CHAPTER 1

Basic Notions of Graph Theory

1.1. The Königsberg Bridges Problem

Graph theory may be said to have begun in the 1736 paper by Leonhard Euler (1707-1783) devoted to the Königsberg bridge problem. In the town of Königsberg (now Kaliningrad in western Russia), there were two islands and seven bridges connected as shown in the figure below. The challenge was to leave home and to traverse each bridge exactly once and return home.

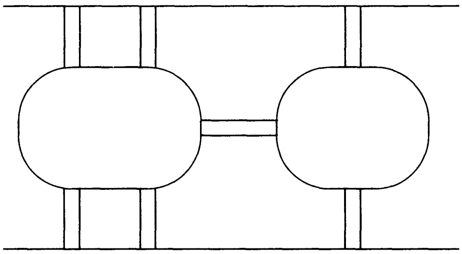

FIGURE 1.1. The bridges of Königsberg

Euler constructs a *graph* corresponding to the problem as follows (see Figure 1.2). The two sides of the river and the two islands are represented by *vertices* or *points* in the plane. They are joined if there is a bridge between them.

The resulting graph has *multiedges* according as the number of bridges between the two points. The Königsberg bridge problem reduces to a *circuit* through the graph which traverses each edge only once. Euler reasoned that if there is such a circuit in the graph, the *valence* of each vertex, or the number of edges coming out of any vertex must be even (see Figure 1.3).

In the Königsberg bridge graph, the valence of each vertex is odd and hence, no such circuit exists. This example illustrates many of the basic notions of graph theory which we take up in the next section.

S. M. Cioabă and M. R. Murty, *A First Course in Graph Theory and Combinatorics*

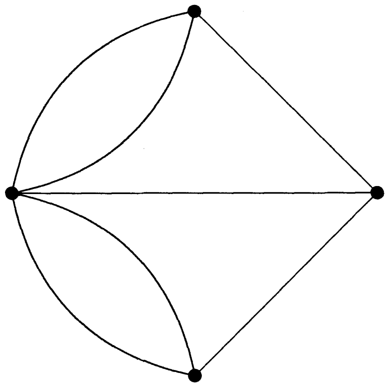

FIGURE 1.2. A graph representation of the bridges of Königsberg

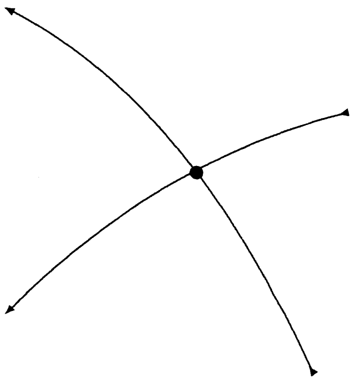

FIGURE 1.3. A vertex in an Eulerian cycle

1.2. What is a Graph?

A **graph** X is a pair (V, E) consisting of a set of **vertices** $V = V(X)$ and **edges** $E = E(X)$ that associates with each edge two vertices (not necessarily distinct) called its **endpoints**. A **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having the same pair of endpoints. A graph is called **simple** if it has no loops or multiple edges. When u and v are endpoints of an edge, we say they are **adjacent** or are **neighbours**. The **valence** or **degree** of a vertex is the number of edges coming out of it. We denote the valence or the degree of the vertex x by $d(x)$. A vertex is said to be **odd** or **even** according as $d(x)$ is odd or even. A graph is said to be **finite** if the vertex and edge sets are finite. We will be treating only finite graphs here.

The graph in Figure 1.2 has multiple edges and thus, is not a simple graph. It has one vertex of valence 5 and three of valence 3. All of its vertices are odd. It has no loops. Our first theorem of graph theory is obvious.

THEOREM 1.2.1. *For a finite graph X,*

$$\sum_{x \in V} d(x) = 2|E(X)|.$$

COROLLARY 1.2.2. *In any finite graph, the number of odd vertices is even.*

An **independent set** or **stable set** in a graph is a subset of vertices no two of which are adjacent. The **complete graph** on n vertices is a simple graph in which any two distinct vertices are adjacent. We denote this graph by the notation K_n. A graph is called **bipartite** if the vertex set can be written as the union of two disjoint independent sets. The bipartite graph $K_{r,s}$ is the simple bipartite graph whose vertex set is a disjoint union of two independent sets of size r and s with every element in the first set adjacent to every element in the second set. The n-**cycle** denoted C_n is the graph on n vertices $v_1, ..., v_n$ with only the adjacency relation $(v_i, v_{i+1}) \in E(C_n)$ for $1 \leq i \leq n$ where we interpret v_{n+1} as v_1.

