CHAPTER 8

Matching Theory

8.1. The Marriage Theorem

A **matching** of a graph X is a collection of edges of X which are pairwise disjoint. The vertices incident to the edges of a matching M are **saturated** by M. A **perfect matching** is a matching that saturates all the vertices of X.

Given a bipartite graph X with bipartite sets A and B, we would like to know when there is a matching such that each element of A is matched to an element of B uniquely, i.e., a matching that saturates A. Thus, a matching is a one-to-one map $f : A \to B$ such that $(a, f(a))$ is an edge of the bipartite graph X.

This question arises in many "real life" contexts: A could be a set of jobs a company would like to fill and B could be a set of candidates applying for the jobs. We would join $a \in A$ to $b \in B$ if b is qualified to do job a. Then the matching question is whether all the jobs can be filled. In another example, A could be a set of patients and B could be a set of drugs. Some patients being allergic to certain drugs, one would like to match each patient to a drug the patient is not allergic to such that each drug is taken by at most one subject.

This question was formulated in "matrimonial terms" and solved by Philip Hall (1904-1982) in 1935. His theorem goes under the appellation of the 'marriage theorem'. Suppose we have a set of n girls and n boys. We would like to match each girl to a boy she likes. Under what conditions can we match all the girls? We can encode this information as a bipartite graph X, with A being the set of girls, B the set of boys. We join vertex $a \in A$ to $b \in B$ if a likes b. Clearly, for a matching to be possible, each girl must like at least one boy. If we have a situation where two girls like only one boy, then we have a problem and the matching question cannot be solved.

More generally, a necessary condition is that for any subset S of A, if we let $N(S)$ be the set of boys liked by some girl in S, then we need $|N(S)| \geq |S|$. Hall's theorem is that this obvious necessary condition

S. M. Cioabă and M. R. Murty, *A First Course in Graph Theory and Combinatorics*

is also sufficient. This is one of the simplest, yet powerful, theorems in mathematics with far-reaching applications.

THEOREM 8.1.1 (Marriage Theorem). *Let X be a bipartite graph with partite sets A and B. There exists a matching that saturates A if and only if for every subset S of A, we have*

$$|N(S)| \geq |S|$$

where $N(S)$ is the set of neighbours of S.

PROOF. The proof is by induction on the number of vertices in A. The base case $|A| = 1$ is trivial since a matching that saturates A consists of one edge in this case. Assume now that $|A| \geq 2$.

First suppose that

$$|N(S)| \geq |S| + 1$$

for every proper subset S of A, i.e., a subset $S \subset A$ with $S \neq \emptyset$ and $S \neq A$. By deleting one edge ab of X with $a \in A, b \in B$ (together with the incident vertices a and b) we obtain a bipartite graph Y with parts $A' = A \setminus \{a\}$ and $B' = B \setminus \{b\}$. In this graph, our partite set $A' = A \setminus \{a\}$ has fewer elements than A. Every subset S of A' satisfies Hall's condition $|N(S)| \geq |S|$ and by induction there is a matching that saturates A' in Y. Together with the deleted edge xy, we obtain a matching in X that saturates A. This finishes the proof of this case.

If the condition

$$|N(S)| \geq |S| + 1$$

is not satisfied for all proper subsets of A, then for some proper subset S_0 of A, we have

$$|N(S_0)| = |S_0|.$$

The subgraph X_1 with partite sets S_0 and $N(S_0)$ satisfies Hall's condition and so by induction, we have a matching M_1 that saturates S_0 in X_1. The subgraph X_2 with partite sets $A \setminus S_0$ and $B \setminus N(S_0)$ also satisfies Hall's condition for if some subset $C \subseteq A \setminus S_0$ is such that

$$|N_{X_2}(C)| < |C|,$$

(where the notation $N_{X_2}(C)$ refers to the neighbours of C in X_2) then

$$|N_X(S_0 \cup C)| \leq |N_X(S_0)| + |N_{X_2}(C)| < |S_0| + |C|$$

contrary to Hall's condition. It follows that there is a matching M_2 that saturates $A \setminus S_0$ in X_2. We deduce that $M_1 \cup M_2$ is a matching of X that saturates A. This completes the proof. ■

8.2. Systems of Distinct Representatives

Suppose S is a finite set and $A_1, ..., A_n$ are subsets. When is it possible to choose n distinct elements $a_1, ..., a_n$ with $a_i \in A_i$? The marriage theorem answers this question.

