Chapter 3
Theories of Point Relations and Relational Model Checking

3.1 Introduction

In this chapter we consider logics providing a means of relational reasoning in the theories which refer to individual objects of their domains. There are two relational formalisms for coping with the objects. A logic $\mathsf{RL}_{ax}(\mathbb{C})$ presented in Sect. 3.2 is a purely relational formalism, where objects are introduced through point relations which, in turn, are presented axiomatically with a well known set of axioms. The axioms say that a binary relation is a point relation whenever it is a non-empty right ideal relation with one-element domain. We recall that a binary relation R on a set U is right ideal whenever $R;1 = R$, where $1 = U \times U$. In other words such an R is of the form $X \times U$, for some $X \subseteq U$. We may think of right ideal relations as representing sets, they are sometimes referred to as vectors (see [SS93]). If the domain of a right ideal relation is a singleton set, the relation may be seen as a representation of an individual object. A logic $\mathsf{RL}_{df}(\mathbb{C})$ presented in Sect. 3.3 includes object constants in its language interpreted as singletons. Moreover, associated with each object constant c is a binary relation C, such that its meaning in every model is defined as a right ideal relation with the domain consisting of the single element being the meaning of c. The logic $\mathsf{RL}_{ax}(\mathbb{C})$ will be applied in Sect. 16.5 to the relational representation of some temporal logics. The logic $\mathsf{RL}_{df}(\mathbb{C})$ will be applied in Chap. 15 to the relational representation of the logic for order of magnitude reasoning. In Sects. 3.4 and 3.5 we present the methods of model checking and verification of satisfaction of a formula by some given objects in a finite model, respectively. The methods are based on the development of a relational logic which enables us to replace the problems of model checking and verification of satisfaction by the problems of verification of validity of some formulas of this logic. The logic is obtained from $\mathsf{RL}(1, 1')$-logic by an appropriate choice of object constants and relational constants in its language and by some specific postulates concerning its models. Then, a dual tableau for the logic is obtained from the $\mathsf{RL}(1, 1')$-dual tableau by adapting it to this language and by adding the rules which reflect these specific semantic postulates.

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3.2 Relational Logics with Point Relations Introduced with Axioms

The language of the logics considered in this section includes, apart from the relational constants 1 and $1'$, a family $\mathbb{C}$ of relational constants interpreted as point relations. The language of $\mathsf{RL}_{ax}(\mathbb{C})$-logic is a relational language as presented in Sect. 2.3 such that:

- $\mathbb{RC}_{\mathsf{RL}_{ax}(\mathbb{C})} = \{1', 1\} \cup \mathbb{C}$;
- The relational operations are $-$, $\cup$, $\cap$, $^{-1}$, and ;.

As in the case of RL-logic (see Sect. 2.5), $\mathsf{RL}_{ax}(\mathbb{C})$ represents, in fact, a whole family of logics which possibly differ in the object constants admitted in their languages.

An $\mathsf{RL}_{ax}(\mathbb{C})$-*model* is an $\mathsf{RL}(1, 1')$-model $\mathcal{M} = (U, m)$ such that for every $C \in \mathbb{C}$:

(1) $m(C) \neq \emptyset$,
(2) $m(C) = m(C)\,; m(1)$,
(3) $m(C)\,; m(C)^{-1} \subseteq m(1')$.

An $\mathsf{RL}_{ax}(\mathbb{C})$-model $\mathcal{M} = (U, m)$ is said to be *standard* whenever $m(1')$ is the identity on U.

The conditions (1), (2), and (3) say that relations from $\mathbb{C}$ are point relations. Condition (2) guarantees that every $C \in \mathbb{C}$ is a right ideal relation, and condition (3) together with the remaining axioms says that in the standard models the domains of relations from $\mathbb{C}$ are singleton sets.

$\mathsf{RL}_{ax}(\mathbb{C})$-dual tableau consists of decomposition rules and specific rules of $\mathsf{RL}(1, 1')$-system adjusted to the $\mathsf{RL}_{ax}(\mathbb{C})$-language and the specific rules that characterize relational constants from the set $\mathbb{C}$:

For all object symbols x and y and for every $C \in \mathbb{C}$,

$$(C1) \quad \frac{}{z{-}C\,t} \qquad z, t \text{ are new object variables and } z \neq t$$

$$(C2) \quad \frac{xCy}{xCz, xCy} \qquad z \text{ is any object symbol}$$

$$(C3) \quad \frac{x1'y}{xCz, x1'y \mid yCz, x1'y} \qquad z \text{ is any object symbol}$$

$\mathsf{RL}_{ax}(\mathbb{C})$-axiomatic sets are those of $\mathsf{RL}(1, 1')$ adapted to the $\mathsf{RL}_{ax}(\mathbb{C})$-language.

It is easy to see that the specific rules for relational constants from $\mathbb{C}$ have the property of preservation of formulas built with atomic terms or their complements. Hence, the closed branch property holds.

As usual, an $\mathsf{RL}_{ax}(\mathbb{C})$-set is a finite set of $\mathsf{RL}_{ax}(\mathbb{C})$-formulas such that the first-order disjunction of its members is true in all $\mathsf{RL}_{ax}(\mathbb{C})$-models. Correctness of a rule is defined as in Sect. 2.4.

Proposition 3.2.1.

1. *The* $\mathsf{RL}_{ax}(\mathbb{C})$*-rules are* $\mathsf{RL}_{ax}(\mathbb{C})$*-correct;*
2. *The* $\mathsf{RL}_{ax}(\mathbb{C})$*-axiomatic sets are* $\mathsf{RL}_{ax}(\mathbb{C})$*-sets.*

Proof. Correctness of the rules $(C1)$, $(C2)$, and $(C3)$ follows directly from the semantic conditions (1), (2), and (3), respectively. By way of example, we prove correctness of the rule $(C1)$. Let X be a finite set of $\mathsf{RL}_{ax}(\mathbb{C})$-formulas and let z and t be object variables that do not occur in X and such that $z \neq t$. If X is an $\mathsf{RL}_{ax}(\mathbb{C})$-set, then so is $X \cup \{z{-}Ct\}$. Assume that $X \cup \{z{-}Ct\}$ is an $\mathsf{RL}_{ax}(\mathbb{C})$-set, that is for every $\mathsf{RL}_{ax}(\mathbb{C})$-model $\mathcal{M}$ and for every valuation v in $\mathcal{M}$, $(z,t) \notin m(C)$ or $\mathcal{M}, v \models \varphi$ for some formula $\varphi \in X$. By the assumption on variables z and t, for every $\mathsf{RL}_{ax}(\mathbb{C})$-model $\mathcal{M} = (U, m)$ and for every valuation v in $\mathcal{M}$, either $(a,b) \notin m(C)$ for all $a,b \in U$ or $\mathcal{M}, v \models \varphi$ for some formula $\varphi \in X$. Since in every $\mathsf{RL}_{ax}(\mathbb{C})$-model $m(C_i) \neq \emptyset$, for every $\mathsf{RL}_{ax}(\mathbb{C})$-model $\mathcal{M}$ and for every valuation v in $\mathcal{M}$ there exists $\varphi \in X$ such that $\mathcal{M}, v \models \varphi$. Therefore, X is an $\mathsf{RL}_{ax}(\mathbb{C})$-set. □

The notions of an $\mathsf{RL}_{ax}(\mathbb{C})$-proof tree, a closed branch of such a tree, a closed $\mathsf{RL}_{ax}(\mathbb{C})$-proof tree, and an $\mathsf{RL}_{ax}(\mathbb{C})$-proof of an $\mathsf{RL}_{ax}(\mathbb{C})$-formula are defined as in Sect. 2.4.

