of the students to try to find it. And she received them all equally, whether they had been invented or inspired by an auxiliary peek at a fellow student's work. The essential thing was that the student adopted a production as his own. Who remembers how and from whom he learned the words and most of the knowledge that he uses?

The principal difficulty for the teacher in conducting an a-didactical lesson is maintaining a fragile equilibrium between what is said and not said, what is desired and what assumed, what is suggested and what required.

For the students, the *connaissances* thus emerge from a story resulting from a mixture of truth and fiction. The story told at the end of the adventure by the students and by the teacher assembles these pieces and becomes not just the reality of a class but the legend of the birth of a notion or a concept. The important thing is that that adventure be intriguing and fascinating, that it be possible to engage in it with one's strengths and weaknesses, and above all that it have a meaning and an epistemological and didactical value such that the quality of what is gained justifies and recompenses the efforts, the disappointed hopes, and the vain attempts.

And what happens when for one reason or another the miracle does not happen? Sometimes because of a detail, a fault in the preparation or execution of a delicate sequence of actions produces a fiasco: the materials refuse to follow what appears to be their natural law, damp paper compresses and eliminates all precision of measurement, the water spills (predictably!) over the side of the bowl, the pantograph does not work right or a bird flies into the classroom … everything gets muddled and nobody understands anything, or worse understands the reverse of what was hoped for.

Nothing is lost and often the students not only imagine and understand anyway what was supposed to happen, but sometimes even understand it better than if they had gotten it without the complications. And depending on her personality, the teacher repairs the thread of the story they are in the midst of in her own way, and admits that, like her students, she cannot always get everything right.

By frequently putting the teacher and the students under the obligation of cooperating to make the current action succeed, Situations stimulate, facilitate and guarantee a large part of the learning of the goal knowledge.

Situations of action, formulation and proof (or validation) proceed in principle without the teacher intervening directly in the course of their solution. They are called a-didactical: in them the teacher is not directly teaching any knowledge. But they should most often be proposed by the teacher, who ought at least to "teach" the rules of the game as instructions – the students should simply learn to play, not take the rules as *savoir* to be learned. The teacher *informs* the students and *prescribes* an activity for them. (He introduces the rules to be followed and an objective to aim for as a provisional institution – a convention – in the class.)

At other moments the teacher may intervene to *comment* on the progress of the lesson and to *report* with the students the state of the adventure and its results. Recognizing, organizing, presenting, explaining and leading an evaluation of the *savoirs* aimed at, drawing conclusions from these reports in terms of decisions for following lessons are types of didactical Situations (because what is taught passes through the formulation of the didactical will of the teacher).

Presentation of the Rules of the Game

The teacher transmits the rules of the game, but these rules are means of learning, not *savoirs* to learn. They may be forgotten, but in fact they leave a trace in the form of the conditions of the final *savoir*. The teacher proposes the Situations, which are in charge of advancing the class knowledge. She must present the materials, designate the players (individuals or teams), indicate the goal of the action of the students, the starting position, the activities permitted or not permitted, the final state being sought for and the states that indicate a failure. She might offer a reward – a purely symbolic one – or designate the number of rounds to be played.

If the students are to undertake an action that might bring them some additional information they must envisage a *basic strategy*, which the teacher might possibly suggest. In general, this basic strategy is not the one that is supposed to be found.

It won't work fully, or it is long and messy, and it should swiftly be clear that it needs to be avoided. It simply makes it possible to take the first steps.[4] Note that one must accept as a success – as mathematicians do – the blind trial of all possible cases, or even the presentation of a good solution when the student is unable to explain how he got it. It is enough to show that the solution is valid and sufficient. The rest (commentaries, explanations, etc.) is a legitimate requirement, but it is didactical.

Except for exercises and classical problems, the teachers almost never simply set out the instructions for Situations in a form that had been written up for them. They needed, for example, to play a couple of trial rounds so the students could understand the rules. The ratios between the time spent explaining the rules, the time the students needed to solve it, and the importance of the knowledge that they needed to use to do so were clearly decisive criteria.

The teachers must above all pay attention to the time required for a Situation. If the "Situation" under discussion is such that the students could find the strategy and answer without actually playing a round, then it is just a question and should be treated as such. It is better not to use a game if:

- The rules are harder to teach and understand than the solution
- The solution cannot be found in a reasonable time (then it is just a riddle), or
- It does not require that the student invent an interesting and instructive strategy (then it is just a pastime).

