An Accelerated Newmark Scheme for Integrating the Equation of Motion of Nonlinear Systems Comprising Restoring Elements Governed by Fractional Derivatives

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Abstract. In this paper a new efficient algorithm for numerical integration of the equation of motion of a non linear system with restoring forces governed by fractional derivatives in the time domain is devised. This approach is based on the Grunwald-Letnikov representation of a fractional derivative and on the well known Newmark numerical integration scheme for structural dynamic problems. A Taylor expansion is used at every time step to represent the near past terms of the solution; thus, a dual mesh of the time domain is introduced: the coarse mesh is used for the time integration and the fine mesh is used for the fractional derivative approximation. It is shown that with this formulation the problem yields an equivalent non linear system without fractional terms which involves effective values of mass, damping, and stiffness coefficients as a predictive approach and a correction on the excitation. The major advantage of this approach is that a rather small number of past terms are required for the numerical propagation of the solution; and that the calculation of the effective values of mass, damping, and stiffness is performed only once. Several examples of applications are included.

Keywords: Non linear fractional differential equations, Newmark integration scheme, Taylor expansion, Duffing oscillator.

1 Introduction

Fractional calculus has been successfully applied in many engineering fields. Several papers have been published describing the advantages of using fractional derivatives in order to approximate hysteretic behavior [1,2,3], and base isolation with frequency dependent materials [21,24,25]. The advantage of using fractional derivatives of time lies in their fading memory property [4,5,6]. That is, the fractional derivative of a function in a certain time instant depends on the history of the function and not on a small neighborhood as it is the case of the integer order derivatives. This dependence is computationally inefficient when it comes to numerical evaluation. Specifically, for a small increment forward one must re-evaluate the value of the fractional derivative from the entire history. Using the Grunwald-Letnikov representation, the complexity increases in a quadratic wayyer with the time steps taken. However, due to the fading effect of the Grunwald-Letnikov coefficients which are monotonically decreasing, one can truncate the series after a certain order and achieve a linearly increasing complexity. Padovan [7], suggested another way to lower the computational cost by evaluating the fractional derivative every b number of predetermined steps since the time integration mesh can be fine and the value of fractional derivative would be slowly changing between every time step. Yuan and Agrawal [8], Adolfsson [9] and Ford et all [10], suggested algorithms based on the so-called logarithmic memory principal. When it comes to multi-degree-of -freedom systems the complexity of the integration increases significantly. Schmidt and Gaul [13] implemented an algorithm using finite differences that updates the current fractional derivative value using a so-called transfer function with very good results for multi degree of freedom systems.

The equation of motion of a non linear system with terms governed by fractional derivatives is essentially a multi-term non linear fractional differential equation which accommodates a series of solutions in the time domain. In [17] the general form of linear multi term fractional differential equations is solved and the method can be expanded for the non linear case; in [18] the general non linear multi term fractional differential equation is solved by an algorithm based on the A domain decomposition. In [19] a series of explicit Adams-Bashforth and Adams-Moulton methods were presented for efficient solutions of fractional differential equations. An interesting paper on the pitfalls of fast solvers of fractional differential equations is referenced [22], where points in implementing multi step methods for numerical efficiency are presented. In [23] a selection of algorithms for the estimation of the fractional derivative was given. In [20] the case of the linear spring and non linear fractional derivative terms is considered; efficiency in solving this system was achieved by using the nested mesh variant technique in the convolution integral.

In many respects, the way all these improved algorithms are programmed and implemented is quite complex. For engineering applications where typically the highest derivative appearing is of order two, an algorithm based on commonly used tools is desirable. In this paper, the Newmark time integration scheme is used, the non linearity of the system is readily handled by Newton-Raphson iterations, and the fractional derivative is approximated by the truncated Grunwald-Letnikov representation. The additional efficiency of the algorithm is based on the dual mesh of the time domain and on the continuous Taylor's expansion of the near past terms with respect to the current step. The coarse mesh is used as in the Newmark scheme for time integration, and the fine mesh is used to approximate accurately the fractional derivative at the specific time step. The Taylor expansion up to the second order yields a system with new effective values for the mass and stiffness. Further, a correction must be made to the excitation. In this formulation one must calculate the effective values of the mass, damping and stiffness coefficients that depend on the order of the fractional derivative and on the time step of the coarse time mesh. Then an equivalent second order non linear differential equation with a correction on the excitation that depends on few past terms must be solved.

2 Fractional Derivative Estimation

Prior to the derivation of the new algorithm, a basic mathematical background is presented on the Grunwald-Letnikov representation of fractional derivatives. The GL representation of a fractional derivative of a function x(t) at a point of time t is given by

$$ {}_{GL}D_{0,t}^{a} x(t) = \lim_{h \to 0} h^{-a} \sum_{k=0}^{\infty} (-1)^k \binom{a}{k} x(t-kh) , \tag{1} $$

where alpha is the order of the fractional derivative and 0,t are the terminals as defined in [16] representing the entire history of the function that is taken into account. Equation (1) can now be cast in the form

$$ {}_{GL}D_{0,t}^{a} x(t) = \lim_{h \to 0} h^{-a} \sum_{k=0}^{n} GL_k x(t-kh) , \tag{2} $$

where GL_k are the coefficients and n represents the number of past terms used such as

$$ GL_k = (-1)^k \binom{a}{k} \tag{3} $$

and

$$ GL_n = (-1)^n \binom{a}{n} < threshold . \tag{4} $$

It is easily proven in [12,14] that

$$ GL_{k=0} = 1 , \tag{5} $$

and

$$ GL_k = \frac{\Gamma(k-a)}{\Gamma(-a)\Gamma(k+1)} = \frac{k-a-1}{k} GL_{k-1} . \tag{6} $$

