Chapter 4
Solution of Jacobi's Virial Equation for Conservative Systems

In Chap. 3 we derived Jacobi's virial equation of dynamical equilibrium in the framework of various physical models which are used for describing the dynamics of natural systems. We showed that, instead of the traditional description of a system in co-ordinates and velocities, the problem of dynamics can be studied from the position of an external observer. In this case the system as a whole is described by a compact and elegant equation and is characterized by integral (volumetric) parameters. Such a description of the integral equation does not depend on the choice of the frame of reference. The external observer can estimate by observations only some moments of distribution of mass density, i.e. total mass and energy of a system, which are its integral characteristics. Moreover, in order to solve the problem of a body's motion in the framework of its dynamical equilibrium, we invoked the relationship between its force function and the polar moment of inertia, which is the source of motion. This relationship reveals the nature of the gravitational energy. We also succeeded in reanimating the lost kinetic energy and obtaining both an equation of dynamics and an equation of dynamical equilibrium in the form of the oscillating motion during each period of time and within the whole duration of the system's evolution.

The problem is now to find the general solution of Jacobi's virial equation relative to oscillation and rotation of a body and to apply the solution to study its dynamics. This application is valid for studying the Sun, the Earth, the Moon and other celestial bodies.

In this chapter we show that Jacobi's virial equation provides first of all a solution for the models of natural systems, which have explicit solutions in the framework of the classical many-body problem. We shall give parallel solutions for both the classical and dynamical approaches, and in doing so we shall show that, with the dynamical approach, the solution acquires a new physical meaning. We shall also consider a general case of the solution of Jacobi's virial equation for conservative and dissipative systems.

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4.1 Solution of Jacobi's Virial Equation in Classical Mechanics

The many-body problem is known to be the key problem in classical mechanics and especially in celestial mechanics. A particular example of this is the unperturbed problem of Keplerian motion, when the system consists of only two material points interacting by Newtonian law. The explicit solution of the problem of unperturbed Keplerian motion permits the many-body problem to be solved with some approximation by varying arbitrary constants. In this case the problem of dynamics, for example that of the Solar System, is transferred into the solution of the problem of dynamics of nine pairs of bodies in each of which one body is always the Sun and the second is each of the nine planets forming the system. Considering each planet-sun sub-system, the influence of the other eight planets of the system is taken into account by introducing the perturbation function. By the virial approach we can obtain for the Sun one characteristic period of circulation with respect to the center of mass of the system which will not coincide with any period of the planets. The dynamical approach evidences that the planet's orbital motion is performed by the central body, i.e. by the Sun, by the energy of its outer force field or by the field of the pressure. Each planet interacts with the solar force field by the energy of its own outer force field. The planet's orbit is the certain curve of its equilibrium motion which results from the two interacting fields of pressure. The planet's own oscillation and rotation perform by action of the inner fields of pressure.

Following these brief physical comments on the dynamical equilibrium motion of a planet, we now present two methods of solving the Keplerian problem: the classical and the integral.

4.1.1 The Classical Approach

The traditional way of solving the unperturbed Keplerian problem is excellently described in the university courses for celestial mechanics found in (Duboshin 1978). Here we present only the principle ideas. The method consists in transforming the two-body problem described by the system of equations (3.10) into the one-body problem using six integrals of motion of the center of mass (3.13). The system of equations obtained is sixth order and expresses the change of barycentric co-ordinates of one point with respect to the center of mass of the system as a whole. Let us write it in the form

$$\ddot{x} = -\frac{\mu x}{r^3},$$

$$\ddot{y} = -\frac{\mu y}{r^3}, \tag{4.1}$$

$$\ddot{z} = -\frac{\mu z}{r^3},$$

where μ is the constant depending on the number of the point and for which the second point is equal to

$$\mu = \frac{Gm_1^3}{(m_1+m_2)^2}.$$

We then pass on from that Cartesian system of co-ordinates OXYZ to orbital $\xi\ \eta\ \zeta$, using first integrals of the system of equations (4.1). Those are three integrals of the area,

$$\begin{aligned} y\dot{z} - z\dot{y} &= c_1, \\ z\dot{x} - x\dot{z} &= c_2, \\ x\dot{y} - y\dot{x} &= c_3, \end{aligned} \tag{4.2}$$

the energy integral,

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \frac{2\mu}{r} + h, \tag{4.3}$$

and the Laplacian integrals,

$$\begin{aligned} -\frac{\mu x}{r} + c_3\dot{y} - c_2\dot{z} &= f_1, \\ -\frac{\mu y}{r} + c_1\dot{z} - c_3\dot{x} &= f_2, \\ -\frac{\mu z}{r} + c_2\dot{x} - c_1\dot{y} &= f_3. \end{aligned} \tag{4.4}$$

As these seven integrals are not independent, we conclude that they cannot form a general solution of the system (4.1). In fact there are two relations for these integrals:

$$\begin{aligned} c_1f_1 + c_2f_2 + c_3f_3 &= 0, \\ f_1^2 + f_2^2 + f_3^2 &= \mu^2 + h\left(c_1^2 + c_2^2 + c_3^2\right), \end{aligned}$$

showing that only five of them are independent. But the last integral needed can be found by simple quadrature. Using these integrals we can pass on to the system of orbital co-ordinates $O\xi\eta\zeta$ using the transformation relations (see Fig. 4.1):

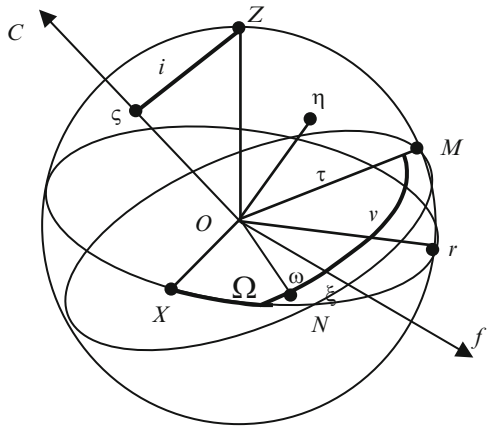

Fig. 4.1 Transition from Cartesian co-ordinate system OXYZ to orbital Oξηζ

$$\xi = \frac{f_1}{f}x + \frac{f_2}{f}y + \frac{f_3}{f}z,$$

$$\eta = \frac{C_2f_3 - C_3f_2}{Cf}x + \frac{C_3f_1 - C_1f_2}{Cf}y + \frac{C_1f_2 - C_3f_1}{Cf}z, \tag{4.5}$$

$$\zeta = \frac{C_1}{C}x + \frac{C_2}{C}y + \frac{C_3}{C}z.$$

The equation of the curve along which the point moves in accordance with (4.1) has the simplest form in the system of initial co-ordinates. The equation is

$$\zeta = 0,$$

$$\mu r = C^2 - f\xi. \tag{4.6}$$

Finally, introducing the polar orbital co-ordinates r and *v*, which are related to the rectangular orbital co-ordinates ξ and η by the expressions (see Fig. 4.2)

$$\xi = r\cos v$$

and

$$\eta = r\sin v,$$

and using the integral of areas

$$r^2v = C,$$

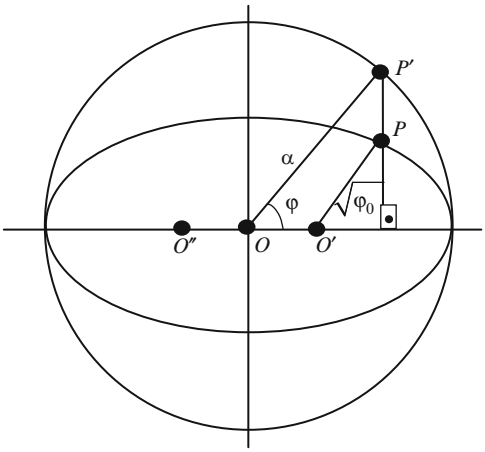

Fig. 4.2 Relationship between the polar and the rectangular co-ordinates

we come to the equation

$$C(t-r)=\left(\frac{C^2}{\mu}\right)^2\int_0^v\frac{dv}{\left(1+\frac{f}{\mu}\cos v\right)^2}. \tag{4.7}$$

