Chapter 22
Dual Tableaux for First-Order Post Logics

22.1 Introduction

Emil Post's doctoral dissertation [Pos21] included a description of an n-valued, functionally complete algebra, for a finite $n \geq 2$. The notion of Post algebra was introduced in [Ros42]. The first algebraic formulation of Post algebras with a family of unary disjoint operations was presented in [Eps60]. In [Rou69, Rou70] an equivalent formulation of the class of Post algebras was given where monotone operations instead of disjoint operations were used. It became a starting point of extensive research and since then various generalizations of Post algebras have been developed, see e.g., [Ras73, Ras85, ER90, ER91]. Post algebras were also applied to a multiple-valued formalization of some logics of programs, see [Ras94].

In this chapter we present a dual tableau for the class of first-order Post logics based on Post algebras of order $n \geq 2$, along the lines of [Sal72, Orł85b].

22.2 Post Algebras of Order *n*

A Post algebra of order $n \geq 2$ is a structure of the form:

$$\mathfrak{P}_n = (P, -, \vee, \wedge, \rightarrow, d_1, \ldots, d_{n-1}, e_0, \ldots, e_{n-1}),$$

where for all $a, b \in P$ the following conditions are satisfied:

(P1) $(P, \vee, \wedge)$ is a distributive lattice;
(P2) $e_0, \ldots, e_{n-1}$ are distinguished elements of P such that e_0 is the smallest element and e_{n-1} is the greatest element of the lattice;
(P3) $(P, -, \vee, \wedge, \rightarrow, e_0, e_{n-1})$ is a Heyting algebra;
(P4) $d_i(a \vee b) = d_i a \vee d_i b$;
(P5) $d_i(a \wedge b) = d_i a \wedge d_i b$;
(P6) $d_i(a \rightarrow b) = (d_1 a \rightarrow d_1 b) \wedge \ldots \wedge (d_i a \rightarrow d_i b)$;
(P7) $d_i(-a) = -d_1 a$;
(P8) $d_i d_j a = d_j a$;

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(P9) $d_i e_j = e_{n-1}$ if $i \leq j$, and $d_i e_j = e_0$ if $i > j$;
(P10) $a = (d_1 a \wedge e_1) \vee \ldots \vee (d_{n-1} a \wedge e_{n-1})$;
(P11) $d_1 a \vee -d_1 a = e_{n-1}$.

Proposition 22.2.1. *Let $\leq$ be the ordering in the lattice $(P, \vee, \wedge)$. Then in any Post algebra of order n, the following hold for all $i, j \in \{1, \ldots, n-1\}$ and for all $a, b \in P$:*

1. $e_0 \leq e_1 \leq \ldots \leq e_{n-1}$;
2. $d_i a \leq d_j a$, *for* $j \leq i$;
3. *Operations d_i are monotone, i.e., if $a \leq b$, then $d_i a \leq d_i b$;*
4. *The set B_P of elements of the form $d_i a$ is closed with respect to operations $-$, $\vee$, and $\wedge$ and the algebra $\mathfrak{B}_{\mathfrak{P}_n} = (B_P, -, \vee, \wedge)$ is a Boolean algebra;*
5. $d_i(a \to b) = \bigwedge_{k=1}^{i} (-d_k a \vee d_k b)$;
6. $-d_i(a \to b) = d_1 a \wedge \bigwedge_{k=1}^{i-1} (d_{k+1} a \vee -d_k b) \wedge -d_i b$.

Infinite meets and joins in $\mathfrak{P}_n$ are denoted by $\bigcap^{\mathfrak{P}_n}$ and $\bigcup^{\mathfrak{P}_n}$, respectively. Similarly, $\bigcap^{\mathfrak{B}_{\mathfrak{P}_n}}$ and $\bigcup^{\mathfrak{B}_{\mathfrak{P}_n}}$ denote the infinite meet and join, respectively, in the Boolean algebra $\mathfrak{B}_{\mathfrak{P}_n}$ determined by $\mathfrak{P}_n$. In [Eps60] the following was proved:

Proposition 22.2.2. *For every element $a \in \mathfrak{P}_n$ and for any indexed family $\{a_t\}_{t \in T}$ of elements of $\mathfrak{P}_n$, the infinite meets and joins satisfy the following conditions:*

1. $a = \bigcup_{t \in T}^{\mathfrak{P}_n} a_t$ *iff* $d_i a = \bigcup_{t \in T}^{\mathfrak{B}_{\mathfrak{P}_n}} d_i a_t$;
2. $a = \bigcap_{t \in T}^{\mathfrak{P}_n} a_t$ *iff* $d_i a = \bigcap_{t \in T}^{\mathfrak{B}_{\mathfrak{P}_n}} d_i a_t$.