THEOREM 8.2.1. *A system of distinct representatives* $a_1, ..., a_n$ *with* $a_i \in A_i$ *can be chosen from a collection* $A_1, ..., A_n$ *of subsets of a set* S *if and only if*

$$|\cup_{i \in I} A_i| \geq |I|$$

for every subset I *of* $\{1, ..., n\}$.

PROOF. Consider the bipartite graph X with partite sets A and B. The vertices of A correspond to the subsets A_i $(1 \leq i \leq n)$ and the vertices of B are the elements of S. We join A_i in A to a vertex $a_j \in B$ if and only if $a_j \in A_i$. Choosing a set of distinct representatives is equivalent to finding a matching in X and the condition of the theorem is precisely Hall's condition. ∎

COROLLARY 8.2.2. *In a bipartite graph* X *with partite sets* A *and* B *there is a matching of* A *if for some* k, *we have* deg $(a) \geq k$ *for all* $a \in A$ *and* deg $(b) \leq k$ *for all* $b \in B$.

PROOF. We verify Hall's condition. For any subset S of A, at least $k|S|$ edges emanate from S. Since deg $(b) \leq k$ for all $b \in B$, these edges must be incident with at least

$$\frac{1}{k}(k|S|) = |S|$$

vertices of B. ∎

EXAMPLE 8.2.3. At a party, if every boy knows at least k girls and every girl knows at most k boys, then it is possible to match every boy with a girl he knows.

EXAMPLE 8.2.4. A **Latin square** is an $n \times n$ array on n symbols such that every symbol appears in each row and each column exactly once. For instance, the multiplication table for a finite group of order n would be an example of a Latin square. A $r \times n$ **Latin rectangle** is a $r \times n$ matrix on n symbols such that every symbol appears once in each row and at most once in each column. The first r rows of a Latin square form a $r \times n$ Latin rectangle.

A classical question is to determine if given a $r \times n$ Latin rectangle that uses the symbols $\{1, 2, ..., n\}$, it is possible to complete it to give a Latin square. The marriage theorem allows us to deduce that we can always do this. We construct a bipartite graph as follows. Let A_i

be the set of elements of $[n]$ **not** used in the i-th column. Choosing a system of distinct representatives for the A_i's would allow us to add one more row that can be inductively completed to produce a Latin square. This can be done if Hall's condition is satisfied. However, the bipartite graph with partite sets A consisting of the A_i's and B consisting of the elements of $[n]$ and we join A_i to $b \in B$ if and only if $b \in A_i$ has the property that deg $(A_i) = n-r$ for all i. Clearly, deg $(b) = n-r$ because each entry has been used exactly once for each row. By Corollary 8.2.2, we are done.

A pair of Latin squares (a_{ij}) and (b_{ij}) are called **orthogonal** if the n^2 pairs (a_{ij}, b_{ij}) are all distinct. For example, the two Latin squares on two elements

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

are not orthogonal since the pair matrix

$$\begin{pmatrix} (1,2) & (2,1) \\ (2,1) & (1,2) \end{pmatrix}$$

is not a matrix of distinct entries.

In the 1780's, Euler showed how to construct $n \times n$ orthogonal Latin squares when n is odd or divisible by 4. He also conjectured that one cannot construct a pair of orthogonal Latin squares for all $n \equiv 2 \pmod 4$. The case $n = 6$ is also known as the **thirty-six officers problem**. It asks if it is possible to arrange 6 regiments of 6 officers each of different ranks in 6×6 square so that no rank or regiment will be repeated in a row or column. In 1900, Gaston Tarry (1843-1913) proved that this problem has no solution by checking all the possible arrangements of symbols.

In 1960, Raj Chandra Bose (1901-1987), Sharadchandra Shankar Shrikhande and Ernest Tilden Parker (1926-1991) showed that Euler's conjecture is false for $n > 6$. This means that $n \times n$ orthogonal Latin squares exist for all $n \geq 3$ except $n = 6$.