The solution of Eq. 4.7 gives the change of function v with respect to time. Repetition of the transformation in the reverse order leads to solution of the problem. In doing this, we obtain the expression for the change of co-ordinates of the material point with respect to the initial data ξ_{10}, η_{10}, ζ_{10}, ξ_{20}, η_{20}, ζ_{20}, $\dot{\xi}_{10}$, $\dot{\eta}_{10}$, $\dot{\zeta}_{10}$, $\dot{\xi}_{20}$, $\dot{\eta}_{20}$, $\dot{\zeta}_{20}$. It is remarkable that if the total energy (4.3) has negative value, then the solution of Eq. 4.7 leads to the Keplerian equation

$$E'-e\sin E'=n(t-\tau), \tag{4.8}$$

where the function v is related to the variable E' by the expression

$$\operatorname{tg}\frac{v}{2}=\sqrt{\frac{1+e}{1-e}}\operatorname{tg}\frac{E}{2},$$

and

$$e=\frac{f}{\mu},\quad n=\frac{\sqrt{\mu}}{a^{3/2}},\quad p=\frac{C^2}{\mu}=a\left(1-e^2\right).$$

Because energy by definition is the property to do work (motion) and can be only a positive value, then the physical meaning of negative total energy which defines the elliptic orbit of a body moving in the central field of the two-body problem should be revealed. In the presented solution of the two-body problem, the left-hand side of the energy integral (4.3) expresses the kinetic energy and the right-hand side

means the potential energy of the mass interaction. The integral of energy (4.3) as a whole, in the co-ordinates and in the velocities, represents the averaged virial theorem, where the potential energy has formally a negative value. Here the physical meaning of the total energy determination consists in comparison of magnitude of the potential and kinetic energy. A negative value of the total energy means that the potential energy exceeds the kinetic one by that value. As it follows from analysis of the inner force field of a self-gravitating body presented in Chap. 2, the potential energy exceeds the kinetic one only in the case of non-uniform distribution of the mass density and cannot be less than it. In the case of equality of both energies the total potential energy is realized into oscillating motion. The excited part of the potential energy is used for rotation of the masses and in the dissipation. The last case is discussed in Chap. 7.

4.1.2 The Dynamic Approach

Let us consider the solution of the problem of unperturbed motion of two material points on the basis of Jacobi's virial equation which in accordance with Eq. 3.16 is written in the form

$$\ddot{\Phi}_0 = 2E_0 - U,$$

where $E_0 = T_0 + U = \text{const}$ is the total energy of the system in a barycentric co-ordinate system;

The Jacobi function Φ_0 is expressed by (3.15):

$$\Phi_0 = \frac{m_1 m_2}{2(m_1 + m_2)}\left[(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2\right],$$

and the potential energy U in accordance with (3.2) is

$$U = \frac{Gm_1 m_2}{\sqrt{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2}}.$$

It is easy to see that between the Jacobi function Φ_0 and the potential energy U exists the relationship

$$|U|\sqrt{\Phi_0} = \frac{G(m_1 m_2)^{3/2}}{\sqrt{2(m_1 + m_2)}} = G^{1/2} m \mu^{3/2} = B = \text{const}, \tag{4.9}$$

where μ is the generalized mass of the two bodies; m is the total mass of the system; B is a constant value.

The relationship (4.9) is remarkable because it is independent of the initial data, i.e. of its co-ordinates and velocities. Being an integral characteristic of the system and dependent only on the total mass and the generalized mass of the two points, the relationship permits Jacobi's virial equation to be transformed to an equation with one variable as follows:

$$\ddot{\Phi}_0 = 2E_0 + \frac{B}{\sqrt{\Phi_0}}. \tag{4.10}$$

We consider the solution of Eq. 4.10 for the case when total energy E_0 has negative value. Introducing $A = -2E_0 > 0$, Eq. 4.10 can be rewritten:

$$\ddot{\Phi}_0 = -A + \frac{B}{\sqrt{\Phi_0}}. \tag{4.11}$$

We apply the method of change of variable for solution of Eq. 4.11 and show that partial solution of two linear equations (Ferronsky et al. 1984a):

$$\left(\sqrt{\Phi_0}\right)'' + \sqrt{\Phi_0} = \frac{B}{A}, \tag{4.12}$$

$$t'' + t = \frac{4B\lambda}{(2A)^{3/2}}, \tag{4.13}$$

which include only two integration constants, is also the solution of Eq. 4.11.

We now introduce the independent variable λ into Eqs. 4.12 and 4.13, where primes denote differentiation with respect to λ. Note that time here is not an independent variable. This allows us to search for the solution of two linear equations instead of solving one non-linear equation. The solution of Eqs. 4.12 and 4.13 can be written in the form

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos(\lambda - \psi)], \tag{4.14}$$

$$t = \frac{4B}{(2A)^{3/2}}[\lambda - \varepsilon\sin(\lambda - \psi)]. \tag{4.15}$$

Let us prove that the partial solution (4.14) and (4.15) differential Eqs. 4.12 and 4.13 is the solution of Eq. 4.10 that is sought. For this purpose we express the first and second derivatives of the function $\sqrt{\Phi_0}$ with respect to λ through corresponding derivatives with respect to time using Eq. 4.15. From (4.15) it follows that

$$\frac{dt}{d\lambda} = \frac{4B}{(2A)^{3/2}}[1 - \varepsilon\cos(\lambda - \psi)]. \tag{4.16}$$

We can replace the right-hand side of the obtained relationship by $\sqrt{\Phi_0}$ from (4.14)

$$\frac{dt}{d\lambda} = \sqrt{\Phi_0}\sqrt{\frac{2}{A}}. \tag{4.17}$$

Transforming the derivative from $\sqrt{\Phi_0}$ with respect to λ into the form

$$\frac{d\sqrt{\Phi_0}}{d\lambda} = \frac{d\sqrt{\Phi_0}}{dt}\frac{dt}{d\lambda} = \frac{\dot{\Phi}_0}{2\sqrt{\Phi_0}}\frac{dt}{d\lambda}$$

and taking into account (4.17), we can write

$$\left(\sqrt{\Phi_0}\right)' = \frac{\dot{\Phi}_0}{\sqrt{2A}}.$$

The second derivative can be written analogously:

$$\left(\sqrt{\Phi_0}\right)'' = \frac{dt}{d\lambda}\frac{d}{dt}\left(\frac{\dot{\Phi}_0}{\sqrt{2A}}\right) = \frac{\ddot{\Phi}_0}{\sqrt{2A}}\sqrt{\Phi_0}\sqrt{\frac{2}{A}} = \frac{\ddot{\Phi}_0\sqrt{\Phi_0}}{A}. \tag{4.18}$$

Putting Eq. 4.18 into (4.12), we obtain

$$\frac{\ddot{\Phi}_0\sqrt{\Phi_0}}{A} + \sqrt{\Phi_0} = \frac{B}{A}.$$

Dividing the above expression by $\sqrt{\Phi_0}/A$, we can finally write

$$\ddot{\Phi}_0 = -A + \frac{B}{\sqrt{\Phi_0}}.$$

This shows that the partial solution of the two linear differential Eqs. 4.12 and 4.13 appears to be the solution of the non-linear Eq. 4.11.

