# Chapter 10
# Dual Tableaux for Many-Valued Logics

## 10.1 Introduction

The development of multiple-valued logic in its modern form began with the work of Jan Łukasiewicz [Łuk20] and Emil Post [Pos20, Pos21]. Since the emergence of computer science as an independent discipline, there have been an extensive interplay and mutual inspiration between the two fields. Apart from its logical and philosophical motivation, multiple-valued logic has applications, among others, in hardware design and artificial intelligence. In the field of hardware design classical propositional logic is used as a tool for specification and analysis of electrical switching circuits with two stable voltage levels. A generalization to a finitely-valued logic allows the analogous applications with possibly many stable states. In artificial intelligence multiple-valued logic provides models of vagueness or uncertainty of information and contributes to the development of formal methods simulating commonsense reasoning. Some recent developments and applications of multiple-valued logics can be found e.g., in [Mal93, Got00, Häh01, FO03].

In this chapter we develop a method of designing a dual tableau for an arbitrary finite-valued propositional logic. We follow a relational approach, however, in this case the relational logic appropriate for the translation of formulas of an $n$-valued logic is not $\mathsf{RL}(1, 1')$, a logic of $n$-ary relations is employed, where $n \geq 2$. The relational operations of the logic include $n$ specific unary relational operators such that $k$th operator, $k \leq n$, applied to a relation selects, in a sense, the $k$th components of the $n$-tuples belonging to that relation and represents them as an $n$-ary relation. We apply the method to three multiple-valued logics: Rosser and Turquette logic, $\mathsf{RT}$ [RT52], symmetric Heyting logic of order $n$, $n \geq 2$, $\mathsf{SH}_n$ [Itu83, IO06], and a finite poset-based generalization of Post logic, $\mathsf{L_T}$ [Ras91]. Decidability of the logic $\mathsf{RT}$ follows from the developments in [Got00, Häh03]. Decidability of the logic $\mathsf{SH}_n$ follows from the results presented in [Itu82]. Decidability of logic $\mathsf{L_T}$ is proved in [Nou99]. The present chapter is based on the developments in [KMO98]. Dual tableaux for many-valued modal logics can be found in [KO01].

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## 10.2 Finitely Many-Valued Logics

In this chapter we consider $n_{\mathsf{L}}$-valued, $n_{\mathsf{L}} \geq 2$, propositional logics, $\mathsf{L}$, whose languages are built from the following pairwise disjoint sets of symbols:

- $\mathbb{V}$ – a countable infinite set of propositional variables;
- $\{o_j : 1 \leq j \leq j_{\mathsf{L}}\}$ – a set of propositional operations where $j_{\mathsf{L}} \geq 1$ and the arity of $o_j$ is $a(j) \geq 1$.

The set of $\mathsf{L}$-formulas is generated from the propositional variables with the propositional operations. The standard many-valued semantics for $\mathsf{L}$ is based on a semantic range $\mathcal{SR}_{\mathsf{L}} = \{0, 1, \ldots, s, \ldots, n_{\mathsf{L}} - 1\}$ consisting of $n_{\mathsf{L}}$ logical values indexed by integers, where $0 < s \leq n_{\mathsf{L}} - 1$. We assume that the values $\{s, \ldots, n_{\mathsf{L}} - 1\}$ are designated, and all the other ones undesignated. Though in notation we identify the values with their natural number indices, in general we do not assume any kind of ordering, in particular, any linear ordering corresponding to that of the natural indices is not necessarily assumed in the set of the logical values.

With the family of propositional operations $\{o_1, \ldots, o_{i_{\mathsf{L}}}\}$, we associate a family of semantic functions $\{f_{o_j} : j = 1, \ldots, i_{\mathsf{L}}\}$, where $f_{o_j}$ maps $\mathcal{SR}_{\mathsf{L}}^{a(j)}$ into $\mathcal{SR}_{\mathsf{L}}$.

An $\mathsf{L}$-model is a structure of the form $\mathcal{M} = (U, m, \{m_k : k \in \mathcal{SR}_{\mathsf{L}}\})$, where $U$ is a non-empty set of states, $m\colon \mathbb{V} \times U \to \mathcal{SR}_{\mathsf{L}}$, and $\{m_k : k \in \mathcal{SR}_{\mathsf{L}}\}$ is a family of meaning functions such that:

- $m_k(p) = \{w \in U : m(p, w) = k\}$, for $p \in \mathbb{V}$;
- $m_k(o_j(\varphi_1, \ldots, \varphi_{a(j)})) = \bigcup_{f_{o_j}(k_1,\ldots,k_{a(j)})=k} (m_{k_1}(\varphi_1) \cap \ldots \cap m_{k_{a(j)}}(\varphi_{a(j)}))$.

Intuitively, $m_k(\varphi)$ is the set of states at which formula $\varphi$ takes value $k$.

The standard notions of satisfaction of a formula at a state in a model, truth in a model, and validity are defined as follows. An $\mathsf{L}$-formula $\varphi$ is satisfied in a model $\mathcal{M}$ at a state $w$, written $\mathcal{M}, w \models \varphi$, whenever $w \in m_k(\varphi)$ for some $k \in \{s, \ldots, n_{\mathsf{L}} - 1\}$; $\varphi$ is true in a model $\mathcal{M}$, written $\mathcal{M} \models \varphi$, if and only if $\mathcal{M}, w \models \varphi$, for all $w \in U$, and it is $\mathsf{L}$-valid whenever it is true in all $\mathsf{L}$-models.

Along these lines we present three examples of multiple-valued logics.

*Rosser-Turquette Logic*

Rosser-Turquette logic $\mathsf{RT}$ is an $n_{\mathsf{RT}}$-valued logic with $\mathcal{SR}_{\mathsf{RT}} = \{0, \ldots, n - 1\}$, designated values are $s, \ldots, n - 1$, where $0 < s \leq n - 1$, and the logical values are linearly ordered consistently with the order of their natural indices. The propositional operations of $\mathsf{RT}$ include the family $\{J_k : k \in \mathcal{SR}_{\mathsf{RT}}\}$ of unary operations. The operations $J_k$ play a special role. Namely, $J_k$ is a unary operation 'selecting' the logical value $k$. Other propositional operations are $\vee$, $\wedge$, and $\neg$.

The respective semantic functions are defined as follows:

- $f_{\vee}(k, l) = \max(k, l)$;
- $f_{\wedge}(k, l) = \min(k, l)$;

- $f_{J_k}(k) = n - 1$, and for $l \neq k$, $f_{J_k}(l) = 0$;
- $f_{\neg}(l) = \max(f_{J_0}(l), \ldots, f_{J_{s-1}}(l))$.

An RT-model is a structure of the form $\mathcal{M} = (U, m, \{m_k : k \in \mathcal{SR}_{\mathsf{RT}}\})$, where $U$ is a non-empty set of states, $m \colon \mathbb{V} \times U \to \mathcal{SR}_{\mathsf{RT}}$, and $\{m_k : k \in \mathcal{SR}_{\mathsf{RT}}\}$ is a family of meaning functions such that:

- $m_k(p) = \{w \in U : m(p, w) = k\}$, for every $k \in \mathcal{SR}_{\mathsf{RT}}$;
- $m_k(\varphi \vee \psi) = \bigcup_{i=0}^{k}((m_k(\varphi) \cap m_i(\psi)) \cup (m_i(\varphi) \cap m_k(\psi)))$;
- $m_k(\varphi \wedge \psi) = \bigcup_{i=k}^{n-1}((m_k(\varphi) \cap m_i(\psi)) \cup (m_i(\varphi) \cap m_k(\psi)))$;
- $m_{n-1}(J_l(\varphi)) = m_l(\varphi)$, $m_0(J_l(\varphi)) = \bigcup_{i \neq l} m_i(\varphi)$, and $m_k(J_l(\varphi)) = \emptyset$, for $k \neq 0, n-1$;
- $m_{n-1}(\neg\varphi) = \bigcup_{i=0}^{s-1} m_i(\varphi)$, $m_0(\neg\varphi) = \bigcup_{i=s}^{n-1} m_i(\varphi)$, and $m_k(\neg\varphi) = \emptyset$, for $k \neq 0, n-1$.