8.3. Systems of Common Representatives

Suppose we are given two collections of subsets $A_1, ..., A_n$ and $B_1, .., B_n$ of a set S. A set of elements $a_1, ..., a_n$ is said to be a **system of common representatives** if $\{a_1, ..., a_n\}$ is a system of distinct representatives for both $A_1, ..., A_n$ and $B_1, ..., B_n$. We consider the problem of when we can find a system of common representatives. In case one of the collections is a partition of S (or even a disjoint collection) this is an immediate consequence of the marriage theorem.

THEOREM 8.3.1. *A system of common representatives exists if and only if the union of any k of the sets A_i is not contained in the union of any $k-1$ of the sets B_j.*

PROOF. We construct a bipartite graph X in which the partite set A corresponds to the sets A_i and the set B correspond to the sets B_j. We join A_i to B_j if $A_i \cap B_j \neq \emptyset$. Clearly the existence of a complete matching is equivalent to the existence of a system of common representatives. The condition of the theorem is precisely Hall's condition. ∎

THEOREM 8.3.2. *Let G be a finite group, H and K subgroups of the same order. Then we can find elements $x_1, ..., x_r$ in G such that*

$$G = Hx_1 \cup Hx_2 \cup \cdots \cup Hx_r = x_1K \cup x_2K \cup \cdots x_rK.$$

PROOF. We apply Theorem 8.3.1 with the A_i's being the right cosets of H and the B_j's being the left cosets of K. Since these cosets are disjoint, the condition of Theorem 8.3.1 is clearly satisfied simply by a cardinality count. Thus, it is possible to choose a system of common representatives and this is precisely the statement of the Theorem. ∎

COROLLARY 8.3.3. *If G is a finite group and H a subgroup, then it is possible to choose $x_1, ..., x_r$ so that $x_1H, ...x_rH$ is a complete set of left cosets of H and $Hx_1, ..., Hx_r$ is a complete set of right cosets of H.*

8.4. Doubly Stochastic Matrices

We now prove a famous theorem in the theory of doubly stochastic matrices using the marriage theorem. This result is the Birkhoff-von Neumann theorem that states that every doubly stochastic matrix is a convex combination of permutation matrices. Recall that a matrix $A = (a_{ij})$ is called **doubly stochastic** if every row sums to 1 and every column sums to 1. Such matrices arise naturally in probability theory. A **permutation matrix** is a doubly stochastic matrix in which a_{ij} is 0 or 1. Thus, every row and every column of a permutation matrix contains a single 1 and the rest of the entries are zero. The set of $n \times n$ permutation matrices forms a group isomorphic to the symmetric group on permutations on n letters.

THEOREM 8.4.1 (Birkhoff 1946, von Neumann 1953). *Every doubly stochastic matrix can be written as a linear combination of permutation matrices.*

PROOF. Let $M = (a_{ij})$ be a doubly stochastic matrix. We define a bipartite graph X with partite sets A and B. The vertices of A will be the rows R_i of A and the vertices of B will be the columns C_j of A. We join a row R_i to a column C_j if $a_{ij} \neq 0$. We claim that this bipartite graph satisfies Hall's condition. Indeed, suppose that $|N(S)| < |S|$ for some subset S of A. Let $|S| = s$. The previous inequality implies that there are s rows R_i with fewer than s neighbours. If we list our rows horizontally, the neighbours are precisely the columns in which the rows have non-zero entries. Adding up all the entries of each row gives a total of s. Doing the same column-wise gives us a sum of $< s$, which is a contradiction. Thus, Hall's condition is satisfied and there is a matching. The existence of a matching means we may select n non-zero entries of M in such a way that each row and each column contains exactly one of them. Of all these non-zero entries, let c_1 be one of least value. Thus, we can write

$$M = c_1 P_1 + R$$

where P_1 is a permutation matrix. Moreover, $(1 - c_1)^{-1} R$ is again a doubly stochastic matrix but with one less non-zero entry. Thus, the proof is completed by inducting on the number of non-zero entries. ■

8.5. Weighted Bipartite Matching

We now consider a weighted bipartite graph $K_{n,n}$ with non-negative weights w_{ij} corresponding to the edge (i, j). Our goal is to find a **maximal transversal**, that is, a matching so that the sum of the weights of the edges in the matching is maximal among all matchings. For the sake of simplicity, we assume that the weights are non-negative integers (which is usually not a restriction in practice). Let $W = (w_{ij})$ be the weight matrix.