The disjoint operations c_i, $i \in \{0, \ldots, n-1\}$, introduced in [Eps60], can be defined in terms of the monotone operations d_i as:

$$c_0 a \stackrel{\text{df}}{=} -d_1 a = -a;$$

$$c_i a \stackrel{\text{df}}{=} d_i a \wedge -d_{i+1} a, \text{for } i \in \{1, \ldots, n-2\};$$

$$c_{n-1} a \stackrel{\text{df}}{=} d_{n-1} a.$$

Then, $c_i a \wedge c_j a = e_0$, for all $i, j \in \{0, \ldots, n-1\}$ such that $i \neq j$. Furthermore, $c_i(e_j) = e_{n-1}$ if $i = j$, and otherwise $c_i(e_j) = e_0$.

22.3 First-Order *n*-Valued Post Logic

The language of an n-valued Post logic, P_n, is a first-order language whose formulas are constructed with the symbols from the following pairwise disjoint sets:

- $\mathbb{OV}_{\mathsf{P}_n}$ – a countable infinite set of individual (object) variables;
- $\{E_0, \ldots, E_{n-1}\}$ – the set of propositional constants;

- $\mathbb{P}^k_{\mathsf{P}_n}$ – a countable set of predicate symbols, where $k \geq 1$;
- $\{\neg, D_1, \ldots, D_{n-1}\}$ – the set of unary propositional operations;
- $\{\vee, \wedge, \rightarrow\}$ – the set of binary propositional operations;
- $\{\forall, \exists\}$ – the set of quantifiers.

As usual, we slightly abuse the notation using the symbols $\neg, \vee, \wedge, \rightarrow$ both for the operations in Post algebras and in the language of the logic.

Atomic P_n-formulas are of the form E_i, for $i \in \{0, \ldots, n-1\}$, or $R(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in \mathbb{OV}_{\mathsf{P}_n}$ and R is a k-ary predicate symbol, $k \geq 1$. The set of P_n-formulas is the smallest set including the set of atomic formulas and closed with respect to propositional operations and quantifiers.

Algebraic semantics of P_n-language is provided by the class of complete Post algebras of order n, $\mathfrak{P}_n = (P, -, \vee, \wedge, \rightarrow, d_1, \ldots, d_{n-1}, e_0, \ldots, e_{n-1})$. Elements $e_0, \ldots, e_{n-1}$ of $\mathfrak{P}_n$ play the role of truth values. Propositional operations correspond to the respective algebraic operations, and quantifiers $\forall$ and $\exists$ correspond to infinite meet and join in $\mathfrak{P}_n$, respectively. Intuitively, a formula $D_i\varphi$ says that the value of φ is not less than e_i. A P_n*-model* is a structure $\mathcal{M} = (U, \mathfrak{P}_n, m)$, where U is a non-empty set, $\mathfrak{P}_n$ is a complete Post algebra of order n, and m is a meaning function such that:

- $m(E_i) = e_i$, for $i \in \{0, \ldots, n-1\}$;
- $m(R) \in \{e_0, \ldots, e_{n-1}\}^{U^k}$, for every k-ary predicate symbol $R \in \mathbb{P}_{\mathsf{P}_n}$, $k \geq 1$.

Thus, m assigns functions from U^k into $\{e_0, \ldots, e_{n-1}\}$ to k-ary predicate symbols. It follows that the meaning of a k-ary predicate is an n-ary characteristic function of a k-ary relation.

Let $\mathcal{M}$ be a P_n-model. A valuation in $\mathcal{M}$ is a function $v\colon \mathbb{OV}_{\mathsf{P}_n} \rightarrow U$ assigning elements of the universe to individual variables. Given a P_n-model $\mathcal{M}$ and a valuation v in $\mathcal{M}$, we define function $val_{\mathcal{M},v}$ that assigns elements of algebra $\mathfrak{P}_n$ to formulas:

