CHAPTER 10

Planar Graphs

10.1. Euler's Formula

A graph is said to be **embedded** in the plane if it can be drawn on the plane so that no two edges intersect. Such a graph is called a **planar graph**. Graphs arising from maps are clearly planar. In fact, planar maps can be characterized as such. Any planar map cuts out the plane into **faces**. To be precise, a maximal region of the plane which does not contain in its interior a vertex of the graph is called a **face**. A finite plane graph has also one unbounded face called the **outer face**. The faces are pairwise disjoint. The basic relation for planar graphs is the following theorem due to Euler.

THEOREM 10.1.1 (Euler, 1758). *If X is a connected planar graph with v vertices, e edges and f faces, then*

$$v - e + f = 2.$$

PROOF. The proof will be by induction on the number of vertices. If $v = 1$, then X is a "bouquet" of loops. If in addition $e = 0$, then $f = 1$ and the formula is true in this case. Each added loop cuts the face into two faces and so increases the face count by 1. So the formula holds in case $v = 1$. For $v > 1$ and X connected, take an edge e_0 which is not a loop and the contraction of X by e_0 gives X/e_0. Contraction does not reduce the number of faces so X/e has $v - 1$ vertices, $e - 1$ edges, and f faces. Since X/e has fewer number of vertices, we can apply the induction hypothesis to get

$$(v-1) - (e-1) + f = 2 = v - e + f = 2$$

which is what we want to prove. ■

If X is not connected, then Euler's formula fails. If X is a planar graph with c connected components, then

$$v - e + f = c + 1.$$

This is easily seen be adding $c-1$ edges (or "bridges") and then applying Euler's formula to this connected graph. Adding the bridges does not alter the face count. Thus, we get

$$v - (e + c - 1) + f = 2$$

from which the formula follows. Euler's formula has many applications. The first is that we can derive some necessary conditions for a graph to be planar.

THEOREM 10.1.2. *If X is a simple planar graph with at least 3 vertices, then $e \leq 3v - 6$. If X is triangle-free, then $e \leq 2v - 4$.*

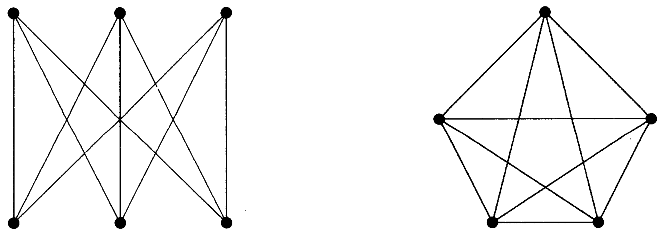
PROOF. It suffices to prove this for connected graphs. Every face must contribute at least three edges. But each edge appears in two faces. Thus, $3f \leq 2e$ and putting this into Euler's formula gives us

$$2 = v - e + f \leq v - e + \frac{2}{3}e$$

which gives the inequality

$$e \leq 3v - 6.$$

If X is triangle-free, then, each face contributes at least four edges. Since each edge appears in two faces, we get $2e \geq 4f$. Putting this back into Euler's formula gives the second inequality. ■



$K_{3,3}$ K_5

FIGURE 10.1

COROLLARY 10.1.3. *The graphs, K_5 and $K_{3,3}$ are non-planar.*

PROOF. If K_5 were planar, then applying the theorem gives10 $\leq$ $15 - 6 = 9$, a contradiction. For $K_{3,3}$, we get $9 \leq 18 - 6 = 12$ which does not give a contradiction if we use the first inequality. However, the bipartite graph has no triangles and so, by the second inequality, we get $9 \leq 8$, which is a contradiction. ■

A famous theorem of Kazimierz Kuratowski (1896-1980) proved in 1930 states that a graph is planar if and only if it can be (edge)-contracted to either K_5 or $K_{3,3}$. Thus, for example, as the Petersen graph (shown in Figure 10.2) can be contracted to K_5 by collapsing the edges connecting the "inside" cycle of 5 vertices to the outer five vertices, it is not planar.

