CHAPTER 2

Recurrence Relations

2.1. Binomial Coefficients

Combinatorics is the study of finite sets. To define finite sets, we need the notion of bijective function. Given two sets X and Y, a function $f : X \to Y$ is **injective** or **one-to-one** if $f(a) \neq f(b)$ for any $a, b \in X$ with $a \neq b$. A function $f : X \to Y$ is **surjective** or **onto** if for any $y \in Y$, there exist $x \in X$ such that $f(x) = y$. A function is **bijective** if it is injective and surjective. A function $f : X \to Y$ is **invertible** if there exists a function $g : Y \to X$ such that $f(x) = y$ if and only if $g(y) = x$. If g exists, it is called the **inverse** of f and it is usually denoted by f^{-1}. We leave as an exercise the fact that a function is bijective if and only if it is invertible.

We say that a set X is finite if there exists an positive integer n and a bijective function $f : X \to \{1, \ldots, n\}$. In this case, we say that X has n elements or it has cardinality n. Also, the empty set $\emptyset$ is the finite set of cardinality 0.

We usually denote a set with n elements by $[n] = \{1, 2, \ldots, n\}$. To a subset A of $[n]$, one can associate its **characteristic vector** $\chi_A \in \{0, 1\}^n$, where $\chi_A(i) = 1$ if $i \in A$ and 0, otherwise.

PROPOSITION 2.1.1. *The number of subsets of a set with n elements is 2^n.*

PROOF. The correspondence $A \to \chi_A$ is a bijection between the subsets of $[n]$ and the vectors in $\{0, 1\}^n$. The result follows easily since there are 2^n vectors in $\{0, 1\}^n$. ∎

One can also use induction on n to prove the previous proposition (see Exercise 2.7.2).

A **permutation** of $[n]$ is a bijective function $f : [n] \to [n]$. The set of all permutations of $[n]$ is denoted by S_n. It is a group called the **symmetric group**. Since $f(1)$ can be chosen in n ways, $f(2)$ in $(n-1)$ ways, ..., $f(n-2)$ in 2 ways and $f(n-1)$ in one way, it follows that the number of permutations is $n(n-1)\ldots 2 \cdot 1$ which will be denoted by $n!$.

S. M. Cioabă and M. R. Murty, *A First Course in Graph Theory and Combinatorics*

We call $i \in [n]$ a **fixed point** for a permutation σ if $\sigma(i) = i$. For $k \geq 2$, the **cycle** $(i_1, \dots, i_k)$ is the permutation $\pi \in S_n$ with $\pi(i_j) = i_{j+1}$ for $j \in [k]$ (here $i_{k+1} = i_1$) and any other $l \neq i_1, \dots, i_k$ is a fixed point of π.

Note that $(i_1, \dots, i_k) = (i_j, i_{j+1}, \dots, i_k, i_1, \dots, i_{j-1})$ for each $j \in [k]$. The **length** of the cycle π is k. A cycle of length 2 is also known as a **transposition**. The **parity** of a permutation $\sigma \in S_n$ equals parity of the number of pairs $1 \leq i < j \leq n$ such that $\sigma(i) > \sigma(j)$. The **signature** of σ is 1 if the parity of σ is even and -1 otherwise.

THEOREM 2.1.2. *Every permutation can be written as a product of disjoint cycles. The representation is unique modulo the order of the factors and the starting points of the cycles.*

PROOF. Let $\sigma \in S_n$. We prove the theorem by induction on the number k of points that are not fixed by the permutation σ.

If $k = 2$, then σ is a transposition which is a cycle of length 2 and we are done.