4.2 Solution of the N-Body Problem in the Framework of Conservative System

After solving Jacobi's virial equation for the unperturbed two-body problem, we come to dynamics of a system of n material particles where $n \to \infty$.

Let us assume that an external observer studying the dynamics of a system of *n* particles in the framework of classical mechanics has the following information. He has the mass of the system, its total and potential energy and can

determine the Jacobi function and its first derivative with respect to time in any arbitrary moment. Then he can use Jacobi's virial Eq. 4.9 and, making only the assumption needed for its solution that $|U|\sqrt{\Phi_0} = B = \text{const}$, may predict the dynamics of the system, i.e. the dynamics of its integral characteristics at any moment of time. The assumption $|U|\sqrt{\Phi_0} = \text{const}$will be considered separately in Chap. 6.

If the total energy E_0 of the system has negative value, the external observer can immediately write the solution of the problem of the Jacobi function change with respect to time in the form of (4.14) and (4.15):

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos(\lambda - \psi)],$$

$$t = \frac{4B}{(2A)^{3/2}}[\lambda - \varepsilon\sin(\lambda - \psi)],$$

where $A = -2E_0$; ε and ψ are constants depending on the initial values of the Jacobi function Φ_0 and its first derivative $\dot{\Phi}_0$ at the moment of time t_0 .

Let us obtain the values of constants ε and ψ, in explicit form expressed through the values Φ_0 and $\dot{\Phi}_0$ at the initial moment of time t_0. For convenience we introduce a new independent variable φ, connected to λ by the relationship соотношением $\lambda - \psi = \varphi$. Then Eqs. 4.14 and 4.15 can be rewritten:

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos\varphi], \tag{4.19}$$

$$t - \frac{4B}{(2A)^{3/2}}\psi = \frac{4B}{(2A)^{3/2}}[\varphi - \varepsilon\sin\varphi]. \tag{4.20}$$

Using Eq. 4.19 we write the expression for φ:

$$\varphi = \arccos\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon} \tag{4.21}$$

and taking into account the equality

$$\frac{d\sqrt{\Phi_0}}{d\lambda} = \frac{d\sqrt{\Phi_0}}{d\varphi},$$

substitute Eq. 4.21 into the expression

$$\frac{\dot{\Phi}_0}{\sqrt{2A}} = \frac{B}{A}\varepsilon\sin\varphi.$$

The last equation can be rewritten finally in the form

$$\frac{\dot{\Phi}_0}{\sqrt{2A}} = \frac{B}{A}\,\varepsilon\sqrt{1-\left(\frac{1-\frac{A}{B}\sqrt{\Phi_0}}{\varepsilon}\right)^2}. \tag{4.22}$$

Equation 4.22 allows us to determine the first constant of integration ε as a function of the initial data Φ_0 and $\dot{\Phi}_0$ at $t = t_0$. Solving Eq. 4.22 with respect to ε after simple algebraic transformation, we obtain

$$\varepsilon = \sqrt{1-\frac{A}{2B^2}\left(-\dot{\Phi}_0 + 4B\sqrt{\Phi_0} - 2A\Phi_0\right)|_{t=t_0}} = \text{const.} \tag{4.23}$$

The second constant of integration ψ can be expressed through the initial data after solving Eq. 4.20 with respect to ψ and change of value φ by its expression from Eq. 4.21. Defining

$$t - \frac{4B}{(2A)^{3/2}}\,\psi = \tau,$$

we obtain

$$-\tau = \left\{\frac{4B}{(2A)^{3/2}}\left[\arccos\frac{1-\frac{A}{B}\sqrt{\Phi_0}}{\varepsilon} - \varepsilon\sqrt{1-\left(\frac{1-\frac{A}{B}\sqrt{\Phi_0}}{\varepsilon}\right)^2}\right] - t\right\}\Big|_{t=t_0} = \text{const.} \tag{4.24}$$

The physical meaning of the integration constants ε, τ, and the parameter $T_v = 8\pi B/(2A)^{3/2}$ can be understood after the definitions

$$T_v = \frac{8\pi B}{(2A)^{3/2}},$$

$$n = \frac{2\pi}{T_v} = \frac{(2A)^{3/2}}{4B},$$

$$a = \frac{B}{A}$$

and rewriting Eqs. 4.19 and 4.20 in the form

$$\sqrt{\Phi_0} = a(1 - \varepsilon\cos\varphi), \tag{4.25}$$

$$M = \varphi - \varepsilon\sin\varphi, \tag{4.26}$$

where $M = n(t - \tau)$.

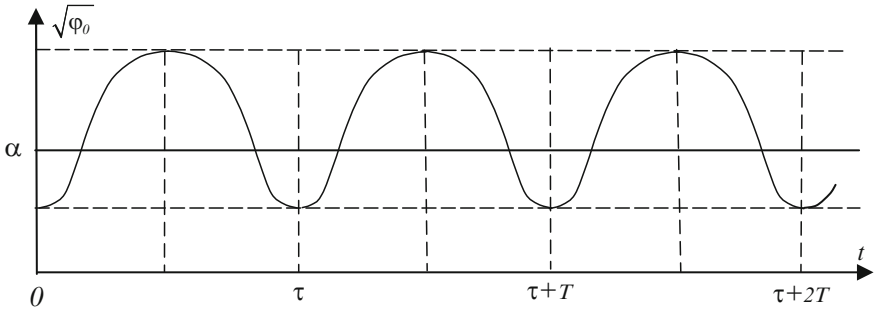

Fig. 4.3 Changes of the Jacobi function over time

The value $\sqrt{\Phi_0}$ draws an ellipse during the period of time $T_v = 8\pi B/(2A)^{3/2}$ (see Fig.4.3). The ellipse is characterized by a semi-major axis a equal to B/A and by the eccentricity ε which is defined by expression (4.23). In the case considered, where $E_0 < 0$, the value ε is changed in time from 0 to 1. The value τ characterizes the moment of time when the ellipse passes the pericentre.

Let us obtain explicit expressions with respect to time for the functions $\sqrt{\Phi_0}$, Φ_0 and $\dot{\Phi}_0$. For this purpose we write Eq. 4.24 in the form of a Lagrangian:

$$F(\varphi) = \varphi - \varepsilon \sin\varphi - M = 0. \tag{4.27}$$

It is known (Duboshin 1978) that by application of Lagrangian formulas we can write in the form of a series the expressions for the root of the Lagrange Eq. 4.27 and for the arbitrary function f which is dependent φ:

$$\varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{d^{k-1}}{dM^{k-1}} \left[\sin^k M\right] = M + \varepsilon \sin M + \frac{\varepsilon^2}{1\cdot 2} \frac{d}{dM} \left[\sin^2 M\right] + \cdots, \tag{4.28}$$

$$f(\varphi) = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{d^{k-1}}{dM^{k-1}} \left[f'(M)\sin^k M\right] = f(M) + \varepsilon f'(M)\sin M + \frac{\varepsilon^2}{1\cdot 2} \frac{d}{dM} \left[f(M)\sin^2 M\right] + \ldots. \tag{4.29}$$