- $val_{\mathcal{M},v}(E_i) = e_i$, for every $i \in \{0, \ldots, n-1\}$;
- $val_{\mathcal{M},v}(R(x_1, \ldots, x_k)) = m(R)(v(x_1), \ldots, v(x_k))$, for every k-ary predicate symbol $R \in \mathbb{P}_{\mathsf{P}_n}$, $k \geq 1$, and for all $x_1, \ldots, x_k \in \mathbb{OV}_{\mathsf{P}_n}$;
- $val_{\mathcal{M},v}(\neg\varphi) = -val_{\mathcal{M},v}(\varphi)$;
- $val_{\mathcal{M},v}(\varphi \vee \psi) = val_{\mathcal{M},v}(\varphi) \vee val_{\mathcal{M},v}(\psi)$;
- $val_{\mathcal{M},v}(\varphi \wedge \psi) = val_{\mathcal{M},v}(\varphi) \wedge val_{\mathcal{M},v}(\psi)$;
- $val_{\mathcal{M},v}(\varphi \rightarrow \psi) = val_{\mathcal{M},v}(\varphi) \rightarrow val_{\mathcal{M},v}(\psi)$;
- $val_{\mathcal{M},v}(D_i\varphi) = d_i val_{\mathcal{M},v}(\varphi)$;
- $val_{\mathcal{M},v}(\forall x\varphi(x)) = \bigcap^{\mathfrak{P}_n}_{u \in U} val_{\mathcal{M},v_u}(\varphi(x))$;
- $val_{\mathcal{M},v}(\exists x\varphi(x)) = \bigcup^{\mathfrak{P}_n}_{u \in U} val_{\mathcal{M},v_u}(\varphi(x))$;

where v_u is the valuation in $\mathcal{M}$ such that $v_u(x) = u$ and $v_u(z) = v(z)$, for all $z \neq x$.

The function $val_{\mathcal{M},v}$ extends to finite sets of formulas. Let $X = \{\varphi_1, \ldots, \varphi_l\}$, $l \leq 1$, be a finite set of P_n-formulas. We define:

$$val_{\mathcal{M},v}(X) \stackrel{\text{df}}{=} val_{\mathcal{M},v}(\varphi_1 \vee \ldots \vee \varphi_l).$$

A P_n-formula φ is said to be true in a model $\mathcal{M}$ whenever for every valuation v in $\mathcal{M}$, $val_{\mathcal{M},v}(\varphi) = e_{n-1}$, and it is said to be e_s*-valid*, $s \in \{1, \ldots, n-1\}$, if for every P_n-model $\mathcal{M}$ and for every valuation v in $\mathcal{M}$, $val_{\mathcal{M},v}(\varphi) \geq e_s$. A formula φ is P_n-valid whenever it is e_{n-1}-valid. The definition of semantics leads to the following observation:

Proposition 22.3.1. *For every* P_n*-formula* φ *and for every* $s \in \{1, \ldots, n-1\}$*, the following conditions are equivalent:*

1. φ *is* e_s*-valid;*
2. $D_s\varphi$ *is* P_n*-valid.*

22.4 Dual Tableaux for Post Logics

Dual tableaux for Post logics consist of the decomposition rules of the following forms:

For all P_n-formulas φ and ψ, for every individual variable x, and for all $i, j \in \{1, \ldots, n-1\}$,

Decomposition Rules for Propositional Operations

$$(\vee)\ \frac{D_i(\varphi \vee \psi)}{D_i\varphi, D_i\psi} \qquad (\neg\vee)\ \frac{\neg D_i(\varphi \vee \psi)}{\neg D_i\varphi \mid \neg D_i\psi}$$

$$(\wedge)\ \frac{D_i(\varphi \wedge \psi)}{D_i\varphi \mid D_i\psi} \qquad (\neg\wedge)\ \frac{\neg D_i(\varphi \wedge \psi)}{\neg D_i\varphi, \neg D_i\psi}$$

$$(\rightarrow)\ \frac{D_i(\varphi \rightarrow \psi)}{\neg D_1\varphi, D_1\psi \mid \ldots \mid \neg D_i\varphi, D_i\psi}$$

$$(\neg \rightarrow)\ \frac{\neg D_i(\varphi \rightarrow \psi)}{D_1\varphi \mid D_2\varphi, \neg D_1\psi \mid \ldots \mid \neg D_i\varphi, \neg D_{i-1}\psi \mid \neg D_i\psi}$$

$$(\neg)\ \frac{D_i(\neg\varphi)}{\neg D_1\varphi} \qquad (\neg\neg)\ \frac{\neg D_i(\neg\varphi)}{D_1\varphi}$$

$$(ij)\ \frac{D_iD_j(\varphi)}{D_j\varphi} \qquad (\neg ij)\ \frac{\neg D_iD_j(\varphi)}{\neg D_j\varphi}$$

Decomposition Rules for Quantifiers

(∀) $\dfrac{D_i \forall x \varphi(x)}{D_i \varphi(z)}$ z is a new individual variable

(¬∀) $\dfrac{\neg D_i \forall x \varphi(x)}{\neg D_i \varphi(z), \neg D_i \forall x \varphi(x)}$ z is any individual variable

(∃) $\dfrac{D_i \exists x \varphi(x)}{D_i \varphi(z), D_i \exists x \varphi(x)}$ z is any individual variable

(¬∃) $\dfrac{\neg D_i \exists x \varphi(x)}{\neg D_i \varphi(z)}$ z is a new individual variable

We observe that any application of the rules of P_n-dual tableau preserves the atomic formulas and their negations.

