Chapter 2
Recent Observations and Understanding Physical Meaning of Jacobi's Virial Equation

Fundamentals of all the planetary and Solar System sciences are tested first of all by the laws of the Earth movement, where the confidence limit to the laws can be checked by observation. More over, all the sense of human being is connected with this planet. As far as the techniques and instruments for observation were developed, then geodesists, astronomers and geophysicists have noticed that in the planet's inertial rotation some irregularities and deviations relative to the accepted standard parameters and hydrostatic state conditions have appeared. Those irregularities that are often called as inaccuracies, number of which is counted by more than dozen, finally were incorporated into two problems, namely, variation of the angular velocity in the daily, monthly, annually and secular time scale, and variation in the poles motion in the same time scales. Just after the problems became obvious and have not find resolution in the frame work of the accepted physical and theoretical conceptions of celestial mechanics the latter lost interest in the problems of the Earth dynamics. In this connection the well known German theoreticians in dynamics, Klein and Sommerfeld, stated that the Earth mechanics appear to be more complicated than the celestial mechanics and represents "*some confused labyrinths of geophysics*" (Klein and Sommerfeld 1903). In order to study irregular velocity of the Earth rotation and the pole motion numerous projects of observation and regular monitoring were organized by the planetary network. As it was always in such cases, the cause of the observed effects was searched in the effects of perturbations coming from the Moon and the Sun, and also in the influence of dynamical effects of the own shells like the atmosphere, the oceans and the liquid core, existence of which is considered by many researchers.

2.1 Dynamical Effects Discovered by Space Study

Artificial satellites, which made a start of space study in the second part of the twentieth century, opened a new page in space sciences. It was determined that the ultimate goal of this scientific program should be an answer to the question of

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the Solar system origin. Investigation of the near Earth cosmic space for solution of geodetic and geophysical problems was in the beginning initiated.

The first geodetic satellites for studying dynamic parameters of the planet were launched almost 50 years ago. They gathered vast amounts of data that significantly improved our knowledge of the inner structure and dynamics of the Earth. They made it a real possibility to evaluate experimentally the correctness of basic physical ideas and hypotheses in astronomy, astrophysics, geophysics, geodesy and geology, and to compare theoretical calculations with observations. Success in this direction was achieved in a short period of time.

On the basis of satellite orbit measurements, the zonal, sectorial and tesseral harmonics of gravitational moments in expansion of the gravitational potential by a spherical function, up to tens, twenties and higher degrees were calculated. The calculations have resulted in an important discovery having far-reaching effects. The obtained results proved the long-held assumption of geophysicists that the Earth does not stay in hydrostatic equilibrium, which, in fact, is the basic principle of the theories of dynamics, figure and inner structure of the planet. The same conclusion was made about the Moon.

This conclusion means that the physical conception of hydrostatic equilibrium state of the Earth which was applied for construction of model of the outer and central force field does not satisfy the observed dynamic effects of gravitational interaction of mass particles and should be revised. But the state of scientific knowledge of this phenomenon has been found to be not ready to cope with such a situation. The story of the condition of hydrostatic equilibrium of the planet begins with Newton's consideration, in his famous work *Philosophiae Naturalis Principia Mathematica*, of the Earth's oblateness problem. The investigation based on hydrostatics was further developed by French astronomer and mathematician Clairaut. Later on the hypothesis of hydrostatic equilibrium was extended to all celestial bodies including stars. The authority of Newton was always so high that any other theories for solution of the problem in dynamics and celestial body structure were never proposed. But in current times the problem has arisen of the cause of the discrepancy between theory and observation and a moment has come to take over this crisis in the study of fundamentals of the Earth sciences. A situation like this happened at the beginning of the twentieth century when radioactive and roentgen radiation was discovered and the corpuscular-wave nature of light was proved. This was the starting point for development of quantum mechanics. We seem now to have a similar situation with respect to the planets motion.

We found a still more serious discrepancy related to the Earth hydrostatic equilibrium, which is as follows (Ferronsky and Ferronsky 2007). It is known that the planet's potential energy is almost 300 times more than the kinetic one represented by the body's rotation. This ratio between the potential and kinetic energy contradicts the requirement of the virial theorem according to which the potential energy of a body in the outer uniform force field should be twice as much of the kinetic one.

From point of view of the observed potential energy the Earth's angular velocity should be about 17 times as much as it is. However, the planet has remained for a long time in an equilibrium state. In fact, the Earth appears to have been deprived of its kinetic energy. Some of the other planets, such like Mars, Jupiter, Saturn, Uranus and Neptune, exhibit the same behavior. But for the Mercury, Venus, our Moon and the Sun, the equilibrium states of which are also accepted as hydrostatic, the potential energy exceeds their kinetic energy by 10^4 times. A logical explanation comes to mind that there is some hidden form of motion of the body's interacting mass particles, together with their respective kinetic energy, which has not been taken previously into account. It is known that the hydrostatic equilibrium condition of a body, being stay in the outer force field, satisfies the requirement of the Clausius virial theorem. The same requirement follows also from the Eulerian equations for a liquid-filled uniform sphere. The virial theorem gives an averaged relationship between the potential and kinetic energies of a body. A periodic component of the energy change there during the corresponding time interval is accepted as a constant value and eliminated from consideration. From this evidence it was not difficult to guess that the hidden form of motion and the source of needed kinetic energy of the Earth and the planets including the Moon and the Sun might be found in that eliminated periodic component. In the problem considered by Newton, that component was absent because of his concept of the central gravitational force field, the total sum of which is equal to zero.

Taking into account the relationship between the Earth's gravitational moments and the gravitational potential observed by the satellites, we came back to derivation of the virial theorem in classical mechanics (see below) and obtained its generalized form of the relationship between the energy and the polar moment of inertia of a body. Doing so, we obtained the equation of dynamical equilibrium of a body in its own force field where the hydrostatic equilibrium is a particular case of a uniform body in its outer force field. The equation establishes a relationship between the potential and kinetic energies of a body by means of energy of oscillation of the polar moment of inertia in the form of the energy conservation law. An analytical expression of the derived new form of the virial theorem, based on the Newton's laws of motion, appeared to be the Jacobi's virial equation. In this case the earlier lost of kinetic energy is found by taking into account the oscillating motion of the interacting mass particles, the integral effect of which is expressed through oscillation of the polar moment of inertia. That effect fits the relationship between the potential and kinetic energies in the classical virial theorem. At the same time a novel physical conception about gravitation and electromagnetic interaction is appeared and mechanism of the energy generation becomes clear. The nature of the gravity forces as a derivative of the body's inner energy appears to be discovered.

The obtained, on the basis of Jacobi dynamics, results related to the problem of the Earth are as follows. We found that the new effect, which creates dynamics of the Earth, is its own inner force field. Earlier, the sum of the inner forces and their moments being effected by the outer central force field were considered as equal to zero. We find that the mass forces of interaction being volumetric ones created the inner force field which appears to be the field of power (energy) pressure. That field, according to its definition, can not be equal to zero. The resultant of the field pressure appears to be a space envelope. The envelope has a spherical shape for a sphere and an elliptic shape for an ellipsoid. It was found that dynamic effects of the body's force field occur in oscillation and rotation of the shells according to Kepler's laws. A body that has a uniform mass density distribution realizes all its kinetic energy of the motion in the form of so-called virial oscillations. It was assumed, earlier, that wave properties of this nature, like oscillations for mass particles in mechanics of bodies, are unessential. We found that virial oscillations of a body initiated by the force field of its own interacting mass particles represent the main part of its kinetic energy. Theories based on hydrostatics ignore that energy. But, as it was noted above, in this case the potential energy of the Earth and other celestial bodies by two or more orders exceeds their kinetic energy represented only in the form of axial rotation of the mass. Such an unusual effect has a simple physical explanation. Still in the beginning of the last century French physicist Louis de Broglie expressed an assumption, proved later on, that any micro-particle including electron, proton, atom and molecule, acquires particle-wave properties. The relationship, discovered by the artificial satellites between changes of the Earth's gravitational potential and the moment of inertia, shows that interaction of the planet's masses takes place on their elementary particle levels. It means that the main form of motion of the interacting mass particles is their oscillation. Continuous 'trembling' of the planet's gravitational field, detected by satellites as the gravitational moments change, is another fact proving the de Broglie idea and extending it to the gravitational interaction of celestial body masses.

