Chapter 11
Dual Tableaux for Information Logics of Plain Frames

11.1 Introduction

Information logics considered here and in the following chapter originated in connection with representation and analysis of data structures known as information systems with incomplete information, introduced in [Lip76], see also [Lip79]. Any such system consists of a collection of objects described in terms of their properties. A property is specified as a pair 'an attribute, a subset of values of this attribute'. Such a form of properties is a manifestation of incompleteness of information. Instead of a single value of an attribute assigned to an object, as is the case in relational database model, here we have a range of values. A disjunctive interpretation of a set of values admits an interpretation that a value is not sufficiently specified, it is only estimated to be in some range. Clearly, also a conjunctive interpretation may be meaningful for some objects. A characterization of objects in terms of attributes and their values induces some relationships among the objects. Typically, they have a form of binary relations and are referred to as information relations derived from an information system. Several families of information relations have been studied in the literature, an extensive catalogue can be found in [DO02]. There are two major classes of these relations: indistinguishability relations and distinguishability relations. The indistinguishability relations reflect degrees of similarity or sameness of objects and distinguishability relations correspond to degrees of dissimilarity or distinctness. Each of these relations is defined in terms of a subset of the attributes of the objects. Thus a distinguishing feature of the relations is that they indicate both which objects are related and also with respect to which of their attributes they are related. In this way the information relations capture a qualitative degree of having a property. Relations of that kind are referred to as relative relations.

Information logics are modal logics where the modal operators are determined by information relations. Depending on a type of the relation, the operators receive various specific, application oriented interpretations. For example, if we consider an indiscernibility relation determined by an attribute i.e., it holds between two objects whenever their sets of values of this attribute are equal, then the necessity

E. Orłowska and J. Golińska-Pilarek, *Dual Tableaux: Foundations, Methodology, Case Studies*, Trends in Logic 33, DOI 10.1007/978-94-007-0005-5_11,
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and possibility operators determined by that relation are the lower and the upper approximation operations, respectively, as considered in rough set theory (see [Paw82, Paw91, DO02]).

In this chapter, first, we recall the fundamental notions concerning information systems with incomplete information, information relations, and operators determined by these relations. Next, we develop relational dual tableaux for some typical information logics with modal operators determined by the information relations both from the group of indistinguishability relations and from the group of distinguishability relations. The models of the logics considered in this chapter are based on what are called plain frames where each of the information relations is determined by the whole set of attributes of an information system. In the following chapter we deal with logics of relative frames, consisting of the families of relations determined by all the finite subsets of the set of attributes of an information system.

11.2 Information Systems

In a formal model of an information system with incomplete information, as introduced by Lipski, information systems are collections of information items that have the form of descriptions of some objects in terms of their properties. An *information system* is a structure of the form $S = (\mathbb{OB}, \mathbb{AT}, (\mathbb{VAL}_a)_{a\in\mathbb{AT}}, f)$ where:

- $\mathbb{OB}$ is a non-empty set of objects;
- $\mathbb{AT}$ is a non-empty set of attributes;
- $\mathbb{VAL}_a$ is a non-empty set of values of the attribute a;
- f is a total function $\mathbb{OB}\times\mathbb{AT} \to \bigcup_{a\in\mathbb{AT}} \mathcal{P}(\mathbb{VAL}_a)$ such that for every $(x,a) \in \mathbb{OB} \times \mathbb{AT}$, $f(x,a) \subseteq \mathbb{VAL}_a$; f is referred to as *an information function*.

Usually, instead of $(\mathbb{OB}, \mathbb{AT}, (\mathbb{VAL}_a)_{a\in\mathbb{AT}}, f)$ the more concise notation, namely $(\mathbb{OB}, \mathbb{AT})$, is used. With that short notation, each attribute $a \in \mathbb{AT}$ is considered as a mapping $a\colon \mathbb{OB} \to \mathcal{P}(\mathbb{VAL}_a)$ that assigns subsets of values to objects. An information system $(\mathbb{OB}, \mathbb{AT})$ is *total* (resp. *deterministic*) whenever for every $a \in \mathbb{AT}$ and for every $x \in \mathbb{OB}$, $f(x,a) \neq \emptyset$ (resp. $card(f(x,a)) \leq 1$, in that case x is said to be a *deterministic object*). If an information system is not deterministic, then it is said to be *nondeterministic*. In nondeterministic information systems descriptions of objects are tuples consisting of subsets of values of attributes. Such a representation is also used in symbolic data analysis, see e.g., [Did87, Did88, Pre97], and in rough set-based data analysis, see e.g., [Orł97a, WDB98, WDG00].

Any set $a(x)$ can be viewed as a set of properties of an object x determined by attribute a. Any such property is referred to as *a-property*. Set $\mathbb{VAL}_a \setminus a(x)$ will be referred to as a set of *negative a-properties*. For example, if attribute a is 'colour' and $a(x) =$ green, then x possesses the property of 'being green'; if a is 'languages spoken', and if a person x speaks Polish (Pl), German (D), and French (F), then $a(x) = \{$Pl, D, F$\}$. If both the set of objects and the set of attributes are finite, then we regard such a system as a data table with rows labeled by objects, and columns labeled by attributes; the entry (x, a) contains the value set $a(x)$ of attribute a for object x.

Any information system $S = (\mathbb{OB}, \mathbb{AT})$ contains also some implicit information about relationships among its objects. These relationships are determined by the properties of objects. Usually, they have the form of binary relations. They are referred to as *information relations derived from an information system*. There are two groups of information relations. The relations that reflect various kinds of 'sameness' or 'similarity' of objects are referred to as *indistinguishability relations*. The relations that indicate 'differences' or 'dissimilarity' of objects are referred to as *distinguishability relations*. Below we present a list of the classes of atomic relations that generate a whole family of information relations.

Indistinguishability Relations

Let $S = (\mathbb{OB}, \mathbb{AT})$ be an information system. For every $A \subseteq \mathbb{AT}$ and for all $x, y \in \mathbb{OB}$ we consider the following binary indistinguishability relations on $\mathbb{OB}$:

- The *strong* (*weak*) *indiscernibility relation* $ind(A)$ (resp. $wind(A)$): $(x, y) \in ind(A)$ (resp. $(x, y) \in wind(A)$) iff for all (resp. for some) $a \in A$, $a(x) = a(y)$;
- The *strong* (*weak*) *similarity relation* $sim(A)$ (resp. $wsim(A)$): $(x, y) \in sim(A)$ (resp. $(x, y) \in wsim(A)$) iff for all (resp. for some) $a \in A$, $a(x) \cap a(y) \neq \emptyset$;
- The *strong* (*weak*) *forward inclusion relation* $fin(A)$ (resp. $wfin(A)$): $(x, y) \in fin(A)$ (resp. $(x, y) \in wfin(A)$) iff for all (resp. for some) $a \in A$, $a(x) \subseteq a(y)$;
- The *strong* (*weak*) *backward inclusion relation* $bin(A)$ (resp. $wbin(A)$): $(x, y) \in bin(A)$ (resp. $(x, y) \in wbin(A)$) iff for all (resp. for some) $a \in A$, $a(y) \subseteq a(x)$;
- The *strong* (*weak*) *negative similarity relation* $nim(A)$ (resp. $wnim(A)$): $(x, y) \in nim(A)$ (resp. $(x, y) \in wnim(A)$) iff for all (resp. for some) $a \in A$, $-a(x) \cap -a(y) \neq \emptyset$, where $-$ is the complement with respect to $\mathbb{VAL}_a$;
- The *strong* (*weak*) *incomplementarity relation* $icom(A)$ (resp. $wicom(A)$): $(x, y) \in icom(A)$ (resp. $(x, y) \in wicom(A)$) iff for all (resp. for some) $a \in A$, $a(x) \neq -a(y)$.

