CHAPTER 13

Hints

Chapter 1. Basic Notions of Graph Theory

1.6.1 Use Corollary 1.2.2.

1.6.2 Use Theorem 1.5.2.

1.6.3 The degree of a vertex in a connected graph with n vertices is between 1 and $n-1$.

1.6.4 Use induction on n.

1.6.5 Use the idea from Theorem 1.5.2.

1.6.6 Consider a path of maximum length.

1.6.7 Modify the proof of Theorem 1.4.1.

1.6.8 The endpoints of each edge have different colours.

1.6.9 Find the maximum number of edges in $K_{a,b}$, when $a+b=n$.

1.6.10 Each C_4 must have two vertices of each colour.

1.6.11 If $x, y \in \{0,1\}^n$, show that the distance between x and y equals the number of positions in which x and y differ. For $x \in \{0,1\}^n$, let $w(x)$ denote the number of 1's in x. Partition the vertices of Q_n according to the parity of $w(x)$.

1.6.12 Use induction on n to calculate the number of vertices. For a vertex $x \in \{0,1\}^n$, calculate its degree.

1.6.13 Use induction on n or for each $x \neq y \in \{0,1\}^n$, count the common neighbours of x and y.

1.6.14 Use the previous hint.

1.6.15 Start with an arbitrary bipartite subgraph with two non-empty colour classes. For each vertex x, if the number of neighbors of x which are contained in its colour class is greater than the number of neighbors of x which are contained in the other colour class, then move x to the other colour class.

1.6.16 Use Theorem 1.4.1.

1.6.17 Prove by contradiction.

1.6.18 Prove by contradiction.

1.6.19 Show first that $\sum_{uv \in E(X)} (d(u)+d(v)) = \sum_{u \in V(X)} d^2(x)$ and use Cauchy-Schwarz inequality.

1.6.20 Use 1.6.19 to show that if a graph has n vertices and at least $\lfloor \frac{n^2}{4} \rfloor$ edges, then it contains a K_3.

Chapter 2. Recurrence Relations

2.7.1 Calculate $\frac{\binom{n}{k+1}}{\binom{n}{k}}$.

2.7.2 Count the subsets containing n and the ones not containing n separately.

2.7.3 Use the relation $(k+1)! - k! = k! \cdot k$

2.7.4 Use the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ or count the number of pairs $\{(K, L) : K \subset [n], |K| = k, L \subset K, |L| = l\}$ in two ways.

2.7.5 Use the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ or count the number of k-subsets of $[n]$ depending on whether or not they contain n.

2.7.6 Use 2.7.5.

2.7.7 Count the number of k-subsets of $[m+n]$ in two ways.

2.7.8 Use the formula $\binom{2n}{n} = \frac{(2n)!}{n!n!}$ and Stirling's formula for $n!$ and $(2n)!$.

2.7.9 With binomial coefficients, if $|A| = k$, then there are 2^{n-k} subsets B with $A \cap B = \emptyset$. Combinatorially, consider the matrix whose rows are the characteristic vectors of A and B.

2.7.10 The number of even subsets is $\binom{n}{0} + \binom{n}{2} + \dots$ and the number of odd subsets is $\binom{n}{1} + \binom{n}{3} + \dots$. Use Newton's binomial formula. For a bijective proof, if n is odd, consider the function $A \to A^c$. If n is even, use the fact that $n-1$ is odd.

2.7.11 If $\omega = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}$, calculate $(1+\omega)^n$ in two different ways.

2.7.12 Use $F_n = F_{n-1} + F_{n-2}$.

2.7.13 Label $n-1$ sides of a convex n-gon with distinct labels. Construct a bijective function between to the triangulations of the convex n-gon by $n-2$ nonintersecting diagonals and the ways of bracketing the sum of $n-1$ terms corresponding to the labeled sides.

2.7.14 Write $n = 1 + 1 + \cdots + 1$ and construct a bijection between the solutions of the given equation and the $(k-1)$-subsets of a set with $n-1$ elements.

2.7.15 Use the formula of $\binom{n}{k}$ for the first inequality. Use induction on n for the second inequality.

2.7.16 Use binomial formula or count in two ways the number of triples (A, x, y) where $A \subset [n]$ and $x, y \in A$.

2.7.17 Use Stirling's formula.

2.7.18 Let $x_k = \max\{x : \binom{x}{k} \le n\}$.

2.7.19 Use the recurrence relation for the Bell numbers.

2.7.20 Find a recurrence formula using the fact the last term of the sum can be 1 or 2.

Chapter 3. The Principle of Inclusion and Exclusion

3.6.1 Use Theorem 3.1.1.

3.6.2 For $k \leq pq$, if $\gcd(k, pq) \neq 1$, then $\gcd(k, p) \neq 1$ or $\gcd(k, q) \neq 1$.

3.6.3 For $i \in [r]$, let $A_i = \{x : x \leq n, p_i | x\}$. Then use Theorem 3.1.1.

3.6.4 Use Theorem 3.1.1.

3.6.5 Use Theorem 3.1.1.

3.6.6 Use induction on n.

3.6.7 Use Theorem 3.2.1.

3.6.8 Use Theorem 3.3.1.

3.6.9 Use Theorem 3.4.2.

3.6.10 The number of permutations with an even number of cycles is $|s(n, 2)| + |s(n, 4)| + \ldots$. The number of permutations with an odd number of cycles is $|s(n, 1)| + |s(n, 3)| + \ldots$.

