Chapter 81
Adaptive Disturbance Rejection Control of Linear Time Varying System

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Abstract A novel adaptive disturbance rejection control scheme for a linear time varying (LTV) system from the perspective of differential algebraic framework is proposed. A numerical differentiator is used to obtain the derivative estimates from the system output, which contain overall dynamics of the system. Combining a local modeling technique and conventional proportional integral differential controller, the proposed control scheme perfectly accommodates disturbances and measurement noises. The convincing simulations validate the proposed control scheme well.

Keywords Linear time varying system · Numerical differentiator · Adaptive control · Disturbance rejection

81.1 Introduction

The case of LTV systems is important since one or some parameters of the real physical systems are time varying. Further more, the control of nonlinear system is, usually, accomplished by linearizing this system around a given trajectory

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which renders an LTV system [1]. Researchers have made a great number of contributions on linear time varying (LTV) systems [2] from 1960s'. In the literature [3] and its related literatures, a number of adaptive control schemes for LTV systems has been proposed. Most of these adaptive control schemes stemmed from the mature control theories of linear time invariant (LTI) systems, and have been used in engineering successfully. In this chapter, we propose a new adaptive control law for the LTV system via a differential algebraic observer, which is constructed by a new differentiator. The closed-loop error dynamics are the nature of LTI, and all signals in the closed-loop system are uniformly ultimately bounded (UUB). The main advantage of the proposed method lies in the excellent performance in the presence of disturbances and measurement noise.

81.2 Preliminary

We briefly present the method of numerical differentiation, which is proposed by Fliess and Mboup. Further information can be found in [4–7]. Consider an analytical real-valued signal $x(t)$, which has a truncated Taylor expansion $x_N(t) = \sum_{k=1}^{N} c_k t^k/k!$ at $t = 0$ without loss of generality. The expansion satisfies $d^{N+1}x_N(t)/dt^{N+1} = 0$, which is transformed into s domain, we therefore obtain

$$s^{N+1}x_N(s) = s^N x_N(0) + s^{N-1}x^{(1)}(0) + \cdots + x^{(n)}(0) \tag{81.1}$$

Multiply both sides of Eq. 81.1 by operator $\Pi_k^{N,n} = \frac{d^{n+k}}{ds^{n+k}} \frac{1}{s} \frac{d^{N-n}}{ds^{N-n}}$ a direct estimation of $x^{(n)}(0)$can be acquired as

$$x_N^{(n)}(0) = s^{\nu+n+k+1} \frac{(-1)^{n+k}}{(n+k)!(N-n)!} \frac{1}{s^\nu} \prod_k^{N,n} (s^{N+1}y(s)) \tag{81.2}$$

where $\nu = N + 1 + \mu, \mu \geq 0$. Let $N = n$, and read Eq. 81.2 in time domain thereby [7]

$$\tilde{x}^{(n)}(0) = x_N^{(n)}(0) = \frac{\gamma(n,k,\mu)}{(-T)^n} \int_0^t \frac{d^n}{d\tau^n} \{\tau^{k+n}(1-\tau)^{\mu+n}\} y(\tau) d\tau \tag{81.3}$$

where $\gamma(n,k,\mu) = (\mu + k + 2n + 1)!/[(\mu + n)!(k + n)!]$. The boundedness of the derivative estimate above is demonstrated in the following lemma.

Lemma 1 *For* $0 < t < \varepsilon$, *by using* Eq. 81.3 *the estimate error* $\|e_{x^{(n)}}\| = \|\tilde{x}^{(n)}(0) - x^{(n)}(0)\| < \delta$ *with* δ *is a sufficiently small positive constant.*

Proof The estimate error consists of two parts due to truncated error $R_N(t) = O(t^{N+1})$ and measurement noise $n(t)$. For truncated error when $t \to 0$ or $N \to +\infty$, the term of $O(t^{N+1})$ becomes negligible. For measurement noise, we have a reasonable assumption that $n(t) \in L_2$ is bounded fluctuated function of t with higher frequency. For $0 < t < \varepsilon$, there exist positive constant δ_{R_N}, δ_N and δ_h such that $||R_N(t)\ || < \delta_{R_N}$ and $\|\ n(t)\ \| < \delta_N$, meanwhile, $\|h(t)\| = \left\| \frac{\gamma(n,k,\mu)}{(-T)^n} \frac{d^n}{dt^n} \{t^{k+n}(1-t)^{\mu+n}\} \right\| \le \delta_h$. Rewrite Eq. 81.3 as

$$\begin{aligned} x^{(n)}(0) &= \int_0^t h(\tau)[x_N(\tau) + R_N(\tau) + n(\tau)]d\tau \\ &= x_N^{(n)}(0) + \int_0^t h(\tau)R_N(\tau)d\tau + \int_0^t h(\tau)n(\tau)d\tau \end{aligned}$$