If A is a singleton set, then the respective strong and weak relations coincide. Intuitively, two objects are strongly A-indiscernible whenever all of their sets of a-properties determined by the attributes $a \in A$ are the same. Objects are weakly A-indiscernible whenever their properties determined by some members of A are the same. Objects are strongly A-similar (resp. weakly A-similar) whenever all (resp. some) of the sets of their properties determined by the attributes from A are not disjoint, in other words the objects share some properties. Strong (resp. weak) information inclusions hold between the objects whenever their all (resp. some) corresponding sets of properties are included in each other. Strong (resp. weak) negative similarity relation holds between objects whenever they share some negative properties with respect to all (resp. some) attributes. Strong (resp. weak) incomplementarity relation holds between objects whenever a-properties of one object do not coincide with negative a-properties of the other one, for all (resp. some) attributes.

Important applications of the information relations from the indiscernibility group are related to the representation of approximations of subsets of objects in information systems. If $R(A)$ is one of these relations, where A is a subset of $\mathbb{AT}$ and X is a subset of $\mathbb{OB}$, then the *lower* $R(A)$*-approximation of* X, $L_{R(A)}(X)$, and the *upper* $R(A)$*-approximation of* X, $U_{R(A)}(X)$, are defined as follows:

$$L_{R(A)}(X) = \{x \in \mathbb{OB} : \text{ for all } y \in \mathbb{OB}, (x, y) \in R(A) \text{ implies } y \in X\};$$
$$U_{R(A)}(X) = \{x \in \mathbb{OB} : \text{ there exists } y \in \mathbb{OB}, (x, y) \in R(A) \text{ and } y \in X\}.$$

In the rough set theory (see [Paw91]), where a relation $R(A)$ is a strong indiscernibility relation, we obtain the following hierarchy of definability of sets. A subset X of $\mathbb{OB}$ is said to be:

- *A-definable* iff $L_{ind(A)}(X) = X = U_{ind(A)}(X)$;
- *Roughly A-definable* iff $L_{ind(A)}(X) \neq \emptyset$ and $U_{ind(A)}(X) \neq \mathbb{OB}$;
- *Internally A-indefinable* iff $L_{ind(A)}(X) = \emptyset$;
- *Externally A-indefinable* iff $U_{ind(A)}X = \mathbb{OB}$;
- *Totally A-indefinable* iff it is internally and externally A-indefinable.

The other application of the above information relations is related to modeling uncertain knowledge acquired from information about objects collected in an information system. Let X be a subset of $\mathbb{OB}$. The sets of *A-positive* ($POS_A(X)$), *A-borderline* ($BOR_A(X)$), and *A-negative* ($NEG_A(X)$) *instances of* X are as follows:

$$POS_A(X) = L_{ind(A)}(X);$$
$$BOR_A(X) = U_{ind(A)}(X) - L_{ind(A)}(X);$$
$$NEG_A(X) = \mathbb{OB} - U_{ind(A)}(X).$$

The elements of $POS_A(X)$ can be seen as the members of X up to properties from A. The elements of $NEG_A(X)$ are not the members of X up to properties from A.

Knowledge about a set X of objects that can be discovered from an information system can be modelled as $K_A(X) = POS_A(X) \cup NEG_A(X)$. Intuitively, A-knowledge about X consists of those objects that are either A-positive instances of X or A-negative instances of X.

We say that A-knowledge about X is:

- *Complete* if $K_A(X) = \mathbb{OB}$, otherwise *incomplete*;
- *Rough* if $POS_A(X)$, $BOR_A(X)$, and $NEG_A(X)$ are non-empty;
- *Pos-empty* if $POS_A(X) = \emptyset$;
- *Neg-empty* if $NEG_A(X) = \emptyset$;
- *Empty* if it is pos-empty and neg-empty.

Distinguishability Relations

Let $S = (\mathbb{OB}, \mathbb{AT})$ be an information system. For every $A \subseteq \mathbb{AT}$ and for all $x, y \in \mathbb{OB}$ we consider the following binary distinguishability relations on $\mathbb{OB}$:

- The *strong* (*weak*) *diversity relation* $div(A)$ (resp. $wdiv(A)$): $(x, y) \in div(A)$ (resp. $(x, y) \in wdiv(A)$) iff for all (resp. for some) $a \in A$, $a(x) \neq a(y)$;
- The *strong* (*weak*) *right orthogonality relation* $rort(A)$ (resp. $wrort(A)$): $(x, y) \in rort(A)$ (resp. $(x, y) \in wrort(A)$) iff for all (resp. for some) $a \in A$, $a(x) \subseteq -a(y)$;
- The *strong* (*weak*) *left orthogonality relation* $lort(A)$ (resp. $wlort(A)$): $(x, y) \in lort(A)$ (resp. $(x, y) \in wlort(A)$) iff for all (resp. for some) $a \in A$, $-a(x) \subseteq a(y)$;
- The *strong* (*weak*) *right negative similarity relation* $rnim(A)$ (respectively $wrnim(A)$): $(x, y) \in rnim(A)$ (resp. $(x, y) \in wrnim(A)$) iff for all (resp. for some) $a \in A, a(x) \cap -a(y) \neq \emptyset$;
- The *strong* (*weak*) *left negative similarity relation* $lnim(A)$ (respectively $wlnim(A)$): $(x, y) \in lnim(A)$ (resp. $(x, y) \in wlnim(A)$) iff for all (resp. for some) $a \in A, -a(x) \cap a(y) \neq \emptyset$;
- The *strong* (*weak*) *complementarity relation* $com(A)$ (resp. $wcom(A)$): $(x, y) \in com(A)$ (resp. $(x, y) \in wcom(A)$) iff for all (resp. for some) $a \in A$, $a(x) = -a(y)$.

Intuitively, objects are strongly A-diverse (resp. weakly A-diverse) if all (resp. some) of the sets of their properties determined by members of A are different. The objects are strongly A-right orthogonal (resp. weakly A-right orthogonal) whenever all (resp. some) of the sets of their properties determined by attributes from A are disjoint. The objects are strongly A-left orthogonal (resp. weakly A-left orthogonal) whenever all (resp. some) of their a-properties, for $a \in A$, are exhaustive i.e., $a(x) \cup a(y) = \mathbb{VAL}_a$. Two objects are right or left strongly (resp. weakly) A-negatively similar whenever some properties of one of them are not the properties of the other, for all (resp. some) attributes from A. The objects are strongly (resp. weakly) A-complementary whenever their respective sets of properties are complements of each other, for all (resp. some) attributes from A.

Distinguishability relations can be applied to a non-numerical modelling of degrees of dissimilarity. Diversity relations are applied, among others, in the algorithms for finding cores of sets of attributes. Let an information system $(\mathbb{OB}, \mathbb{AT})$ be given and let A be a subset of $\mathbb{AT}$. We say that an attribute $a \in A$ is *indispensable* in A whenever $ind(A) \neq ind(A - \{a\})$, that is there are some objects such that a is the only attribute from A that can distinguish between them. A *reduct* of A is a minimal subset A' of A such that every $a \in A'$ is indispensable in A' and $ind(A') = ind(A)$. The *core* of A is defined as $CORE(A) = \bigcap\{A' \subseteq \mathbb{AT} : A' \text{ is a reduct of } A\}$. For any pair x, y of objects we define the *discernibility set* $D_{xy} = \{a \in \mathbb{AT} : (x, y) \in div(\{a\})\}$. It is proved in [SR92] that $CORE(A) = \{a \in A : \text{ there are } x, y \in \mathbb{OB} \text{ such that } D_{xy} = \{a\}\}$.