Thus, equation (6) justifies the fading memory effect of the fractional derivatives since the scalar that multiplies the GL coefficient of step k, yields

$$\frac{k-a-1}{k} < 1. \tag{7}$$

The above described method to approximate the fractional derivative of a function is known as the G1 algorithm and is extensively discussed in [14]. Next, using equation (5) and (7) it is easily deduced that the second coefficient is always the order of the fractional differentiation with a negative sign. A quite accurate estimate can be obtained by using small h [11,14,15,16]. The smaller the h becomes, the larger is the number of past terms that are involved in the estimation of the fractional derivative. It will be shown later that, in the process of solving the non linear fractional differential equation in the time domain, the time step for the time integration is much larger than the time step h for the derivative approximation. Due to this kind of dual meshing of the time domain, useful computational advantages can be obtained.

3 Dual Mesh of the Time Domain

The time domain is discretized in two separate meshes; one coarse mesh for the Newmark time integration scheme, and one fine mesh for the fractional derivative estimation. The integration step is of length Δt and is shown in Figure 1 as the interval between the tall vertical lines. The step of length h is the fine mesh and is the interval between the short vertical lines, each time step of length Δt includes p steps of length h, thus Δt/h=p, Figure 1 helps show the dual meshing technique.

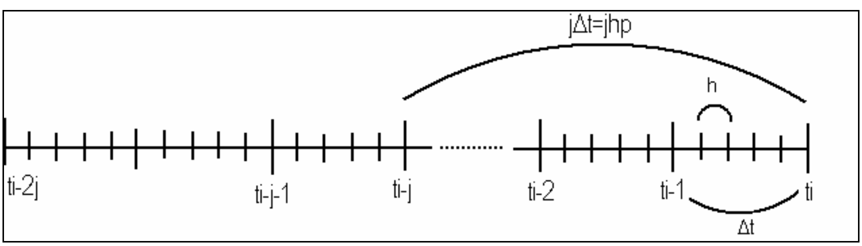

Fig. 1. Time axis discritization, with both the integration time step and the fractional derivative estimation mesh

The accelerated algorithm is based on representing the past terms needed for the Grunwald-Letnikov approximation of the fractional derivative at one point by the Taylor expansion of the same point in time. Since, the Newmark scheme determines the displacement velocity and acceleration at each time step, a Taylor expansion can be performed for at least up to second order terms. However, the range of an accurate approximation is limited and is shown in Figure 1 as jΔt. Therefore, it is assumed that a small number j of previous terms can be accurately captured by a Taylor expansion, and since several past terms are needed for an accurate approximation, more than one Taylor expansions are needed. The overall number of past terms needed for the fractional derivative estimation is denoted by kj and, thus, k Taylor expansions are needed.

4 Accelerated Algorithm

A non linear fractional differential equation representing the equation of motion of a single degree of freedom system under an arbitrary excitation is given below. Specifically,

$$m\ddot{x} + cD_{0,t}^{a}x + q(x) = f(t), \tag{8}$$

with time integration step of length,

$$\Delta t = \frac{T}{N}, \tag{9}$$

where T is the overall time interval of integration and N is the total number of steps to be considered. Clearly considering the equation of motion at two consecutive time instants one derives

$$m\Delta\ddot{x}_i + c({}_{GL}D_{0,t_i}^{a}x_i - {}_{GL}D_{0,t_{i-1}}^{a}x_{i-1}) + \Delta q(x)_i = \Delta f_i, \tag{10}$$

where,

$$\Delta x_i = x_{i+1} - x_i. \tag{11}$$

Assuming a constant stiffness coefficient during this small time step equation (10) yields,

$$m\Delta\ddot{x}_i + c({}_{GL}D_{0,t_i}^{a}x_i - {}_{GL}D_{0,t_{i-1}}^{a}x_{i-1}) + k_i\Delta x_i = \Delta f_i. \tag{12}$$

Note that the time step Δ t is much larger than the step h of the fine mesh and therefore, $\frac{\Delta t}{h} = p$ is of the order 5~20. Next, adopting the Grunwald-Letnikov representation of the fractional derivative and using equation (2) in the fine mesh one derives the form,

$$x_i^{(a)} = h^{-\alpha}\begin{bmatrix} GL_0 & \dots & GL_{k\bullet j\bullet p} \end{bmatrix}\begin{bmatrix} x_i \\ \dots \\ x_{i-p} \\ \dots \\ x_{i-2p} \\ \dots \\ x_{i-k\bullet j\bullet p} \end{bmatrix} \tag{13}$$

where, p is the number of past terms of length h in a time integration step of length Δt. The number j represents the previous time steps of length Δt that can be approximated accurately by a backwards Taylor expansion using the displacement, velocity and acceleration at a certain time step i. Note that, the number k represents the overall chunks of j time steps that must be taken into consideration to accurately approximate the fractional derivative at a given point. Proceeding to approximating the jph past terms which are of time length jΔt, using a Taylor expansion one can obtain those terms utilizing the displacement, velocity and acceleration at a given time step i. The Taylor backwards expansion yields the first jΔt past terms with respect to the current step x_i. Specifically,

$$x_{i-1} = x_i - h \cdot x_i^{(1)} + h^2 / 2 \cdot x_i^{(2)} + O(h^3) \tag{14.a}$$

$$x_{i-2} = x_i - 2h \cdot x_i^{(1)} + 4h^2 / 2 \cdot x_i^{(2)} + O(h^3) \tag{14.b}$$

$$x_{i-3} = x_i - 3h \cdot x_i^{(1)} + 9h^2 / 2 \cdot x_i^{(2)} + O(h^3) \tag{14.c}$$