The dynamical approach to solve the problem under consideration allowed the authors to expand the body's potential energy on its normal, tangential and dissipative components. The differential equations that determine the main body's dynamical parameters, namely its oscillation and rotation, were written. A rigorous solution of the equations was considered on the basis for bodies with spherical and axial symmetry. The solutions of problems relating to rotation, oscillation, obliquity and oblateness of a body's orbit and itself was considered on the basis of the general solution of dynamics of a self-gravitating body in its own force field. It was found that precession and wobbling of the Earth and irregularity of its rotation depends on effects of the polar and equatorial oblateness and the separate rotation of the planet's, the Sun's and the Moon's shells. The induced outer force field of a body follows rotation of the resultant envelope of the shells, but with some delay because of the finite velocity of the energy propagation in the induced outer force field. Also the problems of inner structure of the Earth, the nature of the planet's electromagnetic field and mechanism of the energy generation were considered. The presented theory is applicable not only to the planets and satellites, but also to the stars, where hydrostatic equilibrium is considered as an equation of state. Finally, the theory opens a way to understand the physics of gravitation as the internal power (energy) pressure which occurs at mass interaction on the level of the molecules, atoms and nuclei and elementary mass particles.

The obtained new results which have common relation to all celestial bodies are presented in the corresponding chapters and sections of the book.

2.2 Interpretation of Satellite Orbits and Failure of Hydrostatic Equilibrium of the Earth and the Moon

We recall briefly the conditions of the Earth hydrostatic equilibrium. By definition the hydrostatics is a branch of the hydromechanics, which studies the equilibrium of a liquid and gas and the effects of a stationary liquid on immersed bodies relative to the chosen reference system. For a liquid equilibrated relative to a rigid body, when its velocity of motion is equal to zero and the field of densities is steady the equation of state follows from the Eulerian and Navier–Stokes equations in the form (Sedov 1970)

$$\mathrm{grad}\, p = \rho F, \tag{2.1}$$

where p is the pressure; ρ is the density; F is the mass force.

In the Cartesian system of reference Eq. 2.1 is written as

$$\frac{\partial p}{\partial x} = \rho F_x, \qquad \frac{\partial p}{\partial y} = \rho F_y, \qquad \frac{\partial p}{\partial z} = \rho F_z \tag{2.2}$$

If the outer mass forces are absent, i.e. $F_x = F_y = F_z = 0,$ then

$$\mathrm{grad}\ p = 0.$$

In this case, in accordance with the Pascal's law the pressure in all liquid points will be the same.

For the uniform incompressible liquid, when $\rho = \mathrm{const}$, its equilibrium can be only in the potential field of the outer forces. For general case of incompressible liquid and potential field of the outer forces from (2.1) one has

$$dp = \rho dU, \tag{2.3}$$

where U is the forces potential.

It follows from Eq. 2.3, that for an equilibrated liquid in the potential force field its density and pressure appear to be a function only of the potential U.

For a gravity force field, when in the steady-state liquid only these forces act, one has

$$F_x = F_y = 0, \quad F_z = -g, \quad U = -gz + \mathrm{const} \quad \text{and} \quad p = p(z), \rho = \rho(z).$$

Here the surfaces of constant pressure and density appear as horizontal planes. Then Eq. 2.3 is written in the form

$$\frac{dp}{dz} = -\rho g < 0. \tag{2.4}$$

It means that with elevation the pressure falls and with depth it grows. From here it follows that

$$p - p_0 = -\int_{z_0}^{z} \rho g dz = -\rho g(z - z_0) \tag{2.5}$$

where g is the acceleration of the gravity force.

If a spherical vessel is filled in by incompressible liquid and rotates around its vertical axis with constant angular velocity ω, then for determination of the equilibrated free surface of the liquid in Eq. 2.2 the centrifugal inertial forces should be introduced in the form

$$\frac{\partial p}{\partial x} = \rho\,\omega^2 x,$$

$$\frac{\partial p}{\partial x} = \rho\,\omega^2 y, \tag{2.6}$$

$$\frac{\partial p}{\partial x} = -\rho g$$

From here, for the rotating body with radius $r^2 = x^2 + y^2$, one finds

$$p = -\rho g z + \frac{\rho \omega^2 r^2}{2} + C. \tag{2.7}$$

For the points on the free surface $r = 0$, $z = z_{\,o}$ one has $p = p_o$. Then

$$C = p_0 + \rho g z_0, \tag{2.8}$$

$$p = p_0 + \rho g(z_0 - z) + \frac{\rho \omega r^2}{2}. \tag{2.9}$$

The equation of the liquid free surface, where $p = p_o$, has a paraboloidal shape

$$z - z_0 = \frac{\omega^2 r^2}{2g}. \tag{2.10}$$

Above relations determine the principal physical conditions and equations of hydrostatic equilibrium of a liquid. They remain a basis for the modern dynamics and theory of the Earth figure. The attempt to harmonize these conditions with the planet's motion conditions has failed, which was proved by observation. It will be shown below that the main obstacle for such harmonization is rejection of the planet's inner force field without which the hydrostatics is unable to provide the equilibrium between the body's interacted forces as Newton's third law requires. The Earth is a self-gravitating body. Its matter moves in the own force field which is generated by elementary mass particle interaction. The mass density distribution, rotation and oscillation of the body's shells result from the inner force field. And the orbital motion of the planet is controlled by interaction of the outer force fields of the planet and the Sun.

Let us look for more specific effects determining the absence of the Earth hydrostatic equilibrium and more realistic conditions of its equilibrium based on the results of the Earth's satellite orbit motion.

The initial factual material for the problem study is presented by the observed orbit elements of the geodetic satellites which move on perturbed Kepler's orbits. The satellite motion is fixed by means of observational stations located within zones of a visual height range of 1,000–2,500 km, which is optimal for the planet's gravity field study. It was found that the satellite's perturbed motion at such a close distance from the Earth's surface is connected with the non-uniform distribution of mass density, the consequences of which are the non-spherical shape in the figure and the corresponding non-uniform distribution of the outer gravity field around the planet. These non-uniformities cause corresponding changes in trajectories of the satellite's motion, which are fixed by tracking stations. Thus, distribution of the Earth's mass density determines an adequate equipotential trajectory in the planet's gravity field, which follows the satellite. The main goal of the geodetic satellites launched under different angles relative to the equatorial plane is in measurement of all deviations in the trajectory from the unperturbed Kepler's orbit.