The algorithm to find a maximal matching that we now describe is called the **Hungarian algorithm**. It was first discovered by Harold Kuhn in 1955 and later revised by James Munkres in 1957. The algorithm is based on the work of two Hungarian mathematicians Denes König (1884-1944) and Jenö Egerváry (1891-1958) and Kuhn named it the Hungarian algorithm in their honour.

The goal of finding a maximal matching is facilitated by supplementary "weights". We say a collection of numbers $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_n)$ is a **weighted cover** for W if

$$w_{ij} \leq u_i + v_j \qquad \forall 1 \leq i, j \leq n.$$

The **cost** of a cover is defined as

$$c(u, v) := \sum_i u_i + \sum_j v_j.$$

LEMMA 8.5.1. *For any matching M and any weighted cover, we have*

$$c(u,v) \geq w(M)$$

where $w(M)$ is defined as the sum of the weights of the edges in M. Moreover, $c(u,v) = w(M)$ if and only if M is a matching with maximal weight.

PROOF. The first part of the lemma is clear simply by summing over all the edges of the matching the inequality

$$w_{ij} \leq u_i + v_j.$$

Thus, there is no matching with weight greater than $c(u,v)$ for any cover and the maximal weight is at most the minimal cost of a cover. If $c(u,v) = w(M)$, then we must have the equality

$$w_{ij} = u_i + v_j$$

for all edges of the matching and this must be a matching of maximal weight. ■

This lemma is the basis of the Hungarian algorithm. As we mentioned before, we suppose w_{ij} are non-negative integers and this is not any stringent restriction. We begin by choosing an arbitrary cover, which can easily be done simply by choosing u_i to be the largest weight in the i-th row and v_i to be zero. Clearly,

$$w_{ij} \leq u_i + v_j$$

is satisfied with this choice. Next, we form a bipartite graph $X_{u,v} = (A, B)$ where the vertices of A are the rows of the matrix W and the vertices of B are the columns. We join row i to column j if and only if $w_{ij} = u_i + v_j$. If we have a perfect matching in this graph, we are done by the lemma. Otherwise, Hall's condition is not satisfied and so there is a set of m rows "adjacent" to fewer than m columns. If for each of these rows, we decrease u_i by 1 and increase v_j by 1, and thus get a new sequence $u'_1, ..., u'_n$ and $v'_1, ..., v'_n$, the inequality

$$w_{ij} \leq u'_i + v'_j$$

is satisfied. To see this, note that if i, j are not related this is clear since we have the strict inequality $w_{ij} < u_i + v_j$. If i, j are related then the sum $u_i + v_j$ has not changed. We have thus obtained a new cover whose cost is smaller than the earlier one simply because Hall's condition is violated. The claim is that this converges to the minimal cost and thus the maximal weight transversal. This is clear since we must arrive at a matching for otherwise, we can lower the cost of the cover and this cannot go on endlessly.

To see how to work this algorithm in practice, it is best to use matrices. We illustrate this to determine a maximal transversal in the matrix

$$\begin{pmatrix} 4 & 1 & 6 & 2 & 3 \\ 5 & 0 & 3 & 7 & 6 \\ 2 & 3 & 4 & 5 & 8 \\ 3 & 4 & 6 & 3 & 4 \\ 4 & 6 & 5 & 8 & 6 \end{pmatrix}.$$

We will write the cost covers above the columns and along the rows. The initial cost cover is obtained by simply taking the largest weight in each row. We write the matrix whose entries are $u_i + v_j - w_{ij}$ alongside:

$$\begin{array}{c} \\ 6 \\ 7 \\ 8 \\ 6 \\ 8 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array} \\ \begin{pmatrix} 2 & 5 & 0 & 4 & 3 \\ 2 & 7 & 4 & 0 & 1 \\ 6 & 5 & 4 & 3 & 0 \\ 3 & 2 & 0 & 3 & 2 \\ 4 & 2 & 3 & 0 & 2 \end{pmatrix} \end{array}.$$

This gives rise to the “equality subgraph” :