*Symmetric Heyting Logic of Order n,* $\mathsf{SH}_n$

The formulas of logic $\mathsf{SH}_n$, $n \geq 2$, are constructed from propositional variables with the operations $\wedge, \vee, \to, \neg, \sim$, and with a family $\{\sigma_i\}_{i=1,\ldots,n-1}$ of unary operations. The semantic range for logic $\mathsf{SH}_n$ is:

$$\mathcal{SR}_{\mathsf{SH}_n} = \left\{(x, y) : x, y \in \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\right\}\right\}.$$

We can treat the set $\left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\right\}$ as a symmetric Heyting algebra

$$((n), \vee, \wedge, \to, \neg, \sim, 0, 1),$$

where $(n) = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\right\}$, and

$$x \vee y = \max(x, y) \qquad x \wedge y = \min(x, y) \qquad \sim x = 1 - x$$

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases} \qquad \neg x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

We define the unary operations $\{\sigma_i\}_{i=1,\ldots,n-1}$ in this algebra by:

$$\sigma_i\left(\frac{x}{n-1}\right) = \begin{cases} 1 & \text{if } x \geq n - i \\ 0 & \text{otherwise} \end{cases}$$

Then the semantic functions for the propositional operations in $\mathcal{SR}_{\mathsf{SH}_n}$ are given by:

- $f_{\vee}((x, y), (x', y')) = (x \vee x', y \vee y')$;
- $f_{\wedge}((x, y), (x', y')) = (x \wedge x', y \wedge y')$;
- $f_{\to}((x, y), (x', y')) = (x \to x', y \to y')$;
- $f_{\neg}((x, y)) = (x \to 0, y \to 0)$;

- $f_{\sim}((x,y)) = (1-y, 1-x)$;
- $f_{\sigma_i}((x,y)) = (\sigma_i(x), \sigma_i(y))$.

An $\mathsf{SH}_n$-model is a structure of the form $\mathcal{M} = (U, m, \{m_{(k,l)} : (k,l) \in \mathcal{SR}_{\mathsf{SH}_n}\})$, where $U$ is a non-empty set of states, $m\colon \mathbb{V} \times U \to \mathcal{SR}_{\mathsf{SH}_n}$, and $\{m_{(k,l)} : (k,l) \in \mathcal{SR}_{\mathsf{SH}_n}\}$ is a family of meaning functions reflecting the intended interpretation of the operations. For example, the functions for the implication and negation $\sim$ are defined as follows:

- $m_{(1,1)}(\varphi \to \psi) = \bigcup_{x \leq x',\ y \leq y'}(m_{(x,y)}(\varphi) \cap m_{(x',y')}(\psi))$;
- $m_{(1,l)}(\varphi \to \psi) = \bigcup_{x \leq x',\ y > l}(m_{(x,y)}(\varphi) \cap m_{(x',l)}(\psi))$;
- $m_{(k,1)}(\varphi \to \psi) = \bigcup_{x > k,\ y \leq y'}(m_{(x,y)}(\varphi) \cap m_{(k,y')}(\psi))$;
- $m_{(k,l)}(\varphi \to \psi) = \bigcup_{x > k,\ y > l}(m_{(x,y)}(\varphi) \cap m_{(x',y')}(\psi))$, where $k, l \neq 1$;
- $m_{(k,l)}(\sim \varphi) = m_{(1-l,1-k)}(\varphi)$.

*The Logic* $\mathsf{L_T}$

The logic $\mathsf{L_T}$ is a certain version of the logics introduced in [CH73], see also [Ras91]. The logic is based on a generalization of Post algebras investigated in [CHR89] where a chain of Post constants is replaced by a finite partially ordered set. The elements of this set are the indices of the unary Post operations but, in contrast with the classical Post logic, they do not have syntactic counterparts in the language.

$\mathsf{L_T}$ is a propositional logic whose formulas are built from propositional variables with the operations $\vee, \wedge, \to, \neg$, and a family $\{d_t\}_{t \in \mathsf{T}}$ of unary operations, where $\mathsf{T}$ is a finite set partially ordered by a relation $\leq$. The semantic range for the logic $\mathsf{L_T}$ is the set of all the increasing subsets of $\mathsf{T}$, that is:

$$\mathcal{SR}_{\mathsf{L_T}} = \{s \subseteq \mathsf{T} : \text{ for all } x, y,\ x \in s \text{ and } x \leq y \text{ imply } y \in s\} \cup \{\emptyset\}.$$

The semantic functions providing meaning of the propositional operations are defined as follows:

- $f_{\vee}(s, s') = s \cup s'$;
- $f_{\wedge}(s, s') = s \cap s'$;
- $f_{\to}(s, s') = \bigcup\{u \in \mathcal{SR}_{\mathsf{L_T}} : s \cap u \subseteq s'\}$;
- $f_{\neg}(s) = s \to \emptyset$;
- $f_{d_t}(s) = \begin{cases} \mathsf{T} & \text{if } t \in s, \\ \emptyset & \text{otherwise.} \end{cases}$

The only distinguished element of $\mathcal{SR}_{\mathsf{L_T}}$ is the set $\mathsf{T}$.

In an $\mathsf{L_T}$-model $\mathcal{M} = (U, m, \{m_k : k \in \mathcal{SR}_{\mathsf{L_T}}\})$, the meaning functions for operations $d_t$ are defined as follows, for $l \in \mathcal{SR}_{\mathsf{L_T}}$:

- $m_{\mathsf{T}}(d_t(\varphi)) = \bigcup_{t \in l} m_l(\varphi)$;
- $m_{\emptyset}(d_t(\varphi)) = \bigcup_{t \notin l} m_l(\varphi)$;
- $m_l(d_t(\varphi)) = \emptyset$, for $l \neq \emptyset, \mathsf{T}$.

## 10.3 Relational Formalization of Finitely Many-Valued Logics

In the relational formalization of many-valued logics we apply the standard method of interpreting formulas of an $n_{\mathsf{L}}$-valued logic $\mathsf{L}$ as $n_{\mathsf{L}}$-ary relations. The vocabulary of the language of relational logic $\mathsf{RL_L}$ adequate for logic $\mathsf{L}$ consists of symbols from the following pairwise disjoint sets:

- $\mathbb{RV}_{\mathsf{RL_L}}$ – a countable infinite set of relational variables representing $n_{\mathsf{L}}$-ary relations;
- $\{o_j : 1 \leq j \leq i_{\mathsf{L}}\} \cup \{J_t : 0 \leq t \leq n_{\mathsf{L}} - 1\}$ – the set of relational operations, where $o_j$ is of the arity $a(j)$, for $j = 1, \ldots, i_{\mathsf{L}}$, and every $J_t$ is unary.

We slightly abuse the notation here by denoting the relational operations of $\mathsf{RL_L}$ with the same symbols as the operations of $\mathsf{L}$.

The set $\mathbb{RT}_{\mathsf{RL_L}}$ of relational terms is the smallest set that includes $\mathbb{RV}_{\mathsf{RL_L}}$ and is closed with respect to all the relational operations. The set of formulas is simply the set of relational terms $\mathbb{RT}_{\mathsf{RL_L}}$. An $\mathsf{RL_L}$-formula is said to be *indecomposable* whenever it is of the form $J_t(P)$, for some $t \in \{0, \ldots, n_{\mathsf{L}} - 1\}$ and $P \in \mathbb{RV}_{\mathsf{RL_L}}$. A finite set of $\mathsf{RL_L}$-formulas is indecomposable whenever all of its formulas are indecomposable.

