Chapter 5
Two-Sided Fractional Derivatives

5.1 Motivation

In previous chapters the causal and anti-causal fractional derivatives were presented. An application to shift-invariant linear systems was studied. Those derivatives were introduced into four steps:

1. Use as starting point the Grünwald–Letnikov differences and derivatives.
2. With an integral formulation for the fractional differences and using the asymptotic properties of the Gamma function obtain the generalised Cauchy derivative.
3. The computation of the integral defining the generalised Cauchy derivative is done with the Hankel path to obtain regularised fractional derivatives.
4. The application of these regularised derivatives to functions with Laplace transform, we obtain the Liouville fractional derivative and from this the Riemann–Liouville and Caputo, two-step derivatives.

Here we will repeat the procedures for the centred (two-sided) derivatives. As we enhanced in Chap. 2, the GL derivative and those obtained from it impose preferable directions of the independent variable. We said there that the forward derivative was causal. However, there are many physical space dependent phenomena without any privilegiate direction. This means that we need a derivative suitable for these situations. To motivate the appearance of another derivative we are going to consider the following problem: which is the autocorrelation of the output of a fractional differintegrator when the input is white noise?

Assume that $x(t)$ is a stationary white noise process with $\sigma^2\delta(t)$ as its autocorrelation function. The autocorrelation of the output is given by:

$$R_X^{\alpha}(t_1, t_2) = \lim_{h\to 0+} \frac{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{\alpha}{k}(-1)^{k-n}\binom{\alpha}{n}R_f[t_1 - t_2 - (k-n)h]}{h^{2\alpha}}$$

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With a change in the summation variable, it is not hard to show that,

$$R_X^{\alpha}(t_1, t_2) = R_X^{\alpha}(t_1 - t_2) = \sigma^2 \lim_{h \to 0+} \frac{\sum_{k=-\infty}^{\infty} R_{\alpha}(n)\delta[t_1 - t_2 - nh]}{h^{2\alpha}} \tag{5.1}$$

where $R_{\alpha}(n)$ is the discrete autocorrelation of the binomial coefficient sequence

$$R_{\alpha}(n) = \sum_{i=0}^{\infty} h_i \cdot h_{i+n} \tag{5.2}$$

with

$$h_n = (-1)^n \binom{\alpha}{n} u_n \tag{5.3}$$

The computation of its autocorrelation function is slightly involved. Inserting (5.3) into (5.2), we obtain

$$R_{\alpha}(n) = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} (-1)^{i+k} \binom{\alpha}{i+n} \quad n \geq 0 \tag{5.4}$$

or

$$R_{\alpha}(n) = (-1)^k \sum_{i=0}^{\infty} \binom{\alpha}{i} \binom{\alpha}{i+n} \quad n \geq 0 \tag{5.5}$$

Let us introduce the Gauss Hypergeometric function [1]

$${}_2F_1(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c_k)} \frac{z^k}{\cdot k!} \tag{5.6}$$

where $c \neq 0, -1, -2, \ldots$ and $(a)_k$ is the Pochhammer symbol. The series (5.6) is convergent for $|z| \leq 1$, if $c - a - b > 0$.[1]

As:

$$\binom{\alpha}{i} = \frac{(-1)^i (-\alpha)_i}{i!} \tag{5.7}$$

and attending to

$$(i + k)! = (i + 1)_k i! \tag{5.8}$$

and

[1] If $0 < b < c$ and $|\arg(1 - z)| < \pi$, that function can be represented by the Euler integral: ${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b) \cdot \Gamma(c-b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - zt)^{-a} \mathrm{d}t$

$$(-\alpha)_{i+k} = (-\alpha)_i(-\alpha+i)_k \tag{5.9}$$

we obtain

$$R_\alpha(n) = (-1)^n \cdot \binom{\alpha}{n} \cdot {}_2F_1(\alpha, -\alpha+n, n+1, 1) \quad n \geq 0 \tag{5.10}$$

Using the Gauss relation:

$${}_2F_1(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad c-a-b>0 \tag{5.11}$$

we obtain after some simple manipulations:

$$R_\alpha(n) = (-1)^k \frac{\Gamma(1+2\alpha)}{\Gamma(\alpha+n+1)\Gamma(\alpha-n+1)} \tag{5.12}$$

that is an even function as expected. For $n = 0$, we obtain the power:

$$P_\alpha = \frac{\Gamma(1+2\alpha)}{[\Gamma(\alpha+1)]^2} \tag{5.13}$$

that is positive if $1+2\alpha > 0$, or $\alpha > -1/2$. This means that, only for those values, we may be led to a stationary stochastic process. If $\alpha = -1/2$, the process has an infinite power and can be considered as wide sense stationary. With this procedure we arrive at:

$$R_X^\alpha(\tau) = \lim_{h\to 0} \frac{\Gamma(2\alpha+1)}{h^{2\alpha}} \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{\Gamma(\alpha-k+1)\Gamma(\alpha+k+1)} \delta(\tau-kh) \tag{5.14}$$

So, the autocorrelation function of the forward α-order derivative of white noise suggests us the introduction of a new (centred) derivative similar to the GL derivative but that is two-sided in the sense of using past and future. We will proceed accordingly to the following steps:

1. Introduction of the general framework for the central (two-sided) differences, considering two cases that we will be called type 1 and type 2 differences. These are generalisations of the usual central differences for even and odd positive orders respectively.
2. Limit computation as in the usual Grünwald–Letnikov derivatives.
3. For those differences, suitable integral representations were introduced. From these representations we can obtain the derivative integral formulations by using the properties of the Gamma function. The integration is performed over two infinite lines that "close at infinite" to form a closed path. Two generalisations of the usual Cauchy derivative definition are obtained that agree with it when α is an even or an odd positive integer, respectively.
4. The computation of those integrals over a two straight lines path leads to generalisations of the Riesz potentials.

5. The most interesting feature of the obtained relations lies in the summation formulae for the Riesz potentials.

We will test the coherence of the proposed framework by applying them to the complex exponential. The results show that they are suitable for functions with Fourier transform. The formulation agrees also with Okikiolu [2] studies. Special cases are studied and some properties presented.

