CHAPTER 7

Enumeration under Group Action

7.1. The Orbit-Stabilizer Formula

Let G be a group and X a set. We say G **acts** on X if there is a map $G \times X \to X$ (usually denoted by $(g, x) \mapsto g \cdot x$) satisfying the following axioms for all $x \in X$:

(1) $1 \cdot x = x$, where 1 denotes the identity of G;
(2) $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$.

Here are a few examples.

(1) If G is a group and H is a subgroup, let X be the set of left cosets of H in G. Then G acts on X via $g(aH) = (ga)H$.
(2) If G is a group and we let X be G itself, then G acts on itself via conjugation: $g \cdot x = gxg^{-1}$.
(3) Let p be prime and $G = \mathbf{Z}/p\mathbf{Z}$ be the additive group of residue classes $[a]$ mod p. Let X be the set of all p-tuples $(x_1, x_2, ..., x_p)$ where $x_i \in \{1, 2, ..., n\}$. Since G is cyclic, it suffices to define how $[1]$ acts on X. We put

$$[1] \cdot (x_1, x_2, ..., x_p) = (x_p, x_1, ..., x_{p-1}).$$

In other words, $[1]$ acts like a shift operator, shifting the coordinates by one component.
(4) Let n be a natural number and $G = \mathbf{Z}/n\mathbf{Z}$. Let X be the set of all n-tuples $(x_1, ..., x_n)$ where $x_i \in \{1, 2, ..., \lambda\}$. We define

$$[1] \cdot (x_1, x_2, ..., x_n) = (x_n, x_1, ..., x_{n-1}).$$

We can view the set X as all the possible "necklaces" formed by using beads of λ colours. This perspective will be useful in later applications.

It will be convenient to simplify our notation slightly. Instead of writing $g \cdot x$, we will simply write gx, when it is clear that $g \in G$ and $x \in X$. An action of G on X determines an equivalence relation on X as follows. Namely, we will write $x \sim y$ if there is an element $g \in G$ such that $gx = y$. Thus, if $gx = y$ then $x = g^{-1}y$ and $y \sim x$. Since $1x = x$, this means that $x \sim x$. Also, it is easy to check that $x \sim y$ and

$y \sim z$ implies $x \sim z$. Therefore, $\sim$ defines an equivalence relation on X. Consequently, we can partition X into equivalence classes, which we call **orbits**. More precisely, if we use the notation Gx to signify the set

$$\{gx : g \in G\}$$

then it is clear that the equivalence classes consist of sets of the form Gx_i for various x_i's.

If G and X are finite, it is natural to ask how many elements are there in each orbit and how many equivalence classes there are. We begin with the first question. We begin by listing the $|G|$ elements

$$gx : \quad g \in G \tag{7.1.1}$$

and ask how many times an element gets repeated. Indeed, $gx = hx$ if and only if $h^{-1}gx = x$, that is if and only if $h^{-1}g$ fixes x.

This leads to the notion of the **stabilizer** of x, denoted G_x, and defined as the set of elements of G fixing x. It is easy to see that the stabilizer of x is a subgroup of G for any $x \in X$. In the context above, we see that $gx = hx$ if and only if $h^{-1}g$ lies in G_x. In other words, $gx = hx$ if and only if $gG_x = hG_x$. Thus, in the listing (7.1.1), each element is repeated the same number of times, namely $|G_x|$ times so that the number of distinct elements is $[G : G_x]$. As the set X is partitioned into its orbits, we see that there are elements x_i's so that

$$X = \cup_{i=1}^{t} Gx_i.$$

For each subgroup H of G we define fix(H) to be the set of H-fixed points of X. That is

$$\text{fix}(H) = \{x \in X : \quad hx = x \quad \forall h \in H\}.$$

If $g \in G$, we simply write fix(g) for the set of elements fixed by the subgroup generated by g. From the above relation, we separate those x_i's for which Gx_i consists of singleton sets. In other words, we obtain:

THEOREM 7.1.1 (Orbit-Stabilizer formula). *If G is a finite group acting on a finite set X, we have*

$$|X| = |\text{fix}(G)| + \sum_{G_{x_i} \neq G} [G : G_{x_i}].$$

This formula is of central importance in mathematics and has numerous applications. For instance, in the case a group G acts on itself via conjugation, we get:

COROLLARY 7.1.2 (The class equation). *Let G act on itself via conjugation. Let $Z(G) = \{g : gx = xg, \forall x \in G\}$ denote its* **center** *and $C(x) = \{g \in G : gx = xg\}$ be the* **centralizer** *of x in G. Then,*

$$|G| = |Z(G)| + \sum_{x \notin Z(G)} [G : C(x)].$$

PROOF. We see immediately that x is a G-fixed point if and only if $x \in Z(G)$. Moreover, the stabilizer of any element x is $C(x)$. The formula is now immediate from the orbit-stabilizer formula applied to this specific case. ■

If we apply the orbit-stabilizer formula to Example 3 above, we see that on one hand, we have n^p elements in X and on the other, the set of fixed elements is easily seen to be of size n. Now every summand in the sum is p since $\mathbf{Z}/p\mathbf{Z}$ has no non-trivial subgroups. We recover the following result:

THEOREM 7.1.3 (Fermat's little theorem). *If p is a prime number, then p divides $n^p - n$ for each integer n.*

A less trivial application by considering the following situation. Let G be a group of order n and consider

$$X = \{(x_1, ..., x_p) : \quad x_1 \cdots x_p = 1, \quad x_i \in G\}.$$

The size of X is n^{p-1} since we may choose each of $x_1, ..., x_{p-1}$ in n ways, then x_p is uniquely determined by the equation

$$x_1 \cdots x_p = 1.$$

We let the additive group $\mathbf{Z}/p\mathbf{Z}$ act on X by setting

$$[1] \cdot (x_1, ..., x_p) = (x_p, x_1, ..., x_{p-1}).$$

Note that the set of fixed points consists of elements $(x, x, ..., x)$ with $x^p = 1$. If p is a prime divisor of n, the orbit-stabilizer formula immediately gives that the number of fixed points is divisible by p. Since $\text{fix}(G) \neq \emptyset$ (why?), it follows that G has an element of order p. This is usually referred to as Cauchy's theorem. We record this as:

COROLLARY 7.1.4 (Cauchy, 1845). *If G is a group of order n and p is a prime dividing n, then G has an element of order p.*

However, much more is true. Cauchy's theorem was generalized by Peter Ludwig Sylow (1832-1918) in 1872. Almost all work on finite groups use Sylow's theorems. The class equation enables us to deduce the first Sylow theorem, namely:

COROLLARY 7.1.5 (Sylow's First Theorem). *If G is a group of order n and p^k is a prime power dividing n, then G has a subgroup of order p^k.*

PROOF. We proceed by induction on $|G|$. If $|G| = 2$, the theorem is true.

Let $|G| = p^r m$, where $r \geq k$ and m and p are coprime. If $x \in G$ and p^k divides $|C(x)|$, then we are done by induction.

Otherwise, because every summand in the sum occurring in the class equation is divisible by p, we deduce that p divides the order of the center $Z(G)$. By Cauchy's theorem, $Z(G)$ has an element x of order p. The subgroup generated by x in G is normal since $x \in Z(G)$. The quotient $G/\langle x\rangle$ has order divisible by p^{k-1} and by induction has a subgroup $H/\langle x\rangle$ of order p^{k-1}. By the correspondence theorem, H is a subgroup of G of order p^k, as desired. ∎

We remark that all of the Sylow theorems can be derived by considering appropriate group action. Recall the notion of a p-Sylow subgroup. If p^k is the largest power of a prime number p dividing the order of G, and P is a subgroup of order p^k, we call P a p-**Sylow subgroup** of G. The **normalizer** of a subgroup H of G is $N(H) = \{g : g \in G, gHg^{-1} = H\}$.

COROLLARY 7.1.6 (Sylow's Second Theorem). *Let G be a finite group of order n and P a p-Sylow subgroup of G. Let X be the set of p-Sylow subgroups of G and let P act on X via conjugation. Then, P is the only fixed point under this action. Thus, the number of p-Sylow subgroups is $\equiv 1(\mod p)$ and all of the p-Sylow subgroups are conjugates of P. Moreover, any p-subgroup of G is contained in some conjugate of P.*

