CHAPTER 12

Regular Graphs

12.1. Eigenvalues of Regular Graphs

Recall that a k-regular graph is one in which every vertex has degree k. Thus, every row sum (and hence every column sum) of its adjacency matrix A is k. We have seen (see Exercise 4.5.1) that k is an eigenvalue of A. Moreover, it is easy to see that all the eigenvalues λ satisfy $|\lambda| \leq k$. Indeed, let $v = (x_1, ..., x_n)^t$ be an eigenvector with eigenvalue λ. Then

$$\lambda v = Av$$

implies that

$$\lambda x_i = \sum_{(i,j)\in E} x_j.$$

Without loss of generality, we may suppose $|x_1| = \max_i |x_i|$. Then,

$$|\lambda||x_1| \leq k|x_1|,$$

from which we infer $|\lambda| \leq k$. A similar argument shows that if X is connected, then the multiplicity of $\lambda_0 = k$ is one. In fact, the same argument shows that the multiplicity of $\lambda_0 = k$ is the number of connected components of X. To see this, let $v = (x_1, ..., x_n)^t$ be an eigenvector corresponding to the eigenvalue k and without loss of generality, suppose $|x_1|$ is maximal as before. We may also suppose $x_1 > 0$. Then,

$$kx_1 = \sum_{(1,j)\in E} x_j \leq kx_1$$

which means that there is no cancelation in the sum and all the x_j's are equal to x_1.

Thus, if X is a connected k-regular graph, we may arrange the eigenvalues as

$$k = \lambda_0(X) > \lambda_1(X) > \cdots > \lambda_n(X) \geq -k.$$

It is not difficult to show that $-k$ is an eigenvalue of X if and only if X is bipartite, in which case, its multiplicity is again equal to the number of connected components.

S. M. Cioabă and M. R. Murty, *A First Course in Graph Theory and Combinatorics*

Indeed, we have already observed (see Theorem 4.3.1) that the eigenvalues of the adjacency matrix of a bipartite graph occur in pairs λ_i, λ_j with $\lambda_i = -\lambda_j$. To show that if $-k$ is an eigenvalue of a connected k-regular graph X, that X must be bipartite, we let $(x_1, ..., x_n)$ be an eigenvector corresponding to $-k$. Then,

$$-kx_i = \sum_{j=1}^{n} a_{ij}x_j$$

implies

$$k|x_i| \leq \sum_{j=1}^{n} a_{ij}|x_j| \leq k|x_i|$$

if i is an index such that $|x_i|$ is maximal among the absolute values of the components of $(x_1, ..., x_n)$. The above inequality implies that we must have $|x_i| = |x_j|$ for any j adjacent to i. Since the graph is connected, this must be true of every component. Since the eigenvector is non-zero, each component must be strictly positive or strictly negative. Now let A be the vertices i such that $x_i > 0$ and B the vertices where $x_i < 0$. We can now show that A and B are independent sets. Indeed, if $x_i > 0$, then the relation

$$-kx_i = \sum_{j=1}^{n} a_{ij}x_j$$

shows that if we let a_i be the number of vertices in A adjacent to i and b_i the number of vertices adjacent to i in B, then

$$a_i - b_i = -k.$$

But $a_i + b_i = k$ so we deduce $2a_i = 0$. Hence, if $-k$ is an eigenvalue of a k-regular graph, then X is bipartite.

Any eigenvalue $\lambda_i \neq \pm k$ is referred to as a non-trivial eigenvalue. We denote by $\lambda(X)$ the maximum of the absolute values of all the non-trivial eigenvalues. We will see in the next sections that $\lambda(X)$ has closed connections with the structure of X.

12.2. Diameter of Regular Graphs

Recall that we defined a metric on a connected graph by defining the distance $d(x, y)$ for $x, y \in V$ as the minimal length amongst all the paths from x to y. The **diameter** of a connected graph was then the maximum value of the distance function. We begin by deriving an estimate for the diameter involving $\lambda(X)$ due to Fan Chung. If A is the adjacency matrix, then the (x, y)-th entry of A^r is the number of paths from x to y of length r. Hence, if m is the diameter of X, then every entry of A^m is strictly positive.

Let $n = |V|$ and $u_0, u_1, ..., u_{n-1}$ be an orthonormal basis of eigenvectors of A with corresponding eigenvalues $\lambda_0, ..., \lambda_{n-1}$ respectively. We may take $u_0 = u/\sqrt{n}$ where $u = (1, 1, ..., 1)$ as defined earlier. We can write

$$A = \sum_{i=0}^{n-1} \lambda_i u_i u_i^t.$$

More generally,

$$A^r = \sum_{i=0}^{n-1} \lambda_i^r u_i u_i^t.$$

In particular, we see that the (x, y)-th entry of A^m is

$$= \sum_i \lambda_i^m (u_i u_i^t)_{x,y}$$

which is

$$\geq \frac{k^m}{n} - \left| \sum_{i \geq 1} \lambda_i^m (u_i)_x (u_i)_y \right|.$$