5.2 Integer Order Two-Sided Differences and Derivatives

We introduce Δ_c as finite two-sided (centred) difference defined by

$$\Delta_c f(t) = f(t + h/2) - f(t - h/2) \tag{5.15}$$

By repeated application, we have:

$$\Delta_e^N f(z) = \sum_{k=-N/2}^{N/2} (-1)^{N/2-k} \frac{N!}{(N/2+k)!(N/2-k)!} f(t - kh) \tag{5.16}$$

when N is even, and

$$\Delta_o^N f(z) = \sum_{k=-N/2}^{N/2^*} (-1)^{N/2-k} \frac{N!}{(N/2+k)!(N/2-k)!} f(t - kh) \tag{5.17}$$

if N is odd and where the $\sum_{k=-N/2}^{N/2^*}$ means that the summation is done over half-integer values. Using the Gamma function, we can rewrite the above formulae in the format stated as follows.

Definition 5.1 Let N be a positive even integer. We define a centred difference by:

$$\Delta_e^N f(t) = (-1)^{N/2} \sum_{k=-N/2}^{N/2} (-1)^k \frac{\Gamma(N+1)}{\Gamma(N/2+k+1)\Gamma(N/2-k+1)} f(t - kh) \tag{5.18}$$

Definition 5.2 Let N be a positive odd integer. We define a two-sided difference by:

$$\Delta_o^N f(t) = (-1)^{(N+1)/2} \sum_{k=-(N-1)/2}^{(N+1)/2} \times (-1)^k \frac{\Gamma(N+1)}{\Gamma((N+1)/2-k+1)\Gamma((N-1)/2+k+1)} f(t - kh + h/2) \tag{5.19}$$

with these definitions we are able to define the corresponding derivatives.

Definition 5.3 Let N be a positive even integer. We define an even order two-sided derivative by:

$$D_e^N f(t) = \lim_{h\to 0} \frac{\Delta_e^N f(t)}{h^N}$$
$$= \lim_{h\to 0} \frac{(-1)^{N/2}}{h^N} \sum_{k=-N/2}^{N/2} (-1)^k \frac{\Gamma(N+1)}{\Gamma(N/2+k+1)\Gamma(N/2-k+1)} f(t-kh) \tag{5.20}$$

Definition 5.4 Let N be a positive odd integer. We define an odd order two-sided derivative by:

$$D_o^N f(t) = \lim_{h\to 0} \frac{\Delta_o^N f(t)}{h^N} = \lim_{h\to 0} \frac{(-1)^{(N+1)/2}}{h^N}$$
$$\sum_{k=-(N-1)/2}^{(N+1)/2} (-1)^k \frac{\Gamma(N+1)}{\Gamma((N+1)/2-k+1)\Gamma((N-1)/2+k+1)} f(t-kh+h/2) \tag{5.21}$$

Both derivatives (5.20) and (5.21) coincide with the usual derivative Nth order derivative.

5.3 Integral Representations for the Integer Order Two-Sided Differences

The result stated in (5.20) can be interpreted in terms of the residue theorem leading to an integral representation for the difference. Assume that $f(z)$ is analytic inside and on a closed integration path that includes the points $t = z - kh$, $h \in C$, with $k = -N/2, -N/2 + 1, \ldots, -1, 0, 1, \ldots, N/2 - 1, N/2$. Then

$$\Delta_e^N f(z) = \frac{(-1)^{N/2} N!}{2\pi i h} \int_{C_c} f(z+w) \frac{\Gamma\left(\frac{-w}{h}+1\right)}{\Gamma\left(\frac{-w}{h}+\frac{N}{2}+1\right)} \frac{\Gamma\left(\frac{w}{h}\right)}{\Gamma\left(\frac{w}{h}+\frac{N}{2}+1\right)} \mathrm{d}w \tag{5.22}$$

To prove this result, remark that Eq. 5.20 can be considered as $1/2\pi i \sum$ residues in the computation of the integral of a function with poles at $t = z - kh$ (Fig. 5.1).

We can make a translation and consider poles at kh. As it can be seen by direct verification, we have

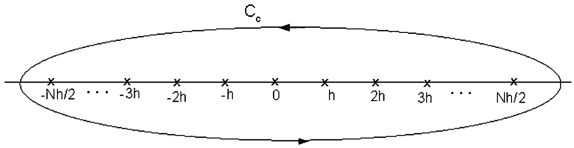


Fig. 5.1 Integration path and poles for the integral representation of integer even order differences

$$\sum_{k=-N/2}^{N/2} \frac{N!(-1)^{N/2-k}}{(N/2+k)!(N/2-k)!} f(t-kh)$$
$$= \frac{N!}{2\pi i h} \int\limits_{C_c} \frac{f(z+w)}{\prod_{k=0}^{N/2}\left(\frac{w}{h}-k\right)\prod_{k=1}^{N/2}\left(\frac{w}{h}+k\right)} \mathrm{d}w \qquad (5.23)$$

Introducing the Pochhammer symbol, we can rewrite the above formula as:

$$\Delta_e^N f(z) = \frac{(-1)^{N/2} N!}{2\pi i h} \int\limits_{C_c} \frac{f(z+w)\left(\frac{-w}{h}\right)}{\left(\frac{w}{h}\right)_{N/2+1}\left(\frac{-w}{h}\right)_{N/2+1}} \mathrm{d}w \qquad (5.24)$$

Attending to the relation between the Pochhammer symbol and the Gamma function:

$$\Gamma(z+n) = (z)_n \Gamma(z) \qquad (5.25)$$

we can write (5.22).

It is easy to test the coherency of (5.22) relatively to (5.20), by noting that the Gamma function $\Gamma(z)$ has poles at the negative integers ($z = -n$, $n \in Z^+$). The corresponding residues are equal to $(-1)^n/n!$. Both the Gamma functions have infinite poles, but outside the integration path they cancel out and the integrand is analytic.