Following the general method of proving soundness presented in Sect. 2.6, Proposition 3.2.1 implies:

Proposition 3.2.2. *Let* φ *be an* $\mathsf{RL}_{ax}(\mathbb{C})$*-formula. If* φ *is* $\mathsf{RL}_{ax}(\mathbb{C})$*-provable, then it is* $\mathsf{RL}_{ax}(\mathbb{C})$*-valid.*

Corollary 3.2.1. *Let* φ *be an* $\mathsf{RL}_{ax}(\mathbb{C})$*-formula. If* φ *is* $\mathsf{RL}_{ax}(\mathbb{C})$*-provable, then it is true in all standard* $\mathsf{RL}_{ax}(\mathbb{C})$*-models.*

A branch b of an $\mathsf{RL}_{ax}(\mathbb{C})$-proof tree is said to be $\mathsf{RL}_{ax}(\mathbb{C})$-complete whenever it is closed or it satisfies $\mathsf{RL}_{ax}(\mathbb{C})$-completion conditions which consist of the completion conditions of $\mathsf{RL}(1, 1')$-dual tableau and the following:

For every $C \in \mathbb{C}$ and for all object symbols x and y,

Cpl($C1$) There exist object variables z and t such that $z{-}Ct \in b$, obtained by an application of the rule $(C1)$;
Cpl($C2$) If $xCy \in b$, then for every object symbol z, $xCz \in b$, obtained by an application of the rule $(C2)$;
Cpl($C3$) If $x1'y \in b$, then for every object symbol z, either $xCz \in b$ or $yCz \in b$, obtained by an application of the rule $(C3)$.

The notions of a complete $\mathsf{RL}_{ax}(\mathbb{C})$-proof tree and an open branch of an $\mathsf{RL}_{ax}(\mathbb{C})$-proof tree are defined as in RL-logic (see Sect. 2.5).

Let b be an open branch of an $\mathsf{RL}_{ax}(\mathbb{C})$-proof tree. We define a branch structure $\mathcal{M}^b = (U^b, m^b)$ with $U^b = \mathbb{OS}_{\mathsf{RL}_{ax}(\mathbb{C})}$ in a similar way as in $\mathsf{RL}(1, 1')$-logic, that is $m^b(R) = \{(x,y) \in U^b \times U^b : xRy \notin b\}$, for every $R \in \mathbb{RA}_{\mathsf{RL}_{ax}(\mathbb{C})}$.

Proposition 3.2.3 (Branch Model Property). *For every open branch b of an* $\mathsf{RL}_{ax}(\mathbb{C})$*-proof tree, the branch structure* $\mathcal{M}^b$ *is an* $\mathsf{RL}_{ax}(\mathbb{C})$*-model.*

Proof. We have to show that meaning function m^b satisfies conditions (1), (2), and (3) of $\mathsf{RL}_{ax}(\mathbb{C})$-models.

For (1), by the completion condition Cpl($C1$), there are $z, t \in U^b$ such that $z{-}Ct \in b$. Thus, $zCt \notin b$, since otherwise b would be closed. Hence, $(z, t) \in m^b(C)$.

For (2), note that $m^b(1) = U^b \times U^b$ implies $m^b(C) \subseteq m^b(C)\,;m^b(1)$. Assume there exists $z \in U^b$ such that $(x, z) \in m^b(C)$ and $(z, y) \in m^b(1)$, that is $xCz \notin b$ and $z1y \notin b$. Suppose $(x, y) \notin m^b(C)$. Then $xCy \in b$. By the completion condition Cpl($C2$), for every $z \in U^b$, $xCz \in b$, a contradiction.

The proof of (3) is similar. □

Since the branch model $\mathcal{M}^b$ is defined in a standard way (see Sect. 2.6, p. 44), the satisfaction in branch model property can be proved as in $\mathsf{RL}(1, 1')$-logic (see Sects. 2.5 and 2.7). Hence, completeness of $\mathsf{RL}_{ax}(\mathbb{C})$-dual tableau follows.

Proposition 3.2.4. *Let φ be an* $\mathsf{RL}_{ax}(\mathbb{C})$*-formula. If φ is true in all standard* $\mathsf{RL}_{ax}(\mathbb{C})$*-models, then it is* $\mathsf{RL}_{ax}(\mathbb{C})$*-provable.*

The proof of this theorem can be obtained applying the general method described in Sect. 2.6 (p. 44), see also Propositions 2.5.6 and 2.7.8.

Corollary 3.2.2. *Let φ be an* $\mathsf{RL}_{ax}(\mathbb{C})$*-formula. If φ is* $\mathsf{RL}_{ax}(\mathbb{C})$*-valid, then it is* $\mathsf{RL}_{ax}(\mathbb{C})$*-provable.*

Due to Proposition 3.2.2 and 3.2.4, we get:

Theorem 3.2.1 (Soundness and Completeness of $\mathsf{RL}_{ax}(\mathbb{C})$). *Let φ be an* $\mathsf{RL}_{ax}(\mathbb{C})$*-formula. Then the following conditions are equivalent:*

1. *φ is* $\mathsf{RL}_{ax}(\mathbb{C})$*-valid;*
2. *φ is true in all standard* $\mathsf{RL}_{ax}(\mathbb{C})$*-models;*
3. *φ is* $\mathsf{RL}_{ax}(\mathbb{C})$*-provable.*

3.3 Relational Logics with Point Relations Introduced with Definitions

The language of $\mathsf{RL}_{df}(\mathbb{C})$-logic is a relational language with relational constants which are explicitly defined in such a way that in the standard models they are right ideal relations with singleton domains. For that purpose we include in the language the object constants interpreted as elements of the models. Thus, the vocabulary of $\mathsf{RL}_{df}(\mathbb{C})$-language is a relational language as defined in Sect. 2.3 such that:

- $\mathbb{RC}_{\mathsf{RL}_{df}(\mathbb{C})} = \{1', 1\} \cup \mathbb{C}$;
- $\mathbb{OC}_{\mathsf{RL}_{df}(\mathbb{C})}$ is a set of object constants, which includes the set $\{c_C : C \in \mathbb{C}\}$ of object constants needed for definitions of point relations.

An $\mathsf{RL}_{df}(\mathbb{C})$-*model* is an $\mathsf{RL}(1,1')$-model $\mathcal{M} = (U, m)$ such that for every $C \in \mathbb{C}$ the following hold:

- $m(c_C) \in U$;
- $m(C) = \{(x, y) \in U \times U : (x, m(c_C)) \in m(1')\}$.

The following proposition shows that for the formulas of $\mathsf{RL}_{ax}(\mathbb{C})$-language the notions of validity in the logics $\mathsf{RL}_{ax}(\mathbb{C})$ and $\mathsf{RL}_{df}(\mathbb{C})$ coincide.

Proposition 3.3.1. *For every $\mathsf{RL}_{ax}(\mathbb{C})$-formula φ, the following conditions are equivalent:*

1. *φ is $\mathsf{RL}_{ax}(\mathbb{C})$-valid;*
2. *φ is $\mathsf{RL}_{df}(\mathbb{C})$-valid.*

Proof. Let φ be an $\mathsf{RL}_{ax}(\mathbb{C})$-formula.