Let us assume that X is not bipartite (so that $-k$ is not an eigenvalue. Then, by the Cauchy-Schwarz inequality,

$$\left| \sum_{i \geq 1} \lambda_i^m (u_i)_x (u_i)_y \right| \leq \lambda(X)^m \left(\sum_{i \geq 1} (u_i)_x^2 \right)^{1/2} \left(\sum_{i \geq 1} (u_i)_y^2 \right)^{1/2},$$

which is easily seen to be

$$\leq \lambda(X)^m (1 - (u_0)_x^2)^{1/2} (1 - (u_0)_y^2)^{1/2} \leq \lambda_1^m (1 - 1/n).$$

Thus, (x, y)-th entry of A^m is always positive if

$$\frac{k^m}{\lambda(X)^m} > n - 1.$$

If X is bipartite, it is easy to see that we get

$$\frac{2k^m}{\lambda(X)^m} > n - 1.$$

In other words, we have proved

THEOREM 12.2.1. *Let X be a k-regular graph with n vertices and diameter m. If X is not bipartite, then*

$$m < \frac{\log(n-1)}{\log(k/\lambda(X))}.$$

If X is bipartite, then we have the sharper inequality

$$m < \frac{\log[(n-1)/2]}{\log(k/\lambda(X))}.$$

This inequality also shows that regular graphs with small $\lambda(X)$, have small diameter. In communication theory, one requires the network to have small diameter for efficient operation. Note that the diameter of a connected, k-regular graph X on n vertices is always at least $\frac{\log(n-1)-2}{\log k}$ (see Exercise 5.5.20). The best upper bound obtained from the previous result is about twice as large as this lower bound.

At this point, a natural question is how small can $\lambda(X)$ be ? The following elementary observation about the eigenvalue $\lambda(X)$ is worth making. Observe that the eigenvalues of A^2 are simply the squares of the eigenvalues of A. On the other hand, the trace of A^2 is simply kn for a k-regular graph X. Thus, if X is not bipartite,

$$k^2 + (n-1)\lambda(X)^2 \geq kn$$

which gives the inequality

$$\lambda(X) \geq \left(\frac{n-k}{n-1}\right)^{1/2} \sqrt{k}.$$

If X is bipartite, then

$$2k^2 + (n-2)\lambda(X)^2 \geq nk,$$

in which case

$$\lambda(X) \geq \left(\frac{n-2k}{n-2}\right)^{1/2} \sqrt{k}.$$

If we think of k as fixed and $n \to \infty$, then we see that

$$\lim_{n\to\infty} \lambda(X) \geq \sqrt{k}.$$

An asymptotic version of a theorem of Alon and Bopanna from 1986 asserts that

$$\text{(12.2.1)} \qquad \operatorname{liminf}_{n\to\infty}\lambda(X_{n,k}) \geq 2\sqrt{k-1}$$

where the limit is taken over k-regular graphs with n going to infinity. Several proofs of this result exist in the literature. A sharper version was derived by Nilli in 1991.

THEOREM 12.2.2. *Suppose that X is a k-regular graph. Assume that the diameter of X is $\geq 2b+2 \geq 4$. Then*

$$\lambda_1(X) \geq 2\sqrt{k-1} - \frac{2\sqrt{k-1}-1}{b}.$$

Let us make the following observation. If $m = d(u, v)$ is the diameter of X, then the number of paths from u of length m is $\leq k^m$ and as each such path has $m + 1$ vertices, we deduce that the number of vertices n satisfies the inequality

$$n \leq (m+1)k^m.$$

Thus, if k is fixed and $n \to \infty$, then the diameter also tends to infinity. In particular, Theorem 12.2.2 implies inequality (12.2.1) since $\lambda(X) \geq \lambda_1(X)$.

We preface our proof of Theorem 12.2.2 by recalling the Rayleigh-Ritz Theorem from Chapter 6, Section 6.8. Let A be a symmetric matrix (a similar analysis applies to Hermitian matrices). Let $\lambda_{\max}$ and λ_{min} be the largest and smallest eigenvalues of A respectively. Then, we have

$$\lambda_{\max} = \max_{v \neq 0} \frac{(Av, v)}{(v, v)}$$

and

$$\lambda_{\min} = \min_{v \neq 0} \frac{(Av, v)}{(v, v)}.$$

Now let $L(X)$ denote the space of real-valued functions on X. We can equip the vector space $L(X)$ with an inner product by defining

$$(f, g) = \sum_{x \in X} f(x)g(x).$$

We can view the adjacency matrix as acting on $L(X)$ via the formula

$$(Af)(x) = \sum_{(x,y) \in E(X)} f(y).$$

For a connected k-regular graph, $\lambda_0 = k$ is an eigenvalue of multiplicity 1 and the corresponding eigenspace is the set of constant functions. Hence, we can decompose our space as

$$L(X) = \mathbf{R} f_0 \oplus L_0(X)$$

where $f_0 \equiv 1$ and $L_0(X)$ is the space of functions orthogonal to f_0. Thus, we can consider A as operating on $L_0(X)$. By the Rayleigh-Ritz theorem,

$$\lambda_1(X) = \max_{\substack{f \neq 0 \\ (f, f_0) = 0}} \frac{(Af, f)}{(f, f)}.$$

Since we want a lower bound for $\lambda_1(X)$, it is natural to consider the matrix $\Delta = kI - A$ whose eigenvalues are easily seen to be $k - \lambda_i$

$(0 \leq i \leq n-1)$. (Δ is a discrete analogue of the classical Laplace operator.) Thus,

$$k - \lambda_1(X) = \min_{\substack{f \neq 0 \\ (f, f_0)=0}} \frac{(\Delta f, f)}{(f, f)}.$$

The strategy now is to find an appropriate function f, obtain an upper bound for (f,f) and a lower bound for $(\Delta f, f)$. We can now prove Theorem 12.2.2.

