Chapter 1
Why These Adventures?

This book is intended to be read by teachers, researchers in education, mathematicians, and anyone else who is curious about what educational research has to say about the teaching of mathematics. It centers around a set of lessons on rational and decimal numbers. The lessons came into existence to validate the Theory of Situations, a basic tenet of which is that children can best learn a mathematical concept by being put into a very carefully designed situation where achieving some goal requires them to invent or discover the concept, and their prior knowledge enables them to do so.

The core of the book is the day-by-day journal of a fifth grade class in which the teacher reports every stage of what she presented in 65 lessons on rational and decimal numbers, and what happened with it. The journal was originally produced to enable two parallel classes to reproduce the lessons. The lesson sequence was conceived in 1972–1973 and was considered stable by 1975–1976. Enriched by various observations made by succeeding teachers, the sequence was officially reproduced every year in two parallel classes until 1999.

This lesson sequence is one of a number realized in the COREM (*Centre d'Observations et de Recherches sur l'Enseignement des Mathématiques*), a school set up specifically for observation supporting mathematics education research. A description of the school and its functioning can be found in Chap. 3, while Chap. 4 provides the origins of its conception as a research necessity. The lessons carried out there played a central role in the development of *Didactique* – a program of scientific research in mathematics education whose structure is unique to France, but whose contributions are valid and valuable everywhere that mathematics is taught.

Untangling the web of ideas, experiments, discoveries, hypotheses and proofs involved in a new teaching project is a long, perilous and debatable task. What we have to tell is thus the tale of three adventures.

One is the adventure of researchers opening up a new territory. It is certainly interesting, but it is complex and breaks with too many concepts and venerable habits of thought to be easily accepted without the support of the observations that

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provide its experimental foundations. Some of the many research results produced in a variety of fields:

- Elementary school students are able to construct, understand and practice fundamental mathematical concepts, using the modern mathematical and epistemological organization of those concepts.
- Factors that cannot be formally and directly evaluated, such as things left unsaid and knowledge that cannot be expressed or has not been decided, play an essential role in the elaboration, manifestation, learning and teaching of recognizable knowledge. Methods that eliminate the action of such factors are less effective: when they are used, only the students who are capable of filling in on their own what was left unsaid in the texts that are taught can make progress.
- Radical constructivism does not work as a general model: Institutionalization is indispensable.

The adventure of the researchers constitutes Chap. 4.

The second adventure is that of the teachers, recounted in Chap. 3. It is also captivating. They threw themselves into a scientific episode that was fascinating for them, but strange to them and very much of a disruption of their standard work as teachers. To appreciate that adventure, the reader needs to bear in mind that the teachers whose actions are being directed and recorded in the tale that follows were stepping from a familiar and comfortable terrain into a completely new teaching world in which many of the familiar landmarks had been removed or disguised.

But above all we are eager to introduce the reader to the adventures of the students as they took part in the classes, recounted in Chap. 2. What were the conditions in which they produced and learned some difficult mathematics? In what ways were their mathematical activities closest to the activities of mathematicians? To know that we had to make the conditions of their work explicit, with precision, as they were planned and as they were realized by the teachers. Likewise we had to make their reactions clear – the significant ones that made it possible to pursue the process. The "didactical files" that we have translated provide the best mechanism for following the adventure step by step from the point of view of the students. They were established in order for teachers – the original ones or their successors – to be able to reproduce the lessons. They were reproduced at least 50 times with completely similar results.

The texts that we present or describe in Chap. 2 were thus carefully designed to enable the lessons exactly as described to be reproduced in their original context. On the other hand, they were not designed for the lessons to be exported. They were carried out in an institution that was specifically created to permit this kind of experiment to be carried out in conditions that were secure for the teachers and for the students. This is especially true in that the options we chose were not those that we would recommend for development. They absolutely do not prefigure a curriculum to be developed in ordinary classes. Their sole objective was to provide scientific answers to some essential questions.

If this does not provide a model that can transfer directly into the day-to-day life of a teacher, what does it offer? It offers encouragement and hope, by directly demonstrating forms of teaching and learning that mathematics educators and philosophers have been trying for two centuries to promote. We hope it will encourage all the people who aspire to improve teaching itself, which is currently suffering seriously from the increasing divergence between society's requirements for education and its obstinate refusal to call into question obsolete ideologies and inappropriate scientific practices. Yes, students can learn mathematics, and learn it well, by taking part in mathematical activity. Not only that, but they can thoroughly enjoy doing so. What else is at the heart of all of our endeavors?