Using Eq. 4.29, we write expressions for $\cos\varphi$, $\cos^2\varphi$ and $\sin\varphi$ in the form of a Lagrangian series of parameter ε power:

$$\begin{aligned} \cos\varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{d^{k-1}}{dM^{k-1}} \left[(-1)\sin M \sin^k M\right] &= \cos M + \varepsilon(-1)\sin M \sin(M) \\ + \frac{\varepsilon^2}{1\cdot 2} \frac{d}{dM} \left[(-1)\sin M \sin^2 M\right] + \cdots &= \cos M - \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\cos 2M \\ - \frac{3}{4}\varepsilon^3 \cos M + \frac{3}{8}\varepsilon^2 \cos 3M + \cdots & \end{aligned} \tag{4.30}$$

$$\cos^2\varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{d^{k-1}}{dM^{k-1}} \left[(-2)\sin M \cos M \sin^k M\right] = \cos^2 M$$
$$+ \ \varepsilon(-2)\sin M \cos M \sin M + \frac{\varepsilon^2}{1\cdot 2}\frac{d}{dM}\left[(-2)\sin M \cos M \sin^2 M\right] + \cdots$$
$$= \cos^2 M - 2\ \varepsilon \sin^2 M \cos M + \frac{\varepsilon^2}{2}(-2)\left(3\sin^2 M \cos^2 M - \sin^4 M\right) + \cdots \tag{4.31}$$

$$\sin\varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{d^{k-1}}{dM^{k-1}} [\cos M \sin M] = \sin M + \ \varepsilon \cos M \sin M$$
$$+ \frac{\varepsilon^2}{1\cdot 2}\frac{d}{dM}\left[\cos M \sin^2 M\right] + \ldots = \sin M + \ \varepsilon \cos M \sin M \tag{4.32}$$
$$+ \frac{\varepsilon^2}{1\cdot 2}\left[2\sin M \cos^2 M - \sin^3 M\right] + \cdots$$

We write the expressions for $\sqrt{\Phi_0}$, Φ_0, $\dot{\Phi}_0$ using Eqs. 4.25 and 4.26 in the form

$$\sqrt{\Phi_0} = a(1 - \ \varepsilon \cos\varphi), \tag{4.33}$$

$$\Phi_0 = a^2\left(1 - 2\ \varepsilon \cos\varphi + \ \varepsilon^2 \cos^2\varphi\right), \tag{4.34}$$

$$\dot{\Phi}_0 = \sqrt{\frac{2}{A}}\ \varepsilon B \sin\varphi. \tag{4.35}$$

Substituting into (4.33)–(4.35) the expressions for $\cos\varphi$, $\cos^2\varphi$ and $\sin\varphi$ in the form of the Lagrangian series (4.30)–(4.32) we obtain

$$\sqrt{\Phi_0} = \frac{B}{A}\left[1 + \frac{\varepsilon^2}{2} + \left(-\ \varepsilon + \ \frac{3}{8}\ \varepsilon^3\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{3}{8}\ \varepsilon^3 \cos 3M + \cdots\right], \tag{4.36}$$

$$\Phi_0 = \frac{B^2}{A^2}\left[1 + \frac{3}{2}\ \varepsilon^2 + \left(-2\ \varepsilon + \frac{\varepsilon^3}{4}\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{\varepsilon^3}{4}\cos 3M + \cdots\right], \tag{4.37}$$

$$\dot{\Phi}_0 = \sqrt{\frac{2}{A}}\ \varepsilon B\left[\sin M + \frac{1}{2}\ \varepsilon \sin 2M + \frac{\varepsilon^2}{2}\sin M\left(2\cos^2 M - \sin^2 M\right) + \ldots\right]. \tag{4.38}$$

The series of Eqs. 4.36–4.38 obtained are put in order of increased power of parameter ε and are absolutely convergent at any value of M in the case when the parameter ε satisfies the condition

$$\varepsilon < \bar{\varepsilon} = 0,6627\ldots, \tag{4.39}$$

where $\bar{\varepsilon}$ is the Laplace limit.

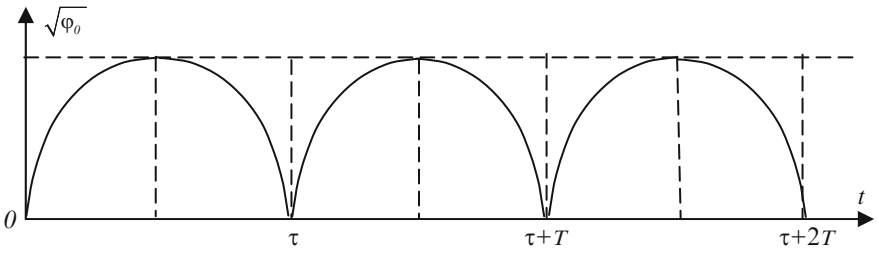

Fig. 4.4 Changes of the value $\sqrt{\Phi_0}$ in time at $\varepsilon = 1$

In some cases it is convenient to expand the values $\sqrt{\Phi_0}$, Φ_0, $\dot{\Phi}_0$ in the form of a Fourier series, using conventional methods (see, for example, Duboshin 1978). Figure 4.4 demonstrates the changes of $\sqrt{\Phi_0}$ in time at $\varepsilon = 1$.

It is also possible to consider the case solution of Jacobi's virial equation for $E_0 = 0$ and $E_0 > 0$. The reader can find here without difficulty a full analogy of these results as well as the solution of the two-body problem.

4.3 Solution of Jacobi's Virial Equation in Hydrodynamics

Let us consider the solution of the problem of the dynamics of a homogeneous isotropic gravitating sphere in the framework of traditional hydrodynamics and the virial approach we have developed.

4.3.1 The Hydrodynamic Approach

The sphere is assumed to have radius R_0 and be filled by an ideal gas with ρ_0. We assume that at the initial time the field of velocities which has the only component is described by equation

$$u = H_0 r, \tag{4.40}$$

where u is the radial component of the velocity of the sphere's matter at the distance r from the center of mass; H is independent of the quantity r and equal to H_0 at time t_0 .

We also assume that the motion of the matter of the sphere goes on only under action of the forces of mutual gravitational interaction between the sphere particles. In this case the influence of the pressure gradient is not taken into account, assuming that the matter of the sphere is sufficiently diffused. Then the symmetric spherical shells will move only under forces of gravitational attraction and will not coincide. In this case the mass of the matter of any sphere shell will keep its constant value and the condition (4.40) will be satisfied at any moment of time, and constant H should be dependent on time.

Under those conditions the Eulerian system of equations (3.28) can be written in the form

$$\rho \frac{\partial u}{\partial t} + \rho(u\nabla)u = \rho \frac{\partial U_G}{\partial r}, \tag{4.41}$$

where $\rho(t)$ is the density of the matter of the sphere at the moment of time t; u is the radial component of the velocity of matter at distance r from the sphere's center; U_G is the Newtonian potential for the considered point of the sphere.

The expression for the Newtonian potential U_G (3.29) can be written as follows:

$$U_G = G\frac{4}{3}\pi\rho r^2, \tag{4.42}$$

and the continuity equation will be

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial U}{\partial r} = 0. \tag{4.43}$$

Within the framework of the traditional approach, the problem is to define the sphere radius R and the value of the constant H at any moment of time, if the radius R_0, density ρ_0 and the value of the constant H_0 at the initial moment of time t_0 are given. If we know the values H(t) and R(t), we can then obtain the field of velocities of the matter within the sphere which is defined by Eq. 4.40, and also the density ρ of matter at any moment of time, using the relationship

$$\frac{4}{3}\pi R_0^3 \rho_0 = \frac{4}{3}\pi R^3 \rho = \text{const} = m.$$

Hence the formulated problem is reduced to identification of the law of motion of a particle which is on the surface of the sphere and within the field of attraction of the entire sphere mass $m = 4/3\ \pi\ \rho_0 {R_0}^3$.