(1. $\rightarrow$ 2.) Assume that φ is $\mathsf{RL}_{ax}(\mathbb{C})$-valid. Suppose φ is not $\mathsf{RL}_{df}(\mathbb{C})$-valid, that is there exists an $\mathsf{RL}_{df}(\mathbb{C})$-model $\mathcal{M} = (U, m)$ such that $\mathcal{M} \not\models \varphi$. Consider a model $\mathcal{M}' = (U, m')$ with the same universe as $\mathcal{M}$ and such that $m'(R) = m(R)$, for every $R \in \mathbb{RA}_{\mathsf{RL}_{ax}(\mathbb{C})}$. Model $\mathcal{M}'$ is an $\mathsf{RL}_{ax}(\mathbb{C})$-model. Indeed, $m'(C)$ is a non-empty right ideal binary relation on U, hence conditions (1) and (2) from definition of $\mathsf{RL}_{ax}(\mathbb{C})$-models in Sect. 3.2 are satisfied. Moreover, by the definition of $m(C)$, if $(x, z) \in m'(C)$ and $(y, z) \in m'(C)$, then by symmetry and transitivity of 1', $(x, y) \in m'(1')$. Therefore, the condition (3) is satisfied. Clearly, models $\mathcal{M}$ and $\mathcal{M}'$ satisfy the same $\mathsf{RL}_{ax}(\mathbb{C})$-formulas. Thus, by the assumption, $\mathcal{M}' \not\models \varphi$, a contradiction.

(2. $\rightarrow$ 1.) Now, assume that φ is $\mathsf{RL}_{df}(\mathbb{C})$-valid. Suppose that φ is not $\mathsf{RL}_{ax}(\mathbb{C})$-valid, that is there exists an $\mathsf{RL}_{ax}(\mathbb{C})$-model $\mathcal{M} = (U, m)$ such that $\mathcal{M} \not\models \varphi$. Note that by condition (3) from definition of $\mathsf{RL}_{ax}(\mathbb{C})$-models, for every relational constant $C \in \mathbb{C}$, if x and y belong to the domain of $m(C)$, then $(x, y) \in m(1')$. We construct a model $\mathcal{M}' = (U, m')$ with the same universe as in $\mathcal{M}$ as follows: $m'(R) = m(R)$, for every $R \in \mathbb{RA}_{\mathsf{RL}_{ax}(\mathbb{C})}$, and $m'(c_C)$ is defined as an arbitrary element from the domain of $m(C)$. Now, by the above definition and condition (3), it follows that $m'(C) = \{x \in U : (x, m'(c_C)) \in m'(1')\} \times U$, hence model $\mathcal{M}'$ is an $\mathsf{RL}_{df}(\mathbb{C})$-model satisfying the same $\mathsf{RL}_{ax}(\mathbb{C})$-formulas as $\mathcal{M}$. Therefore, by the assumption, $\mathcal{M}' \not\models \varphi$, a contradiction. □

$\mathsf{RL}_{df}(\mathbb{C})$-dual tableau consists of the rules and axiomatic sets of $\mathsf{RL}(1,1')$-dual tableau adjusted to $\mathsf{RL}_{df}(\mathbb{C})$-language and the specific rules that characterize relational constants from the set $\mathbb{C}$:

For every $C \in \mathbb{C}$ and for all object symbols x and y,

$$(CD1) \quad \frac{xCy}{x1'c_C,\, xCy}$$

$$(CD2) \quad \frac{x{-}Cy}{x{-}1'c_C,\, x{-}Cy}$$

The notions of $\mathsf{RL}_{df}(\mathbb{C})$-set, correctness of a rule, an $\mathsf{RL}_{df}(\mathbb{C})$-proof tree, a closed branch of an $\mathsf{RL}_{df}(\mathbb{C})$-proof tree, a closed $\mathsf{RL}_{df}(\mathbb{C})$-proof tree, and $\mathsf{RL}_{df}(\mathbb{C})$-provability are defined in a standard way as in Sect. 2.4.

Proposition 3.3.2.

1. *The* $\mathsf{RL}_{df}(\mathbb{C})$*-rules are* $\mathsf{RL}_{df}(\mathbb{C})$*-correct;*
2. *The* $\mathsf{RL}_{df}(\mathbb{C})$*-axiomatic sets are* $\mathsf{RL}_{df}(\mathbb{C})$*-sets.*

Proof. By way of example, we show correctness of the specific rules for relational constants from the set $\mathbb{C}$. It is easy to see that correctness of the rule $(CD1)$ follows from the property: for every $\mathsf{RL}_{df}(\mathbb{C})$-model $\mathcal{M} = (U, m)$, if $(x, m(c_C)) \in m(1')$, then for every $y \in U$, $(x, y) \in m(C)$. Correctness of the rule $(CD2)$ follows from the property: if $(x, m(c_C)) \notin m(1')$, then for every $y \in U$, $(x, y) \notin m(C)$. □

Due to the above proposition, we have:

Proposition 3.3.3. *Let φ be an* $\mathsf{RL}_{df}(\mathbb{C})$*-formula. If φ is* $\mathsf{RL}_{df}(\mathbb{C})$*-provable, then it is* $\mathsf{RL}_{df}(\mathbb{C})$*-valid.*

Corollary 3.3.1. *Let φ be an* $\mathsf{RL}_{df}(\mathbb{C})$*-formula. If φ is* $\mathsf{RL}_{df}(\mathbb{C})$*-provable, then it is true in all standard* $\mathsf{RL}_{df}(\mathbb{C})$*-models.*

To prove completeness of $\mathsf{RL}_{df}(\mathbb{C})$-dual tableau we define, as usual, the branch structure and we prove that branch model property and satisfaction in branch model property are satisfied. Note that in view of Fact 2.5.1 and since any application of the rules $(CD1)$ and $(CD2)$ to a set X of formulas preserves the formulas of X built with atomic terms or their complements, for every branch b of an $\mathsf{RL}_{df}(\mathbb{C})$-proof tree whenever an atomic formula xRy and the formula $x{-}Ry$ appear in b, then the branch is closed. As stated in Sects. 2.5 and 2.7, the same holds for the remaining rules.

A branch b of an $\mathsf{RL}_{df}(\mathbb{C})$-proof tree is said to be $\mathsf{RL}_{df}(\mathbb{C})$-complete whenever it is closed or it satisfies $\mathsf{RL}_{df}(\mathbb{C})$-completion conditions which consist of the completion conditions of $\mathsf{RL}(1, 1')$-system adjusted to $\mathsf{RL}_{df}(\mathbb{C})$-language and the following completion conditions determined by the specific rules for relational constants from $\mathbb{C}$.

For every $C \in \mathbb{C}$ and for all object symbols x and y,

Cpl$(CD1)$ If $xCy \in b$, then $x1'c_C \in b$, obtained by an application of the rule $(CD1)$;

Cpl$(CD2)$ If $x{-}Cy \in b$, then $x{-}1'c_C \in b$, obtained by an application of the rule $(CD2)$.

The notions of a complete $\mathsf{RL}_{df}(\mathbb{C})$-proof tree and an open branch of an $\mathsf{RL}_{df}(\mathbb{C})$-proof tree are defined as usual (see Sect. 2.5).