Similarly to the above development, we have[2]:

$$\Delta_o^N f(z) = \frac{(-1)^{(N+1)/2} N!}{2\pi i h} \int\limits_{C_c} f(z+w) \frac{\Gamma\left(-\frac{w}{h}+\frac{1}{2}\right) \quad \Gamma\left(\frac{w}{h}+\frac{1}{2}\right)}{\Gamma\left(\frac{-w}{h}+\frac{N}{2}+1\right)\Gamma\left(\frac{w}{h}+\frac{N}{2}+1\right)} \mathrm{d}w \qquad (5.26)$$

To prove this, we proceed as above. By direct verification, we have

$$\sum_{k=-N/2}^{N/2} (-1)^{N/2-k} \binom{N}{N/2-k} f(z-kh)$$
$$= \frac{N!}{2\pi i h} \int\limits_{C_c} \frac{f(z+w)}{\prod_{k=0}^{(N-1)/2}\left(\frac{w}{h}-k-\frac{1}{2}\right)\prod_{k=1}^{(N-1)/2}\left(\frac{w}{h}+k+\frac{1}{2}\right)} \mathrm{d}w \qquad (5.27)$$

and

[2] Figure 5.2 shows the integration path and corresponding poles.

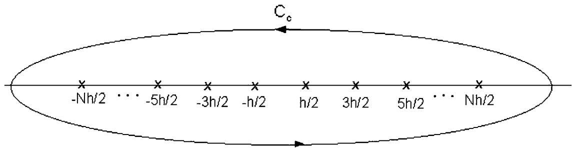


Fig. 5.2 Integration path and poles for the integral representation of integer odd order differences

$$\Delta_o^N f(z) = \frac{(-1)^{(N+1)/2} N!}{2\pi i h} \int\limits_{C_c} \frac{f(z+w)}{\left(\frac{w}{h}+\frac{1}{2}\right)_{(N+1)/2} \left(-\frac{w}{h}+\frac{1}{2}\right)_{(N+1)/2}} \mathrm{d}w \tag{5.28}$$

that leads immediately to (5.26)

5.4 Fractional Central Differences

We are going to consider two types of fractional central differences. Let $\alpha > -1$, $h \in R^+$ and $f(t)$ a Fig. 5.2 complex variable function.

Definition 5.5 We define a type 1 fractional difference by:

$$\Delta_{c_1}^\alpha f(t) = \sum_{-\infty}^{+\infty} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\alpha/2 - k + 1)\Gamma(\alpha/2 + k + 1)} f(t - kh) \tag{5.29}$$

Definition 5.6 We define a type 2 fractional difference by[3]:

$$\Delta_{c_2}^\alpha f(t) = \sum_{-\infty}^{+\infty} \frac{(-1)^k \Gamma(\alpha+1) f(t - kh + h/2)}{\Gamma[(\alpha+1)/2 - k + 1]\Gamma[(\alpha-1)/2 + k + 1]} \tag{5.30}$$

Remark that we did not insert any power of (-1). Although it may be useful in some problems to keep it, we found better to remove it due to the relation with the Riesz potentials that we will obtain later.

With the following relation [3][4]:

$$\sum_{-\infty}^{+\infty} \frac{1}{\Gamma(a-k+1)\Gamma(b-k+1)\Gamma(c+k+1)\Gamma(d+k+1)}$$
$$= \frac{\Gamma(a+b+c+d+1)}{\Gamma(a+c+1)\Gamma(b+c+1)\Gamma(a+d+1)\Gamma(b+d+1)} \tag{5.31}$$

valid for $a + b + c + d > -1$, it is not very hard to show that:

[3] Here we assume that α is also non zero.

[4] See page 123.

$$\Delta_{c_1}^{\beta}\{\Delta_{c_1}^{\alpha} f(t)\} = \Delta_{c_1}^{\alpha+\beta} f(t) \tag{5.32}$$

and

$$\Delta_{c2}^{\beta}\{\Delta_{c_2}^{\alpha} f(t)\} = -\Delta_{c_1}^{\alpha+\beta} f(t) \tag{5.33}$$

while

$$\Delta_{c_2}^{\beta}\{\Delta_{c_1}^{\alpha} f(t)\} = \Delta_{c_2}^{\alpha+\beta} f(t) \tag{5.34}$$

provided that $\alpha + \beta > -1$. In particular, $\alpha + \beta = 0$, and the relations (5.32) and (5.33) show that when $|\alpha| < 1$ and $|\beta| < 1$ the inverse differences exist and can be obtained by using formulae (5.29) and (5.30). We must remark that the zero order type 1 difference is the identity operator and is obtained from (5.29). The zero order type 2 difference will be considered later.

5.5 Integral Representations for the Fractional Two-Sided Differences

Let us assume that $f(z)$ is analytic in a region of the complex plane that includes the real axis. To obtain the integral representations for the previous differences we follow here the procedure used in Chap. 3. We only have to give interpretations to (5.29) and (5.30) in terms of the residue theorem. For the first case, we must remark that the poles must lie at nh, $n \in Z$. This leads easily to

$$\Delta_{C_1}^{\alpha} f(t) = \frac{\Gamma(\alpha+1)}{2\pi i j h} \int_{C_c} f(z+w) \frac{\Gamma\left(\frac{-w}{h}+1\right) \qquad \Gamma\left(\frac{w}{h}\right)}{\Gamma\left(\frac{-w}{h}+\frac{\alpha}{2}+1\right)\Gamma\left(\frac{w}{h}+\frac{\alpha}{2}+1\right)} \mathrm{d}w \tag{5.35}$$

The integrand function has infinite poles at every nh, with $n \in Z$. The integration path must consist of infinite lines above and below the real axis closing at the infinite. The easiest situation is obtained by considering two straight lines near the real axis, one above and the other below (see Fig. 5.3).

