Chapter 17
Nonlinear Retarded Integral Inequalities for Discontinuous Functions and Its Applications

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Abstract It is well-known that integral inequality for continuous function is an important tool for studying the existence, uniqueness, boundedness, stability and other qualitative properties of solutions of differential equations and integral equations. The integral inequality for discontinuous function is an important tool for studying impulsive differential equations as well. To study the estimations of solution of nonlinear retarded impulsive integral equation, firstly retarded integral inequalities including the nonlinear composite function of discontinuous function are established, next the estimations of the unknown function of the integral inequalities are given by the methods of replacement, enlargement, differential, integral, segmentation, mathematical induction. Finally, the estimations obtained here are used to give the estimation of the solution of a class of nonlinear impulsive differential equation.

Keywords Retarded integral inequality · Discontinuous function · Estimation

17.1 Introduction

Gronwall-Bellman type inequality which furnishes explicit bounds on unknown function have become an important tool in the study of the existence, boundedness, stability and other qualitative properties of solutions of differential and integral

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equations. Some results related to Gronwall-Bellman type inequality can be found in [1–4]. In recent years much attention has been given to the analogous inequalities and their applications for discontinuous functions, some recent works can be found in [5–8] and some references therein. In 2010, Li et al. [4] obtained the explicit bound to the unknown function of the following inequalities.

$$u^2(t) \le k(t) + 2\int_0^{\alpha(t)} [M_1 f_1(t,s)u(s) + N_1 g_1(t,s)u^2(s)]ds$$
$$+ 2\int_0^{t} [M_2 f_2(t,s)u(s) + N_2 g_2(t,s)u^2(s)]ds, t \ge 0$$

On the basis of the above inequality, we establish a new class of Gronwall-Bellman type inequality for discontinuous function, this result furnish a handy tool for the study of the conditions of boundedness, stability by Lyapunov, practical stability by Chetaev for the solutions of impulsive differential and integro-differential systems.

17.2 Conclusion

Throughout this paper, $\mathbb{R}$ denotes the set of real number, $t_0 \ge 0$ is given number. $\mathbb{R}_+ := (0, \infty)$, $I_i := [t_{i-1}, t_i)$, $i = 1, 2, \cdots$.

17.2.1 A. Conclusion 1

Theorem 1 *Let us consider a nonnegative piecewise continuous function* $u(t)$ *at* $t \ge t_0 \ge 0$, *with the first kind of discontinuity at the points* $t_i (t_0 < t_1 < t_2 \ldots, \lim_{i\to\infty} t_i = \infty)$, *which satisfies the retarded integral inequality for discontinuous function*

$$u^m(t) \le k(t) + 2\int_{\alpha(t_0)}^{\alpha(t)} [M_1 f_1(t,s)u^{\frac{m}{2}}(s) + N_1 g_1(t,s)u^m(s)]ds$$
$$+ 2\int_{t_0}^{t} [M_2 f_2(t,s)u^{\frac{m}{2}} + N_2 g_2(t,s)u^m(s)]ds + \sum_{t_0 < t_i < t} \beta_i u(t_i - 0) \quad (17.1)$$

where $\beta_i \geq 0$, $m > 0$, $M_i \geq 0$, $N_i \geq 0$, $i = 1, 2$ *are given constants*, $k : \mathbb{R}_+ \to (0, \infty)$ *is a continuous and nondecreasing function*, $\alpha \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ *is a nondecreasing function with* $\alpha(t) \leq t$, $\alpha(t_i) = t_i$, $i = 0, 1, 2, \cdots$, $\lim_{t\to\infty} \alpha(t) = \infty$, $f_i, g_i \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ *are nondecreasing on* t, $\partial_t f_i(t,s)$, $\partial_t g_i(t,s) \in C(\mathbb{R}_+^2, \mathbb{R}_+)$, $i = 1, 2$. *Then the function* $u(t)$ *will satisfy the estimation*

$$u(t) \leq \sqrt[m]{k(t)} \left[e^{\int_{t_i}^{t} R_i(s)ds} \left(c_i + \int_{t_i}^{t} Q_i(s) e^{-\int_{t_i}^{s} R_i(\tau)d\tau} ds \right) \right]^{\frac{2}{m}} \qquad (17.2)$$