In the proposition below some of the properties satisfied by information relations derived from an information system are listed. We recall that a binary relation R on a set U is:

- A weakly reflexive relation whenever for all $x, y \in U$, $(x, y) \in R$ implies $(x, x) \in R$;
- A tolerance relation whenever it is reflexive and symmetric;
- A 3-transitive relation whenever for all $x, y, z, t \in U$, if $(x, z) \in R$, $(z, t) \in R$, and $(t, y) \in R$, then $(x, y) \in R$.

For any property α of R by property co-α of R we mean that $-R$ has the property α.

Proposition 11.2.1. *For every information system $S = (\mathbb{OB}, \mathbb{AT})$, for every $A \subseteq \mathbb{AT}$, the following hold:*

1. *$ind(A)$ is an equivalence relation;*
2. *$sim(A)$ and $nim(A)$ are weakly reflexive and symmetric;*
3. *if S is total, then $sim(A)$ is a tolerance relation;*
4. *$fin(A)$ and $bin(A)$ are reflexive and transitive;*
5. *$icom(A)$ is symmetric and if $A \neq \emptyset$, then $icom(A)$ is reflexive; for every $a \in \mathbb{AT}$, $icom(a)$ is co-3-transitive;*
6. *$wind(A)$ is a tolerance relation and for every $a \in \mathbb{AT}$, $wind(a)$ is transitive;*
7. *$wsim(A)$ is a tolerance relation;*
8. *$wnim(A)$ is weakly reflexive and symmetric;*
9. *$wicom(A)$ is reflexive, symmetric and co-3-transitive;*
10. *$wfin(A)$ and $wbin(A)$ are reflexive; for every $a \in \mathbb{AT}$, $wfin(a)$ and $wbin(a)$ are transitive;*
11. *$div(A)$ is symmetric; if $A \neq \emptyset$, then $div(A)$ is irreflexive; for every $a \in \mathbb{AT}$, $div(a)$ is co-transitive;*
12. *$rort(A)$ is symmetric; if $A \neq \emptyset$, then $rort(A)$ is irreflexive;*
13. *$lort(A)$ is co-weakly reflexive and symmetric;*
14. *$com(A)$ is symmetric and 3-transitive; if $A \neq \emptyset$, then $com(A)$ is irreflexive;*
15. *$rnim(A)$ and $lnim(A)$ are irreflexive, for every $A \neq \emptyset$; for every $a \in \mathbb{AT}$, $rnim(a)$ and $lnim(a)$ are co-transitive;*
16. *$wdiv(A)$ is irreflexive, symmetric and co-transitive;*
17. *$wrort(A)$ is symmetric; if S is total, then $wrort(A)$ is irreflexive;*
18. *$wlort(A)$ is co-weakly reflexive and symmetric;*
19. *$wcom(A)$ is irreflexive and symmetric; for every $a \in \mathbb{AT}$, $wcom(a)$ is 3-transitive;*
20. *$wrnim(A)$ and $wlnim(A)$ are irreflexive and co-transitive.*

The following proposition states some relationships between information relations of different kinds:

Proposition 11.2.2. *For every information system $S = (\mathbb{OB}, \mathbb{AT})$, for every $A \subseteq \mathbb{AT}$, and for all $x, y, z \in \mathbb{OB}$, the following hold:*

1. *$(x, y) \in sim(A)$ and $(x, z) \in fin(A)$ imply $(z, y) \in sim(A)$;*

2. $(x, y) \in ind(A)$ *implies* $(x, y) \in fin(A)$*;*
3. $(x, y) \in fin(A)$ *and* $(y, x) \in fin(A)$ *imply* $(x, y) \in ind(A)$.

Observe that definitions of information relations include both an information on which objects are related and with respect to which attributes they are related. Relations of that kind are referred to as *relative relations*, they are relative to a subset of attributes. It follows that the formal systems for reasoning about these relations should refer to the structures with relative relations. For that purpose we define a class of *relative frames* which have the form:

$$(U, (R^1_P)_{P \in \mathcal{P}_{fin}(\mathrm{Par})}, \ldots, (R^n_P)_{P \in \mathcal{P}_{fin}(\mathrm{Par})}),$$

where the relations in each family $(R^i_P)_{P \in \mathcal{P}_{fin}(\mathrm{Par})}$ are indexed with finite subsets of a non-empty set Par of parameters. Intuitively, the elements of Par are representations of the attributes of an information system. A *plain frame* is the frame where all the relations are understood as being determined by the whole set of attributes of an information system.

A modal approach to reasoning about incomplete information resulted in various modal systems which are now called information logics. The first logics of that family are defined in [Orł82] published later as [Orł83] and in [Orł84, OP84, Orł85a]. We refer the reader to [DO02] for a comprehensive survey of information logics and to [DG00a, DG00b, DS02b, Bal02, DO07] for some more recent developments. In information logics the elements of the universes of the models are thought of as objects in an information system. This interpretation is quite different from the usual interpretation postulated in modal logics, where the elements of a model represent states (or possible worlds) in which formulas may be true or false.

11.3 Information Logics NIL and IL

The languages of most popular information logics with semantics of plain frames are multimodal languages whose symbols are included in the following pairwise disjoint sets:

- $\mathbb{V}$ – a set of propositional variables possibly including also a propositional constant D interpreted as a set of deterministic objects;
- $\{\leq, \geq, \sigma, \equiv\}$ – a set of relational constants, where $\leq, \geq, \sigma, \equiv$ are the abstract counterparts to the relations of inclusions, similarity, and indiscernibility derived from an information system, respectively;
- $\{\neg, \vee, \wedge, [\leq], [\geq], [\sigma], [\equiv]\}$ – a set of propositional operations.

The set of formulas of a given logic L based on such a language is defined as usual in modal logics (see Sect. 7.3). An L-*frame* is a modal frame of the form $\mathcal{F} = (U, Rel)$, where $Rel \subseteq \{\leq, \geq, \sigma, \equiv\}$. As usual, relations in a frame are denoted with the same symbols as the corresponding relational constants in the language. The relations from Rel are referred to as the accessibility relations. In various information logics the frames satisfy some postulates. Typical conditions on relations in the frames of logics associated with information systems with incomplete information are among the following:

(I1) $\leq = \geq^{-1}$;
(I2) $\leq$ is reflexive and transitive;
(I3) σ is weakly reflexive and symmetric;
(I4) σ is reflexive and symmetric;
(I5) $\equiv$ is an equivalence relation;

For all $x, x', y, y' \in U$,

(I6) If $(x, y) \in \sigma$, $(x, x') \in \leq$ and $(y, y') \in \leq$, then $(x', y') \in \sigma$;
(I7) If $(x, y) \in \sigma$ and $(x, z) \in \leq$, then $(z, y) \in \sigma$;
(I8) If $y \in D$ and $(x, y) \in \leq$, then $x \in D$;
(I9) If $x \in D$ and $(x, y) \in \sigma$, then $(x, y) \in \leq$;
(I10) If $(x, y) \in \equiv$, then $(x, y) \in \leq$;
(I11) If $x, y \in D$ and $(x, y) \in \sigma$, then $(x, y) \in \equiv$;
(I12) If $(x, y) \in \leq$ and $(y, x) \in \leq$, then $(x, y) \in \equiv$;
(I13) If $x \notin D$, then there is $y \in U$ such that $(x, y) \notin \leq$.