$$x_{i-jp} = x_i - jph \cdot x_i^{(1)} + j^2 p^2 h^2 / 2 \cdot x_i^{(2)} + O(h^3) \tag{14.d}$$

where the number inside the brackets represents the order of the derivative and the power of the time step h respectively. The set of Equations (14) can be cast in a matrix form, and assuming that the $O(h^3)$ can be truncated since the mesh of the time axis for the estimation of the fractional derivative is fine and the past terms $jph << 1$ is quite small, yields

$$\begin{bmatrix} x_i \\ x_{i-1} \\ x_{i-2} \\ x_{i-3} \\ \cdots \\ x_{i-jp+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -h & h^2/2 \\ 1 & -2h & 4h^2/2 \\ 1 & -3h & 9h^2/2 \\ 1 & \cdots & \cdots \\ 1 & -(jp-1)h & (jp-1)^2 h^2/2 \end{bmatrix} \begin{bmatrix} x_i \\ x_i^{(1)} \\ x_i^{(2)} \end{bmatrix}. \tag{15}$$

In the same manner the displacements from the step i-jp to the i-2jp+1 can be cast in a matrix form in terms of the displacement, velocity and acceleration of the i-jp step. Equation (16) yields the relationship of these past terms with the Taylor expansion matrix, herein called connectivity matrix, and the displacement, velocity and acceleration of the given step. The same approach can be followed for all the past terms until the i-kjp term. However, it is possible to include higher order derivatives than the acceleration. This can be proven quite helpful in approximating more past terms. Thus, the connectivity matrix of the remaining past terms will have one more column if one selects to include the jerk in the computation.

$$\begin{bmatrix} x_{i-jp} \\ x_{i-jp-1} \\ x_{i-jp-2} \\ x_{i-jp-3} \\ \cdots \\ x_{i-2jp+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -h & h^2/2 & -h^3/6 \\ 1 & -2h & 4h^2/2 & -8h^3/6 \\ 1 & -3h & 9h^2/2 & -27h^3/6 \\ 1 & \cdots & \cdots & \cdots \\ 1 & -(jph-1) & (jp-1)^2 h^2/2 & -(jp-1)^3 h^3/6 \end{bmatrix} \begin{bmatrix} x_{i-j} \\ x_{i-j}^{(1)} \\ x_{i-j}^{(2)} \\ x_{i-j}^{(3)} \end{bmatrix} \tag{16}$$

Note, that the connectivity matrix is of dimensions jp x 3 for the first p terms, and the connectivity matrix is of dimensions jp x 4 for the rest of the past terms if someone includes the jerk. Note also that the connectivity matrix is constant through out the time integration and it is once built in the beginning. These constant connectivity matrices are called H_0 and H. It can be readily seen that every set of j past steps of length Δt can be obtained in terms of the matrices H_0, H and the displacement, velocity and acceleration of the corresponding time steps. Substituting equation (15), (16) etc into equation (13) one obtains

$$x_i^{(a)} = h^{-\alpha}\begin{bmatrix}1 & .. & GL_{jp-1}\end{bmatrix}\begin{bmatrix}H_0\end{bmatrix}\begin{bmatrix}x_i \\ x_i^{(1)} \\ x_i^{(2)}\end{bmatrix} + h^{-\alpha}\begin{bmatrix}GL_{jp} & .. & GL_{2jp-1}\end{bmatrix}\begin{bmatrix}H\end{bmatrix}\begin{bmatrix}x_{i-j} \\ x_{i-j}^{(1)} \\ x_{i-j}^{(2)} \\ x_{i-j}^{(3)}\end{bmatrix} +$$

$$.. + h^{-\alpha}\begin{bmatrix}GL_{(k-1)jp} & .. & GL_{kjp-1}\end{bmatrix}\begin{bmatrix}H\end{bmatrix}\begin{bmatrix}x_{i-(k-1)j} \\ x_{i-(k-1)j}^{(1)} \\ x_{i-(k-1)j}^{(2)} \\ x_{i-(k-1)j}^{(3)}\end{bmatrix}. \tag{17}$$

Note that the GL coefficients are fixed after the order of the fractional derivative alpha is fixed and the connectivity matrices H_0 and H are fixed after the time axis is divided by choosing h, Δt, j and k. The vector matrix multiplication will produce a vector 1x3 and a vector 1x4 therefore equation (17) yields

$$x_i^{(a)} = \begin{bmatrix}D_{01} & D_{02} & D_{03}\end{bmatrix}\begin{bmatrix}x_i \\ x_i^{(1)} \\ x_i^{(2)}\end{bmatrix} + \begin{bmatrix}D_{11} & D_{12} & D_{13} & D_{14}\end{bmatrix}\begin{bmatrix}x_{i-j} \\ x_{i-j}^{(1)} \\ x_{i-j}^{(2)} \\ x_{i-j}^{(3)}\end{bmatrix} + \dots$$

$$\dots + \begin{bmatrix}D_{(k-1)1} & D_{(k-1)2} & D_{(k-1)3} & D_{(k-1)4}\end{bmatrix}\begin{bmatrix}x_{i-(k-1)j} \\ x_{i-(k-1)j}^{(1)} \\ x_{i-(k-1)j}^{(2)} \\ x_{i-(k-1)j}^{(3)}\end{bmatrix}. \tag{18}$$