The satellite orbits data for solving the Earth's oblateness problem are interpreted on the basis of the known (in celestial mechanics) theory of expansion of the gravity potential of a body, the structure and the shape of which do not much differ from the uniform sphere. The expression of the expansion, by spherical functions, recommended by the International Union of Astronomy, is the following equation (Grushinsky 1976):

$$U(r,\varphi,\lambda) = \frac{GM}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_e}{r}\right)^n P_n(\sin\varphi) + \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_e}{r}\right)^n P_{nm}(\sin\varphi)(C_{nm}\cos m\,\lambda + S_{nm}\sin m\,\lambda)\right], \quad (2.11)$$

where r, φ and λ are the heliocentric polar co-ordinates of an observation point; G is the gravity constant; M and R_e are the mass and the mean equatorial radius of the Earth; P_n is the Legendre polynomial of n order; $P_{nm}(\sin\varphi)$ is the associated spherical functions; J_n, C_{nm}, S_{nm} are the dimensionless constants characterizing the Earth's shape and gravity field.

The first terms of Eq. 2.11 determine the zero approximation of Newton's potential for a uniform sphere. The constants J_n, C_{nm}, S_{nm} represent the dimensionless gravitational moments, which are determined through analyzing the satellite orbits. The values J_n express the zonal moments, and C_{nm} and S_{nm} are the tesseral moments. In the case of hydrostatic equilibrium of the Earth as a body of rotation, in the expression of the gravitational potential (2.11) only the even *n*-zonal moments J_n are rapidly decreased with growth, and the odd zonal and all tesseral moments turn into zero, i.e.

$$U = \frac{GM}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos\ \theta) - \sum_{n=3}^{\infty} J_n\left(\frac{R_e}{r}\right)^{n+1} P_n(\cos\ \theta)\right], \qquad (2.12)$$

where θ is the angle of the polar distance from the Earth's pole.

Here the constant J_2 represents the zonal gravitational moment, which characterizes the axial planet's oblateness and makes the main contribution to correction of the unperturbed potential. That constant determines the dimensionless coefficient of the moment of inertia relative to the polar axis and equal to

$$J_2 = \frac{C - A}{MR_e^2}, \qquad (2.13)$$

where C and A are the Earth's moments of inertia with respect to the polar and equatorial axes accordingly, and R_e is the equatorial radius.

For expansion by spherical functions of the Earth's gravity forces potential, the rotation of which is taken to be under action of the outer inertial forces, but not of its own force field, the centrifugal force potential is introduced into Eq. 2.12. Then for the hydrostatic condition with the even zonal moments J_n one has

$$W = \frac{GM}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos\theta) - \sum_{n=3}^{\infty} J_n\left(\frac{R_e}{r}\right)^{n+1} P_n(\cos\theta)\right] + \frac{\omega^2 r^2}{3}[1 - P_2(\cos\theta)], \qquad (2.14)$$

where W is the potential of the body of rotation; $\omega^2 r$ is the centrifugal force. The first two terms and the term of the centrifugal force in Eq. 2.14 express the normal potential of the gravity force

$$W = \frac{GM}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos\theta)\right] + \frac{\omega^2 r^2}{3}[1 - P_2(\cos\theta)]. \qquad (2.15)$$

The potential (2.15) corresponds to the spheroid's surface which within oblateness coincides with the ellipsoid of rotation. Rewriting term $P_2(\cos\theta)$ in this equation through the sinus of the heliocentric latitude and the angular velocity – through the geodynamic parameter q, one can find the relationship of the Earth's oblateness ε with the dynamic constant J_2. Then the equation of the dynamic oblateness ε is obtained in the form (Grushinsky 1976; Melchior 1972)

$$\varepsilon = \frac{3}{2} J_2 + \frac{q}{2}, \tag{2.16}$$

where the geodynamic parameter q is the ratio of the centrifugal force to the gravity force at the equator

$$q = \frac{\omega^2 R}{GM/R^2}. \tag{2.17}$$

Dynamic parameter J_2, found by satellite observation in addition to the oblateness calculation, is used for determination of a mean value of the Earth's moment of inertia. For this purpose the constant of the planet's free precession is also used, which represents one more observed parameter expressing the ratio of the moments of inertia in the form:

$$H = \frac{C - A}{C}. \tag{2.18}$$

This is the theoretical base for interpretation of the satellite observations. But its practical application gave very contradictory results (Grushinsky 1976; Melchior 1972; Zharkov 1978). In particular, the zonal gravitation moment calculated by means of observation was found to be $J_2 = 0.0010827$, from where the polar oblateness $\varepsilon = 1/298.25$ appeared to be short of the expected value and equal to 1/297.3. The all zonal moments J_n, starting from J_3, which relate to the secular perturbation of the orbit, were close to constant value and equal, by an order of magnitude, to the square of the oblateness i.e., ~$(1/300)^2$ and slowly decreasing with an increase of n. The tesseral moments C_{nm} and S_{nm} appeared to be not equal to zero, expressing the short-term nutational perturbations of the orbit. In the case of hydrostatic equilibrium of the Earth at the found value of J_2, the polar oblateness ε should be equal to 1/299.25. On this basis the conclusion was made that the Earth does not stay in hydrostatic equilibrium. The planet's deviation from the hydrostatic equilibrium evidenced that there is a bulge in the planet's equatorial region with amplitude of about 70 m. It means that the Earth body is forced by normal and tangential forces which develop corresponding stresses and deformations. Finally, by the measured tesseral and sectorial harmonics, it was directly confirmed that the Earth has an asymmetric shape with reference to the axis of rotation and to the equatorial plane.

Because the Earth does not stay in hydrostatic equilibrium, then the above described initial physical fundamentals for interpretation of the satellite observations should be recognized as incorrect and the related physical concepts cannot explain the real picture of the planet's dynamics.

The question is raised of how to interpret the obtained actual data and where the truth should be sought. First of all we should verify correctness of the oblateness interpretation and the conclusion about the Earth's equatorial bulge. It is known from observation that the Earth is a triaxial body (Grushinsky 1976). Theoretical application of the triaxial Earth model was not considered because it contradicts the hydrostatic equilibrium hypothesis. But after it was found that the hydrostatic equilibrium is absent, the alternative with the triaxial Earth should be considered first.

Let us analyze Eq. 2.16. It is known from the observation data, that the constant of the centrifugal oblateness q is equal to

$$q = \frac{\omega_3^2}{GM/R^3} = \left(\frac{1}{17.01}\right)^2 = \frac{1}{289.37}. \tag{2.19}$$

Determine a difference between the centrifugal oblateness constant q and the polar oblateness ε' found by the satellite orbits, assuming that the desired value has a relationship with the perturbation caused by the equatorial ellipsoid

$$\begin{aligned} \varepsilon' &= \frac{a-c}{a} - \frac{b-c}{a} = \frac{a-b}{a} = \frac{1}{289.37} - \frac{1}{298.25} = \frac{1}{9720} \\ &= 1.713\left(\frac{1}{289.37}\right)^2, \end{aligned} \tag{2.20}$$

where a, b and c are the semi-axes of the triaxial Earth.