The equation of motion for a particle on the surface of the sphere, which follows from Eq. 4.41 after transforming the Eulerian co-ordinates into a Lagrangian, has the form

$$\frac{d^2R}{dt^2} = -G\frac{m}{R^2}. \tag{4.44}$$

It is necessary to determine the law of change of R(t), resolving Eq. 4.44 at the initial data:

$$R(t_0) = R_0, \tag{4.45}$$

$$\left.\frac{dR}{dt}\right|_{t=t_o} = H_0 R_0.$$

We reduce the order of Eq. 4.44. To do so we multiply it by dR/dt:

$$\frac{dR}{dt}\frac{d^2R}{dt^2} = -\frac{dR}{dt}\frac{Gm}{R^2}$$

and integrate with respect to time:

$$\int_{t_0}^{t} \frac{1}{2}\frac{d}{dt}\left(\dot{R}\right)^2 = \int_{t_0}^{t} \frac{d}{dt}\left(\frac{Gm}{R}\right)dt.$$

After integration we obtain

$$\frac{1}{2}\dot{R}^2 - \frac{1}{2}\dot{R}_0^2 = \frac{Gm}{R} - \frac{Gm}{R_0}$$

or

$$\frac{1}{2}\dot{R}^2 = \frac{Gm}{R} + k, \tag{4.46}$$

where the constant k is determined as

$$\begin{aligned} k &= \frac{1}{2}\dot{R}_0^2 - \frac{Gm}{R_0} = \frac{1}{2}H_0^2R_0^2 - G\frac{4\pi}{3}\rho_0\frac{R_0^3}{R_0} \\ &= \frac{1}{2}H_0^2R_0^2\left[1 - \frac{8\pi}{3}\frac{G\rho_0}{H_0^2}\right] = \frac{1}{2}H_0^2R_0^2[1-\Omega] = \text{const} \end{aligned} \tag{4.47}$$

Here the quantity $\Omega = \rho_o/\rho_{cr}$, where $\rho_{cr} = 3H_0{}^2/8\pi G$.

Note that Eq. 4.46 obtained after reduction of the order of the initial Eq. 4.44 is in its substance the energy conservation law. Equation 4.46 permits the variables to be divided and can be rewritten in the form

$$\int_{R_0}^{R} \frac{dR}{\sqrt{\frac{2Gm}{R} + 2k}} = \int_{t_0}^{t} dt. \tag{4.48}$$

The plus sign before the root is chosen assuming that the sphere at the initial time is expanding, i.e., $H_0 > 0$.

The differential Eq. 4.46 has three different solutions at $k = 0, k > 0$ and $k < 0$ depending on the sign of the constant k, which is in its turn defined by the value of the parameter Ω at the initial moment of time. First we consider the case when $k = 0$ which relates, by analogy with the Keplerian problem, to the parabolic model at

$k = 0$. Equation 4.46 is easily integrated and for the expression case, i.e.,$\dot{R}>0$, we obtain

$$\dot{R}^2 = \frac{2Gm}{R},$$
$$\dot{R} = \frac{(2Gm)^{1/2}}{R^{1/2}},$$

from which it follows that

$$R^{1/2}dR = (2Gm)^{1/2}dt$$

or

$$\frac{2}{3}R^{3/2} = (2Gm)^{1/2}t + \text{const.} \tag{4.49}$$

We choose as initial counting time $t = 0$, the moment when $R = 0$. In this case the integration constant disappears:

$$R = \left(\frac{9}{2}Gm\right)^{1/3} t^{2/3}. \tag{4.50}$$

The density of the matter changes in accordance with the law

$$\rho(t) = \frac{m}{\frac{4}{3}\pi R^3} = \frac{1}{6\pi G t^2}, \tag{4.51}$$

and the quantity H(t), as a consequence of (4.50), has the form

$$H(t) = \frac{\dot{R}}{R} = \frac{2}{3}\frac{1}{t}. \tag{4.52}$$

For the case when $k > 0$, which corresponds to so-called hyperbolic motion, the solution of Eq. 4.46 can be written in parametric form (Zeldovich and Novikov 1967)

$$R = \frac{Gm}{2k}(ch\eta - 1), \tag{4.53}$$

$$t = \frac{Gm}{(2k)^{3/2}}(\{ch\eta - \eta),$$

where the constants of integration in (4.53) have been chosen so that $t = 0$, $\eta = 0$ at $R = 0$.

Finally we consider the case when $k < 0$, which corresponds to elliptic motion. At $k < 0$ the expansion of the sphere cannot continue for unlimited time and the expansion phase should be changed by attraction of the sphere.

The explicit solution of Eq. 4.46 at $k < 0$ can be written in parametric form (Zeldovich and Novikov 1967)

$$R = \frac{Gm}{2|k|}(1 - ch\eta), \tag{4.54}$$

$$t = \frac{Gm}{(2|k|)^{3/2}}(\eta - sh\eta).$$

The maximum radius of the sphere is determined from Eq 4.46 on the condition $dR/dt = 0$ and equals

$$R_{max} = \frac{Gm}{|E|}. \tag{4.55}$$

The time needed for expansion of the sphere from $R_0 = 0$ at $t_0 = 0$ to R_{max} is

$$t_{max} = \frac{\pi Gm}{(2|k|)^{3/2}}. \tag{4.56}$$

So the sphere should make periodic pulsations with period T_p equal to

$$T_p = \frac{2\pi Gm}{(2|k|)^{3/2}}. \tag{4.57}$$

The considered solution has important cosmologic applications.

4.3.2 The Virial Approach

We shall limit ourselves by formal consideration of the same problem in the framework of the condition of the dynamical equilibrium of a self-gravitating body based on the solution of Jacobi's virial equation, which we discussed earlier.

As shown in Chap. 3, Jacobi's virial equation (3.50), derived from Eulerian equations (3.28), is valid for the considered gravitating sphere. It was written in the form

$$\ddot{\Phi} = 2E - U, \tag{4.58}$$

where Φ is the Jacobi function for a homogeneous isotropic sphere and is defined by

$$\Phi = \frac{1}{2}\int_0^R 4\,\pi r^2 \rho r^2 dr = \frac{2\pi\rho R^5}{5} = \frac{3}{10}mR^2. \tag{4.59}$$

The potential gravitational energy of the matter of the sphere is expressed as

$$U = -4\pi G \int_0^R r\rho(r)m(r)dr = -\frac{16\pi^2}{15} G\rho^2 R^2 = -\frac{3}{5} G\frac{m^2}{R}. \tag{4.60}$$

The total energy of the sphere E will be equal to the sum of the potential U and kinetic T energies.

The kinetic energy T is expressed as

$$T = \frac{1}{2}\int_0^R 4\ \pi u^2 \rho r^2 dr = \frac{1}{2}\int_0^R 4\ \pi H^2 r^2 \rho r^2 dr = \frac{4\pi\rho H^2 R^5}{10} = \frac{3}{10} m H^2 R^2. \tag{4.61}$$

For a homogeneous isotropic gravitating sphere, the constancy of the relationship between the Jacobi function (4.59) and the potential energy (4.60) can be written:

$$|U|\sqrt{\Phi} = B = \frac{3}{5} G\frac{m^2}{R}\sqrt{\frac{3}{10} m R^2} = \frac{1}{\sqrt{2}}\left(\frac{3}{5}\right)^{3/2} G m^{3/2}, \tag{4.62}$$

where B has constant value because of the conservation law of mass m of the considered sphere.