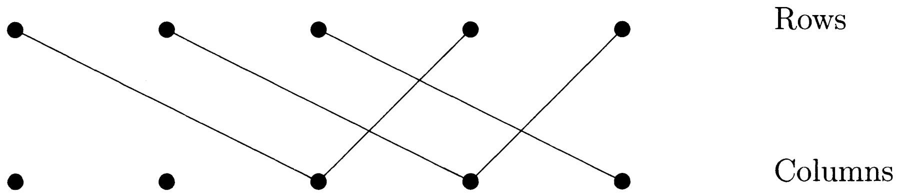

FIGURE 8.1

We can decrease u_i's by 1 and increase v_3, v_4, v_5 by 1 and re-write the matrix whose entries are $u_i + v_j - w_{ij}$ given by this new cover:

$$\begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 5 \\ 7 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 0 & 1 & 1 & 1 \end{array} \\ \begin{pmatrix} 1 & 4 & 0 & 4 & 3 \\ 1 & 6 & 4 & 0 & 1 \\ 5 & 4 & 4 & 3 & 0 \\ 2 & 1 & 0 & 3 & 2 \\ 3 & 1 & 3 & 0 & 2 \end{pmatrix} \end{array}$$

and we draw the equality subgraph again getting the same graph as before. Thus, we can reduce all the u_i's by 1 and increase the v_3, v_4, v_5

by 1. Repeating the process once more gives:

$$\begin{pmatrix} 0 & 3 & 0* & 4 & 3 \\ 0* & 5 & 4 & 0 & 1 \\ 4 & 3 & 4 & 3 & 0* \\ 1 & 0* & 0 & 3 & 2 \\ 2 & 0 & 3 & 0* & 2 \end{pmatrix}$$

where we have indicated a transversal by an asterisk. Since we have found a transversal, we can determine the cost as the sum of the u_i's and v_j's which we see to be 31.

If we were interested in a minimal transversal, all we need to do is to take the maximum M of all the entries and replace our weights w_{ij} by $M - w_{ij}$ and repeat the above algorithm.

8.6. Matchings in General Graphs

In a bipartite graph X with bipartite sets A and B, the marriage theorem gives a necessary and sufficient condition for the existence of a matching that saturates A. For general graphs, the following theorem gives a necessary and sufficient condition for the existence of a perfect matching. It was proved by William Tutte (1917-2002) in 1947. Tutte was one of the leading mathematicians in graph theory and combinatorics. In 1935, he began his studies at Cambridge in chemistry, but soon after he became interested in mathematics. During World War II, he worked at Bletchley Park as a code breaker and he was able to deduce the structure of a German encryption machine using only some intercepted encrypted messages.

An **odd component** of a graph H is a component of H with an odd number of vertices. Let odd(H) denote the number of odd components of H.

THEOREM 8.6.1 (Tutte 1947). *A graph X contains a perfect matching if and only if*

$$\text{odd}(X \setminus S) \leq |S| \tag{8.6.1}$$

for each $S \subset V(X)$.

PROOF. If X has a perfect matching and S is a subset of vertices of X, then each odd component of $X \setminus S$ has a vertex adjacent to a vertex in S. This means odd$(X \setminus S) \leq |S|$.

The proof of sufficiency is more complicated. We start it here and invite the reader to complete it.

Assume that condition (8.6.1) is satisfied for all $S \subset V(X)$. Note that by adding edges to X, condition (8.6.1) is preserved (Prove this).

The theorem is true unless there exists a graph X that has no perfect matchings and adding any missing edges would create a graph with a perfect matching.

Let X be such a graph. We will obtain a contradiction by showing that X actually contains a perfect matching.

Let C denote the set of vertices whose degree is $|V(X)| - 1$. If $X \setminus C$ is formed by disjoint complete graphs, then one can find a perfect matching easily. The case when $X \setminus C$ is not a union of disjoint cliques is left as an exercise. ∎

Tutte's theorem was later extended by Claude Berge (1926-2002) in 1958. Berge was one of the leading mathematicians in graph theory and combinatorics in the last century. His result gives a formula for $\nu(X)$ which is the size of a largest matching of a general graph. By size we mean the number of edges in the matching.