Next, forming the difference of the fractional derivatives from two consecutive time steps yields

$$\Delta x_i^{(a)} = \begin{bmatrix}D_{01} & D_{02} & D_{03}\end{bmatrix}\begin{bmatrix}\Delta x_i \\ \Delta x_i^{(1)} \\ \Delta x_i^{(2)}\end{bmatrix} + \begin{bmatrix}D_{11} & D_{12} & D_{13} & D_{14}\end{bmatrix}\begin{bmatrix}\Delta x_{i-j} \\ \Delta x_{i-j}^{(1)} \\ \Delta x_{i-j}^{(2)} \\ \Delta x_{i-j}^{(3)}\end{bmatrix} + ..$$

$$\dots + \begin{bmatrix}D_{(k-1)1} & D_{(k-1)2} & D_{(k-1)3} & D_{(k-1)4}\end{bmatrix}\begin{bmatrix}\Delta x_{i-(k-1)j} \\ \Delta x_{i-(k-1)j}^{(1)} \\ \Delta x_{i-(k-1)j}^{(2)} \\ \Delta x_{i-(k-1)j}^{(3)}\end{bmatrix}. \tag{19}$$

Further, combining equation (19) with equation (12) yields

$$(m + cD_{03})\Delta \ddot{x}_i + cD_{02}\Delta \dot{x}_i + (cD_{01} + k)\Delta x_i = \Delta f_i - \Delta f_{correction}. \tag{20}$$

where

$$\Delta f_{correction} = c\begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \end{bmatrix}\begin{bmatrix} \Delta x_{i-j} \\ \Delta x_{i-j}^{(1)} \\ \Delta x_{i-j}^{(2)} \\ \Delta x_{i-j}^{(3)} \end{bmatrix} + \ldots \tag{21}$$

$$\ldots + c\begin{bmatrix} D_{(k-1)1} & D_{(k-1)2} & D_{(k-1)3} & D_{(k-1)4} \end{bmatrix}\begin{bmatrix} \Delta x_{i-(k-1)j} \\ \Delta x_{i-(k-1)j}^{(1)} \\ \Delta x_{i-(k-1)j}^{(2)} \\ \Delta x_{i-(k-1)j}^{(3)} \end{bmatrix}.$$

The equation of motion is integrated in time using the Newmark time integration scheme where, mass, damping and stiffness are substituted by the effective values shown in equation (20). The solution of this equivalent second order differential equation can be considered as a prediction, and a correction in the same time step of the excitation is due to the additional past terms. Note that the number k is of the order of 3~5. The above algorithm needs a small number of past terms for every integration step in time, and the complete information at these steps such as displacement, velocity and acceleration. However, the steps needed are separated by a number of j steps. There is a convenient way to avoid the saving of the velocity and acceleration at each time step by approximating these quantities using the displacements at neighboring points. However, since the velocity and the acceleration are readily given by the Newmark algorithm it is deemed appropriate to describe the method avoiding the jerk and the approximation through the displacements of the steps. Therefore, for simplification purposes the correction is considered to be given by the equation

$$\Delta f_{correction} = c\begin{bmatrix} D_{11} & D_{12} & D_{13} \end{bmatrix}\begin{bmatrix} \Delta x_{i-j} \\ \Delta x_{i-j}^{(1)} \\ \Delta x_{i-j}^{(2)} \end{bmatrix} + \dots$$

$$\dots + c\begin{bmatrix} D_{(k-1)1} & D_{(k-1)2} & D_{(k-1)3} \end{bmatrix}\begin{bmatrix} \Delta x_{i-(k-1)j} \\ \Delta x_{i-(k-1)j}^{(1)} \\ \Delta x_{i-(k-1)j}^{(2)} \end{bmatrix}. \tag{22}$$

Next, note that the saving of the velocity and acceleration every j steps is needed along with the displacements to advance the algorithm. This can be proven to be quite powerful for the implementation of the algorithm in multi-degree-of-freedom systems. As Figure 1 shows, the jp steps of the fractional derivative estimation are approximated by the Taylor backwards expansion with respect to the displacement, velocity and acceleration of the i^{th} step. The multiplication of the first jp GL coefficients with the connectivity matrix H produces an equivalent mass, damping and stiffness coefficient matrix which can be added to the original equation of motion turning the fractional non linear differential equation into a non linear equation without fractional terms. The solution of this equation can be called the predictor step. Clearly, this equation takes into account only a limited number of past terms, specifically jp terms of time duration jph=jΔt. The correction step comes from the adjustment of the excitation by a term representing the rest of the past terms associated with the displacement, velocity and acceleration of previous time steps. Once again this is accomplished by a set of equivalent mass, damping and stiffness coefficients that are calculated once in the beginning as a product of the H connectivity matrix and the GL coefficients vector.

5 Numerical Example

As an example, consider a simple linear system of unit mass under a sinusoidal excitation with the equation of motion

$$\ddot{x} + 0.5D_{0,t}^{0.5}x + 4x = \sin(2t)\,. \tag{23}$$

Dividing the time domain in Δt of length 0.02 sec, h of 0.001 sec and therefore Δt/h=p=20, and estimating that the max Δt for Newmark is T/10=0.2962 where T is the natural period of the system, one can obtain the number j= Δt max/Δt=15 steps. In 15 coarse steps that are approximated by a Taylor expansion there are 15x20=300 fine past terms, where $h^{-\alpha}GL_{300} \approx 10^{-3}$. The connectivity matrix

H_0 is constructed from equation (15) and the vector of dimensions 1x3 representing the correction on the mass, damping and stiffness is calculated by the multiplication $GLxH_0$ and yields $D_0 = [1.0314 \quad 0.3084 \quad -0.0155]$. Therefore, by utilizing equation (20), the equation to be solved now becomes

$$(0.5 \bullet (-0.0155) + 1)\Delta \ddot{x} + 0.5 \bullet 0.3084 \Delta \dot{x} + (4 + 0.5 \bullet 1.0314)\Delta x = \Delta \sin(2t) - \Delta f \tag{24}$$

where Δf is the rest of the past terms given by equation (22).