An $\mathsf{RL_L}$-model is a structure $\mathcal{M} = (U \cup \{\emptyset\}, m, \{m_k : k \in \mathcal{SR}_{\mathsf{L}}\})$ such that $U$ is a non-empty set, $m: \mathbb{RT}_{\mathsf{RL_L}} \to \mathcal{P}((U \cup \{\emptyset\})^{n_{\mathsf{L}}-1})$ and for every $k \in \mathcal{SR}_{\mathsf{L}}$, $m_k: \mathbb{RT}_{\mathsf{RL_L}} \to \mathcal{P}(U \cup \{\emptyset\})$ are meaning functions such that the following conditions are satisfied:

- $m_k(P) \subseteq U \cup \{\emptyset\}$ and $m(P) = m_0(P) \times \cdots \times m_{n_{\mathsf{L}}-1}(P)$, for $P \in \mathbb{RV}_{\mathsf{RL_L}}$;
- If $m(P) = P_0 \times \cdots \times P_{n_{\mathsf{L}}-1}$, then:

  (1) For any $i$, $0 \leq i \leq n_{\mathsf{L}} - 1$, $P_i \in \mathcal{P}(U \cup \{\emptyset\}) \setminus \{\emptyset\}$, i.e., $P_i$ is a non-empty subset of $U \cup \{\emptyset\}$,
  (2) If $i \neq j$, then $P_i \cap P_j \in \{\emptyset, \{\emptyset\}\}$, i.e., $P_i \cap P_j$ is either empty or equals $\{\emptyset\}$,
  (3) $U \subseteq \bigcup_{k=0}^{n_{\mathsf{L}}-1} P_k$;

- For any $j$ and $t$ the operations $o_j$ and $J_t$ are interpreted as functions on relations on $(U \cup \{\emptyset\})^{n_{\mathsf{L}}-1}$ such that:

  (i) For all terms $P^1, \ldots, P^{a(j)}$ such that $m\left(P^l\right) = \left(m_0\left(P^l\right) \times \cdots \times m_{n_{\mathsf{L}}-1}\left(P^l\right)\right)$, we have:
  $$m\left(o_j\left(P^1, \ldots, P^{a(j)}\right)\right) = Q_0 \times \cdots \times Q_{n_{\mathsf{L}}-1},$$
  where
  $$Q_k = \bigcup_{f_{o_j}(g_1, \ldots, g_{a(j)}) = k} m_{g_1}\left(P^1\right) \cap \cdots \cap m_{g_{a(j)}}\left(P^{a(j)}\right)$$
  if the above union is non-empty, and $Q_k = \{\emptyset\}$ otherwise;

(ii) For any relational term $P$ such that $m(P) = m_0(P) \times \cdots \times m_{n_{\mathsf{L}}-1}(P)$ we have:

$$m(J_t(P)) = Q_0 \times \cdots \times Q_{n_{\mathsf{L}}-1},$$

where

$$Q_{n_{\mathsf{L}}-1} = m_t(P), \; Q_0 = \bigcup_{l \neq t} m_l(P), \; \text{ and } Q_i = \{\emptyset\} \text{ for } i \neq 0, n_{\mathsf{L}} - 1.$$

Note that $m(J_t(P))$ partitions the set $U$ into the sets $m_t(P)$ and $U - m_t(P)$.

An $\mathsf{RL_L}$-formula $P$ is said to be true in an $\mathsf{RL_L}$-model $\mathcal{M}$, written $\mathcal{M} \models P$, whenever $\bigcup_{i=s}^{n_{\mathsf{L}}-1}(m_i(P)) = U \cup \{\emptyset\}$. A formula $P$ is $\mathsf{RL_L}$-valid whenever it is true in all $\mathsf{RL_L}$-models.

**Proposition 10.3.1.** $\mathsf{RL_L}$-*models are well-defined.*

*Proof.* We have to show that $\mathcal{P}((U \cup \{\emptyset\})^{n_{\mathsf{L}}-1})$ is closed under the interpretations of operations $o_j$ and $J_t$, that is for all relational terms $P, P^1, \ldots, P^{a(j)}$, the relations $m\left(o_j\left(P^1, \ldots, P^{a(j)}\right)\right)$ and $m(J_t(P))$ satisfy the conditions (1), (2), and (3). The preservation of condition (1) is quite obvious, so we concentrate on conditions (2) and (3).

Assume $m(P^r) = m_0(P^r) \times \cdots \times m_{n_{\mathsf{L}}-1}(P^r)$ satisfy the conditions (1), (2), and (3), for $r = 1, \ldots, a(j)$. Let $Q = m\left(o_j\left(P^1, \ldots, P^{a(j)}\right)\right)$, where $Q = Q_0 \times \cdots \times Q_{n_{\mathsf{L}}-1}$. Let $0 \leq l, k \leq n_{\mathsf{L}} - 1$, and $l \neq k$. Then either at least one of $Q_k$, $Q_l$ is $\{\emptyset\}$, and then obviously $Q_k \cap Q_l$ is either $\emptyset$ or $\{\emptyset\}$ or

$$Q_k = \bigcup_{f_{o_j}(g_1,\ldots,g_{a(j)})=k} m_{g_1}\left(P^1\right) \cap \ldots \cap m_{g_{a(j)}}\left(P^{a(j)}\right),$$

$$Q_l = \bigcup_{f_{o_j}(h_1,\ldots,h_{a(j)})=l} m_{h_1}\left(P^1\right) \cap \ldots \cap m_{h_{a(j)}}\left(P^{a(j)}\right),$$

hence

$$Q_k \cap Q_l = \bigcup_{f_{o_j}(g_1,\ldots,g_{a(j)})=k} \; \bigcup_{f_{o_j}(g_1,\ldots,g_{a(j)})=l} m_{g_1}\left(P^1\right) \cap \ldots \cap m_{g_{a(j)}}\left(P^{a(j)}\right) \cap$$

$$\cap\, m_{h_1}\left(P^1\right) \cap \ldots \cap m_{h_{a(j)}}\left(P^{a(j)}\right).$$

By our assumption about $m(P^r)$'s we have $m_s(P^r) \cap m_t(P^r) = \emptyset$ or $\{\emptyset\}$, for $s \neq t$. On the other hand, in each of the summands above $g_r = h_r$, for $r = 1, \ldots, a(j)$ would imply

$$k = f_{o_j}(g_1, \ldots, g_{a(j)}) = f_{o_j}(h_1, \ldots, h_{a(j)}) = l$$

which is a contradiction. Thus $g_r \neq h_r$, for some $r$, so $m_{g_r}(P^r) \cap m_{h_r}(P^r) \in \{\emptyset, \{\emptyset\}\}$. Obviously, this means that the same holds for each summand in $Q_k \cap Q_l$, and in consequence for the latter intersection, therefore $Q$ satisfies the condition (2).

To prove that $Q$ satisfies condition (3), consider any $u \in U$. Since all the $P^r$'s satisfy condition (3), then for each $r$, $1 \leq r \leq a(j)$, there exists $k_r$, $0 \leq k_r \leq n_{\mathsf{L}} - 1$ such that $u \in m_{k_r}(P^r)$. Obviously, $u \in m_{k_1}(P^1) \cap \ldots \cap m_{k_{a(j)}}(P^{a(j)})$. Therefore, for $k = f_{o_j}(k_1, \ldots, k_{a(j)})$, we obtain:

$$u \in \bigcup_{f_{o_j}(g_1,\ldots,g_{a(j)})=k} m_{g_1}(P^1) \cap \ldots \cap m_{g_{a(j)}}\left(P^{a(j)}\right).$$

Since $u$ is in the above union, the union is non-empty. Thus, by the definition it equals $Q_k$. Therefore, $u \in \bigcup_{l=0}^{n_{\mathsf{L}}-1} Q_l$. Since $u$ is an arbitrary element of $U$, $Q$ satisfies condition (3).

Assume that $m(P) = m_0(P) \times \cdots \times m_{n_{\mathsf{L}}-1}(P)$ satisfies conditions (1), (2), and (3). Let $Q = m(J_t P)$, where $Q = Q_0 \times \cdots \times Q_{n_{\mathsf{L}}-1}$. Then obviously, $Q$ satisfies condition (2), because $Q_1 = \{\emptyset\}$ for $l \neq 0, n_{\mathsf{L}} - 1$ and $Q_0 \cap Q_{n_{\mathsf{L}}} = (\bigcup_{l \neq t} m_l(P)) \cap m_t(P)$. Since for $l \neq t$ the intersection $m_l(P) \cap m_t(P)$ is either $\emptyset$ or $\{\emptyset\}$ by the assumption on $m(P)$, the same holds for the union representing $Q_0 \cap Q_{n_{\mathsf{L}}}$. Finally, $Q$ satisfies also condition (3), since:

$$\bigcup_{i=0}^{n_{\mathsf{L}}} Q_i = \bigcup_{l \neq k} m_l(P) \cup \{\emptyset\} \cup m_t(P) \subseteq \bigcup_{i=0}^{n_{\mathsf{L}}-1} m_i(P) \supseteq U,$$

by the assumption on $m(P)$. □

The translation of $\mathsf{L}$-formulas into relational terms starts with a one-to-one assignment $\tau' \colon \mathbb{V} \to \mathbb{RV}_{\mathsf{RL_L}}$ of relational variables to the propositional variables. Then, the translation $\tau$ of formulas is defined by:

$$\tau(o_j(\varphi_1, \ldots, \varphi_{a(j)})) = o_j(\tau(\varphi_1), \ldots, \tau(\varphi_{a(j)})).$$

**Proposition 10.3.2.** *For every* $\mathsf{L}$*-formula* $\varphi$ *and for every* $\mathsf{L}$*-model* $\mathcal{M}$ *there exists an* $\mathsf{RL_L}$*-model* $\mathcal{M}'$ *such that* $\mathcal{M} \models \varphi$ *iff* $\mathcal{M}' \models \tau(\varphi)$.

