Chapter 3
Can Modelling Be Taught and Learnt? Some Answers from Empirical Research

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Abstract This chapter deals with empirical findings on the teaching and learning of mathematical modelling, with a focus on grades 8–10, that is, 14–16-year-old students. The emphasis lies on the actual behaviour of students and teachers in learning environments with modelling tasks. Most examples in this chapter are taken from our own empirical investigations in the context of the project DISUM. In the first section, the terms used in this chapter are recollected from a cognitive point of view by means of examples, and reasons are summarised why modelling is an important and also demanding activity for students and teachers. In the second section, examples are given of students' difficulties when solving modelling tasks, and some important findings concerning students dealing with modelling tasks are presented. The third section concentrates on teachers; examples of successful interventions are given, as well as some findings concerning teachers treating modelling examples in the classroom. In the fourth section, some implications for teaching modelling are summarised, and some encouraging (though not yet fully satisfying) results on the advancement of modelling competency are presented.

1 A Cognitive View on Mathematical Modelling

In this chapter, the actual dealing of students and teachers with modelling tasks is to be investigated. In order to describe, interpret and explain what is happening not only on the surface but also in teachers' and students' minds, a cognitive view on modelling is necessary. Hence, when clarifying some basic notions in this first section, this is done from a cognitive point of view.

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The following first example is meant to set the scene:

Example 1: "Giant's shoes"

In a sports centre on the Philippines, Florentino Anonuevo Jr. polishes a pair of shoes. They are, according to the Guinness Book of Records, the world's biggest, with a width of 2.37 m and a length of 5.29 m.

Approximately how tall would a giant be for these shoes to fit? Explain your solution.

This is a mathematical *modelling* task since the essential demand of the task is to *translate* between reality and mathematics (make assumptions on how the height of a man is related to the size of his shoes, establish appropriate mathematical relationships, interpret results of calculations and check the validity of these results). *Reality* means the "rest of the world" (Pollak 1979) outside mathematics, including nature, society, other scientific disciplines or everyday life.

That such modelling tasks are very difficult for many students is shown by a solution of "Giant's shoes" obtained by a pair of grade 9 students in a laboratory session. They multiplied width and length and thus reached the answer "The giant would be 12.53 m tall", as shown in Fig. 3.1.

This kind of solution is rather common and was observed many times in our investigations (see also Sect. 2), not only with weaker students but also with students from Gymnasium (the high ability track in the German school system). So, a pair of grade 9 Gymnasium students applied the Pythagorean Theorem in "Giant's shoes" and thus got to the answer 33.6 m. Also in this solution, no check was carried out concerning units (in both cases, the unit of the calculated result would have been m² instead of m).

This example and all the following ones are taken from the project *DISUM* ("*D*idaktische *I*nterventionsformen für einen *s*elbständigkeitsorientierten aufgabengesteuerten *U*nterricht am Beispiel *M*athematik", in English "Didactical intervention modes for mathematics teaching oriented towards self-regulation and directed by tasks"; see Blum and Leiß 2008 for a description of this project). DISUM is an interdisciplinary project between mathematics education (W. Blum), pedagogy

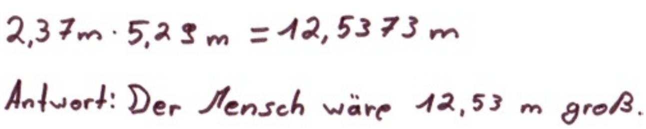

Fig. 3.1 Students' solution of task "Giant's shoes"

(R. Messner, both University of Kassel) and educational psychology (R. Pekrun, University of Munich), which aims at investigating how students and teachers deal with cognitively demanding modelling tasks and what effects various learning environments for modelling have on students' competency development. The focus in DISUM is on grades 8–10 (14–16-year-olds), which will also be the focus of this chapter.

The DISUM examples are all "medium-size" modelling tasks which can be solved within one lesson. The spectrum of tasks suitable for teaching is, of course, much bigger, reaching from straightforward standard applications to authentic modelling problems or complex modelling projects where the data collection alone takes several hours or days (compare, for instance, the "modelling weeks" presented in Kaiser and Schwarz 2006).

