CHAPTER 4

Matrices and Graphs

4.1. Adjacency and Incidence Matrices

Given a graph X, we associate two matrices to encode its information. The first is the **adjacency matrix** A or sometimes denoted A_X or $A(X)$. If n is the number of vertices of X, then A is an $n \times n$ matrix whose (i, j)-th entry is the number of edges between i and j. In case X is a simple graph, this is simply a $(0, 1)$ matrix whose i, j-th entry is 1 or 0 according as i is joined to j.

THEOREM 4.1.1. *The (i, j)-th entry of A^m is the number of walks of length m from i to j.*

PROOF. We prove this by induction. For $m = 1$, this is clear from the definition. Suppose we have proved it for A^j for $j \leq m - 1$. Write $A^r = (a_{i,j}^{(r)})$. Since $A^m = A^{m-1} \cdot A$, we have

$$a_{ij}^{(m)} = \sum_{k=1}^{n} a_{ik}^{(m-1)} a_{kj}.$$

Clearly, the number of paths from i to j of length m is

$$\sum_{k=1}^{n} (\#\text{of paths from } i \text{ to } k \text{ of length } m-1)\, a_{kj}.$$

By induction, the number of paths from i to k of length $m - 1$ is $a_{ik}^{(m-1)}$ which proves the theorem. ∎

There is another matrix M called the **incidence matrix** of the graph. If X has n vertices and e edges, then M is a an $n \times e$ matrix defined as follows. The (i, j)-th entry is 1 if the vertex v_i is incident to the edge e_j, and 0 otherwise. The relationship between this matrix and the adjacency matrix is given by the easily verified equation

$$MM^t = D + A$$

where D is the diagonal matrix consisting of the vertex degrees.

S. M. Cioabă and M. R. Murty, *A First Course in Graph Theory and Combinatorics*

4.2. Graph Isomorphism

An **isomorphism** between two graphs X and Y is a bijection f between the vertex set of X and the vertex set of Y such that uv is an edge of X if and only if $f(u)f(v)$ is an edge of Y. The reader is invited to show that the graphs in Figure 4.1 are isomorphic. We will usually study isomorphism in the context of simple graphs. A moment's reflection shows that applying a permutation to both the rows and columns of the adjacency matrix of a graph X has the effect of reordering the vertices of X. A **permutation matrix** is a square 0, 1 matrix which has precisely one entry 1 in each row and each column and 0's elsewhere.

THEOREM 4.2.1. *The graphs X and Y are isomorphic if and only if there is a permutation matrix P such that*

$$PA_XP^{-1} = A_Y.$$

We begin by reviewing some elementary facts from linear algebra about matrices and their characteristic polynomials. Given a square matrix A, its **characteristic polynomial** is $\det(\lambda I - A)$. The roots of this polynomial are called **eigenvalues** of A. If λ is an eigenvalue and v is an eigenvector so that $Av = \lambda v$, then v is called an **eigenvector** corresponding to λ. Thus, for two graphs to be isomorphic, it

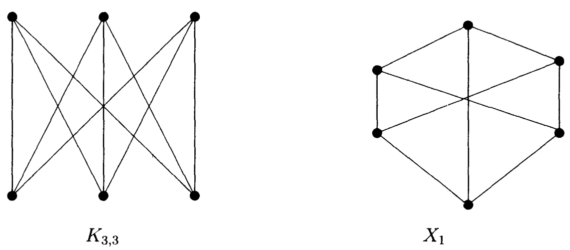

FIGURE 4.1

is necessary that their adjacency matrices have the same eigenvalues. However, this is not a sufficient condition for isomorphism. Consider the graph obtained from C_4 by adding an isolated vertex. This graph has the same eigenvalues as $K_{1,4}$, but it is obviously not isomorphic to $K_{1,4}$. See also Exercise 4.5.15 and Exercise 4.5.16.

EXAMPLE 4.2.2. Let us compute the characteristic polynomial of the n by n matrix J whose i,j-th entry is 1 for all $1 \leq i, j \leq n$. Clearly, it is a singular matrix (that is, its determinant is zero because the rows are linearly dependent). Any eigenvector $v = (x_1, ..., x_n)$ with eigenvalue λ satisfies $Jv = \lambda v$ so that

$$x_1 + \cdots + x_n = \lambda x_i,$$

for all $1 \leq i \leq n$. Clearly, $\lambda = n$ is an eigenvalue and $v = (1, 1, ..., 1)$ is a corresponding eigenvector. On the other hand, the subspace of vectors $v = (x_1, ..., x_n)$ satisfying the equation

$$x_1 + \cdots + x_n = 0$$

has dimension $n - 1$ and these vectors correspond to eigenvalue zero. Thus, the characteristic polynomial is $(\lambda - n)\lambda^{n-1}$.