Evaluation in A-didactical Situations

A-didactical situations mobilize knowledge that the students are in the process of learning. Thus they constitute an opportunity for the teacher to evaluate the acquisition of that knowledge. But this evaluation is not summative, it is formative. The student carries on without thinking about it if he succeeds. If not, he simply notices that things are not working and either fixes it himself or calls on the teacher, who can record the fact, but with no immediate consequences. The teacher also goes on some indications that the student is unaware of: what the latter does and says must be interpreted. The student's knowledge evolves differently from that of students in situations where the learning is parceled out and the evaluations match the parcels. Nadine Brousseau's excellent descriptions that keep us in contact with the students in the class in Chap. 2 could not include the mass of individual and collective observations that she collected and decoded instantaneously to understand the state of the Situation and evaluate its consequences in order to decide whether to intervene immediately, or delay intervening, or not intervene at all.

[4]This method is comparable to the attempts to prove Fermat's conjecture before the twentieth century. Working with a fixed, particular value of *n* was clearly not going to advance the general solution at all, but there was always the hope that the examples would provide some useful reflections.

The precision, quality and dependability of the Situations made it possible to reduce evaluation to observation of how they actually developed and the participation of each student. This development was reproduced regularly each year with the same results, which contributed greatly to reducing the fears of both the teachers and the students to an acceptable level.

Obsolescence

Situations, whatever they are, are adventures for the students each new year, but for the teachers they tend to become rituals. The teachers' memories lead them to construct a simplified and stereotyped image of the development of the Situation.[5] They are then unable to respond in a differentiated and opportune way to the actual events that occur, or even to let them occur. The Situation becomes a classical lesson that then loses its suggestive properties and thus becomes far too heavy for a minimal profit.

In trying unconsciously to have the students reproduce a stereotyped development, the teacher tries to prevent the difficulties observed in the years before. She intervenes more and more directly in the behavior of the students and the Situation becomes purely didactical. She tends to transform high level objects (for example those corresponding to high levels in Bloom's taxonomy) into algorithms. For the student, the situation loses its suggestive qualities and becomes the execution of a sequence of instructions, a simple task.

The relatively unpredictable character of a-didactical Situations helps the teacher fight this tendency. Nonetheless, Nadine Brousseau notes that she had to make an effort to maintain her capacity to deal with diverse but equivalent manifestations of the same knowledge. For example, she feels that it was in the end a good thing that she had to accept and monitor reasoning about fractions and also commensurations, which were fairly unfamiliar to her. It is important to distinguish between what is justified for the students and what is obvious to the teacher.

The complexity of the evaluation, interpretation and management of the a-didactical phases of the acquisition of knowledge may explain the evolution of the practices towards exclusively didactical methods as the pressure of evaluations mounts. A heavy tendency has been observed since the 1970s to replace the phases of acquisition of *connaissances* – which by definition should always precede the teaching of *savoirs* and the evaluation of the ensemble (which had developed greatly in the previous century under the influence of great pedagogues) – by direct instruction of answers to questions on standardized tests. Nicely aligned with naive popular beliefs, this practice greatly simplifies didactical decisions (start over, increase the pressure, eliminate comprehension in favor of reproduction, discriminate among the students, individualize, formalize, etc.), the knowledge needed to make those decisions and their justification with the population. But no observable improvement in results of teaching has resulted. As standard evaluation becomes more and

[5]This phenomenon is being studied under the name of "Obsolescence of Situations."

more present, with higher and higher stakes, it tends to produce the progressive disappearance of all the activities that traditionally preceded or accompanied the construction of knowledge.

Even though they cannot be evaluated in a standard way, *connaissances* are nonetheless indispensible. They should always accompany the teaching of *savoirs* and their evaluation. Training for standardized tests by giving standardized tests (teaching by worksheets) destroys the coherence both of the mathematics and of the class.

Isolated Evaluation of *Savoirs* and Constant Evaluation; the Necessity of the Uncertain and the Implicit

The evaluation of *savoirs*, the kind whose outcome is known to both the teacher and the students, is only justified in the case where one or the other of the protagonists has available not just the means of judging the results but also the possibility of using this outcome as a basis for worthwhile decisions. Otherwise it is simply a question of unjustified and unhealthy pressure, by definition unproductive.