Why is modelling so difficult for students? In particular, because of the cognitive demands of modelling tasks; modelling involves translating between mathematics and reality in both directions, and for that, appropriate mathematical ideas ("Grundvorstellungen", see Blum 1998; Hofe 1998) as well as real-world knowledge are necessary. In addition, modelling is inseparably linked with other mathematical competencies (Blomhøj and Jensen 2007; Niss 2003), in particular designing and applying problem solving strategies, reading texts as well as working mathematically (reasoning, calculating, …). Helpful for cognitive analyses of modelling tasks are models of the "modelling cycle" which show typical ways of solving such tasks. In literature, there is a considerable variety of such models (see Borromeo Ferri 2006 for an overview). In the DISUM project, a seven-step model proved particularly helpful (Fig. 3.2, taken from Blum and Leiß 2007).

The following example (Blum and Leiß 2006) is meant to illustrate this model in some more detail.

Example 2: "Filling up"

Mrs. Stone lives in Trier, 20 km away from the border of Luxemburg. To fill up her VW Golf she drives to Luxemburg where immediately behind the border there is a petrol station. There you have to pay 1.10 Euro for one litre of petrol whereas in Trier you have to pay 1.35 Euro.

Is it worthwhile for Mrs. Stone to drive to Luxemburg? Give reasons for your answer.

The first step is to understand the given problem situation, that is the problem solver has to construct a *situation model* which here involves at least two gas stations

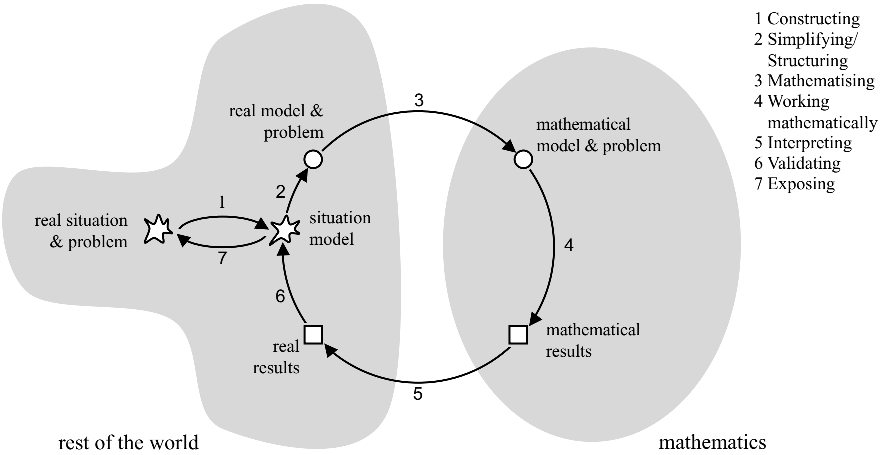

Fig. 3.2 DISUM model of the modelling process

and the 20 km connection. The second step is to structure the situation by bringing certain variables into play, especially tank volume and consumption rate of the Golf, and to simplify the situation by defining what "worthwhile" should mean, leading to a *real model* of the situation. In the standard model, "worthwhile" means only "minimising the costs of filling up and driving". Mathematisation, the third step, transforms the real model into a *mathematical model* which consists here of certain equations, perhaps with variables. The fourth step is working mathematically (calculating etc.), which yields *mathematical results*. In step five, these are interpreted in the real world as *real results,* ending up in a recommendation for Mrs. Stone of what to do. A validation of these results, step six, may show that it is appropriate or necessary to go round the loop a second time, for instance in order to take into account more factors such as time or air pollution. Dependent on which factors have been chosen, the recommendations for Mrs. Stone might be quite different. The seventh and final step is an exposure of the final solution.

This particular model of the modelling process comes from two sources. The notion of "situation model" has its origin in the research on texts (Kintsch and Greeno 1985; Staub and Reusser 1995; Verschaffel et al. 2000), whereas the other components stem from applied mathematical problem solving (Burghes 1986; Burkhardt 2006; Pollak 1979). There are several advantages of this model: step one – a particularly individual construction process, the first cognitive barrier for students – is separated, and all other steps are also essential stages in students' actual modelling processes and potential cognitive barriers, though generally not in linear order (for more details see Sect. 2).