EXAMPLE 4.2.3. Let us determine the characteristic polynomial of the complete graph K_n. The adjacency matrix of K_n is $J - I$ with J as in Example 4.2.2 and I is the identity matrix of order n. Now let us recall that if A has eigenvalue μ then $\mu + c$ is an eigenvalue of $A + cI$ because $\det(\lambda I - (A + cI)) = \det((\lambda - c)I - A)$. Thus, the eigenvalues of $J - I$ are $n - 1$ and -1 with multiplicity 1 and $n - 1$ respectively. Therefore, the characteristic polynomial of the complete graph on n vertices is $[\lambda - (n - 1)](\lambda + 1)^{n-1}$.

EXAMPLE 4.2.4. Let us determine the characteristic polynomial of the bipartite graph $K_{r,s}$. Since the adjacency matrix has form

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ & & \cdots & & & & \cdots & \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ & & \cdots & & & & \cdots & \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

it has rank 2. Recall now that the **rank** of a square matrix is equal to the number of non-zero eigenvalues counted with multiplicity. As our matrix has trace zero and this is also equal to the sum of the eigenvalues, we deduce that $A_{m,n}$ has only two non-zero eigenvalues λ_1, λ_2 with $\lambda_1 = -\lambda_2 = b$ (say). Moreover, each of these has multiplicity 1. Thus, the characteristic polynomial is (with $n = r + s$)

$$\lambda^{n-2}(\lambda^2 - b^2).$$

We can actually determine b more precisely. If we look at the definition of the characteristic polynomial as $\det(\lambda I - A_{r,s})$, we see that the coefficient of λ^{n-2} can be arrived at as follows. From the determinant expression, we must choose $(n - 2)$ diagonal entries and the other two entries must come from non-zero entries in order to contribute to the coefficient. This can also be seen from the formula for the determinant. The permutations that contribute must necessarily fix $(n-2)$ letters and thus correspond to transpositions. The remaining positions contribute $-a_{i,j}$ and $-a_{j,i}$ for some i, j. Since the graph is bipartite, there are rs non-zero contributions of this form. This means b^2 must be rs. Thus, the characteristic polynomial is

$$\lambda^{n-2}(\lambda - \sqrt{rs})(\lambda + \sqrt{rs}).$$

This can also be deduced in another (simpler) way. As we observed, the number of closed walks of length 2 is equal to the trace of the square of the adjacency matrix. In our bipartite case, this is clearly $2rs$, which must necessarily equal the sum of the squares of the eigenvalues, which is $2b^2$. Thus, $b^2 = rs$.

4.3. Bipartite Graphs and Matrices

The eigenvalues of bipartite graphs have the following interesting property.

THEOREM 4.3.1. *If X is bipartite, and λ is an eigenvalue with multiplicity m, then $-\lambda$ is also an eigenvalue of multiplicity m.*

PROOF. Since X is bipartite, we may arrange our rows and columns of $A = A_X$ according to the partite sets so that A has the following form

$$A = \begin{pmatrix} O & B \\ B^t & O \end{pmatrix}$$

where B is a $0, 1$ matrix. If λ is an eigenvalue with eigenvector

$$v = \begin{pmatrix} x \\ y \end{pmatrix},$$

partitioned according to the partite sets. We have

$$\lambda v = Av = \begin{pmatrix} By \\ B^t x \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

so that $By = \lambda x$ and $B^t x = \lambda y$. Let

$$v' = \begin{pmatrix} x \\ -y \end{pmatrix}.$$

Then

$$Av' = \begin{pmatrix} -By \\ B^t x \end{pmatrix} = \begin{pmatrix} -\lambda x \\ \lambda y \end{pmatrix} = -\lambda \begin{pmatrix} x \\ -y \end{pmatrix} = -\lambda v'.$$

Thus, v' is an eigenvector with eigenvalue $-\lambda$. Also, m independent eigenvectors corresponding to λ give m independent eigenvectors corresponding to $-\lambda$. This completes the proof. ∎

We can now characterize bipartite graphs by the shape of the characteristic polynomial.