*Proof.* Let $\varphi$ be an $\mathsf{L}$-formula and let $\mathcal{M} = (U, m, \{m_k : k \in \mathcal{SR}_{\mathsf{L}}\})$ be an $\mathsf{L}$-model. We define the corresponding $\mathsf{RL_L}$-model $\mathcal{M}' = (U' \cup \{\emptyset\}, m', \{m'_k : k \in \mathcal{SR}_{\mathsf{L}}\})$ as follows:

- $U' = U$;
- For a relational variable $P$ such that $\tau'(p) = P$ we define $m'_k(P) = m_k(p) \cup \{\emptyset\}$ and $m'(P) = m'_0(P) \times \cdots \times m'_{n_{\mathsf{L}}-1}(P)$;
- $m'$ extends to all the compound relational terms as in $\mathsf{RL_L}$-models.

We show that for every $\mathsf{L}$-formula $\varphi$, condition (1) $m'_k(\tau(\varphi)) = m_k(\varphi) \cup \{\emptyset\}$ holds. Then, models $\mathcal{M}$ and $\mathcal{M}'$ clearly satisfy the proposition.

We prove (1) by induction on the complexity of formulas. If $\varphi = p$, for $p \in \mathbb{V}$, then (1) holds by the definition of $m'_k(\tau(p))$. Let $\varphi = o_j(p^1, \ldots, p^{a(j)})$. Then $m'_k(\tau(\varphi)) = m'_k(o_j(\tau(p^1), \ldots, \tau(p^{a(j)})))$. By the definition of the model $\mathcal{M}'$, $m'_k(o_j(\tau(p^1), \ldots, \tau(p^{a(j)})))$ equals:

$$\bigcup_{f_{o_j}(k_1,\ldots,k_{a(j)})=k} (m'_{k_1}(\tau(p^1)) \cap \ldots \cap m'_{k_{a(j)}}(\tau(p^{a(j)})))$$

if the union is non-empty, otherwise it equals $\{\emptyset\}$. If the latter holds, then $m_k(\varphi) = \emptyset$, hence $m'_k(\tau(\varphi)) = \{\emptyset\} = m_k(\varphi) \cup \{\emptyset\}$. If the union is non-empty, then by the induction hypothesis, we obtain:

$$\begin{aligned}
&\bigcup_{f_{o_j}(k_1,\ldots,k_{a(j)})=k} (m'_{k_1}(\tau(p^1)) \cap \ldots \cap m'_{k_{a(j)}}(\tau(p^{a(j)})))\\
&= \bigcup_{f_{o_j}(k_1,\ldots,k_{a(j)})=k} ((m_{k_1}(p^1) \cup \{\emptyset\}) \cap \ldots \cap (m_{k_{a(j)}}(p^{a(j)})) \cup \{\emptyset\})\\
&= \left( \bigcup_{f_{o_j}(k_1,\ldots,k_{a(j)})=k} (m_{k_1}(p^1) \cap \ldots \cap m_{k_{a(j)}}(p^{a(j)})) \right) \cup \{\emptyset\} = m_k(\varphi) \cup \{\emptyset\}.
\end{aligned}$$

Therefore $m'_k(\tau(\varphi)) = m_k(\varphi) \cup \{\emptyset\}$. □

In a similar way we can prove:

**Proposition 10.3.3.** *For every* L*-formula* $\varphi$ *and for every* $\mathsf{RL}_\mathsf{L}$*-model* $\mathcal{M}$ *there exists an* L*-model* $\mathcal{M}'$ *such that* $\mathcal{M}' \models \varphi$ *iff* $\mathcal{M} \models \tau(\varphi)$.

The above two propositions lead to:

**Theorem 10.3.1.** *Let* $\varphi$ *be an* L*-formula. Then the following conditions are equivalent:*

1. $\varphi$ *is* L*-valid;*
2. $\tau(\varphi)$ *is* $\mathsf{RL}_\mathsf{L}$*-valid.*

*Relational Formalization of Rosser–Turquette Logic* RT

The vocabulary of the language of the relational logic $\mathsf{RL}_{\mathsf{RT}}$ consists of the set of relational variables, two binary operations $\vee$ and $\wedge$, and unary operations $\neg$ and $J_t$, for $t = 0, \ldots, n_{\mathsf{RT}} - 1$.

In $\mathsf{RL}_{\mathsf{RT}}$-models we define the meaning functions in a standard way with the following clauses for the compound terms:

- $m_k(P \vee Q) = \bigcup_{l=0}^{k} (m_k(P) \cap m_l(Q) \cap m_l(P) \cap m_k(Q))$ if this union is non-empty, and $m_k(P \vee Q) = \{\emptyset\}$ otherwise;
- $m_k(P \wedge Q) = \bigcup_{l=k}^{n_{\mathsf{RT}}-1} (m_k(P) \cap m_l(Q) \cap m_l(P) \cap m_k(Q))$ if this union is non-empty, and $m_k(P \wedge Q) = \{\emptyset\}$ otherwise;

- $m_0(\neg P) = \bigcup_{l=s}^{n_{\mathsf{RT}}-1} m_l(P)$, $m_{n_{\mathsf{RT}}-1}(\neg P) = \bigcup_{l=0}^{s-1} m_l(P)$, and for $l \neq 0, n_{\mathsf{RT}}-1$, $m_l(\neg P) = \{\emptyset\}$;
- For any relational term $P$ such that $m(P) = m_0(P) \times \cdots \times m_{n_{\mathsf{RT}}-1}(P)$ we have:

$$m(J_k(P)) = Q_0 \times \cdots \times Q_{n_{\mathsf{RT}}-1},$$

where

$$Q_{n_{\mathsf{RT}}-1} = m_k(P),\ Q_0 = \bigcup_{l \neq k} m_l(P),\ \text{ and } Q_i = \{\emptyset\} \text{ for } i \neq 0, n_{\mathsf{RT}}-1.$$

*Relational Formalization of Symmetric Heyting Logics* $\mathsf{SH}_n$

The vocabulary of the relational logic $\mathsf{RL}_{\mathsf{SH}_n}$ consists of relational variables and the relational operations which are the direct counterparts to the propositional operations of the $\mathsf{SH}_n$ logics. $\mathsf{RL}_{\mathsf{SH}_n}$-models are defined so that the meaning of compound relational terms reflects properties of the corresponding propositional operations. For example, for any relational term $P$ such that:

$$m(P) = \times_{(k,l) \in \mathcal{SR}_{\mathsf{SH}_n}} m_{(k,l)}(P)$$

we have

$$m(\sigma_i(P)) = \times_{(k,l) \in \mathcal{SR}_{\mathsf{SH}_n}} R_{(k,l)},$$

where $\times_i A_i$ denotes the direct product of sets $A_i$ and:

$R_{(k,l)} = \{\emptyset\}$ if either $k \neq 0, 1$ or $l \neq 0, 1$,
$R_{(1,1)} = \bigcup_{x \geq n-i,\ y \geq n-i} m_{(\frac{x}{n-1}, \frac{y}{n-1})}(P)$
if the above union is non-empty, and $R_{(1,1)} = \{\emptyset\}$ otherwise,

$R_{(1,0)} = \bigcup_{x \geq n-i,\ y < n-i} m_{(\frac{x}{n-1}, \frac{y}{n-1})}(P)$
if the above union is non-empty, and $R_{(1,0)} = \{\emptyset\}$ otherwise,

$R_{(0,1)} = \bigcup_{x < n-i,\ y \geq n-i} m_{(\frac{x}{n-1}, \frac{y}{n-1})}(P)$
if the above union is non-empty, and $R_{(0,1)} = \{\emptyset\}$ otherwise,

$R_{(0,0)} = \bigcup_{x < n-i,\ y < n-i} m_{(\frac{x}{n-1}, \frac{y}{n-1})}(P)$
if the above union is non-empty, and $R_{(0,0)} = \{\emptyset\}$ otherwise.