Regarding to the second case, the poles are located now at the half integer multiples of h, which leads to

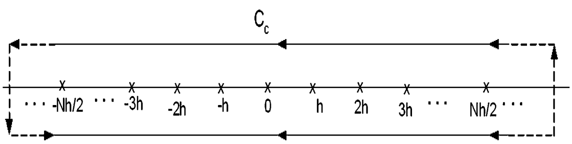


Fig. 5.3 Path and poles for the integral representation of type 1 differences

$$\Delta_{C_2}^{\alpha} f(t) = \frac{\Gamma(\alpha+1)}{2\pi i j h} \int_{C_c} f(z+w) \frac{\Gamma\left(-\frac{w}{h}+\frac{1}{2}\right)}{\Gamma\left(\frac{-w}{h}+\frac{\alpha}{2}+1\right)} \frac{\Gamma\left(\frac{w}{h}+\frac{1}{2}\right)}{\Gamma\left(\frac{w}{h}+\frac{\alpha}{2}+1\right)} \mathrm{d}w \tag{5.36}$$

These integral formulations will be used in the following section to obtain the integral formulae for the central derivatives generalising the Cauchy derivative for the two-sided case. We could consider the poles as lying over any straight line as we did with the forward and backward cases in Chap. 3. However, this may not be very important. So we will work over the real axis.

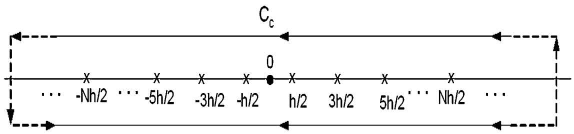


Fig. 5.4 Path and poles for the integral representation of type 2 differences

5.6 The Fractional Two-Sided Derivatives

To obtain fractional central derivatives we proceed as usually [4–8]: divide the fractional differences by h^{α} ($h \in R^{+}$) and let $h \to 0$. For the first case and assuming again that $\alpha > -1$, we obtain:

$$\begin{aligned} D_{c_1}^{\alpha} f(t) &= \lim_{h \to 0} \frac{\Delta_{c_1}^{\alpha} f(t)}{h^{\alpha}} \\ &= \lim_{h \to 0} \frac{\Gamma(\alpha+1)}{h^{\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)^k}{\Gamma(\alpha/2-k+1)\Gamma(\alpha/2+k+1)} f(t-kh) \end{aligned} \tag{5.37}$$

that we will call type 1 two-sided fractional derivative.

For the second case and assuming also that $\alpha \neq 0$, we obtain the type 2 two-sided fractional derivative given by

$$\begin{aligned} D_{c_2}^{\alpha} f(t) &= \lim_{h \to 0} \frac{\Delta_{c_2}^{\alpha} f(t)}{h^{\alpha}} \\ &= \lim_{h \to 0} \frac{\Gamma(\alpha+1)}{h^{\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)^k f(t-kh+h/2)}{\Gamma[(\alpha+1)/2-k+1]\Gamma[(\alpha-1)/2+k+1]} \end{aligned} \tag{5.38}$$

Formulae (5.37) and (5.38) generalise the positive integer order central derivatives to the fractional case, although there should be an extra factor $(-1)^{\alpha/2}$ in the first case and $(-1)^{(\alpha+1)/2}$ in the second case that we removed, as referred before.

5.7 Integral Formulae

To obtain the integral formulae for the derivatives we must substitute the integral formulae (5.35) and (5.36) into (5.37) and (5.38) respectively and permute there the limit and integral operations. With this permutation we must compute the limit of two quotients of Gamma functions. As it is well known, the quotient of two gamma functions $\frac{\Gamma(s+a)}{\Gamma(s+b)}$ has the expansion

$$\frac{\Gamma(s+a)}{\Gamma(s+b)} = s^{a-b}\left[1+\sum_{1}^{N} c_k s^{-k} + O(s^{-N-1})\right] \tag{5.39}$$

as $|s| \to \infty$, uniformly in every sector that excludes the negative real half-axis. When h is very small

$$\frac{\Gamma(w/h+a)}{\Gamma(w/h+b)} \approx (w/h)^{a-b}[1+h\cdot\varphi(w/h)] \tag{5.40}$$

where φ is regular near the origin. Accordingly to the above statement, the branch cut line used to define a function on the right hand side in (5.40) is the negative real half axis. Similarly, we

$$\frac{\Gamma(-w/h+a)}{\Gamma(-w/h+b)} \approx (w/h)^{a-b}[1+h\cdot\varphi(-w/h)] \tag{5.41}$$

but now, the branch cut line is the positive real axis. With these results, we obtain generalisations of the Cauchy integral formulation for the type 1 derivative given by

$$D^{\alpha}_{C_1} f(t) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_{C_c} f(z+w)\frac{1}{(w)_l^{\alpha/2+1}(-w)_r^{\alpha/2}}\,\mathrm{d}w \tag{5.42}$$

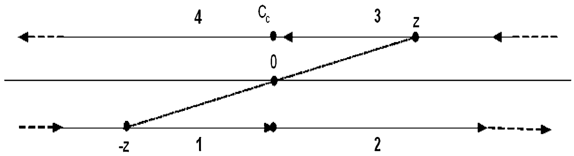
while for the type 2 derivative is

$$D^{\alpha}_{C_2} f(t) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_{C_c} f(z+w)\frac{1}{(w)_l^{(\alpha+1)/2}(-w)_r^{(\alpha+1)/2}}\,\mathrm{d}w \tag{5.43}$$

The subscripts "l" and "r" mean respectively that the power functions have the left and right half real axis as branch cut lines.

Now, we are going to compute the above integrals for the special case of straight line paths. Let us assume that the distance between the horizontal straight lines in Figs. 5.3 and 5.4 is $2\varepsilon(h)$ that decreases to zero with h. In Fig. 5.5 we show the different segments used for the computation of the above integrals. If we assume that the two straight lines are infinitely near, we have for the type 1 derivative:

Fig. 5.5 Segments for the computation of the integrals (5.42) and (5.43)



$$\int_1 = -\frac{\Gamma(\alpha+1)}{2\pi i}\int_0^\infty f(z-x)\frac{1}{x^{\alpha+1}e^{-i\alpha\pi/2}e^{-i\pi}}e^{-i\pi}\,dx$$

$$\int_2 = \frac{\Gamma(\alpha+1)}{2\pi i}\int_0^\infty f(z+x)\frac{1}{x^{\alpha+1}e^{i\alpha\pi/2}}\,dx$$

$$\int_3 = -\frac{\Gamma(\alpha+1)e^{-i\alpha\pi/2}}{2\pi i}\int_0^\infty f(z+x)\frac{1}{x^{\alpha+1}e^{-i\alpha\pi/2}}\,dx$$

$$\int_4 = \frac{\Gamma(\alpha+1)}{2\pi i}\int_0^\infty f(z-x)\frac{1}{x^{\alpha+1}e^{i\alpha\pi/2}}e^{i\pi}dx$$

where the integer numbers refer the straight-line segment used in the computation. Joining the four integrals, we obtain:

$$D^\alpha_{C_1}f(t) = -\frac{\Gamma(\alpha+1)\sin(\alpha\pi/2)}{\pi}\int_0^\infty f(z-x)\frac{1}{x^{\alpha+1}}\,dx$$
$$-\frac{\Gamma(\alpha+1)\sin(\alpha\pi/2)}{\pi}\int_0^\infty f(z+x)\frac{1}{x^{\alpha+1}}\,dx$$

or

$$D^\alpha_{C_1}f(t) = -\frac{\Gamma(\alpha+1)\sin(\alpha\pi/2)}{\pi}\int_{-\infty}^\infty f(z-x)\frac{1}{|x|^{\alpha+1}}\,dx \tag{5.44}$$

As α is not an odd integer and using the reflection formula of the gamma function we obtain

$$D^\alpha_{C_1}f(t) = -\frac{1}{2\Gamma(-\alpha)\cos(\alpha\pi/2)}\int_{-\infty}^\infty f(z-x)\frac{1}{|x|^{\alpha+1}}\,dx \tag{5.45}$$

When $-1<\alpha<0$, it is the so called Riesz potential [8], for $0<\alpha<1$, it is the corresponding inverse operator.

For the type 2 case, we compute again the integrals corresponding to the four segments to obtain:

$$\int_1 = -\frac{\Gamma(\alpha+1)}{2\pi i}\int_0^\infty f(z-x)\frac{1}{x^{\alpha+1}e^{-i(\alpha+1)\pi/2}}e^{i\pi}dx,$$

$$\int_2 = \frac{\Gamma(\alpha+1)}{2\pi i}\int_0^\infty f(z+x)\frac{1}{x^{\alpha+1}e^{i(\alpha+1)\pi/2}}dx$$

$$\int_3 = -\frac{\Gamma(\alpha+1)e^{-i\alpha\pi/2}}{2\pi i}\int_0^\infty f(z+x)\frac{1}{x^{\alpha+1}e^{-i(\alpha+1)\pi/2}}dx$$

$$\int_4 = \frac{\Gamma(\alpha+1)}{2\pi i}\int_0^\infty f(z-x)\frac{1}{x^{\alpha+1}e^{-i(\alpha+1)\pi/2}}e^{i\pi}dx$$

Joining the four integrals, we obtain:

$$D_{c_2}^{\alpha}f(t) = \frac{\Gamma(\alpha+1)\sin[(\alpha+1)\pi/2]}{\pi}\int_0^\infty f(z-x)\frac{1}{x^{\alpha+1}}dx$$
$$-\frac{\Gamma(\alpha+1)\sin[(\alpha+1)\pi/2]}{\pi}\int_0^\infty f(z+x)\frac{1}{x^{\alpha+1}}dx$$

As the last integral can be rewritten as:

$$\int_0^\infty f(z+x)\frac{1}{x^{\alpha+1}}dx = \int_{-\infty}^0 f(z-x)\frac{1}{(-x)^{\alpha+1}}dx$$

we obtain

$$D_{c_2}^{\alpha}f(t) = -\frac{1}{2\Gamma(-\alpha)\sin(\alpha\pi/2)}\int_{-\infty}^{\infty} f(z-x)\frac{\mathrm{sgn(x)}}{|x|^{\alpha+1}}dx \tag{5.46}$$

that is the modified Riesz potential [8], when $-1 < \alpha < 0$, when $0 < \alpha < 1$, it is the corresponding inverse operator. Both potentials (5.45) and (5.46) were studied also by Okikiolu [2]. These are essentially convolutions of a given function with two acausal (neither causal nor anti-causal) operators.

Letting $F(\omega)$ be the Fourier transform of $f(t)$ and, as the Fourier transform of $\frac{1}{2\Gamma(-\alpha)\cos(\alpha\pi/2)}|t|^{-\alpha-1}$ is given by $|\omega|^{\alpha}$ we conclude that:

$$F[D^{\alpha}_{c_1} f(t)] = |\omega|^{\alpha} F(\omega) \tag{5.47}$$

Similarly, as the Fourier transform of $\frac{-\mathrm{sgn}(t)}{(\alpha+1)2\Gamma(-\alpha-1)\cos[(\alpha+1)\pi/2]}|t|^{-\alpha-1}$ is given by $-j|\omega|^{\alpha}\mathrm{sgn}(\omega)$ [2], we conclude that:

$$F[D^{\alpha}_{c_2} f(t)] = -j|\omega|^{\alpha}\mathrm{sgn}(\omega)F(\omega) \tag{5.48}$$

It is interesting to use the type 1 derivative with $\alpha = 2M + 1$ and the type 2 with $\alpha = 2M$. This will be done later.

Relations (5.47) and (5.48) generalise a well known property of the Fourier transform.