The differences between the major and minor equatorial semi-axes can be found from Eq. 2.20. If the major semi-axis is taken in accordance with recommendation of the International Union of Geodesy and Geophysics as a = 6,378,160 m, then the minor equatorial semi-axis *b* can be equal to:

$$a - b = 6378160/9720 = 656\ \text{m}; \quad b = 6377504\,\text{m}.$$

There is a reason now to assume, that the value of equatorial oblateness $\varepsilon' = 1/9{,}720$ is a component in all the zonal gravitation moments J_n, related to the secular perturbations of the satellite orbits including J_2. They are perturbed both by the polar and the equatorial oblateness of the Earth. Experimental result of dependence of the satellite precession of equinoxes on the orbit angle to the equatorial plane proves the above statement. This effect ought to be expected because it was known long ago from observation that the Earth is a triaxial body. If our conclusion is true, then there is no ground for discussion about the equatorial bulge. And also the problem of the hydrostatic equilibrium is closed automatically because in this case the Earth is not a figure of rotation; and the nature of the observing fact of rotation of the Earth should be looked for rather in the action of its own inner force field but not in the effects of the inertial forces. As to the nature of the Earth's oblateness, then for its explanation later on the effects of perturbation arising during separation of the Earth's shells by mass density differentiation and separation of the Earth itself from the Protosun will be considered. In particular, the effect of heredity in creation of the body's oblateness is evidenced by the ratio of kinetic energy of the Sun and the Moon expressed through the ratio of square frequencies of oscillation ε'' of their polar moments of inertia, which is close to the planet's equatorial oblateness:

$$\varepsilon'' = \frac{\omega_c^2}{\omega_\pi^2} = \frac{\left(10^{-4}\right)^2}{\left(0.96576 \cdot 10^{-2}\right)^2} = \left(\frac{1}{96.576}\right)^2 = 1.73\left(\frac{1}{289.3}\right)^2,$$

where $\omega_c = 10^{-4}\ s^{-1}$ and $\omega_л = 0.96576 \cdot 10^{-2}\ s^{-1}$ are the frequencies of oscillation of the Sun's and the Moon's polar moment of inertia correspondingly.

By observation the Moon is also a triaxial body. In addition, the retrograde motion of the nodes of the Earth, the Moon and the artificial satellites is registered and is explained by rotation of the bodies' orbits. Later on it will be shown, that the above remarkable phenomenon is explained by rotation of the body's inner masses together with their gravity fields, the periods of which are equal to the periods of the precession of their oblique axes. The observed body rotation is valid only for the upper shells, which were separated during mass density differentiation in their own force fields and stay in that field in a suspended state of equilibrium.

The most prominent effect, which was discovered by investigation of the geodetic satellite orbits, is the fact of a physical relationship between the Earth's mean (polar) moment of inertia and the induced outer gravity field. That fact without exaggeration can be called a fundamental contribution to understanding the nature of the planet's self-gravity. The planet's moment of inertia is an integral characteristic of the mass density distribution. Calculation of the gravitational moments based on measurement of elements of the satellite orbits is the main content of satellite geodesy and geophysics. Short-periodic perturbations of the gravity field fixed at revolution of a satellite around the Earth, the period of which is small compared to the planet's period, provides evidence about oscillation of the moment of inertia or, to be more correct, about oscillating motion of the interacting mass particles. It will be shown, that oscillating motion of the interacting particles forms the main part of a body's kinetic energy and the moment of inertia itself is the periodically changing value.

Oscillation of the Earth's moment of inertia and also the gravitational field is fixed not only during the study by artificial satellites. Both parameters have also been registered by surface seismic investigations. Consider briefly the main points of these observations.

The study of the Earth's eigenoscillation started with Poisson's work on oscillation of an elastic sphere, which was considered in the framework of the theory of elasticity. In the beginning of the twentieth century Poisson's solution was generalized by Love in the framework of the problem solution of a gravitating uniform sphere of the Earth's mass and size. The calculated values of periods of oscillation were found to be within the limit of some minutes to 1 h.

In the middle of the twentieth century during the powerful earthquakes in 1952 and 1960 in Chile and Kamchatka an American team of geophysicists headed by Beneoff, using advanced seismographs and gravimeters, reliably succeeded in recording the an entire series of oscillations with periods from 8.4 min up to 57 min. Those oscillations in the form of seismograms have represented the dynamical effects of the interior of the planet as an elastic body, and the gravimetric records have shown the "tremor" of the inner gravitational field (Zharkov 1978). In fact, the effect of the simultaneous action of the potential and kinetic energy in the Earth's interior was fixed by the above experiments.

About 1,000 harmonics of different frequencies were derived by expansion of the line spectrum of the Earth's oscillation. These harmonics appear to be integral characteristics of the density, elastic properties and effects of the gravity field, i.e. of the potential and kinetic energy of separate volumetric parts of the non-uniform planet. As a result two general modes of the Earth's oscillations were found by the above spectral analysis, namely, spherical with a vector of radial direction and torsion with a vector perpendicular to the radius.

From the point of view of the existing conception about the planet's hydrostatic equilibrium, the nature of the observed oscillations was considered to be a property of the gravitating non-uniform (regarding density) body in which the pulsed load of the earthquake excites elementary integral effects in the form of elastic gravity quanta (Zharkov 1978). Considering the observed dynamical effects of earthquakes, geophysicists came close to a conclusion about the nature of the oscillating processes in the Earth's interior. But the conclusion itself still has not been expressed because it continues to relate to the position of the planet's hydrostatic equilibrium.

2.3 Imbalance Between the Earth's Potential and Kinetic Energy

We discovered the most likely serious cause, for which even formulation of the problem of the Earth's dynamics based on the hydrostatic equilibrium is incorrect. The point is that the ratio of kinetic to potential energy of the planet is equal to ~1/300, i.e. the same as its oblateness. Such a ratio does not satisfy the fundamental condition of the virial theorem, the equation of which expresses the hydrostatic equilibrium condition. According to that condition the considered energies' ratio should be equal to 1/2. Taking into account that kinetic energy of the Earth is presented by the planet's inertial rotation, then assuming it to be a rigid body rotating with the observed angular velocity $\omega_r = 7.29 \cdot 10^{-5}\ s^{-1}$, the mass $M = 6 \cdot 10^{24}$ kg, and the radius $R = 6.37 \cdot 10^6$ m, the energy is equal to:

$$T_e = 0.6MR^2\,\omega_r^2 = 0.6 \cdot 6 \cdot 10^{24} \cdot \left(6.37 \cdot 10^6\right)^2 \left(7.29 \cdot 10^{-5}\right)^2$$
$$= 7.76 \cdot 10^{29}\text{J} = 7.76 \cdot 10^{36}\text{erg}.$$

The potential energy of the Earth at the same parameters is

$$U_e = 0.6 \cdot GM^2/R \ = 0.6 \cdot 6.67 \cdot 10^{-11} \cdot \left(6 \cdot 10^{24}\right)^2/6.37 \cdot 10^6$$
$$= 2.26 \cdot 10^{32}\mathrm{J} \ = 2.26 \cdot 10^{39}\mathrm{erg}.$$

The ratio of the kinetic and potential energy comprises

$$\frac{T_e}{U_e} = \frac{7.76 \cdot 10^{29}}{2.26 \cdot 10^{32}} = \frac{1}{291}.$$

One can see that the ratio is close to the planet's oblateness. It does not satisfy the virial theorem and does not correspond to any condition of equilibrium of a really existing natural system because, in accordance with the third Newton's law, equality between the acting and the reacting forces should be satisfied. The other planets, the Sun and the Moon, the hydrostatic equilibrium for which is also accepted as a fundamental condition, stay in an analogous situation. Since the Earth in reality exists in equilibrium and its orbital motion strictly satisfies the ratio of the energies, then the question arises where the kinetic energy of the planet's own motion has disappeared. Otherwise the virial theorem for the Earth is not valid. Moreover, if one takes into account that the energy of inertial rotation does not belong to the body, then the Earth and other celestial bodies equilibrium problem appears to be out of discussion.