*Relational Formalization of the Logic* $\mathsf{L_T}$

The formulas of relational logic $\mathsf{RL}_{\mathsf{L_T}}$ are built from relational variables with the relational operations which are the direct counterparts of the propositional operations of the logic $\mathsf{L_T}$. $\mathsf{RL_{L_T}}$-models are defined in a standard way, so that the meaning of compound relational terms reflects properties of the corresponding propositional operations. For example, if $R = P \rightarrow Q$, and $m(Z) = \times_{k \in \mathcal{SR}_{\mathsf{L_T}}} m_k(Z)$, for $Z = P, Q, R$, then we have:

$$m_k(R) = \bigcup_{\bigcup\{u \in \mathcal{SR}_{\mathsf{L_T}} : l \cap u \subseteq t\} = k} m_l(P) \cap m_t(Q),$$

if the above union is non-empty, and $m_k(R) = \{\emptyset\}$ otherwise.

## 10.4 Dual Tableaux for Finitely Many-Valued Logics

Let $\mathsf{L}$ be a finitely many-valued logic. A relational dual tableau for $\mathsf{L}$ consist of axiomatic sets of formulas and decomposition rules that apply to finite sets of formulas. The notion of a rule is defined as in Sect. 2.4.

Decomposition rules have the following forms:

$(J\!-\!in)$ $\quad \dfrac{R}{J_s(R), \ldots, J_{n_{\mathsf{L}}-1}(R)}$

where $R$ is not of the form $J_l(Q)$

for any $l \in \mathcal{SR}_{\mathsf{L}}$ and for any $\mathsf{RL_L}$-formula $Q$

$(o_j)_1$ $\quad \dfrac{K, J_t(o_j(R_1, \ldots, R_i))}{K, H, J_{t_1}(R_1) \mid \ldots \mid K, H, J_{t_i}(R_i)}$

for all $t_1, \ldots, t_i$ such that $f_{o_j}(t_1, \ldots, t_i) = t$, where

$i = a(j)$, $H = J_t(o_j(R_1, \ldots, R_i))$, and

for some $l \in \{1, \ldots, i\}$ sequence $K$ of formulas

does not contain $J_{t_l}(R_l)$

$(o_j)_2$ $\quad \dfrac{K, J_t(o_j(R_1, \ldots, R_i))}{K}$

where $K = J_{t_1}(R_1), \ldots, J_{t_i}(R_i)$ and $f_{o_j}(t_1, \ldots, t_i) = t$

$(J_0(J_l))$ $\quad \dfrac{J_0(J_l(R))}{J_0(R), \ldots, J_{l-1}(R), J_{l+1}(R), \ldots, J_{n_{\mathsf{L}}-1}(R)}$

for any $l \in \mathcal{SR}_{\mathsf{L}}$ such that $l \neq 0$

$(J_0(J_0))$ $\quad \dfrac{J_0(J_0(R))}{J_1(R), \ldots, J_{n_{\mathsf{L}}-1}(R)}$

$(J_{n_{\mathsf{L}}-1}(J_l))$ $\quad \dfrac{J_{n_{\mathsf{L}}-1}(J_l(R))}{J_l(R)}$, for any $l \in \mathcal{SR}_{\mathsf{L}}$

$(J_t(J_l))$ $\quad \dfrac{K, J_t(J_l(R))}{K}$

for any $l \in \mathcal{SR}_{\mathsf{L}}$ and for any $t \neq 0, n_{\mathsf{L}} - 1$,

and for any sequence $K$ of $\mathsf{RL_L}$-formulas.

A set of $\mathsf{RL_L}$-formulas is said to be an $\mathsf{RL_L}$-axiomatic set whenever it includes either of the following sets:

For any relational term $R$,

- $\{J_0(R), \ldots, J_{n_{\mathsf{L}}-1}(R)\}$;
- $\{J_0(J_t(J_l(R)))\}$, for $t \neq 0, n_{\mathsf{L}} - 1$.

A finite set $X = \{P_1, \ldots, P_n\}$ of $\mathsf{RL_L}$-formulas is said to be $\mathsf{RL_L}$-set whenever for every $\mathsf{RL_L}$-model $\mathcal{M}$, $\bigcup_{j=1}^{n} \bigcup_{i=s}^{n_{\mathsf{L}}-1} m_i(P_j) = U \cup \{\emptyset\}$. Correctness of a rule is defined in a similar way as in the relational logics of classical algebras of binary relations (see Sect. 2.4).

**Proposition 10.4.1.**

1. *The* $\mathsf{RL_L}$*-rules are* $\mathsf{RL_L}$*-correct;*
2. *The* $\mathsf{RL_L}$*-axiomatic sets are* $\mathsf{RL_L}$*-sets.*

*Proof.* By way of example, we prove $\mathsf{RL_L}$-correctness of the rule $(J_{n_{\mathsf{L}}-1}(J_l))$. First, note that by the definition of $\mathsf{RL_L}$-models we have:

$$m_{n_{\mathsf{L}}-1}(J_{n_{\mathsf{L}}-1}(J_l(R))) = m_{n_{\mathsf{L}}-1}(J_l(R)) = m_l(R),$$

$$m_i(J_{n_{\mathsf{L}}-1}(J_l(R))) = m_i(J_l(R)) = \{\emptyset\}, \text{ for any } i \neq 0, n_{\mathsf{L}} - 1.$$

Therefore, $\bigcup_{k=s}^{n_{\mathsf{L}}-1} m_k(J_{n_{\mathsf{L}}-1}(J_l(R))) = \bigcup_{k=s}^{n_{\mathsf{L}}-1} m_k(J_l(R)) = \{\emptyset\} \cup m_l(R)$. Thus, for every $\mathsf{RL_L}$-model $\mathcal{M}$, $\mathcal{M} \models J_{n_{\mathsf{L}}-1}(J_l(R))$ iff $\mathcal{M} \models J_l(R)$. Hence, the correctness follows. □

The notions of an $\mathsf{RL_L}$-proof tree and $\mathsf{RL_L}$-provability of an $\mathsf{RL_L}$-formula are defined as in Sect. 2.4.

**Theorem 10.4.1 (Soundness and Completeness of $\mathsf{RL_L}$).** *Let $\varphi$ be an $\mathsf{RL_L}$-formula. Then, the following conditions are equivalent:*

1. *$\varphi$ is $\mathsf{RL_L}$-valid;*
2. *$\varphi$ is $\mathsf{RL_L}$-provable.*

*Proof.* The implication 2. → 1. follows from Proposition 10.4.1, hence $\mathsf{RL_L}$-dual tableau is sound. Moreover, it can be easily proved that every $\mathsf{RL_L}$-proof tree is finite. Assume that $\varphi$ is $\mathsf{RL_L}$-valid. Suppose $\varphi$ does not have a closed $\mathsf{RL_L}$-proof tree. Let us consider a non-closed $\mathsf{RL_L}$-proof tree for $\varphi$. This tree has to contain a branch $b$ which ends with a non-axiomatic set $\Delta$ of $\mathsf{RL_L}$-formulas. By the construction of the tree, each element of $\Delta$ is of the form $J_t(P)$, for some $P \in \mathbb{RV}_{\mathsf{RL_L}}$ and

$t \in \{0, \ldots, n_{\mathsf{L}} - 1\}$, since otherwise we could apply to $\Delta$ one of the rules. Define the branch structure $\mathcal{M}^b = (U^b \cup \{\emptyset\}, m^b, \{m^b_k : k \in \mathcal{SR}_{\mathsf{L}}\})$ as follows:

- $U^b = \{w\}$;
- For any relational variable $P$,

$$m^b_k(P) = \begin{cases} \{w\} & \text{if } k = \min\{l : J_l(P) \notin \Delta\}, \\ \{\emptyset\} & \text{otherwise;} \end{cases}$$

- $m^b_k$ extends to all compound relational terms as in $\mathsf{RL_L}$-models.