THEOREM 8.6.2 (Berge 1958). *For a graph X,*

$$\nu(X) = \frac{1}{2}\left(n - \max_{S \subset V(X)} (\text{odd}(X \setminus S) - |S|)\right).$$

8.7. Connectivity

Recall that a graph X is called connected if any two of its vertices are connected by a path. A graph is **disconnected** if it is not connected. A component of X is a maximal connected subgraph of X. This notions can be extended as follows. The **vertex-connectivity** $\kappa(X)$ of X equals the minimum size of a subset of vertices of X whose deletion disconnects X. The **edge-connectivity** $\kappa'(X)$ of X equals the minimum size of a subset of edges of X whose deletion disconnects X. Thus, a graph is connected if and only if its (vertex- or edge-)connectivity is non-zero. By convention, $\kappa(K_n) = \kappa'(K_n) = n - 1$. In general, the following inequalities hold in any connected graph.

LEMMA 8.7.1. *If X is a connected graph, then*

$$1 \leq \kappa(X) \leq \kappa'(X) \leq \delta(X)$$

where $\delta(X)$ denotes the minimum degree of X.

PROOF. If X is connected, then obviously $\kappa(X) \geq 1$. Also, if x is a vertex of X whose degree equals $\delta(X)$, then deleting the $\delta(X)$ edges incident to x disconnects the graph X. Thus, $\kappa'(X) \leq \delta(X)$.

If $X = K_n$ or if $\kappa'(X) = 1$, then the inequality $\kappa(X) \leq \kappa'(X)$ holds as well. Assume that X is not a complete graph and $\kappa(X) \geq 2$. Let $x_1y_1, \ldots, x_ky_k$ be a set of $k = \kappa'(X)$ edges whose removal disconnects

X. If removing $\{x_1, \ldots, x_k\}$ disconnects X, then $\kappa(X) \leq k = \kappa'(X)$ and we are done. Otherwise, it means that the degree of each x_i is at most k which implies that $\kappa(X) \leq k$. ■

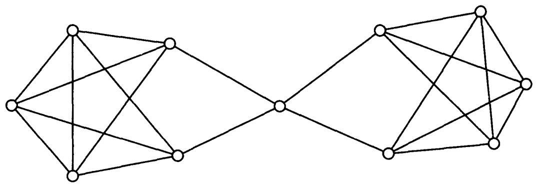

FIGURE 8.2. A 4-regular graph with $\kappa = 1$ and $\kappa' = 2$

A graph X is called k-connected if $\kappa(X) \geq k$. This means that the deletion of any $k-1$ vertices of X will not disconnect X. Similarly, X is called k-edge-connected if $\kappa'(X) \geq k$. Thus, a graph is 1-connected if and only if it is connected. The following result provides a necessary and sufficient condition 2-connectivity. We leave its proof as an exercise.

THEOREM 8.7.2. *A graph X is 2-connected if and only if any two vertices of X lie on a common cycle.*

The fundamental result involving graph connectivity was proved by Karl Menger (1902-1985) in 1927. Menger's theorem is an example of a min-max theorem. Given a graph X and two vertices $x \neq y$ of X, let $\kappa(x, y)$ denote the minimum number of vertices of X whose removal separates x from y. Also, two paths from x to y are called **independent** if they have only x and y in common.

THEOREM 8.7.3. *(a) Let x and y be two distinct nonadjacent vertices of a graph X. Then $\kappa(x, y)$ equals the minimum number of independent paths from x to y.*
(b) Let x and y be two vertices of X. Then the minimal number of edges whose removal separates x from y equals the minimum number of edge-disjoint paths from x to y.

PROOF. One inequality is obvious. If there are r independent paths from x to y, then deleting exactly one internal vertex from each path will separate x from y. The other inequality is left as an exercise. ■

Menger's theorem gives the following necessary and sufficient for a graph to be k-connected or k-edge-connected.

COROLLARY 8.7.4. *(a) For $k \geq 2$, a graph X is k-connected if and only if it has at least two vertices and there are k independent paths between any two vertices.*
(b) For $k \geq 2$, a graph X is k-edge-connected if and only if it has at least two vertices and there are k edge-disjoint paths between any two vertices.

Menger's theorem is a very powerful result with many consequences in discrete mathematics. The interested reader may try to apply it to prove the Marriage Theorem for example.

8.8. Exercises

EXERCISE 8.8.1. A building contractor advertises for a bricklayer, a carpenter, a plumber and a toolmaker; he has five applicants - one for the job of bricklayer, one for the job of carpenter, one for the jobs of bricklayer and plumber, and two for the jobs of plumber and toolmaker. Can the jobs be filled? In how many ways?