Let b be an open branch of an $\mathsf{RL}_{df}(\mathbb{C})$-proof tree. We define a branch structure $\mathcal{M}^b = (U^b, m^b)$ as follows:

- $U^b = \mathbb{OS}_{\mathsf{RL}_{df}(\mathbb{C})}$;
- $m^b(c) = c$, for every $c \in \mathbb{OC}_{\mathsf{RL}_{df}(\mathbb{C})}$;

- $m^b(R) = \{(x, y) \in U^b \times U^b : xRy \notin b\}$, for every $R \in \mathbb{RV}_{\mathsf{RL}_{df}(\mathbb{C})} \cup \{1, 1'\}$;
- $m^b(C) = \{x \in U^b : (x, c_C) \in m^b(1')\} \times U^b$, for every $C \in \mathbb{C}$;
- m extends to all the compound relational terms as in $\mathsf{RL}_{df}(\mathbb{C})$-models.

Directly from the above definition we get:

Proposition 3.3.4 (Branch Model Property). *Let b be an open branch of an $\mathsf{RL}_{df}(\mathbb{C})$-proof tree. Then $\mathcal{M}^b$ is an $\mathsf{RL}_{df}(\mathbb{C})$-model.*

Proposition 3.3.5 (Satisfaction in Branch Model Property). *For every open branch b of an $\mathsf{RL}_{df}(\mathbb{C})$-proof tree and for every $\mathsf{RL}_{df}(\mathbb{C})$-formula φ, if $\mathcal{M}^b, v^b \models \varphi$, then $\varphi \notin b$.*

Proof. By way of example, we prove that the proposition holds for formulas of the form xCy and $x{-}Cy$.

Assume $(x, y) \in m^b(C)$, that is $(x, c_C) \in m^b(1')$. Then $x1'c_C \notin b$. Suppose $xCy \in b$. By the completion condition Cpl($CD1$), $x1'c_C \in b$, a contradiction.

Now, assume $(x, y) \in m^b(-C)$, that is $(x, c_C) \notin m^b(1')$. Suppose $x{-}Cy \in b$. By the completion condition Cpl($CD2$), $x{-}1'c_C \in b$. Thus $x1'c_C \notin b$, hence $(x, c_C) \in m^b(1')$, contradiction. □

Finally, we get:

Proposition 3.3.6. *Let φ be an $\mathsf{RL}_{df}(\mathbb{C})$-formula. If φ is true in all standard $\mathsf{RL}_{df}(\mathbb{C})$-models, then it is $\mathsf{RL}_{df}(\mathbb{C})$-provable.*

The proof of the above proposition follows the general method described in Sect. 2.6 (p. 44), see also Propositions 2.5.6 and 2.7.8.

Corollary 3.3.2. *Let φ be an $\mathsf{RL}_{df}(\mathbb{C})$-formula. If φ is $\mathsf{RL}_{df}(\mathbb{C})$-valid, then it is $\mathsf{RL}_{df}(\mathbb{C})$-provable.*

Due to Proposition 3.3.3 and 3.3.6, we have:

Theorem 3.3.1 (Soundness and Completeness of $\mathsf{RL}_{df}(\mathbb{C})$). *Let φ be an $\mathsf{RL}_{df}(\mathbb{C})$-formula. Then the following conditions are equivalent:*

1. *φ is $\mathsf{RL}_{df}(\mathbb{C})$-valid;*
2. *φ is true in all standard $\mathsf{RL}_{df}(\mathbb{C})$-models;*
3. *φ is $\mathsf{RL}_{df}(\mathbb{C})$-provable.*

3.4 Model Checking in Relational Logics

The $\mathsf{RL}(1, 1')$-dual tableau can also be used for model checking in relational logics, besides verification of validity and entailment. Let $\mathcal{M} = (U, m)$ be a fixed standard $\mathsf{RL}(1, 1')$-model with a finite universe U and let $\varphi = xRy$ be an $\mathsf{RL}(1, 1')$-formula, where R is a relational term and x, y are any object symbols. For the simplicity of the presentation we assume that R does not contain 1 or $1'$, although the presented method applies to all $\mathsf{RL}(1,1')$-terms. In order to obtain a relational formalism appropriate for representing and solving the problem '$\mathcal{M} \models \varphi$?', we consider an instance $\mathsf{RL}_{\mathcal{M},\varphi}$ of the logic $\mathsf{RL}(1,1')$. Its language provides a code of model $\mathcal{M}$ and formula φ, and in its models the syntactic elements of φ are interpreted as in the model $\mathcal{M}$.

The vocabulary of language of the logic $\mathsf{RL}_{\mathcal{M},\varphi}$ consists of symbols from the following pairwise disjoint sets:

- $\mathbb{OV}_{\mathsf{RL}_{\mathcal{M},\varphi}}$ – a countable infinite set of object variables;
- $\mathbb{OC}_{\mathsf{RL}_{\mathcal{M},\varphi}} = \mathbb{OC}^{\mathbf{0}}_{\mathsf{RL}_{\mathcal{M},\varphi}} \cup \mathbb{OC}^{1}_{\mathsf{RL}_{\mathcal{M},\varphi}}$ – a finite set of object constants, where $\mathbb{OC}^{\mathbf{0}}_{\mathsf{RL}_{\mathcal{M},\varphi}} = \{c_{\mathsf{a}} : \mathsf{a} \in U\}$ is such that if $\mathsf{a} \neq \mathsf{b}$, then $c_{\mathsf{a}} \neq c_{\mathsf{b}}$, $\mathbb{OC}^{1}_{\mathsf{RL}_{\mathcal{M},\varphi}} = \{c \in \mathbb{OC}_{\mathsf{RL}(1,1')} : c \text{ occurs in } \varphi\}$ and $\mathbb{OC}^{1}_{\mathsf{RL}_{\mathcal{M},\varphi}} \cap \mathbb{OC}^{\mathbf{0}}_{\mathsf{RL}_{\mathcal{M},\varphi}} = \emptyset$;
- $\mathbb{RC}_{\mathsf{RL}_{\mathcal{M},\varphi}} = \mathbb{S} \cup \{1, 1'\}$ – the set of relational constants, where $\mathbb{S}$ is the set of all the atomic subterms of R;
- $\{-, \cup, \cap, ;, ^{-1}\}$ – the set of relational operations.

An $\mathsf{RL}_{\mathcal{M},\varphi}$-model is a pair $\mathcal{N} = (W, n)$, where:

- $W = U$;
- $n(c) = m(c)$, for every $c \in \mathbb{OC}^{1}_{\mathsf{RL}_{\mathcal{M},\varphi}}$;
- $n(c_{\mathsf{a}}) = \mathsf{a}$, for every $c_{\mathsf{a}} \in \mathbb{OC}^{\mathbf{0}}_{\mathsf{RL}_{\mathcal{M},\varphi}}$;
- $n(S) = m(S)$, for every $S \in \mathbb{S}$;
- $n(1), n(1')$ are defined as in standard $\mathsf{RL}(1,1')$-models;
- n extends to all the compound terms as in $\mathsf{RL}(1,1')$-models.