THEOREM 4.3.2. *The following statements are equivalent.*

(1) *X is bipartite;*
(2) *The eigenvalues of X occur in pairs λ_i, λ_j such that $\lambda_i = -\lambda_j$;*
(3) *The characteristic polynomial of X is a polynomial in λ^2;*
(4) *for any positive integer t, $\sum_{i=1}^{n} \lambda_i^{2t-1} = 0$ where the sum is over the eigenvalues (with multiplicity) of A_X.*

PROOF. The fact that (1) implies (2) was done in the previous theorem. The equivalence of (2) and (3) is clear since $(\lambda - \lambda_i)(\lambda - \lambda_j) = (\lambda^2 - a)$ with $a = \lambda_i^2$. It is also clear that (2) implies (4) since the eigenvalues occur in pairs with opposite signs and so they cancel each other in the sum in (4). To see that (4) implies (1), we recall that the (i, j)-th entry of A_X^{2t-1} counts the number of paths of length $2t - 1$ from vertex i to vertex j. In particular, the diagonal entries count the number of closed walks of this length. But the sum of the diagonal entries is the total number of closed paths of this length and (4) says this sum is zero. Thus, X has no closed paths of odd length. By Theorem 1.5.1, X is bipartite. ∎

4.4. Diameter and Eigenvalues

Recall from linear algebra the notion of a **minimal polynomial** of a matrix. By the Cayley-Hamilton theorem (or by the fact that $1, A, A^2, ..., A^{n^2}$ are linearly dependent via dimension considerations) we deduce that A satisfies some monic polynomial equation. Among all, there is one of minimal degree which is necessarily unique (by the division algorithm). The degree of this minimal polynomial is equal to the number of distinct eigenvalues of A.

Indeed, this is easy to see in the case of real symmetric matrices that we are dealing with. By the **spectral theorem** all the eigenvalues of a real symmetric matrix are real and there is a basis of eigenvectors. If we let

$$g(\lambda) = \prod_{i=1}^{r} (\lambda - \lambda_i)$$

where the λ_i range over the distinct eigenvalues of A, then $g(A) = 0$. To see this, it suffices to see how $g(A)$ operates on set of basis vectors.

We take the basis of eigenvectors and see that this is immediate (as the various factors commute) and we may write

$$g(A) = \prod_{i=1}^{r} (A - \lambda_i I).$$

If there were a polynomial h of smaller degree with $h(A) = 0$, we see that it must divide the polynomial $g(\lambda)$ and must consist of a product of terms of the form $\lambda - \lambda_i$ for some proper subset of subscripts. But then, the eigenvector v_j corresponding to an eigenvalue λ_j that is omitted in the product will not be annihilated by $h(A)$.

Recall that the distance $d(u, v)$ between vertices u, v equals the shortest length of a path connecting u and v. Now we define the **diameter** of a graph X as

$$\operatorname{diam}(X) = \max_{u,v \in V(X)} d(u, v)$$

where the maximum is over all possible pairs of vertices.

THEOREM 4.4.1. *If* $\operatorname{diam}(X) < \infty$, *then the diameter is strictly less than the number of distinct eigenvalues of* X.

PROOF. Let A be the adjacency matrix of X. Then A satisfies a polynomial of degree r if and only if some non-zero linear combination of $A^0, A^1, ..., A^r$ is zero. Since the number of distinct eigenvalues is equal to the degree of the minimal polynomial, we need only show that $A^0, A^1, ..., A^k$ are linearly independent when $k \leq \operatorname{diam}(X)$. Let $k = \operatorname{diam}(X)$ and choose v_i, v_j so that the distance between v_i and v_j equals k. By counting walks from v_i to v_j we see that the i, j-th entry of A^k is not zero. But the (i, j)-th entry of A^t for $t < k$ is zero because $d(v_i, v_j) = k$. Therefore, A^k is not a linear combination of A^t for $t < k$. Hence, the degree of the minimal polynomial is strictly greater than $\operatorname{diam}(X)$. ∎

The examples of previous section show that this result is sharp. For instance, in the case of the complete graph K_n, the diameter is equal to 1 and the number of distinct eigenvalues is 2. In the case of the bipartite graph $K_{r,s}$, we have diameter 2 and the number of distinct eigenvalues is 3. There are many other classes of graphs X whose number of distinct eigenvalues equals $1 + \operatorname{diam}(X)$.

4.5. Exercises

EXERCISE 4.5.1. Determine the eigenvalues of P_4 and C_5.