PROOF. Suppose Q is another p-Sylow subgroup fixed by P. Then, $gQg^{-1} = Q$ for all $g \in P$. Take $x \in P\backslash Q$. Then, x is in the normalizer $N(Q)$. But $N(Q)$ contains Q and the coset xQ is not Q. As the quotient $N(Q)/Q$ has order coprime to p, the coset xQ has order k coprime to p. Thus, for some k, $x^k \in Q$ with $(k,p) = 1$. But x has order equal to some prime power p^b (say). So we can find integers u, v so that $ku + p^b v = 1$. Hence, $x = x^{ku+p^b v} \in Q$, contrary to hypothesis. As the set X is partitioned into orbits under the action of P, we deduce immediately that the number of elements of X is $\equiv 1 \pmod p$. Now let Y be the set of conjugates of P. Let H be a p-subgroup of G. Then H acts on Y. If H fixes an element Q of Y, then H is in the normalizer of Q. If H is not contained in Q, then the argument above gives us a contradiction. Thus every p-subgroup H is contained in some conjugate of P. In particular, if H is another p-Sylow subgroup, this means that it is conjugate to P. This completes the proof. ∎

A **p-group** is a group whose order is a power of p where p is a prime number. We remark that any p-group G has subgroups of all orders dividing $|G|$. Indeed, the class equation implies the non-triviality of the center. By Cauchy's theorem, we may take an element z in the center of order p and consider the quotient $G/\langle z\rangle$. By induction, this has subgroups of all orders dividing $|G|/p$ which by the correspondence theorem give subgroups of the required order in G. For an arbitrary group G, and any prime power p^t dividing $|G|$, we deduce that G has subgroups of order p^t. Moreover, one can show that the number of these subgroups is $\equiv 1(\mod p)$, but we leave this as an exercise.

Given a finite group G of order n, and a subgroup H of G, we can partition G into the cosets of H from which we see **Lagrange's theorem**, namely that the order of any subgroup is a divisor of the order of G. The converse is not true, as is seen by considering the alternating group A_4 on 4 letters. These are the even permutations of S_4 and one can list the elements:

$$(1),(1\,2)(3\,4),(1\,3)(2\,4),(1\,4)(2\,3),(1\,2\,3),(1\,3\,2),(1\,2\,4),(1\,4\,2)$$

$$(2\,3\,4),(2\,4\,3),(3\,4\,1),(3\,1\,4).$$

If A_4 had a subgroup H of order 6, then this subgroup is necessarily normal which means that the square of any element of A_4 lies in H. In particular, the square of any 3-cycle g is in H. But $g = (g^2)^2$ lies in H so that all 3-cycles must lie in H, a contradiction since there are 8 3-cycles. The virtue of Sylow theory is that it shows that the converse of Lagrange's theorem holds for prime powers dividing the order of the group.

7.2. Burnside's Lemma

It is possible to derive a formula for the number of equivalence classes under a group action. This is called Burnside's lemma as William Burnside (1852-1927) wrote about it in 1900. The result was known before Burnside mentioned it as it appears in the works of Augustin Louis Cauchy (1789-1857) in 1845 and of Ferdinand Georg Frobenius (1849-1917) in 1887.

We will apply the next result to the problem of counting necklaces encountered in the previous chapter.

THEOREM 7.2.1 (Burnside's lemma). *If G is a finite group acting on a set X, the number of equivalence classes is*

$$\frac{1}{|G|}\sum_{g\in G}|\text{fix}(g)|.$$

In other words, the number of equivalence classes is the average number of fixed points.

PROOF. The equivalence class of an element x of X is the orbit of x. Thus, if $w(x)$ is $1/|Gx|$, we see that the number of equivalence classes is

$$\sum_{x\in X} w(x).$$

On the other hand, this is

$$\sum_{x\in X}\frac{1}{|G|}|G_x| = \frac{1}{|G|}\sum_{x\in X}\sum_{g\in G_x} 1 = \frac{1}{|G|}\sum_{x\in X}\sum_{g\in G:\, gx=x} 1.$$

By interchanging the sum, we find this is

$$\frac{1}{|G|}\sum_{g\in G}\sum_{x\in X: gx=x} 1 = \frac{1}{|G|}\sum_{g\in G}|\text{fix}(g)|.$$