Thus, we came to the problem of the Earth equilibrium from two positions. From one side, the planet by observation does not stay in hydrostatic equilibrium, and from the other side, it does not stay in general mechanical equilibrium because there is no reaction forces to counteract to the acting potential forces. The answer to both questions is given below while deriving an equation of the dynamical equilibrium of the planet by means of generalization of the classical virial theorem.

2.4 Generalization of Classical Virial Theorem

The main methodological question arises: in what kind state of equilibrium the Earth exists? The answer to the question results from the generalized virial theorem for a self-gravitating body, i.e. the body which itself generates the energy for its own motion by interaction of the constituent particles having innate moments. The guiding effect which we use here is the observed by artificial satellite functional relationship between changes in the outer gravity field of the Earth and its mean (polar) moment of inertia. The deep physical meaning of this relationship is as follows. The observed planet's polar moment of inertia is an integral (volumetric) parameter, which represents not fixed interacted mass particles, but expresses changes in their motion under the inner body's energy. The Clausius' virial theorem represents relationship between the potential and kinetic energy in averaged form for a non-interacted (ideal) gaseous cloud of particles or a uniform body which stay in the outer force field. In order to generalize the theorem for a uniform and non-uniform body staying in the own force field we introduce there the volumetric moments of interacted particles, taking into account their volumetric nature. Moreover, the interacted mass particles of a continuous medium generate volumetric forces (pressure or capacity of energy) and volumetric moments, which, in fact, produce the motion in the form of oscillation and rotation of matter. The oscillating form of motion of the Earth and other celestial bodies is the dominating part of their kinetic energy which up to now has not been taken into account. We wish to fill in this gap in dynamics of celestial bodies.

The classic virial theorem is the analytical expression of the hydrostatic equilibrium condition and follows from Newton's and the Euler's equations of motion. Let us recall its derivation in accordance with classical mechanics (Goldstein 1980).

Consider a system of mass points, the location of which is determined by the radius vector $\mathbf{r}_i$ and the force $\mathbf{F}_i$ including the constraints. Then equations of motion of the mass points through their moments $\mathbf{p}_i$ can be written in the form

$$\dot{p}_i = F_i, \tag{2.21}$$

The value of the moment of momentum is

$$Q = \sum_i p_i \cdot r_i,$$

where the summation is done for all masses of the system. The derivative with respect to time from that value is

$$\frac{dQ}{dt} = \sum_i \dot{r}_i \cdot p_i + \sum_i \dot{p}_i \cdot r_i. \tag{2.22}$$

The first term in the right hand side of (2.22) is reduced to the form

$$\sum_i \dot{r}_i \cdot p_i = \sum_i m_i \cdot \dot{r}_i \cdot \dot{r}_i = \sum_i m_i v_i^2 = 2T,$$

where T is the kinetic energy of particle motion under action of the forces $\mathbf{F}_i$. The second term in the Eq. 2.22 is

$$\sum_i \dot{p}_i \cdot r_i = \sum_i F_i \cdot r_i.$$

Now Eq. 2.22 can be written as

$$\frac{d}{dt} \sum_i p_i \cdot r_i = 2T + \sum_i F_i \cdot r_i. \tag{2.23}$$

The mean values in (2.23) within the time interval τ are found by their integration from 0 to t and division by τ:

$$\frac{1}{r}\int_{0}^{t}\frac{dQ}{dt}dt = \overline{\frac{dQ}{dt}} = \overline{2T} + \overline{\sum_{1} F_{1}\cdot r_{1}}$$

or

$$\overline{2T} + \overline{\sum_{i} F_{i}\cdot r_{i}} = \frac{1}{\tau}[Q(\tau) - Q(0)]. \qquad (2.24)$$

For the system, in which the co-ordinates of mass point motion are repeated through the period τ, the right hand side of Eq. 2.24 after its averaging is equal to zero. If the period is too large, then the right hand side becomes a very small quantity. Then, the expression (2.24) in the averaged form gives the following relation

$$-\overline{\sum_{i} F_{i}\cdot r_{i}} = 2T, \qquad (2.25)$$

or in mechanics it is written in the form

$$2T = -U$$

Equation 2.25 is known as the virial theorem, and its left hand side is called the virial of Clausius (German *virial* is from the Latin *vires* which means forces). The virial theorem is a fundamental relation between the potential and kinetic energy and is valid for a vide range of natural systems, the motion of which is provided by action of different physical interactions of their constituent particles. Clausius proved the theorem in 1870 when he solved the problem of work of the Carnot thermal machine, where the final effect of the water vapor pressure (the potential energy) was connected with the kinetic energy of the piston motion. The water vapor was considered as a perfect gas. And the mechanism of the potential energy (the pressure) generation at the coal burning in the firebox was not considered and was not taken into account.

The starting point in the above-presented derivation of virial theorem in mechanics is the moment of the mass point system, the nature of which is not considered both in mechanics and by Clausius. By Newton's definition the moment "*is the measure of that determined proportionally to the velocity and the mass*". The nature of the moment by his definition is "*the innate force of the matter*". By his understanding that force is an inertial force, i.e. the motion of a mass continues with a constant velocity.

The observed (by satellites) relationship between the potential and the kinetic energy of the gravitation field and the Earth's moment of inertia evidences, that the kinetic energy of the interacted mass particle motion, which is expressed as a volumetric effect of the planet's moment of inertia, is not taken into account. The evidence of that was given in the previous Sect. 2.3 in the quantitative calculation of a ratio between the kinetic and potential energies, equal to ~1/300.

In order to correct the contradiction, the kinetic energy of motion of the interacted particles should be taken into account in the derived virial theorem. Because of any mass has volume the momentum **p** should be written in volumetric form:

$$p_i = \sum_i m_i \dot{r}_i. \tag{2.26}$$

Now the volumetric moment of momentum acquires the wave nature and is presented as

$$Q = \sum_i p_i \cdot r_i = \sum_i m_i \cdot \dot{r}_i \cdot r_i = \frac{d}{dt}\left(\sum_i \frac{m_i r_i^2}{2}\right) = \frac{1}{2}\dot{I}_p \tag{2.27}$$

where I_p is the polar moment of inertia of the system of interacted particles (for the sphere it is equal to 3/2 of the axial moment).

The derivative from that value with respect to time is

$$\frac{dQ}{dt} = \frac{1}{2}\ddot{I}_p = \sum_i \dot{r}_i \cdot p_i + \sum_i \dot{p}_i \cdot r_i. \tag{2.28}$$

The first term in the right hand part of (2.28) remains without change

$$\sum_i \dot{r}_i \cdot p_i = \sum_i m_i \cdot \dot{r}_i \cdot \dot{r}_i = \sum_i m_i v_i^2 = 2T. \tag{2.29}$$

The second term represents the potential energy of the system

$$\sum_i \dot{p}_i \cdot r_i = \sum_i F_i \cdot r_i = U. \tag{2.30}$$

Equation 2.28 is written now in the form

$$\frac{1}{2}\ddot{I}_p = 2T + U. \tag{2.31}$$

Expression (2.31) represents a generalized equation of the virial theorem for a mass particle system interacted by the Newton's law. Here in the left hand side of (2.31) the ignored up to now inner kinetic energy of interaction of the mass particles appears. Solution of Eq. 2.31 gives a variation of the polar moment of inertia within the period τ. For a conservative (uniform with respect to density) system averaged expression (2.28) by integration from 0 to t within time interval τ gives

$$\frac{1}{T}\int_{0}^{t}\frac{dQ}{dt}dt = \overline{\frac{dQ}{dt}} = 2\overline{T} + \overline{U} = \dot{I}_p. \qquad (2.32)$$

Equation 2.32 at $\ddot{I}_p = 0$ (conservative system) gives $\dot{I}_p = E = \text{const.}$, where E is the total system's energy. It means that the interacted mass particles of the system move. with constant velocity. In the case of dissipative system, Eq. 2.32 is not equal to zero and the interacted mass particles move with acceleration. Now the ratio between the potential and kinetic energy has a value in accordance with the Eq. 2.31. Kinetic energy of the interacted mass particles in the form of oscillation of the polar moment of inertia in that equation is taken into account. And now in the frame of the law of energy conservation the ratio of the potential to kinetic energy of a celestial body has a correct value.