The evaluation that plays an essential role in the playing out of the lessons is the one that the teacher and students engage in separately. It takes the ambiguous form of assessments, encouragements, questions, funny faces, etc., using a complex and delicate system of communication.

The Play of the Real and the Fictional

The curriculum gives the teachers the canvas and the means to make present a story that constitutes a sort of epistemology of each notion. But if they apply the program of work without discernment they will devote a considerable amount of time to episodes that are of no interest to the students and/or not productive of much learning. One means of regulation at their disposition is the passage from real mode to fictional mode and vice versa. Students can understand a lot of information without its having to be imbedded in an actual action on their part and a fortiori in a situation that can be complex and difficult to put in action. When a Situation that demands a certain intellectual and material investment has its effect, the students suddenly understand the notion sought for. They imagine the possibilities of what follows, anticipate the didactical intentions of the teacher, and often negotiate with the teacher to abandon the action phase. Their relationship with the Situation becomes imaginary. The imaginary mode makes it possible to save a lot of time – but if the students have made a mistake there is none of the feedback that a real Situation would have given them. Explanations may then be long, tangled, delicate and hazardous. Actually checking things out physically frequently meets with resistance from the students; re-checking an idea in a real Situation meets with even stronger resistance. The students ask the teacher to hand them the answer. That is why real, costly Situations must be rare, fascinating and productive of emotions, questions, etc.

For that, the teacher absolutely must retain control over the passage from one to the other: the real mode is slower but surer and the imaginary mode is faster and more productive. They are alternately indispensible. The teacher must maintain an optimal balance between a reasonable speed and a reasonable comprehension by the set of students.

In this dilemma the teacher needs the consent of all of the students. Those who have found a solution or think they have should wait until the others have had a reasonable time to carry out their own actions; those who are having trouble should feel pushed to do it right by the fact that the others are waiting. The acquisition of knowledge is a collective effort, like a culture. Individualism makes the work of the teacher and the students very difficult. Emulation stimulates but does not help, while cooperation helps, especially when the work of some depends on the work of others.

The Inexpressible, the Said and the Unsaid

The conduct of Situations is much more sensitive than classical lessons to the maintenance of a suitable equilibrium between what is or should be said and what is not, or should not be. Knowledge and invention or learning are the means of reducing the uncertainty presented by the Situations. The teacher must at all times monitor the part of the help that he offers the student to help her advance and the part that he leaves for her. In this paradoxical relationship where the teacher must say everything about what he wants to teach except for the most important thing: what he wants to teach the student to do and think herself, the unsaid and complicity play a considerable role. Any attempt to clarify everything and require immediate describable and measurable results like those in a notary's contract immediately condemns the teaching and learning project to irremediable failure. Furthermore it is abusive. A-didactical Situations let the teacher stand beside the student and follow her efforts, leaving to the Situation itself the task of criticizing them. And students accord far more importance to things they have flushed out themselves than to things procured for them without any effort on their part.

Further Aspects of the Teachers' Adventures

This chapter presents only the didactical part of the adventure of the teachers engaged in this COREM project. Other aspects would also have been of interest, among them the work of the teacher: the preparation of ordinary lessons, that of "experimental" lessons, lesson development as a team, difficult lessons, evaluations, commentaries at the moment, seminars, mathematical training, relations with other disciplines, their point of view about the students, and also especially their relations with the students and their parents, and the authorities, also their relations with the other teachers.

Their roles as teachers and educators were complicated terribly by the requirements of the research: having always in “their” classroom two colleagues with whom they had to share the attention, the obedience and the affection of the children; the obligation to coordinate with and thus to explain to the colleagues the intentions and results of the lessons; the intrusive attention of the researchers, the hide-and-seek game with what needed to be understood of their intentions, their implicit or explicit requests and what they could not say because knowing that a result is anticipated makes it almost impossible to avoid intentionally or unconsciously working towards it. The subtlety of these relationships created and was sustained by intellectual complicity.

The teachers had the responsibility of defending the interest of their students. They had the last word whenever that was an issue. Researchers and teachers strove to conceive at all costs situations that would teach what was necessary for the students, but the designs were such that the scientific conclusions were never at any time based on failures of the students.