A valuation in $\mathcal{N}$ is a function v: $\mathbb{OS}_{\mathsf{RL}_{\mathcal{M},\varphi}} \to W$ such that $v(c) = n(c)$, for every $c \in \mathbb{OC}_{\mathsf{RL}_{\mathcal{M},\varphi}}$. Observe that any valuation v in model $\mathcal{N}$ restricted to $\mathbb{OS}_{\mathsf{RL}_{\mathcal{M},\varphi}} \setminus \mathbb{OC}^{\mathbf{0}}_{\mathsf{RL}_{\mathcal{M},\varphi}}$ is a valuation in model $\mathcal{M}$. Moreover, the above definition implies that for every $S \in \mathbb{S}$ and for all $x, y \in \mathbb{OS}_{\mathsf{RL}_{\mathcal{M},\varphi}}$, $\mathcal{N}, v \models xSy$ iff $\mathcal{M}, v \models xSy$. Therefore, it is easy to prove that $n(R) = m(R)$. Note also that the class of $\mathsf{RL}_{\mathcal{M},\varphi}$-models has exactly one element up to isomorphism. Therefore, $\mathsf{RL}_{\mathcal{M},\varphi}$-validity of xRy is equivalent to its truth in a single $\mathsf{RL}_{\mathcal{M},\varphi}$-model $\mathcal{N}$, that is the following holds:

Proposition 3.4.1. *The following statements are equivalent:*

1. $\mathcal{M} \models \varphi$;
2. φ *is* $\mathsf{RL}_{\mathcal{M},\varphi}$*-valid.*

Dual tableau for the logic $\mathsf{RL}_{\mathcal{M},\varphi}$ includes all the rules and axiomatic sets of $\mathsf{RL}(1,1')$-system adapted to the language of $\mathsf{RL}_{\mathcal{M},\varphi}$ and, in addition, the rules and axiomatic sets listed below.

Specific Rules for Model Checking

For every $S \in \mathbb{S}$, for any object symbols x and y, and for any $\mathtt{a}, \mathtt{b} \in U$,

$$(-S\mathtt{ab}) \quad \frac{x-Sy}{x1'c_{\mathtt{a}}, x-Sy \mid y1'c_{\mathtt{b}}, x-Sy} \quad \text{for } (\mathtt{a},\mathtt{b}) \notin m(S)$$

$$(1') \quad \frac{}{x-1'c_{\mathtt{1}} \mid \dots \mid x-1'c_{\mathtt{n}}} \quad \text{whenever } \mathbb{OC}^{0}_{\mathsf{RL}_{\mathcal{M},\varphi}} = \{c_{\mathtt{1}}, \dots, c_{\mathtt{n}}\}, \mathtt{n} \geq 1$$

$$(\mathtt{a} \neq \mathtt{b}) \quad \frac{}{c_{\mathtt{a}}1'c_{\mathtt{b}}} \quad \text{for } \mathtt{a} \neq \mathtt{b}$$

Observe that these rules preserve the formulas built with atomic terms and their complements.

Specific Axiomatic Sets for Model Checking

$\{c_{\mathtt{a}}Sc_{\mathtt{b}}\}$, for any $\mathtt{a}, \mathtt{b} \in U$ such that $(\mathtt{a},\mathtt{b}) \in m(S)$;
$\{c_{\mathtt{a}}-Sc_{\mathtt{b}}\}$, for any $\mathtt{a}, \mathtt{b} \in U$ such that $(\mathtt{a},\mathtt{b}) \notin m(S)$.

The rules $(-S\mathtt{ab})$ and the axiomatic sets reflect the meaning of atomic subterms of R, while the rules $(1')$ and $(\mathtt{a} \neq \mathtt{b})$ guarantee that the universe of an $\mathsf{RL}_{\mathcal{M},\varphi}$-model is of the same cardinality as that of $\mathcal{M}$.

The notions of $\mathsf{RL}_{\mathcal{M},\varphi}$-set and $\mathsf{RL}_{\mathcal{M},\varphi}$-correctness of a rule are defined as in Sect. 2.4.

Proposition 3.4.2. *The rules* $(-S\mathtt{ab})$, $(1')$, *and* $(\mathtt{a} \neq \mathtt{b})$ *are* $\mathsf{RL}_{\mathcal{M},\varphi}$*-correct.*

Proof. For the rule $(-S\mathtt{ab})$, let $\mathtt{a}, \mathtt{b} \in U$ be such that $(\mathtt{a},\mathtt{b}) \notin m(S)$ and let X be any finite set of $\mathsf{RL}_{\mathcal{M},\varphi}$-formulas. Assume $X \cup \{x1'c_{\mathtt{a}}, x-Sy\}$ and $X \cup \{y1'c_{\mathtt{b}}, x-Sy\}$ are $\mathsf{RL}_{\mathcal{M},\varphi}$-sets. Suppose $X \cup \{x-Sy\}$ is not $\mathsf{RL}_{\mathcal{M},\varphi}$-set, that is for some valuation v in $\mathcal{N}$, $(v(x), v(y)) \in m(S)$. It follows from the assumption that the valuation v satisfies $v(x) = \mathtt{a}$ and $v(y) = \mathtt{b}$. Since $(\mathtt{a},\mathtt{b}) \notin m(S)$, $(v(x), v(y)) \notin m(S)$, a contradiction. On the other hand, if $X \cup \{x-Sy\}$ is an $\mathsf{RL}_{\mathcal{M},\varphi}$-set, then so are $X \cup \{x1'c_{\mathtt{a}}, x-Sy\}$ and $X \cup \{y1'c_{\mathtt{b}}, x-Sy\}$.

For the rule $(1')$, note that for every $x \in \mathbb{OS}_{\mathsf{RL}_{\mathcal{M},\varphi}}$ and for every valuation v in $\mathcal{N}$, there exists $c \in \mathbb{OC}^{0}_{\mathsf{RL}_{\mathcal{M},\varphi}}$ such that $v(x) = n(c)$, hence the rule $(1')$ is $\mathsf{RL}_{\mathcal{M},\varphi}$-correct.

The rule $(\mathtt{a} \neq \mathtt{b})$ is correct, since for all $\mathtt{a}, \mathtt{b} \in U$, if $\mathtt{a} \neq \mathtt{b}$, then $n(c_{\mathtt{a}}) \neq n(c_{\mathtt{b}})$. □

Validity of specific axiomatic sets follows directly from the definition of semantics of $\mathsf{RL}_{\mathcal{M},\varphi}$.

The notions of an $\mathsf{RL}_{\mathcal{M},\varphi}$-proof tree, a closed branch of such a tree, a closed $\mathsf{RL}_{\mathcal{M},\varphi}$-proof tree, and an $\mathsf{RL}_{\mathcal{M},\varphi}$-proof of an $\mathsf{RL}_{\mathcal{M},\varphi}$-formula are defined as in Sect. 2.4.

The notions of a complete branch of an $\mathsf{RL}_{\mathcal{M},\varphi}$-proof tree and a complete $\mathsf{RL}_{\mathcal{M},\varphi}$-proof tree are defined as in Sect. 2.5. The completion conditions are those of $\mathsf{RL}(1,1')$-dual tableau adapted to the language of $\mathsf{RL}_{\mathcal{M},\varphi}$ and the conditions listed below:

For all object symbols x and y, for every $S \in \mathbb{S}$, and for all $\mathtt{a},\mathtt{b} \in U$ such that $(\mathtt{a},\mathtt{b}) \notin m(S)$,

Cpl($-S\mathtt{ab}$) If $x{-}Sy \in b$, then either $x1'c_{\mathtt{a}} \in b$ or $y1'c_{\mathtt{b}} \in b$, obtained by an application of the rule ($-S\mathtt{ab}$);

Cpl($1'$) There exists $c \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$ such that $x{-}1'c \in b$, obtained by an application of the rule ($1'$);

Cpl($\mathtt{a} \neq \mathtt{b}$) $c_{\mathtt{a}}1'c_{\mathtt{b}} \in b$, for all $\mathtt{a},\mathtt{b} \in U$ such that $\mathtt{a} \neq \mathtt{b}$, obtained by an application of the rule ($\mathtt{a} \neq \mathtt{b}$).