Then

$$\begin{aligned} \|e_{x^n}\| &= \left\|\tilde{x}^{(n)}(0) - x^{(n)}(0)\right\| \\ &= \left\| \int_0^t h(\tau)R_N(\tau)d\tau + \int_0^t h(\tau)n(\tau)d\tau \right\| \le \left\| \int_0^t h(\tau)R_N(\tau)d\tau \right\| \\ &\quad + \left\| \int_0^t h(\tau)n(\tau)d\tau \right\| \le \delta_h(\delta_{R_N} + \delta_n)t \\ &= \delta \end{aligned}$$

with $\delta > 0$.

Remark 1 Derivative estimation given by Eq. 81.3 are not of asymptotic nature [5]. One hand, from the proof above, there has $\delta \to 0$ when $t \to 0$, thus, as long as t is small enough, δ will be a sufficiently small positive constant. On the other hand, the differentiator functions as a low-pass filter, which will attenuate those fast fluctuated noises. However, the performance of noise rejection will degrade when $t \to 0$. Thus the choice of time window t is a compromise result.

Remark 2 The derivative estimate at time 0 is obtained from Eq. 81.3 is based on the observation of $y(t)$ on the time interval $I_{0+}^t = [0, t]$, and this is not causal. In order to obtain a causal estimate, we replace $y(\tau)$ by $-y(t - t\tau)$ in Eq. 81.3 henceforth a causal estimate $\tilde{x}^{(n)}(t)$ based on the observation on the time interval $I_{t-}^t = [0, t]$. We can simply move the estimate from t to any $T \ge 0$ by a heaviside function [7].

81.3 Main Results

Problem Statement. Consider the tracking control of a uniformly controllable and uniformly observable single input single output (SISO) LTV system, which is nth order system and characterized by

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ a_0(t) & a_1(t) & \dots & a_{n-1}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ d(t) \end{bmatrix} \\ y = x_1 + n \end{cases} \triangleq \begin{cases} \dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}u + \mathbf{D}(t) \\ y = \mathbf{C}\mathbf{x} + n \end{cases} \tag{81.4}$$

where $\mathbf{x} \in R^n, y \in R, u \in R; \mathbf{A}(t)$ is a time dependent matrix with corresponding dimensions, $B \in R^n, C \in R^n; d$ is the external disturbances, w_2 is the measurement noise. The definitions of uniform controllability and uniform observability for LTV system can be found in [2]. For the convenience of analysis we have are following assumptions

Assumption 1 $\mathbf{A}(t)$ and $\mathbf{D}(t)$ are continuous and uniformly bounded such that $\|\mathbf{A}(t)\| \le M_A$ with $M_A > 0$ and $\|\mathbf{D}(t)\| \le M_D$ with $M_D > 0$ for all $t > 0$.

Assumption 2 $\|\dot{\mathbf{A}}(t)\| \le \delta_A$ with $\delta_A > 0$ and $\|\mathbf{D}(t)\| \le \delta_D$ with $\delta_D > 0$ for all $t > 0$.

It is to note that $\|\bullet\|$ is defined as a spectral norm of a matrix in here and the following paper.

Local Modeling. We rewrite Eq. 81.4 as an equivalent form

$$\begin{cases} x^{(n)} = \sum_{i=0}^{n-1} a_i(t) x^{(i)} + bu + d = f(\mathbf{x}; t) + bu + d \\ y = x + n \end{cases} \tag{81.5}$$

where $\mathbf{x} = \begin{bmatrix} x & \dot{x} & \dots & x^{(n-1)} \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$. Let $F(t) = f(\mathbf{x}; t) + d$, then we have $F(t) = x^{(n)} - bu$. Thank to the sampling technique, we can model F at a time instant k as $F_k = x_k^{(n)} - bu_k$ to avoid the algebraic loop, where $(\bullet)_k$ stands for the value of $(\bullet)$ at time instant k. By using Eq. 81.3 the nth order derivative of x can be obtained from the observation of output y. Thus, the local model of F can be written as

$$\tilde{F}_k = \tilde{y}_k^{(n)} - bu_{k-1} \tag{81.6}$$

Since $\tilde{y}_k^n$ can be estimated well even in noisy environment, $\tilde{F}_k$ consists of the overall dynamics of the system at time instant k, including the external disturbance d.