5.8 Coherence of the Definitions

5.8.1 *Type 1 Derivative*

We want to test the coherence of the results by considering functions with Fourier transform. To perform this study, we only have to find the behaviour of the defined derivatives for $f(t) = \mathrm{e}^{-j\omega t}$, $t,\omega \in R$. In the following we will consider non integer orders greater than -1. We start by considering the type 1 derivative. From (5.29) we obtain

$$\Delta^{\alpha}_{c_1} \mathrm{e}^{j\omega t} = \mathrm{e}^{-j\omega t} \sum_{-\infty}^{+\infty} \frac{(-1)^n \Gamma(\alpha+1)}{\Gamma(\alpha/2 - n + 1)\Gamma(\alpha/2 + n + 1)} \mathrm{e}^{j\omega n h} \tag{5.49}$$

where we recognize the discrete-time Fourier transform of $R_b(n)$,[5] given by:

$$R_b(n) = \frac{(-1)^n \Gamma(\alpha+1)}{\Gamma(\alpha/2 - n + 1)\Gamma(\alpha/2 + n + 1)} \tag{5.50}$$

As before, this function is the discrete autocorrelation of

$$h_n = \frac{(-\alpha/2)_n}{n!} u_n \tag{5.51}$$

where u_n is the discrete unit step Heaviside function. As the binomial series is convergent over the unit circle excepting the point $z = 1$, the discrete-time Fourier transform of h_n is:

$$H(\mathrm{e}^{j\omega}) = FT[h_n] = (1 - \mathrm{e}^{-j\omega h})^{\alpha/2} \tag{5.52}$$

and the discrete-time Fourier transform of $R_b(n)$

[5] In purely mathematical terms it is a Fourier series with $R_b(n)$ as coefficients.

$$S(e^{j\omega}) = \lim_{z\to e^{j\omega h}} (1-z^{-1})^{\alpha/2}(1-z)^{\alpha/2} = (1-e^{-j\omega h})^{\alpha/2}(1-e^{j\omega h})^{\alpha/2}$$
$$= \left|e^{j\omega h/2} - e^{-j\omega h/2}\right|^{\alpha} = |2\sin(\omega h/2)|^{\alpha} \tag{5.53}$$

So,

$$|2\sin(\omega h/2)|^{\alpha} = \sum_{-\infty}^{+\infty} \frac{(-1)^n \Gamma(\alpha+1)}{\Gamma(\alpha/2-n+1)\Gamma(\alpha/2+n+1)} e^{j\omega nh} \tag{5.54}$$

We write, then:

$$\Delta_{c_1}^{\alpha} e^{-j\omega t} = e^{-j\omega t} |2\sin(\omega h/2)|^{\alpha} \tag{5.55}$$

So, there is a linear system with frequency response given by:

$$H_{\Delta 1}(\omega) = |2\ \sin(\omega h/2)|^{\alpha} \tag{5.56}$$

that acts on a signal giving its type 1 central fractional difference. Dividing (5.56) by $h^{\alpha}(h \in R^{+})$ and computing the limit as $h \to 0$, it comes:

$$H_{D1}(\omega) = |\omega|^{\alpha} \tag{5.57}$$

As α is not an even integer:

$$|\omega|^{\alpha} = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)^n \Gamma(\alpha+1)}{\Gamma(\alpha/2-n+1)\Gamma(\alpha/2+n+1)} e^{j\omega nh} \tag{5.58}$$

valid for $\alpha > -1$. The inverse Fourier Transform of $|\omega|^{\alpha}$ is given by Okikiolu [2]:

$$FT^{-1}[|\omega|^{\alpha}] = \frac{1}{2\Gamma(-\alpha)\cos(\alpha\pi/2)} |t|^{-\alpha-1} \tag{5.59}$$

and we obtain the impulse response:

$$h_{D1}(t) = \frac{1}{2\Gamma(-\alpha)\cos(\alpha\pi/2)} |t|^{-\alpha-1} \tag{5.60}$$

leading to

$$D_{C_1}^{\alpha} f(t) = \frac{1}{2\Gamma(-\alpha)\cos(\alpha\pi/2)} \int_{-\infty}^{+\infty} f(\tau)|t-\tau|^{-\alpha-1} d\tau \tag{5.61}$$

that is coincides with (5.45). Relations (5.52) and (5.53) allow us to conclude that the type 1 central derivative is equivalent to the application of the $\alpha/2$ order forward (or backward) derivative twice: one with increasing time and the other with reverse time.

5.8.2 Type 2 Derivative

A similar procedure allows us to obtain

$$\Delta_{c_2}^{\alpha} \mathrm{e}^{-j\omega t} = \mathrm{e}^{-j\omega t} \mathrm{e}^{-j\omega h/2} \sum_{-\infty}^{+\infty} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma[(\alpha+1)/2-k+1]\Gamma[(\alpha-1)/2+k+1]} \mathrm{e}^{j\omega kh} \tag{5.62}$$

In order to maintain the coherence with the usual definition of discrete-time Fourier transform, we change the summation variable, obtaining

$$\Delta_{c_2}^{\alpha} \mathrm{e}^{-j\omega t} = \mathrm{e}^{-j\omega t} \mathrm{e}^{-j\omega h/2} \sum_{-\infty}^{+\infty} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma[(\alpha+1)/2+k+1]\Gamma[(\alpha-1)/2-k+1]} \mathrm{e}^{-j\omega kh} \tag{5.63}$$

Now, the coefficients of the above Fourier series are the cross-correlation, $R_{bc}(k)$, between

$$h_n = \frac{(-a)_n}{n!} u_n \tag{5.64}$$

and

$$g_n = \frac{(-b)_n}{n!} u_n \tag{5.65}$$

with $a = (\alpha + 1)/2$ and $b = (\alpha - 1)/2$. Let $S_{bc}(\mathrm{e}^{j\omega})$ be the discrete-time Fourier transform of the cross-correlation, $R_{bc}(k)$:

$$S_{bc}(\mathrm{e}^{j\omega}) \ = \ FT[R_{bc}(k)] \tag{5.66}$$

$R_{bc}(k)$ being a correlation, we conclude easily that $S_{bc}(\mathrm{e}^{j\omega})$ is given by:

$$S_{bc}(\mathrm{e}^{j\omega}) = \lim_{z \to \mathrm{e}^{j\omega h}} (1 - z^{-1})^{(\alpha+1)/2} (1-z)^{(\alpha-1)/2} \tag{5.67}$$

$$= (1 - \mathrm{e}^{-j\omega h})^{(\alpha+1)/2} (1 - \mathrm{e}^{j\omega h})^{(\alpha+1)/2} (1 - \mathrm{e}^{j\omega h})^{-1} \tag{5.68}$$

We write, then:

$$\Delta_{c_2}^{\alpha} \mathrm{e}^{-j\omega t} = \mathrm{e}^{j\omega t} |2\sin(\omega h/2)|^{\alpha+1} [2j\sin(\omega h/2)]^{-1}$$