The total energy E of the sphere also has a constant value:

$$E = T + U = \frac{A}{2}. \tag{4.63}$$

Then, if the total energy of the sphere has a negative value, Jacobi's virial equation can be written in the form:

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}. \tag{4.64}$$

Let us consider the conditions under which the total energy of the system will have a negative value. For this purpose we write it explicitly:

$$E = T + U = -\frac{16}{15}\ \pi^2 G\rho^2 R^5 + \frac{2\pi\rho H^2 R^5}{5} = \frac{2}{5}\pi\rho H^2 R^5\left[1 - \frac{8\pi G\rho}{3H^2}\right]. \tag{4.65}$$

It is clear from Eq. 4.65 that the total energy E has a negative value, when $\rho > \rho_c$, where $\rho_c = 3H^2/8\pi G$.

The general solution of Eq. 4.64 has the form of Eqs. 4.14 and 4.15:

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos(\lambda - \psi),] \tag{4.66}$$

$$t = \frac{4B}{(2A)^{3/2}}[\,\lambda - \varepsilon \sin(\,\lambda - \psi)], \tag{4.67}$$

where ε and ψ are constants dependent on the initial values of the Jacobi function Φ_0 and its first derivative $\dot{\Phi}_0$ at the moment of time t_o. The constants ε and ψ are determined by Eqs. 4.23 and 4.24 accordingly.

If we express all the constants in Eq. 4.23:

$$\varepsilon = \sqrt{1 - \frac{A}{2B^2}\left(-\dot{\Phi}_0 + 4B\sqrt{\Phi_0} - 2A\Phi_0\right)}\Big|_{t = t_0} = \text{const}. \tag{4.68}$$

through mass m of the system, it is not difficult to see that

$$-\dot{\Phi}_0^2 + 4B\sqrt{\Phi_0} - 2A\Phi_0 = 0.$$

Then the constant ε will be equal to zero. Hence the solutions (4.28) and (4.29) coincide with the solution (4.54), which was obtained in the framework of the traditional hydrodynamic approach. In this case the period of eigenpulsations of the Jacobi function (the polar moment of inertia) of the sphere $T = 8\pi R/(2A)^{3/2}$ will be equal to the period of change of its radius $T_p = 2\pi Gm/(2|k|)^{3/2}$ obtained from Eq. 4.54.

4.4 The Hydrogen Atom as a Quantum Mechanical Analogue of the Two-Body Problem

Let us consider the problem concerning the energy spectrum of the hydrogen atom, which is a unique example of the complete conformity of the analytical solution with experimental results. The problem consists of a study of all the forms of motion using the postulates of quantum mechanics and based on the solution of Jacobi's virial equation.

The classical Hamiltonian in the two-body problem is written as

$$H = \frac{\bar{p}_1^2}{2m_1} + \frac{\bar{p}_2^2}{2m_2} + U(|\bar{r}_1 - \bar{r}_2|), \tag{4.69}$$

where

$$\bar{p}_1 = \frac{\partial H}{\partial \dot{\vec{r}}_1} = m_1 \dot{\bar{r}}_1,$$

$$\bar{p}_2 = \frac{\partial H}{\partial \dot{\vec{r}}_2} = m_2 \dot{\bar{r}}_2,$$

which after separation of the center of mass can be transformed into the form

$$H = \frac{\bar{P}^2}{2M} + \frac{\bar{p}^2}{2m} + U(r), \tag{4.70}$$

where $r = |\bar{r}_1 - \bar{r}_2|$ is the distance between two particles and

$$P = M\dot{\bar{R}}; \quad p = m\dot{\bar{r}}; \quad M = m_1 + m_2;$$

$$R = \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{m_1 + m_2}; \quad m = \frac{m_1 m_2}{m_1 + m_2}.$$

We obtain the Hamiltonian operator for the quantum mechanical two-body problem through changing the pulses and radii by the corresponding operators with the communication relations

$$[\hat{p}_i, \hat{p}_0] = -i\hbar\ \delta_{ik},$$

$$[\hat{p}_i, \hat{r}_k] = -i\hbar\ \delta_{ik}.$$

Then

$$\hat{H} = -\frac{\hbar^2}{2M}\Delta_R - \frac{\hbar^2}{2m}\Delta_r + \hat{U}(r).$$

The wave function$u(\bar{r}_1,\bar{r}_2) = \varphi(\overline{R})\psi(r)$, which satisfies the Schrödinger equation

$$\hat{H}u = \varepsilon u,$$

describes the motion of the inertia center (the free motion of the particle of mass m_c is described by the function $\varphi(R)$ and the motion of the particle of mass m in the U(r) is described by the wave function $\Psi(r)$). Subsequently we consider only the wave function of the motion of particle m.
The Schrödinger equation

$$\Delta\Psi + \frac{2m}{\hbar^2}[E - U(r)]\Psi = 0$$

written here for the stationary state in a central symmetrical field in spherical coordinates, has the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\Psi}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2\Psi}{\partial\varphi^2}\right] + \frac{2m}{\hbar^2}[E - U(r)]\ \psi = 0. \tag{4.71}$$

Using the Laplacian operator $\hat{\ell}^2$:

$$\hat{\ell}^2 = \left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2}{\partial\varphi^2}\right],$$

we obtain

$$\frac{\hbar^2}{2m}\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{\hat{\ell}^2}{r^2}\Psi\right] + U(r)\Psi = E\Psi.$$

The operators $\hat{\ell}^2$ and $\hat{\ell}_z$ ($\hat{\ell}_z = -i\partial/\partial\varphi$) commutate with the Hamiltonian $\hat{H}(r)$ and therefore there are common eigenfunctions of the operators $\hat{H}$, $\hat{\ell}^2$ и $\hat{\ell}_z$. We consider only such solutions of Schrödinger equations. This condition determines the dependence of the function Ψ on the angles

$$\Psi(r, \Theta, \varphi) = R(r)Y_{\ell k}(\Theta, \varphi),$$

where the quantity $Y_{\ell k}(\Theta, \varphi)$ is determined by the expression

$$Y_{\ell k}(\Theta, \varphi) = \frac{1}{\sqrt{2\,\pi}}e^{ik\varphi}(-1)^k i^\ell \sqrt{\frac{(2\ell+1)(\ell-k)!}{2(\ell+k)!}}P_\ell^k(\cos\Theta),$$

and $P_\ell^k(\cos\Theta)$ is the associated Legendre polynomial, which is

$$P_\ell^k(\cos\Theta) = \frac{1}{2^\ell \ell!}\sin^k\Theta\frac{d^{r+\ell}}{d\cos\Theta^{r+\ell}}\left(\cos^2\Theta - 1\right)^\ell.$$

Since

$$\hat{\ell}^2 Y_{\ell k} = \ell(\ell+1)Y_{\ell k},$$

we obtain for the radial part of the wave function R(r)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right) - \frac{\ell(\ell+1)}{r^2}R + \frac{2m}{\hbar^2}[E - U(r)]R = 0. \tag{4.72}$$

Equation 4.72 does not contain the value $\ell_z = m$, i.e. at the given ℓ the energy level E corresponds to $2\ell + 1$ states differing by the value ℓ_z.
The operator

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{d\Psi}{dr}\right).$$

is equivalent to the expression

$$\frac{1}{r}\frac{d^2}{dr^2}(rR)$$

and thus it is convenient to make the change of variables, assuming that

$$X(r) = rR(r).$$

So that Eq. 4.71 can be rewritten in the form

$$\frac{d^2X}{dr^2} - \frac{\ell(\ell+1)}{r^2}X + \frac{2m}{\hbar^2}[E - U(r)]X = 0. \tag{4.73}$$

We now consider the demand following from the boundary conditions and related to the behavior of the wave function X(r). At $r \to 0$ and the potentials satisfying the condition

$$\lim U(r)r^2 = 0, \tag{4.74}$$

$$r \to 0;$$

only the first two terms play an important role in Eq. 4.73. $X(r) \sim r^v$ and we obtain

$$v(v-1) = \ell(\ell+1).$$

This equation has roots $v_1 = \ell + 1$ and $v_2 = -\ell$.