With this model as a background, *modelling competency* can be defined (see Niss et al. 2007) as the ability to construct and to use mathematical models by carrying out those various modelling steps appropriately as well as to analyse or to compare given models. It is a natural hypothesis that these modelling steps correspond to *sub-competencies* (Kaiser 2007; Maaß 2006) of modelling. The main goal of teaching is that students develop modelling competency with – using the notions of Niss et al. (see Blomhøj and Jensen 2007; Jensen 2007; Niss 2003) – a degree of coverage, a radius of action and a technical level as extensive as possible.

Why is modelling so important for students? Mathematical models and modelling are everywhere around us, often in connection with powerful technological tools. Preparing students for responsible citizenship and for participation in societal developments presupposes modelling competency. More precisely (compare Blum and Niss 1991), mathematical modelling is meant to:

- Help students' to better understand the world.
- Support mathematics learning (motivation, concept formation, comprehension, retaining).
- Contribute to the development of various mathematical competencies and appropriate attitudes.
- Contribute to an adequate picture of mathematics.

By modelling, mathematics becomes more meaningful for learners (this is, of course, not the only possibility for that). Underlying all these justifications of modelling are the main goals of mathematics teaching in secondary schools (Niss 1996). The goals correspond to different *perspectives* on modelling in the sense of Kaiser et al. (2006). For realising these goals and, in particular, developing modelling competency with students, a large variety of modelling tasks has to be treated.

There is a tendency in several countries to include more mathematical modelling in the curriculum. In Germany, for instance, mathematical modelling is one of six compulsory competencies in the new national "Education Standards" for mathematics. However, in everyday mathematics teaching in most countries, there is still only little modelling. Mostly "word problems" are treated where, after "undressing" the given context, the essential aim is exercising mathematics. For competency development and for learning support also word problems are legitimate and helpful; it is only important to be honest about the true nature of reality-oriented tasks and problems. However, word problems are not at all sufficient for fulfilling all goals intended with modelling. Why is the situation in schools like this, why are there only so few modelling examples in everyday classrooms, why do we find such a gap between the educational debate (and even official curricula), on the one hand, and classroom practice, on the other hand? The main reason is certainly that modelling is difficult also for teachers; as real-world knowledge is needed, teaching becomes more open and less predictable, and all the competencies required from students have, of course, to be acquired by the teachers themselves (see, e.g., Burkhardt 2004; DeLange 1987; Freudenthal 1973; Ikeda 2007; Pollak 1979).

2 How Do Students Deal with Modelling Tasks?

Studies such as PISA (see, e.g., OECD 2005, 2007) have shown several times: modelling tasks are difficult for students all around the world. Analyses carried out by the PISA Mathematics Expert Group (see Turner et al. in press) have shown that the difficulty of modelling tasks can be substantially explained by the inherent cognitive complexity of these tasks, measured by the necessary competencies.

All potential cognitive barriers are empirically observable, specific for individual tasks and individual students (see also, e.g., Galbraith and Stillman 2006). In the following, I will show some typical examples of *students' difficulties* with modelling tasks, taken from DISUM studies.

- *Step 1 constructing*: See the introductory example "Giant's shoes"; this is an instance of the well-known superficial solution strategy "Ignore the context, just extract all data from the text and do something with these according to a familiar schema" which in everyday classrooms is very often successful for solving word problems (for impressive examples of this strategy, see Baruk 1985 or Verschaffel et al. 2000).
- *Step 2 simplifying*: This is an authentic solution of example 2 "Filling up": "*You cannot know if it is worthwhile since you don't know what the Golf consumes. You also don't know how much she wants to fill up*". Obviously, the student has constructed an appropriate situation model, but he is not used to making assumptions.

The next few examples of difficulties relate to a third modelling example.