We consider two information logics with semantics of plain frames: the logic NIL introduced in [OP84, Vak87] and the logic IL introduced in [Vak89].

The set of relational constants of NIL is $\{\leq, \geq, \sigma\}$. $\mathbb{V}$ consists of propositional variables. The set of propositional operations is $\{\neg, \vee, \wedge, [\leq], [\geq], [\sigma]\}$. The set of NIL-formulas is defined as described in Sect. 7.3.

The NIL-*models* are the structures $(U, \leq, \geq, \sigma, m)$ satisfying the conditions of the definition of models from Sect. 7.3 and conditions (I1), (I2), (I4), and (I6), for all $x, y, z \in U$.

The satisfaction of NIL-formulas by states in a model is defined as in Sect. 7.3, in particular, for formulas built with modal operations we have:

For $T \in \{\leq, \geq, \sigma\}$,

$$\mathcal{M}, s \models [T]\varphi \text{ iff for all } s' \in U, (s, s') \in T \text{ implies } \mathcal{M}, s' \models \varphi.$$

In [Dem00] the following theorem is proved:

Theorem 11.3.1.

1. *The logic* NIL *is decidable;*
2. NIL*-satisfiability is* PSPACE*-complete.*

Moreover, in [Vak87] the following is proved:

Theorem 11.3.2 (Informational representability of NIL). *For every* NIL*-model* $(U, \leq, \geq, \sigma, m)$*, there is a total information system* $\mathcal{S}$ *such that the relations of forward inclusion, backward inclusion, and similarity derived from* $\mathcal{S}$ *coincide with* $\leq, \geq$*, and* σ*, respectively.*

The logic IL is intended to be a tool for reasoning about indiscernibility, similarity, and forward inclusion, and about relationships between them. The set of relational constants of IL is $\{\equiv, \leq, \sigma\}$. $\mathbb{V}$ is a countably infinite set including propositional variables and the propositional constant D which is intuitively interpreted as a set of deterministic objects of an information system. The set of propositional operations is $\{\neg, \vee, \wedge, [\equiv], [\leq], [\sigma]\}$. The set of IL-formulas is defined as usual.

The IL-*models* are the structures of the form $(U, \equiv, \leq, \sigma, D, m)$ satisfying the conditions of the definition of models from Sect. 7.3 and such that $m(D) = D$ and the conditions (I2), (I3), (I5), and (I7), ..., (I13) are satisfied. Satisfaction of IL-formulas by states in a model is defined as in Sect. 7.3.

The following theorem can be found in [DO02].

Theorem 11.3.3.

1. *The logic* IL *is decidable;*
2. IL*-satisfiability problem is* PSPACE*-hard.*

11.4 Relational Formalization of Logics NIL and IL

Let L be a logic with semantics of plain frames. The language of relational logics $\mathsf{RL_L}$ appropriate for expressing L-formulas is $\mathsf{RL}(1, 1')$-language with relational constants representing the accessibility relations from L-models and with propositional constants of L which will be interpreted appropriately as relations (see Sect. 7.4). For the sake of simplicity, we denote these relational constants with the same symbols as in L. An $\mathsf{RL_L}$-structure is of the form $(U, \{T : T \in \mathbb{RC}_{\mathsf{RL_L}} \setminus \{1, 1'\}\}, m)$, where (U, m) is an $\mathsf{RL}(1, 1')$-model and T is a binary relation on U such that $m(T) = T$, for every $T \in \mathbb{RC}_{\mathsf{RL_L}} \setminus \{1, 1'\}$. An $\mathsf{RL_L}$-model is an $\mathsf{RL_L}$-structure that satisfies all the constraints posed on the relational constants in the L-models. The translation of modal formulas of information logics into relational terms is defined as in Sect. 7.4.

More precisely, the language of the relational logic $\mathsf{RL_{NIL}}$ appropriate for a relational representation of logic NIL is $\mathsf{RL}(1, 1')$-language with the set of relational constants $\mathbb{RC}_{\mathsf{RL_{NIL}}} = \{\leq, \geq, \sigma, 1, 1'\}$. An $\mathsf{RL_{NIL}}$-structure is of the form $\mathcal{M} = (U, \leq, \geq, \sigma, m)$, where (U, m) is an $\mathsf{RL}(1, 1')$-model and for every $T \in \{\leq, \geq, \sigma\}$, T is a binary relation on U such that $m(T) = T$. An $\mathsf{RL_{NIL}}$-model is an $\mathsf{RL_{NIL}}$-structure such that the relations $\leq$, $\geq$, and σ satisfy the conditions (I1), (12), (I4), and (I6).

The language of the relational logic $\mathsf{RL_{IL}}$ appropriate for a relational representation of logic IL is $\mathsf{RL}(1, 1')$-language with $\mathbb{RC}_{\mathsf{RL_{IL}}} = \{\leq, \equiv, \sigma, D, 1, 1'\}$. An $\mathsf{RL_{IL}}$-structure is of the form $\mathcal{M} = (U, \leq, \equiv, \sigma, D, m)$, where (U, m) is an $\mathsf{RL}(1, 1')$-model and for every $T \in \{\leq, \equiv, \sigma, D\}$, T is a binary relation on U such that $m(T) = T$. An $\mathsf{RL_{IL}}$-model is an $\mathsf{RL_{IL}}$-structure such that the relations $\leq$, $\equiv$, σ, and D satisfy the conditions (I2), (I3), (I5), (I7), (I10), (12), and the following:

(I8') If $(y, z) \in D$ and $(x, y) \in \leq$, then $(x, z) \in D$;
(I9') If $(x, z) \in D$ and $(x, y) \in \sigma$, then $(x, y) \in \leq$;

(I11') If $(x, z), (y, z) \in D$ and $(x, y) \in \sigma$, then $(x, y) \in \equiv$;
(I13') If $(x, z) \notin D$, then there is $y \in U$ such that $(x, y) \notin \leq$.

Observe that according to the convention established in Sect. 7.4 constant D in $\mathsf{RL}_{\mathsf{IL}}$-logic represents a right ideal relation which is a counterpart to the constant D of IL. Note also that in RL_{L}-models we list explicitly all relations corresponding to accessibility relations from L-models and we denote them with the same symbols as the corresponding constants in the language.

The models of RL_{L} such that $1'$ is interpreted as identity are referred to as standard RL_{L}-models.

By Theorem 7.4.1, we get:

Theorem 11.4.1. *Let* L *be a logic with semantics of plain frames. Then for every* L*-formula* φ *and for all object variables* x *and* y*, the following conditions are equivalent:*

1. φ *is* L*-valid;*
2. $x\tau(\varphi)y$ *is* RL_{L}*-valid.*

Dual tableaux for the logics NIL and IL in their relational formalizations are constructed as follows. We add to $\mathsf{RL}(1, 1')$-dual tableau the rules corresponding to the constraints on relations that are assumed in the models of these logics. In the following list the rule (rI#) corresponds to the condition (I#), for $\# \in \{1, 6, 7, 8', 9', 10, 11', 12, 13'\}$:

For all object variables x and y,

(wref σ) $$\frac{x\sigma x}{x\sigma z, x\sigma x}$$ z is any object variable

(rI1 $\subseteq$) $$\frac{x \leq y}{y \geq x, x \leq y}$$ (rI1 $\supseteq$) $$\frac{x \geq y}{y \leq x, x \geq y}$$

(rI6) $$\frac{x\sigma y}{z\sigma t, x\sigma y \mid z \leq x, x\sigma y \mid t \leq y, x\sigma y}$$ z, t are any object variables

(rI7) $$\frac{x\sigma y}{z\sigma y, x\sigma y \mid z \leq x, x\sigma y}$$ z is any object variable

(rI8') $$\frac{xDy}{zDy, xDy \mid x \leq z, xDy}$$ z is any object variable

(rI9') $$\frac{x \leq y}{xDz, x \leq y \mid x\sigma z, x \leq y}$$ z is any object variable

(rI10) $$\frac{x \leq y}{x \equiv y, x \leq y}$$ (rI12) $$\frac{x \equiv y}{x \leq y, x \equiv y \mid y \leq x, x \equiv y}$$

(rI11') $\dfrac{x \equiv y}{xDz, x \equiv y \mid yDz, x \equiv y \mid x\sigma y, x \equiv y}$ z is any object variable

(rI13') $\dfrac{xDy}{x \leq z, xDy}$ z is a new object variable

The specific NIL-rules are (ref $\leq$), (tran $\leq$), (ref σ), (sym σ), which are the instances of the corresponding rules presented in Sect. 6.6 (see also Sect. 7.4), and in addition (rI1$\subseteq$), (rI1$\supseteq$), and (rI6).

