Chapter 10
Spectra of Discrete Multi-Splitting Waveform Relaxation Methods to Determining Periodic Solutions of Linear Differential-Algebraic Equations

Xiaolin Lin, Liming Liu, Hong Wei, Yuan Sang, Yumei Wang and Ronghui Lu

Abstract This chapter proposed spectra of discrete multi-splitting waveform relaxation (DMSWR) method to determine the periodic solutions of linear differential-algebraic equations. Based on the spectral radius of the derived operator by decoupled process, we obtained some convergent conditions for DMSWR method. The DMSWR method is an acceleration technique of the periodic waveform relaxation. A numerical example in circuit simulation is provided to further confirm the theoretical analysis and also to show that the multi-splitting technique can effectively accelerate the convergent performance of the iterative process.

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10.1 Introduction

We have known that waveform relaxation (WR) method is a basic and efficient iteration technique for solving ordinary differential equations (ODEs) and differential-algebraic equations (DAEs) either in initial value problems or two point boundary problems in engineering applications, such as circuit simulation and mechanical modeling. In fact WR was first proposed to simulate MOS VLSI circuits [1–3]. Numerical algorithms incorporated with WR are relaxation-based methods and they are suitable for scientific computations of transient responses for very large dynamic systems. Many researchers have given convergence conditions of WR [4–9] and multi-splitting waveform relaxation (MSWR) [10, 11].

The resulted iterative systems with periodic constraint can be numerically solved by the sophisticated codes of DAEs or ODEs on boundary problem in public domain. In WR method, there are many decouple techniques such as Jacobian Iteration, Gauss–Seidel Iteration and so on. In order to accelerate the speed of convergence of WR, we present the multi-splitting waveform relaxation (MSWR) method [10], it is a novel splitting technique in engineering applications.

Consider the DAEs as the following:

$$\begin{cases} M\dot{x}(t) + Ax(t) + By(t) = f_1(t), & x(\mathbf{0}) = x(T), \\ Cx(t) + Ny(t) = f_2(t), \quad t \in [\mathbf{0}, T]. & \end{cases} \tag{10.1}$$

where M and N are, respectively, $n_1 \times n_1$ and $n_2 \times n_2$ nonsingular matrices, A is an $n_1 \times n_1$ matrix, B is an $n_1 \times n_2$ matrix, C is an $n_2 \times n_1$ matrix, $f_1(t) \in R^{n_1}$ and $f_2(t) \in R^{n_2}$ $(t \in [\mathbf{0}, T])$ are two known input functions with period T, $x(t) \in R^{n_1}$ and $y(t) \in R^{n_2}$ $(t \in [\mathbf{0}, T])$ are to be computed. It is also obvious that $y(\mathbf{0}) = N^{-1}(f_2(\mathbf{0}) - Cx(\mathbf{0}))$ is for (10.1). Further, $y(\mathbf{0}) = y(T)$ is resulted from $x(\mathbf{0}) = x(T)$ and $f_2(\mathbf{0}) = f_2(T)$. We assume that the boundary condition on periodic solutions of (10.1) means that the condition $x(\mathbf{0}) = x(T)$ implies $\dot{x}(\mathbf{0}) = \dot{x}(T)$ and $y(\mathbf{0}) = y(T)$.

Let $M = M_{1l} - M_{2l}$, $A = A_{1l} - A_{2l}$, $B = B_{1l} - B_{2l}$, $C = C_{1l} - C_{2l}$, $N = N_{1l} - N_{2l}$ $(l = \mathbf{1}, \mathbf{2}, \ldots, L)$ and $\left(x^{(0)}(\cdot), y^{(0)}(\cdot)\right)^T$ is a given initial guess. Now, we present the MSWR algorithm to compute the steady-state periodic response over one period for (10.1). The MSWR algorithm of (10.1) is:

$$\begin{cases} M_{1l}\dot{x}^{k,l}(t)+A_{1l}x^{k,l}(t)+B_{1l}y^{k,l}(t)=M_{2l}\dot{x}^{(k-1)}(t)+A_{2l}x^{(k-1)}(t)+B_{2l}y^{(k-1)}(t)+f_1(t), \\ C_{1l}x^{k,l}(t)+N_{1l}y^{k,l}(t)=C_{2l}x^{(k-1)}(t)+N_{2l}y^{(k-1)}(t)+f_2(t), \\ x^{k,l}(\mathbf{0})=x^{k,l}(T),\ \ y^{k,l}(\mathbf{0})=y^{k,l}(T),\ \ l=\mathbf{1},\mathbf{2},\ldots,L, \\ x^{(k)}(t)=\sum_{l=1}^{L}E_l x^{k,l}(t),\ \ y^{(k)}(t)=\sum_{l=1}^{L}\tilde{E}_l y^{k,l}(t),\ \ t\in[\mathbf{0},T],\ k=\mathbf{1},\mathbf{2},\ldots. \end{cases} \tag{10.2}$$

where we suppose that M_{1l} and $N_{1l}(l = \mathbf{1}, \mathbf{2}, \ldots, L)$ are nonsingular, El and $\tilde{E}_l(l = \mathbf{1}, \mathbf{2}, \ldots, L)$ are non-negative diagonal matrix and $\sum_{l=1}^{L} E_l = I$, also $\sum_{l=1}^{L} \tilde{E}_l = I$ in this chapter. In order to preserve the consistency of the boundary conditions for every periodic iteration an initial guess $\left(x^{(0)}(t), y^{(0)}(t)\right)^T$ in (10.2) should satisfy $\left(x^{(0)}(\mathbf{0}), y^{(0)}(\mathbf{0})\right)^T = \left(x^{(0)}(T), y^{(0)}(T)\right)^T$ and $\dot{x}(\mathbf{0}) = \dot{x}(T)$. For any constant guess, the required boundary conditions are naturally held. Often, for a linear system we only consider its MSWR solutions in $C([\mathbf{0}, T], C^n)$ or $L^2([\mathbf{0}, T], C^n)$, here $n = n_1 + n_2$. This treatment can greatly simplify the theoretical analyzes on the MSWR. The convergence behaviors of the MSWR are mainly decided by the corresponding MSWR operators in these functions spaces and decouple process.

The WR solutions of initial value problems of equations as in (10.1) were reported in [12]. The expressions of spectra and pseudo-spectra for their WR operators were also clearly understood [13]. However, so far as we known, most of these theoretical convergence results are about the WR, and there are few chapters to theoretically analyze the spectra of the DMSWR operator for linear dynamic systems in the WR literatures. In this chapter we discuss the DMSWR operator derived from (10.2) where an analytic expression of its spectra is obtained. A finite-difference method is then used to solve the decoupled systems (10.2) in our test examples. The results of paper [6] are the special case of our results in this chapter. At same tine, a numerical example in circuit simulation is provided to further confirm the theoretical analysis and also to show that the multi-splitting technique can effectively accelerate the convergent performance of the iterative process.

10.2 Spectra of DMSWR Operators and Finite-Difference for Solving MSWR Solutions

In this section we consider the discrete case of Sect. 10.2 and give a finite-difference formula for solving the MSWR solution of (10.1).

10.2.1 Spectra of DMSWR Operators

Now we discuss the application of linear multi-step method in the MSWR algorithm (10.2). For this purpose, let us fix the time increment $\tau = T/N$ and discretize (10.2) by a linear multi-step method, where its characteristic polynomials are $a(\xi)$ and $b(\xi)$, i.e., $a(\xi) = \sum_{j=0}^{m} \alpha_j \xi^j$ and $b(\xi) = \sum_{j=0}^{m} \beta_j \xi^j$, to obtain

$$\begin{cases} \frac{1}{\tau} M_{1l} \sum\limits_{j=0}^{m} \alpha_j x_{p-m+j}^{k,l} + A_{1l} \sum\limits_{j=0}^{m} \beta_j x_{p-m+j}^{k,l} + B_{1l} \sum\limits_{j=0}^{m} \beta_j y_{p-m+j}^{k,l} \\ \quad = \frac{1}{\tau} M_{2l} \sum\limits_{j=0}^{m} \alpha_j x_{p-m+j}^{(k-1)} + A_{2l} \sum\limits_{j=0}^{m} \beta_j x_{p-m+j}^{(k-1)} + B_{2l} \sum\limits_{j=0}^{m} \beta_j y_{p-m+j}^{(k-1)} + \sum\limits_{j=0}^{m} \beta_j (f_1)_{p-m+j} \\ C_{1l} x_p^{k,l} + N_{1l} y_p^{k,l} = C_{2l} x_p^{(k-1)} + N_{2l} y_p^{(k-1)} + (f_2)_p, \qquad l = 1, 2, \ldots, L \\ x_p^{(k)} = \sum\limits_{l=1}^{L} E_l x_p^{k,l}, \qquad y_p^{(k)} = \sum\limits_{l=1}^{L} \tilde{E}_l y_p^{k,l}, \qquad p = 0, \pm 1, \pm 2, \ldots, \qquad k = 1, 2, \ldots \end{cases} \tag{10.3}$$