This completes the proof. ∎

COROLLARY 7.2.2. *The number of conjugacy classes in a group is*

$$\frac{1}{|G|}\sum_{g\in G}|C(g)|.$$

PROOF. The number of fixed points of $g\in G$ is precisely $|C(g)|$. ∎

Let us apply this to the problem of counting necklaces. Each necklace of length n formed out of beads of λ colours can be viewed as a sequence $(a_1, ..., a_n)$ with $a_i \in \{1, 2, ...\lambda\}$. Two necklaces are considered the same if the two sequences representing them are the same after a shift. In other words, $\mathbf{Z}/n\mathbf{Z}$ acts on the sequences and the number of necklaces is precisely the number of equivalence classes under this action. Now, how many fixed points does an element r of $\mathbf{Z}/n\mathbf{Z}$ have? A sequence $(a_1, ..., a_n)$ is fixed r if and only if

$$a_{i+tr} = a_i$$

for all t and all i. In other words,

$$a_{i+u} = a_i$$

for all i and all u lying in the subgroup generated by r in $\mathbf{Z}/n\mathbf{Z}$. Since $\mathbf{Z}/n\mathbf{Z}$ is cyclic, any subgroup is also cyclic so the number of fixed points of r is $\lambda^{n/o(r)}$ where $o(r)$ is the order of r mod n. Recall that in any cyclic group of order n, the number of elements of order $d|n$ is precisely $\phi(d)$. Thus, the number of necklaces is

$$\frac{1}{n}\sum_{r}\lambda^{n/o(r)} = \frac{1}{n}\sum_{d|n}\phi(d)\lambda^{n/d} = \frac{1}{n}\sum_{d|n}\phi(n/d)\lambda^{d}.$$

7.3. Pólya Theory

George Pólya (1887-1985) was one of the most influential mathematicians of the 20th century.

The action of a group G on a set X can be viewed as a map

$$G \to \mathrm{Sym}(X)$$

where we send each element $g \in G$ to the permutation $x \mapsto gx$ since $gx = gy$ implies $x = y$ by the axioms of action. In this way, we may view each element of G as a permutation and so we can consider its cycle decomposition as a product of disjoint cycles. Suppose g has c_1 cycles of length 1, c_2 cycles of length 2 ..., c_n cycles of length n where $n = |X|$. The **cycle index** of g is defined to be the monomial

$$x_1^{c_1}x_2^{c_2}\cdots x_n^{c_n}$$

which we symbolically denote by x^g. The **cycle index** of G is defined to be the polynomial

$$P_G(x) = \frac{1}{|G|}\sum_{g\in G} x^g.$$

The situation can be looked at in another way. If G acts on X and we have a map $f : X \to Y$, we may view Y as a set of **colours**. Then, the action of G on X induces an action of G on $\mathrm{Map}(X;Y)$, the set of maps from X to Y as follows:

$$(g \cdot f)(x) = f(g^{-1}x).$$

It is important to check that this is indeed an action: we have for $x \in X$,

$$[(gh)f](x) = f((gh)^{-1}x) = f(h^{-1}g^{-1}x).$$

On the other hand,

$$[g(hf)](x) = (hf)(g^{-1}x) = f(h^{-1}g^{-1}x),$$

as desired. Burnside's lemma immediately implies the following.

THEOREM 7.3.1 (Pólya). *Let X and Y be finite sets and G act on X. The number of orbits of G on* $\mathrm{Map}(X;Y)$ *is*

$$\frac{1}{|G|}\sum_{k=1}^{\infty} c_k(G)|Y|^k,$$

where $c_k(G)$ is the number of elements of G with exactly k disjoint cycles in their cycle decomposition.

REMARK 7.3.2. Notice that this number is simply $P_G(|Y|,|Y|,\ldots)$.

PROOF. To apply Burnside's lemma, we must count the number of fixed points of an element g on $\mathrm{Map}(X;Y)$. That is, we must count the number of maps $f : X \to Y$ such that $gf = f$. This means that f is constant on each orbit of g. The number of orbits is the number of disjoint cycles in the cycle decomposition of g. We may assign values of f arbitrarily on each orbit, so the final count is given as stated in the theorem. ∎

If we let Y denote the set of λ colours of beads, and X denotes the set $\{1,2,\ldots,n\}$, then a sequence $(a_1,\ldots,a_n)$ of length n can then be viewed as a map f from X to Y. As the group $\mathbf{Z}/n\mathbf{Z}$ acts on the co-ordinates in the obvious way by shifting, this induces an action on $\mathrm{Map}(X;Y)$. We see then that the maps that correspond to distinct necklaces are equivalence classes of maps under this induced action.