A set of P_n-formulas is said to be P_n-axiomatic whenever it includes either of the following sets of formulas:

For every P_n-formula φ,

(Ax1) $\{D_i(E_j)\}$, for $1 \leq i \leq j$;
(Ax2) $\{\neg D_i(E_j)\}$, for $i > j \geq 0$;
(Ax3) $\{D_i \varphi, \neg D_j \varphi\}$, for $1 \leq i \leq j$.

Proposition 22.4.1. *Let $X = \{\varphi_1, \ldots, \varphi_l\}$, $l \geq 1$, be a finite set of P_n-formulas, let $\mathcal{M}$ be a P_n-model, and let v be a valuation in $\mathcal{M}$. Then, for every P_n-decomposition rule for a propositional operation of the form $\frac{\Phi}{\Phi_1 | \ldots | \Phi_t}$, $t \geq 1$, the following holds:*

$$val_{\mathcal{M},v}(X) \vee val_{\mathcal{M},v}(\Phi) = \bigwedge_{i=1}^{t} (val_{\mathcal{M},v}(X) \vee val_{\mathcal{M},v}(\Phi_i)).$$

Proof. By way of example, we prove the statement for the rules (¬∨), (→), (¬ij).

(¬∨) By axioms (P1) and (P4) of Post algebras and since $\mathfrak{B}_{\mathfrak{P}_n}$ is a Boolean algebra, we obtain:

$$\begin{aligned}
&(val_{\mathcal{M},v}(X) \vee val_{\mathcal{M},v}(\neg D_i \varphi)) \wedge (val_{\mathcal{M},v}(X) \vee val_{\mathcal{M},v}(\neg D_i \psi)) \\
&= val_{\mathcal{M},v}(X) \vee (val_{\mathcal{M},v}(\neg D_i \varphi) \wedge val_{\mathcal{M},v}(\neg D_i \psi)) \\
&= val_{\mathcal{M},v}(X) \vee (-d_i val_{\mathcal{M},v}(\varphi) \wedge -d_i val_{\mathcal{M},v}(\psi)) \\
&= val_{\mathcal{M},v}(X) \vee -(d_i val_{\mathcal{M},v}(\varphi) \vee d_i val_{\mathcal{M},v}(\psi)) \\
&= val_{\mathcal{M},v}(X) \vee -d_i val_{\mathcal{M},v}(\varphi \vee \psi) \\
&= val_{\mathcal{M},v}(X) \vee val_{\mathcal{M},v}(\neg D_i (\varphi \vee \psi)).
\end{aligned}$$

$(\rightarrow)$ By the axiom (P1) and the condition 5. of Proposition 22.2.1, we have:

$$val_{\mathcal{M},v}(X) \vee val_{\mathcal{M},v}(D_i(\varphi \rightarrow \psi))$$
$$= val_{\mathcal{M},v}(X) \vee \bigwedge_{k=1}^{i} (-d_k val_{\mathcal{M},v}(\varphi) \vee d_k val_{\mathcal{M},v}(\psi))$$
$$= \bigwedge_{k=1}^{i} (val_{\mathcal{M},v}(X) \vee -d_k val_{\mathcal{M},v}(\varphi) \vee d_k val_{\mathcal{M},v}(\psi)).$$

$(\neg ij)$ By axiom (P8) we have:

$$val_{\mathcal{M},v}(X) \vee val_{\mathcal{M},v}(\neg D_i D_j \varphi) = val_{\mathcal{M},v}(X) \vee -d_j val_{\mathcal{M},v}(\varphi)$$
$$= val_{\mathcal{M},v}(X) \vee val_{\mathcal{M},v}(\neg D_j \varphi).$$

The proofs for the remaining rules are similar. □

Proposition 22.2.1 (1.) and (4.) imply:

Proposition 22.4.2. *Let φ and ψ be P_n-formulas and let $i, j \in \{1, \ldots, n-1\}$. Then, for every P_n-model $\mathcal{M}$ and for every valuation v in $\mathcal{M}$, the following hold:*

1. *$val_{\mathcal{M},v}(D_i\varphi \vee D_i\psi) = e_{n-1}$ iff $val_{\mathcal{M},v}(D_i\varphi) = e_{n-1}$ or $val_{\mathcal{M},v}(D_i\psi) = e_{n-1}$;*
2. *If $i \leq j$, then $val_{\mathcal{M},v}(D_i\varphi) \vee val_{\mathcal{M},v}(\neg D_j\varphi) = e_{n-1}$.*

Proof. For 1., note that for every $i \in \{1, \ldots, n-1\}$ and for every formula φ, either $val_{\mathcal{M},v}(D_i\varphi) = e_{n-1}$ or $val_{\mathcal{M},v}(D_i\varphi) = e_0$, hence by the definition of semantics, 1. follows.