In the above algorithm we assume that $a(\xi)$ and $b(\xi)$ have no common roots where $a(1) = 0$ and $\dot{a}(1) = b(1)$. In practical codes one adopts a convergent linear multi-step method to solve DAEs of (10.2). A special case of the linear multi-step method is the backward differentiation formula (BDF) where $b(\xi) = \xi^m$. The m-step constant BDF method converges to $O(\tau^m)$ for $m < 7$ (see [1]).

Let $x_\tau^{k,l}$ and $y_\tau^{k,l}$ stand for the infinite sequences $\{x_p^{k,l}\}_{p=-\infty}^{\infty}$ and $\{y_p^{k,l}\}_{p=-\infty}^{\infty}$ for all $l = 1, 2, \ldots, L$ and similarly let $x_\tau^{(k)}$, $y_\tau^{(k)}$, $x_\tau^{(k-1)}$, $y_\tau^{(k-1)}$, $(f_1)_\tau$ and $(f_2)_\tau$ stand for the infinite sequences $\{x_p^{(k)}\}_{p=-\infty}^{\infty}$, $\{y_p^{(k)}\}_{p=-\infty}^{\infty}$, $\{x_p^{(k-1)}\}_{p=-\infty}^{\infty}$, $\{y_p^{(k-1)}\}_{p=-\infty}^{\infty}$, $\{(f_1)_p\}_{p=-\infty}^{\infty}$ and $\{(f_2)_p\}_{p=-\infty}^{\infty}$. These infinite sequences are N-periodic, for example it means that $x_{p+N}^{(k)} = x_p^{(k)} (p = 0, \pm 1, \pm 2, \ldots)$ for the sequence $\{x_p^{(k)}\}_{p=-\infty}^{\infty}$. Now we simply rewrite (10.3) as

$$\begin{cases} \frac{1}{\tau} aM_{1l} x_\tau^{k,l} + bA_{1l} x_\tau^{k,l} + bB_{1l} y_\tau^{k,l} = \frac{1}{\tau} aM_{2l} x_\tau^{(k-1)} + bA_{2l} x_\tau^{(k-1)} + bB_{2l} y_\tau^{(k-1)} + b(f_1)_\tau \\ C_{1l} x_\tau^{k,l} + N_{1l} y_\tau^{k,l} = C_{2l} x_\tau^{(k-1)} + N_{2l} y_\tau^{(k-1)} + (f_2)_\tau, \qquad l = 1, 2, \ldots, L \\ x_\tau^{(k)} = \sum\limits_{l=1}^{L} E_l x_\tau^{k,l}, \qquad y_\tau^{(k)} = \sum\limits_{l=1}^{L} \tilde{E}_l y_\tau^{k,l}, \qquad k = 1, 2, \ldots \end{cases} \tag{10.4}$$

where we denote the infinite sequences $\left\{\sum_{j=0}^{m} \alpha_j M_s x_{p-m+j}^{(r)}\right\}_{p=-\infty}^{\infty}$, $\left\{\sum_{j=0}^{m} \beta_j A_s x_{p-m+j}^{(r)}\right\}_{p=-\infty}^{\infty}$ and $\left\{\sum_{j=0}^{m} \beta_j B_s y_{p-m+j}^{(r)}\right\}_{p=-\infty}^{\infty}$ by $aM_s x_\tau^{(r)}$, $bA_s x_\tau^{(r)}$ and $bB_s y_\tau^{(r)}$.