Each experimental lesson was an adventure for the whole community, students, teachers and researchers, and there was no need to add to the hazards supplementary recompense or a fortiori sanctions. The quantity of observations on the complex set of scientific questions in course was abundant for everyone. The community appropriated them and kept them available for the next adventures. So-called “ordinary” lessons might equally well harvest the most obvious and certain conclusions of the current research or be directly and narrowly inspired by the most down-at-the-heels pedagogical or didactical models. What was called a “basic lesson” was composed in the most classic manner of: mathematical terms, presentation, text, examples, exercises, explanations, problem, applications. But starting early on the teams of teachers often had to modify them to benefit from the opportunities created by the experimental lessons. They had that liberty, and felt free to exercise it on the basis of lines that they found solid and practical. And the researchers in their turn exercised a certain vigilance. But not one ideological or systematic slippage, not one hasty generalization based on one or two “successes”, not one general rule had the right to be indulged. Filmed observations are a cruel threat to that kind of slippage.

Also of interest is how the teachers accustomed themselves to a new vocabulary for concepts that they already knew, and for new concepts sometimes behind the same vocabulary, and also how it was necessary to fight against “didactical permeability”, that is, the uncontrolled and regrettable penetration of scientific didactical (or psychological, or other) vocabulary into the exchanges of the teachers with their students.

And also, how did one get into this establishment? Teachers, researchers — who was chosen, or rejected, and why? How were things worked out? How could the administration accept such a singular teaching environment? Did it cause any difficulties? These adventures are part of the same story, but they would need another book!

In any case, to this day no one has done justice to these teachers for the treasures that they inventoried and put at our disposal. The adventures that this book describe were hose of all the members of a unique institution. In the course of 25 years it involved more than 250 adults and nearly 2000 students.

The Mathematical Organization of the Curriculum

We have now looked at the curriculum from the perspective of the students and the teachers. One important vantage point that remains is that of the mathematical foundations on which the whole sequence is constructed. The mathematical plan underlying the curriculum is that of a rigorously mathematical construction of the positive rational numbers in the modern sense: it is axiomatic and based on formal mathematical structures. But this plan is also subject to a complementary set of conditions of epistemological and didactical origin. Thus in *Part One* (the first three modules) the positive rational numbers are introduced as a set of numbers designed to measure lengths, masses and volumes using an arbitrary unit, and to provide, by a calculation, the results of the physical operations of addition, subtraction, and multiplication, and of division by a whole number.

In the *second part* (Modules 4–7) the children become conscious of the difficulties of effectively putting fractions in order, of estimating their differences, of locating them and of keeping up the habits developed in dealing with the natural numbers for all the customary operations of measurement. And it is they themselves who choose what mathematicians call the *decimal number filter* to “represent” the rational numbers, or more accurately, to approximate them with a manageable precision. The game they play gives this search the meaning of “finding a decimal number as close as is necessary to represent a given fraction.” They end up having available the remaining necessary operation – successive divisions of natural numbers – as a unique operation that looks like division but will only be recognized as such after some other adventures. The repetition of the operations and of the reasoning about calculations gives the students, even the less swift ones, a chance to carry out a number of useful and instructive operations.

Classical curricula treat the decimal environment as an obvious extension of the practices of natural measurement and are content here to teach algorithms without mathematical content, as simple conventions. In these curricula the mathematics appears after the fact, simply as a commentary, or else as a refinement that breaks with the previously inculcated practices. In those conditions it can only strike most students as casual remarks of no particular interest.

The *third part* (Modules 8–11) introduces rational numbers as functions and as scalar ratios. The lesson on the Puzzle makes this introduction the object of a new adventure on the conditions for the conservation of ratios. They thus define linearity by a non-mysterious criterion: the image of a sum needs to be the sum of the images, in contrast to the traditional reference to proportionality, which is more mysterious and always gives some students trouble. The study of geometrical forms swiftly provides the occasion for extending the practices they have been using with rational and decimal numbers as measurements to a set of functions. Working on putting the enlargements in order makes them revisit rational and decimal numbers and think about changes of unit and reciprocal mappings. The structure thus constructed is that of fractions as scalar operators.