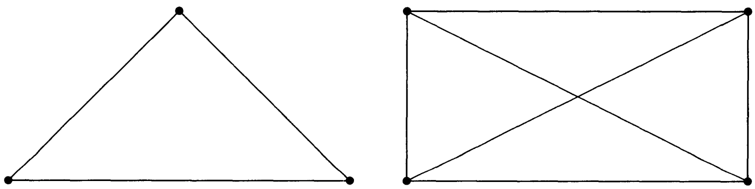

FIGURE 1.4. $K_3 = C_3$ and K_4

Until 1976, one of the most famous unsolved problems of mathematics was the four colour conjecture. This conjecture says that every map can be properly coloured using only four colours, where a proper colouring means that no two adjacent regions should be coloured the same. It was finally solved in 1976 by Kenneth Appel and Wolfgang Haken using extensive computer verification. For some, this is not satisfying and so the search is still on for a more conceptual and clearer solution.

The problem has a long history. It was first posed in a letter of October 23, 1852 from Augustus de Morgan (1806-1871) to William Rowan Hamilton (1805-1865). It was asked by one of de Morgan's students Frederick Guthrie who later attributed to his brother Francis

Guthrie. In 1878, Arthur Cayley (1821-1895) announced the problem to the London Mathematical Society and Alfred Bray Kempe (1849-1922) published a "proof" in 1879. In 1890, Percy John Heawood (1861-1955) indicated there was a gap in Kempe's proof and gave a simple proof that "five colours suffice". This is called the Five Colour Theorem which we will prove later in the book.

Graphs arise in diverse contexts and many of the "real world" problems can be formulated graph-theoretically. An important problem that arises in practice is the following scheduling one. Suppose we have n timetable slots in which to schedule r classes. We want a timetabling so that no student has a conflict. We can create a graph on r vertices, each vertex denoting a class. We join two vertices if they have a common student. We want to "colour" the graph using n colours so that no two adjacent vertices have the same colour. The **chromatic number** of a graph X, written $\chi(X)$, is the minimum number of colours needed to colour the vertices so that no two adjacent vertices have the same colour.

Bipartite graphs arise in job assignment questions. Suppose we have m jobs and n people, but not all people are qualified to do the job. Can we make job assignments so that all the jobs are done? Each job is filled by one person and each person can hold at most one job. Thus, we can create a bipartite graph consisting of n people and m jobs and join a person to a job if the person can do the respective job.

Sometimes, we can assign "weights" to edges and this facilitates discussion of "routing problems". Suppose we have a road network. The edges will correspond to road segments and the weights can be the distances between various points of the network. Questions concerning the shortest path from point a to point b can be formulated in terms of finding the graph geodesic between vertex a and vertex b.

1.3. Mathematical Induction and Graph Theory Proofs

In many proofs of theorems in graph theory and combinatorics, we will require the principle of **mathematical induction** which we recall below. Suppose we have a sequence of propositions $\{P_n\}$ indexed by the natural numbers that we would like to prove. We begin by verifying that P_1 is true. If we can prove that P_k for $k \leq n$ implies P_{n+1} for every n, then all of the propositions are established.

The simplest illustration of this is the following. Notice that

$$1^3 + 2^3 = 9 = 3^2$$

$$1^3 + 2^3 + 3^3 = 36 = 6^2$$

$$1^3 + 2^3 + 3^3 + 4^3 = 100 = 10^2$$

a pattern first noticed by Aryabhata in 5th century India. He showed that in general that

$$s_n := 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

and he did this essentially by the principle of mathematical induction. So the proposition P_n is that s_n is given by the formula above. For $n = 1$ it is clear. Assume we have proved it for each $k \leq n$. Then,

$$s_{n+1} = s_n + (n+1)^3$$

and by the induction hypothesis,

$$s_n = \left(\frac{n(n+1)}{2}\right)^2$$

so that we obtain

$$s_{n+1} = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = (n+1)^2\left(\frac{n^2}{4} + (n+1)\right)$$

$$= (n+1)^2\left(\frac{n^2 + 4n + 4}{4}\right) = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

as required.