Solving this equation without any correction and comparing it to the readily derivable frequency domain solution one can assess the accumulating error from Figure 2.

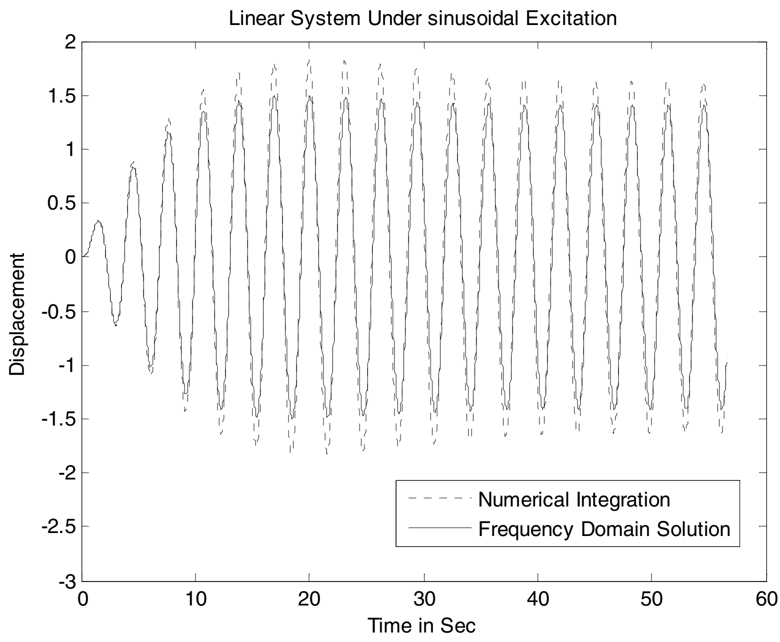

Fig. 2. Frequency domain solution vis a vis numerical integration without correction step, p=20, j=15, k=1.

It is easily seen the 300 past terms are not enough to capture the steady state response as well as the transient response. Calculating next the D_1 and D_2 vectors that are derived as the product of the next 600 GL coefficients with the H_0 avoiding the use of the jerk, one obtains

$$D_1 = [-0.3025 \quad 0.0374 \quad -0.0034] \tag{25}$$

and

$$D_2 = [-0.1339 \quad 0.0180 \quad -0.0017]. \tag{26}$$

These vectors multiplied by $\begin{bmatrix} \Delta x_{i-15} \\ \Delta x_{i-15}^{(1)} \\ \Delta x_{i-15}^{(2)} \end{bmatrix}$ and $\begin{bmatrix} \Delta x_{i-30} \\ \Delta x_{i-30}^{(1)} \\ \Delta x_{i-30}^{(2)} \end{bmatrix}$ respectively will correct the solution by taking into account another 600 past terms. Figure 3 shows the solution with correction of 2 past terms and specifically of the past terms x_{i-15} and x_{i-30}.

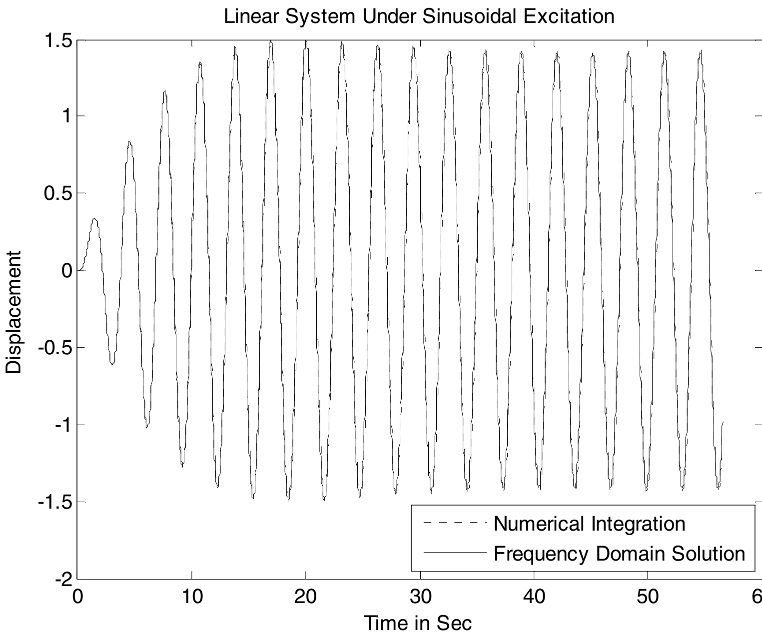

Fig. 3. Frequency domain solution vis a vis numerical integration with correction step of two past terms, p=20, j=15, k=3.

Figure 4 shows the solution with correction step of three past terms and specifically of the past terms x_{i-15}, x_{i-30} and x_{i-45}. Calculating the vector D_3 one obtains