For 2., observe that:

$$val_{\mathcal{M},v}(D_i\varphi) \vee val_{\mathcal{M},v}(\neg D_j\varphi) = d_i(val_{\mathcal{M},v}(\varphi)) \vee -d_j(val_{\mathcal{M},v}(\varphi)).$$

If $val_{\mathcal{M},v}(\varphi) = e_k$, for some $k \in \{1, \ldots, n\}$ such that $1 \leq k < i \leq j$, then $d_i(val_{\mathcal{M},v}(\varphi)) \vee -d_j(val_{\mathcal{M},v}(\varphi)) = d_i e_k \vee (-d_j e_k) = e_0 \vee (-e_0) = e_0 \vee e_{n-1} = e_{n-1}$. If $val_{\mathcal{M},v}(\varphi) = e_k$ for some $k \in \{1, \ldots, n\}$ such that $1 \leq i \leq k < j$, then $d_i(val_{\mathcal{M},v}(\varphi)) \vee -d_j(val_{\mathcal{M},v}(\varphi)) = d_i e_k \vee (-d_j e_k) = e_{n-1} \vee (-e_0) = e_{n-1} \vee e_{n-1} = e_{n-1}$. If $val_{\mathcal{M},v}(\varphi) = e_k$ for some $k \in \{1, \ldots, n\}$ such that $1 \leq i \leq j \leq k$, then $d_i(val_{\mathcal{M},v}(\varphi)) \vee -d_j(val_{\mathcal{M},v}(\varphi)) = d_i e_k \vee (-d_j e_k) = e_{n-1} \vee (-e_{n-1}) = e_{n-1} \vee e_0 = e_{n-1}$. □

As usual, a P_n-set is a finite set of P_n-formulas such that the disjunction of its members is true in all P_n-models. P_n-correctness of a rule is defined in a similar way as in the logic F (see Sect. 1.3), i.e., a rule is P_n-correct whenever it preserves and reflects P_n-validity.

Proposition 22.4.3.

1. *The* P_n*-rules are* P_n*-correct;*
2. *The* P_n*-axiomatic sets are* P_n*-sets.*

Proof. For 1., observe that correctness of decomposition rules for propositional operations follows from Proposition 22.4.1. Correctness of decomposition rules for quantifiers follows from Proposition 22.2.2 and Proposition 22.4.2(1.).

2. follows from Proposition 22.4.2(2.). □

In order to prove e_s-validity of a P_n-formula φ, we built a P_n-decomposition tree for the formula $D_s\varphi$. As usual, each node of the tree includes all the formulas of its predecessor node, possibly except for those which have been transformed by a rule. A node of the tree does not have successors whenever its set of formulas includes a P_n-axiomatic subset or none of the rules is applicable to it. The notions of a closed branch of a P_n-proof tree, a closed P_n-proof tree, and P_n-provability are defined as in Sect. 1.3.

Proposition 22.4.4 (Closed Branch Property). *Let* φ *be an atomic* P_n*-formula and let* $1 \leq i \leq j$*. For every branch* b *of a* P_n*-proof tree, if* $D_i\varphi \in b$ *and* $D_j\varphi \in b$*, then* b *is closed.*

Proof. Let φ be an atomic P_n-formula, let $1 \leq i \leq j$, and let b be a branch of a P_n-proof tree. Observe that the rules of P_n-dual tableau, in particular the specific rules, guarantee that if formulas $D_i\varphi$ and $D_j\varphi$ belong to the branch b, then there is a node of that branch which includes both of them. Thus, branch b contains an axiomatic set of formulas, hence it is closed. □

A branch b of a P_n-proof tree is complete whenever it is closed or it satisfies the following completion conditions:

For all P_n-formulas φ and ψ, for every individual variable x, and for all $i, j \in \{1, \ldots, n-1\}$,

Cpl($\vee$) (resp. Cpl($\neg\vee$)) If $D_i(\varphi \vee \psi) \in b$ (resp. $\neg D_i(\varphi \vee \psi) \in b$), then both $D_i\varphi \in b$ and $D_i\psi \in b$ (resp. $\neg D_i\varphi \in b$ and $\neg D_i\psi \in b$), obtained by an application of the rule ($\vee$) (resp. ($\neg\vee$));

Cpl($\wedge$) (resp. Cpl($\neg\wedge$)) If $D_i(\varphi \wedge \psi) \in b$ (resp. $\neg D_i(\varphi \wedge \psi) \in b$), then either $D_i\varphi \in b$ or $D_i\psi \in b$ (resp. $\neg D_i\varphi \in b$ or $\neg D_i\psi \in b$), obtained by an application of the rule ($\wedge$) (resp. ($\neg\wedge$));