Control Law. For the tracking problem of Eq. 81.4, let the desired trajectory of the output y_d be smooth enough and differentiable, then $\mathbf{x}_d = \begin{bmatrix} y_d & \dot{y}_d & \ldots & y_d^{(n-1)} \end{bmatrix}$. The output y and its finite order derivatives can be estimated by Eq. 81.3, then we have $\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{y} & \tilde{\dot{y}} & \ldots & \tilde{y}^{(n-1)} \end{bmatrix}$. On the basis of the local model, we propose our disturbance rejection controller as the following

$$u = \frac{1}{b}\left[-\tilde{F}_k + x_d^{(n)} - \mathbf{k}^T(\tilde{\mathbf{x}} - \mathbf{x}_d)\right] \tag{81.7}$$

where $\tilde{F}_k$ is given by Eq. 81.6, $\mathbf{k} \in R^n$ is the vector of designed parameters, and $x_d^{(n)} = y_d^{(n)}$. The stability of our control scheme will be discussed in the following subsection.

Stability. The following stability theorem regarding to the control law defined by Eq. 81.7 is stated.

Theorem 1 *Consider the system governed by* Eq. 81.4 *and consider assumptions* 1 and 2 *are satisfied. If the control law is provided by* Eq. 81.7 *and the differential algebraic observer is given by* Eq. 81.6, *then all signals in the closed-loop system are UUB.*

Proof Substitute Eq. 81.7 into Eq. 81.4 resulting in the closed-loop system $x^{(n)} - x_d^n = F(\mathbf{x};t) - \tilde{F}_k - \mathbf{k}^T(\tilde{\mathbf{x}} - \mathbf{x}_d)$. Let $\mathbf{e} = \mathbf{x} - \mathbf{x}_d$ be the tracking error, $\boldsymbol{\varepsilon}_x = \tilde{\mathbf{x}} - \mathbf{x}$ the estimate error of state, and $\varepsilon_F = \tilde{F}_k - F(t)$ the model error of our local model. Rewrite the closed-loop equation as

$$\dot{\mathbf{e}} = \boldsymbol{\Lambda}_c\mathbf{e} + \mathbf{b}_c\left[\varepsilon_F + \mathbf{k}^T\boldsymbol{\varepsilon}_x\right] \tag{81.8}$$

where $\boldsymbol{\Lambda}_c = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 1 \\ -k_n & -k_{n-1} & \ldots & -k_1 \end{bmatrix}, \quad \mathbf{b}_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$

Let $\varepsilon = \varepsilon_F + \mathbf{k}^T\boldsymbol{\varepsilon}_x$. We first prove ε is bounded. From Lemma 1, there exits a $\xi_{y^{(n)}} > 0$ which make $\left\|\tilde{y}^{(n)} - y^{(n)}\right\| < \xi_{y^{(n)}}$. Then the estimate error of x satisfies $\|\varepsilon_x\| = \left\|\begin{bmatrix} \xi_y & \xi_{\dot{y}} & \ldots & \xi_{y^{(n-1)}} \end{bmatrix}^T\right\| < M_{ex}$. Similarly, we have $\|\boldsymbol{\varepsilon}_{\dot{x}}\| = \left\|\tilde{\dot{x}} - \dot{x}\right\| < M_{e\dot{x}}$. Here M_{ex} and $M_{e\dot{x}}$ are small positive constants, respectively.