EXERCISE 4.5.2. Show that a graph X with n vertices is connected if and only if $(A + I_n)^{n-1}$ has no zero entries, where A is the adjacency matrix of X.

EXERCISE 4.5.3. For a simple graph X with e edges, t_3 triangles and adjacency matrix A, show that

$$\mathrm{tr}(A) = 0, \quad \mathrm{tr}(A^2) = 2e, \quad \mathrm{tr}(A^3) = 6t_3.$$

EXERCISE 4.5.4. If X is a bipartite graph with e edges and λ is an eigenvalue of X, show that

$$|\lambda| \leq \sqrt{e}.$$

EXERCISE 4.5.5. Let X be a simple graph with n vertices and e edges. If λ is an eigenvalue of the adjacency matrix A of X, show that

$$|\lambda| \leq \sqrt{\frac{2e(n-1)}{n}}.$$

EXERCISE 4.5.6. If two non-adjacent vertices of a graph X are adjacent to the same set of vertices, show that its adjacency matrix has eigenvalue 0.

EXERCISE 4.5.7. The **eccentricity** of a vertex u in a graph X is the maximum of $d(u, v)$ as v ranges over the vertices of X. The minimum of all the possible eccentricities is called the **radius**, denoted $\mathbf{rad}(X)$, of the graph X. Show that if X is connected, then

$$\mathrm{rad}(X) \leq \mathrm{diam}(X) \leq 2\mathrm{rad}(X).$$

EXERCISE 4.5.8. Let R be a commutative ring and $A_1, ..., A_k$ be $n \times n$ matrices. We define a **generalized commutator** as

$$[A_1, ..., A_k] = \sum_{\sigma \in S_k} (\mathrm{sgn}\, \sigma) A_{\sigma(1)} ... A_{\sigma(k)}.$$

When $k = 2n$, show that

$$[A_1, ..., A_k] = 0.$$

This is a classical theorem of Shimson Avraham Amitsur (1921-1994) that can be proved using Euler circuits in digraphs.

EXERCISE 4.5.9. Show that the graphs in Figure 4.1 are isomorphic by presenting an explicit isomorphism.

EXERCISE 4.5.10. Let M be the incidence matrix of a simple graph X. Prove that

$$MM^t = D + A$$

where A is the adjacency matrix of X and D is a diagonal matrix consisting of the degrees of the vertices of X.

EXERCISE 4.5.11. In a simple graph X, we choose an **orientation** by assigning a direction to each edge. The modified incidence matrix N is defined as follows. Its rows are parameterized by the vertices v_i and the columns by the edges e_j, as before. The i,j-th entry of N is $+1$ if v_i is the tail of e_j, -1 if it is the head and zero otherwise. Prove that

$$NN^t = D - A_X.$$

EXERCISE 4.5.12. The **Laplacian matrix** of a graph X is $D - A$. Show that the smallest eigenvalue of the Laplacian is 0. If X is connected, then 0 has multiplicity 1 for the Laplacian.

EXERCISE 4.5.13. If X is k-regular, then λ is an eigenvalue of the its adjacency matrix if and only if $k - \lambda$ is an eigenvalue of its Laplacian matrix.

EXERCISE 4.5.14. Prove that $\lambda^4 + \lambda^3 + 2\lambda^2 + \lambda + 1$ cannot be the characteristic polynomial of an adjacency matrix of any graph.

EXERCISE 4.5.15. Determine the characteristic polynomial of the cycle C_4.

EXERCISE 4.5.16. Let Y denote the graph obtained from a graph X by adding an isolated vertex. Show that $P_Y(\lambda) = \lambda P_X(\lambda)$. If $X = C_4$, compare P_Y with $P_{K_{1,4}}$.

EXERCISE 4.5.17. The **odd girth** of a graph X is the shortest length of an odd cycle. If X and Y have the same eigenvalues, then they have the same odd girth.

EXERCISE 4.5.18. The **line graph** $L(X)$ (see also Chapter 11)of a graph X the edges of X as vertices, two edges e and f of X being adjacent in $L(X)$ if they have common endpoint in X. Show that if N is the incidence matrix of X, then the adjacency matrix of $L(X)$ is $N^tN - 2I_m$, where m is the number of edges of X.

EXERCISE 4.5.19. If X is k-regular and λ is an eigenvalue of the adjacency matrix of X, then $k + \lambda - 2$ is an eigenvalue of the line graph of X.

EXERCISE 4.5.20. Any eigenvalue of a line graph is greater than or equal to -2.