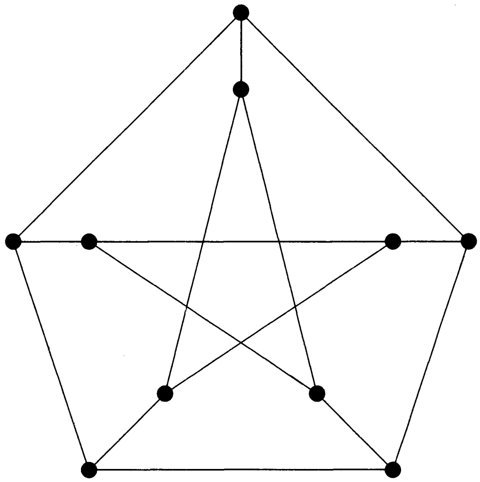


FIGURE 10.2. Petersen graph

THEOREM 10.1.4. *Every simple planar graph X contains a vertex of degree at most five.*

PROOF. If every vertex has degree at least six, then $2e \geq 6v$ which implies $e \geq 3v$. However, Theorem 10.1.2 implies $e \leq 3v - 6$ which is a contradiction. ∎

Now, we can prove the six-colour theorem:

THEOREM 10.1.5 (The six colour theorem). *Every map can be properly coloured using six colours.*

PROOF. We proceed by induction on the number of vertices (or regions) of the planar graph associated with the map. Suppose that all planar graphs with fewer than $n - 1$ vertices are 6-colourable. By Theorem 10.1.4, X contains a vertex of degree 5 or less. By induction, $X - v$ is 6-colourable and as v has degree 5 or less, we can colour it with one of the six colours not used on any of its adjacent vertices. ∎

10.2. The Five Colour Theorem

The four colour theorem has a colourful history! It states that any planar graph can be coloured using four colours. Since K_4 is planar, and has chromatic number 4, we see that four colours are necessary. To prove that this is sufficient is more difficult. The four colour conjecture was first formulated by Francis Guthrie on October 23, 1852. Guthrie was a student at University College London where he studied under Augustus de Morgan (1806-1871). When Guthrie asked de Morgan, he did not know how to prove it and wrote to Sir William Rowan Hamilton (1805-1865) in Dublin if he knew. It seems that Guthrie graduated and then studied law. After practicing as a barrister, he went to South Africa in 1861 as a professor of mathematics. After a few mathematical papers, he switched to the field of botany.

In the meanwhile, de Morgan circulated Guthrie's question to many mathematicians. Arthur Cayley, who learned of the question from de Morgan in 1878, posed it as a formal unsolved problem to the London Mathematical Society on 13 June, 1878. On 17 July 1879, Alfred Kempe (1849-1922), a London barrister and amateur mathematician announced in *Nature* that he had a proof. Kempe had studied under Cayley, and at Cayley's suggestion, submitted his paper to the *American Journal of Mathematics* in 1879. We will discuss Kempe's "proof" below. Apparently, Kempe received great acclaim for his work. He was elected Fellow of the Royal Society and served as its treasurer for many years. In 1912, he was knighted. The error in his "proof" was discovered in 1890 by Percy John Heawood (1861-1955), a lecturer in Durham, England. In his paper, Heawood showed how to salvage the proof and prove that every map is 5-colourable. We will now prove the following theorem due to Heawood.

THEOREM 10.2.1 (Heawood, 1890). *Any planar graph is* 5*-colourable.*

PROOF. We will prove the theorem by induction on the number of vertices. Let X be a planar graph on n vertices. The base case $n = 1$ is obvious. Assume $n \geq 2$. By Theorem 10.1.4, there is a vertex x of degree at most 5. The graph $Y = X \setminus \{x\}$ is also planar and by induction, Y can be coloured using at most 5 colours. If the degree of x is 4 or less, then x an be coloured with a colour not used for any of its adjacent vertices. This way, we can obtain a proper colouring of X with at most 5 colours. So we may suppose that x has degree 5. If any two of the neighbours of x get the same colour, then the previous argument shows how one can colour X with at most 5 colours. Let us label the neighbours of x as p, q, r, s, t and say that they are coloured in

Y with $1, 2, 3, 4, 5$ respectively, by the induction hypothesis. Denote by $X_{i,j}$ the subgraph of Y whose vertices are coloured with colour with i and j. Now consider $X_{1,3}$. Both p and r belong to $X_{1,3}$. If they lie in two distinct components, then, we may interchange colours 1 and 3 in the component containing r with the result that p and r are coloured using colour 1. Then, we can colour x using colour 3. If however, p and r lie in the same connected component of $X_{1,3}$, then this means there is a chain of vertices with alternating colours 1 and 3 from p to r. Now consider $X_{2,4}$. Both q and s belong to this subgraph. Again, if q and s lie in distinct connected components, we may interchange colours 2 and 4 in one of the components and free up one colour and use that to colour x. If q and s do not lie in the same connected component, then there is a path of alternating colours from q to s. But this path must cross the path from p to r and this would violate planarity. Thus, the second possibility cannot arise which means that we can use the same colour on q and s and thus, colour X with 5 colours. This finishes the proof. ∎