$$\forall t \in [t_i, t_{i+1}], \quad i = 0, 1, 2, \ldots,$$

where

$$R_j(s) := N_1 g_1(s, \alpha(s))\alpha'(s) + N_2 g_2(s,s) + \int_{\alpha(t_k)}^{\alpha(s)} N_1 \partial_s g_1(s,\tau)d\tau + \int_{t_k}^{s} N_2 \partial g_2(s,\tau)d\tau \qquad (17.3)$$

$$Q_j(s) := M_1 \tilde{f}_1(s, \alpha(s))\alpha'(s) + M_2 \tilde{f}_2(s,s) + \int_{\alpha(t_k)}^{\alpha(s)} M_1 \partial_s \tilde{f}_1(s,\tau)d\tau + \int_{t_k}^{s} M_2 \partial_s \tilde{f}_2(s,\tau)d\tau \qquad (17.4)$$

$$j = 0, 1, 2, \ldots, i-1,\ s \in [t_k, t_{k+1}), k = i, s \in [t_i, t),$$

and

$$\tilde{f}_1(t,s) = \frac{f_1(t,s)}{\sqrt{k(s)}} \tilde{f}_2(t,s) = \frac{f_2(t,s)}{\sqrt{k(s)}} \qquad (17.5)$$

$$c_0 = 1 \qquad (17.6)$$

$$c_{\mathrm{i}} = \frac{\beta_{\mathrm{i}}}{\frac{m-1}{m}(t_i)} \left[e^{\int_{t_{i-1}}^{t_i} R_{i-1}(s)ds} \left(c_{i-1} + \int_{t_{i-1}}^{i} Q_{i-1}(s)^{-\int_{t_{i-1}}^{s} R_{i-1}(\tau)\mathrm{d}\tau} ds \right) \right]^{\frac{2}{m}}$$
$$+ \left[\int_{t_{i-1}}^{i} R_{i-1}(\mathrm{s})ds \left(c_{i-1} + \int_{t_{i-1}}^{i} Q_{i-1}(\mathrm{s})^{-\int_{t_{i-1}}^{s} R_{i-1}(\tau)d\tau} ds \right) \right]^2 i = 1, 2, \ldots \qquad (17.7)$$

Proof Taking into account the inequality of (17.1), we get

$$\frac{u^m(t)}{k(t)} \le 1 + 2\int_{\alpha(t_0)}^{\alpha(t)} \left[M_1 f_1(t,s)\frac{u^{\frac{m}{2}}(s)}{k(s)} + N_1 g_1(t,s)\frac{u^m(s)}{k(s)} \right] ds$$
$$+2\int_{t_0}^{t} \left[M_2 f_2(t,s)\frac{u^{\frac{m}{2}}(s)}{k(s)} + N_2 g_2(t,s)\frac{u^m(s)}{k(s)} \right] ds + \sum_{t_0<t_i<t} \beta_i \frac{u(t_{\rm i}-0)}{k(t_{\rm i})}, \forall t \ge t_0. \tag{17.8}$$

Let

$$W(t) := \frac{u^m(t)}{k(t)}, \tag{17.9}$$

from (17.5) and (17.8), we have

$$W(t) \le 1 + 2\int_{\alpha(t_0)}^{\alpha(t)} \left[M_1 \frac{f_1(t,s)}{\sqrt{k(s)}} W^{\frac{1}{2}}(s) + N_1 g_1(t,s) W(s) \right] ds$$
$$+ 2\int_{t_0}^{t} \left[M_2 \frac{f_2(t,s)^{\frac{1}{2}}}{\sqrt{k(s)}}(s) + N_2 g_2(t,s) W(s) \right] ds + \sum_{t_0<t_i<t} \beta_i \frac{W^{\frac{1}{m}}(t_{\rm i}-0)}{k^{\frac{m-1}{m}}(t_{\rm i})}$$
$$= 1 + 2\int_{\alpha(t_0)}^{\alpha(t)} \left[M_1 \widetilde{f_1}(t,s)\, W^{\frac{1}{2}}(s) + N_1 g_1(t,s) W(s) \right] ds$$
$$+ 2\int_{t_0}^{t} \left[M_2 \widetilde{f_2}(t,s)^{\frac{1}{2}}(s) + N_2 g_2(t,s) W(s) \right] ds + \sum_{t_0<t_i<t} \beta_i \frac{^{\frac{1}{m}}(t_i-0)}{^{\frac{m-1}{m}}(t_i)}, \forall t \ge t_0 \tag{17.10}$$