Example 3: "Fire-brigade"

Die Münchner Feuerwehr hat sich im Jahr 2004 ein neues Drehleiter-Fahrzeug angeschafft. Mit diesem kann man über einem am Ende der Leiter angebrachten Korb Personen aus großen Höhen retten. Dabei muss das Feuerwehrauto laut einer Vorschrift 12 m Mindestabstand vom brennenden Haus einhalten.

Technical data of the engine:

Fahrzeugtyp:	Daimler Chrysler AG Econic 18/28 LL - Diesel
Baujahr:	2004
Leistung:	205kw (279 PS)
Hubraum:	6374 cm³
Maße des Fahrzeug:	Länge 10m Breite 2,5m Höhe 3,19m
Maße der Leiter:	30m Länge
Leergewicht:	15540kg
Gesamtgewicht:	18000 kg

From which maximal height can the Munich fire-brigade rescue persons with this engine?

- *Step 3 mathematising*: Often, after a successful construction of a real model of the problem situation in "fire-brigade", students forget to include the height of the engine into their model.
- Step 4, the *intra-mathematical part*, may, of course, be arbitrarily difficult. Step 5 is usually less difficult; here is an example:
- *Step 5 interpreting*: After correctly carrying out the first three modelling steps and successfully applying Pythagoras' theorem, a student's final answer was "*The ladder is 27.49 m long if it is extended"*. Apart from the meaningless accuracy and the usual mistake of ignoring the engine's height, the student has obviously forgotten what his calculation actually meant.
- *Step 6 validating*: The introductory example "Giant's shoes" also provides an example of a missing validation since it is obvious that someone has to be more than only two-and-a-half times as tall as his shoe length (or can giants look like this?).

Particularly interesting are students' individual modelling routes during the process of solving modelling tasks. The notion of *modelling route* (Borromeo Ferri 2007) is used to describe a specific modelling process in detail, referring to the various steps of the modelling cycle (with the above model of the modelling cycle as a powerful analytical instrument). As Borromeo Ferri's analyses have shown, all these steps can actually be observed, though generally not in the same linear order (for detailed analyses of modelling processes, see also Leiß 2007 and Matos and Carreira 1997). There seem to be preferences of students for working more within mathematics or more within reality, depending on the individual *thinking styles* (for this notion, see Borromeo Ferri 2004); details are reported in Borromeo Ferri and Blum (2010).

Seeing students successfully performing certain modelling steps and having difficulties with other steps points again to the supposition that these steps correspond to sub-competencies of a global modelling competency. It is a particularly challenging open research question to establish a theoretically and empirically based *competence model* for mathematical modelling. Essential parts of such a model will be to identify distinct sub-competencies, to differentiate between various cognitive levels of such sub-competencies, and to set up connections between sub-competencies, modelling competency as a whole and other competencies such as reading. The proficiency levels identified in the context of PISA mathematics can be interpreted as a first attempt towards such a competence model (see OECD 2005, p. 260 ff). Another attempt was made in the context of the German Education Standards. Roughly speaking, the following five levels were identified:

- Applying simple standard models.
- Direct modelling from familiar contexts.
- Few-step modelling.
- Multi-step modelling.
- Complex modelling or evaluating models.

In the following, I will mention some more empirical findings concerning students' dealing with modelling tasks. An important observation is related to strategies. In most cases, there is no conscious use of *problem-solving strategies* by students.

This explains many of the observed difficulties since it is known from several studies that strategies (meta-cognitive activities) are helpful also for modelling (Burkhardt and Pollak 2006; Kramarski et al. 2002; Matos and Carreira 1997; Schoenfeld 1994; Stillman and Galbraith 1998; Tanner and Jones 1993) for an overview see Greer and Verschaffel in Blum and Leiß 2007). To put it more sharply: There are many indications that meta-cognitive activities are not only helpful but even necessary for the development of modelling competency. Indispensable for this to happen is an appropriate support by the teacher (see Sect. 3).