Definition For an N-periodic sequence w_τ, its discrete Fourier coefficients are

$$\tilde{w}_p = \frac{1}{N}\sum_{q=1}^{N} w_q e^{-ipq(2\pi/N)}, \quad p = 0, \pm 1, \pm 2, \ldots$$

By use of Definition, we know that $w_\tau = \sum_{q=1}^{N-1} \tilde{w}_q \varepsilon_{\tau,q}$, where $\varepsilon_{\tau,q} = \left\{e^{ipq(2\pi/N)}\right\}_{p=-\infty}^{\infty}$.

Condition (**S**) For the characteristic polynomials $a(\xi)$ and $b(\xi)$, we assume that the matrix

$$\begin{pmatrix} \frac{1}{\tau}\frac{a}{b}(\xi^q)M_{1l} + A_{1l} & B_{1l} \\ C_{1l} & N_{1l} \end{pmatrix}^{-1} \quad q = 0, 1, \ldots, N-1 \tag{10.5}$$

exist for the splitting matrices $M_{1l}, A_{1l}, B_{1l}, C_{1l}$ and N_{1l} $(l = 1, 2, \ldots, L)$ in which $\xi = e^{i(2\pi/N)}$.

Let $z_\tau^{(r)} = [(x_\tau^{(r)})^T, (y_\tau^{(r)})^T]^T$, if Condition (S) holds, for any fixed k the solution of (10.4) can be written as

$$z_\tau^{(k)} = \mathrm{K}_\tau z_\tau^{(k-1)} + \varphi_\tau, \tag{10.6}$$

here

$$\mathrm{K}_\tau z_\tau = \sum_{q=0}^{N-1} \begin{pmatrix} \sum_{l=1}^{L} E_l(\frac{1}{\tau}\frac{a}{b}(\xi^q)M_{1l} + A_{1l}) & \sum_{l=1}^{L} E_l B_{1l} \\ \sum_{l=1}^{L} \tilde{E}_l C_{1l} & \sum_{l=1}^{L} \tilde{E}_l N_{1l} \end{pmatrix}^{-1} \begin{pmatrix} \sum_{l=1}^{L} E_l(\frac{1}{\tau}\frac{a}{b}(\xi^q)M_{2l} + A_{2l}) & \sum_{l=1}^{L} E_l B_{2l} \\ \sum_{l=1}^{L} \tilde{E}_l C_{2l} & \sum_{l=1}^{L} \tilde{E}_l N_{2l} \end{pmatrix} \tilde{z}_q \varepsilon_{\tau,q}$$

and

$$\varphi_\tau = \sum_{q=0}^{N-1} \begin{pmatrix} \sum_{l=1}^{L} E_l(\frac{1}{\tau}\frac{a}{b}(\xi^q)M_{1l} + A_{1l}) & \sum_{l=1}^{L} E_l B_{1l} \\ \sum_{l=1}^{L} \tilde{E}_l C_{1l} & \sum_{l=1}^{L} \tilde{E}_l N_{1l} \end{pmatrix}^{-1} \tilde{f}_q \varepsilon_{\tau,q}$$

in which $\tilde{f}_q = [(\tilde{f}_1)_q^T, (\tilde{f}_2)_q^T]^T$. With the same approach given in [14], we can get the following theorem (we omit the proof here).

Theorem 1 Under Condition (S) the spectral set of the DMSWR operator K_τ in (10.6) is

$$\sigma(\mathrm{K}_\tau) = \cup\left\{ \sigma\left(\begin{pmatrix} \sum_{l=1}^{L} E_l\left(\frac{\mathbf{1}}{\tau}\frac{a}{b}(\xi^q)M_{1l} + A_{1l}\right) & \sum_{l=1}^{L} E_l B_{1l} \\ \sum_{l=1}^{L} \tilde{E}_l C_{1l} & \sum_{l=1}^{L} \tilde{E}_l N_{1l} \end{pmatrix}^{-1} \begin{pmatrix} \sum_{l=1}^{L} E_l\left(\frac{\mathbf{1}}{\tau}\frac{a}{b}(\xi^q)M_{2l} + A_{2l}\right) & \sum_{l=1}^{L} E_l B_{2l} \\ \sum_{l=1}^{L} \tilde{E}_l C_{2l} & \sum_{l=1}^{L} \tilde{E}_l N_{2l} \end{pmatrix} \right) : q = \mathbf{0}, \mathbf{1}, \ldots, N - \mathbf{1} \right\} \tag{10.7}$$

where $\xi = e^{i(2\pi/N)}$.