Denote $v(t)$ by

$$v(t) := 2\int_{\alpha(t_0)}^{\alpha(t)} \left[M_1 \widetilde{f_1}(t,s)\, W^{\frac{1}{2}}(s) + N_1 g_1(t,s) W(s) \right] ds$$
$$+ 2\int_{t_0}^{t} \left[M_2 \widetilde{f_2}(t,s)^{\frac{1}{2}}(s) + N_2 g_2(t,s) W(s) \right] ds \tag{17.11}$$

In the following, we shall prove the estimation (17.2).
Firstly, we consider the case $t \in I_1$, by (17.11), we have

$$W(t) \le 1 + v(t), \qquad W^{\frac{1}{2}}(t) \le \sqrt{1+v(t)} \tag{17.12}$$

By the assumptions on $f_i,\ g_i$ andα, we see that $v(t)$ is nondecreasing on $\mathbb{R}_+$. Hence, from (17.11) and (17.12), we have

$$\begin{aligned}
v'(t) &= 2\alpha'(t)\Big[M_1\tilde{f}_1(t,\alpha(\mathrm{t}))\,W^{\frac{1}{2}}(\alpha(t)) + N_1 \mathrm{g}_1(t,\alpha(t))W(\alpha(t))\Big] \\
&\quad + 2\Big[M_2\tilde{f}_2(t,t)\,W^{\frac{1}{2}}(t) + N_2 \mathrm{g}_2(t,t)W(t)\Big] \\
&\quad + 2\int\limits_{\alpha(t_0)}^{\alpha(t)} \Big[M_1\,\partial_t\tilde{f}_1(t,s)^{\frac{1}{2}}(s) + N_1\,\partial_t\, g_1(t,s)W(s)\Big]ds \\
&\quad + 2\int\limits_{t_0}^{t} \Big[M_2\,\partial_t\tilde{f}_2(t,s)\,W^{\frac{1}{2}}(s) + N_2\,\partial_t\,\mathrm{g}_2(t,s)W(s)\Big]ds \\
&\le 2\sqrt{1+v(t)}[M_1\,\widetilde{f_1}(t,\alpha(t))\alpha'(t) + M_2\,\widetilde{f_2}(t,t) + \int\limits_{\alpha(t_0)}^{\alpha(t)} M_1\,\partial_t\,\widetilde{f_1}(t,s)d(s) \\
&\quad + \int\limits_{t_0}^{t} M_2\,\partial_t f_2(t,s)ds] + 2(1+v(t))[N_1 g_1(t,\alpha(t))\alpha'(t) \\
&\quad + N_2 g_2(t,t) + \int\limits_{\alpha(t_0)}^{\alpha(t)} N_1\,\partial_t\, g_1(t,s)ds + \int\limits_{t_0}^{t} N_2\,\partial_t\,\mathrm{g}_2(t,s)ds]
\end{aligned} \tag{17.13}$$

By the definition of $R_j(t)$ in (17.3) and $Q_j(t)$ in (17.4), from (17.13), we obtain

$$v'(t) \le 2Q_0(t)\sqrt{1+v(t)} + 2R_0(t)(1+v(t)),$$

i.e.