$$D_3 = [-0.0691 \quad 0.0128 \quad -0.0016]. \tag{27}$$

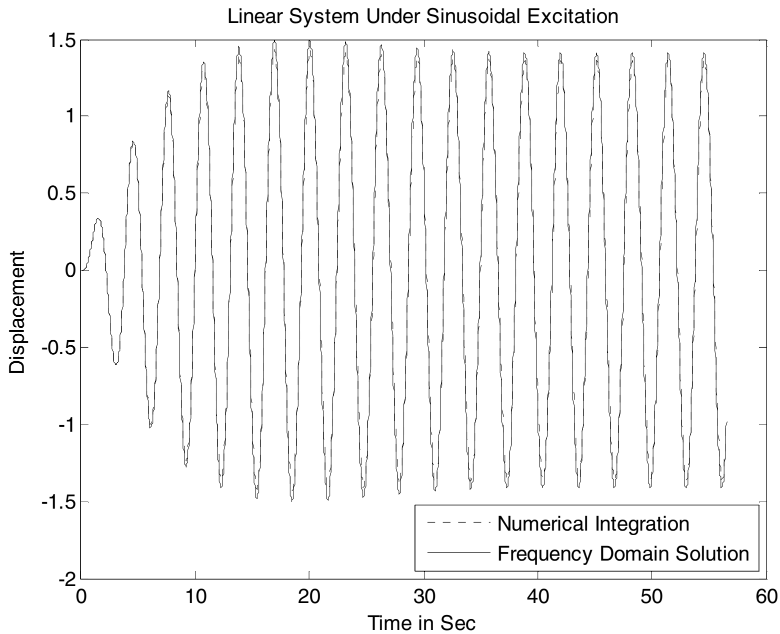

Fig. 4. Frequency domain solution vis a vis numerical integration with correction step of three past terms, p=20, j=15, k=4.

6 Numerical Results for Earthquake Excitation

Consider next the earthquake excitation of ElCentro California 1940 USA, and a single degree of freedom oscillator with restoring forces governed by fractional derivatives. Proceed in implementing the described algorithm to obtain the system response. In this regard, Figure 5 shows the El Centro accelerogram.

The coarse time mesh for the integration in time is of length Δt=0.02 seconds and the fine mesh is h=0.001 and h=0.004 seconds, thus p=20 and p=5. The range of the Taylor expansion is equal to 3Δt, 4Δt, 5Δt and 6Δt thus j=3,4,5 and 6. Since four past steps have been used in every simulation k=5. The range of the Taylor expansion can be obtained as in the previous numerical example, thus assuming that the max Δt for Newmark is T/10 where T is the natural period of the system, one can obtain the number j as j=Δt max/Δt. It can be argued that the choice of the number j representing the range of an acceptable approximation by the Taylor expansion can be rigorously addressed. However such an attempt would be a quite laborious process and is currently out of the scope of this article. Empirically, however it is seen that the first jΔt seconds including the jp past terms must have a significantly high threshold. That is equation (4) for n=jp must have a much larger value than n=kjp which is the threshold for the entire representation range. This is

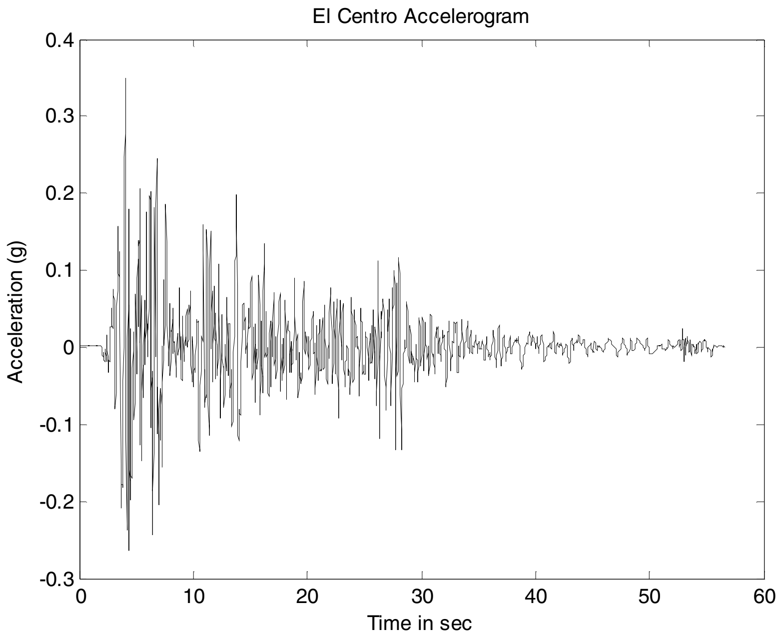

Fig. 5. The El-Centro Accelerogram; g is the gravitational acceleration.

readily explained, by the fact that the correcting terms must account for larger correction than the correction achieved by the first Taylor expansion and the changing of the mass damping and stiffness values. Results are presented for linear systems, and non linear systems of the Duffing type. The same values of fractional derivatives have been used for both the linear and non linear cases and quite large values of the non linearity strength ε have been considered. The system considered herein has an equation of motion given by equation (8) with a Duffing kind nonlinearity

$$m\ddot{x} + cD_{0,t}^{\alpha}x + kx(1 + \varepsilon x^2) = f(t). \tag{28}$$

The results of solving equation (25) with the proposed algorithm are compared to the benchmark algorithm results which uses the truncated Grunwald-Letnikov representation. Pertinent results for various values of fractional derivatives are shown in Figures 7, 9 and 11, in addition for linear systems this approach is verified through comparison with the frequency domain solution and the results are shown in Figures 6, 8 and 10.