10.2.2 Finite-Difference for Solving MSWR Solutions

In this section we compute the iterative waveforms $\left[\left(\dot{x}^{(k)}\right)^T(t), \left(\dot{y}^{(k)}\right)^T(t)\right]^T$ $(k = \mathbf{1}, \mathbf{2}, \ldots)$ in (10.2) at m + 1 time-points, $t_0 = \mathbf{0}, t_1, t_2, \ldots, t_m = T$, with a constant step-size τ. For any fixed k we approximate the derivatives $\dot{x}^{(k)}$ and $\dot{x}^{(k-1)}$ in (10.2) with the implicit Euler method. As a simple case of the linear multi-step method presented in Sect. 10.3.1, we now may write out the iterative matrix for discrete waveforms without using the discrete Fourier series technique. We will follow this form to do our computations in the next section. For the purpose, we denote that

$$X^{(r)} = \left[\left(x^{(r)}\right)^T(t_1), \ldots, \left(x^{(r)}\right)^T(t_m)\right]^T \in R^{mn_1},$$

$$Y^{(r)} = \left[\left(y^{(r)}\right)^T(t_1), \ldots, \left(y^{(r)}\right)^T(t_m)\right]^T \in R^{mn_2},$$

$$F_1 = \left[f_1^T(t_1), f_1^T(t_2), \ldots, f_1^T(t_m)\right]^T \in R^{mn_1},$$

$$F_2 = \left[f_2^T(t_1), f_2^T(t_2), \ldots, f_2^T(t_m)\right]^T \in R^{mn_2}.$$

It is mentioned here that the order of discrete equations is different from that of Sect. 10.2.1 for the differential part and the algebraic part. By $x^{(r)}(t_0) = x^{(r)}(t_m)$ the discrete MSWR form of (10.2) is

$$\begin{cases} H_{1l}X^{k,l} + H_{2l}Y^{k,l} = J_{1l}X^{(k-1)} + J_{2l}Y^{(k-1)} + \tau F_1 \\ H_{3l}X^{k,l} + H_{4l}Y^{k,l} = J_{3l}X^{(k-1)} + J_{4l}Y^{(k-1)} + \tau F_2 \\ l = \mathbf{1}, \mathbf{2}, \ldots, L \\ X^{(k)} = \sum_{l=1}^{L} E_l X^{k,l}, Y^{(k)} = \sum_{l=1}^{L} \tilde{E}_l Y^{k,l}, \quad k = \mathbf{1}, \mathbf{2}, \ldots \end{cases} \tag{10.8}$$

where

$$H_{1l} = \begin{pmatrix} M_{1l} + \tau A_{1l} & 0 & & \cdots & & 0 & -M_{1l} \\ -M_{1l} & \ddots & & & & & 0 \\ 0 & \ddots & M_{1l} + \tau A_{1l} & & & & \\ & & -M_{1l} & \ddots & & & \vdots \\ \vdots & & & \ddots & M_{1l} + \tau A_{1l} & & \\ & & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & -M_{1l} & M_{1l} + \tau A_{1l} \end{pmatrix},$$

$$H_{2l} = \begin{pmatrix} \tau B_{1l} & & \\ & \ddots & \\ & & \tau B_{1l} \end{pmatrix}, \quad H_{3l} = \begin{pmatrix} C_{1l} & & \\ & \ddots & \\ & & C_{1l} \end{pmatrix}, \quad H_{4l} = \begin{pmatrix} N_{1l} & & \\ & \ddots & \\ & & N_{1l} \end{pmatrix}$$

and

$$J_{1l} = \begin{pmatrix} M_{2l} + \tau A_{2l} & 0 & & \cdots & & 0 & -M_{2l} \\ -M_{2l} & \ddots & & & & & 0 \\ 0 & \ddots & M_{2l} + \tau A_{2l} & & & & \\ & & -M_{2l} & \ddots & & & \vdots \\ \vdots & & & \ddots & M_{2l} + \tau A_{2l} & & \\ & & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & -M_{2l} & M_{2l} + \tau A_{2l} \end{pmatrix},$$

$$J_{2l} = \begin{pmatrix} \tau B_{2l} & & \\ & \ddots & \\ & & \tau B_{2l} \end{pmatrix}, \quad J_{3l} = \begin{pmatrix} C_{2l} & & \\ & \ddots & \\ & & C_{2l} \end{pmatrix}, \quad J_{4l} = \begin{pmatrix} N_{2l} & & \\ & \ddots & \\ & & N_{2l} \end{pmatrix}.$$