Cpl($\rightarrow$) If $D_i(\varphi \rightarrow \psi) \in b$, then there exists $k \in \{1, \ldots, i\}$ such that both $\neg D_k\varphi \in b$ and $D_k\psi \in b$, obtained by an application of the rule ($\rightarrow$);

Cpl($\neg\rightarrow$) If $\neg D_i(\varphi \rightarrow \psi) \in b$, then either $D_1\varphi \in b$ or $\neg D_i\psi \in b$ or there exists $k \in \{2, \ldots, i\}$ such that both $D_k\varphi \in b$ and $\neg D_{k-1}\psi \in b$, obtained by an application of the rule ($\neg\rightarrow$);

Cpl($\neg$) If $D_i(\neg\varphi) \in b$, then $\neg D_1\varphi \in b$, obtained by an application of the rule ($\neg$);

Cpl($\neg\neg$) If $\neg D_i(\neg\varphi) \in b$, then $D_1\varphi \in b$, obtained by an application of the rule ($\neg\neg$);

Cpl(ij) If $D_i D_j(\varphi) \in b$, then $D_j\varphi \in b$, obtained by an application of the rule (ij);
Cpl($\neg ij$) If $\neg D_i D_j(\varphi) \in b$, then $\neg D_j\varphi \in b$, obtained by an application of the rule $(\neg ij)$;
Cpl($\forall$) (resp. Cpl($\neg\exists$)) If $D_i\forall x\varphi(x) \in b$ (resp. $\neg D_i\exists x\varphi(x) \in b$), then for some individual variable z, $D_i\varphi(z) \in b$ (resp. $\neg D_i\varphi(z) \in b$), obtained by an application of the rule ($\forall$) (resp. $\neg\exists$));
Cpl($\exists$) (resp. Cpl($\neg\forall$)) If $D_i\exists x\varphi(x) \in b$ (resp. $\neg D_i\forall x\varphi(x) \in b$), then for every individual variable z, $D_i\varphi(z) \in b$ (resp. $\neg D_i\varphi(z) \in b$), obtained by an application of the rule ($\exists$) (resp. ($\neg\forall$)).

The notions of a complete P_n-proof tree and an open branch of a P_n-proof tree are defined as in F-logic (see Sect. 1.3).

Let b be an open branch of a P_n-proof tree. We define a branch structure $\mathcal{M}^b = (U^b, \mathfrak{P}_n, m^b)$ as follows:

- $U^b = \mathbb{OV}_{\mathsf{P}_n}$;
- $m^b(E_i) = e_i$;
- For all $x_1, \ldots, x_k \in \mathbb{OV}_{\mathsf{P}_n}$ and for every k-ary predicate symbol $R \in \mathbb{P}_{\mathsf{P}_n}$, $k \geq 1$:

$$m^b(R)(x_1,\ldots,x_k) = \begin{cases} e_{i-1} & \text{if } i \text{ is the smallest element of } \{1,\ldots,n-1\} \\ & \text{such that } D_i R(x_1,\ldots,x_k) \in b \\ & \\ e_{n-1} & \text{if for all } i < n,\ D_i R(x_1,\ldots,x_k) \notin b. \end{cases}$$

Clearly, a branch structure $\mathcal{M}^b$ is a P_n-model. Therefore, the branch model property holds. Let v^b be the identity valuation in $\mathcal{M}^b$. Now, we show the satisfaction in branch model property:

Proposition 22.4.5 (Satisfaction in Branch Model Property). *Let b be an open branch of a P_n-proof tree. Then, for every P_n-formula φ and for every $i \in \{1,\ldots,n-1\}$, the following hold:*

1. *If $D_i\varphi \in b$, then $val_{\mathcal{M}^b,v^b}(D_i\varphi) < e_{n-1}$;*
2. *If $\neg D_i\varphi \in b$, then $val_{\mathcal{M}^b,v^b}(\neg D_i\varphi) < e_{n-1}$.*

Proof. The proof is by induction on the complexity of formulas. First, we prove that 1. and 2. hold for atomic P_n-formulas.

If $D_i(E_j) \in b$, then $i > j$, since otherwise b would be closed. Thus, $val_{\mathcal{M}^b,v^b}(D_i(E_j)) = e_0 < e_{n-1}$. Assume that $D_i R(x_1,\ldots,x_k) \in b$, for some predicate symbol R and for some individual variables $x_1,\ldots,x_k$. Then, we have $val_{\mathcal{M}^b,v^b}(D_i R(x_1,\ldots,x_k)) = d_i val_{\mathcal{M}^b,v^b}(R(x_1,\ldots,x_k))$ and, by the definition of $m^b(R)$, $val_{\mathcal{M}^b,v^b}(R(x_1,\ldots,x_k)) = e_j$, for some $j < i$. Thus, $d_i val_{\mathcal{M}^b,v^b}(R(x_1,\ldots,x_k)) = d_i e_j = e_0 < e_{n-1}$.