Another important result concerns the transfer of knowledge. We know from several studies in the frame of *situated cognition* that learning is always dependent on the specific learning context, and hence a simple transfer from one situation to others cannot be expected (Brown et al. 1989; De Corte et al. 1996; Niss 1999). This holds for the learning of mathematical modelling in particular, so modelling has to be learnt specifically. Therefore, a sufficiently broad variation of contexts (real-world situations as well as mathematical domains) by the teacher is necessary, as well as making transfers between situations and domains explicitly conscious for students.

A global remark: Several studies have shown that mathematical modelling *can* be learnt in certain environments, in spite of all the difficulties associated with the teaching and learning of modelling (Abrantes 1993; Galbraith and Clathworthy 1990; Kaiser-Messmer 1987; Maaß 2007; see also Sect. 4). The decisive variable for successful teaching seems to be "quality teaching." This will be addressed in the next section.

3 How Do Teachers Treat Modelling in the Classroom?

Concerning mathematics teaching and learning, the perhaps most important finding is one that may sound rather trivial but is not at all trivial (Antonius et al. 2007; Pauli and Reusser 2000): Teachers are indispensable, there is a fundamental distinction between students working independently with teacher's support and students working alone. Meta-analyses (e.g., Lipowsky 2006) have shown that teachers really matter a lot for students' mathematics learning, more than other variables such as class size or type of school. What makes the difference is, of course, the way of teaching. There is extensive empirical evidence that teaching effects can at most be expected on the basis of *quality mathematics teaching*. What could that mean? Here is the working definition we use in DISUM (compare, e.g., Blum and Leiß 2008):

- A *demanding orchestration* of teaching the mathematical subject matter (by giving students vast opportunities to acquire mathematical competencies and making connections within and outside mathematics).
- Permanent *cognitive activation* of the learners (by stimulating cognitive and meta-cognitive activities, fostering students' independence and handling mistakes constructively).
- An effective and learner-oriented *classroom management* (by varying methods flexibly, using time effectively, separating learning and assessment, etc.).

For quality teaching, it is crucial that a permanent balance between (minimal) teacher guidance and (maximal) students' independence is maintained, according to Maria Montessori's famous hundred-year-old maxim: "Help me to do it by myself" (see the "principle of minimal support", Aebli 1985). In particular, when students are dealing with mathematical tasks, this balance can be achieved best by individual, adaptive, independence-preserving teacher interventions. In a modelling context, often strategic interventions are most adequate, that means interventions which give hints to students on a meta-level ("Imagine the real situation clearly!", "Make a sketch!", "What do you aim at?", "How far have you got?", "What is still missing?", "Does this result fit to the real situation?", etc.). In everyday mathematics teaching, those quality criteria are often violated. In particular, teachers' interventions are mostly not independence-preserving, and there is nearly no stimulation of students' solution strategies.

Learning environments for modelling are generated by appropriate modelling tasks in a general sense. Here are a few well-tried proposals from literature:

- "Sense-making by meaningful tasks" (Freudenthal 1973; Verschaffel et al. 2000).
- "Model-eliciting activities" by challenging tasks (Lesh and Doerr 2003).
- "Authentic tasks" (Kaiser and Schwarz 2010; Palm 2007).

And more generally (in the words of Alsina 2007): "Less chalk, less words, less symbols – more objects, more context, more actions". Often helpful in such modelling contexts are suitable technological aids (Henn 2007).

Classroom observations (see, e.g., Leikin and Levav-Waynberg 2007) show that the teacher's own favourite solution of a given task is often imposed on the students through his interventions, mostly without even noticing it, also due to an insufficient knowledge of the richness of the "task space" on the teacher's side. However, we know that it is important to encourage various individual solutions (Hiebert and Carpenter 1992; Krainer 1993; Schoenfeld 1988), also to match different thinking styles of students, and particularly as a basis for retrospective reflections after the students' presentations. To this end, it is necessary for teachers to have an intimate knowledge of the cognitive demands of given tasks. In the project COACTIV (see Krauss et al. 2008), we have found that the teacher's ability to produce multiple solutions of tasks is one significant predictor of his students' achievement gains.