The specific IL-rules are (ref $\leq$), (tran $\leq$), (wref σ), (sym σ), (ref $\equiv$), (sym $\equiv$), (tran $\equiv$), (rI7), (rI8'), (rI9'), (rI10), (rI11'), (rI12), and (rI13').

As in RL-logic, an $\mathsf{RL_L}$-set is a finite set of $\mathsf{RL_L}$-formulas such that the first-order disjunction of its members is true in all $\mathsf{RL_L}$-models. If $\mathcal{K}$ is a class of $\mathsf{RL_L}$-structures, then the notion of a $\mathcal{K}$-set is defined in a similar way. Correctness of a rule is defined as in the logic RL (see Sects. 2.4 and 2.5).

Theorem 11.4.2 (Correspondence). *Let* L *be an information logic satisfying some of the conditions (I1), ..., (I13) and let* $\mathcal{K}$ *be a class of* $\mathsf{RL_L}$*-structures* $\mathcal{M} = (U, Rel, m)$*, for* $Rel \subseteq \{\leq, \geq, \sigma, \equiv\}$*. Then:*

1. *A relation* $R \in Rel$ *is reflexive (resp. weakly reflexive, symmetric, transitive) if and only if the rule (ref R) (resp. (wref R), (sym R), (tran R)) is* $\mathcal{K}$*-correct;*
2. *Every* $\mathsf{RL_L}$*-structure of* $\mathcal{K}$ *satisfies the condition (I#) iff the rule (rI#) is* $\mathcal{K}$*-correct, where* $\# \in \{1, 6, 7, 8', 9', 10, 11', 12, 13'\}$.

Proof. 1. can be proved in a similar way as Theorem 6.6.1. By way of example, we show 2. for the condition (I6).

Assume that every structure of $\mathcal{K}$ satisfies this condition. Then preservation of validity from the upper set to the bottom sets is obvious. Let X be any finite set of $\mathsf{RL_L}$-formulas. Assume $X_1 = X \cup \{z\sigma t, x\sigma y\}$, $X_2 = X \cup \{z \leq x, x\sigma y\}$, and $X_3 = X \cup \{t \leq y, x\sigma y\}$ are $\mathcal{K}$-sets. Suppose $X \cup \{x\sigma y\}$ is not a $\mathcal{K}$-set, that is there exist an $\mathsf{RL_L}$-structure $\mathcal{M}$ and a valuation v in $\mathcal{M}$ such that $\mathcal{M}, v \not\models x\sigma y$. Since X_1, X_2, X_3 are $\mathcal{K}$-sets, the model $\mathcal{M}$ and the valuation v satisfy $(v(z), v(t)) \in \sigma$, $(v(z), v(x)) \in \leq$, and $(v(t), v(y)) \in \leq$. By the condition (I6), $(v(x), v(y)) \in \sigma$, a contradiction.

Now, assume the rule (rI6) is $\mathcal{K}$-correct. Let $X \stackrel{\text{df}}{=} \{z{-}\sigma t, z{-}\leq x, t{-}\leq y\}$. Then $X \cup \{z\sigma t, x\sigma y\}$, $X \cup \{z \leq x, x\sigma y\}$, and $X \cup \{t \leq y, x\sigma y\}$ are $\mathcal{K}$-sets. Thus, by the assumption, $X \cup \{x\sigma y\}$ is a $\mathcal{K}$-set. Therefore, for every $\mathsf{RL_L}$-structure $\mathcal{M}$ in $\mathcal{K}$ and for every valuation v in $\mathcal{M}$, if $\mathcal{M}, v \models z\sigma t$, $\mathcal{M}, v \models z \leq x$, and $\mathcal{M}, v \models t \leq y$, then $\mathcal{M}, v \models x\sigma y$. □

The above proposition implies that all the specific rules of $\mathsf{RL_L}$-dual tableau are $\mathsf{RL_L}$-correct. Correctness of all the remaining rules can be proved as in $\mathsf{RL}(1, 1')$-dual tableau (see Sects. 2.5 and 2.7), thus we get:

Proposition 11.4.1.

1. *The* NIL*-rules are* $\mathsf{RL_{NIL}}$*-correct;*
2. *The* IL*-rules are* $\mathsf{RL_{IL}}$*-correct.*

It is known that conditions (I12) and (I13) are not expressible in the language of logic IL, hence the completeness proof for its Hilbert-style axiomatization requires a special technique referred to as copying (see [Vak89]). As it is shown above, in the case of relational formalization the rules corresponding to (I12) and (I13) can be explicitly given. They enable us to prove constraints (I12) and (I13), respectively, directly from their representation in the language of $\mathsf{RL}_{\mathsf{IL}}$.

The notions of an RL_{L}-proof tree, a closed branch of such a tree, a closed RL_{L}-proof tree, and RL_{L}-provability are defined as in Sect. 2.4. A branch b of an RL_{L}-proof tree is complete whenever it is closed or it satisfies the completion condition of $\mathsf{RL}(1, 1')$-dual tableau adjusted to the language of RL_{L} and the completion conditions corresponding to the rules that are specific for RL_{L}.

The completion conditions determined by the rules (ref R) for $R \in \{\leq, \sigma, \equiv\}$, (sym R) for $R \in \{\sigma, \equiv\}$, and (tran R) for $R \in \{\leq, \equiv\}$ are the instances of the completion conditions presented in Sect. 6.6.

For all object variables x and y,

Cpl(wref σ) If $x\sigma x \in b$, then for every object variable z, $x\sigma z \in b$, obtained by an application of the rule (wref σ);

Cpl(rI1 $\subseteq$) If $x \leq y \in b$, then $y \geq x \in b$, obtained by an application of the rule (rI1 $\subseteq$);

Cpl(rI1 $\supseteq$) If $x \geq y \in b$, then $y \leq x \in b$, obtained by an application of the rule (rI1 $\supseteq$);

Cpl(rI6) If $x\sigma y \in b$, then for all object variables z and t, either $z\sigma t \in b$ or $z \leq x \in b$ or $t \leq y \in b$, obtained by an application of the rule (rI6);

Cpl(rI7) If $x\sigma y \in b$, then for every object variable z, either $z\sigma y \in b$ or $z \leq x \in b$, obtained by an application of the rule (rI7);

Cpl(rI8') If $xDy \in b$, then for every object variable z, either $zDy \in b$ or $x \leq z \in b$, obtained by an application of the rule (rI8');

Cpl(rI9') If $x \leq y \in b$, then for every object variable z, either $xDz \in b$ or $x\sigma z \in b$, obtained by an application of the rule (rI9');

Cpl(rI10) If $x \leq y \in b$, then $x \equiv y \in b$, obtained by an application of the rule (rI10);

Cpl(rI11') If $x \equiv y \in b$, then for every object variable z, either $xDz \in b$, $yDz \in b$ or $x\sigma y \in b$, obtained by an application of the rule (rI11');

Cpl(rI12) If $x \equiv y \in b$, then either $x \leq y \in b$ or $y \leq x \in b$, obtained by an application of the rule (rI12);

Cpl(rI13) If $xDy \in b$, then for some object variable z, $x \leq z \in b$, obtained by an application of the rule (rI13).