PROOF. Let $u, v \in G$ be such that $d(u,v) \geq 2b+2$. For $i \geq 0$, define sets

$$U_i = \{x \in G : d(x,u) = i\}$$
$$V_i = \{x \in G : d(x,v) = i\}.$$

Then, the sets $U_0, U_1, ..., U_b, V_0, V_1, ..., V_b$ are disjoint, for otherwise, by the triangle inequality we get $d(u,v) \leq 2b$ which is a contradiction. Moreover, no vertex of

$$U = \cup_{i=0}^{b} U_i$$

is adjacent to

$$V = \cup_{i=0}^{b} V_i$$

for otherwise $d(u,v) \leq 2b+1$ which is a contradiction. For each vertex in U_i, at least one lies in U_{i-1} and at most $q = k-1$ lie in U_{i+1} (for $i \geq 1$). Thus,

$$|U_{i+1}| \leq q|U_i|.$$

By the same logic, $|V_{i+1}| \leq q|V_i|$. By induction, we see that $|U_b| \leq q^{(b-i)}|U_i|$ and $|V_b| \leq q^{(b-i)}|V_i|$. We will set $f(x) = f_i$ for $x \in U_i$, $f(x) = g_i$ for $x \in V_i$ and zero otherwise, with the f_i and g_i to be chosen later. Now,

$$(f,f) = A + B$$

where

$$A = \sum_{i=0}^{b} f_i^2 |U_i|$$

and

$$B = \sum_{i=0}^{b} g_i^2 |V_i|.$$

By the inequalities derived above, we get

$$(f,f) \geq \sum_{i=0}^{b} f_i^2 q^{-(b-i)} |U_b| + \sum_{i=0}^{b} g_i^2 q^{-(b-i)} |V_b|.$$

We now choose $f_0 = \alpha$, $g_0 = \beta$, $f_i = \alpha q^{-(i-1)/2}$ and $g_i = \beta q^{-(i-1)/2}$ for $i \geq 1$. Thus,

$$(f, f) \geq (\alpha^2 + \beta^2)\left(1 + b\frac{|V_b|}{q^{b-1}}\right).$$

We choose α and β so that $(f, f_0) = 0$.

Now we derive an upper bound for $(\Delta f, f)$. Note that

$$\frac{1}{2}\sum_{(x,y)\in E} (f(x) - f(y))^2 = k(f, f) - (Af, f) = (\Delta f, f)$$

by an easy calculation. Let A_U denote the sum

$$\frac{1}{2}\sum_{\substack{(x,y)\in E \\ x \text{ or } y \in U}} (f(x) - f(y))^2$$

and let A_V be defined similarly. If we partition according to the contribution from each U_i and keep in mind that each $x \in U_i$ has at most $q = k - 1$ neighbours in U_{i+1}, we obtain

$$A_U \leq \sum_{i=1}^{b-1} |U_i| q \left(q^{-(i-1)/2} - q^{-i/2}\right)^2 \alpha^2 + |U_b| q \cdot q^{-(b-1)}\alpha^2.$$

This is easily computed to be

$$= (\sqrt{q} - 1)^2 \left(|U_1| + |U_2| q^{-1} + \cdots + |U_{b-1}| q^{-(b-2)} + |U_b| q^{-(b-1)}\right)\alpha^2$$

$$+\alpha^2(2\sqrt{q} - 1)|U_b| q^{-(b-1)}.$$

Consequently,

$$A_U \leq (\sqrt{q} - 1)^2 (A - \alpha^2) + (2\sqrt{q} - 1)\frac{A - \alpha^2}{b}$$

which is less than

$$\left(1 + q - 2\sqrt{q} + \frac{2\sqrt{q} - 1}{b}\right) A.$$

Similarly,

$$A_V < \left(1 + q - 2\sqrt{q} + \frac{2\sqrt{q} - 1}{b}\right) B.$$

Combining these inequalities gives

$$k - \lambda_1(X) < 1 + q - 2\sqrt{q} + \frac{2\sqrt{q} - 1}{b}$$

which proves the theorem. ■

12.3. Ramanujan Graphs

The previous theorem motivates the definition of a **Ramanujan graph**. A k-regular graph is said to be **Ramanujan** if

$$\lambda(X) \leq 2\sqrt{k-1}.$$

This notion was introduced by Lubotzky, Phillips and Sarnak in a fundamental paper from 1986 in which they constructed infinite families of k-regular Ramanujan graphs whenever $k-1$ is a prime power. The graphs were named after Srinivasan Ramanujan (1887-1920) because the construction obtained by Lubotzky, Phillips and Sarnak and independently by Margulis, used deep number theoretic results related a conjecture of Ramanujan.