A Few Words by the Anglophone Author

The content of this book is completely international. The activities of the children, the decisions of the teachers and the explorations of the researchers are part of a fabric of mathematics education that increasingly is spreading worldwide. However, a certain amount of the background for the teaching project that is central to the book is unfamiliar to most readers outside of France, and knowing the background of the book itself may help enrich the reading of it, so as a lead-in to a book that is very much a joint effort we will present a few paragraphs that are specifically a Warfield production.

First an Introduction to All Three Authors

Guy Brousseau has had a long and notable career in mathematics education research, for which the most telling evidence is probably his having been awarded the first Felix Klein Award from the International Commission on Mathematics Instruction, in recognition of "the essential contribution Guy Brousseau has given to the development of mathematics education as a scientific field of research, through his theoretical and experimental work over four decades, and [of] the sustained effort he has made throughout his professional life to apply the fruits of his research to the mathematics education of both students and teachers."[1] His background, determination and reflections, combined with some favorable circumstances, led him to conceive of, create and sustain both a wide-ranging program of coherent, flexible and scientifically based research and the necessary institutions, including a school, to carry out and develop that research. The program has been successful thanks to the help of numerous collaborators whom Brousseau managed to interest in his projects, and to the encouragement and support that he was given. In particular, it was at the school he helped create that the curriculum here described was taught for many years, starting in the early seventies.

Nadine Brousseau's career was in elementary school teaching, and she was among the initial teachers in the research school. This was ideal for two reasons: she was able to confer with her husband long and deeply about the intentions and plans for the lessons, and the results and implications of what happened when she taught

[1] http://www.mathunion.org/icmi/other-activities/awards/past-recipients/the-felix-klein-medal-for-2003/

them. Her contribution was irreplaceable and decisive. In addition, she kept extremely good records, both of the proposed lessons (including the elements added when the two Brousseau's continued their discussions long after their fellow researchers and teachers had gone home) and of the class response to them. Her notes became the functional memory of the project, and her present memories enhance and enrich the recorded ones.

This author (Virginia Warfield) came onto the scene considerably later. In the course of a career that combined mathematics and interesting ways to teach it at both elementary and university levels, I had become increasingly interested in mathematics education as a field. A fortunate sequence of events led me to the work of Guy Brousseau and to the discovery that it was very little known in the English speaking world. My first work was with Nicolas Balacheff who, with translating and co-editing by Martin Cooper, Rosamund Sutherland and myself, published Brousseau's *Theory of Didactical Situations in Mathematics* (Brousseau, 1997).

My work on that book resulted in a partnership with Brousseau himself from which so far a number of articles and talks have emerged, as well as a small introductory book. Four of the articles were a series in the Journal of Mathematical Behavior (Brousseau, Brousseau, & Warfield, 2004, 2007, 2008, 2009), covering separate parts of the Rational and Decimal Number curriculum under discussion here. Eventually we decided that the articles needed to be assembled and expanded into a book.

As should be clear, this thoroughly asymmetrical set of positions leads to some variation in the meaning of the word "we". Since, on the other hand, the variation produces no ambiguities, we (in this case all three authors) have decided to leave it.

Next the Background of the Teaching Project Itself: How and Why It Came to Exist

Part of that background begins in the 1960s, when a substantial international group of mathematics education researchers agreed to the need for more serious, coordinated, collaborative research. In France, part of the response to this need was the establishment of a number of IREM's – Research Institutes for Mathematics Teaching. Guy Brousseau was an enthusiastic supporter of this development, and was instrumental in bringing a very early IREM to the University of Bordeaux, where he was on the faculty. He felt, though, that although an IREM was necessary, it was not sufficient for the level of scientific focus he envisioned. To achieve that level, he spent a lot of time and a huge amount of energy which jointly paid off in the creation of the COREM (Center for Observation and Research on Mathematics Teaching). This center took the form of a school, the École Michelet, which was a regular public school in a blue collar district on the edge of Bordeaux equipped with a carefully constructed set of research arrangements. On the physical side, the arrangements consisted of an observation classroom in which classes would occasionally be held – often enough so that the students found them routine. The classroom was equipped with a multitude of video cameras and enough space for observers to sit unobtrusively. Other arrangements were far more complex, involving an extra teacher at each level and an agreement among the teachers, administrators and researchers setting out the responsibilities and rights of each. Nothing involving that many humans could possibly glide smoothly through the years, but the fundamental idea proved robust, and the École Michelet functioned as a rich resource for researchers for two and a half decades.