If $\neg D_i(E_j) \in b$, then $i \leq j$, since otherwise b would be closed. Thus, $val_{\mathcal{M}^b,v^b}(\neg D_i(E_j)) = -e_{n-1} = e_0 < e_{n-1}$. Assume $\neg D_i R(x_1,\ldots,x_k) \in b$, for some predicate symbol R and for some individual variables $x_1, \ldots, x_k$. Then, by the closed branch property, for all $j \leq i$, $D_j R(x_1, \ldots, x_k) \not\in b$. Thus, by the definition of $m^b(R)$, $val_{\mathcal{M}^b,v^b}(R(x_1, \ldots, x_k)) = e_j$, for some $j \geq i$. Hence, $val_{\mathcal{M}^b,v^b}(\neg D_i R(x_1, \ldots, x_k)) = -d_i e_j = -e_{n-1} = e_0 < e_{n-1}$.

Now, we prove that 1. and 2. hold for negations of atomic P_n-formulas. Let φ be an atomic P_n-formula.

Observe that $val_{\mathcal{M}^b,v^b}(D_i(\neg\varphi)) = -d_1 val_{\mathcal{M}^b,v^b}(\varphi) = val_{\mathcal{M}^b,v^b}(\neg D_1\varphi)$, due to axiom (P7). Assume that $D_i(\neg\varphi) \in b$. Then, by the completion condition Cpl($\neg$), $\neg D_1\varphi \in b$. By the induction hypothesis, $val_{\mathcal{M}^b,v^b}(\neg D_1\varphi) < e_{n-1}$, therefore $val_{\mathcal{M}^b,v^b}(D_i(\neg\varphi)) < e_{n-1}$. The statement 2. can be proved in a similar way.

Now, we prove that 1. and 2. hold for compound P_n-formulas. Assume that 1. and 2. hold for formulas ψ, ψ_1, ψ_2, and their negations.

Let $\varphi = D_j\psi$. Assume $D_i D_j \psi \in b$. Then, by the completion condition Cpl(ij), $D_j\psi \in b$. By the induction hypothesis, $val_{\mathcal{M}^b,v^b}(D_j\psi) < e_{n-1}$. By axiom (P8) of Post algebras, $val_{\mathcal{M}^b,v^b}(D_i D_j\psi) = val_{\mathcal{M}^b,v^b}(D_j\psi) < e_{n-1}$.

Let $\varphi = \psi_1 \wedge \psi_2$. Assume $D_i(\psi_1 \wedge \psi_2) \in b$. Then, by the completion condition Cpl($\wedge$), either $D_i\psi_1 \in b$ or $D_i\psi_2 \in b$. By the induction hypothesis, $val_{\mathcal{M}^b,v^b}(D_i(\psi_1)) < e_{n-1}$ or $val_{\mathcal{M}^b,v^b}(D_i\psi_2) < e_{n-1}$. Suppose that $val_{\mathcal{M}^b,v^b}(D_i(\psi_1 \wedge \psi_2)) = e_{n-1}$. Then, $val_{\mathcal{M}^b,v^b}(D_i\psi_1) = e_{n-1}$ and $val_{\mathcal{M}^b,v^b}(D_i\psi_2) = e_{n-1}$, a contradiction.

Let $\varphi = \psi_1 \rightarrow \psi_2$. Assume $D_i(\psi_1 \rightarrow \psi_2) \in b$. Then, by the completion condition Cpl($\rightarrow$), there exists $k \in \{1, \ldots, i\}$ such that both $\neg D_k\psi_1 \in b$ and $D_k\psi_2 \in b$. By the induction hypothesis, $-d_k val_{\mathcal{M}^b,v^b}(\psi_1) < e_{n-1}$ and $d_k val_{\mathcal{M}^b,v^b}(\psi_2) < e_{n-1}$. Thus, $-d_k val_{\mathcal{M}^b,v^b}(\psi_1) \vee d_k val_{\mathcal{M}^b,v^b}(\psi_2) < e_{n-1}$. Suppose that $val_{\mathcal{M}^b,v^b}(D_i(\psi_1 \rightarrow \psi_2)) = e_{n-1}$. Note that the following holds: $val_{\mathcal{M}^b,v^b}(D_i(\psi_1 \rightarrow \psi_2)) = \bigwedge_{k=1}^{i}(d_k val_{\mathcal{M}^b,v^b}(\psi_1) \rightarrow d_k val_{\mathcal{M}^b,v^b}(\psi_2)) = e_{n-1}$ iff for every $k \in \{1, \ldots, i\}$, $d_k val_{\mathcal{M}^b,v^b}(\psi_1) \rightarrow d_k val_{\mathcal{M}^b,v^b}(\psi_2) = e_{n-1}$. Hence, by Proposition 22.2.1(5.), $-d_k val_{\mathcal{M}^b,v^b}(\psi_1) \vee d_k val_{\mathcal{M}^b,v^b}(\psi_2) = e_{n-1}$, a contradiction.