An open branch of an $\mathsf{RL}_{\mathcal{M},\varphi}$-proof tree is defined as in Sect. 2.5. A branch structure $\mathcal{N}^b = (W^b, n^b)$ is defined as follows:

- $W^b = \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$;
- $n^b(c) = c_{\mathtt{a}}$, where $\mathtt{a} = n(c)$, for every $c \in \mathbb{OC}^1_{\mathsf{RL}_{\mathcal{M},\varphi}}$;
- $n^b(c) = c$, for every $c \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$;
- $n^b(S) = \begin{cases} \{(c_{\mathtt{a}}, c_{\mathtt{b}}) \in W^b \times W^b : c_{\mathtt{a}}Sc_{\mathtt{b}} \notin b\}, & \text{if } S \in \{1, 1'\} \\ \{(c_{\mathtt{a}}, c_{\mathtt{b}}) \in W^b \times W^b : (\mathtt{a},\mathtt{b}) \in m(S)\}, & \text{if } S \in \mathbb{S}; \end{cases}$
- n^b extends to all the compound terms as in $\mathsf{RL}(1,1')$-models.

Similarly as in $\mathsf{RL}(1,1')$-logic it is easy to prove that $n^b(1')$ and $n^b(1)$ are an equivalence relation and a universal relation, respectively. Therefore, $\mathcal{N}^b$ is an $\mathsf{RL}(1,1')$-model. Note that $\mathcal{N}^b$ is not necessarily an $\mathsf{RL}_{\mathcal{M},\varphi}$-model, since $n^b(1')$ may not be the identity.

Let $v^b\colon \mathbb{OS}_{\mathsf{RL}_{\mathcal{M},\varphi}} \to W^b$ be a valuation in $\mathcal{N}^b$ such that:

$v^b(c) = n^b(c)$, for $c \in \mathbb{OC}_{\mathsf{RL}_{\mathcal{M},\varphi}}$;
$v^b(x) = c_{\mathtt{a}}$, where $c_{\mathtt{a}} \in \mathbb{OC}_{\mathsf{RL}_{\mathcal{M},\varphi}}$ is such that $x1'c_{\mathtt{a}} \notin b$, for $x \in \mathbb{OV}_{\mathsf{RL}_{\mathcal{M},\varphi}}$.

The valuation v^b is well defined, that is for every $x \in \mathbb{OV}_{\mathsf{RL}_{\mathcal{M},\varphi}}$, there exists exactly one $c \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$ such that $x1'c \notin b$. Indeed, by the completion condition Cpl($1'$), there exists $c \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$ such that $x{-}1'c \in b$. So $x1'c \notin b$. Suppose there exist two different $c_{\mathtt{a}}, c_{\mathtt{b}} \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$ such that $x1'c_{\mathtt{a}} \notin b$ and $x1'c_{\mathtt{b}} \notin b$. By the completion condition Cpl($\mathtt{a} \neq \mathtt{b}$), $c_{\mathtt{a}}1'c_{\mathtt{b}} \in b$. Then, by the completion conditions Cpl($1'1$) and Cpl($1'2$), $x1'c_{\mathtt{a}} \in b$ or $x1'c_{\mathtt{b}} \in b$, a contradiction.

Proposition 3.4.3 (Satisfaction in Branch Model Property). *For every open branch b of an $\mathsf{RL}_{\mathcal{M},\varphi}$-proof tree and for every $\mathsf{RL}_{\mathcal{M},\varphi}$-formula ψ, if $\mathcal{N}^b, v^b \models \psi$, then $\psi \notin b$.*

Proof. First, we need to show that the proposition holds for formulas xSy and $x{-}Sy$, where $S \in \mathbb{S}$.

Let $\psi = xSy$, for some $S \in \mathbb{S}$. Assume $\mathcal{N}^b, v^b \models xSy$. Let $\mathtt{a}, \mathtt{b} \in U$ be such that $v^b(x) = c_\mathtt{a}$ and $v^b(y) = c_\mathtt{b}$, that is $x1'c_\mathtt{a} \notin b$ and $y1'c_\mathtt{b} \notin b$. Since $\mathcal{N}^b, v^b \models xSy$, $(c_\mathtt{a}, c_\mathtt{b}) \in n^b(S)$, hence $(\mathtt{a}, \mathtt{b}) \in m(S)$. Thus $c_\mathtt{a}Sc_{c_\mathtt{b}} \notin b$, otherwise b would be closed. Suppose $xSy \in b$. By the completion conditions for the rules (1′1) and (1′2) presented in Sect. 2.7, at least one of the following holds: $x1'c_\mathtt{a} \in b$ or $y1'c_\mathtt{b} \in b$ or $c_\mathtt{a}Sc_{c_\mathtt{b}} \in b$, a contradiction.

Let $\psi = x{-}Sy$, for some $S \in \mathbb{S}$. Assume $\mathcal{N}^b, v^b \models x{-}Sy$. Let $\mathtt{a}, \mathtt{b} \in U$ be such that $v^b(x) = c_\mathtt{a}$ and $v^b(y) = c_\mathtt{b}$, that is $x1'c_\mathtt{a} \notin b$ and $y1'c_\mathtt{b} \notin b$. Since $\mathcal{N}^b, v^b \models x{-}Sy$, $(c_\mathtt{a}, c_\mathtt{b}) \notin n^b(S)$. Therefore $(\mathtt{a}, \mathtt{b}) \notin m(S)$. Suppose $x{-}Sy \in b$. Then by the completion condition Cpl($-S\mathtt{ab}$), either $x1'c_\mathtt{a} \in b$ or $y1'c_\mathtt{b} \in b$, a contradiction.

The rest of the proof is similar to the proofs of the analogous propositions for logics RL and $\mathsf{RL}(1, 1')$ (see Proposition 2.5.5 and 2.7.6). □

Since $n^b(1')$ is an equivalence relation on W^b, we may define the quotient structure $\mathcal{N}_q^b = (W_q^b, n_q^b)$:

- $W_q^b = \{\|c\| : c \in W^b\}$, where $\|c\|$ is the equivalence class of $n^b(1')$ determined by c;
- $n_q^b(c) = \|n^b(c)\|$, for every $c \in \mathbb{OC}_{\mathsf{RL}_{\mathcal{M},\varphi}}$;
- $n_q^b(S) = \{(\|c_\mathtt{a}\|, \|c_\mathtt{b}\|) \in W_q^b \times W_q^b : (c_\mathtt{a}, c_\mathtt{b}) \in n^b(S)\}$, for every $S \in \mathbb{RC}_{\mathsf{RL}_{\mathcal{M},\varphi}}$;
- n_q^b extends to all the compound terms as in $\mathsf{RL}(1, 1')$-models.

Proposition 3.4.4 (Branch Model Property). *Let b be an open branch of an $\mathsf{RL}_{\mathcal{M},\varphi}$-proof tree. Then models $\mathcal{N}_q^b$ and $\mathcal{N}$ are isomorphic.*

Proof. Since constants $c \in \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}}$ are uniquely assigned to the elements of model $\mathcal{M}$, $card(W_q^b) = card(W)$. Let $f : W \to W^b$ be a function defined as $f(\mathtt{a}) \stackrel{\text{df}}{=} \|c_\mathtt{a}\|$, for every $\mathtt{a} \in W$. By the definition of model $\mathcal{N}_q^b$, the function f is an isomorphism between $\mathcal{N}_q^b$ and $\mathcal{N}$. □

Let v_q^b be a valuation in $\mathcal{N}_q^b$ defined as $v_q^b(x) = \|v^b(x)\|$, for every $x \in \mathbb{OS}_{\mathsf{RL}_{\mathcal{M},\varphi}}$. As in RL-logic, it is easy to show that the sets of formulas satisfied in $\mathcal{N}^b$ and $\mathcal{N}_q^b$ by valuations v^b and v_q^b, respectively, coincide. Moreover, since $\mathcal{N}_q^b$ and $\mathcal{N}$ are isomorphic, they satisfy exactly the same formulas. Now the completeness of $\mathsf{RL}_{\mathcal{M},\varphi}$ can be proved following Theorems 2.5.1 and 2.7.2.