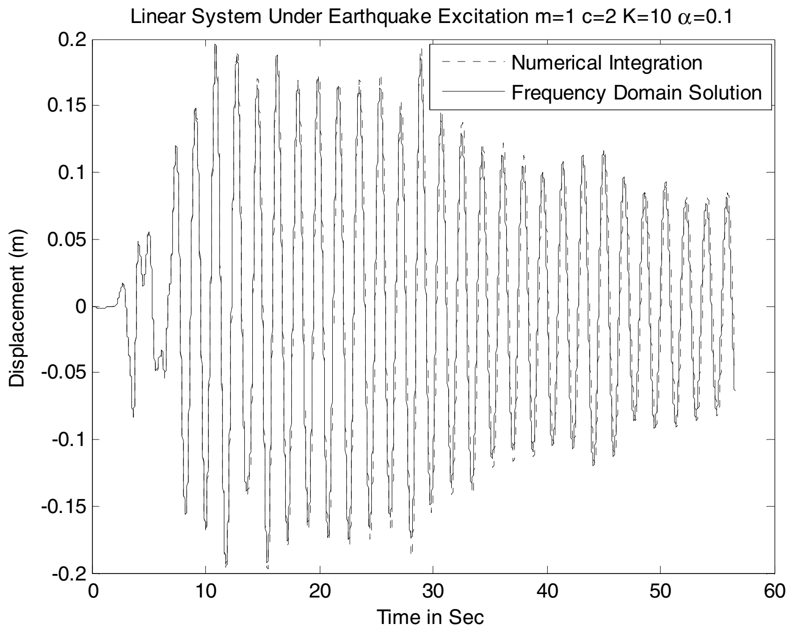

Fig. 6. Frequency domain solution vis a vis numerical integration with correction step of four past terms p=20, j=5, k=5.

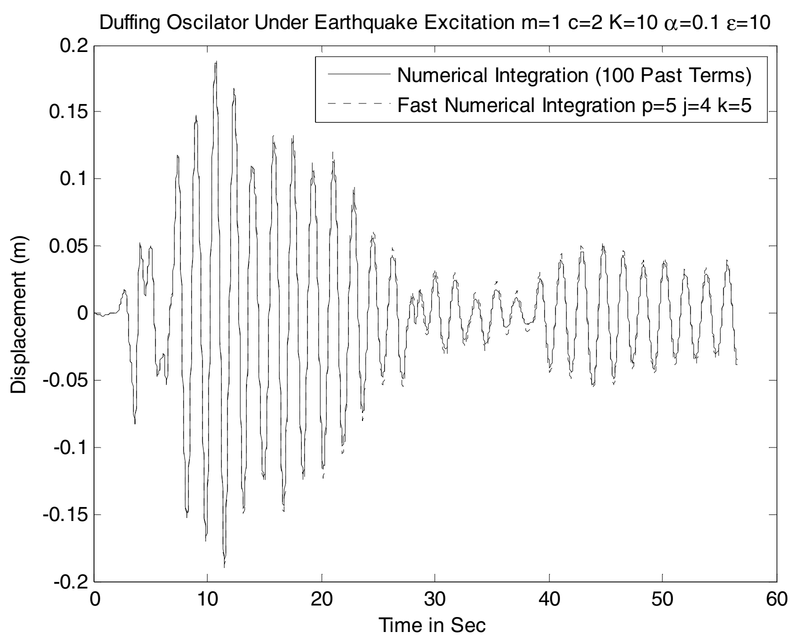

Fig. 7. Numerical solution with past terms corresponding to GL coefficients of the order 10^{-3} vis a vis the enhanced numerical integration algorithm

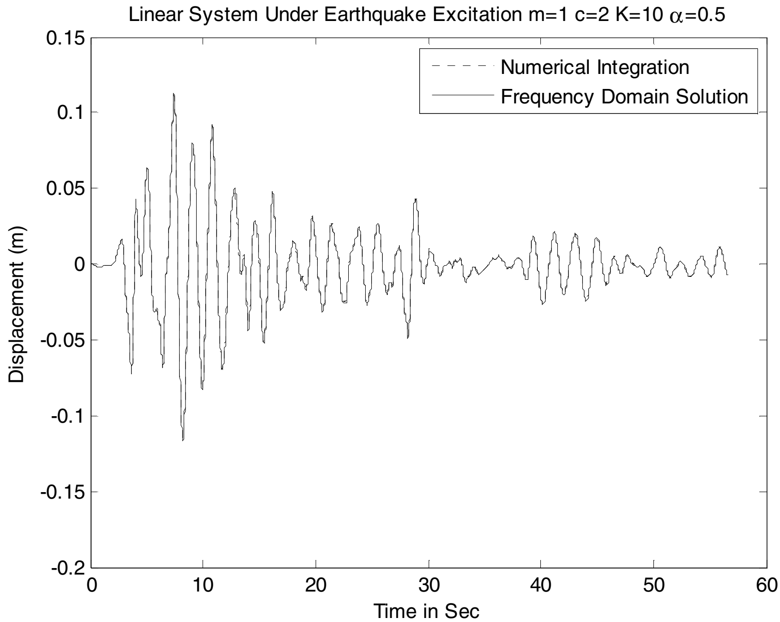

Fig. 8. Frequency domain solution vis a vis numerical integration with correction step of four past terms p=20, j=9, k=5.

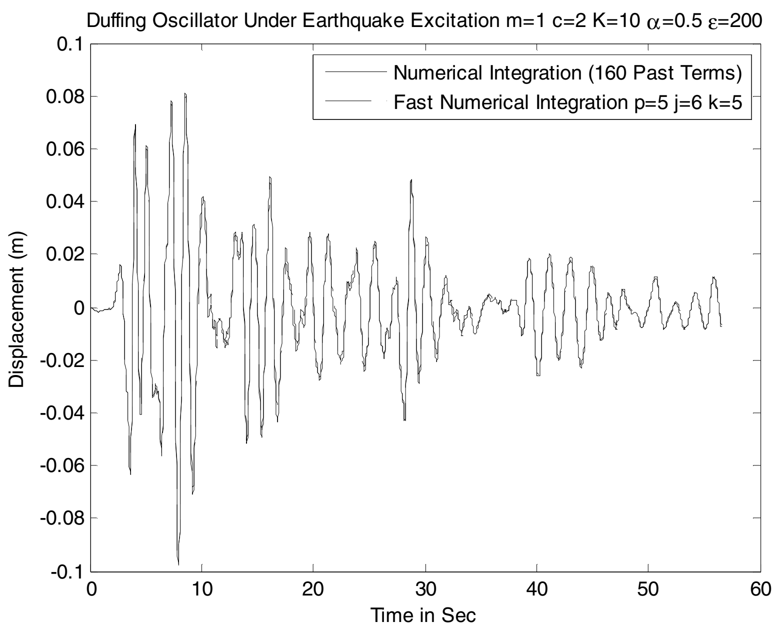

Fig. 9. Numerical solution with past terms corresponding to GL coefficients of the order 10^{-3} vis a vis the enhanced numerical integration algorithm