Let $E_l \in R^{m(n_1+n_2)}$ $(l = 1, 2, \ldots, L)$ be non-negative diagonal matrix and $\sum_{l=1}^{L} E_l = I$. For any fixed step-size τ, the convergence condition of the above iterative algorithm can be concluded in the following theorem.

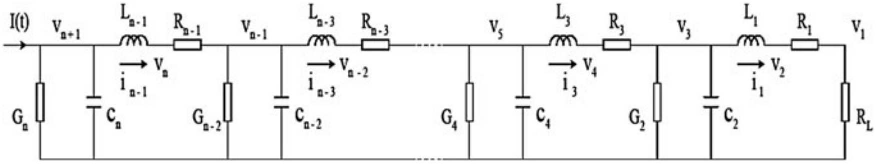

Fig. 10.1 A linear periodic DAEs circuit with n even

Theorem 2 The DMSWR algorithm (10.8) is convergent if

$$\rho\left(\sum_{l=1}^{L} E_l \begin{pmatrix} H_{1l} & H_{2l} \\ H_{3l} & H_{4l} \end{pmatrix}^{-1} \begin{pmatrix} J_{1l} & J_{2l} \\ J_{3l} & J_{4l} \end{pmatrix}\right) < \mathbf{1} \tag{10.9}$$

10.3 Numerical Experiments

We define that the iterative error is the sum of the squared difference of successive waveforms taken over all time-points.

10.3.1 Example One

Example one is a test circuit shown in Fig. 10.1 where n is 10. This circuit is taken from [15]. It is a general form of uniformly dissipative low-pass ladder filter circuit with a current-source input and a voltage output.The circuit equations have a form as (10.1) where

$$x(t) = [i_1(t), v_3(t), i_5(t), v_7(t), i_9(t), v_{11}(t)]^T$$

$$\text{and} \quad y(t) = [v_1(t), v_2(t), v_4(t), v_6(t), v_8(t), v_{10}(t)]^T \in R^6,$$

$f_1(t) = [\mathbf{0}, \ldots, \mathbf{0}, I(t)]^T \in R^{10}$ and $f_2(t) = [\mathbf{0}, \ldots, \mathbf{0}]^T \in R^6$, for any given$t \in [\mathbf{0}, T]$.

Now, M, A, B, C and N in (10.1) are some concrete matrices. For $M, A \in R^{10\times 10}$ we have $M = diag(L_1, C_2, L_3, C_4, \ldots, L_9, C_{10})$ and $A = diag(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_5)$ where

$$\tilde{A}_i = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & G_{2i} + R_{2i+1}^{-1} \end{pmatrix} \quad (i = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}), \quad \tilde{A}_5 = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & G_{10} \end{pmatrix}$$

The matrix $B \in R^{10\times 6}$ and $C \in R^{6\times 10}$ are

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_5^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -R_7^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -R_9^{-1} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_3^{-1} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_5^{-1} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -R_7^{-1} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_9^{-1} & -1 & 0 \end{pmatrix}.$$

For $N \in R^{6\times 6}$ we have $N = diag(\tilde{N}_1, \tilde{N}_2)$,where $\tilde{N}_2 = diag(R_3^{-1}, R_5^{-1}, R_7^{-1}, R_9^{-1}) \in R^{4\times 4}$ and $\tilde{N}_1 = \begin{pmatrix} R_1^{-1} + R_2^{-1} & -R_1^{-1} \\ -R_1^{-1} & R_1^{-1} \end{pmatrix}$.

We seek its periodic responses by the MSWR algorithm. In our computations we use the discrete algorithm (10.8). For simplicity we let n = 10 and $T = \mathbf{2}\pi$, all circuit parameters are set to be one. The boundary values satisfy $x(\mathbf{0}) = x(\mathbf{2}\pi)$ and $y(\mathbf{0}) = -N^{-1}Cx(\mathbf{0})(= y(\mathbf{2}\pi))$.