Another part of the background has roots that can be traced back through the generations, but came to the foreground in the 1960s under the title of constructivism. The title stems from the underlying tenet that knowledge is constructed in the human mind rather than absorbed by it. Applications of that tenet range from the radical constructivist belief that absolutely no information should be conveyed to students directly, to the naïve conviction that having children manipulate some physical objects that an adult can see to represent a mathematical concept will result in the children understanding the concept itself. Guy Brousseau had studied many of them, but while he found many interesting points, he felt that so far there was a serious lack of solid research in support of the theory itself. With his fellow researchers he therefore set himself the goal of taking some serious piece of mathematics and proving that in certain conditions the children – all the children, together – could create, understand, learn, use and love that mathematics. Accompanying that goal was the goal of studying the conditions themselves.

Clearly the mathematics to be used for this experiment had to be both significant and challenging. After some consideration he made a choice that will resonate with elementary teachers worldwide: fractions, or more properly, rational and decimal numbers. He had, in fact, some reservations about whether rational numbers should be taught at all, but they were firmly part of the national. They had a further virtue: the experimental curricula he had in mind for the very youngest classes introduced them to numbers in such a way as to permit the construction of all the epistemological and mathematical bases of the fundamental numerical structures. Part of the objective was to prepare them for much later studies – reflective, mathematical and formal studies starting at the first year of the secondary level aimed directly at mastering basic symbolic, algebraic and analytic instruments. The study of rational and decimal numbers provided a point of articulation between these two projects.

Having made this choice, he then spent a lot of energy and time doing research into the different mathematical aspects of both the rational numbers and the decimal numbers, as well as possible ways of generating them. He also looked into the history of how each has been taught in different cultures and historical contexts. One of his conclusions was that a major source of learning difficulty is that although rational numbers are used in several very distinct ways – among others as measurement (3/5 cm), as a proportion (this thing is 3/5 as long as that thing), and as an operation (take 3/5 of this quantity) – they are generally taught as if all the meanings were equivalent. The result is that the student must accept many things simply on the basis that the teacher says so, and in the long run has no coherent foundation for the concepts. This conclusion led to the mathematical structure of the curriculum presented here. By way of a roadmap, we will sketch the resulting order here. A more mathematical description will be found in Chap. 3. For a considerably more detailed description of both the background and the decision procedure, see Chap. 4 of the Theory of Didactical Situations in Mathematics (Brousseau, 1997).

The first lessons are taken up entirely with commensuration[2] and its consequences. The children first work with different thicknesses of paper and realize that even though they cannot measure a single sheet, they can distinguish the papers by specifying how many sheets it takes to make up 2 cm, or alternatively how thick 50 sheets are. Deep familiarity with that idea paves the way for developing an understanding of equivalence and the basic operations. That understanding is solidified with some work generalizing the results to measuring weights of nails, volumes of glasses and lengths of carefully selected strips of paper.

The following set of lessons works with decimal numbers. In a series of challenges to find smaller and smaller intervals around some rational number, the class discovers the virtues and some of the working principles of using numbers whose denominator is a power of ten. Once they are secure with that, they begin to use decimal notation for these convenient objects.

With their grasp of rational and decimal numbers as measurements now reasonably solidified, the students then progress to a more active aspect, using them first to enlarge a tangram-like puzzle, then to enlarge and reduce a variety of items. The rest of the curriculum is devoted to deepening mathematical connections, broadening applications and enlivening problem-solving using these concepts.

The remaining element of background concerns the format for the learning adventure itself. Brousseau, in the course of teaching elementary school for several years, reading voraciously and maintaining on-going lively discussions with an array of people that included teachers, university professors, psychologists, linguists, teacher educators, administrators and even a priest had developed his own take on constructivism, which took the form that he eventually called the *Theory of Situations*. His idea was that for children to learn a concept they should be put into a Situation (a very carefully orchestrated classroom situation or sequence of situations) in which in order to resolve some problem or win some game they would need to invent the concept in question. He was strongly committed to this theory, but had an equally strong commitment to the principle that before people were asked to accept it they should be presented with solid research validating it. This pair of commitments helped fuel his drive to create the COREM. Once it was created, his first goal was to design research to test the theory. At the heart of that research was the curriculum that provided the adventure of Chap. 2.

One final note: this curriculum is sufficiently enticing, both mathematically and pedagogically, to give the impression that it should and could be simply picked up and transplanted into other classrooms. This was not Brousseau's intention in producing it, and he warns repeatedly and vigorously against that illusion. It does indeed illustrate a wonderful kind of teaching and learning, and it provides thought-provoking insights and ideas with direct or indirect application to the classroom. On the other hand, the many iterations of successful use of the curriculum itself were all

[2]Commensuration is the measurement of things in comparison to each other rather than in terms of a unit.

carried out with the extraordinary support provided by the COREM, and Brousseau feels strongly that an attempt to use it without that support would be likely to have disastrous consequences.