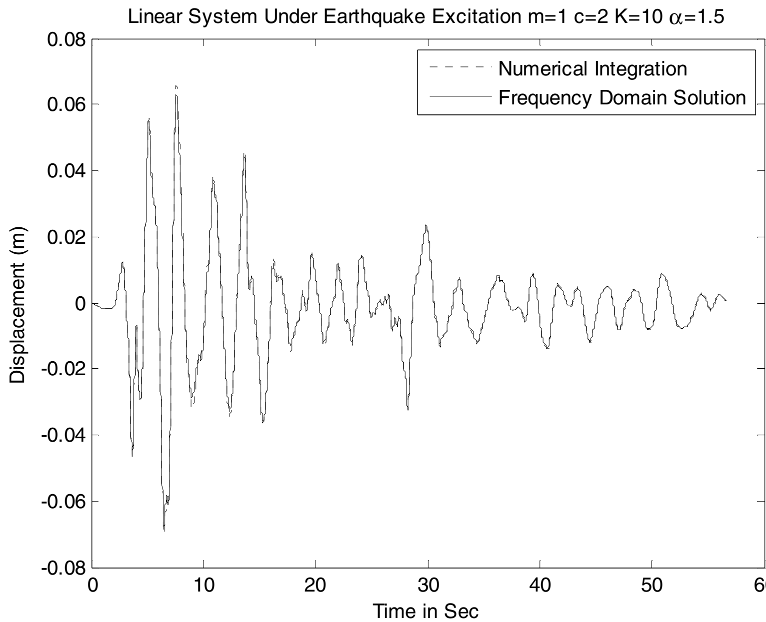

Fig. 10. Exact solution vis a vis numerical integration with correction step of four past terms p=20, j=9, k=5.

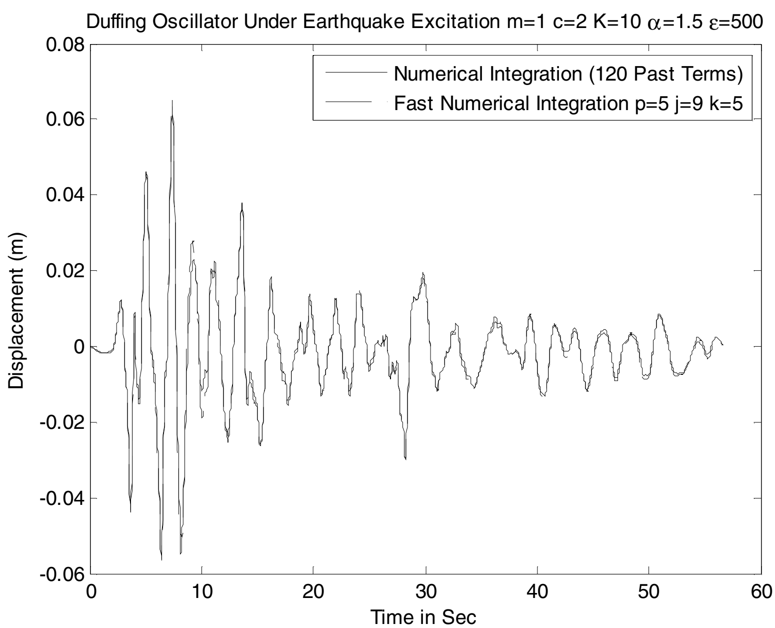

Fig. 11. Numerical solution with past terms corresponding to GL coefficients of the order 10^{-3} vis a vis the enhanced numerical integration algorithm

7 Concluding Remarks

Efficient determination of the response of systems of single or multi –degree-of-freedom endowed with restoring terms governed by fractional derivatives has been pursued in this paper. For this purpose the Grunwald-Letnikov representation of the fractional derivative has been adopted; this representation involves several past terms of the process itself. Thus, the numerical evaluation of fractional derivatives requires a high number of computations due to their memory effect.

A modified Newmark algorithm that includes a correction on the excitation has been used to numerically solve the equation. It has been shown that a dual meshing technique in the time domain in conjunction with the Taylor expansion and the Grunwald-Letnikov fractional derivative representation yields an efficient Newmark time integration scheme for the determination of the response of non linear oscillators comprising fractional derivative terms. In this context, the governing equation of motion can be transformed into a second order differential equation using new effective values of mass, damping, and stiffness without any fractional terms involved. The effective values are calculated once by a vector representing the GL coefficients and the connectivity matrix representing the Taylor expansion. Next, this equation is solved for an excitation corrected by the contribution of few past terms. The new scheme has been used to obtain simulation results for linear systems under seismic excitation which have been validated by the solution derived by frequency domain techniques. For similar excitations, it has been shown that the new scheme can be effectively implemented for determining the dynamic response of nonlinear systems endowed with fractional derivative terms.

Acknowledgements

The partial support of this work by a grant from the National Science Foundation-USA is greatly appreciated.

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