3.6.11 Use the counting idea from the proof of Theorem 3.5.3.

3.6.12 Use inclusion and exclusion. If you have m red cards and n blue cards, how many k-elements subsets are there consisting only of red cards.

3.6.13 $|s(n, 1)|$ equals the number of permutations with exactly one cycle or use the recurrence relation.

3.6.14 Use the definition of $S(n, k)$.

3.6.15 Use the definition of $S(n, k)$.

3.6.16 Consider $f'(t)$ and use the recurrence relation $s(n) = s(n-1) + (n-1)s(n-2)$.

3.6.17 Calculate $(e^t - 1)g(t)$.

3.6.18 Use 3.6.17.

3.6.19 Use 3.6.17 and 3.6.18.

3.6.20 Calculate $e^X f(X)$.

Chapter 4. Matrices and Graphs

4.5.1 Determine the characteristic polynomials of P_4 and C_5.

4.5.2 Recall that $A^k_{i,j}$ equals the number of walks of length k from i to j and use the binomial theorem.

4.5.3 Use Theorem 4.1.1.

4.5.4 Use Theorem 4.1.1.

4.5.5 Use Exercise 4.5.3 and the Cauchy-Schwarz inequality.

4.5.6 Consider the rows corresponding to the two vertices.

4.5.7 For any three vertices i, j, k, $d(i, k) \leq d(i, j) + d(j, k)$, where $d(x, y)$ is the length of a shortest path from x to y.

4.5.8 Use the directed version of Theorem 1.4.1 to prove the result for matrices having all entries equal to 0 except one entry which equals 1.

4.5.9 Use induction on n.

4.5.10 Use the definition of M and A.

4.5.11 Use the definition of N and A.

4.5.12 Multiply the Laplacian matrix by the all one vector and use 4.5.11.

4.5.13 Use the definition of the Laplacian matrix.

4.5.14 Look at the coefficient of λ^3.

4.5.15 Calculate the eigenvalues of C_4.

4.5.16 Consider the adjacency matrix of Y.

4.5.17 If the odd girth of X is $2r+1$, then calculate in two ways the number of closed cycles of length $2s+1$ for $s \leq r-1$.

4.5.18 For any two edges e, f of X, calculate the (e, f)-entry of the matrix N^tN.

4.5.19 If μ is an eigenvalue of NN^t, then μ is an eigenvalue of N^tN.

4.5.20 Show that any eigenvalue of NN^t is positive.

Chapter 5. Trees

5.5.1 A tree has no cycles.

5.5.2 A connected graph has no cycles if and only if it has $n-1$ edges.

5.5.3 Use induction on n.

5.5.4 Use induction on n.

5.5.5 Use 5.5.1.

5.5.6 Use 5.5.1.

5.5.7 Use 5.5.6.

5.5.8 Use 5.5.7.

5.5.9 Use the principle of inclusion and exclusion.

5.5.10 Use 5.5.9.

5.5.11 Use the matrix-tree theorem.

5.5.12 Use induction on n.

5.5.13 Use Kruskal's algorithm.

5.5.14 If i has degree d and j is adjacent to i, consider the furthest point k from j such that $d(i, k) = d(j, k) + 1$.

5.5.15 Show that the maximum degree is 2.

5.5.16 Consider a spanning tree of X.

5.5.17 Use the matrix-tree theorem.

5.5.18 Use the matrix-tree theorem.

5.5.19 The number of vertices is less than the number of vertices of a k-regular tree of height D.

5.5.20 Consider a path of maximum length in the tree.

Chapter 6. Möbius Inversion and Graph Colouring

6.9.1 Use the definition of a poset.

6.9.2 Use the definition of the Hasse diagram.

6.9.3 Use Theorem 6.1.1.

6.9.4 Use 6.9.1.

6.9.5 Use the definition of the Möbius function.

6.9.6 Use the definition of a linear ordering.

6.9.7 Construct a graph whose vertices are stations with two stations adjacent if their distance is less than 150 miles.

6.9.8 The sum of the coefficients is related to the value of the chromatic polynomial at 1. Use induction for the second part.