We derive two important theorems below by the method of mathematical induction. The first concerns parity of cycles in graphs. The second characterizes Eulerian graphs.

A **walk** in a graph is a sequence $v_0, e_1, v_1, ..., e_k, v_k$ of vertices v_i and edges e_i such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. We sometimes refer to the walk as a v_0, v_k walk to indicate the initial and final points of the walk. The **length** of a walk is the number of edges in it. We say a walk is **odd** or **even** according as the length of the walk is odd or even respectively. A **trail** is a walk with no repeated edge. A **path** is a walk with no repeated vertex. Thus, a path is also a trail. The **distance** $d(u, v)$ between vertices u and v equals the shortest length of a u, v path. A **circuit** is a closed trail. A **cycle** is a closed path. We speak of odd or even paths, trails, cycles, circuits according as their lengths are odd or even respectively.

A graph is said to be **connected** if any two of its vertices are joined by a path. Any graph can be partitioned into its connected components.

LEMMA 1.3.1. *Every closed odd walk contains an odd cycle.*

PROOF. We use induction on the length ℓ of the closed walk W. For $\ell = 1$, a closed walk of length one clearly is also a cycle of length one. So there is nothing to prove. Now suppose that the assertion has been established for odd walks of length $< \ell$. If W has no repeated vertices, then W itself is a closed cycle. Otherwise, we may suppose that a vertex v is repeated in W. We can think of the walk as starting from v and view W as two v, v walks W_1 and W_2 (say). The length of W is the sum of the lengths of W_1 and W_2. As the length of W is odd, one of W_1 or W_2 must have odd length which is necessarily smaller than the length of W. By induction, this odd walk must have an odd cycle. ∎

1.4. Eulerian Graphs

A graph is called **Eulerian** if it has a closed trail containing all edges. A beautiful theorem of Leonhard Euler is the following result.

THEOREM 1.4.1. *A graph is Eulerian if and only if it is connected and all vertices have even degree.*

REMARK 1.4.2. It seems that Euler did not give a complete proof in his 1741 paper. The first complete published proof was given by Karl Hierholzer (1840-1871) in a posthumous article in 1873. The graph we drew to model the problem did not appear in print until 1894.

Before we prove the previous theorem, we need the following lemma.

LEMMA 1.4.3. *If every vertex of a graph X has degree at least 2, then X contains a cycle.*

PROOF. Let P be a maximal path in X. Let u be an endpoint of P. Since P is maximal, every neighbour of u must already be a vertex of P otherwise, P can be extended. Since u has degree at least 2, it has a neighbour v in $V(P)$ via an edge not in P (see Figure 1.5). The edge uv completes a cycle with the portion of P from v to u. ∎

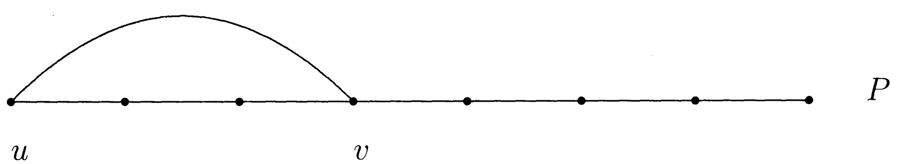

FIGURE 1.5

PROOF. (Theorem 1.4.1) The necessity is clear from what we have said before. We prove sufficiency by induction on the number of edges m of X. If $m = 0$, there is nothing to prove. Since X has even degrees, every vertex of X has degree at least 2. By Lemma 1.4.3, X contains a cycle C. Let X' be the graph obtained from X by deleting the edges of the cycle C. Since C has 0 or 2 edges at each vertex, each component of X' is a connected graph whose degrees are all even. By induction, each component of X' has an Eulerian circuit. We combine these circuits with C to get an Eulerian circuit of X as follows. We travel along C and when we encounter a component of X' for the first time, we go through the Eulerian circuit of that component. This completes the proof. ■

This theorem can be generalized to directed graphs (or **digraphs**). In this context, the theorem on the existence of an Eulerian circuit can be suitably generalized and has interesting algebraic and combinatorial applications (see Exercise 4.5.8).