Expression (2.31) appears to be an equation of dynamical equilibrium of a self-gravitating body (star, planet, satellite). The hydrostatic equilibrium is absent here because the interacted particles are continuously moving by use of inner energy. Integral effect of the moving particles is fixed by the satellite orbits in the form of changing zonal, sectorial and tesseral gravitational moments. For derivation of the generalized virial theorem we used the potential energy generated by interacted particles of the initial moment (2.26). The initial moments form the inner, or "*innate*" by Newton's definition, energy of the body which has an inherited origin.

Thus, we obtained a differential equation of the second order (2.31) which describes the body dynamics and its dynamical equilibrium.

The virial Eq. 2.31 was obtained by Jacobi already one and a half century ago from the Newton's equations of motion in the form (Jacobi 1884)

$$\ddot{\phi} = U + 2T \qquad (2.33)$$

wher где Φ is the Jacobi's function (the polar moment of inertia).

Jacobi has not considered physical meaning of his equation. He assumed that because of two independent variables Φ and U in the equation it can not be resolved.

We succeeded to find an empirical relationship between the two variables and obtained at first an approximate and later on rigorous solution of the equation (Ferronsky et al. 1978, 1987; Ferronsky 2005). The relationship is proved by means of the satellite observation.

Let us tray to explain the cause of discrepancy between the geometric (static) and dynamic oblateness of the Earth. The reason is as follows. The planet's moment of inertia (polar or axial) has changing in time value. The polar moment of inertia of a self-gravitating body has a functional relation with the potential energy, the generation of which results by interaction of the mass particles in regime of periodic oscillations. The hydrostatic equilibrium of a body does not express the real dynamic processes because of loss of energy of the interacting particle oscillation. Because of that it was not possible to understand the nature of the energy. The main part of the body's kinetic energy of the body's oscillation was also lost. As to the rotational motion of the body shells, it appears only in the case of the non-uniform distribution of the mass density. The contribution of rotation to the total body's kinetic energy covers its very small part.

The cause of the accepted incorrect ratio between the Earth potential and kinetic energy lies in the body's hydrostatic equilibrium. Clairaut's equation (1.20), derived for the planet's hydrostatic equilibrium state and applied to determine the geometric oblateness, because of the above discussed reason, has no functional relationship between the force function and the moment of inertia. Therefore for the Earth dynamics problem the equation gives only a first approximation. In formulation of the Earth oblateness problem, Clairaut accepted the Newton's model of action of the centripetal forces from the surface of the planet to its geometric center. In such a physical conception the total effect of the inner force field becomes equal to zero. Below in Sect. 2.5 it will be shown, that the force field of the continuous body's interacted masses represents volumetric pressure, but not a field of vector forces. That is the cause, why the accepted postulate related to the planet's inertial rotation is physically incorrect. It was proved in electrodynamics that the force is not acceptable to be a measure of particle interaction.

The question is raised about how was it happened, that geodynamic problems and first of all the problem of stability of the Earth motion up to now were solved without knowing the planet's kinetic energy. The probable explanation of that seems to lie in the history of the development of science. In Kepler's problem and in the Newton's two body problem solution the transition from the averaged parameters of motion to the real conditions is provided through the mean and the eccentric anomalies, which by geometric procedures indirectly take into account the above energy of motion. In the Earth figure problem this procedure of Kepler is not applicable. Therefore, the so called "inaccuracies" in the Earth motion appear to be the regular dynamic effects of a self-gravitating body, and the hydrostatic model in the problem is irrelevant. The hydrostatic model was accepted by Newton for the other problem, where just this model allowed discovery and formulation the general laws of the planets motion around the Sun. The Newton's centripetal forces in principle satisfy the Kepler's condition when the distance between bodies is mach more than their size accepted as mass points. Such model gives a first approximation in the problem solution. Kepler's laws express the real picture of the planets and satellites motion around their parent bodies in averaged parameters. All the deviations of those averaged values related to the outer perturbations are not considered as it was done in the Clausius' virial theorem for the perfect gas.

Newton solved the two body problem, which has been already formulated by Kepler. The solution was based on the heliocentric world system of Copernicus, on the Galilean laws of inertia and free fall in the outer force field and on Kepler's laws of the planet's motion in the central force field considered as a geometric plane task. The goal of Newton's problem was to find the force by which the planet's motion is resulted. His centripetal attraction and the inertial forces in the two body problem satisfy Kepler's laws.

As it was mentioned, Newton understood the physical meaning of his centripetal or attractive forces as a pressure, which is accepted now like a force field. But by his opinion, for mathematical solutions the force is a more convenient instrument. And in the two body problem the force-pressure is acting from the center (of point) to the outer space.

It is worth to discuss briefly the Newton's preference given to the force but not to the pressure. In mechanics the term "*mass point*" is understood as a geometric point of space, which has no dimension but possesses a finite mass. In physics a small amount of mass is called by the term "*particle*", which has a finite value of size and mass. But very often physicists use models of particles, which have neither size nor mass. A body model like mass point has been known since ancient times. It is simple and convenient for mathematical operations. The point is an irreplaceable geometric symbol of a reference point. The physical point, which defines inert mass of a volumetric body, is also suitable for operations. But the interacted and physically active mass point creates a problem. For instance, in the field theory the point value is taken to denote the charge, the meaning of which is not better understood than is the gravity force. But it is considered often there, that the point model for mathematical presentation of charges is not suitable because operations with it lead to zero and infinite values. Then for resolution of the situation the concept of charge density is introduced. The charge is presented as an integral of density for the taken volume and by this way the solving problem is resolved.

The point model in the two-body problem allowed reduction of it to the one-body problem and for a spherical body of uniform density to write the main seven integrals of motion. In the case when a body has a finite size, then not the forces but the pressure becomes an effect of the body particle interaction. The interacted body's mass particles form a volumetric gravitational field of pressure, the strength of which is proportional to the density of each elementary volume of the mass. In the case of a uniform body, the gravitational pressure should be also uniform within the whole volume. The outer gravitational pressure of the uniform body should be also uniform at the given radius. The non-uniform body has a non-uniform gravitational pressure of both inner and outer field, which has been observed in studying the real Earth field. Interaction of mass particles results in their collision, which leads to oscillation of the whole body system. In general if the mass density value is higher then the frequency of body oscillation has also higher vale.

It was known from the theory of elasticity, that in order to calculate the stress and the deformation of a beam from a continuous load, the latter can be replaced by the equivalent lumped force. In that case the found solution will be approximate because the beam's stress and deformation will be different. The question is what degree of approximation of the solution and what kind of the error is expected. Volumetric forces are not summed up by means of the parallelogram rule. Volumetric forces by their nature can not be reduced for application either to a point, or to a resultant vector value. Their actions are directed to the 4π space and they form inner and outer force field. The force field by its action is proportional to action of the energy. This is because the force is the derivative of the energy.