In view of the Alon-Bopanna theorem, these graphs are extremal with respect to the property of trying to minimize $\lambda(X)$ in the class of all k-regular graphs. Given $k \geq 3$, the explicit construction of an infinite family of k-regular Ramanujan graphs is still a major unsolved problem for any given k. So far, such constructions have been possible using deep results from algebraic geometry and number theory and only when $k-1$ is a prime power. For example, no one has been able to construct an infinite family of 7-regular Ramanujan graphs.

The complete graph K_n is an $(n-1)$-regular Ramanujan graph. Also, the cycle graph C_n is a 2-regular Ramanujan graph.

In section 4, we will construct a family of regular graphs using group theory and determine explicitly the eigenvalues of the adjacency matrix in terms of group characters. This will allow us to construct some explicit examples of Ramanujan graphs.

12.4. Basic Facts about Groups and Characters

A group G is a set together with a binary operation $*$ (say) satisfying the following axioms:

(1) $a, b \in G$ implies $a * b \in G$ (closure);
(2) $a, b, c \in G$ implies $(a * b) * c = a * (b * c)$ (associativity);
(3) there is an element called the **identity** $e \in G$ such that $a * e = e * a = a$ for all $a \in G$ (identity element);
(4) for any $a \in G$, there is a $b \in G$ so that $a * b = b * a = e$ (inverses); we write a^{-1} to denote the inverse of a.

If in addition to this, $a * b = b * a$ for all $a, b \in G$, we say that G is abelian or commutative. When G is finite, we call the size of G the **order** of G. Note also that in a group, we have the **cancelation law**: $a * b = a * c$ implies $b = c$ since we can multiply both sides on the left by a^{-1}. Warning: if $a * b = c * a$, we cannot necessarily conclude that $b = c$.

See example 6 below. The cancelation law also shows that the identity element is unique because if there were two e, e' say, then $a = ae = ae'$ and we deduce $e = e'$.

The reason for studying groups in the abstract is that many scientific discoveries can be formulated in the language of group theory. In addition, the fundamental particles in the heart of the atom seem to know everything about non-abelian groups! In fact, the character theory of certain subgroups of the group $GL_2(\mathbf{C})$ (see example 6 below) led to the discovery of new sub-atomic particles in the early 20th century.

Here are some examples of groups.

(1) $\mathbf{Z}$ under addition.
(2) $\mathbf{Z}$ under multiplication is not a group since there are no inverses.
(3) $\mathbf{R}^*$, non-zero reals under multiplication.
(4) $\mathbf{C}^*$, non-zero complex numbers under multiplication.
(5) $\mathbf{C}$ and $\mathbf{R}$ under addition.

All of these are examples of infinite abelian groups.

(6) $GL_2(\mathbf{R})$, or $GL_2(\mathbf{C})$ the collection of 2×2 invertible matrices with entries in $\mathbf{R}$ or $\mathbf{C}$ is a group under multiplication.

These are infinite non-abelian groups. Notice that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We **cannot** cancel the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

from both sides of the equation!

(7) $\mathbf{Z}/n\mathbf{Z}$ under addition consists of residue classes modulo n. This is a finite abelian group of order n.
(8) $\mathbf{Z}/6\mathbf{Z}$ with multiplication is not a group since the residue class 2 does not have an inverse.
(9) $(\mathbf{Z}/p\mathbf{Z})^*$ is the set of coprime residue classes mod p, with p prime. This is a finite abelian group of order $p - 1$.

To indicate $a * b$ we sometimes drop the $*$ and simply write ab with no cause for confusion. There is a general tendency to use the multiplicative notation for writing the group law although there is non universal convention about this. Part of the reason for this is to emphasize that the groups we are dealing with need not be abelian. There is also a tendency to use the symbol 1 to denote the identity element (and 0 when we write the group additively).

(10) The symmetries of the equilateral triangle, namely rotation by 60 degrees denoted r and a flip about the vertical axis f generates a non-abelian group of order 6. This group is isomorphic to the group of permutations on 3 letters.

THEOREM 12.4.1. *If G is a finite abelian group of order n, then $g^n = 1$ for any element $g \in G$.*

PROOF. Let $g_1, ..., g_n$ be the distinct elements of G. The elements

$$gg_1, gg_2, ..., gg_n$$

are also distinct and therefore must be all of the elements of the group. Thus,

$$g_1 \cdots g_n = gg_1 \cdots gg_n = g^n(g_1 \cdots g_n)$$

and canceling by $(g_1 \cdots g_n)$, we deduce the result. ■

This theorem can be thought of as a generalization of Fermat's little theorem which says that if p is prime and a is coprime to p, then

$$a^{p-1} \equiv 1(\bmod p).$$

Theorem 1 is true for non-abelian groups also and is due to Lagrange.