Let L be an information logic. The notions of a complete branch of an RL_{L}-proof tree, a complete RL_{L}-proof tree, and an open branch of an RL_{L}-proof tree are defined as in RL-logic (see Sect. 2.5). In order to prove completeness, we need to define a branch model and to show the three theorems of Table 7.1.

All the rules listed above (see p. 226) guarantee that whenever a branch of an RL_{L}-proof tree contains two formulas one of which is built with an atomic term and the other with its complement, then the branch can be closed. Thus, the closed branch property can be proved as the Proposition 2.8.1.

Let b be an open branch of an $\mathsf{RL_L}$-proof tree. The branch structure has the form $\mathcal{M}^b = (U^b, (\#^b)_{\#\in\{\leq,\geq,\sigma,\equiv,D\}}, m^b)$, where $U^b = \mathbb{OV}_{\mathsf{RL_L}}$, for every relational constant R, $m^b(R) = \{(x,y) \in U^b \times U^b : xRy \notin b\}$, $\#^b = m^b(\#)$ for every $\# \in \{\leq,\geq,\sigma,\equiv,D\}$, and m^b extends to all the compound relational terms as in $\mathsf{RL}(1,1')$-models. To prove that branch structures satisfy all the conditions that are assumed in the models of a given logic, we employ the corresponding completion conditions.

Proposition 11.4.2 (Branch Model Property). *Let* L *be an information logic that satisfies some of the conditions among (I1), ..., (I13). The branch structure* $\mathcal{M}^b$ *determined by an open branch b of an* $\mathsf{RL_L}$*-proof tree is an* $\mathsf{RL_L}$*-model.*

Proof. By way of example, we prove that the branch structure $\mathcal{M}^b = (U^b, \leq^b, \equiv^b, \sigma^b, m^b)$ determined by an open branch b of an $\mathsf{RL_{IL}}$-proof tree satisfies the condition (I12). Assume $(x,y) \in \leq^b$ and $(y,x) \in \leq^b$, that is $x \leq y \notin b$ and $y \leq x \notin b$. Suppose $(x,y) \notin \equiv^b$. Then $x \equiv y \in b$. By the completion condition Cpl(rI12), either $x \leq y \in b$ or $y \leq x \in b$, a contradiction. □

Since the branch models are defined in a standard way, that is for any relational constant R, $m^b(R)$ is defined as in the completeness proof of $\mathsf{RL}(1,1')$-dual tableau (see Sects. 2.5 and 2.7), the satisfaction in branch model property can be proved as in $\mathsf{RL}(1,1')$-logic. Thus, we get:

Theorem 11.4.3 (Soundness and Completeness of $\mathsf{RL_{NIL}}$ and $\mathsf{RL_{IL}}$).

1. $\mathsf{RL_{NIL}}$*-dual tableau is sound and complete;*
2. $\mathsf{RL_{IL}}$*-dual tableau is sound and complete.*

Finally, by the above theorem and Theorem 11.4.1, we obtain:

Theorem 11.4.4 (Relational Soundness and Completeness of NIL and IL). *Let* L $\in$ {NIL, IL}*. Then for every* L*-formula* φ *and for all object variables x and y, the following conditions are equivalent:*

1. φ *is* L*-valid;*
2. $x\tau(\varphi)y$ *is* $\mathsf{RL_L}$*-provable.*

Example. Let us consider the NIL-formula φ and IL-formula ψ:

$$\varphi = p \to [\leq]\langle\geq\rangle p,$$

$$\psi = [\leq]p \to [\equiv]p.$$

The formula φ (resp. ψ) is true in a modal frame $(U, \leq, \geq)$ (resp. $(U, \leq, \equiv)$) provided that $\leq^{-1}\subseteq\geq$ (resp. $\equiv\subseteq\leq$).

For the sake of simplicity, let us denote $\tau(p)$ by P. According to the translation presented in Sect. 7.4 (see p. 147), the relational representations of these formulas are:

$$\tau(\varphi) = -P \cup -(\leq \,;-(\geq \,; P)),$$
$$\tau(\psi) = --(\leq \,;-P) \cup -(\equiv \,;-P).$$

Figure 11.1 presents an $\mathsf{RL}_{\mathsf{NIL}}$-proof of the formula $x\tau(\varphi)y$ which shows NIL-validity of φ, and Fig. 11.2 presents an $\mathsf{RL}_{\mathsf{IL}}$-proof of the formula $x\tau(\psi)y$ that shows IL-validity of ψ.

As in standard modal logics, the relational logic $\mathsf{RL}(1, 1')$ can be used for verification of entailment, model checking in finite models, and verification of satisfaction of a given formula by some objects in a finite model (see Sects. 7.6, 7.7, and 7.8).

Let L be a logic with semantics of plain frames. In order to verify entailment we apply the method presented in Sect. 2.11, that is, first, we translate L-formulas

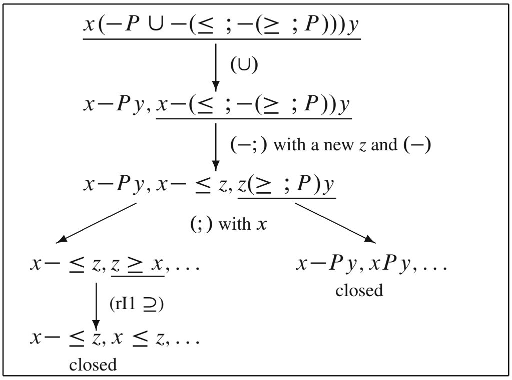

Fig. 11.1 A relational proof of $p \rightarrow [\leq]\langle\geq\rangle p$

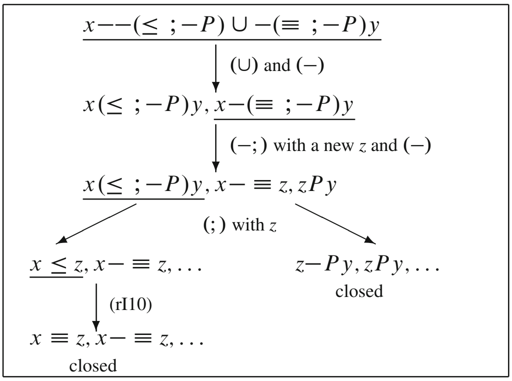

Fig. 11.2 A relational proof of $[\leq]p \rightarrow [\equiv]p$

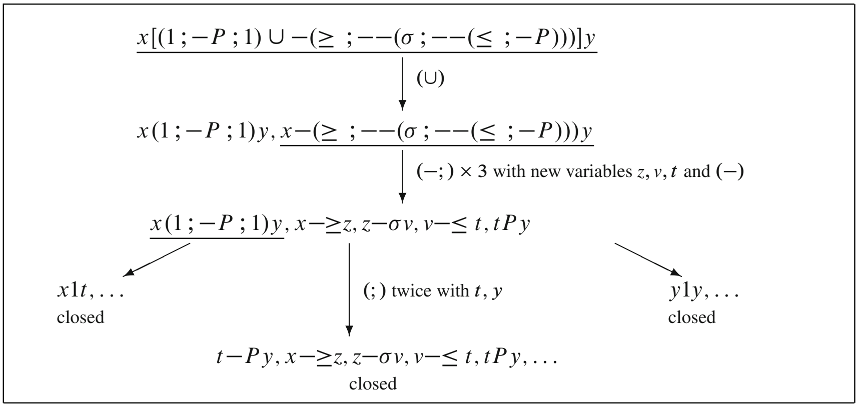

Fig. 11.3 An $\mathsf{RL}_{\mathsf{NIL}}$-proof showing that p entails $[\geq][\sigma][\leq]p$

in question into terms of the relational logic RL_{L}, and then we use the method of verification of entailment for RL_{L}-logic as it is shown in Sect. 2.11.