Kempe's "proof" performed this colour reversal technique twice and this leads to difficulties as Heawood pointed out. Here is Kempe's argument. As before, we proceed by induction on the number of vertices. Let x be a vertex of degree at most 5. If the degree of x is at most 4, then an argument as in Heawood's proof can be applied (and we leave this as an exercise to the reader). However, the proof breaks down when degree of x is 5 for the following reasons. Label the vertices adjacent to x as p, q, r, s, t and let us suppose that induction gave the colouring of vertices as shown in Figure 10.3. If there is no $2, 3$ colour chain between

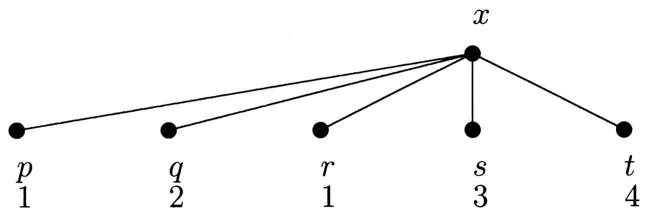


FIGURE 10.3

q and s, we can carry a colour reversal to free the colour 2 (say) for vertex x. If there is no $2, 4$ colour chain between q and t, we can carry out a colour reversal to free the colour 2 for vertex x. It looks as if we therefore have a situation indicated in Figure 10.4. Since there cannot be a $1, 3$ colour chain between p and s, a colour reversal can paint the

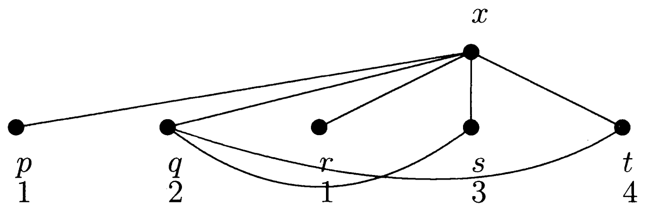


FIGURE 10.4

vertex p with colour 3. Since there cannot be a $1,4$ colour chain between r and t a colour reversal will paint the vertex r with colour 4. So it looks as if colour 1 is freed and we can use it to colour x. However, there is a gap in the reasoning. In the figure below, carrying out the reversals as indicated above will paint p and r with colour 1. Indeed,

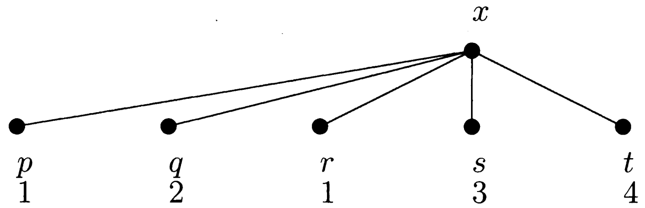


FIGURE 10.5

the colour reversal argument is valid for changing the colour of p and the colour of r. However, simultaneously changing the colour of p and r leads to difficulties. This is essentially what Heawood observed as the gap in Kempe's proof. He was able to salvage the argument to deduce the five colour theorem as we indicated above.

10.3. Colouring Maps on Surfaces of Higher Genus

As we mentioned earlier, $K_{3,3}$ is not a planar graph since we cannot draw it on the plane without intersecting edges. If however, we tried to draw it on a torus, then it is possible to draw the graph without crossing of edges as it can be verified easily. A celebrated theorem of Möbius (1870) is that any compact (orientable) surface is homeomorphic to a sphere with g handles. The **genus** of the surface is denoted g. A torus, for example, has genus one since it is homeomorphic to a sphere with one handle.

One can show that any graph X can be embedded in some compact orientable surface. The minimal genus of the surface for which this can be done is called the **genus of the graph**. For example, the genus of $K_{3,3}$ is 1.

For graphs embedded on a surface of genus g, Euler's formula generalizes as follows. A **face** is defined as before, as a maximal region cut out by the graph which contains no vertex of the graph in its interior. We state without proof the following result.

THEOREM 10.3.1 (Euler's formula). *If G is a connected graph of genus g, then*

$$v - e + f = 2 - 2g.$$

Using Euler's formula, we can prove as before that any simple graph of genus g has at most $3(v - 2 + 2g)$ edges. This is the analogue that a planar graph has at most $3(v - 2)$ edges. As before, summing up the degrees gives

$$2e \leq 6(v - 2 + 2g)$$

so that there has to be at least one vertex of degree

$$\leq \frac{6(v - 2 + 2g)}{v}.$$

This is the analog of the result for planar graphs which says there is at least one vertex of degree at most five. Now we can prove:

THEOREM 10.3.2 (Heawood, 1890). *Any graph X of genus g can be coloured with*

$$\left\lceil \frac{7 + \sqrt{1 + 48g}}{2} \right\rceil$$

colours provided $g > 0$. Here $\lceil x \rceil$ denotes the smallest integer larger than or equal to x.