The requirement of normalization of the wave function is incompatible with the values $v = -\ell$ at $\ell \neq 0$ because the normalization integral

$$\int_0^\infty |X_r^2(r)dr|$$

will be divergent for the discrete spectrum, and the condition

$$\int \Psi(\lambda, \xi)\Psi(\lambda, X)d\lambda = \delta(X - \xi)$$

does not hold for the continuous spectrum.

At $\ell = 0$ the boundary conditions are determined by the demand for the finiteness of the mean value of the kinetic energy which is satisfied only at $v = 1$. So, when the condition (4.74) is satisfied, then the wave function of a particle is everywhere finite and at any ℓ

$$X(0) = 0.$$

Let us consider the energy spectrum and the wave function of the bounded states of a system of two charges. The bounded states exist only in the case of the attracted particles. Such a system defines the properties of the hydrogen atom and hydrogen-like ions.

The equation for the radial wave function is

$$\frac{d^2R}{dt^2}+\frac{2}{r}\frac{dR}{dr}-\frac{\ell(\ell+1)}{r^2}R+\frac{2m}{\hbar^2}\left(E+\frac{\alpha}{r}\right)R=0, \tag{4.75}$$

where $\alpha = Ze^2$ is constant, characterizing the potential; e is the electron charge; Z is the whole number equal to the nucleus charge in the charge units.

The constants e^2, m and $\hbar$ allow us to construct the value with the dimension of length

$$a_0=\frac{\hbar^2}{me^2}=0.529\cdot 10^{-8}\text{cm}$$

known as the Bohr radius, and the time

$$t_0=\frac{\hbar^3}{me^4}=0.242\cdot 10^{-11}\text{s}.$$

These quantities define the typical space and time scale for describing a system, and it is therefore convenient to use these units as the basic system of atomic units. Equation 4.75 in atomic units (at $Z = 1$) takes the form

$$\frac{d^2R}{dt^2}+\frac{2}{r}\frac{dR}{dr}-\frac{\ell(\ell+1)}{r^2}R+2\left(E+\frac{1}{r}\right)R=0. \tag{4.76}$$

At $E < 0$ the motion is finite and the energy spectrum is discrete. We need the solutions (4.76) quadratically integrable with r^2. Let us introduce the specification

$$n=\frac{1}{\sqrt{-2E}}\qquad \rho=\frac{2r}{n}.$$

Equation 4.76 can be written as

$$\frac{d^2R}{dt^2}+\frac{2}{\rho}\frac{dR}{d\rho}+\left[\frac{n}{\rho}-\frac{1}{4}-\frac{\ell(\ell+1)}{\rho^2}\right]R=0. \tag{4.77}$$

We find the asymptotic forms of the radial function R(r). At $\rho \to \infty$ and omitting the terms $\sim \rho^{-1}$ and $\sim \rho^{-2}$ in (4.77), we obtain

$$\frac{d^2R}{d\rho^2}=\frac{R}{4}.$$

Therefore at high values of ρ, $R \propto e^{\pm\rho/2}$. The normalization demand is satisfied only by $R(\rho) \propto e^{-\rho/2}$. The asymptotic forms at $r \to 0$ have already been determined.

Substituting

$$R(\rho) = \rho^{\ell} e^{-\rho/2}\, \omega(\rho),$$

Equation 4.77 is reduced to the form

$$\rho \frac{d^2\omega}{d\rho^2} + (2\ell + 2 - \rho)\frac{d\omega}{d\rho} + (n - \ell - 1)\,\omega = 0. \tag{4.78}$$

To solve this equation in the limit of $\rho = 0$, we substitute $\omega(\rho)$ in the form of a power series

$$\omega(\rho) = 1 + \frac{(0-v)}{(0+\lambda)}\rho + \frac{(0-v)(1-v)}{(0+\lambda)(1+\lambda)}\frac{\rho^2}{2!} + \frac{(0-v)(1-v)(2-v)}{(0+\lambda)(1+\lambda)(2+\lambda)}\frac{\rho^3}{3!} + \ldots, \tag{4.79}$$

where $\lambda = 2\ell + 2$ and $-v = -n + \ell + 1$.

At $\rho \to \infty$, the function $\omega(\rho)$ should increase, but not faster than the limiting power ρ. Then $\omega(\rho)$ has to be a polynomial of v power. So, $-n + \ell + 1 = -k$, and $n = \ell + 1 + k$ ($k = 0, 1, 2, \ldots$) at a given value of ℓ. Hence, using the definition for n, we can find the expression for the energy spectrum

$$E_n = -\frac{1}{2n^2}. \tag{4.80}$$

The number n is called the principal quantum number. In general units it has the form

$$E = -Z^2 \frac{me^4}{2\hbar^2 n^2}. \tag{4.81}$$

This formula was obtained by Bohr in 1913 on the basis of the old quantum theory, by Pauli in 1926 from matrix mechanics, and by Schrödinger in 1926 by solving the differential equations.

Let us solve the problem of the spectrum of the hydrogen atom using the equation of dynamical equilibrium of the system. In Chap. 3 we obtained Jacobi's virial equation for a quantum mechanical system of particles whose interaction is defined by the potential being a homogeneous function of the co-ordinates. This equation in the operator form is

$$\ddot{\hat{\Phi}} = 2\hat{H} - \hat{U}v \tag{4.82}$$

where $\hat{\Phi}$ is the operator of the Jacobi function, which, for the hydrogen atom, is written

$$\hat{\Phi} = \frac{1}{2}\mathrm{m}\hat{\mathrm{r}}^2, \tag{4.83}$$

The Hamiltonian operator is

$$\hat{\mathrm{H}} = -\frac{\hbar^2}{2\mathrm{m}}\Delta_{\mathrm{r}} + \hat{\mathrm{U}} \tag{4.84}$$

and the operator of the function of the potential energy for the hydrogen atom is

$$\hat{\mathrm{U}} = -\frac{\mathrm{e}^2}{\mathrm{r}}. \tag{4.85}$$

We solve the problem with respect to the eigenvalues of Eq. 4.82, using the main idea of quantum mechanics. For this we use the Schrödinger equation

$$\hat{\mathrm{H}}\Psi = \mathrm{E}\Psi$$

and rewrite Eq. 4.82 in the form

$$\ddot{\hat{\Phi}} = 2\mathrm{E} - \hat{\mathrm{U}}v \tag{4.86}$$

This equation includes two (unknown in the general case) operator functions $\hat{\Phi}$ and $\hat{\mathrm{U}}$. In the case of the interaction, the potential is determined by the relation (4.85), and we can use a combination of the operators $\hat{\Phi}$ and $\hat{\mathrm{U}}$ in the form

$$|\hat{\mathrm{U}}|\sqrt{\hat{\Phi}} = \frac{\mathrm{e}^2\mathrm{m}^{1/2}}{\sqrt{2}} = \mathrm{B}. \tag{4.87}$$

We now transform (4.86) into the form which was considered in classical mechanics:

$$\ddot{\hat{\Phi}} = 2\mathrm{E} + \frac{\mathrm{B}}{\sqrt{\hat{\Phi}}}. \tag{4.88}$$