Since $\Delta$ is not an axiomatic set, $\{l : J_l(P) \notin \Delta\} \neq \emptyset$. Therefore, for every $P \in \mathbb{RV}_{\mathsf{RL_L}}$, there exists exactly one $i \in \{0, \ldots, n_{\mathsf{L}} - 1\}$ such that $m_i(P) = \{w\}$. Hence, it can be easily proved that $\mathcal{M}^b$ is an $\mathsf{RL_L}$-model. Moreover, if $J_l(P) \in \Delta$ for some $l \in \{0, \ldots, n_{\mathsf{L}} - 1\}$, then by the definition of the branch structure $m_l(P) = \{\emptyset\}$. Therefore, if $J_l(P) \in \Delta$, then $\bigcup_{k=s}^{n_{\mathsf{L}}-1} m_k(J_l(P)) = \{\emptyset\}$, so $\mathcal{M}^b \not\models J_l(P)$. Hence, $\Delta$ is not $\mathsf{RL_L}$-valid. Since every node of the branch $b$ is obtained from its predecessor node by means of some $\mathsf{RL_L}$-rule, and the rule preserves and reflects validity, and since $\Delta$ is not $\mathsf{RL_L}$-valid, the formula $\varphi \in b$ is not $\mathsf{RL_L}$-valid, a contradiction. □

Finally, by the above theorem and Theorem 10.3.1, we obtain:

**Theorem 10.4.2 (Relational Soundness and Completeness of L).** *Let $\varphi$ be an L-formula. Then, the following conditions are equivalent:*

1. *$\varphi$ is L-valid;*
2. *$\tau(\varphi)$ is $\mathsf{RL_L}$-provable.*

*Dual Tableaux for Rosser–Turquette Logics*

$\mathsf{RL_{RT}}$-dual tableau consists of the rules ($J$-in), $(J_0(J_l))$, $(J_{n_{\mathsf{RT}}-1}(J_l))$, $(J_k(J_l))$, and the rules of introduction and elimination of disjunction, conjunction, and negation:

$$(\vee\text{-}in)_1 \quad \frac{J_t(P \vee Q)}{J_t(P \vee Q), J_t(P) \mid J_t(P \vee Q), J_l(Q)}$$
for $0 \leq l \leq t$

$$(\vee\text{-}in)_2 \quad \frac{J_t(P \vee Q)}{J_t(P \vee Q), J_l(P) \mid J_t(P \vee Q), J_t(Q)}$$
for $0 \leq l \leq t$

$$(\vee\text{-}el) \quad \frac{K, J_t(P \vee Q)}{K}$$
$K = J_0(P), J_0(Q), \ldots, J_t(P), J_t(Q)$

$$(\wedge\text{-}in)_1 \quad \frac{J_t(P \wedge Q)}{J_t(P \wedge Q), J_t(P) \mid J_t(P \wedge Q), J_l(Q)}$$
for $t \leq l \leq n_{\mathsf{RT}} - 1$

$$(\wedge{-}in)_2 \quad \frac{J_t(P \wedge Q)}{J_t(P \wedge Q), J_l(P) \mid J_t(P \wedge Q), J_t(Q)}$$

for $t \leq l \leq n_{\mathsf{RT}} - 1$

$$(\wedge{-}el) \quad \frac{K, J_t(P \wedge Q)}{K}$$

$K = J_t(P), J_t(Q), \ldots, J_{n_{\mathsf{RT}}-1}(P), J_{n_{\mathsf{RT}}-1}(Q)$

$$(\neg{-}in) \quad \frac{J_t(\neg R)}{J_i(R), \ldots, J_{n_{\mathsf{RT}}-1-t}(R)}$$

for $t = 0, n_{\mathsf{RT}} - 1$, where $i = \begin{cases} s, & \text{if } t = 0 \\ s-1, & \text{if } t = n_{\mathsf{RT}} - 1 \end{cases}$

$$(\neg{-}el) \quad \frac{K, J_t(\neg R)}{K}$$

$$K = \begin{cases} \text{any set of } \mathsf{RL}_{\mathsf{RT}}\text{-formulas}, & \text{if } t \neq 0, n_{\mathsf{RT}} - 1 \\ J_0(R), \ldots, J_{s-1}(R), & \text{if } t = n_{\mathsf{RT}} - 1 \\ J_s(R), \ldots, J_{n_{\mathsf{RT}}-1}(R), & \text{if } t = 0 \end{cases}$$

*Dual Tableaux for Symmetric Heyting Logics* $\mathsf{SH}_n$

We present $\mathsf{RL}_{\mathsf{SH}_n}$-rules for the operations $\sigma_i$, $i = 1, \ldots, n-1$:
For any sequence $K$ of $\mathsf{RL}_{\mathsf{SH}_n}$-formulas,

$$(J_{(k,l)}\sigma_i{-}el)_1 \quad \frac{K, J_{(k,l)}(\sigma_i(R))}{K}, \quad \text{if either } k \neq 0, 1 \text{ or } l \neq 0, 1$$

$$(J_{(k,l)}\sigma_i{-}in)_1 \quad \frac{K, J_{(1,1)}(\sigma_i(R))}{K, J_{(1,1)}(\sigma_i(R)), J_{(\frac{x}{n-1}, \frac{y}{n-1})}(R)}$$

for $x \geq n-i$, $y \geq n-i$, if $J_{(\frac{x}{n-1}, \frac{y}{n-1})}(R) \notin K$

$$(J_{(k,l)}\sigma_i{-}in)_2 \quad \frac{K, J_{(1,0)}(\sigma_i(R))}{K, J_{(1,0)}(\sigma_i(R)), J_{(\frac{x}{n-1}, \frac{y}{n-1})}(R)}$$

for $x \geq n-i$, $y < n-i$, if $J_{(\frac{x}{n-1}, \frac{y}{n-1})}(R) \notin K$

$$(J_{(k,l)}\sigma_i{-}in)_3 \quad \frac{K, J_{(0,1)}(\sigma_i(R))}{K, J_{(0,1)}(\sigma_i(R)), J_{(\frac{x}{n-1}, \frac{y}{n-1})}(R)}$$

for $x < n-i$, $y \geq n-i$, if $J_{(\frac{x}{n-1}, \frac{y}{n-1})}(R) \notin K$

$(J_{(k,l)}\sigma_i{-}in)_4 \quad \dfrac{K, J_{(0,0)}(\sigma_i(R))}{K, J_{(0,0)}(\sigma_i(R)), J_{(\frac{x}{n-1},\frac{y}{n-1})}(R)}$

for $x < n-i$, $y < n-i$, if $J_{(\frac{x}{n-1},\frac{y}{n-1})}(R) \notin K$

$(J_{(k,l)}\sigma_i{-}el)_2 \quad \dfrac{K, J_{(k,l)}(\sigma_i(R))}{K}$

for $(k,l) \in \{0,1\}^2$, if for any $x, y$, $J_{(\frac{x}{n-1},\frac{y}{n-1})}(R) \in K$

*Dual Tableaux for Logics* $\mathsf{L_T}$

We present $\mathsf{RL_{L_T}}$-rules for the operations $d_t$:

For any sequence $K$ of $\mathsf{RL_{L_T}}$-formulas,

$(J_l(d_t){-}el)_1 \quad \dfrac{K, J_l(d_t(R))}{K}$, for $l \neq \emptyset, \mathsf{T}$

$(J_l(d_t){-}in)_1 \quad \dfrac{K, J_{\mathsf{T}}(d_t(R))}{K, J_{\mathsf{T}}(d_t(R)), J_l(R)}$

for $l \in \mathcal{SR}_{\mathsf{L_T}}$ such that $t \in l$, if $J_l(R) \notin K$

$(J_l(d_t){-}in)_2 \quad \dfrac{K, J_{\emptyset}(d_t(R))}{K, J_{\emptyset}(d_t(R)), J_l(R)}$

for $l \in \mathcal{SR}_{\mathsf{L_T}}$ such that $t \notin l$, if $J_l(R) \notin K$

$(J_l(d_t){-}el)_2 \quad \dfrac{K, J_k(d_t(R))}{K}$

for $k \in \{\emptyset, \mathsf{T}\}$, if $J_l(R) \in K$ for every $l \in \mathcal{SR}_{\mathsf{L_T}}$

## 10.5 Three-Valued Logics

In this section we present examples of dual tableau proofs in three-valued instances of Rosser–Turquette logic, symmetric Heyting logic, and logic $\mathsf{L_T}$.