Assume that $k \geq 3$. Let $i \in [n]$ such that $\sigma(i) \neq i$. Denote by l the smallest integer such that $\sigma^l(i) = i$. Then, $\pi = (i, \sigma(i), \dots, \sigma^{l-1}(i))$ is a cycle of length l. We leave as an exercise for the reader to prove that the number of points that are not fixed by $\sigma\pi^{-1}$ is less than k. By applying the induction hypothesis to $\sigma\pi^{-1}$, the theorem follows. ∎

For any integer k with $0 \leq k \leq n$, define the **binomial coefficient** $\binom{n}{k}$ as the number of subsets with k elements (or k-subsets) of $[n]$.

PROPOSITION 2.1.3.

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}.$$

PROOF. It is obvious that $\binom{n}{1} = n$ for each $n \geq 1$. Let us count the number of pairs (A, x), where $A \subseteq [n], |A| = k$ and $x \in A$. There are $\binom{n}{k}$ such A's and each has k elements. Thus, the answer is $k\binom{n}{k}$. On the other hand, if we count the x's first, we have n choices. For each x, there are $\binom{n-1}{k-1}$ subsets A such that $A \subseteq [n], x \in A$. This is because each such A is of the form $B \cup \{x\}$, where $B \subset [n] \setminus \{x\}$ and $|B| = k-1$. Thus, the answer we get now is $n\binom{n-1}{k-1}$. Hence, $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$.

Replacing n by $n-1, n-2, \dots, n-k+2$, we obtain

$$\binom{n-i}{k-i} = \frac{n-i}{k-i}\binom{n-i-1}{k-i-1}$$

for $i = 0, 1, \ldots, k-2$. Multiplying all these equations together, we get

$$\binom{n}{k}\prod_{i=1}^{k-2}\binom{n-i}{k-i} = \frac{n(n-1)\ldots(n-k+1)}{k!}\prod_{i=1}^{k-2}\binom{n-i}{k-i}.$$

Simplifying the previous equality, we obtain

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}$$

as claimed. ∎

When n is an integer, an easy to remember formula for $\binom{n}{k}$ is $\frac{n!}{k!(n-k)!}$. One can use these results to determine which binomial coefficient $\binom{n}{k}$ is the largest when $0 \le k \le n$ (see Exercise 2.7.1).

The originators of combinatorics came from the East and the main stimulus came from the Hindus. The formulae for the number of permutations on n elements and the number of k-subsets of $[n]$ were known to Bhaskara around 1150. Special cases of these formulae were found in texts dating back to the second century BC.

The following theorem is often attributed to Blaise Pascal (1623-1662) who knew this result as it appeared in a posthumous pamphlet published in 1665. It appears that the result was known to various mathematicians preceding Pascal such as the 3rd century Indian mathematician Pingala.

THEOREM 2.1.4 (Binomial Theorem). *For any positive integer n,*

$$(x+a)^n = \sum_{k=0}^{n}\binom{n}{k}x^k a^{n-k}.$$

PROOF. Writing $(x+a)^n$ as $(x+a)(x+a)\ldots(x+a)$, we notice that the number of times the term $x^k a^{n-k}$ appears, equals the number of ways of choosing k brackets (for x) from the n factors of the product. That is exactly $\binom{n}{k}$. ∎

Sir Isaac Newton (1643-1727) was one of the greatest mathematicians of the world. His contributions in mathematics, physics and astronomy are deep and numerous. In 1676, Newton showed that a similar formula holds for real n. Newton's formula involves infinite series and it will be discussed in the Catalan number section.

If $f, g : \mathbb{N} \to \mathbb{R}$, we say $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. James Stirling (1692-1770) was a Scottish mathematician who showed that

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

This is usually called Stirling's formula. It appears in *Methodus Differentialis* which Stirling published in 1730. Abraham de Moivre (1667-1754) also knew this result around 1730.

2.2. Derangements

The term *reccurence* is due to Abraham de Moivre (1722). A sequence satisfies a recurrence relation when each term of the sequence is defined as a function of the preceding terms. In many counting questions, it is more expedient to obtain a recurrence relation for the combinatorial quantity in question. Depending on the nature of this recurrence, one is then able to determine in some cases, an explicit formula, and in other cases, where explicit formulas are lacking, some idea of the growth of the function. We will give several examples in this chapter.