From the assumption 1 and 2, $F(t)$ is continuous and bounded, meanwhile its derivative respect to time is bounded, thus $\forall t, \forall k, \exists t_2$ such that $\|F(t) - F(k)\| = \left\|\dot{F}(t_2)(t-k)\right\| \le \delta_{\dot{F}}\|t-k\|$. Consequently,

$$\begin{aligned}\|e_F\| = \left\|\tilde{F}_k - F(t)\right\| &= \left\|\tilde{F}_k - (F_k + \dot{F}(t_2)(t-k))\right\| \le \left\|\tilde{F}_k - F_k\right\| + \left\|\dot{F}(t_2)(t-k)\right\| \\ &\le \left\|\tilde{\dot{x}}_k - bu_{k-1} - (\dot{x} - bu_k)\right\| + \delta_{\dot{F}}\|t-k\| \le \left\|\tilde{\dot{x}}_k - \dot{x}\right\| + \|b(u_k - u_{k-1})\| \\ &+ \delta_{\dot{F}}\|t-k\| \le M_{e\dot{x}} + \delta_u + \delta_A T_s + \delta_{w_1}T_s = M_F\end{aligned} \tag{81.9}$$

Here we assume $\|u_k - u_{k-1}\| \le \delta_u$ with $\delta_u > 0$, which is reasonable for the system. We henceforth have $\left\|\varepsilon_F + \mathbf{k}^T \boldsymbol{\varepsilon}_x\right\| \le \|-Ke_x\| + \|e_F\| \le kM_{ex} + M_F \le M$ with $M > 0$, i.e., the term $\varepsilon = \varepsilon_F + \mathbf{k}^T \boldsymbol{\varepsilon}_x$ is bounded. For the closed-loop system (81.8), the completely solution is

$$\mathbf{e}(t) = \exp(-\boldsymbol{\Lambda}_c t)\mathbf{e}(t_0) + \int_{t_0}^{t} \exp[-\boldsymbol{\Lambda}_c(t-\tau)]\mathbf{b_c}\varepsilon(\tau)d\tau \tag{81.10}$$

If $\mathrm{Re}(\lambda_i) > 0$, λ_i stand for the eigenvalues of the matrix $\boldsymbol{\Lambda}_c$, there exist finite positive constant ρ (in fact $\rho = \max\|\lambda_i\|$) such that the transition matrices $\|\exp(-\boldsymbol{\Lambda}_c t)\| \le \exp(-\rho t)$.

Hence, the solution of the closed-loop error dynamics satisfies

$$\begin{aligned}\|\mathbf{e}(t)\| &= \left\|\exp(-\boldsymbol{\Lambda}_\mathbf{c} t)\mathbf{e}(t_0) + \int_{t_0}^{t} \exp[-\boldsymbol{\Lambda}_\mathbf{c}(t-\tau)]\mathbf{b}_c\varepsilon(\tau)d\tau\right\| \\ &\le \|\exp(-\boldsymbol{\Lambda}_\mathbf{c} t)\|\|\mathbf{e}(t_0)\| + M\left\|\int_{t_0}^{t} \exp[-\boldsymbol{\Lambda}_\mathbf{c}(t-\tau)]d\tau\right\| \le \exp(-\rho t)\|\mathbf{e}(t_0)\| \\ &+ M\int_{t_0}^{t} \exp(-\rho t)d\tau \le \exp(-\rho t)\|\mathbf{e}(t_0)\| + [M\exp(-\rho t_0) \\ &- M\exp(-\rho t)]/\rho \le \|\mathbf{e}(t_0)\| + M\exp(-\rho t_0)/\rho < \infty\end{aligned} \tag{81.11}$$

According to the definition of UUB, the error $\mathbf{e}$ will eventually converge into a hyper ball including the origin, and it is UUB.

81.4 Illustrative Example

In order to validate our control scheme, a simple example is presented here. Consider an LTV SISO system with standard form of observability

$$\begin{cases} x = \begin{bmatrix} 0 & 1 \\ -(1+0.5\cos t) & -(1-0.5\sin t) \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ \mathrm{sign}(\sin 0.5t) \end{bmatrix} \\ y = [1 \quad 0]x + w \end{cases} \tag{81.12}$$

where $w = N(0, 0.005)$ is the measurement noise. According to the output Eq. 81.9, there have $\tilde{x} = \begin{bmatrix} \tilde{y} & \dot{\tilde{y}} \end{bmatrix}^T$ and $\dot{\tilde{x}} = \begin{bmatrix} \dot{\tilde{y}} & \ddot{\tilde{y}} \end{bmatrix}^T$. By using the control law of Eq. 81.7, we design the controller as the following form