For example, in NIL-logic the formula p entails $[\geq][\sigma][\leq]p$. For the sake of simplicity, denote $\tau(p)$ by P. Then:

$$\tau([\geq][\sigma][\leq]p) = -(\geq\ ;--(\sigma\,;--(\leq\ ;-P))).$$

We need to show that $P = 1$ implies $-(\geq\ ;--(\sigma\,;--(\leq\ ;-P))) = 1$. According to Proposition 2.2.1 (7.), it suffices to show that the formula:

$$x[(1\,;-P\,;1)\cup-(\geq\ ;--(\sigma\,;--(\leq\ ;-P)))]y$$

is $\mathsf{RL}_{\mathsf{NIL}}$-provable. Figure 11.3 presents an $\mathsf{RL}_{\mathsf{NIL}}$-proof of this formula.

11.5 Information Logic CI and Its Relational Formalization

The logic presented in this section and its dual tableau originated in [DO00a]. The language of the logic CI of complementarity and incomplementarity is a multimodal language with symbols from the following pairwise disjoint sets:

- $\mathbb{V}$ – a countable infinite set of propositional variables;
- $\{R, S\}$ – a set of relational constants;
- $\{\neg, \vee, \wedge, [R], [[S]]\}$ – a set of propositional operations.

A CI-frame is a structure (U, R, S) such that:

- U is a non-empty set;
- R is a symmetric and 3-transitive relation on U;

- S is a reflexive relation on U;
- $R \cup S = U \times U$ and $R \cap S = \emptyset$.

Models based on CI-frames are defined as usual (see Sect. 7.3).

Satisfaction of a formula is defined as in Sect. 7.3, that is the clauses for formulas with modal operations are:

- $\mathcal{M}, s \models [R]\varphi$ iff for every $s' \in U$, if $(s, s') \in R$, then $\mathcal{M}, s' \models \varphi$;
- $\mathcal{M}, s \models [[S]]\varphi$ iff for every $s' \in U$, if $\mathcal{M}, s' \models \varphi$, then $(s, s') \in S$.

Observe that although neither irreflexivity of R nor symmetry of S are assumed explicitly, irreflexivity of R is guaranteed by reflexivity of S, and symmetry of S is guaranteed by symmetry of R, since $R = -S$. Irreflexivity of R is not expressible in a modal language with a single accessibility relation. The relational dual tableau for CI will enable us to prove $1' \subseteq -R$. The proof is presented in Fig. 11.5. Symmetry of S is expressible with formula $p \rightarrow [[S]]\,[[S]]p$. Its proof is presented in Fig. 11.6.

The language of the relational logic corresponding to CI is $\mathsf{RL}(1, 1')$-language endowed with the relational constants R and S. $\mathsf{RL}_{\mathsf{CI}}$-models are structures of the form $\mathcal{M} = (U, R, S, m)$, where (U, m) is an $\mathsf{RL}(1, 1')$-model and R and S are binary relations on U that provide the interpretation of the corresponding relational constants and satisfy the above conditions assumed in CI-frames.

The translation of CI-formulas into relational terms of the logic $\mathsf{RL}_{\mathsf{CI}}$ is defined as in Sect. 7.4, that is the translation of formulas with modal operations is:

- $\tau([R]\varphi) \stackrel{\text{df}}{=} -(R\,;-\tau(\varphi))$;
- $\tau([[S]]\varphi) \stackrel{\text{df}}{=} -(-S\,;\tau(\varphi))$.

By Theorem 7.4.1, we obtain:

Theorem 11.5.1. *For every* CI*-formula* φ *and for all object variables* x *and* y*, the following conditions are equivalent:*

1. φ *is* CI*-valid;*
2. $x\tau(\varphi)y$ *is* $\mathsf{RL}_{\mathsf{CI}}$*-valid.*

$\mathsf{RL}_{\mathsf{CI}}$-dual tableau includes the rules and axiomatic sets of $\mathsf{RL}(1, 1')$-dual tableau adjusted to $\mathsf{RL}_{\mathsf{CI}}$-language, rules (ref S) and (sym R) which are the instances of the rules presented in Sect. 6.6 and, in addition, it contains the specific rules and axiomatic sets of the following forms:

For all object variables x, y, z, and t,

$$(\text{dis } R, S) \quad \frac{}{xRy \mid xSy}$$

$$(\text{3-tran } R) \quad \frac{xRy}{xRz, xRy \mid zRt, xRy \mid tRy, xRy} \quad z, t \text{ are any object variables}$$

The rule (dis R, S) is a specialized cut rule. An alternative deterministic form of such rules is discussed in Sect. 25.9.

Specific $\mathsf{RL}_{\mathsf{CI}}$-axiomatic sets are those that include the subset $\{xRy, xSy\}$, for any object variables x and y.

Proposition 11.5.1.

1. *The* $\mathsf{RL}_{\mathsf{CI}}$*-rules are* $\mathsf{RL}_{\mathsf{CI}}$*-correct;*
2. *The* $\mathsf{RL}_{\mathsf{CI}}$*-axiomatic sets are* $\mathsf{RL}_{\mathsf{CI}}$*-sets.*

Proof. The proof of correctness of the rules (ref S), (sym R), and (3-tran R) follows the proof of Theorem 6.6.1. Now, we show correctness of the rule (dis R, S). Let X be a finite set of $\mathsf{RL}_{\mathsf{CI}}$-formulas. The preservation of validity from the upper set to the bottom sets is obvious. Assume that $X \cup \{xRy\}$ and $X \cup \{xSy\}$ are $\mathsf{RL}_{\mathsf{CI}}$-sets. Suppose X is not $\mathsf{RL}_{\mathsf{CI}}$-set. Then there exist an $\mathsf{RL}_{\mathsf{CI}}$-model $\mathcal{M} = (U, R, S, m)$ and a valuation v in $\mathcal{M}$ such that for every $\varphi \in X$, $\mathcal{M}, v \not\models \varphi$. Thus, by the assumption, $\mathcal{M}, v \models xRy$ and $\mathcal{M}, v \models xSy$, hence $(v(x), v(y)) \in R$ and $(v(x), v(y)) \in S$. Therefore, $R \cap S \neq \emptyset$. However, in all $\mathsf{RL}_{\mathsf{CI}}$-models, R and S are disjoint, a contradiction.

Since in every $\mathsf{RL}_{\mathsf{CI}}$-model $\mathcal{M} = (U, R, S, m)$, $R \cup S = U \times U$, for every valuation v in $\mathcal{M}$, $\mathcal{M}, v \models xRy$ or $\mathcal{M}, v \models xSy$, hence $X \cup \{xRy, xSy\}$ is an $\mathsf{RL}_{\mathsf{CI}}$-set, for every set X of $\mathsf{RL}_{\mathsf{CI}}$-formulas. □

An alternative representation of the constraint $R \cup S = U \times U$ can be provided by a rule in the $\mathsf{RL}_{\mathsf{CI}}$-dual tableau. This issue is discussed in Sect. 25.9, see also Sect. 25.6.