We can retrieve our result about the necklace count from the previous section in the following way. First, we must determine the cycle structure of a residue class r viewed as a permutation. Clearly, all orbits have the same length and if $o(r)$ denotes the order of r, then each orbit has size $o(r)$ and the number of disjoint cycles is $n/o(r)$. Hence, the number of elements of $\mathbf{Z}/n\mathbf{Z}$ with exactly k cycles is zero unless $k|n$, in which case it is the number of elements of order n/k. The number of such elements is $\phi(n/k)$, as we saw before.

Now suppose we have the dihedral group D_n acting on the necklace sequences. Thus, if we present D_n as

$$\langle r, f : r^n = 1, f^2 = 1, frf = r^{-1}\rangle.$$

We could try to count the number of equivalence classes by using Burnside's formula. To use Burnside's formula, we have to count the number of fixed points of each element of D_n. It is better to use the cycle index polynomial to determine the number of equivalence classes. We illustrate this as follows.

Firstly, let us have a geometric view of the dihedral group. It is to be viewed as the group of symmetries of a regular n-gon. If we fix any vertex, and bisect the interior angle subtended at that vertex, we can view the element f as the flip of the polygon about this axis. We can view the elements fr^j as flips about the axis determined by the other points. If n is odd, each of these elements fixes one vertex and transposes pairs of vertices which are mirror images about that axis. Thus, the cycle structure of fr^j is that it is a product of one one-cycle and $(n-1)/2$ transpositions. Thus, in the case of n odd, the cycle index polynomial is easily seen to be

$$\frac{1}{2n}\left(\sum_{d|n}\phi(d)x_d^{n/d}+nx_1x_2^{(n-1)/2}\right).$$

Now we consider the case n even. As noted above, there are two axes of symmetry. The elements fr^j with j odd correspond to flipping through an axis through a vertex. In this case, it is seen that the opposite vertex is also fixed. In this way, we see the cycle decomposition is a product of $(n-2)/2$ transpositions and 2 1-cycles. If j is even, there are no fixed points and the cycle decomposition of fr^j is simply a product of $n/2$ transpositions. In this case, the cycle index polynomial is

$$\frac{1}{2n}\left(\sum_{d|n}\phi(d)x_d^{n/d}+\frac{n}{2}x_1^2x_2^{(n-2)/2}+\frac{n}{2}x_2^{n/2}\right).$$

Pólya's theorem now tells us that the number of equivalence classes of maps is $P_G(\lambda,\lambda,\ldots)$ where λ is the number of elements of Y. This shows:

THEOREM 7.3.3. *Under the action of the dihedral group, the number of distinct necklaces of length n formed using beads of λ colours is*

$$\frac{1}{2}\left(\sum_{d|n}\phi(n/d)\lambda^d+\lambda^{(n+1)/2}\right)$$

if n is odd and

$$\frac{1}{2}\left(\sum_{d|n}\phi(n/d)\lambda^d+\frac{1}{2}\lambda^{(n+2)/2}+\frac{1}{2}\lambda^{n/2}\right)$$

if n is even.

We conclude this section with one application of Pólya theory to chemistry. It seems that the historic origins of the theory are rooted in problems arising in chemistry.

The methane molecule has chemical composition CH_4 where C denotes a carbon atom and H is a hydrogen atom. This molecule has tetrahedral shape and the H_4 indicates that there are 4 atoms of hydrogen in the molecule positioned at the vertices of the tetrahedron, with the carbon atom at the centroid. The problem is to determine how many different molecules can be formed by replacing the hydrogen atoms with one of bromine, chlorine or fluorine. This question can be re-interpreted in the context of the colouring problems considered by Pólya theory.

Indeed, the group of symmetries of the regular tetrahedron is A_4, the alternating group on 4 letters. To see this, observe that we can rotate the tetrahedron about the center of any face and each of these correspond to 3-cycles, one for each face. This gives us a total of 8 3-cycles in the group of symmetries. There is one more symmetry given by a rotation by 180 degrees about the axis joining the center of opposite sides. This is easily seen to be a product of two transpositions and there are 3 such permutations. Together with the identity, we have the full group of symmetries.

It is now straightforward to write down the cycle index polynomial of the action of A_4 on the vertices of the regular tetrahedron. From the discussion above, we have

$$P_{A_4}(x_1, x_2, x_3, x_4) = \frac{1}{12}\left(x_1^4 + 8x_1x_3 + 3x_2^2\right).$$

The number of different molecules is then seen to be $P_{A_4}(3,3,3,3) = 15$. If the group of symmetries are not taken into account, we have $3^4 = 81$ ways of placing the atoms of bromine, chlorine or fluorine at the vertices of the tetrahedron. However, many of them clearly give the same molecule.