6.9.9 Use the definition of the chromatic polynomial.

6.9.10 Colour $X \vee Y$ using the colourings of X and Y.

6.9.11 Use induction on n and Theorem 6.5.1.

6.9.12 Every connected graph has a spanning tree.

6.9.13 The vertex of degree 3 can be coloured in λ ways, then the vertex of degree 1 can be coloured with $\lambda - 1$ colours.

6.9.14 Use the definition of the chromatic polynomial.

6.9.15 Use the Rayleigh-Ritz theorem.

6.9.16 Use ideas from the proof of Theorem 6.5.1.

6.9.17 Partition the vertex set into $\chi(X)$ independent sets and count the edges between them.

6.9.18 Use the definition of the chromatic number.

6.9.19 Let A_i denote the family of k-subsets whose smallest element is i.

6.9.20 Use the principle of inclusion and exclusion.

Chapter 7. Enumeration under Group Action

7.4.1 Use the definition of a group action.

7.4.2 Use the definition of a group action.

7.4.3 Use the definition of a group homomorphism.

7.4.4 Use the definition of a group action.

7.4.5 Use Pólya's Theorem.

7.4.6 Use Pólya's Theorem.
7.4.7 Use the definition of the cycle index polynomial.
7.4.8 Use Pólya's Theorem.
7.4.9 A graph on n vertices has at most $\binom{n}{2}$ edges.
7.4.10 Use the definition of the cycle index polynomial.
7.4.11 Use the results in the last section.
7.4.12 Use the results in the last section.
7.4.13 Use the definition of the cycle index polynomial.
7.4.14 Use the results in the last section.
7.4.15 Remember that an automorphism of P_n is a bijection function $f : V(P_n) \to V(P_n)$ such that $xy \in E(P_n)$ if and only if $f(x)f(y) \in E(P_n)$.
7.4.16 Use the definition of the cycle index polynomial.
7.4.17 Use Pólya's Theorem.
7.4.18 Remember that an automorphism of a graph X is a bijection function $f : V(X) \to V(X)$ such that $xy \in E(X)$ if and only if $f(x)f(y) \in E(X)$.
7.4.19 Use the definition of the cycle index polynomial.
7.4.20 Use Pólya's Theorem.

Chapter 8. Matching Theory

8.8.1 Use Hall's Theorem 8.1.1.
8.8.2 Use Hall's Theorem 8.1.1.
8.8.3 Use Hall's Theorem 8.1.1.
8.8.4 Use induction on t.
8.8.5 Use Exercise 8.8.4 and induction on n.
8.8.6 Use Theorem 8.3.1.
8.8.7 Use the Hungarian algorithm.
8.8.8 Use Tutte's Theorem 8.6.1.
8.8.9 Use Tutte's Theorem 8.6.1.
8.8.10 Use induction on n
8.8.11 Use Birkhoff-von Neumann Theorem 8.4.1.
8.8.12 Add a proper number of vertices to A , join them to B and use Hall's Theorem 8.1.1.
8.8.13 Replace each vertex in A by an independent set of proper size and use Hall's Theorem 8.1.1.
8.8.14 Use Hall's Theorem 8.1.1.
8.8.15 If $X \setminus C$ is not a disjoint union of clique, then there are vertices x, y, z, w such that xy, xz are edges and yz and xw are not edges of X. Use the perfect matchings of $X \cup yz$ and $X \cup xw$ to construct a perfect matching for X.

8.8.16 Use Tutte's Theorem 8.6.1.
8.8.17 Consider the symmetric difference of two perfect matchings.
8.8.18 Use Tutte's Theorem 8.6.1.
8.8.19 Use Hall's Theorem 8.1.1.
8.8.20 Follow the proof of Hall's Theorem 8.1.1.

Chapter 9. Block Designs

9.7.1 Use the results in the first section.
9.7.2 Use the results in the first section.
9.7.3 Use the results in the first section.
9.7.4 Use the definition of the Möbius function.
9.7.5 Consider a design whose points are the students.
9.7.6 Generalize the construction from the second section.
9.7.7 The rank is the maximum number of independent rows or columns.
9.7.8 Use Theorem 9.3.1.
9.7.9 Use Theorem 9.4.1.
9.7.10 Use Theorem 9.3.4.
9.7.11 Use Theorem 9.6.1.
9.7.12 Remove the last $d-1$ entries from each codeword.
9.7.13 Use the definition of the Fano plane and Figure 9.1.
9.7.14 Use the definition of a Steiner triple system.
9.7.15 Use the definition of $d(x,z)$.
9.7.16 Use Theorem 9.2.1.
9.7.17 Use the definition of a $2-(v,k,\lambda)$ design.
9.7.18 Use double counting.
9.7.19 Any two distinct blocks have at most $t-1$ points in common.
9.7.20 Prove by contradiction.