A group is called **cyclic** if there is an element g_0 such that every element of the group is of the form g_0^m for some integer m. For instance, $\mathbf{Z}$ is a cyclic group under addition with generator 1. As any cyclic group is countable, the group of non-zero reals under multiplication and the group of additive reals are not cyclic groups. The group of residue classes mod n under addition, is a cyclic group with generator being the residue class 1. Any coprime residue class will also serve as a generator.

A character χ of a group G is a map

$$\chi : G \rightarrow \mathbf{C}^*$$

such that $\chi(ab) = \chi(a)\chi(b)$. It is an example of a **homomorphism**. The character that sends every element to the element 1 is called the **trivial character**. Notice that any character of a group must take the identity element to 1 because $\chi(1^2) = \chi(1) = \chi(1)^2$ and the only non-zero complex number z satisfying $z^2 = z$ is $z = 1$. Another thing to notice is that $\chi(a^{-1}) = \chi(a)^{-1}$ since $1 = \chi(aa^{-1}) = \chi(a)\chi(a^{-1})$ from which the result is immediate. By Theorem 1, we deduce that if G is a finite group of order n,then $\chi(g)$ must be an n-th root of unity since $1 = \chi(g^n) = \chi(g)^n$.

The basic idea of character theory is that to understand the abstract group G, we map into something concrete like the multiplicative group of complex numbers and see how the image looks like to deduce what G looks like. It turns out that if G is a finite abelian group of order n, then there are exactly n distinct characters that one can construct. The set of characters in turn forms a group under multiplication of characters. Indeed, we define for two characters χ and ψ, the product character

$$(\chi\psi)(a) := \chi(a)\psi(a).$$

We call this the character group of G and denote it by $\hat{G}$. The identity element of $\hat{G}$ is the trivial character. The character inverse to χ is χ^{-1} defined by

$$\chi^{-1}(a) = \chi(a)^{-1}.$$

In the case of the additive group of residue classes mod n, all of the characters are given by

$$\chi_j(a) = e^{2\pi ija/n}, \quad j = 0, 1, 2, ..., (n-1).$$

Notice that χ_0 is the trivial character.

12.5. Cayley Graphs

There is a simple procedure to constructing k-regular graphs using group theory. This can be described as follows. Let G be a finite group and S a k-element subset of G. We suppose that S is *symmetric* in the sense that $s \in S$ implies $s^{-1} \in S$. Now construct the graph $X(G, S)$ by having the vertex set to be the elements of G the (x, y) is an edge if and only if $xy^{-1} \in S$.

The eigenvalues of the Cayley graph are easily determined as follows. The cognoscenti will recognize that it is the classical calculation of the Dedekind determinant in number theory.

THEOREM 12.5.1. *Let G be a finite abelian group and S a symmetric subset of G of size k. Then the eigenvalues of the adjacency matrix of $X(G, S)$ are given by*

$$\lambda_\chi = \sum_{s \in S} \chi(s)$$

as χ ranges over all the irreducible characters of G.

REMARK 12.5.2. Notice that for the trivial character, we have $\lambda_1 = k$. If we have for all $\chi \neq 1$

$$\left| \sum_{s \in S} \chi(s) \right| < k$$

then the graph is connected by our earlier remarks. Thus, to construct Ramanujan graphs, we require

$$\left| \sum_{s \in S} \chi(s) \right| \leq 2\sqrt{k-1}$$

for every non-trivial irreducible character χ of G. This is the strategy employed in many of the explicit construction of Ramanujan graphs.

PROOF. For each irreducible character χ, let v_χ denote the vector $(\chi(g) : g \in G)$. Let $\delta_S(g)$ equal 1 if $g \in S$ and zero otherwise and denote by A the adjacency matrix of $X(G, S)$. Then,

$$(Av_\chi)_x = \sum_{g \in S} \delta_S(xg^{-1})\chi(g).$$

By replacing xg^{-1} by s, and using the fact that S is symmetric, we obtain

$$(Av_\chi)_x = \chi(x)\left(\sum_{s \in S} \chi(s)\right)$$

which shows that v_χ is an eigenvector with eigenvalue

$$\sum_{s \in S} \chi(s)$$

which completes the proof. ∎

As mentioned above, this calculation is reminiscent of the Dedekind determinant formula in number theory. Recall that this formula computes $\det A$ where A is the matrix whose (i, j)-th entry is $f(ij^{-1})$ for any function f defined on the finite abelian group G of order n. The determinant is

$$\prod_\chi \left(\sum_{g \in G} f(g)\chi(g)\right).$$

The proof is analogous to the calculation in the proof of Theorem 3 and we leave it to the reader. As an application, it allows us to compute the determinant of a circulant matrix. For instance, we can compute the characteristic polynomial of the complete graph. Indeed, it is not hard to see that by taking the additive cyclic group of order n and setting $f(0) = -\lambda$, $f(a) = 1$ for $a \neq 0$, we obtain that the characteristic polynomial is

$$(-1)^n(\lambda - (n-1))(\lambda - 1)^{n-1}$$

by the Dedekind determinant formula. As the complete graph of order n is an $(n-1)$-regular graph, we see immediately from the above calculation that it is a Ramanujan graph.