We begin with the problem of counting the number of permutations σ of S_n without any fixed points. These are permutations with the property that $\sigma(i) \neq i$ for all $1 \leq i \leq n$. Such permutations are called **derangements**. The first appearance of this problem is in 1708 in a book on games of chance *Essay d'Analyse sur les Jeux de Hazard* by Pierre Rémond de Montmort (1678-1719).

Let d_n be the number of derangements on $[n]$. We will obtain a recurrence relation for it as follows. For such a derangement, we know that $\sigma(n) = i$ for some $1 \leq i \leq n-1$. We fix such an i and count the number of derangements with $\sigma(n) = i$. Since there are $n-1$ choices for i, the final tally is obtained by multiplying this number by $n-1$. If σ is a derangement with $\sigma(n) = i$, we consider two cases. If $\sigma(i) = n$, then σ restricted to

$$\{1, 2, ..., n\} \backslash \{i, n\}$$

is a derangement on $n-2$ letters and the number of such is d_{n-2}. If $\sigma(i) \neq n$, let j be such that $\sigma(j) = n$, with $i \neq j$. Thus, if we define σ' by setting

$$\sigma'(k) = \sigma(k), \quad \text{for } 1 \leq k \leq n-1,\ k \neq j$$

and $\sigma'(j) = i$, we see that σ' is a derangement on $n-1$ letters. Conversely, if σ' is a derangement on $n-1$ letters and $\sigma'(j) = i$, we can extend it to a derangement on n letters by setting $\sigma(j) = n$ and $\sigma(n) = i$. Thus, we get the recurrence

THEOREM 2.2.1.

$$d_n = (n-1)(d_{n-1} + d_{n-2}).$$

Now we will prove by induction that:

THEOREM 2.2.2. *For $n \geq 1$,*

$$d_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}.$$

PROOF. Indeed, if $n = 1$, it is clear that $d_1 = 0$ and for $n = 2$, $d_2 = 1$. If we let $f(n)$ denote the right hand side of the above equation, we will show that $f(n)$ satisfies the same recursion as d_n with the same initial conditions, thereby establishing the result. Thus, $(n-1)(f(n-1) + f(n-2))$ equals

$$\begin{aligned}
&(n-1)! \sum_{j=0}^{n-2} \frac{(-1)^j}{j!} \left((n-1)\left(1 + \frac{(-1)^{n-1}}{(n-1)!}\right) + 1 \right) \\
&= n! \sum_{j=0}^{n-2} \frac{(-1)^j}{j!} + \frac{(-1)^{n-1}}{(n-1)!}(n-1) \\
&= f(n)
\end{aligned}$$

as desired. ∎

Let us observe that

$$\lim_{n\to\infty} \frac{d_n}{n!} = \frac{1}{e}.$$

In fact, we can make this more precise. As the series is alternating we begin by noting that if a_n is a decreasing sequence of positive real numbers tending to zero, then,

$$|\sum_{j=0}^{\infty}(-1)^j a_j - \sum_{j=0}^{n}(-1)^j a_j| \leq |a_{n+1} - (a_{n+2} - a_{n+3}) - \cdots| \leq a_{n+1}.$$

Thus,

$$\left| e^{-1} - \frac{d_n}{n!} \right| < \frac{1}{(n+1)!}.$$

Denoting by $\lfloor x \rfloor$ the largest integer less than or equal to x, the previous equation implies the following result.

THEOREM 2.2.3. *For $n \geq 1$,*

$$d_n = \lfloor n!/e + 1/2 \rfloor.$$

PROOF. By our remarks above,

$$|d_n - n!/e| \leq \frac{1}{n+1} < \frac{1}{2}$$

for $n \geq 1$. As d_n is a non-negative integer, it is uniquely determined by this inequality as the nearest integer to $n!/e$.