Consider a three-valued Rosser–Turquette logic $\mathsf{RT}_{(3,1)}$ with $\mathcal{SR}_{\mathsf{RT}_{(3,1)}} = \{0, 1, 2\}$ where 1 and 2 are the designated values. Let $\varphi$ be the following $\mathsf{RT}_{(3,1)}$-formula:

$$\varphi = \neg J_1(\neg p).$$

Its translations into $\mathsf{RL}_{\mathsf{RT}_{(3,1)}}$-term is:

$$\tau(\varphi) = \neg J_1(\neg P),$$

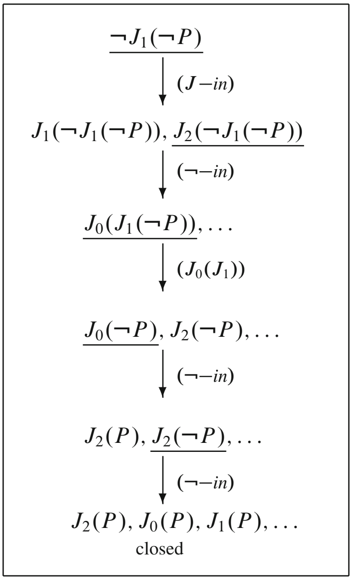


**Fig. 10.1** An $\mathsf{RL}_{\mathsf{RT}_{(3,1)}}$-proof of $\mathsf{RT}_{(3,1)}$-formula $\neg J_1(\neg p)$

where $\tau'(p) = P$. $\mathsf{RT}_{(3,1)}$-validity of this formula is equivalent with $\mathsf{RL}_{\mathsf{RT}_{(3,1)}}$-provability of its translation. Figure 10.1 presents an $\mathsf{RL}_{\mathsf{RT}_{(3,1)}}$-proof of $\tau(\varphi)$.

Now, consider a three-valued Rosser–Turquette logic $\mathsf{RT}_{(3,2)}$ with $\mathcal{SR}_{\mathsf{RT}_{(3,2)}} = \{0, 1, 2\}$, where 2 is the only designated value. Let $\psi$ be the following $\mathsf{RT}_{(3,2)}$-formula:

$$\psi = J_0(p) \vee J_1(p) \vee J_2(p).$$

Its translation into $\mathsf{RL}_{\mathsf{RT}_{(3,2)}}$-term is:

$$\tau(\psi) = J_0(P) \vee J_1(P) \vee J_2(P),$$

where $\tau'(p) = P$. Figure 10.2 presents an $\mathsf{RL}_{\mathsf{RT}_{(3,2)}}$-proof of $\tau(\psi)$ which shows $\mathsf{RT}_{(3,2)}$-validity of the formula $\psi$.

Now, consider symmetric Heyting logic of order 3, $\mathsf{SH}_3$, with $(1,1)$ as the only designated value. The semantic range for the logic $\mathsf{SH}_3$ is $\mathcal{SR}_{\mathsf{SH}_3} = \{(x, y) : x, y \in \{0, \frac{1}{2}, 1\}\}$. Among the rules of $\mathsf{RL}_{\mathsf{SH}_3}$-dual tableau are the rules of the form:

$$(\rightarrow)_{\mathsf{SH}_3} \quad \frac{J_{(1,1)}(R \rightarrow Q)}{J_{(t_1,t_2)}(R), J_{(1,1)}(R \rightarrow Q) \mid J_{(t'_1,t'_2)}(Q), J_{(1,1)}(R \rightarrow Q)}$$

for any $t_1, t_2, t'_1, t'_2 \in \mathcal{SR}_{\mathsf{SH}_3}$ such that $t_1 \leq t'_1$ and $t_2 \leq t'_2$

$$(\sim)_{\mathsf{SH}_3} \quad \frac{J_{(k,l)}(\sim R)}{J_{(1-l,1-k)}(R)} \qquad \text{for any } k, l \in \mathcal{SR}_{\mathsf{SH}_3}$$

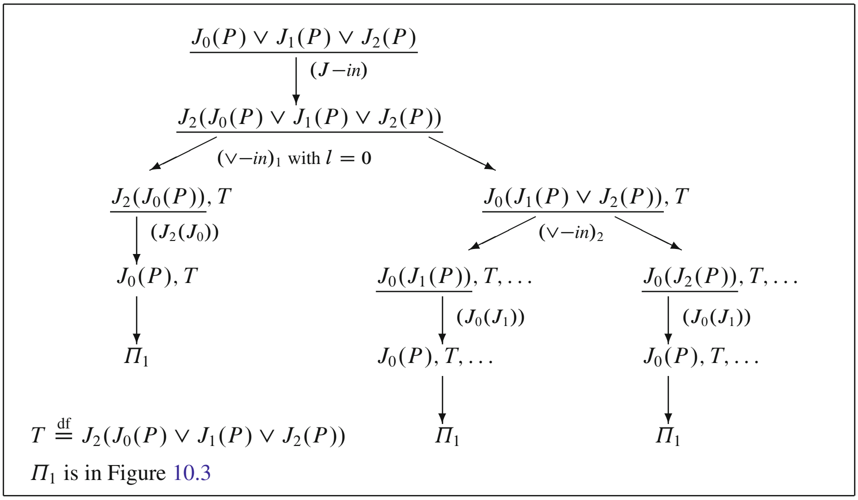


**Fig. 10.2** An $\mathsf{RL}_{\mathsf{RT}_{(3,2)}}$-proof of $\mathsf{RT}_{(3,2)}$-formula $J_0(p) \vee J_1(p) \vee J_2(p)$.