If G is an abelian group and S is a subset of G, we can define another set of graphs $Y(G, S)$ called *sum graphs* as follows. The vertices consist of elements of G and (x, y) is an edge if $xy \in S$. Arguing as before, we can show

THEOREM 12.5.3. *Let G be an abelian group. For each character χ of G, the eigenvalues of $Y(G, S)$ are given as follows. Define*

$$e_\chi = \sum_{s \in S} \chi(s).$$

If $e_\chi = 0$, then v_χ and v_χ^{-1} are both eigenvectors with eigenvalues zero. If $e_\chi \neq 0$, then

$$|e_\chi| v_\chi \pm e_\chi v_{\chi^{-1}}$$

are two eigenvectors with eigenvalues $\pm|e_\chi|$.

Using this theorem, Winnie Li constructed Ramanujan graphs in the following way. Let $\mathbf{F}_q$ denote the finite field of q elements. Let $G = \mathbf{F}_{q^2}$ and take for S the elements of G of norm 1. This is a symmetric subset of G and the Cayley graph $X(G, S)$ turns out to be Ramanujan. The latter is a consequence of a theorem of Deligne estimating Kloosterman sums.

These results allow us to construct Ramanujan graphs by estimating character sums.

There is a generalization of these results to the non-abelian context. This is essentially contained in a paper by Diaconis and Shahshahani. Using their results, one can easily generalize the Dedekind determinant formula as follows (and which does not seem to be widely known). Let G be a finite group and f a class function on G. Then the determinant of the matrix A whose rows (and columns) are indexed by the elements of G and whose (i, j)-th entry is $f(ij^{-1})$ is given by

$$\prod_\chi \left(\frac{1}{\chi(1)} \sum_{g \in G} f(g)\chi(g) \right)^{\chi(1)}$$

with the product over the distinct irreducible characters of G.

The following theorem is due to Diaconis and Shahshahani.

THEOREM 12.5.4. *Let G be a finite group and S a subset which is stable under conjugation. Let A be the adjacency matrix of the graph $X(G, S)$ (where $u, v \in G$ are adjacent if and only if $uv^{-1} \in S$). Then the eigenvalues of A are given by*

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s)$$

as χ ranges over all irreducible characters of G. Moreover, the multiplicity of λ_χ is $\chi(1)^2$.

We remark that the λ_χ in the above theorem need not be all distinct. For example, if there is a non-trivial character χ which is trivial on S, then the multiplicity of the eigenvalue $|S|$ is at least $1 + \chi(1)^2$.

PROOF. We essentially modify the proof of Diaconis and Shahshahani to suit our context. We consider the group algebra $\mathbf{C}[G]$ with basis vectors e_g with $g \in G$ and multiplication defined as usual by $e_g e_h = e_{gh}$. We define the linear operator Q by

$$Q = \sum_{s \in S} e_s = \sum_{g \in G} \delta_S(g) e_g$$

which acts on $\mathbf{C}[G]$ by left multiplication. The matrix representation of Q with respect to the basis vectors e_g with $g \in G$ is precisely the adjacency matrix of $X(G, S)$ as is easily checked. If r denotes the left regular representation of G on $\mathbf{C}[G]$, we find that the action of

$$r(A) = \sum_{s \in S} r(s)$$

on $\mathbf{C}[G]$ is identical to Q. Moreover, $\mathbf{C}[G]$ decomposes as

$$\mathbf{C}[G] = \oplus_\rho V_\rho$$

where the direct sum is over non-equivalent irreducible representations of G and the subspace V_ρ is a direct sum of deg ρ copies of the subspace W_ρ corresponding to the irreducible representation ρ. The result is now clear from basic facts of linear algebra. ∎

12.6. Expanders

For any subset A of the vertex set of a graph X, we may define the **edge-boundary** of A, denoted ∂A by

$$\partial A = \{xy \in E(X) : x \in A, y \notin A\}.$$

That is, the edge-boundary of A consists of the edges which are incident to precisely one vertex of A. The **edge-expansion** $h(X)$ of X equals the minimum of $\frac{|\partial A|}{|A|}$, where the minimum is taken over all subsets A of the vertex set of X of order at most $\frac{|V(X)|}{2}$. As many combinatorial invariants, the edge-expansion of a graph is hard to compute.

Let c be positive real number. A k-regular graph X with n vertices is called a c-**expander** if

$$h(X) \geq c. \tag{12.6.1}$$

A very important problem is constructing infinite families of k-regular c-expanders for fixed $k \geq 3$ and some $c > 0$. Expander graphs play an important role in computer science and the theory of communication networks. These graphs arise in questions about designing networks that connect many users while using only a small number of switches. Our interest in them lies in the fact the theory of c-expanders can be related to the eigenvalue questions of the previous section. This is done in the next theorem.

THEOREM 12.6.1 (Alon-Milman, Dodziuk). *Let X be a k-regular graph. Then*

$$\frac{k-\lambda_1(X)}{2} \leq h(X) \leq \sqrt{2k(k-\lambda_1(X))}.$$

PROOF. We prove only the first inequality, the second inequality is slightly more complicated.