Chapter 10. Planar Graphs

10.4.1 If f_i is the number of faces of length i, then $2e = \sum_i if_i \geq \gamma f$.
10.4.2 The girth is the length of the shortest cycle.
10.4.3 Use Heawood's Theorem 10.3.2.
10.4.4 Use Exercise 10.4.1 or Kuratowski's Theorem.
10.4.5 Find a plane drawing of $K_{3,3} \setminus e$ without any crossings.
10.4.6 Find a plane drawing of $K_5 \setminus f$ without any crossings.
10.4.7 Show that $K_{4,4}$ without a perfect matching is isomorphic to the 3-dimensional cube graph Q_3.

10.4.8 Join any two points by an edge if and only if their distance in the plane is 1. Show this graph is planar.

10.4.9 A nonplanar graph has crossing at least 1.

10.4.10 Use Exercises 10.4.9 and 10.4.1.

10.4.11 Use Exercise 10.4.10 and find a drawing of K_6 with 3 crossings.

10.4.12 Use the definition of outerplanar graphs. For K_4, assume it is outerplanar and derive a contradiction.

10.4.13 Assume $K_{2,3}$ is outerplanar and derive a contradiction. Find a plane drawing of $K_{2,3}$.

10.4.14 Use a greedy colouring.

10.4.15 Decompose the polygon into triangles using its diagonals and use Exercise 10.4.14.

10.4.16 Use Exercises 10.4.1 and 10.4.9.

10.4.17 Use induction or the four colour theorem.

10.4.18 The chromatic number of a planar graph is at most 4.

10.4.19 Use induction on n.

10.4.20 Use induction on the number of inside edges to show that $\sum_i (i-2)f_i' = n-2$ and a similar result for the outside edges.

Chapter 11. Edges and Cycles

11.4.1 Use the definition of the line graph.

11.4.2 Show that the Petersen graph without any of its perfect matchings is formed by two cycles of length 5.

11.4.3 Use Exercise 11.4.2.

11.4.4 Use the greedy colouring.

11.4.5 Use the definition of a Hamiltonian cycle.

11.4.6 Use the definition of a Hamiltonian cycle.

11.4.7 Use the definition of a Hamiltonian cycle.

11.4.8 Follow the proof of Theorem 11.2.3.

11.4.9 Use the pigeonhole principle.

11.4.10 Use the pigeonhole principle.

11.4.11 Use induction and the pigeonhole principle.

11.4.12 Use the pigeonhole principle.

11.4.13 Modify the argument which shows that any red-blue edge-colouring of K_6 results in a monochromatic triangle.

11.4.14 Use induction.

11.4.15 Consider the polygon whose vertices are among the five points and which contains all of them inside it.

11.4.16 Use 11.4.16.

11.4.17 Use induction.

11.4.18 The vertex set can be partitioned into 4 cliques.

11.4.19 Use 11.4.18 and the pigeonhole principle.

11.4.20 Use induction.

Chapter 12. Regular Graphs

12.9.1 If A is the adjacency matrix of X, then the adjacency matrix of $\overline{X}$ is $J - I - A$.

12.9.2 If J is a linear combination of powers of A, then $AJ = JA$. Compare the ij-th entry of AJ and JA. If X is k-regular and connected, then the minimal polynomial of A has the form $(\lambda - k)p(\lambda)$. This means each column of $p(A)$ is an eigenvector of A corresponding to the eigenvalue k.

12.9.3 Use 12.9.2.

12.9.4 Calculate A^2 in terms of I, A and J.

12.9.5 If x is an eigenvector of an eigenvalue $\lambda \neq k$, then $Jx = 0$.

12.9.6 If a is a non-square in $\mathbb{F}_q$, then $x \mapsto ax$ is a bijection between squares and non-squares in $\mathbb{F}_q$.

12.9.7 Use 4.5.19.

12.9.8 Use 12.9.1 and 12.9.7 .

12.9.9 Multiply the matrix C by the column vector $[1, \omega^j, \omega^{2j}, \ldots, \omega^{(n-1)j}]^t$, for $j \in \{0, 1, \ldots, n-1\}$.

12.9.10 Find an isomorphism between $L(K_5)$ and the Petersen graph.

12.9.11 Use 12.9.7, 12.9.8 and 12.9.9.

12.9.12 The matrix C can be written as the sum of n circulant matrices.

12.9.13 Calculate the sum of the elements in each row of the adjacency matrix.

12.9.14 Use 12.9.11 and 12.9.12.

12.9.15 Consider the adjacency matrix of C_n.

12.9.16 Use 12.9.14.

12.9.17 Consider the adjacency matrix of M_{2n}.

12.9.18 Use 12.9.14.

12.9.19 Use the definition of a Ramanujan graph.

12.9.20 Use the definition of W.