Next, in the *fourth part* (modules 12 and 13) the search for new uses for linear mappings leads the students to rediscover the everyday uses of fractions (percentages, scales, taxes, etc.) as well as the translation of operations and the raft of specialized vocabularies associated with them (for example "taking a fraction of something" as a way of saying to multiply by it.) This swiftly leads to the study of external linear mappings, that is, mappings between quantities that are of different natures, and hence accompanied by a dimensional equation. Some of them the students already know well (price/quantity), others are new (distance/fuel consumption, speed, density, debt, etc.). We include the classic use of these visits in their role of review, of illustration and enrichment of concepts, of learning exercises, and of initiation into the ordinary use of elementary mathematics.

The competition of problems posed by the students gives them a chance to pose problems and discuss what is interesting about them (and not just to answer them). We try to develop their interest in problems and the culture of problem-setting. This is an occasion for revisiting all the interpretations of division.

The *fifth part* leads the students to consider, use and calculate compositions of linear mappings, their decomposition into natural mappings, and their inverse mappings (they had already encountered reciprocals). They can thus express all of the interpretations of rational numbers – as measurements, ratios and linear mappings – with the same symbols, those of fractions.

A *sixth part* was prepared, but it was never possible to experiment with it, because it would have had to take place in the first or second year of middle school. It consisted first of symmetrizing the additive group by creating negative rational numbers and completing the construction of the field of rational numbers. The introduction of algebraic symbols then made it possible to formalize the definition of certain useful meta-mathematical terms and of the proofs produced spontaneously in primary school.

Mathematical Commentary on Chap. 2

Even reduced to its mathematical structure the curriculum presents numerous points that may appear strange and even difficult to accept, both for teachers and for mathematicians. The former may be suspicious of the mathematical quality of notions brought up so differently from the normal presentation, and the latter may contest them in the name of the didactical culture they remember from their childhood. For example, it is well known that the use in primary school of algebraic notation like $3+4=7$ does not give students the meaning of an equality. It has been demonstrated that this practice develops in the children an erroneous comprehension and use of the $=$ sign that perturbs the mathematical practices of students all the way up to the university (where they are seen to understand and prove the same equation differently depending on the order in which the two members of it are written.) But an attempt to replace this notation with $3+4 \rightarrow 7$ would give rise to great protests from different people for different reasons.

Elementary mathematics is frozen into practices that we are in the habit of regarding as untouchable.

Here is a description in scholarly mathematical language of the activity proposed in Module 15 of the curriculum to these 10-year-old students: they have equipped the set of rational linear mappings with their multiplicative, commutative group structure, distributive over the additive semi-group of natural numbers. It is a jolt to see it written this way, because it gives the impression that the students are going to have to learn this vocabulary. This is not the case. But it makes it possible for the teacher to verify whether among the exercises proposed all of these properties have indeed been used and justified – which does not in the least necessitate any meta-language beyond ordinary language or any other explicit proof than the comprehension of what one has done. This gives a legitimacy to ulterior definitions (the way language justifies the study of grammar).

Whether or not to teach meta-mathematical terms to students is a much debated question.

We have picked out a few of the singularities that solid mathematical and/or didactical reasons have led us to prefer to more classical practices, and we will give an elementary mathematical justification for the teachers that the mathematicians can easily verify.

The Temporary Replacement of Fractions by Commensurations

A stack of T identical sheets of paper has a (whole number) thickness E. (T,E) is an ordered pair. The students first posit that two ordered pairs (T,E) and (T′,E′), each consisting of identical sheets of paper, both consist of sheets of the same thickness of paper if there exists a number a such that $T'=aT$ and $E'=aE$. This condition is sufficient but not necessary. A necessary and sufficient condition is that there exist a and b such that $(aT, aE)=(bT', bE')$. $(aT, aE)=(bT', bE')$ is equivalent to $aTbE'=aEbT'$, and thus to $TE'=T'E$. This will be used explicitly and even proved (without algebra) by the students when they get to commensuration of lengths (module 3).

The ordered pair (7,4) indicates that a stack of seven sheets has a total thickness of 4 mm. The thickness of a sheet is expressed by the commensuration 4/7 or by the fraction 4/7. The words commensuration and fraction are synonyms and share the same symbolic notation. The conceptions are not.

Commensurations use only operations that can be conceived, realized and carried out materially or by calculations in the known domain of the natural numbers. Fractions make it possible to go back to the familiar model of the natural numbers by using an intermediate unit, but one must assume the prior existence of unit fractions, that is, fractions of the form 1/n, which cannot always be easily constructed. Historically, fractions are the concept that has been retained in all cultures.