More generally, the following elements are necessary for teachers to treat modelling adequately:

- Knowledge of task spaces of modelling tasks (including cognitive demands of tasks and own preferences for special solutions).
- Knowledge of a broad spectrum of tasks, also for assessment purposes (concerning assessment see, e.g., Haines and Crouch 2001; Houston 2007; Niss 1993; Vos 2007).
- Ability to diagnose students' difficulties during modelling processes.
- Knowledge of a broad spectrum of intervention modes (Leiß 2007) and ability to use appropriate interventions.
- Appropriate beliefs (Kaiser and Maaß 2007).

(compare also Doerr 2007). It is an interesting open research question in which elements of *teachers' competencies* precisely are necessary and how these elements contribute to successful teaching.

4 Some Ideas for Teaching Modelling

There is, of course, no general "king's route" for teaching modelling. However, some implications of the findings reported in Sects. 2 and 3 are plausible (not spectacular but not at all trivial!).

Implication 1: The criteria for quality teaching (see Sect. 3) have to be considered also for teaching modelling; teachers ought to realise a permanent balance between students' independence and their guidance, in particular by their flexible and adaptive interventions.

Implication 2: In order to reach the goals associated with modelling, a broad spectrum of tasks ought to be used for teaching and for assessment, covering various topics, contexts, (sub-)competencies and cognitive levels.

Implication 3: Teachers ought to support students' individual modelling routes and encourage multiple solutions.

Implication 4: Teachers ought to foster adequate student strategies for solving modelling tasks and stimulate various meta-cognitive activities, especially reflections on solution processes and on similarities between different situations and contexts.

A few more reflections on Implication 4. For modelling tasks, a specific strategic tool is fortunately available, the modelling cycle. The seven-step schema (presented in Sect. 1) is appropriate and even indispensable for research and teaching purposes. For students, the following four-step schema (developed in the DISUM project) called *Solution Plan* is certainly more appropriate:

Step 1. *Understanding task* (Read the text precisely and imagine the situation clearly! What is required from you? Make a sketch!).

Step 2. *Searching mathematics* (Look for the data you need; if necessary, make assumptions! Look for mathematical relations!).

Step 3. *Using mathematics* (Use appropriate mathematical procedures!).

Step 4. *Explaining result* (Round off and link the result to the task! Is your result reasonable? If not, go back to 1! If yes, write down your final answer!).

As can be seen, steps 2 and 3 from the seven-step schema (Fig. 3.1) are united to one step here (step 2), and the same holds for steps 5, 6 and 7 of the seven-step schema (step 4 here). There are some structural similarities of this "Solution Plan" for modelling tasks to George Polya's famous general problem-solving cycle (compare Polya 1957), but this plan is more specific because it is conceived only for modelling tasks. The Solution Plan is not meant as a schema that has to be used by students but as an aid for difficulties that might occur in the course of the solution process. The goal is that students learn to use this plan independently whenever appropriate. Recent experiences have shown that a careful and stepwise introduction of this plan is necessary, as well as repeated exercises in how to use it. If this is taken into account, even students from Hauptschule (the low ability track in the German school system) are able to successfully handle this plan. However, a systematic study into the effects of the Solution Plan is still to be carried out (and is planned for 2011). A related approach is the use of "Worked-out Examples" (for details, see Zöttl et al. this volume).

Finally, I will present some more encouraging empirical results from the DISUM project. We have developed a so-called *operative-strategic* teaching unit for modelling (for grades 8/9, embedded in the unit on the Pythagorean Theorem). The essential guiding principles for this teaching unit were:

- Teaching aiming at students' active and independent knowledge construction (realising the balance between teacher's guidance and students' independence).
- Systematic change between independent work in groups (coached by the teacher) and whole-class activities (especially for comparison of different solutions and retrospective reflections).
- Teacher's coaching based on concrete four-step solutions for all tasks and on individual diagnoses (students did *not* have the Solution Plan, in order to keep the number of variables small enough).