We make a few additional remarks concerning Pólya's theorem. In the special case that $G = S_n$ acting on the set $\{1, 2, ..., n\}$ in the usual way, the cycle index polynomial $P_{S_n}(\lambda, ...\lambda)$ is

$$\frac{1}{n!}\sum_{k=0}^{n} |s(n,k)|\lambda^k$$

where the $s(n,k)$'s denote the Stirling numbers of the first kind. This represents the number of ways of colouring n indistinguishable objects (or balls) using λ colours. This is related to a problem treated earlier by simpler methods. Indeed, this is the same as asking in how many ways we may put n indistinguishable balls into λ boxes. This is the same as the number of solutions of

$$x_1 + x_2 + \cdots + x_\lambda = n$$

with the x_i's non-negative integers. In either interpretation, it is easily seen that the number of ways is

$$\binom{n+\lambda-1}{\lambda-1}.$$

Indeed, if we first consider a collection of n distinguishable balls and we throw into this collection $\lambda-1$ indistinguishable "sticks", then the number ways we can arrange these objects is clearly

$$(n+\lambda-1)!.$$

However, $\lambda-1$ of these objects are identical and can be permuted in $(\lambda-1)!$ ways and so we get our result. Now if we say the balls are also indistinguishable, then we can permute these among themselves in $n!$ ways. In this way, we retrieve an earlier formula, namely,

$$(\lambda+n-1)(\lambda+n-2)\cdots\lambda=\sum_{k=0}^{n}|s(n,k)|\lambda^k.$$

If we change λ to $-\lambda$, we get

$$(\lambda)_n=\sum_{k=0}^{n}s(n,k)\lambda^k.$$

Two further applications of the Pólya theory are amusing. The game of tic-tac-toe involves a 3×3 grid in which the players place alternately x or o until a row, column or diagonal of the same symbols are placed and the game is over. It is interesting to consider how many possible configurations can be seen at any given moment during a game. Or even, one may ask how many possible outcomes are there. This in its generality is too difficult to answer. We will consider a simpler problem. Namely, in how many ways can we colour a 3×3 grid using three colours. We can see that the cyclic group of order 4 operates by rotation on such a grid. If we label the grid as

1	2	3
6	5	4
7	8	9

then a clockwise rotation r is represented by the permutation

$$(1\,3\,9\,7)(2\,4\,8\,6)(5)$$

whereas r^2 is given by

$$(1\,9)(2\,8)(3\,7)(6\,4)(5).$$

Note that r^3 has the same cycle structure as r and so we easily see that the cycle index polynomial is

$$P(x_1, ..., x_9) = \frac{1}{4}\left(x_1^9 + 2x_1x_4^2 + x_1x_2^4\right).$$

A simple calculation shows that the number of colourings with three colours is 4995. Of course, some of these never can represent the final outcome or the shape of the grid during the game. For such a computation, one needs a finer Pólya theory with weights, which we do not consider here.

Let us now consider the problem of colouring the faces of the cube using λ colours. To do this, we begin by considering the group of symmetries of the cube. These can be classified as follows.

(1) the identity element;
(2) rotation by 90 degrees about the axis joining the center of two opposite faces;
(3) rotation by 180 degrees about the same axis;
(4) rotation by 180 degrees about the axis joining the midpoints of two diagonally opposite edges;
(5) rotation by 120 degrees about the axis determined by the diagonal of the cube.

If we think of these symmetries as acting on the faces, and write down the cycle structure, we obtain the following:

(1) 1 element of type 1^6;
(2) 6 elements of type $1^2 4^1$;
(3) 3 elements of type $1^2 2^2$;
(4) 6 elements of type 2^3;
(5) 8 elements of type 3^2.

Thus, we see the group of symmetries has order 24. One can easily see that this group is isomorphic to S_4. We can immediately write down the cycle index polynomial for S_4 acting on the faces of the cube from the above analysis:

$$P_{S_4}(x_1, ..., x_6) = \frac{1}{24}\left(x_1^6 + 6x_1^2x_4 + 3x_1^2x_2^2 + 6x_2^3 + 8x_3^2\right).$$

By Pólya's theorem, the number of ways of colouring the faces of the cube using λ colours is

$$\frac{1}{24}\left(\lambda^6 + 12\lambda^3 + 3\lambda^4 + 8\lambda^2\right).$$

In particular, there are 10 ways of colouring the faces of the cube using 2 colours.