The concept of commensuration could not arise from the manipulation of concrete lengths long enough to permit the use of a sub-unit obtained by folding and repetitions. Hence our use of a unit that is essentially indivisible for children: the millimeter, and of a length for them to measure that is smaller than the unit. This artifice having rendered the use of fractions improbable, the students were able to invent an original solution to a completely concrete problem and thus undertake the exploration of a necessary mathematical knowledge before being able to name it, justify it and recognize it as a familiar concept.

2. Introducing the topology of decimal and rational numbers as we do in the second part of the curriculum looks as if it were an ambitious and useless enterprise.

Tradition offers students an amalgam of various vague structures mingled under the name of "number". We will point out four.

- Natural numbers. They are more or less correctly understood, but the disappearance of analogical instruments of measurement has caused the disappearance of a powerful means of teaching the topology of the natural numbers which served as a basis for that of the decimal and rational numbers.
- The algebraic structure of the positive rational numbers is taught, even though they have been profoundly scarred as a result of hesitations and accidents in their history. Students distinguish them easily from natural numbers because they are written in the form of "fractions", and calculations with fractions are studied. On the other hand, comparisons, the concept of intervals, and complete ordering are ignored. Thus their dense topology, the first simple reason for their existence because they provide a means of attributing a distinct value for every distinct measurement, is neither practiced nor even envisioned.
- To top off this gap, students use the notation of positive decimal numbers, but the way they implicitly conceive of the set of these numbers leads them to make errors in their subsequent mathematical studies. They reason as if there existed a unique natural number n such that if all decimal numbers were multiplied by 10^n they would all become whole numbers (Such a structure is called D_n.) Topologically, thus, they are still just the whole numbers.
- The real decimal numbers are all the rational numbers of the form $m/10^n$, and thus the set of all of them is the union of all the D_n. They can approximate as closely as desired not only any rational number, but also any algebraic or transcendental real number. They lend themselves to algebraic calculations like the rational numbers and to comparisons and ordering like the natural numbers.
- Students and sometimes even teachers use the term decimal number to designate any *decimal expression* of a rational number, be it a decimal rational number like 0.3 (3/10), or a non-decimal rational number like 0.333… (1/3) or even the decimal approximation to an irrational real number like 3.14 for π.

- These mathematical delusions do not prevent the students from being disconcerted when suddenly the division of two well-behaved natural numbers sets out to produce a sort of monstrous number, made up of a visibly infinite sequence of digits.

Part 2 of our curriculum therefore shows the students the coherent project of replacing the antique fractions, inappropriate for analysis and calculation, with decimal numbers that can easily generate the real numbers. The properties of one and another can thus be established by an authentic and instructive mathematical adventure that they will perhaps recognize later on in more sophisticated guise if they study mathematics.

The students compare and calculate a large number of decimal numbers and intervals, imbedded or not, and rapidly develop an expert *connaissance* of the real line, a *connaissance* that the exclusively numerical apparatuses in their environment no longer show naturally.

In Part 3, the three fundamental objects represented in the course of the story by fractions or by decimal numbers remain distinct: *measurement of sizes, ratios,* and *functions.* The operations on these mathematical objects are conceived differently, and the names for each, depending on their particular uses, proliferate. The problem of change of the unit in the reproductions of the Optimist poses an apparently difficult problem for the students. Clearly, as before, the procedure that they will be using and that will work is not taught to them. The students know very well that they will not have to reproduce them alone and in other conditions. What they are doing has a mathematical identity that can be expressed in more advanced terms in later mathematical programs. The issue is not anticipating these advanced *savoirs*, but justifying the use of the necessary instruments that one wants them to learn by a mathematical problem that gives these notions their meaning and their mathematical use.

In Part 4, using the composition of functions in order finally to define multiplication and division of two fractions appears a new challenge to caution and reason. This new case of division gives rise to a cohort, nonetheless already numerous, of different interpretations. In all of the curriculum, divisions are the quick-change artists of this adventure: they keep reappearing in new guises, for apparently similar (or inverse) uses. But the final identification of all these appearances gives the students a very satisfying sense of accomplishment. A cycle of study is achieved, producing a sentiment of simultaneous completion and unity.