Theorem 3.4.1 (Soundness and Completeness of $\mathsf{RL}_{\mathcal{M},\varphi}$). *For every $\mathsf{RL}_{\mathcal{M},\varphi}$-formula ψ, the following conditions are equivalent:*

1. *ψ is $\mathsf{RL}_{\mathcal{M},\varphi}$-valid;*
2. *ψ is $\mathsf{RL}_{\mathcal{M},\varphi}$-provable.*

Due to the above theorem and Proposition 3.4.1, we have:

Theorem 3.4.2 (Model Checking in $\mathsf{RL}(1,1')$). *For every $\mathsf{RL}(1,1')$-formula φ and for every finite standard $\mathsf{RL}(1,1')$-model $\mathcal{M}$, the following statements are equivalent:*

1. $\mathcal{M} \models \varphi$;
2. φ *is* $\mathsf{RL}_{\mathcal{M},\varphi}$*-provable.*

Example. Consider $\mathsf{RL}(1,1')$-formula $\varphi = xR\,;Py$ and the standard $\mathsf{RL}(1,1')$-model $\mathcal{M} = (U, m)$ defined as follows:

- $U = \{\mathtt{a}, \mathtt{b}\}$;
- $m(1) = U \times U$;
- $m(P) = \{(\mathtt{a},\mathtt{a}), (\mathtt{a},\mathtt{b})\}$;
- $m(R) = \{(\mathtt{a},\mathtt{a}), (\mathtt{b},\mathtt{a})\}$;
- $m(1') = \{(\mathtt{a},\mathtt{a}), (\mathtt{b},\mathtt{b})\}$;
- m extends to all the compound terms as in $\mathsf{RL}(1,1')$-models.

We apply the method presented above to checking whether φ is true in $\mathcal{M}$. The vocabulary of $\mathsf{RL}_{\mathcal{M},\varphi}$-language adequate for expressing this problem consists of the following sets of symbols:

- $\mathbb{OV}_{\mathsf{RL}_{\mathcal{M},\varphi}}$ – a countable infinite set of object variables;
- $\mathbb{OC}_{\mathsf{RL}_{\mathcal{M},\varphi}} = \mathbb{OC}^0_{\mathsf{RL}_{\mathcal{M},\varphi}} = \{c_\mathtt{a}, c_\mathtt{b}\}$ – the set of object constants;
- $\mathbb{RC}_{\mathsf{RL}_{\mathcal{M},\varphi}} = \{R, P, 1, 1'\}$ – the set of relational constants;
- $\{-, \cup, \cap, ;, ^{-1}\}$ – the set of relational operations.

An $\mathsf{RL}_{\mathcal{M},\varphi}$-model is the structure $\mathcal{N} = (U, n)$ defined as model $\mathcal{M}$ with the following additional conditions: $n(c_\mathtt{a}) = \mathtt{a}$, $n(c_\mathtt{b}) = \mathtt{b}$.

The specific rules of $\mathsf{RL}_{\mathcal{M},\varphi}$-dual tableau are: $(-R\mathtt{ab})$, $(-R\mathtt{bb})$, $(-P\mathtt{ba})$, $(-P\mathtt{bb})$, $(\mathtt{a} \neq \mathtt{b})$, $(\mathtt{b} \neq \mathtt{a})$, and the rule $(1')$ of the following form:

$$(1') \quad \frac{}{x{-}1'c_\mathtt{a} \,|\, x{-}1'c_\mathtt{b}}$$

Specific $\mathsf{RL}_{\mathcal{M},\varphi}$-axiomatic sets are those including either of the following sets: $\{c_\mathtt{a}Rc_\mathtt{a}\}$, $\{c_\mathtt{b}Rc_\mathtt{a}\}$, $\{c_\mathtt{b}{-}Rc_\mathtt{b}\}$, $\{c_\mathtt{a}{-}Rc_\mathtt{b}\}$, $\{c_\mathtt{a}Pc_\mathtt{a}\}$, $\{c_\mathtt{a}Pc_\mathtt{b}\}$, $\{c_\mathtt{b}{-}Pc_\mathtt{b}\}$ or $\{c_\mathtt{b}{-}Pc_\mathtt{a}\}$.

By Theorem 3.4.2, truth of φ in $\mathcal{M}$ is equivalent to $\mathsf{RL}_{\mathcal{M},\varphi}$-provability of φ. Figure 3.1 presents an $\mathsf{RL}_{\mathcal{M},\varphi}$-proof of φ.

3.5 Verification of Satisfaction in Relational Logics

The logic $\mathsf{RL}(1,1')$ can also be used for verification of satisfaction of a formula in a fixed finite model. Let $\varphi = xRy$ be an $\mathsf{RL}(1,1')$-formula, where R is a relational term and x, y are any object symbols, let $\mathcal{M} = (U, m)$ be a fixed standard $\mathsf{RL}(1,1')$-model with a finite universe U, and let v be a valuation in $\mathcal{M}$ such that

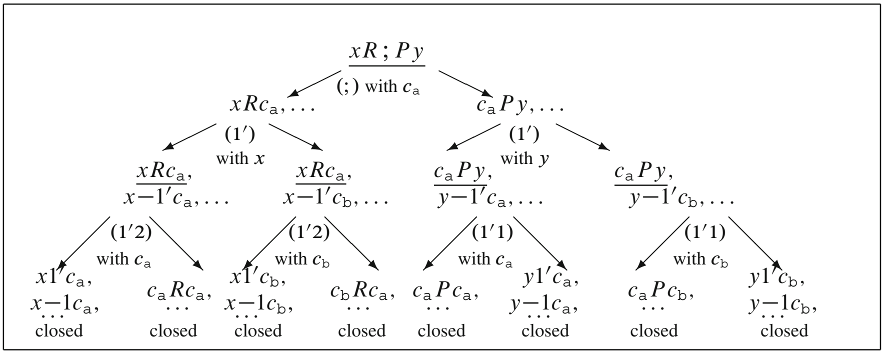

Fig. 3.1 An $\mathsf{RL}_{\mathcal{M},\varphi}$-proof showing that $xR\,;Py$ is true in the model $\mathcal{M}$

$v(x) = \mathtt{a}$ and $v(y) = \mathtt{b}$, for some elements $\mathtt{a}, \mathtt{b} \in U$. Recall that for every object constant c, for every relational model $\mathcal{M} = (U, m)$, and for every valuation v in $\mathcal{M}$, $v(c) = m(c)$ by definition. In particular, if both x and y are object constants, then there is exactly one pair $(\mathtt{a}, \mathtt{b})$ of elements of U such that $v(x) = \mathtt{a}$ and $v(y) = \mathtt{b}$.