The idea is to apply the Rayleigh-Ritz ratio in the following way. As observed in the previous section, let f be a function defined on $V(X)$ that is orthogonal to the constant function f_0. If $L = kI - A$ is the Laplacian matrix of X, then

$$\frac{(Lf,f)}{(f,f)} \geq k - \lambda_1(X)$$

by Rayleigh-Ritz inequality.

Let A be a subset of $V(X)$ of size at most $\frac{|V(X)|}{2}$. If we set

$$f(x) = \begin{cases} |V(X) \setminus A| & \text{if } x \in A \\ -|A| & \text{if } x \notin A \end{cases}$$

then it is easily seen that $(f, f_0) = 0$. On the other hand, a direct calculation shows that

$$(f,f) = |V(X)||A||V(X)\backslash A|.$$

By using the formula

$$(Lf,f) = \frac{1}{2}\sum_{(x,y)\in E}(f(x)-f(y))^2$$

we easily check that

$$(Lf,f) = |X|^2|\partial A|$$

so that by the previous we obtain

$$\frac{|\partial A|}{|A|} \geq (k-\lambda_1(X))\frac{|V(X)\setminus A|}{|A|} \geq \frac{k-\lambda_1(X)}{2}.$$

Since this inequality holds for each $A \subset V(X)$ of size at most $\frac{|V(X)|}{2}$, it follows that $h(X) \geq \frac{k-\lambda_1(X)}{2}$. ∎

The previous theorem shows that making λ_1 as small as possible gives us good expander graphs. By the Alon-Bopanna theorem, we cannot do better than

$$\lambda(X) \leq 2\sqrt{k-1}.$$

Thus, Ramanujan graphs make excellent expanders.

In 1973, Margulis gave the first explicit construction of an infinite family of 8-regular graphs. Given an nonnegative integer m, consider the graph G_m whose vertex set is $\mathbb{Z}_m \times \mathbb{Z}_m$. Each vertex (x,y) of G_m is adjacent exactly to $(x+y,y), (x-y,y), (x,y+x), (x,y-x), (x+y+1,y), (x-y+1,y), (x,y+x+1), (x,y-x+1)$ where all the operations are done modulo m. Varying m produces an infinite family of 8-regular graphs. Margulis showed these graphs are expanders by using results from group representations. In 1981, Gabber and Galil used harmonic analysis to show that any non-trivial eigenvalue of G_m has absolute value at most $5\sqrt{2} \approx 7.05 < 8$.

12.7. Counting Paths in Regular Graphs

If A is the adjacency matrix of X, it is clear that the (x,y)-th coordinate of A^r enumerates the number of paths of length r from x to y. We will be interested in proper paths, that is paths which do not have back-tracking. We are interested in counting the number of proper paths of length r in a k-regular graph. Let A_r denote the matrix whose (x,y)-th entry will be the number of proper paths from x to y. Then, $A_0 = I$ and $A_1 = A$ and clearly

$$A^2 = A_2 + kI$$

since A_2 encodes the number of proper paths of length 2.

Inductively, it is clear that

$$A_1 A_r = A_{r+1} + (k-1)A_{r-1},$$

since the left hand side enumerates paths of length $r+1$ which are extended from proper paths of length r and the right side enumerates first the proper paths of length $r+1$ and proper paths of length $r-1$ which are extended to 'improper' paths of length r.

This recursion allows us to deduce the following identity of formal power series:

Proposition 12.7.1.

$$\left(\sum_{r=0}^{\infty} A_r t^r\right)\left(I - At + (k-1)t^2\right) = (1-t^2)I.$$

12.8. The Ihara Zeta Function of a Graph

Let X be a k-regular graph and set $q = k - 1$. Motivated by the theory of the Selberg zeta function, Ihara was led to make the following definitions and construct the graph-theoretic analogue of it as follows. A proper path whose endpoints are equal is called a *closed geodesic*. If γ is a closed geodesic, we denote by γ^r the closed geodesic obtained by repeating the path γ r times. A closed geodesic which is not the power of another one is called a *prime geodesic*. We define an equivalence relation on the closed geodesics $(x_0, ..., x_n)$ and $(y_0, ..., y_m)$ if and only if $m = n$ and there is a d such that $y_i = x_{i+d}$ for all i (and the subscripts are interpreted modulo n. An equivalence class of a prime geodesic is called a *prime geodesic cycle*. Ihara then defines the zeta function

$$Z_X(s) = \prod_p \left(1 - q^{-s\ell(p)}\right)^{-1}$$

where the product is over all prime geodesic cycles and $\ell(p)$ is the length of p.

Ihara proves the following theorem:

THEOREM 12.8.1. *For $g = (q-1)|X|/2$, we have*

$$Z_X(s) = (1-u^2)^{-g} \det(I - Au + qu^2)^{-1}, \quad u = q^{-s}.$$

Moreover, $Z_X(s)$ satisfies the "Riemann hypothesis" (that is, all the singular points lie on $Re(s) = 1/2$) if and only if X is a Ramanujan graph.