So, there is a linear system with frequency response given by:

$$H_{\Delta 2}(\omega) = |2\sin(\omega h/2)|^{\alpha+1} [2j\sin(\omega h/2)]^{-1} \tag{5.69}$$

that acts on a signal giving its fractional central difference. We can write also

$$|2\sin(\omega h/2)|^{\alpha+1}[2j\sin(\omega h/2)]^{-1}$$
$$= \sum_{-\infty}^{+\infty} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma[(\alpha+1)/2+k+1]\Gamma[(\alpha-1)/2-k+1]} e^{-j\omega kh} \tag{5.70}$$

Dividing (5.69) by h^α ($h \in R^+$) and computing the limit as $h \to 0$, it gives:

$$H_{D2}(\omega) = -j|\omega|^\alpha \operatorname{sgn}(\omega) \tag{5.71}$$

As

$$j\frac{\mathrm{d}|\omega|^{\alpha+1}}{\mathrm{d}\omega} = j(\alpha+1)|\omega|^\alpha \operatorname{sgn}(\omega)$$

and using a well known property of the Fourier transform we obtain from (5.59):

$$h_{D2}(t) = \frac{-\operatorname{sgn}(t)}{(\alpha+1)2\Gamma(-\alpha-1)\cos[(\alpha+1)\pi/2]}|t|^{-\alpha-1} \tag{5.72}$$

or, using the properties of the gamma function

$$h_{D2}(t) = -\frac{\operatorname{sgn}(t)}{2\Gamma(-\alpha)\sin(\alpha\pi/2)}|t|^{-\alpha-1} \tag{5.73}$$

and as previously:

$$D^\alpha_{C_2} f(t) = -\frac{1}{2\Gamma(-\alpha)\sin(\alpha\pi/2)} \int_{-\infty}^{+\infty} f(\tau)|t-\tau|^{-\alpha-1}\operatorname{sgn}(t-\tau)\mathrm{d}\tau \tag{5.74}$$

Relations (5.64), to (5.68) allow us to conclude that the type 2 central derivative is equivalent to the application of the forward (or backward) derivative twice: one with increasing time and order $(\alpha + 1)/2$, and the other with reverse time and order $(\alpha - 1)/2$.

It is interesting to remark that combining (5.57) with (5.71) as

$$H_D(\omega) = H_{D1}(\omega) + jH_{D2}(\omega) \tag{5.75}$$

We obtain a function that is null for $\omega < 0$. This means that the operator defined by (5.74) is the Hilbert transform of that defined in (5.61) and the corresponding "analytic" derivative is given by the convolution of the function at hand with the operator:

$$h_D(t) = \frac{|t|^{-\alpha-1}}{2\Gamma(-\alpha)\cos(\alpha\pi/2)} - j\frac{|t|^{-\alpha-1}\operatorname{sgn}(t)}{2\Gamma(-\alpha)\sin(\alpha\pi/2)} \tag{5.76}$$

This is formally similar to the Riesz–Feller potential definitions [8].

5.8.3 The Integer Order Cases

Let $\alpha = 2\,N$, $N \in Z^{+}$, in the type 1 difference. We obtain:

$$\Delta_{c_1}^{2N} f(t) = \sum_{-N}^{+N} \frac{(-1)^k (2N)!}{(N-k)!(N+k)!} f(t-kh) \tag{5.77}$$

that can be written as

$$\Delta_{c_1}^{2N} f(t) = \sum_{-N}^{+N} (-1)^k \binom{2N}{N-K} f(t-kh) \tag{5.78}$$

A close look into (5.78) shows that aside a factor $(-1)^N$ it is the current $2N$ order central difference, as already known. With $N = 0$, we obtain $f(t)$. Similarly, if α is odd ($\alpha = 2N + 1$), the type 2 difference is equal to current central difference, aside the factor $(-1)^{N+1}$. In fact, we have:

$$\Delta_{c_2}^{2N+1} f(t) = \sum_{-N}^{N+1} \frac{(-1)^k (2N+1)! f(t-kh+h/2)}{(N+1-k)!(N+k)!} \tag{5.79}$$

and

$$\Delta_{c_2}^{2N+1} f(t) = \sum_{-N}^{N+1} (-1)^k \binom{2N+1}{N-k} f(t-kh+h/2) \tag{5.80}$$

In particular, with $N = 0$, we obtain

$$\Delta_{c_2}^{1} f(t) = f(t+h/2) - f(t-h/2).$$

It is interesting to use the central type 1 difference (or, derivative) with $\alpha = 2M + 1$ and the type 2 with $\alpha = 2M$. For the first, $\alpha/2$ is not integer and we can use formulae (5.49) to (5.57). However, they are difficult to manipulate. We found better to use (5.59), but we must avoid the product $\Gamma(-\alpha)\cdot\cos(\alpha\pi/2)$, because the first factor is ∞ and the second is zero. We solve the problem by noting that

$$\frac{1}{2\Gamma(-\alpha)\cos(\alpha\pi/2)} = -\frac{\Gamma(\alpha+1)\cdot\sin(\alpha\pi)}{2\pi\cos(\alpha\pi/2)} = -\frac{\Gamma(\alpha+1)\sin(\alpha\pi/2)}{\pi}$$

assuming the value $-\frac{(2M+1)!(-1)^M}{\pi}$. We obtain finally [9]

$$FT^{-1}\left[|\omega|^{2M+1}\right] = -\frac{(2M+1)!(-1)^M}{\pi}|t|^{-2M-2} \tag{5.81}$$

and the corresponding impulse response:

$$h_{D1}(t) = -\frac{(2M+1)!(-1)^M}{\pi}|t|^{-2M-2} \tag{5.82}$$

Relatively to the second case, $\alpha = 2M$, we can use formula (5.73), provided that we use the relation:

$$-\frac{1}{2\Gamma(-\alpha)\sin(\alpha\pi/2)} = \frac{\Gamma(\alpha+1)\cdot\sin(\alpha\pi)}{2\pi\sin(\alpha\pi/2)} = \frac{\Gamma(\alpha+1)\cos(\alpha\pi/2)}{\pi}$$

to get a factor $\frac{(2M)!(-1)^M}{\pi}$. We obtain then [9]:

$$FT^{-1}\left[|\omega|^{2M}\mathrm{sgn}(\omega)\right] = \frac{\mathrm{sgn}(t)(2M)!(-1)^M}{\pi}|t|^{-2M-1} \tag{5.83}$$

and

$$h_{D2}(t) = \frac{\mathrm{sgn}(t)(2M)!(-1)^M}{\pi}|t|^{-2M-1} \tag{5.84}$$

As we can see, the formulae (5.82) and (5.84) allow us to generalize the Riesz potentials for positive integer orders. However, they do not have inverse.