Let $\varphi = \forall x\psi(x)$. Assume $D_i(\forall x\psi(x)) \in b$. Then, by the completion condition Cpl($\forall$), for some individual variable z, $D_i\psi(z) \in b$. Thus, by the induction hypothesis, $d_i val_{\mathcal{M}^b,v^b}(\psi(z)) < e_{n-1}$. Suppose $val_{\mathcal{M}^b,v^b}(D_i(\forall x\psi(x))) = e_{n-1}$. Then, by the axiom (P9) and Proposition 22.2.2, for every individual variable z, $d_i val_{\mathcal{M}^b,v^b}(\psi(z)) = e_{n-1}$, a contradiction.

The proofs of the remaining cases are similar. □

Theorem 22.4.1 (Soundness and Completeness of P_n). *For every P_n-formula φ and for every $s \in \{1, \ldots, n-1\}$, the following conditions are equivalent:*

1. *φ is e_s-valid;*
2. *$D_s\varphi$ is P_n-provable.*

Proof. Assume φ is e_s-valid. Thus, by Proposition 22.3.1, $D_s\varphi$ is P_n-valid. Then, it must exist a closed P_n-decomposition tree for $D_s\varphi$, since otherwise there is a tree with an open branch, and then, by Proposition 22.4.5(1.), $D_s\varphi$ would not be P_n-valid. Hence, $D_s\varphi$ is P_n-provable. Now, assume that $D_s\varphi$ is P_n-provable, that is there exists a closed P_n-decomposition tree for $D_s\varphi$. Then, Proposition 22.4.3, enables us to prove that $D_s\varphi$ is P_n-valid. Thus, by Proposition 22.3.1, φ is e_s-valid. □

Observe that the reduct of P_n-dual tableau consisting of the decomposition rules for propositional operations is a decision procedure for the propositional Post logic.

Example. Consider the following P_n-formula:

$$\chi = (\varphi \to \psi) \vee (\psi \to \varphi).$$

In order to prove P_n-validity of χ, we show that $D_{n-1}\chi$ is P_n-provable. Figure 22.1 presents its P_n-proof. In this tree all branches end with the sets of the following form: $H(i,j) = \{\neg D_i\varphi, D_i\psi, \neg D_j\psi, D_j\varphi\}$, for $i, j \in \{1,\ldots,n-1\}$. It is easy to prove that $H(i,j)$ is a P_n-axiomatic set, for all $i, j \in \{1,\ldots,n-1\}$. Indeed, if $i \le j$, then $\{D_i\psi, \neg D_j\psi\}$ is axiomatic, and if $i > j$, then $\{D_j\varphi, \neg D_i\varphi\}$ is axiomatic.

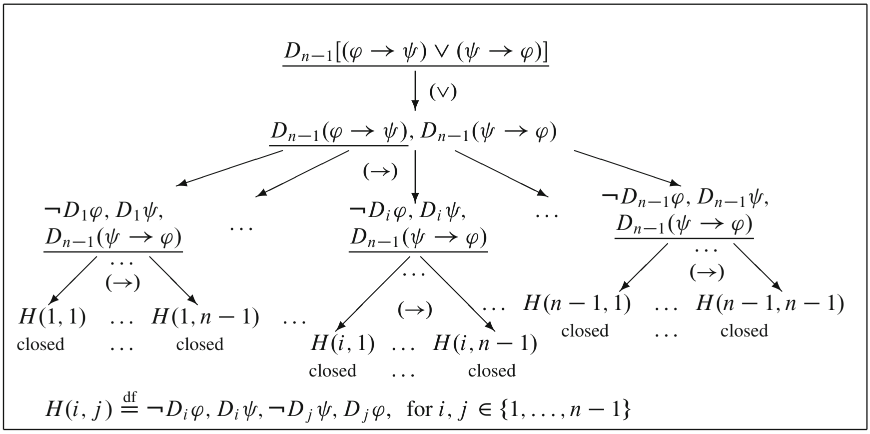

Fig. 22.1 A P_n-proof of the formula $(\varphi \to \psi) \vee (\psi \to \varphi)$