$$\frac{v'(t)}{2\sqrt{1+v(t)}} \le Q_0(t) + R_0(t)\sqrt{1+v(t)},$$

Or equivalently

$$\frac{[1+v(t)]'}{2\sqrt{1+v(t)}} \le Q_0(t) + R_0(t)\sqrt{1+v(t)}, \tag{17.14}$$

from (17.14), we obtain

$$\frac{d(\sqrt{1+v(t)})}{d_t} \le Q_0(t) + R_0(t)\sqrt{1+v(t)}. \tag{17.15}$$

From (17.15), for all$t \in I_1$, we obtain

$$\sqrt{1+v(t)} \le e^{\int_{t_0}^{t} R_0(s)ds}\left(1+\int_{t_0}^{t} Q_0(s)\, e^{-\int_{t_0}^{s} R_0(\tau)d_\tau}\, ds\right). \tag{17.16}$$

By (17.12), we have

$$W(t) \le \left[e^{\int_{t_0}^{t} R_0(s)ds}\left(1+\int_{t_0}^{t} Q_0(s)\, e^{-\int_{t_0}^{s} R_0(\tau)d_\tau}\, ds\right)\right]^2 \tag{17.17}$$

By (17.9), from (17.17), we get $u(t) \le \sqrt[m]{k(t)}$ $\left[e^{\int_{t_0}^{t} R_0(s)ds}(1+\int_{t_0}^{t} Q_0(s)\, e^{-\int_{t_0}^{s} R_0(\tau)d\tau}\, ds)\right]^{\frac{2}{m}}$, Implying that (17.2) is true for $t \in I_1$.

Next, we consider $t \in I_2 = [\mathrm{t}_1, t_2)$. Using the hypotheses on f_i, g_i and α, from (17.10), we have

$$\begin{aligned}
W(t) \le\ & 1 + 2\int_{\alpha(t_0)}^{\alpha(t_1)} [M_1\tilde{f}_1(t,s)W^{\frac{1}{2}}(s) + N_1 g_1(t,s)W(s)]ds \\
& + 2\int_{t_0}^{t_1} [M_2\tilde{f}_2(t,s)W^{\frac{1}{2}}(s) + N_2 g_2(t,s)W(s)]ds \\
& + \beta_1 \frac{W^{\frac{1}{m}}(t_1-0)}{k^{\frac{m-1}{m}}(t_1)} + 2\int_{\alpha(t_1)}^{\alpha(t)} [M_1\tilde{f}_1(t,s)W^{\frac{1}{2}}(s) + N_1 g_1(t,s)W(s)]ds \\
& + 2\int_{t_1}^{t} [M_2\tilde{f}_2(t,s)W^{\frac{1}{2}}(s) + N_2 g_2(t,s)W(s)]ds \\
\le\ & 1 + 2\int_{\alpha(t_0)}^{\alpha(t_1)} [M_1\tilde{f}_1(t_1,s)W^{\frac{1}{2}}(s) + N_1 g_1(t_1,s)W(s)]ds \\
& + 2\int_{t_0}^{t_1} [M_2\tilde{f}_2(t_1,s)W^{\frac{1}{2}}(s) + N_2 g_2(t_1,s)W(s)]ds \\
& + \beta_1 \frac{W^{\frac{1}{m}}(t_1-0)}{k^{\frac{m-1}{m}}(t_1)} + 2\int_{\alpha(t_1)}^{\alpha(t)} [M_1\tilde{f}_1(t,s)W^{\frac{1}{2}}(s) + N_1 g_1(t,s)W(s)]ds
\end{aligned}$$

$$+2\int_{t_1}^{t}[M_2\tilde{f}_2(t,s)W^{\frac{1}{2}}(s)+N_2g_2(t,s)W(s)]ds$$

$$\leq\left[e^{\int_{t_0}^{t_1}R_0(s)ds}\left(1+\int_{t_0}^{t_1}Q_0(s)e^{-\int_{t_0}^{s}R_0(\tau)d\tau}ds\right)\right]^2$$

$$+\frac{\beta_1}{k^{\frac{m-1}{m}}(t_1)}\left[e^{\int_{t_0}^{t_1}R_0(s)ds}\left(1+\int_{t_0}^{t_1}Q_0(s)e^{-\int_{t_0}^{s}R_0(\tau)d\tau}ds\right)\right]^{\frac{2}{m}}$$

$$+2\int_{\alpha(t_1)}^{\alpha(t)}[M_1\tilde{f}_1(t,s)W^{\frac{1}{2}}(s)+N_1g_1(t,s)W(s)]ds$$

$$+2\int_{t_1}^{t}[M_2\tilde{f}_2(t,s)W^{\frac{1}{2}}(s)+N_2g_2(t,s)W(s)]ds. \quad (17.18).$$