The relational formalism appropriate for solving the problem '$(\mathtt{a}, \mathtt{b}) \in m(R)$?' is the logic $\mathsf{RL}_{\mathcal{M},\varphi}$ defined in the previous section with $\varphi = xRy$. Since for every $\mathsf{RL}_{\mathcal{M},\varphi}$-model $\mathcal{N} = (U, n)$, for every valuation v in $\mathcal{N}$, and for every $\mathtt{a} \in U$, $v(c_{\mathtt{a}}) = \mathtt{a}$, the following holds:

Proposition 3.5.1. *The following statements are equivalent:*

1. $(\mathtt{a}, \mathtt{b}) \in m(R)$;
2. $c_{\mathtt{a}} R c_{\mathtt{b}}$ *is* $\mathsf{RL}_{\mathcal{M},\varphi}$*-valid.*

Due to the above proposition and Theorem 3.4.1, we get:

Theorem 3.5.1 (Satisfaction in $\mathsf{RL}(1, 1')$-models). *For every relational term R of $\mathsf{RL}(1, 1')$-language, for every finite standard $\mathsf{RL}(1, 1')$-model $\mathcal{M} = (U, m)$, and for all $\mathtt{a}, \mathtt{b} \in U$, the following statements are equivalent:*

1. $(\mathtt{a}, \mathtt{b}) \in m(R)$;
2. $c_{\mathtt{a}} R c_{\mathtt{b}}$ *is* $\mathsf{RL}_{\mathcal{M},\varphi}$*-provable.*

Example. Consider $\mathsf{RL}(1, 1')$-formula $\varphi = x(P\,; -(R\,;P))y$ and the standard $\mathsf{RL}(1, 1')$-model $\mathcal{M} = (U, m)$ such that:

- $U = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$;
- $m(1) = U \times U$;
- $m(P) = \{(\mathtt{a}, \mathtt{a}), (\mathtt{a}, \mathtt{b}), (\mathtt{a}, \mathtt{c})\}$;
- $m(R) = \{(\mathtt{b}, \mathtt{a}), (\mathtt{c}, \mathtt{c})\}$;
- $m(1') = \{(\mathtt{a}, \mathtt{a}), (\mathtt{b}, \mathtt{b}), (\mathtt{c}, \mathtt{c})\}$;
- m extends to all the compound terms as in $\mathsf{RL}(1, 1')$-models.

Let v be a valuation such that $v(x) = \mathtt{a}$ and $v(y) = \mathtt{b}$. By Theorem 3.5.1 the satisfaction problem 'is the formula $\varphi = x(P\,;-(R\,;P))y$ satisfied in $\mathcal{M}$ by v?' is equivalent to $\mathsf{RL}_{\mathcal{M},\varphi}$-provability of $c_{\mathtt{a}}(P\,;-(R\,;P))c_{\mathtt{b}}$.

$\mathsf{RL}_{\mathcal{M},\varphi}$-dual tableau contains the rules and axiomatic sets of $\mathsf{RL}(1,1')$-proof system adjusted to $\mathsf{RL}_{\mathcal{M},\varphi}$-language, the rules $(-R\mathtt{aa})$, $(-R\mathtt{ab})$, $(-R\mathtt{ac})$, $(-R\mathtt{bb})$, $(-R\mathtt{bc})$, $(-R\mathtt{ca})$, $(-R\mathtt{cb})$, $(-P\mathtt{ba})$, $(-P\mathtt{bb})$, $(-P\mathtt{bc})$, $(-P\mathtt{ca})$, $(-P\mathtt{cb})$, $(-P\mathtt{cc})$, $(\mathtt{a} \neq \mathtt{b})$, $(\mathtt{a} \neq \mathtt{c})$, $(\mathtt{b} \neq \mathtt{c})$, and the rule $(1')$ of the following form:

$$(1') \quad \frac{}{x{-}1'c_{\mathtt{a}} \,|\, x{-}1'c_{\mathtt{b}} \,|\, x{-}1'c_{\mathtt{c}}}$$

The axiomatic sets specific for $\mathsf{RL}_{\mathcal{M},\varphi}$ are those including one of the following sets: $\{c_{\mathtt{b}}Rc_{\mathtt{a}}\}$, $\{c_{\mathtt{c}}Rc_{\mathtt{c}}\}$, $\{c_{\mathtt{a}}Pc_{\mathtt{a}}\}$, $\{c_{\mathtt{a}}Pc_{\mathtt{b}}\}$, $\{c_{\mathtt{a}}Pc_{\mathtt{c}}\}$, $\{c_{\mathtt{a}}{-}Rc_{\mathtt{a}}\}$, $\{c_{\mathtt{a}}{-}Rc_{\mathtt{b}}\}$, $\{c_{\mathtt{a}}{-}Rc_{\mathtt{c}}\}$, $\{c_{\mathtt{b}}{-}Rc_{\mathtt{b}}\}$, $\{c_{\mathtt{b}}{-}Rc_{\mathtt{c}}\}$, $\{c_{\mathtt{c}}{-}Rc_{\mathtt{a}}\}$, $\{c_{\mathtt{c}}{-}Rc_{\mathtt{b}}\}$, $\{c_{\mathtt{b}}{-}Pc_{\mathtt{a}}\}$, $\{c_{\mathtt{b}}{-}Pc_{\mathtt{b}}\}$, $\{c_{\mathtt{b}}{-}Pc_{\mathtt{c}}\}$, $\{c_{\mathtt{c}}{-}Pc_{\mathtt{a}}\}$, $\{c_{\mathtt{c}}{-}Pc_{\mathtt{b}}\}$ or $\{c_{\mathtt{c}}{-}Pc_{\mathtt{c}}\}$.

Figure 3.2 presents an $\mathsf{RL}_{\mathcal{M},\varphi}$-proof of $c_{\mathtt{a}}(P\,;-(R\,;P))c_{\mathtt{b}}$ that shows satisfaction of $x(P\,;-(R\,;P))y$ in the model $\mathcal{M}$ by the valuation v defined above.

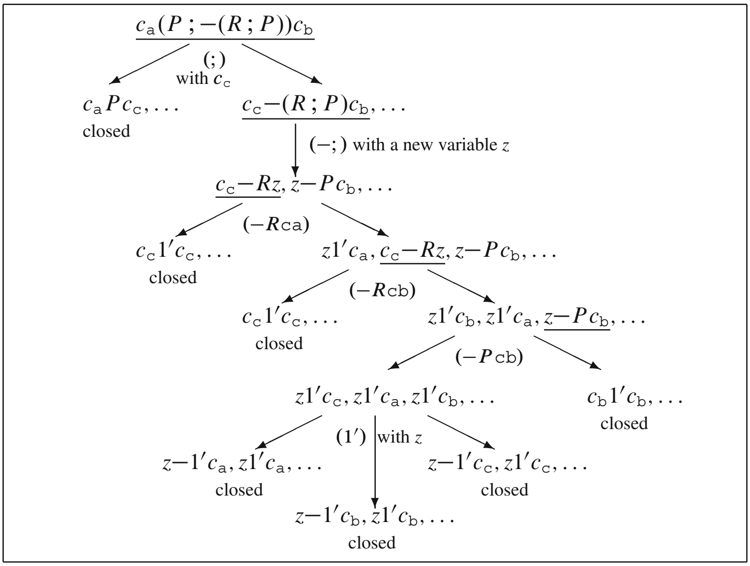

Fig. 3.2 An $\mathsf{RL}_{\mathcal{M},\varphi}$-proof showing that $x(P\,;-(R\,;P))y$ is satisfied in the model $\mathcal{M}$ by the pair $(\mathtt{a},\mathtt{b})$