The centrifugal and Coriolis' forces are also proved to be inertial forces as a consequence of inertial rotation of the body. And the Archimedes force has not found its physical explanation, but it became an observational fact of hydrostatic equilibrium of a body mass immersed into a liquid.

Such is the short story of appearance and development of the hydrostatic equilibrium of the Earth in the outer uniform gravity field. The force of gravity of a body mass is an integral value. In this connection Newton's postulate about the gravity center as a geometric point should be considered as a model for presentation of two interacted bodies, when their mutual distance is much more of the body size. It is shown in the next section, that the reduced physical, but not geometrical, gravity center of a volumetric body is represented by an envelope of the figure, which draws averaged value of radial density distribution of the body.

The problem of dynamics of the Earth as a self-gravitating body, including the figure problem in its formulation and solution needs for a higher degree of approximation. Generalized virial theorem (2.31) satisfies the condition of the Earth dynamical equilibrium state and creates a physical and theoretical basis for farther development of theory. It follows from the theorem that hydrostatic equilibrium state there is the particular case of the dynamics. Solution of problem of the Earth dynamics based on the equation of dynamical equilibrium appears to be the next natural and logistic step from the hydrostatic equilibrium model to a more realistic model without loss of the previous preference.

Below we consider the problem of "decentralization" of the own force field for a self-gravitating body.

2.5 Reduction of Inner Gravitational Field of a Body to the Resultant Envelope of Pressure

As an example, consider the Earth as a self-gravitating sphere with uniform and one-dimensional interacting media. The motion of the Earth proceeds both in its own and in the Sun's force fields. It's known from theoretical mechanics that any motion of a body can be represented by a translation motion of its mass center, rotation around that center and motion of the body mass related to its changes in the shape and structure (Duboshin 1975). In the two-body problem the last two effects are neglected due to their smallness.

In order to study the Earth motion in the own force field the translational (orbital) motion relative to the fixed point (the Sun) should be separated from the two other components of motion. After that one can consider the rotation around the geometric center of the Earth masses under action of the own force field and changes in the shape and structure (oscillation). Such separation is required only for the moment of inertia, which depends on what frame of reference is selected. The force function

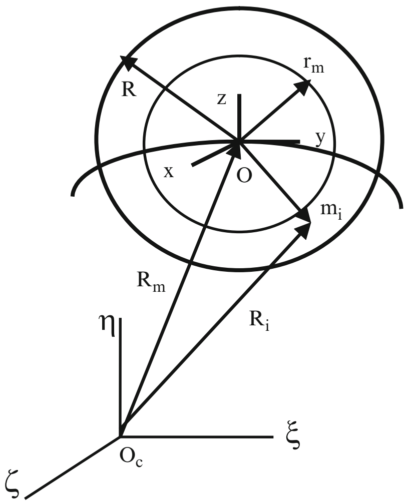

Fig. 2.1 Body motion in own force field

depends on a distance between the interacted masses and does not depend on selection of a frame of reference (Duboshin 1975). The moment of inertia of the Earth relative to the solar reference frame should be split into two parts. The first is the moment of the body mass center relative to the same frame of reference and the second – moment of inertia of the planet's mass relative to the own mass center.

So, set up the absolute Cartesian coordinates $O_c\xi\eta\zeta$ with the origin in the center of the Sun and transfer it to the system Oxyz with the origin in the geometrical center of the Earth's mass (Fig. 2.1).

Then, the moment of inertia of the Earth in the solar frame of reference is

$$I_c = \sum m_i R_i^2, \tag{2.34}$$

where m_i is the Earth mass of particle; R_i is its distance from the origin in the same frame.

The Lagrange's method is applied to separate the moment of inertia (2.34). The method is based on his algebraic identity

$$\left(\sum_{1\leqslant i\leqslant n} a_i^2\right)\left(\sum_{1\leqslant i\leqslant n} b_i^2\right) = \left(\sum_{1\leqslant i\leqslant n} a_i b_i\right)^2 + \frac{1}{2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant j\leqslant n}\left(a_i b_j - b_i a_j\right)^2, \tag{2.35}$$

where a_i and b_i are whichever values; *n* is any positive number.

Jacobi in his "*Vorlesungen über Dynamik*" was the first who performed the mathematical transformation for separation of the moment of inertia of n interacting mass points into two algebraic sums (Jacobi 1884; Duboshin 1975; Ferronsky et al. 1987). It was shown that if we denote (Fig. 2.1)

$$\xi_i = x_i + A; \quad \eta_i = y + B; \quad \zeta_i = z + C;$$

$$\sum m_i = M; \quad \sum m_i \ \xi_i = MA; \quad \sum m_i \ \eta_i = MB; \quad m_i \ \zeta_i = MC, \qquad (2.36)$$

where A, B, C are the coordinates of the mass center in the solar frame of reference. Then, using identity (2.35), one has

$$\sum m_i r_i^2 = \sum m_i \ \xi_i^2 + \sum m_i \ \eta_i^2 + \sum m_i \ \zeta_i^2 = \sum m_i x_i^2 + 2A \sum m_i x_i A^2 \sum m_i$$
$$+ \sum m_i y_i^2 + 2B \sum m_i y_i + B^2 \sum m_i + \sum m_i z_i^2 + 2C \sum m_i z_i + C^2 \sum m_i.$$

Since

$$MA = \sum m_i \ \xi_i = \sum m_i x_i + \sum m_i A = \sum m_i x_i + MA,$$

then $\sum m_i x_i = 0$, and also $\sum m_i y_i = 0$, $\sum m_i z_i = 0$. Now, the moment of inertia (2.34) acquires the form

$$\sum m_i R_i^2 = M\left(A^2 + B^2 + C^2\right) + \sum m_i \left(x_i^2 + y_i^2 + z_i^2\right), \qquad (2.37)$$

where

$$M\left(A^2 + B^2 + C^2\right) = \ M{R^2}_m, \qquad (2.38)$$

$$\sum m_i \left({x_i}^2 + {y_i}^2 + {z_i}^2\right) = M \ {r_m}^2, \qquad (2.39)$$

M is the Earth's mass; R_m and r_m are the radii of inertia of the Earth in the Sun's and the Earth's frame of reference.

Thus, we separated the moment of inertia of the Earth, rotating around the Sun in the inertial frame of reference, into two algebraic terms. The first one (2.38) is the Earth's moment of inertia in the solar reference system $O_c\xi\eta\zeta$. The second term (2.39) presents the moment of inertia of the Earth in the own frame of reference Oxyz. The Earth mass here is distributed over the spherical surface with the reduced radius of inertia r_m. In literature the geometrical center of mass O in the Earth reference system is erroneously identified with the center of inertia and center of gravity of the planet.

For farther consideration of the problem of the Earth's dynamics we accept the polar frame of reference with its origin in center O. Then expression (2.39) for the Earth polar moment of inertia I_p acquires the form

$$I_p = \sum m_i \left({x_i}^2 + {y_i}^2 + {z_i}^2\right) = \sum m_i {r_i}^2 = \ M \ {r_m}^2 \qquad (2.40)$$

Now the reduced radius of inertia r_m, which draws a spherical surface, is

$$ {r_m}^2 = \frac{\sum m_i r_i^2}{M}. \qquad (2.41)$$

Here $M = \sum m_i$ is the Earth's mass relative to own frame of reference.