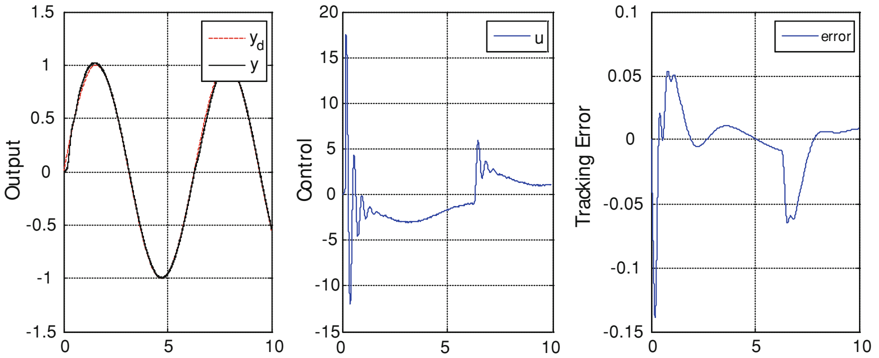

Fig. 81.1 History of output, control and tracking error

$$u = \ddot{y}_d - \left(\tilde{\ddot{y}} - u_{k-1}\right) - k_1\left(\tilde{\dot{y}} - \dot{y}_d\right) - k_0(\tilde{y} - y_d) \tag{81.13}$$

where $y_d = \sin t$ is the desired output trajectory. The design parameters k_1 and k_0 are chosen so as to render the closed-loop characteristic polynomial into a Hurwitz polynomial with desirable roots. In such case, the desired closed-loop equation is set as $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, consequently $k_1 = 2\zeta\omega_n$ and $k_0 = \omega_n^2$.

According to Eq. 81.3 and choosing $N = n, k = 2$ and $\mu = 2$, the estimation of $y, \dot{y}$ and $\ddot{y}$ can be obtained by $\tilde{y} = 30\int_0^1 p_0(\tau)y(T - T\tau)d\tau$, and $\tilde{\dot{y}} = -140\int_0^1 p_1(\tau)y(T - T\tau)/Td\tau$ and $\tilde{\ddot{y}} = 630\int_0^1 p_2(\tau)y(T - T\tau)/T^2 d\tau$, where polynomial $p_i(t)$ respectively are $p_0(t) = t^2 - 2t^3 + t^4$, $p_1(t) = 3t^2 - 12t^3 + 15t^4 - 6t^5$ and $p_2(t) = 12t^2 - 80t^3 + 180t^4 - 168t^5 + 56t^6$

Simulation experiment is conducted with Matlab, the sampling period $T_s = 0.001$ s. The controller parameters set as $\zeta = 1, \omega_n = 5$. Figure 81.1 reveals the proposed method accommodating the external disturbance and system parameters' variation, even in the noisy environment.

81.5 Conclusion

This Chapter has presented a novel control scheme via a differential algebraic framework. An online numerical differentiation technique was introduced for the derivatives estimate, from which a local model of LTV system was established. By using a PID controller, we obtain a closed-loop dynamics of the tracking error, with the nature of linear time invariant. The numerical simulations validate the proposed control scheme is efficient in the control of LTV system, even in the presence of external disturbances and measurement noises.

References

1. Marinescu B (2010) Output feedback pole placement for linear time-varying systems with application to the control of nonlinear systems. Automatica 46:1524–1530
2. Huang R (2007) Output feedback tracking control of nonlinear time-varying systems by trajectory linearization. Russ college of engineering and technology. PhD Thesis, Ohio University
3. Marino R, Tomei P (2003) Adaptive control of linear time-varying systems. Automatica 39:651–659
4. Fliess M, Sira-Ram H (2004) Control via state estimations of some nonlinear systems. In: Proceedings 6th IFAC symposium on nonlinear control systems(NOLCOS 2004), Stuttgart,Germany
5. Fliess M, Join CE, Sira-Ramírez H (2008) Non-linear estimation is easy. IJMIC 4:1–6
6. Mboup M, Join CE, Fliess M (2007) A revised look at numerical differentiation with an application to nonlinear feedback control. In: 2007 mediterranean conference on control and automation Athens, Greece
7. Mboup M, Join CE, Fliess M (2009) Numerical differentiation with annihilators in noisy environment. Numer Algorithms 50(4):1–27