Denote $z(t)$ by

$$z(t)=2\int_{\alpha(t_1)}^{\alpha(t)}[M_1\tilde{f}_1(t,s)W^{\frac{1}{2}}(s)+N_1g_1(t,s)W(s)]ds$$
$$+2\int_{t_1}^{t}[M_2\tilde{f}_2(t,s)W^{\frac{1}{2}}(s)+N_2g_2(t,s)W(s)]ds. \quad (17.19).$$

From (17.7) and (17.19), (17.18) can be written as $W(t)\leq c_1+z(t)$, $W^{\frac{1}{2}}\leq\sqrt{c_1+z(t)}$.

Differentiating $z(t)$, we get

$$z'(t)=2\alpha'(t)\left[M_1\tilde{f}_1(t,\alpha(t))W^{\frac{1}{2}}(\alpha(t))+N_1g_1(t,\alpha(t))W(\alpha(t))\right]$$
$$+\ 2\left[M_2\tilde{f}_2(t,t)W^{\frac{1}{2}}(t)+N_2g_2(t,t)W(t)\right]$$
$$+2\int_{\alpha(t_1)}^{\alpha(t)}\left[M_1\partial_t\tilde{f}_1(t,s)\ W^{\frac{1}{2}}+N_1\partial_tg_1(t,s)W(s)\right]ds$$
$$+2\int_{t_1}^{t}\left[M_2\partial_t\tilde{f}_2(t,s)\ W^{\frac{1}{2}}+N_2\partial_tg_2(t,s)W(s)\right]ds$$

$$\leq 2\sqrt{c_1 + z(t)}\left[M_1 \tilde{f}_1(t, \alpha(t))\alpha'(t) + M_2 \tilde{f}_2(t,t) + \int_{\alpha(t_1)}^{\alpha(t)} M_1 \partial_t \tilde{f}_1(t,s)\right.$$

$$\left. + \int_{t_1}^{t} M_2 \partial_t \tilde{f}_2(t,s)\right] + 2(c_1 + z(t))[N_1 g_1(t, \alpha(t))\alpha'(t) + N_2 g_2(t,t)$$

$$\left. + \int_{\alpha(t_1)}^{\alpha(t)} N_1 \partial_t g_1(t,s) ds + \int_{t_1}^{t} N_2 \partial_t g_2(t,s) ds\right]. \tag{17.20}$$

By the definition of $R_j(t)$ (17.3) and $Q_j(t)$ (17.4), from (17.20), we obtain $z'(t) \leq 2Q_1(t)\sqrt{c_1 + z(t)} + 2R_1(t)(c_1 + z(t))$.

Similar the proof of procedure $t \in [t_0, t_1)$, we can deduce that $u(t) \leq \sqrt[m]{k(t)}\left[e^{\int_{t_1}^{t} R_1(s)ds}\left(c_1 + \int_{t_1}^{t} Q_1(s) e^{-\int_{t_1}^{s} R_1(\tau)d\tau} ds\right)\right]^{\frac{2}{m}}$, for all $t \in [t_1, t_2)$, it implies that (17.2) is true for $t \in [t_1, t_2)$.

In a similar way, for $t \in I_i = [t_i, t_{i+1})$, we can deduce that $u(t) \leq \sqrt[m]{k(t)}\left[e^{\int_{t_i}^{t} R_i(s)ds}\left(c_i + \int_{t_i}^{t} Q_i(s) e^{-\int_{t_i}^{s} R_i(\tau)d\tau} ds\right)\right]^{\frac{2}{m}}$, for all $t \in [t_i, t_{i+1})$. This completes the proof.

Remark 1 (1) When $m = 2, \beta_i = 0$, Theorem 2.1 reduces to Theorem 2.1 of Li et al. [4].