In autumn 2006 (4 classes) and in autumn 2007 (21 classes), we have compared the effects of this "operative-strategic" teaching with a so-called *directive* teaching and with students working totally *alone*, both concerning students' achievement and attitudes. The most important guiding principles for "directive" teaching were:

- Development of common solution patterns by the teacher.
- Systematic change between whole-class teaching, oriented towards a fictive "average student", and students' individual work in exercises.

The students working alone came from those 18 classes that were reduced to 16 learners in advance by means of a standardised mathematical ability test, in order to homogenise the classes for better comparability. Both "operative-strategic" and "directive" teaching were conceived as optimised teaching styles and realised by experienced teachers from a reform project ("SINUS", see Blum and Leiß 2008). All teachers were particularly trained for this purpose. All classes came from Realschule (the medium ability track in Germany). Our study had a classical design (see Fig. 3.3):

Ability test/Pre-test/Treatment (10 lessons with various modelling tasks, including "Filling up" and "Fire-brigade") with accompanying questionnaires/Post-test/Follow-up test (3 months later).

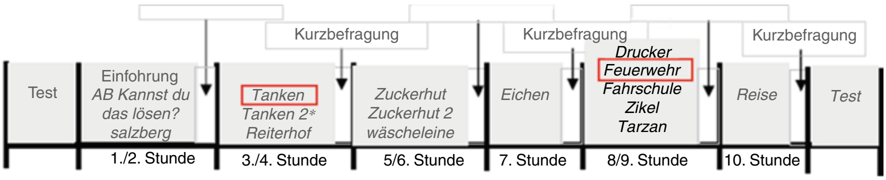

Fig. 3.3 Course of the teaching unit with modelling tasks

In all groups and teaching styles, the same modelling tasks were treated in the same order (see Fig. 3.3).

The tests comprised both modelling tasks and classical mathematical tasks close to the curriculum. According to our knowledge, this study was unique insofar as it was a quasi-experimental study with more than 600 students yielding both quantitative (tests and questionnaires) and qualitative (videos) data. Since two optimised teaching styles were implemented, one could possibly expect no differences between the two treatments concerning students' achievement and attitudes. However, there were remarkable differences. In the following, some important results are reported (more details will be presented in another paper).

Most remarkably: Both students' in "operative-strategic" and in "directive" classes made significant progress (.45 resp. .25 SD), but not so students working alone. The difference in progress was also significant, in favour of the more independence-oriented teaching style, and the progress of these classes was also more enduring than the progress of "directive" classes. The progress of "directive" students was essentially due to their progress in the technical "Pythagorean" tasks. Only "operative-strategic" students made significant progress in their modelling competency. The best results were achieved in those classes where, according to our ratings, the balance between students' independence and teacher's guidance was realised best, with a mixture of different kinds of adaptive interventions and, most importantly to note, with a clear emphasis on meta-cognitive activities (according to Implication 4 above).

However, from a normative point of view, these results are still rather disappointing: The progress after ten hours of teachers' big efforts to train students in modelling is only less than half one standard deviation. In fact, there is a big potential for improving the design:

- Solution Plan for students as well.
- Directive phases also as part of the independence-oriented design, especially in the beginning (teacher as a "model modeller" according to "cognitive apprenticeship").
- More time for practising sub-competencies.

It is the intention of future phases of the DISUM project to investigate these aspects in more detail.

What do these results tell us about the question in the title of this chapter: Can modelling be taught and learnt? The global answer is: There are several indications that modelling *can* be taught and learnt, provided some basic quality principles are fulfilled. Although the teaching units designed so far worldwide can certainly still be improved considerably, we should not wait for future studies before we begin to implement the reported insights into everyday classrooms as well as into teacher education (Lingefjaerd 2007). At the same time, there should be more research since there are still a lot of open questions (compare the lists of research questions in Blum et al. 2002; DaPonte 1993; Niss 2001), among many others the following:

- How can technological devices be appropriately used for developing modelling competency?
- What do competence models for modelling look like?
- Modelling competency has to be built up in long-term learning processes. What is actually achievable regarding long-term competency development?
- How can the interplay between modelling and other competencies be advanced systematically?

Particularly, the final question points to the ultimate goal of mathematics teaching: a comprehensive mathematical education of all students.

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