Taking into account the spherical symmetry of the uniform and one-dimensional Earth, we consider the sphere as a concentric spherical shell with the mass $dm(r) = 4\pi\, r^2\rho(r)dr$. Then the expression (2.41) in the polar reference system can be rewritten in the form

$$r_m{}^2 = \frac{1}{M}\int_0^R r^2 4\,\pi r^2\rho(r)dr = \frac{4\pi R^2}{MR^2}\int_0^R r^4\rho(r)dr, \tag{2.42}$$

or

$$\frac{4\pi r_m^2}{4\,\pi R^2} = \frac{4\pi\int_0^R r^4\rho(r)dr}{MR^2} = \frac{\beta^2 MR^2}{MR^2} = \beta^2, \tag{2.43}$$

from where

$$r_m{}^2 = b^2R^2,$$

where $\rho(r)$ is the law of radial density distribution; R is the radius of the sphere; β^2 is the dimensionless coefficient of the reduced spheroid (ellipsoid) of inertia β^2MR^2.

The value of β^2 depends on the density distribution $\rho(r)$ and is changed within the limits of $1 \geq \beta^2 > 0$. Earlier (Ferronsky et al. 1987) it was defined as a structural form-factor of the polar moment of inertia.

Analogously, the reduced radius of gravity r_g, expressed as a ratio of the potential energy of interaction of the spherical shells with density $\rho(r)$ to the potential energy of interaction of the body mass distributed over the shell with radius R. The potential energy of the sphere is written as

$$U = 4\,\pi G\int_0^R r\rho(r)m(r)dr = \alpha^2\frac{GM^2}{R}, \tag{2.44}$$

from where

$$\alpha^2 = \frac{4\pi G\int_0^R r\rho(r)m(r)dr}{\frac{GM^2}{R}} = \frac{r_g^2}{R^2}, \tag{2.45}$$

where in expressions (2.44) and (2.45) $m(r) = 4\,\pi\int_0^r r^2\rho(r)dr$.

The value of β^2 depends on the density distribution $\rho(r)$ and is changed within the limits of $1 \geq \alpha^2 > 0$. Earlier (Ferronsky et al. 1987) it was defined as a structural form-factor of the force function.

Table 2.1 Numerical values of form factors α^2 and β^2 for radial distribution of mass density and for polytropic models

Density distribution law	α^2	$\beta_\perp^2$	β^2
Radial distribution of mass density			
$\rho(r) = \rho_0$	0.6	0.4	0.6
$\rho(r) = \rho_0(1-r/R)$	0.74	0.27	0.4
$\rho(r) = \rho_0(1-r^2/R^2)$,	0.71	0.29	0.42
$\rho(r) = \rho_0 \exp(1–kr/R)$	0.16 k	$8/k^2$	$12/k^2$
$\rho(r) = \rho_0 \exp(1-kr^2/R^2)$	$\sqrt{\frac{k}{2\pi}}$	1/k	1.5/k
$\rho(r) = \rho_0\, \delta(1-r/R)$	0.5	0.67	1.0
Politrope models			
0	0.6	0.4	0.6
1	0.75	0.26	0.38
1.5	0.87	0.20	0.30
2	1.0	0.15	0.23
3	1.5	0.08	0.12
3.5	2.0	0.045	0.07

Numerical values of the dimensionless form-factors α^2 and β^2 for a number of density distribution laws $\rho(r)$ are given in Table 2.1. The numerical calculations, done by integration of the numerators in Eqs. 2.43–2.45 for the polar moment of inertia and the force function, can be found in the paper (Ferronsky et al. 1978). Note, that value of the polar I_p and axial I_a moments of inertia of one dimensional sphere are related as $I_p = 3/2I_a$.

It follows from Table 2.1 that for a uniform sphere with $\rho(r) = \text{const}$ its reduced radius of inertia coincides with the radius of gravity. Here both dimensionless structural coefficients α^2 and β^2 are equal to 3/5, and the moments of gravitational and inertial forces are equilibrated and because of that the rotation of the mass is absent (Fig. 2.2a).

Thus

$$\frac{r_m^2}{R^2} = \frac{r_g^2}{R^2} = \frac{3}{5}, \tag{2.46}$$

from where

$$r_m = r_g = \sqrt{3/5R^2} = 0.7745966R. \tag{2.47}$$

For a non-uniform sphere at $\rho(r) \neq \mathit{const}$ from Eqs. 2.43–2.45 one has

$$0 < \frac{4\pi r_m^2}{4\ \pi R^2} < \frac{3}{5} < \frac{4\pi r_g^2}{4\ \pi R^2} < 1. \tag{2.48}$$

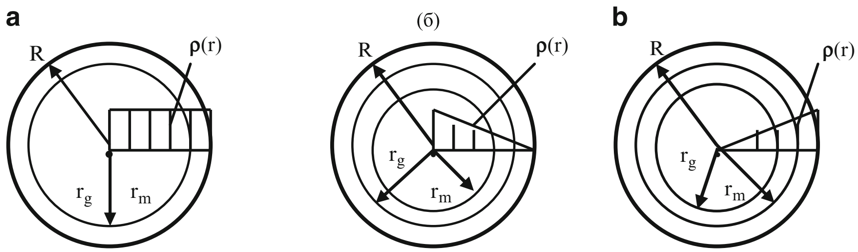

Fig. 2.2 Radius of inertia and radius of gravity for uniform (**a**) and non-uniform sphere with density increased to the center (**b**) and from the center (**c**)

It follows from inequality (2.48) and Table 2.1 that in comparison with the uniform sphere, the reduced radius of inertia of the non-uniform body decreases and the reduced gravity radius increases (Fig. 2.2b). Because of $r_m \neq r_g$ and $r_m < 0.77R < r_g$ the torque appears as a result of an imbalance between gravitational and inertial volumetric forces of the shells. Then from Eq. 2.48 it follows that

$$I_m = I_{mo} - \delta r_{mt} \quad \text{and} \quad r_g = r_{go} + \delta r_{gt} \tag{2.49}$$

where subscripts 0 and t relate to the uniform and non-uniform sphere.

In accordance with (2.48) and (2.49) rotation of shells of a one-dimensional body should be hinged-like and asynchronous. In the case of increasing mass density towards the body surface, then the signs in (2.48) and (2.49) are reversed (Fig. 2.2c). This remark is important because the direction of rotation of a self-gravitating body is function of its mass density distribution.

The main conclusion from the above consideration is that the inner force field of a self-gravitating body is reduced to a closed envelope (spheroid, ellipsoid or more complicated curve) of gravitational pressure, but not to a resulting force passing through the geometric center of the masses. In the case of a uniform body the envelopes have a spherical shape and both gravitational and inertial radii coincide. For a non-uniform body the radius of inertia does not coincide with the radius of gravity, the reduced envelope is closed but has non-spherical (ellipsoidal or any other) shape. Analytical solutions done below justify the above said.

So, we accept the force pressure as an effect of mass particles interaction which is the matter's property to do the work in the form of matter motion.

It follows from this Chapter that physical meaning of the Jacobi's virial equation consists in description of motion of a body (material system) by action of its own force field. This field is formed by the energy of the body's interacting elementary particles and expressed through oscillation of the polar moment of inertia. To the contrary of the failed hydrostatic equilibrium, Jacobi's equation describes the motion in the volumetric forces (energy) and in the volumetric moments (oscillations). The energy here is accepted as the measure of the matter interaction.

We now proceed to derivation of Jacobi's virial equation for the well known physical models of natural systems.