(2) When $k(t) = c, m = 1, M_1 = M_2 = N_1 = 0$, $N_2 = \frac{1}{2}, g(t,s) = v(s)$, Theorem 2.1 reduces to Theorem 1 of Samoilenko and Perestyuk [5].

17.2.2 B. Conclusion 2

Theorem 2 *Let us suppose that a nonnegative piecewise Continuous function* $u(t)$ *at* $t \geq t_0 \geq 0$, *with the first kind of discontinuity at the points* $t_i\left(t_0 < t_1 < t_2 \ldots \lim_{i \to \infty} t_i = \infty\right)$, *which satisfies the integral inequality for discontinuous function*

$$u^m(t) \leq k(t) + \frac{m}{m-n}\int_{a(t_0)}^{a(t)} [M_1 f_1(t,s) u^n(s) + N_1 g_1(t,s) u^n(s) w(u(s))] ds$$

$$+ \frac{m}{m-n}\int_{t_0}^{t} [M_2 f_2(t,s) u^n + N_2 g_2(t,s) u^n(s) w(u(s))] ds + \sum_{t_0 t_i t} \beta_i u^m(t_i - 0), \tag{17.21}$$

where $\beta_i \geq 0, m > n > 0, M_i \geq 0, N_i \geq 0, i = 1,2$ *are given constants* , $k : \mathbb{R}_+ \to (0,\infty)$ *is a continuous and nondecreasing function* $f_i, g_i \in C(\mathbb{R}_+^2, \mathbb{R}_+)$, *are nonincreasing on* $t, \partial_t f_i(t,s), \partial_t g_i(t,s) \in C(\mathbb{R}_+^2, \mathbb{R}_+), i = 1,2, a \in C^1(\mathbb{R}_+^2, \mathbb{R}_+)$ *is a nondecreasing function with* $\alpha(t) \leq t,\ \alpha(t_i) = t_i, \lim_{t\to\infty} \alpha(t) = \infty$, *function* $w(s)$ *satisfies the following class* ς: (1) w *is nondecreasing* ; (2) $w : \mathbb{R}_+ \to \mathbb{R}_+, w(0) = 0$; (3) $w(\alpha\beta) \leq w(\alpha)w(\beta)$. *Then* $\forall t \geq t_0$ *the function* $u(t)$ *will satisfy the estimation*

$$u(t) \leq \sqrt[m]{k(t)} \Bigg\{ \Phi_i^{-1} \Bigg[\Phi_i \Bigg(e_i + \int_{\alpha(t_i)}^{\alpha(t)} M_1 \hat{f}_1(t,s)ds + \int_{t_i}^{t} M_2 \hat{f}_2(t,s)ds \Bigg) + \int_{\alpha(t_i)}^{\alpha(t)} N_1 \hat{g}_1(t,s) w\Big(\sqrt[m]{k(s)}\Big)ds + \int_{t_i}^{t} N_2 \hat{g}_2(t,s) w\Big(\sqrt[m]{k(s)}\Big)ds \Bigg] \Bigg\}^{\frac{1}{m-n}} \tag{17.22}$$

For all $t \in [t_i, t_{i+1}), i = 0,1,2\ldots$, *where*

$$\Phi_i(\mathrm{t}) = \int_{t_i}^{t} \frac{ds}{w(s^{\frac{1}{m-n}})} \lim_{t\to\infty} \Phi_i(t) = \infty i = 0,1,2\ldots \tag{2.23}$$

$$\hat{f}_1(t,s) = \frac{f_1(t,s)}{\sqrt[m]{k^{m-n}(s)}}, \hat{f}_2(t,s) = \frac{f_2(t,s)}{\sqrt[m]{k^{m-n}(s)}},$$

$$\hat{g}_1(t,s) = \frac{g_1(t,s)}{\sqrt[m]{k^{m-n}(s)}}, \ \hat{g}_2(t,s) = \frac{g_2(t,s)}{\sqrt[m]{k^{m-n}(s)}},$$