The completion conditions corresponding to the specific $\mathsf{RL}_{\mathsf{CI}}$-rules (dis R, S) and (3-tran R) are:

For all object variables x, y, z, and t,

Cpl(dis R, S) Either $xRy \in b$ or $xSy \in b$;
Cpl(3-tran R) If $xRy \in b$, then for all object variables z and t, either $xRz \in b$ or $zRt \in b$ or $tRy \in b$.

It can be proved that the rules specific for $\mathsf{RL}_{\mathsf{CI}}$-dual tableau do not violate the closed branch property.

The branch structure $\mathcal{M}^b = (U^b, R^b, S^b, m^b)$ determined by an open branch of an $\mathsf{RL}_{\mathsf{CI}}$-proof tree is defined as usual, that is $U^b = \mathbb{OV}_{\mathsf{RL}_{\mathsf{CI}}}$, $m^b(T) = \{(x, y) \in U^b \times U^b : xTy \notin b\}$, $T^b = m^b(T)$, for $T \in \{R, S\}$, and m^b extends to all the compound relational terms as in $\mathsf{RL}(1, 1')$-models.

Proposition 11.5.2 (Branch Model Property). *Let b be an open branch of an* $\mathsf{RL}_{\mathsf{CI}}$*-proof tree. Then the branch structure $\mathcal{M}^b = (U^b, R^b, S^b, m^b)$ is an* $\mathsf{RL}_{\mathsf{CI}}$*-model.*

Proof. We need to show that R^b is a symmetric and 3-transitive relation on U^b, S^b is a reflexive relation on U^b, and $R^b \cup S^b = U^b \times U^b$ and $R^b \cap S^b = \emptyset$. Symmetry of R^b and reflexivity of S^b can be proved in a similar way as those properties of relation $m^b(1')$ in the completeness proof of $\mathsf{RL}(1, 1')$-logic in Sect. 2.7. The proof of 3-transitivity of R^b is analogous to the proof of transitivity of $m^b(1')$. Now, suppose that $R^b \cup S^b \neq U^b \times U^b$, that is there are object variables x and y such that $(x, y) \notin R^b$ and $(x, y) \notin S^b$. Then $xRy \in b$ and $xSy \in b$. Since $\{xRy, xSy\}$

is an axiomatic set and all the rules preserve formulas built with atomic relational terms, b is closed, a contradiction. Suppose that $R^b \cap S^b \neq \emptyset$, that is there are object variables x and y such that $(x, y) \in R^b$ and $(x, y) \in S^b$. Then $xRy \notin b$ and $xSy \notin b$. By the completion condition Cpl(dis R, S), for all object variables x and y, either $xRy \in b$ or $xSy \in b$, a contradiction. □

Since the branch model is defined in a standard way, the satisfaction in branch model property can be proved as in $\mathsf{RL}(1, 1')$-logic. Therefore, we obtain:

Theorem 11.5.2 (Soundness and Completeness of $\mathsf{RL_{CI}}$). *For every $\mathsf{RL_{CI}}$-formula φ, the following conditions are equivalent:*

1. *φ is $\mathsf{RL_{CI}}$-valid;*
2. *φ is true in all standard $\mathsf{RL_{CI}}$-models;*
3. *φ is $\mathsf{RL_{CI}}$-provable.*

By the above theorem and Theorem 11.5.1, we get:

Theorem 11.5.3 (Relational Soundness and Completeness of CI). *For every CI-formula φ and for all object variables x and y, the following conditions are equivalent:*

1. *φ is CI-valid;*
2. *$x\tau(\varphi)y$ is $\mathsf{RL_{CI}}$-provable.*

Example. Consider a CI-formula:

$$\varphi = [[S]]\neg p \rightarrow [R]p.$$

This formula is true in a CI-frame (U, R, S) because $R \subseteq -S$. For the sake of simplicity, let us denote $\tau(p)$ by P. Then the relational translation of the formula φ is:

$$\tau(\varphi) = (-S\,;-P) \cup -(R\,;-P).$$

Figure 11.4 presents an $\mathsf{RL_{CI}}$-proof of the formula $x(-S\,;-P) \cup -(R\,;-P))y$ which shows CI-validity of φ.

Figure 11.5 presents an $\mathsf{RL_{CI}}$-proof of relational formula $x(-1' \cup -R)y$ which according to Proposition 2.2.1(1.) reflects irreflexivity of relation R.

Now, we consider formula

$$\varphi = p \rightarrow [[S]]\,[[S]]p$$

which reflects symmetry of relation S. Its translation into a relational term is:

$$\tau(\varphi) = -P \cup -(-S\,;-(-S\,;P)).$$

Figure 11.6 depicts an $\mathsf{RL_{CI}}$-proof of $x\tau(\varphi)y$.

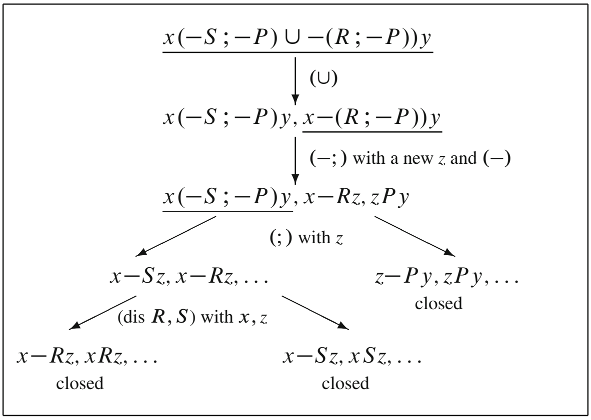

Fig. 11.4 A relational proof of $[\![S]\!]\neg p \rightarrow [R]p$

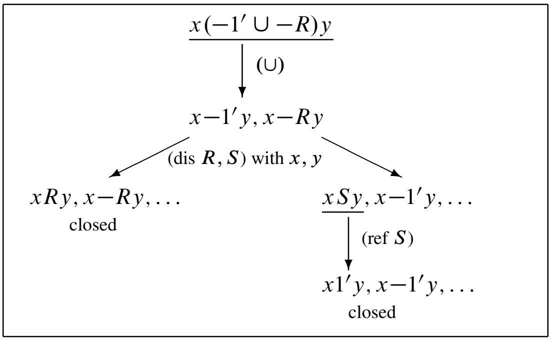

Fig. 11.5 A relational proof of irreflexivity of relation R

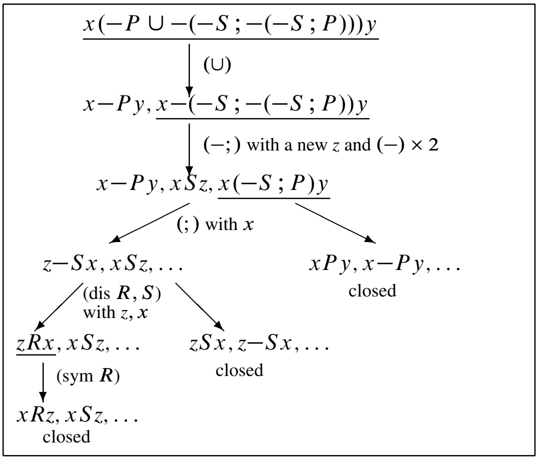

Fig. 11.6 A relational proof of symmetry of relation S