EXERCISE 8.8.2. If in a party, every male knows at least k females and every female knows at most k males, show that it is possible to match every male with a female he knows.

EXERCISE 8.8.3. A **permutation matrix** is a $0,1$ matrix having exactly one 1 in each row and column. Prove that a square matrix of non-negative integers can be expressed as a sum of k permutation matrices if and only if all row sums and column sums are equal to k.

EXERCISE 8.8.4. Let $X = (A, B)$ be a bipartite graph and suppose that A satisfies Hall's condition. Suppose further that each vertex of A is joined to at least t elements of B. Show that the number of matchings that saturate A is at least $t!$ if $t \leq |A|$.

EXERCISE 8.8.5. Show that there are at least $n!(n-1)!\cdots 2!1!$ Latin squares of order n. Show that this quantity is larger than $2^{(n-1)^2}$ for $n \geq 5$.

EXERCISE 8.8.6. There are rs couples in a party. The men are divided into r age groups with s men in each group. The women are divided into r height groups with s women in each group. Show that it is possible to select r couples so that all age groups and all height groups are represented.

EXERCISE 8.8.7. Find a minimum weight transversal in the matrix below.

$$\begin{pmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{pmatrix}.$$

EXERCISE 8.8.8. Determine whether or not the graph in Figure 8.3 has a perfect matching. If not, what is the size of a largest matching ?

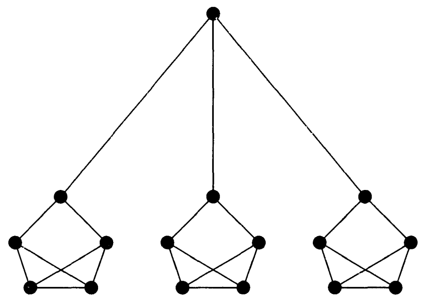

FIGURE 8.3

EXERCISE 8.8.9. For each $k \geq 2$, construct a k-regular graph on an even number vertices containing no perfect matchings. For each $k \geq 3$, construct k-regular graphs X such that $1 \leq \kappa(X) < \kappa'(X) < k$.

EXERCISE 8.8.10. Show that in the complete graph K_{2n} the number of perfect matchings is $(2n)!/2^n n!$.

EXERCISE 8.8.11. Let $W = (w_{ij})$ an $n \times n$ matrix of non-negative weights. Define a function f on the set of $n \times n$ doubly stochastic matrices by setting for $A = (a_{ij})$,

$$f(A) = \sum_{i,j} a_{ij} w_{ij}$$

where the summation is over all indices i, j. Show that f attains its maximum value at a permutation matrix.

EXERCISE 8.8.12. Let $t \geq 0$ be an integer. If X is bipartite graph with bipartite sets A and B such that $|N(S)| \geq |S| - t$ for each $S \subset A$, then X contains a matching that saturates $|A| - t$ vertices of A.

EXERCISE 8.8.13. Let $t \geq 1$ be an integer. If X is bipartite graph with bipartite sets A and B such that $|N(S)| \geq t \cdot |S|$ for each $S \subset A$, then each $a \in A$ has a set S_a of t neighbours in B with $S_a \cap S_{a'} = \emptyset$ for each $a \neq a' \in A$.

EXERCISE 8.8.14. Let A be a matrix with entries 0 or 1. Show that the minimum number of rows and columns that contain all the 1's of A equals the maximum number of 1's in A, no two on the same row or column.

EXERCISE 8.8.15. Finish the proof of Theorem 8.6.1.

EXERCISE 8.8.16. Show that any 3-regular graph with no bridges contains a perfect matching.

EXERCISE 8.8.17. Prove that every tree has at most one perfect matching.

EXERCISE 8.8.18. Show that a tree T has a perfect matching if and only if $\mathrm{odd}(T \setminus x) = 1$ for any vertex x of T.

EXERCISE 8.8.19. Let X be a bipartite graph with bipartite sets A and B such that $|N(S)| > |S|$ for each $S \subset A$. Show that for any edge e of X, there exists a matching that contains e and saturates A.

EXERCISE 8.8.20. Let $V_1, \ldots, V_n$ be subsets of a vector space V. Then $V_1, \ldots, V_n$ has a linearly independent system of distinct representatives if and only if

$$\dim(\cup_{i \in I} V_i) \geq |I|$$

for each $I \subset [n]$.