7.4. Exercises

EXERCISE 7.4.1. Show that the examples from the first section are actions of groups.

EXERCISE 7.4.2. Let G be a finite group acting on a finite set X. For each $g \in G$, define $\sigma_g(x) = g \cdot x$ for each $x \in X$. Show that σ_g is a permutation of X.

EXERCISE 7.4.3. Show that the map

$$g \mapsto \sigma_g$$

is a group homomorphism from G into $\mathrm{Sym}(X)$ which is the group of permutations of the set X.

EXERCISE 7.4.4. Let G be a group acting on a set X and H a group acting on a set Y. Assume that X and Y are disjoint and let $U = X \cup Y$. For $g \in G, h \in H$, define

$$(g, h) \cdot x := g \cdot x \,\mathrm{if}\, x \in X$$

and

$$(g, h) \cdot y := h \cdot y \,\mathrm{if}\, y \in Y.$$

Show that this defines an action of $G \times H$ on U.

EXERCISE 7.4.5. Determine the number of ways in which four corners of a square can be coloured using two colours. It is permissible to use single colour on all four corners.

EXERCISE 7.4.6. In how many ways can you colour the four corners of a square using three colours ?

EXERCISE 7.4.7. If $X = [3]$, define an action of S_3 on X by $\sigma \cdot i = \sigma(i)$ for $i \in X$ and $\sigma \in S_3$. Calculate the cycle index polynomial $P_{S_3}(x_1, x_2, x_3)$.

EXERCISE 7.4.8. In how many ways can you colour the vertices of an equilateral triangle so that at least two colours are used ?

EXERCISE 7.4.9. What is the number of graphs on 4 vertices ? What is the number of nonisomorphic graphs on 4 vertices ?

EXERCISE 7.4.10. Let G and H be finite groups acting on finite sets X and Y. Assume that X and Y are disjoint. By Exercise 7.4.3, we can define an action of $G \times H$ on $X \cup Y$. If P_G and P_H indicate the cycle index polynomials of G acting on X and H acting on Y respectively, show that the cycle index polynomial of $G \times H$ acting on $X \cup Y$ is $P_G P_H$.

EXERCISE 7.4.11. How many striped flags are there having six stripes (of equal width) each of which can be coloured red, white or blue ?

EXERCISE 7.4.12. What if we change the number of stripes to n and the number of colours to q ?

EXERCISE 7.4.13. Let S_n acting on the set $X = [n]$ in the usual way (as in Exercise 7.4.1). Let P_{S_n} be the cycle index polynomial. Prove that P_{S_n} is the coefficient of z^n in the power series expansion of

$$\exp(zx_1 + z^2x_2/2 + z^3x_3/3 + \dots).$$

EXERCISE 7.4.14. We say that $\sigma \in S_n$ has cycle type $(c_1, \dots, c_n)$ if σ has precisely c_i cycles of length i in its unique decomposition as a product of disjoint cycles. Show that the number of permutations of type $(c_1, c_2, \dots, c_n)$ is

$$\frac{n!}{1^{c_1}c_1!2^{c_2}c_2!\dots n^{c_n}c_n!}.$$

EXERCISE 7.4.15. Let P_n denote the path on n vertices. What is the automorphism group $\mathrm{Aut}(P_n)$ of P_n ?

EXERCISE 7.4.16. What is the cycle index polynomial of $\mathrm{Aut}(P_n)$ acting on the vertex set of P_n ?

EXERCISE 7.4.17. In how many ways can we colour the vertices of P_n using λ colours, up to the symmetry of $\mathrm{Aut}(P_n)$?

EXERCISE 7.4.18. Consider the graph X on 5 vertices obtained from the complete graph K_5 by deleting two edges incident to the same vertex. What is the automorphism group $\mathrm{Aut}(X)$ of X ?

EXERCISE 7.4.19. Let X be the graph from Exercise 7.4.18. What is the cycle index polynomial of $\mathrm{Aut}(X)$ acting on the vertex set of X ?

EXERCISE 7.4.20. In how many ways can we colour the vertices of X using λ colours, up to the symmetry of $\mathrm{Aut}(X)$?