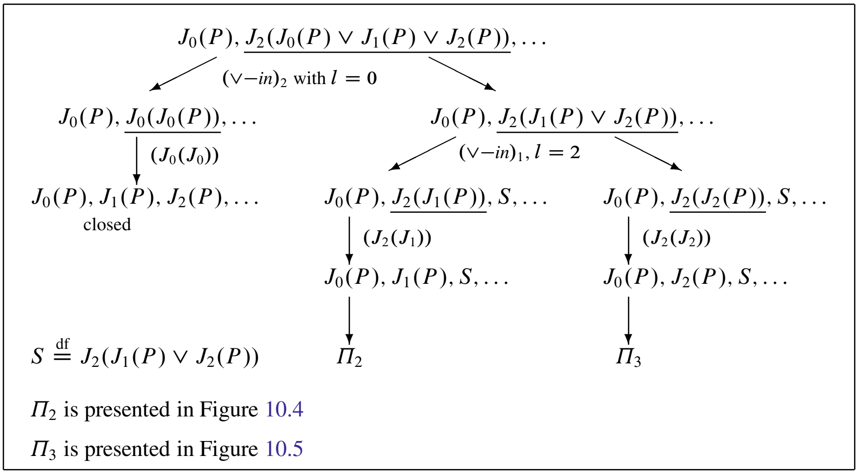


**Fig. 10.3** A subtree $\Pi_1$

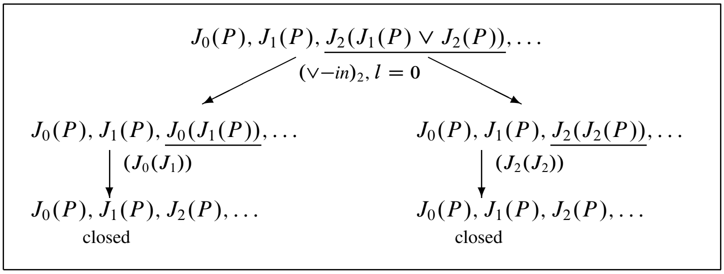


**Fig. 10.4** A subtree $\Pi_2$

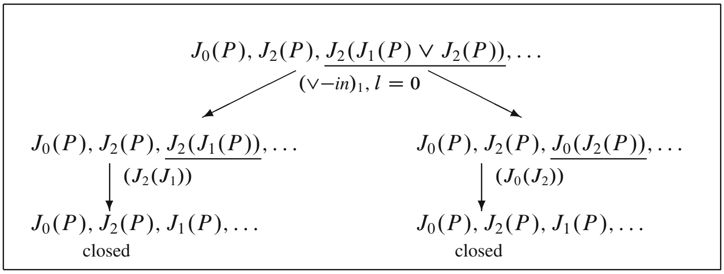


**Fig. 10.5** A subtree $\Pi_3$

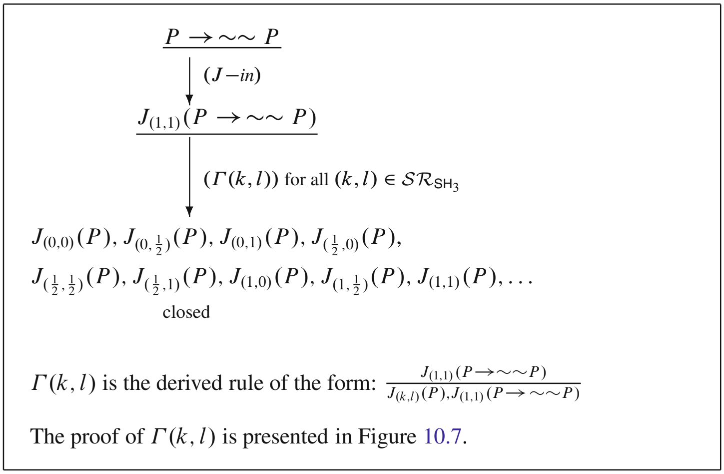


**Fig. 10.6** An $\mathsf{RL}_{\mathsf{SH}_3}$-proof of $\mathsf{SH}_3$-formula $p \to\sim\sim p$

These rules are the instances of the scheme $(o_j)_1$ presented in Section 10.4. Note that the result of an application of the rule $(\sim)_{\mathsf{SH}_3}$ to the formula $J_{(k,l)}(\sim\sim R)$ is the formula $J_{(1-l,1-k)}(\sim R)$, while the result of an application of the rule $(\sim)_{\mathsf{SH}_3}$ to $J_{(1-l,1-k)}(\sim R)$ is the formula $J_{(k,l)}(R)$. Thus, we can introduce the derived rule of the form:

$$(\sim\sim)_{\mathsf{SH}_3} \qquad \frac{J_{(k,l)}(\sim\sim R)}{J_{(k,l)}(R)} \qquad \text{for any } k,l \in \mathcal{SR}_{\mathsf{SH}_3}$$

Let $\varphi$ be an $\mathsf{SH}_3$-formula $p \to\sim\sim p$. The translation of $\varphi$ into $\mathsf{RL}_{\mathsf{SH}_3}$-term is $\tau(\varphi) = P \to\sim\sim P$, where $\tau'(p) = P$. Figure 10.6 presents an $\mathsf{RL}_{\mathsf{SH}_3}$-proof of $\tau(\varphi)$ that shows $\mathsf{SH}_3$-validity of $\varphi$.

Observe that in a diagram of Fig. 10.7 an application of a derived rule $(\sim\sim)_{\mathsf{SH}_3}$ to $J_{(k,l)}(\sim\sim P)$ results in the node which has the same formulas as those obtained by an application of rule $(\to)_{\mathsf{SH}_3}$ to formula $J_{(1,1)}(P \to\sim\sim P)$. Therefore, we identify the two nodes.

As the last example, consider logic $\mathsf{L}_\mathsf{T}$ with $\mathsf{T} = \{0, 1\}$. The semantic range for this logic is $\mathcal{SR}_{\mathsf{L}_\mathsf{T}} = \{\emptyset, \{1\}, \{0, 1\}\}$.

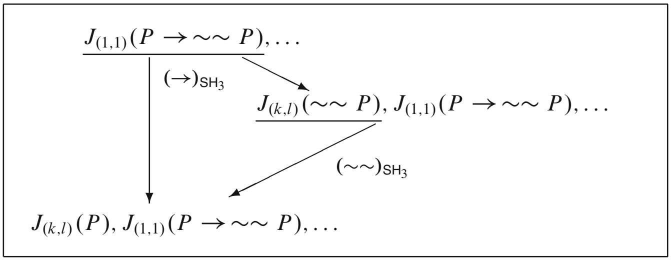


**Fig. 10.7** An $\mathsf{RL}_{\mathsf{SH}_3}$-proof of a derived rule $\Gamma(k,l) = \frac{J_{(1,1)}(P \to \sim\sim P)}{J_{(k,l)}(P), J_{(1,1)}(P \to \sim\sim P)}$.

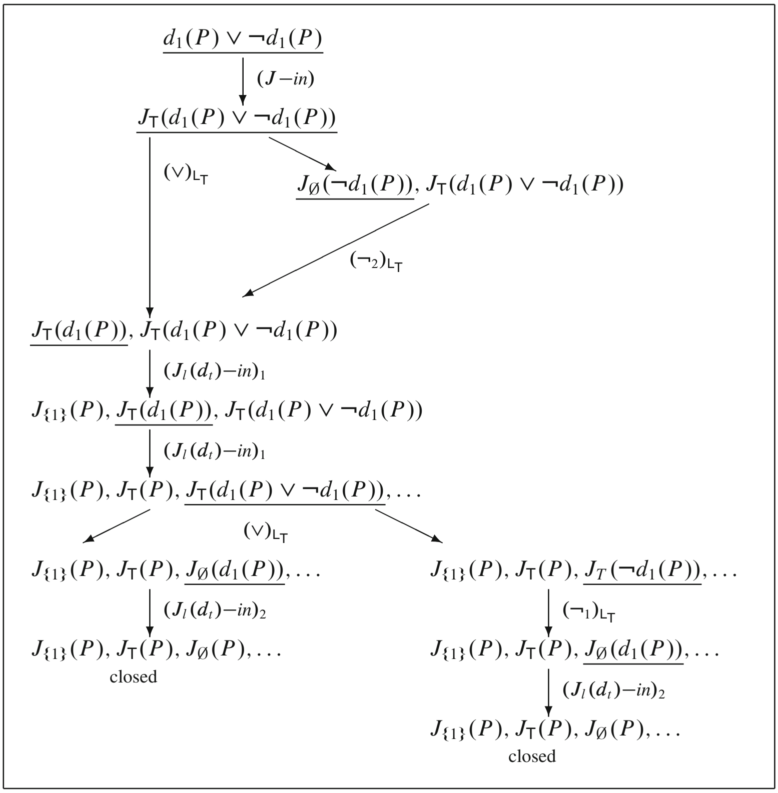


**Fig. 10.8** An $\mathsf{RL}_{\mathsf{L_T}}$-proof of $\mathsf{L_T}$-formula $d_1(p) \vee \neg d_1(p)$

Among the rules of $\mathsf{RL}_{\mathsf{L_T}}$-dual tableau are the rules of the form:

$$(\vee)_{\mathsf{L_T}} \quad \frac{J_t(R \vee Q)}{J_{t_1}(R), J_t(R \vee Q) \mid J_{t_2}(Q), J_t(R \vee Q)}$$

for any $t, t_1, t_2 \in \mathcal{SR}_{\mathsf{L_T}}$ such that $t_1 \cup t_2 = t$

$$(\neg_1)_{\mathsf{L_T}} \quad \frac{J_{\mathsf{T}}(\neg R)}{J_{\emptyset}(R)} \qquad (\neg_2)_{\mathsf{L_T}} \quad \frac{J_{\emptyset}(\neg R)}{J_{\mathsf{T}}(R)}$$

These rules follow the scheme $(o_j)_1$ presented in Sect. 10.4.

Let $\varphi$ be the following $\mathsf{L_T}$-formula:

$$\varphi = d_1(p) \vee \neg d_1(p).$$

The translation of $\varphi$ into $\mathsf{RL_{L_T}}$-term is:

$$\tau(\varphi) = d_1(P) \vee \neg d_1(P),$$

where $\tau'(p) = P$. Figure 10.8 presents $\mathsf{RL_{L_T}}$-proof of $\tau(\varphi)$, which proves $\mathsf{L_T}$-validity of $\varphi$.