We leave as an easy exercise for the reader to show that the nearest integer to x is $[x + 1/2]$. ■

This result means that the probability that a random permutation in S_n is a derangement is about $\frac{1}{e}$. We give a different proof of the formula for the number of derangements using inclusion and exclusion in Chapter 3.

2.3. Involutions

We now want to count the number of elements of order 2 in the symmetric group S_n. Such an element is called an **involution**. Recall that any permutation is a product of disjoint cycles and the order of the permutation is the least common multiple of the cycle lengths. Thus, if the permutation has order 2, then all the cycles must be of length 1 or 2. Let $s(n)$ be the number of such involutions. We partition these involutions into two groups: those that fix n and those that do not. The number fixing n is clearly $s(n-1)$. If σ is an involution not fixing n, then $\sigma(n) = i$ (say) for some $1 \leq i \leq n-1$. But then we must necessarily have $\sigma(i) = n$ as σ is a product of 1-cycles or 2-cycles (transpositions). Thus, σ restricted to

$$\{1, 2, ..., n-1\} \backslash \{i\}$$

is an involution on $n-1$ letters. There are $s(n-2)$ such elements and $n-1$ choices for i, so we get the recurrence

THEOREM 2.3.1. *Let $s(n)$ be the number of involutions in S_n. Then*

$$s(n) = s(n-1) + (n-1)s(n-2).$$

We can derive a modest amount of information from this recurrence, though our results will not be as sharp as what we obtained for d_n, the number of derangements in S_n. We have:

THEOREM 2.3.2. (1) *$s(n)$ is even for all $n > 1$.*
(2) *$s(n) > \sqrt{n!}$ for all $n > 1$.*

PROOF. Clearly, $s(1) = 1$ and $s(2) = 2$ and the assertion is true for $n = 2$. From the recurrence (or directly) we see that $s(3) = 4$. Consequently, applying induction to the recurrence, one can show easily that $s(n)$ is even. We will also apply induction to prove the second part of the theorem. Again, for $n = 2$ and $n = 3$, the inequality is clear. Suppose we have established the inequality for numbers $< n$. Then, by induction,

$$s(n) > \sqrt{(n-1)!} + (n-1)\sqrt{(n-2)!} \geq (\sqrt{(n-1)!})(1 + \sqrt{n-1}).$$

To complete the proof, we need to show

$$1 + \sqrt{n-1} \geq \sqrt{n}.$$

But this is clear by squaring both sides of the inequality. ∎

2.4. Fibonacci Numbers

The **Fibonacci numbers** are defined recursively as follows. $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The following problem led Fibonacci to consider these numbers. Suppose we start with a pair of rabbits, one male and one female. At the end of each month, every female produces one new pair of rabbits (one male and one female). The question that Leonardo Pisano Fibonacci (1170-1250) asked was: how many pairs will there be in one year ? This problem appears in 1202 in his book *Liber abaci* which also introduced the use of Arabic numerals into Europe.

It is easy to see that the number of pairs after n months will be exactly F_n. How can we find a formula for F_n ?

The Fibonacci numbers satisfy a **linear recurrence relation with constant coefficients**. These are recurrence relations of the following form:

$$y_n = a_1 y_{n-1} + a_2 y_{n-2} + \cdots + a_k y_{n-k}$$

where $k \geq 1$ is a fixed integer and $a_1, a_2, \ldots, a_k$ are all constant (they do not depend on n).

To find a general formula for y_n, we must solve the characteristic equation

$$x^k = a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_k.$$

If this equation has distinct solutions, then y_n is going to be a linear combination of the n-th powers of these solutions. Using the initial k values of the sequence $(y_n)_n$, one can find the exact formula for y_n.

If the previous equation has multiple solutions, a formula for y_n can be determined as follows. If α is a solution with multiplicity r, then one can check $\alpha^n, n\alpha^n, \ldots, n^{r-1}\alpha^n$ are all solutions of the characteristic equation. We can write y_n as a linear combination of such solutions and use the initial values of the sequence $(y_n)_n$ to determine a precise formula.