For example one, we use the Jacobi splitting to split the matrices M and N, i.e., M_1 and N_1 are diagonal matrices of M and N if we adopt the symbols in (10.2). The matrices $B1$ and $C1$are

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in R^{10\times 6},$$

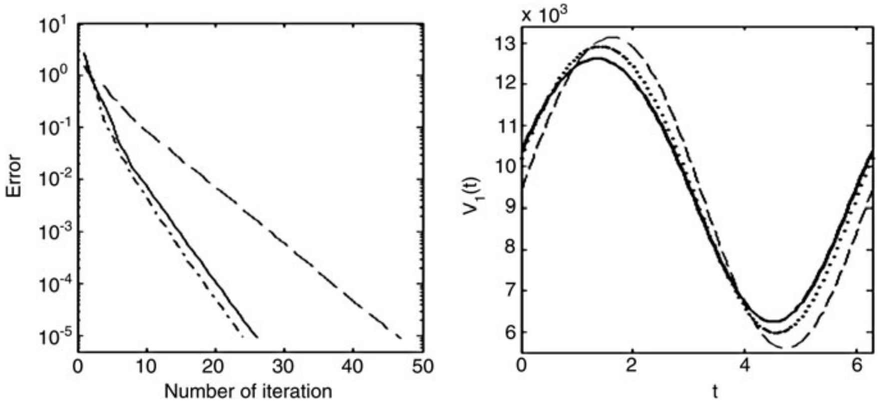

Fig. 10.2 Computed results for Example One. *Left* DMSWR iterations (*dashed line*, *solid line*, and *point line* for Jacobi splitting, Gauss–Sediel splitting and MSWR, respectively). *Right* approximate waveforms (k = 20 for Jacobi splitting, k = 12 for Gauss–Sediel splitting and k = 6 for DMSWR) of the voltage $v_1(t)$ (*solid line*)

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \in R^{6\times 10}.$$

For the matrix A we have two ways to treat its splitting, for $l = \mathbf{1}$, we simply do not split A, i.e., $A_1 = A$; for $l = \mathbf{2}$, we split A as

$$A_1 = \begin{pmatrix} 2 & & & & \\ 1 & \ddots & & & \\ & \ddots & 2 & & \\ & & \ddots & \ddots & \\ & & & 1 & 2 \end{pmatrix} \in R^{10\times 10}$$

Let $\xi = i\zeta(\zeta \in R)$, the spectral of $\sigma(\mathrm{K})$ and $\sigma(\tilde{\mathrm{K}}_\infty) = \overline{\cup\{\sigma(\mathrm{K}(i\zeta)) : \zeta \in R\}}$ can been calculated for $E_1 = \mathbf{0.6}\boldsymbol{I}$ and $E_2 = \mathbf{0.4}\boldsymbol{I}$ in which $p = \mathbf{0}, \pm\mathbf{1}, \ldots, \pm\mathbf{50}$ and$\zeta = \mathbf{0}, \pm\mathbf{0.1}, \ldots, \pm\mathbf{49.9} \pm \mathbf{50}$.

To compute the MSWR solution of the system, we let the input function $I(t) = I(t + \mathbf{2}\pi)$ satisfy $I(t) = \begin{cases} t, & \mathbf{0} \le t \le \mathbf{0.5}\pi; \\ \mathbf{0.5}\pi, & \mathbf{0.5}\pi \le t \le \mathbf{1.5}\pi; \\ (\mathbf{2}\pi - t), & \mathbf{1.5}\pi \le t \le \mathbf{2}\pi. \end{cases}$

The time-step is $\mathbf{0.02}\pi$ sec and the initial guess is the zero function. The convergence results and three approximate waveforms for the voltage $v_1(t)$ are shown in Fig. 10.2.

10.4 Conclusions

We have successfully deduced an analytic expression of the spectral set on the DMSWR operator for a linear system of DAEs under a normal periodic constraint. The convergent conditions of the DMSWR algorithm on periodic solutions can be conveniently chosen from this useful expression, namely the DMSWR algorithm converges to the exact periodic response if the supremum value of spectral radii for a series of complex matrices is less than one. The convergent condition of the chapter is necessary and sufficient for the DMSWR algorithm.

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