It is interesting to study the situation defined by $\alpha = 0$ with the type 2 derivative. From (5.74) and (5.84) and noting that $t = |t|\cdot\mathrm{sgn}(t)$, we obtain:

$$D^0_{C_2}f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(z-x)\frac{1}{x}\mathrm{d}x \tag{5.85}$$

that is the Hilbert transform of $f(t)$.

These results allow us to conclude that:

1. Both type 1 (5.37) and type 2 (5.38) derivatives are defined and meaningful for real orders greater than -1.
2. When the order is an even (odd) integer, type 1 (type 2) derivative is aside a sign equal to the common derivative with the same order.
3. For the same orders, these derivatives cannot be expressed by the Riesz potentials (5.61) and (5.74), because the factors before the integrals are zero.

5.8.4 Other Properties of the Central Derivatives

From the relations (5.32), (5.33), and (5.34) we obtain easily:

$$D^{\beta}_{c_1}\{D^{\alpha}_{c_1}f(t)\} = D^{\alpha+\beta}_{c_1}f(t) \tag{5.86}$$

and

$$D^{\beta}_{c_2}\{D^{\alpha}_{c_2}f(t)\} = -D^{\alpha+\beta}_{c2}f(t) \tag{5.87}$$

while

$$D_{c_3}^{\beta}\{D_{c_3}^{\alpha}f(t)\} = D_{c_3}^{\alpha+\beta}f(t) \tag{5.88}$$

again with $\alpha + \beta > -1$. We conclude:

- If $|\alpha| < 1$ and $|\beta| < 1$ the fractional derivative has always an inverse.
- We can generate the Hilbert transform of a given function with derivations of different types and symmetric orders.

5.9 On the Existence of a Inverse Riesz Potential

In current literature [8] the Riesz potentials are only defined for negative orders verifying $-1 < \alpha < 0$. However, our formulation is valid for every $\alpha > -1$. This means that we can define those potentials even for positive orders. However, we cannot guaranty that there is always an inverse for a given potential. The theory presented in Sect. 5.2 allows us to state that:

- The inverse of a given potential, when existing, is of the same type: the inverse of the type k ($k = 1,2$) potential is a type k potential.
- The inverse of a given potential exists iff its order α verifies $|\alpha| < 1$.
- The order of the inverse of an α order potential is a $-\alpha$ order potential.
- The inverse can be computed both by (5.37) [respectively (5.38)] and by (5.61) [respectively (5.74)].

This is in contradiction with the results stated in Samko et al. [8] about this subject and will have implications in the solution of differential equations involving two-sided derivatives.

5.9.1 Some Computational Issues

In practical applications, we may need to compute a two-sided derivative of a function for which a closed form is not available and we are obliged to truncate the summation or the integral. This leads to an error. We can obtain a bound for such error, by considering a bounded function—$|f(t)| < M$—known inside an interval that we will assume to be symmetric relatively to the origin, $[-L, L]$, only by simplicity. We are going to consider the type 1 case. The other is similar. From (5.61), we conclude that the error is bounded:

$$E < \frac{M}{|\Gamma(-\alpha)\cos(\alpha\pi/2)|}\int_{L}^{\infty}\frac{1}{x^{\alpha+1}}\mathrm{d}x = \frac{ML^{-\alpha}}{|\Gamma(-\alpha)\cos(\alpha\pi/2)|} = \frac{|\Gamma(\alpha+1)|}{\pi}ML^{-\alpha} \tag{5.89}$$

This result is similar to the one stated by Podlubny [10] in connection with the called there "short-memory" principle. A similar result can be obtained for the summation in (5.37). However, here we have an error bound that is function of h. From the properties of the gamma functions, we obtain easily:

$$\frac{(-1)^k}{\Gamma(\alpha/2-k+1)} = -\frac{\sin(\alpha\pi/2)}{\pi}\Gamma(-\alpha/2+k) \tag{5.90}$$

$$\frac{(-1)^k}{\Gamma(\alpha/2-k+1)\Gamma(\alpha/2+k+1)} = -\frac{\sin(\alpha\pi/2)}{\pi}\frac{\Gamma(-\alpha/2+|k|)}{\Gamma(\alpha/2+|k|+1)} \tag{5.91}$$

When $|k|<1$ is high enough, we can use (5.39) again, to obtain

$$\left|\frac{(-1)^k}{\Gamma(\alpha/2-k+1)\Gamma(\alpha/2+k+1)}\right| \sim \frac{1}{\pi}|k|^{-\alpha-1} \tag{5.92}$$

This leads to an error:

$$E(h) \sim \frac{|\Gamma(\alpha+1)|}{\pi}\sum_{L+1}^{+\infty}|k/h|^{-\alpha-1}h \tag{5.93}$$

and leads to (5.89) again.

5.10 Conclusions

We introduced a general framework for defining the fractional central differences and consider two cases that are generalisations of the usual central differences. These new differences led to central derivatives similar to the usual Grünwald–Letnikov ones. For those differences, we presented integral representations from where we obtained the derivative integrals, similar to Cauchy, by using the properties of the Gamma function. The computation of those integrals led to generalisations of the Riesz potentials. The most interesting feature lies in the summation formulae for the Riesz potentials. To test the coherence of the proposed definitions we applied them to the complex exponential. The results show that they are suitable for functions with Fourier transform. Some properties of these derivatives were presented.

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