Let us try to use this method to find a formula for F_n. Since the recurrence relation is $F_n = F_{n-1} + F_{n-2}$, it follows that the characteristic equation is $x^2 = x + 1$. The solutions of this equation are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. We obtain that $F_n = c\alpha^n + d\beta^n$, where c and d are constants to be determined.

Since $1 = F_0 = c + d$ and $1 = F_1 = c\alpha + d\beta$, we obtain that $c = \frac{\sqrt{5}+1}{2\sqrt{5}}$ and $d = \frac{\sqrt{5}-1}{2\sqrt{5}}$. We deduce that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

2.5. Catalan Numbers

Eugéne Charles Catalan (1814-1894) was born in Bruges, Belgium. He defined the numbers which bear his name today, while counting the number of ways of decomposing a convex n-gon into triangles by $n-2$ non-intersecting diagonals. Around the same time, the Catalan numbers were also studied by Johann Andreas von Segner (1704-1777), Leonhard Euler (1707-1783) and Jacques Binet (1786-1856).

The Catalan numbers have many combinatorial interpretations and arise in branches of mathematics and computer science. There are at least 66 combinatorial interpretations of Catalan numbers (see Exercise 6.19 in Richard Stanley's *Enumerative Combinatorics, Volume 2*).

Here we will define the Catalan number C_n as the number of ways we can bracket a sum of n elements so that it can be calculated by adding two terms at a time. For example, for $n = 3$, we have

$$((a+b)+c) \quad \text{and} \quad (a+(b+c)).$$

Thus, $C_3 = 2$.

For $n = 4$, we have $C_4 = 5$ since there are five ways of bracketing a sum with 4 terms:

$$\begin{aligned}&(((a+b)+c)+d),\\&((a+(b+c))+d),\\&(a+((b+c)+d)),\\&(a+(b+(c+d))),\\&((a+b)+(c+d)).\end{aligned}$$

We can obtain a recurrence for C_n as follows. Any bracketed expression is of the form

$$E_1 + E_2$$

where E_1 is a bracketed expression containing i terms (say) and E_2 is a bracketed expression containing $n-i$ terms. By our definition, there are C_i choices for E_1 and C_{n-i} choices for E_2, so we get

$$C_n = \sum_{i=1}^{n-1} C_i C_{n-i}.$$

It may be that Segner was the first to notice this recurrence relation and Euler was the first to solve it (see Chapter 6 in *Enumerative Combinatorics, Volume 2* by Richard Stanley). Notice that this recurrence is more complicated than the one for d_n or $s(n)$ derived in the previous sections in that the recurrence uses all of the previous C_i's for its determination.

In order to determine a nice formula for the Catalan numbers, we use the theory of generating functions. To an infinite sequence $(a_n)_{n\geq 0}$ we associate the following **formal power series**:

$$\sum_{n\geq 0} a_n t^n.$$

We regard such series as algebraic objects without any interest in their convergence. We say two series are equal if their coefficient sequences are identical. We define addition and subtraction as follows

$$\sum_{n\geq 0}(a_n \pm b_n)t^n = \sum_{n\geq 0} a_n t^n \pm \sum_{n\geq 0} b_n t^n.$$

The multiplication is defined similarly to the one for polynomials.

$$\sum_{n\geq 0} a_n t^n \cdot \sum_{n\geq 0} b_n t^n = \sum_{n\geq 0} c_n t^n$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. We can also differentiate formal power series the same way as one would do for polynomials.

$$\left(\sum_{n\geq 0} a_n t^n\right)' = \sum_{n\geq 1} n a_n t^{n-1}.$$

The standard functions of analysis are defined as formal power series by their usual Taylor series. For example,

$$e^t = \sum_{n\geq 0} \frac{t^n}{n!}.$$

The following equation is a definition of $(1+t)^\alpha$

$$(1+t)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} t^n$$

where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$ for any real number α. If α is a non-negative integer, then this is just Theorem 2.1.4 since $\binom{\alpha}{n} = 0$ for $n > \alpha$. For α real, the equation above will be regarded here as a definition. An alternative approach would be to define $(1+t)^\alpha$ for any rational α by using the exponent laws (which hold for power series) and then prove that its Taylor series has the claimed form. This was done by Newton.