Introductory Remarks by Guy Brousseau

I am very grateful to Virginia Warfield, who has worked hard – and made me work hard – for 20 years to make accessible to the American public the texts of one of our most sophisticated instruments of research. It has the most innocent of appearances as a curriculum – the chronicle of an adventure, programmed down to its details, that the students and their teachers lived and above all that others succeeded in reliving identically. An adventure for the students in the sense that the curriculum gives them the sense of having a lot of space for initiatives, experiments and personal reasoning with goals that seem to them objective and that they are able to believe the teacher does not know … but an adventure also for the teachers who always wonder whether the Situations, even though minutely calibrated and reproduced year after year, will really once again permit them to achieve the desired results: the learning in common of a common mathematical culture shared by all of the students in the class. The cost of the apparent freedom of the students is a no less apparent drastic reduction in the freedom of the teachers.

This curriculum was not made to be used in other classes. The sole purpose of the reproducibility was to consolidate the scientific observations that we needed in order to test certain hypotheses. The lessons had above all the property of making apparent the enormous complexity of the acts of teaching: that of the conception, to be sure, but even more that of the carrying out of the lessons. The fact that teaching is a complex activity and passably mysterious is accepted in theory by our societies – but they don't really know what that means! They absolutely do not take the complexity into account when it comes to studying the work of teaching. They intervene authoritatively in the educational system on the basis of grossly erroneous conceptions. They are not even capable of identifying the specific field of science: the need is to understand a phenomenon and they look only at the actors. The consistency and validity of the concepts in question need to be verified, and instead they look only at their use and market value!

The COREM that we called our "Didactron" was a center for anthropological observation: with their consent, we observed as anthropologists the life of a tribe of teachers. Believe me, this is not an easy approach, even for those taking part in it. One among the collaborators and teachers of the COREM was my wife Nadine Brousseau née Labesque, who played an important role in all the steps of the project. She helped me as a collaborator to study didactical versions of the Situations, and as a teacher to present them with her colleagues to the pupils in the school Michelet de Talence for 14 years before her retirement. She also helped with the work of redaction of the script prepared in common and the transcription of remarks and observations. She wrote the first stage of our manual "Rationnels et décimaux dans la scolarité obligatoire" (Rationals and Decimals in Basic School) published by the IREM of Bordeaux. This text, which was produced in 1985, was reserved for researchers in *Didactique*. More than 2,000 copies were sold.

Another invaluable collaborator was Denise Nedelec (known as Denise Greslard), who experimented with the curriculum protocol from 1987 to 1999 with great care and dependable success, and made many fine observations.

We thought the teachers would want to eliminate these lessons after 2 or 3 years, as soon as we had sufficiently observed the phenomenon of the obstacle. Among many challenges was the fact that the least interaction with the students obligated the teachers to interpret their declarations, put out in the system of commensurations, by translating them into the teacher's own knowledge system, that of fractions, and then make reciprocal translations to continue the lesson. Knowing that even though the results are the same the proofs are often different in the two systems it is easy to see that the mathematical exercises produced a lot of stress for them and made the role of the culture in mathematical activity palpable, often cruelly so. Our observations in this context largely confirmed what we had seen of the difficulties of students as they pass from one system to the other.

We were therefore extremely surprised at the end of the experiment when the teachers expressed their desire to keep these lessons in the curriculum despite these difficulties. This reaction led us to understand that in certain cases jumps in complexity can be highly effective. The classical approach is to deconstruct material to be learned so as to keep the amount of information delivered by each lesson more or less constant and optimal. Our experiments demonstrated that in certain particular circumstances this rule can be violated to very good effect. Most of the rules of teaching as practiced are only valid in the absence of deeper and more specific knowledge about the conditions of teaching.

I hope that this gives our reader an idea of what we are offering in this work. I ask them to extend us some credit and to search for good questions before searching for answers. Video recordings of some of these 65 lessons, realized in the course of the 25 years of the COREM, are collected at the ViSA site (Vidéos de Situations d'enseignement et d'Apprentissage http://visa.inrp.fr/visa) to which researchers have access. In addition, all the homework and exercises of all the students from 50 realizations of these lessons can be consulted at the University Jaime 1 de Castillon (Spain) which can make copies of them (made anonymous).

Our curriculum presents a wide variety of types of lessons. Each one has its role and its necessity. But there is absolutely no pedagogical, didactical or epistemological message hidden in them – only questions and occasions to reflect and make discoveries yourself. Try! These are not riddles. Sometimes I give my answers. Compare them to your experiences. The curious could, if they like, launch themselves into a study of the Theory of Situations. So if something astonishes you, ask yourself questions, whether it has to do with the conception, with the conduct of the lesson or with the result of the lessons. Ask us your questions, and we will think about them with you.