PROOF. (Sketch) We assume that the zeta function has the shape given and show that it satisfies the Riemann hypothesis if and only if X is Ramanujan. Let $\phi(z) = \det(zI - A)$ be the characteristic polynomial of A. If we set $z = (1 + qu^2)/u$, then the singular points of the $Z_X(s)$ arise from the zeros of $\phi(z)$. Since

$$u = \frac{z \pm \sqrt{z^2 - 4q}}{2q}$$

and any zero of ϕ is real (because A is symmetric), we deduce that

$$\frac{z\overline{u}}{\overline{u}} = \frac{(1+qu^2)\overline{u}}{u\overline{u}} = \frac{\overline{u} + q|u|^2 u}{|u|^2}$$

is also real. Thus, the numerator is real and so, we must have

$$q|u|^2 = 1,$$

which is equivalent to the assertion of the theorem. ∎

12.9. Exercises

EXERCISE 12.9.1. If X is a k-regular graph with eigenvalues $k = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$, determine the eigenvalues of the complement of X.

EXERCISE 12.9.2. A graph X is regular and connected if and only if J is a linear combination of powers of the adjacency matrix A of X.

EXERCISE 12.9.3. Let $k = \lambda_0 > \lambda_1 > \cdots > \lambda_{s-1}$ be the distinct eigenvalues of the adjacency matrix A of a k-regular connected graph X with n vertices. Show that

$$J = \frac{n}{\prod_{i=1}^{s-1}(k - \lambda_i)} \cdot \prod_{i=1}^{s-1} (A - \lambda_i I_n).$$

EXERCISE 12.9.4. A graph X is **strongly regular** with **parameters** (n, k, a, c) if it is k-regular, every pair of adjacent vertices has a common neighbours and every pair of non-adjacent vertices has c common neighbours. Show that the adjacency matrix A of a strongly regular graph X with parameters (n, k, a, c) satisfies the equation

$$A^2 - (a - c)A - (k - c)I = cJ.$$

EXERCISE 12.9.5. Calculate the eigenvalues of a strongly regular graph X with parameters (n, k, a, c).

EXERCISE 12.9.6. Let $q \equiv 1 \pmod 4$ be a power of a prime. The **Paley graph** $\mathbb{P}_q$ has vertices the elements of the field $\mathbb{F}_q$ with x adjacent to y if $x - y$ is a square in $\mathbb{F}_q$. Show that $\mathbb{P}_5 = C_5$ and that $\mathbb{P}_q$ is a strongly regular graph with parameters $\left(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}\right)$.

EXERCISE 12.9.7. Calculate the eigenvalues of the line graph $L(K_n)$ of the complete graph K_n.

EXERCISE 12.9.8. Calculate the eigenvalues of the complement of the line graph of K_n.

EXERCISE 12.9.9. An $n \times n$ matrix C is called a **circulant matrix** if row i of C is obtained from the first row of C by a cyclic shift of $i - 1$ steps for each $i \in [n]$. Let Z be the $n \times n$ circulant matrix whose first row is $[0, 1, 0, \ldots, 0]$. Show that the eigenvalues of Z are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$.

EXERCISE 12.9.10. Show that the Petersen graph is isomorphic to the complement of the line graph of K_5.

EXERCISE 12.9.11. Calculate the eigenvalues of the Petersen graph.

EXERCISE 12.9.12. Let C be an $n \times n$ circulant matrix whose first row is $[c_1, c_2, \ldots, c_n]$. Show that

$$C = \sum_{i=1}^{n} c_i Z^{i-1},$$

where Z is the $n \times n$ circulant matrix whose first row is $[0, 1, 0, \ldots, 0]$.

EXERCISE 12.9.13. A **circulant graph** is a graph X whose adjacency matrix is a circulant matrix. Show that a circulant graph is regular.

EXERCISE 12.9.14. If $[0, c_2, \ldots, c_n]$ is the first row of the adjacency matrix C of a circulant graph X, show that the eigenvalues of C are

$$\lambda_s = \sum_{i=2}^{n} a_i \omega^{(i-1)s},$$

for $s \in \{0, 1, \ldots, n-1\}$ and $\omega = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$.

EXERCISE 12.9.15. Show that the cycle C_n with n vertices is a circulant graph.

EXERCISE 12.9.16. Calculate the eigenvalues of the cycle C_n.

EXERCISE 12.9.17. The **Möbius ladder** M_{2n} is the 3-regular graph on $2n$ vertices which is obtained from the cycle C_{2n} by joining each pair of opposite vertices. Show that the Möbius ladder is a circulant graph.

EXERCISE 12.9.18. Show that the eigenvalues of the Möbius ladder M_{2n} are

$$\lambda_s = 2\cos\left(\frac{\pi s}{n}\right) + (-1)^s,$$

for $s \in \{0, 1, \ldots, 2n-1\}$.

EXERCISE 12.9.19. Determine which of the graphs $L(K_n), \overline{L(K_n)}$ and M_{2n} are Ramanujan.

EXERCISE 12.9.20. Let X be a graph with n vertices and let $w_{i,j}(r)$ denote the number of walks of length r between the vertices i and j of X. If W is the matrix whose (i,j)-th entry is

$$W_{i,j} = \sum_{r=1}^{\infty} w_{i,j}(r) x^r$$

show that

$$W(I_n - xA) = I_n,$$

where A is the adjacency matrix of X.