$e_0 = 1$,

$$e_i = (1+\beta_i)\sqrt[m]{k(t)} \Bigg\{ \Phi_{i-1}^{-1} \Bigg[\Phi_{i-1} \Bigg(e_{i-1} + \int_{\alpha(t_{i-1})}^{\alpha(t_i)} M_1 \hat{f}_1(t,s)ds + \int_{\alpha(t_i-1)}^{\alpha(t_i)} N_1 \hat{g}_1(t,s) w(\sqrt[m]{k(s)})ds + \int_{t_{i-1}}^{t_i} M_2 \hat{f}_2(t,s)ds \Bigg) + \int_{t_{i-1}}^{t_i} N_2 \hat{g}_2(t,s) w(\sqrt[m]{k(s)})ds \Bigg] \Bigg\}^{\frac{m}{m-n}}, i = 0,1,2\ldots$$

17.2.3 C. Conclusion 3

In this section, we will show that our results are useful in proving the boundedness of solutions of impulsive differential system. We consider an impulsive system as follows:

$$\begin{aligned} \frac{dx^m(t)}{dt} &= F(t, x(t), x(\alpha(t))\ t \neq t_i \\ \Delta x|_{t=t_i} &= I_i(x) \\ x(t_0^+) &= x_0 \end{aligned} \tag{17.24}$$

where $x \in \mathbb{R}^k$, $F \in C(\mathbb{R}^{2k+1}, \mathbb{R}^k)$, k is a given natural number, $I_i(x) \in C(\mathbb{R}^k, \mathbb{R}^k)$, $\alpha \in C^1(\mathbb{R}^k, \mathbb{R}^k)$ is a nondecreasing function with $\alpha(t) \leq t$, $t \geq t_0 \geq 0$, $\alpha(t_i) = t_i$ $t_{i-1} < t_i, \forall i = 1, 2, \ldots$, $\lim_{i\to\infty} t_i = \infty$. Let us assume that F, I_i satisfy the following conditions:

$$(a)\ ||F(t,x)|| \leq f_1(t)||x(t)||^n + f_2(t)||x(\alpha(t))||^n + g_1(t)||x(t)||^n w(||x(t)||) + g_2(t)||x(\alpha(t))||^n w(||x(\alpha(t))||),$$

(b) $||I_i(x)|| \leq \beta_i ||x||^m$, where $\beta_i \geq 0$ are constants, $i = 1, 2\ldots,$ $f_1, f_2, g_1, g_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$, $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a nondecreasing function with $w(t) > 0$ for $t > 0$.

Corollary 1 *Under assumptions of the conditions* (a) and (b), *all solutions* $x(t)$ *of the system* (17.24) *have the estimation*

$$\left\|x(t)\right\| \leq \left\{\Phi_i^{-1}\left[\Phi_i\left(e_i + \frac{m-n}{m}\left(\int_{t_i}^{t} f_1(s)ds + \int_{\alpha(t_i)}^{\alpha(t)} \frac{f_2(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\right)\right) + \frac{m-n}{m}\left(\int_{t_i}^{t_i} g_1(s)ds + \int_{\alpha(t_i)}^{\alpha(t)} \frac{g_2(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\right)\right]\right\}^{\frac{1}{m-n}} \tag{17.25}$$

for all $t \in [t_i, t_{i+1}), i = 0, 1, 2\ldots,$ *where* $\Phi_i(t), i = 0, 1, 2, \ldots,$ *are defined by* (2.23), $e_0 = x_0$,

$$e_j = (1+\beta_j)\left\{\Phi_{j-1}^{-1}\left[\Phi_{j-1}\left(e_{j-1} + \frac{m-n}{m}\left(\int_{t_{j-1}}^{t_j} f_1(s)ds + \int_{\alpha(t_{j-1})}^{\alpha(t_j)} \frac{f_2(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\right)\right) + \frac{m-n}{m}\left(\int_{t_{j-1}}^{t_j} g_1(s)ds + \int_{\alpha(t_{j-1})}^{\alpha(t_j)} \frac{g_2(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\right)\right]\right\}^{\frac{m}{m-n}}, j = 1, 2\ldots, i.$$

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