Equation 4.88 is a consequence of Eq. 4.86 when the Schrödinger equation and the relationship (4.87) are satisfied. Its solution for the bounded state, i.e. when total energy E is determined in parametric form, can be written

$$\sqrt{\Phi} = \frac{\mathrm{B}}{2|\mathrm{E}|}(1 - \varepsilon\cos\varphi), \tag{4.89}$$

$$\varphi - \varepsilon \sin \varphi = M, \tag{4.90}$$

where the parameter M is defined by the relation

$$M = \frac{(4|E|)^{3/2}}{4B}(t - \tau), \tag{4.91}$$

where ε and τ are integration constants and where

$$\varepsilon = \sqrt{1 - \frac{AC}{2B^2}},$$

$$C = -\dot{\hat{\Phi}}_0^2 + 4B\sqrt{\hat{\Phi}_0} - 2A\hat{\Phi}_0.$$

Moreover, the solution can be written in the form of Fourier and Lagrange series. Thus, the expression (4.37) describes the expansion of the operator $\hat{\Phi}$ into a Lagrange series including the accuracy of ε^3, and has the form

$$\hat{\Phi}_0 = \frac{B^2}{A^2}\left[1 + \frac{3}{2}\varepsilon^2 + \left(-2\varepsilon + \frac{\varepsilon^3}{4}\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{\varepsilon^3}{4}\cos 3M + ...\right] \tag{4.92}$$

Using the general expression for the mean values of the observed quantities in quantum mechanics

$$<\Psi|\hat{\Phi}|\Psi> = \bar{\Phi}\nu$$

and taking into account that the mean value of the Jacobi function of the hydrogen atom should be different from zero, we find that our system has multiple eigen-frequencies $v_n = nv_o$ with respect to the basic v_o which corresponds to the period

$$T_\nu = \frac{8\pi B}{(4|E|)^{3/2}}. \tag{4.93}$$

In accordance with the expression

$$E_n = \hbar\,\omega_n = \frac{\hbar 2\pi n}{T_0} \tag{4.94}$$

each of these frequencies corresponds to the energy level E_n of the hydrogen atom. We substitute the expression (4.93) for T_ν into Eq. 4.94 and resolve it in relation to E_n:

$$|E_n| = \frac{\hbar 2\pi n(4|E_n|)^{3/2}}{8\,\pi B} = \frac{\hbar n(4|E_n|)^{3/2}}{\frac{4e^2m^{1/2}}{\sqrt{2}}} = \frac{\hbar n 2\sqrt{2|E_n|}}{e^2m^{1/2}}. \tag{4.95}$$

The expression obtained by Bohr follows from (4.95):

$$E_n = \frac{e^4 m}{2\hbar^2 n^2}. \tag{4.96}$$

This equation solves the problem.

4.5 Solution of a Virial Equation in the Theory of Relativity (Static Approach)

We consider now the solution of Jacobi's virial equation in the framework of the theory of relativity, showing its equivalence to Schwarzschild's solution.

Let us write down the known expression for the radius of curvature of space-time as a function of mass density:

$$\frac{1}{R^2} = \frac{8\pi}{3}\frac{G\rho}{c^2}, \tag{4.97}$$

where R is the curvature radius; ρ is the mass density; G is the gravitational constant and c is the velocity of light.

Equation 4.97 can also be rewritten in the form

$$\rho R^2 = \frac{3}{8\pi}\frac{c^2}{G}. \tag{4.98}$$

If the product ρR^2 in Eq. 4.98 is the Jacobi function ($\Phi = \rho R^2$ is the density of the Jacobi function) then, from (4.98):

$$\Phi = \frac{3}{8\pi}\frac{c^2}{G}. \tag{4.99}$$

and it follows that the Jacobi function is a fundamental constant for the Universe. (In general relativity, the spatial distance does not remain invariant. Therefore, instead of this the Gaussian curvature is used, which has the dimension of the universe distance and is the invariant or, more precisely, the covariant.)

The constancy of the Jacobi function in this case reflects the smoothness of the description of motion in general relativity. The oscillations relative to this smooth motion described by Jacobi's equation are the gravitational waves and horizons, in particular the collapse and all types of singularity up to the process of condensation of matter in galaxies, stars etc.

Now we can show that Schwarzschild's solution in general relativity is equivalent to the solution of Jacobi's equation when $\ddot{\Phi} = 0$. Let us write the expression for the energy-momentum tensor

$$T_i^k = (\rho + p)u_i u^k + p\delta_i^k. \tag{4.100}$$

In the corresponding co-ordinate system, we obtain

$$u^i = \left(0, 0, 0, \frac{1}{\sqrt{-g_{00}}}\right), \tag{4.101}$$

where $\rho = \rho(r)$ and $p = p(r)$.

The independent field equations are written

$$G_1^1 = T_1^1, \quad G_0^0 = T_0^0, \tag{4.102}$$

$$R^{-2} = \frac{1}{3}G\rho c^2.$$

The expression for the metric is written in the form

$$ds^2 = \frac{dr^2}{1 - \dfrac{r^2}{R^2}} + r^2(d\Omega)^2 - \left\{A - B\sqrt{1 - \frac{r^2}{R^2}}\right\}^2 c^2 r^2, \tag{4.103}$$

where

$$\frac{dr^2}{1 - \dfrac{r^2}{R^2}} + r^2(d\Omega)^2.$$

is the spatial element.

In this case the expression for the volume occupied by the system is written

$$V = \int_0^r \int_0^\pi \int_0^{2\pi} \frac{r^2 \sin\Theta}{\sqrt{1 - \dfrac{r^2}{R^2}}} dr d\Theta d\Psi = \frac{4\pi R^3}{3}\left[\arcsin\frac{r}{R} - \frac{r}{R}\sqrt{1 - \frac{r^2}{R^2}}\right] \tag{4.104}$$

It can be easily verified that the right-hand side of Eq. 4.104 coincides with solution (4.14) and (4.15) of the equation of virial oscillations (4.11) at $\ddot{\Phi} = 0$, i.e.,

$$\arcsin x - x\sqrt{1 - x^2} = \arccos\left(\frac{\dfrac{A}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \dfrac{AC}{2B^2}}}\right) - \sqrt{1 - \frac{AC}{2B^2}} \times \sqrt{1 - \left(\frac{\dfrac{A}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \dfrac{AC}{2B^2}}}\right)^2}. \tag{4.105}$$

In fact, Eq. 4.105 is satisfied for

$$x = \frac{\frac{A}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \frac{AC}{2B^2}}} \text{ and } x = \sqrt{1 - \frac{AC}{2B^2}},$$

i.e.

$$\frac{A}{B}\sqrt{\Phi} - 1 = 1 - \frac{AC}{2B^2}, \text{ or } \frac{AC}{2B^2} + \frac{A\sqrt{\Phi}}{B} = 2.$$

At $\ddot{\Phi} = 0$, the parameter of virial oscillations,

$$e = \sqrt{1 - \frac{AC}{2B^2}} \text{ and } \sqrt{\Phi} = \frac{B}{A},$$

so the last condition is satisfied.

Schwarzschild's solution is rigorous and unique for Einstein's equation for a static model of a system with spherical symmetry.

Since this solution coincides with the solution of virial oscillations at the same conditions, the solutions (4.14) and (4.15) of Eq. 4.11, obtained in this chapter, should be considered rigorous. Thus we can conclude that the constancy of the product $U\sqrt{\Phi}$ in the framework of the static system model is proven. In Chap. 6 we will come back to this condition and will obtain another proof of the same very important relationship which is applied for study of the Earth's dynamics.