We encode the recurrence for the Catalan numbers in a **generating function** as follows. Let

$$F(t) = \sum_{n=0}^{\infty} C_n t^n$$

where we set $C_0 = 0$ and $C_1 = 1$. Let us compute the coefficient of t^n in $F(t)^2$ for $n \geq 2$. It is equal to

$$\sum_{i=1}^{n-1} C_i C_{n-i} = C_n$$

since $C_0 = 0$. Thus,

$$F(t)^2 = F(t) - t.$$

This is a quadratic equation in $F(t)$ which we can solve using the familiar formula for solving quadratic equations:

$$F(t) = \frac{1 \pm \sqrt{1-4t}}{2}.$$

We must determine which "sign" will give us the correct solution for $F(t)$. We choose the minus sign because $F(0) = 0$. Thus,

$$F(t) = \frac{1 - \sqrt{1-4t}}{2}.$$

We can use the binomial theorem to determine the C_n's explicitly. Indeed, the coefficient of t^n on the right hand side of the above expression for $F(t)$ is easily seen to be

$$-\frac{1}{2}\binom{1/2}{n}(-4)^n$$

which simplifies to the following result.

THEOREM 2.5.1.

$$C_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

We can use Stirling's formula to determine the asymptotic behaviour of C_{n+1}. Indeed, by Stirling's formula,

$$n! \sim \sqrt{2\pi n}(n/e)^n,$$

so that

$$C_{n+1} \sim \frac{2^{2n}}{(n+1)\sqrt{\pi n}},$$

from which we see that it has exponential growth.

2.6. Bell Numbers

Eric Temple Bell (1883-1960) was born in Aberdeen, Scotland. He was the president of the Mathematical Association of America between 1931 and 1933.

The n-th Bell number, denoted by B_n, is the number of partitions of an n-element set. A partition of $[n]$ is a collection of pairwise disjoint non-empty subsets $B_1, \ldots, B_k$ (called blocks) whose union is $[n]$. By convention, $B_0 = 1$. The partitions of $[2]$ are $\{1\} \cup \{2\}$ and $\{1,2\}$. The partitions of $[3]$ are $\{1\} \cup \{2\} \cup \{3\}, \{1,2\} \cup \{3\}, \{1,3\} \cup \{2\}, \{2,3\} \cup \{1\}$ and $\{1,2,3\}$. Thus, $B_1 = 1$, $B_2 = 2$ and $B_3 = 5$. We will derive a recurrence relation for the Bell numbers. Of the partitions of $[n]$, we consider the block to which n belongs. Clearly, such a block can be written as $\{n\} \cup Y$ for some subset Y of $\{1, 2, ..., (n-1)\}$. If this block has k elements, then Y is a subset of $k-1$ elements. The number of ways of choosing Y is $\binom{n-1}{k-1}$. The remaining elements can be partitioned in B_{n-k} ways. Thus, we obtain

$$B_n = \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-k}.$$

We can use this recurrence to write down an **exponential generating function**:

$$G(t) = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

Then,

$$G'(t) = \sum_{n=1}^{\infty} \frac{B_n}{(n-1)!} t^{n-1} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{t^{k-1}}{(k-1)!} \frac{B_{n-k} t^{n-k}}{(n-k)!}.$$

The sum on the right hand side is easily seen to be

$$e^t G(t).$$

Thus,

$$G(t) = A e^{e^t}$$

for some constant A. Since $G(0) = 1$, we must have $A = e^{-1}$. This proves:

THEOREM 2.6.1.

$$\sum_{n=0}^{\infty} \frac{B_n t^n}{n!} = e^{e^t - 1}.$$

We can use this theorem to derive an explicit formula for B_n as follows. The right hand side of the above equation can be expanded as

$$\frac{1}{e}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{j^n t^n}{n!j!}$$

and on comparing the coefficients of t^n we obtain:

THEOREM 2.6.2.

$$B_n = \frac{1}{e}\sum_{j=0}^{\infty}\frac{j^n}{j!}.$$

2.7. Exercises

EXERCISE 2.7.1. If $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, show that

$$\binom{n}{k} \leq \binom{n}{k+1}.$$

EXERCISE 2.7.2. Prove by induction on n that $[n]$ has 2^n subsets.

EXERCISE 2.7.3. Show that

$$1 \cdot 1! + 2 \cdot 2! + \ldots n \cdot n! = (n+1)! - 1.$$

EXERCISE 2.7.4. Show that

$$\binom{n}{k}\binom{k}{l} = \binom{n}{l}\binom{n-l}{k-l}$$

for each $n \geq k \geq l \geq 0$.

EXERCISE 2.7.5. Show that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

EXERCISE 2.7.6. Show that

$$\binom{n+k+1}{k} = \sum_{i=0}^{n}\binom{n+i}{i}.$$

EXERCISE 2.7.7. Show that

$$\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}.$$

EXERCISE 2.7.8. Show that

$$\frac{2^{2n}}{2n+1} < \binom{2n}{n} < 2^{2n}$$

and use Stirling's formula to prove that

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}.$$

EXERCISE 2.7.9. Give a solution using binomial coefficients and a direct combinatorial solution to the following question: How many pairs (A, B) of subsets of $[n]$ are there such that $A \cap B = \emptyset$?

EXERCISE 2.7.10. Show that the number of even subsets of $[n]$ equals the number of odd subsets of $[n]$. Give two proofs, one using binomial formula, and one using a direct bijection. Calculate the sum of the sizes of all even (odd) subsets of $[n]$.

EXERCISE 2.7.11. Let n be an integer, $n \geq 1$. Let s_i denote the number of subsets of $[n]$ whose order is congruent to $i \pmod 3$ for $i \in \{0, 1, 2\}$. Determine s_0, s_1, s_2 in terms of n.

EXERCISE 2.7.12. Prove by mathematical induction that

$$\sqrt{5}F_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

for $n \geq 0$.

EXERCISE 2.7.13. Show that the number of distinct ways of triangulating a convex n-gon by $n-2$ nonintersecting diagonals equals C_{n-1}.

EXERCISE 2.7.14. Show that the number of solutions of the equation

$$x_1 + \cdots + x_k = n$$

in positive integers ($x_i > 0$ for each i) is $\binom{n-1}{k-1}$.

EXERCISE 2.7.15. Show that for each n and $k, 1 \leq k \leq n$

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} < \left(\frac{en}{k}\right)^k.$$

EXERCISE 2.7.16. Calculate

$$\sum_{A \subseteq [n]} |A|^2.$$

EXERCISE 2.7.17. Calculate

$$\lim_{n\to\infty} \sqrt[n]{\sum_{k=0}^{n} \binom{n}{k}^t}$$

when t is a real number.

EXERCISE 2.7.18. Let k be a non-negative integer number. Show that any non-negative integer number n can be written uniquely as

$$n = \binom{x_k}{k} + \binom{x_{k-1}}{k-1} + \cdots + \binom{x_2}{2} + \binom{x_1}{1}$$

where $0 \leq x_1 < x_2 < \cdots < x_k$.

EXERCISE 2.7.19. Let B_n denote the n-th Bell number. Show that $B_n < n!$ for each $n \geq 3$.

EXERCISE 2.7.20. Determine the number of ways of writing a positive integer n as a sum of ones and twos.