REMARK 10.3.3. Notice that if $g = 0$ were allowed in the formula, then we deduce the four colour theorem.

PROOF. Let

$$c = \left\lceil \frac{7 + \sqrt{1 + 48g}}{2} \right\rceil.$$

If X has at most c vertices, we are done. So suppose that $v > c$. If we can show that every simple graph of genus g has a vertex of degree at most $c - 1$, then we can use an induction argument as before to complete the proof. Notice that

$$c^2 - 7c + (12 - 12g) \geq 0$$

so that

$$c - 1 \geq 6 + \frac{12(g - 1)}{c}.$$

Thus, from the remark before the statement of the theorem, we have that X has a vertex of degree at most

$$6 + \frac{12(g-1)}{v} \leq 6 + \frac{12(g-1)}{c} \leq c - 1$$

as desired. ∎

REMARK 10.3.4. Notice that $g \geq 1$ is used in a crucial way in the inequalities at the end of the proof.

For a long time, it was an outstanding problem to determine the genus of the complete graph. The **complete graph conjecture**, proved in 1968 by Gerhard Ringel and J.W.T. Youngs, states that the genus of K_n is

$$\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

10.4. Exercises

EXERCISE 10.4.1. The **girth** of a graph is the length of its shortest cycle (that is, closed path). Use Euler's formula to show that if X is a planar graph with girth γ, and v vertices, then the number of edges e of X satisfies the inequality

$$e \leq \frac{\gamma}{\gamma - 2}(v-2).$$

EXERCISE 10.4.2. Determine the girth of the Petersen graph (Figure 10.2) and use the previous question to deduce that it is not a planar graph.

EXERCISE 10.4.3. Let X be a graph with chromatic number $\chi(X) > 3$. Show that the genus $g(X)$ of a graph X satisfies the inequality

$$g(X) \geq \frac{1}{12}\left(\chi(X)^2 - 7\chi(X) + 12\right).$$

Deduce that for $n \geq 5$, the genus of the complete graph K_n is at least

$$\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

EXERCISE 10.4.4. Determine all r, s such that $K_{r,s}$ is a planar graph.

EXERCISE 10.4.5. Show that $K_5 \setminus e$ is planar for any edge e of K_5.

EXERCISE 10.4.6. Show that $K_{3,3} \setminus f$ is planar for any edge f of $K_{3,3} \setminus f$.

EXERCISE 10.4.7. Let G be the graph obtained from $K_{4,4}$ by deleting a perfect matching. Is G planar ?

EXERCISE 10.4.8. Let S be a set of n points in the plane such that the distance between any two of them is at least 1. Show that there are at most $3n-6$ pairs x, y such that the distance between x and y is 1.

EXERCISE 10.4.9. The **crossing number** of a graph X is the minimum number of crossings in a drawing of X in the plane. What are the crossing numbers of K_5 and $K_{3,3}$?

EXERCISE 10.4.10. Let X be a graph with n vertices and e edges. If k is the maximum number of edges in a planar subgraph of X, show that the crossing number of X is at least $e-k$. Prove that the crossing number of X is at least $e-3n+6$. If X has no triangles, then the crossing number is at least $e-2n+4$.

EXERCISE 10.4.11. Show that the crossing number of K_6 is 3.

EXERCISE 10.4.12. A planar graph X is **outerplanar** if it has a drawing with every vertex on the boundary of the unbounded face. Show that any cycle is outerplanar. Show that K_4 is planar, but not outerplanar.

EXERCISE 10.4.13. Show that $K_{2,3}$ is planar, but not outerplanar.

EXERCISE 10.4.14. Any outerplanar graph is 3-colourable.

EXERCISE 10.4.15. An art gallery is represented by a polygon with n sides. Show that it is possible to place $\lfloor \frac{n}{3} \rfloor$ guards such that every point interior to the polygon is visible to some guard. Construct a polygon that can be guarded by precisely $\lfloor \frac{n}{3} \rfloor$ guards.

EXERCISE 10.4.16. What is the crossing number of the Petersen graph ?

EXERCISE 10.4.17. Prove that every outerplanar graph has a vertex of degree at most 2.

EXERCISE 10.4.18. Show that every planar graph decomposes into two bipartite graphs.

EXERCISE 10.4.19. For any $n \geq 4$, construct a planar graph with n vertices and chromatic number 4.

EXERCISE 10.4.20. Let X be a planar graph with a Hamiltonian cycle C. If X has f_i' faces of length i inside C and f_i'' faces of length i outside C, then

$$\sum_i (i-2)(f_i' - f_i'') = 0.$$