1.5. Bipartite Graphs

We will use Lemma 1.3.1 to prove the following theorem of König (1936). Dénes König (1884-1944) studied at Budapest and Göttingen. His book *Theorie der endlichen und unendlichen Graphen* - "Theory of finite and infinite graphs" which appeared in 1936 is considered to be the first monograph in graph theory and contributed greatly to the growing interest in this subject.

THEOREM 1.5.1. *A graph X is bipartite if and only if it has no odd cycle.*

PROOF. We first show necessity. Every walk alternates between the two sets of a bipartition. So every return to the original partite set happens after an even number of steps. Hence, X has no odd cycle. For the converse, let X be a graph with no odd cycle. Let U be a non-trivial component of X and u a vertex in it. For each $v \in V(U)$ let $f(v)$ be the minimum length of a u, v-path. Since U is connected, $f(v)$ is defined of every $v \in V(U)$. Let

$$A = \{v \in V(U) : f(v) \,\text{is even}\}$$

and

$$B = \{v \in V(U) : f(v) \,\text{is odd}\}.$$

An edge v, v' within A or B would create a closed odd walk. By Lemma 1.3.1, X would contain an odd cycle, contrary to assumption. Thus, A and B are independent sets. Clearly, $X = A \cup B$ so X is bipartite. ■

We conclude this section with a simple result concerning the degrees of the vertices of a bipartite graph. The bipartite version of the Theorem 1.2.1 is the following result.

THEOREM 1.5.2. *If X is a bipartite graph with colour sets A and B, then*

$$\sum_{a\in A} d(a) = \sum_{b\in B} d(b) = |E(X)|.$$

PROOF. By counting the number of edges of X in two different ways, the result follows immediately. ∎

1.6. Exercises

EXERCISE 1.6.1. Is there a simple graph of 9 vertices with degree sequence

$$3, 3, 3, 3, 5, 6, 6, 6, 6?$$

EXERCISE 1.6.2. Is there a bipartite graph of 8 vertices with degrees

$$3, 3, 3, 5, 6, 6, 6, 6?$$

EXERCISE 1.6.3. In a simple graph with at least two vertices, show that there are at least two vertices with the same degree.

EXERCISE 1.6.4. Show that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

and

$$1 + 3 + \cdots + (2n-1) = n^2.$$

EXERCISE 1.6.5. A **directed graph** (or **digraph**) is a graph X together with a function assigning to each edge, an **ordered** pair of vertices. The first vertex is called the **tail** of the edge and the second is called the **head**. To each vertex v, we let $d^+(v)$ be the number of edges for which v is the tail and $d^-(v)$ the number for which it is the head. We call $d^+(v)$ the **outdegree** and $d^-(v)$ the **indegree** of v. Prove that

$$\sum_v d^+(v) = \sum_v d^-(v) = \#E(X)$$

where the sum is over the vertex set of X.

EXERCISE 1.6.6. In any digraph, we define a **walk** as a sequence

$$v_0, e_1, v_1, e_2, \ldots, e_k, v_k$$

with v_{i-1} the tail of e_i and v_i its head. The analogous notions of trail, path, circuit and cycle are easily extended to digraphs in the obvious way. If X is a digraph such that the outdegree of every vertex is at least one, show that X contains a cycle.

EXERCISE 1.6.7. An **Eulerian trail** in a digraph is a trail containing all the edges. An **Eulerian circuit** is a closed trail containing all the edges. Show that a digraph X contains an Eulerian circuit if and only if $d^+(v) = d^-(v)$ for every vertex v and the underlying graph has at most one component.

EXERCISE 1.6.8. Determine for what values of $m \geq 1$ and $n \geq 1$ is $K_{m,n}$ Eulerian.

EXERCISE 1.6.9. What is the maximum number of edges in a connected, bipartite graph of order n ?

EXERCISE 1.6.10. How many 4-cycles are in $K_{m,n}$?

EXERCISE 1.6.11. Let Q_n be the **n-dimensional cube** graph. Its vertices are all the n-tuples of 0 and 1 with two vertices being adjacent if they differ in precisely one position. Show that Q_n is connected and bipartite.

EXERCISE 1.6.12. Show that Q_n has 2^n vertices and $n2^{n-1}$ edges.

EXERCISE 1.6.13. How many 4-cycles are in Q_n ?

EXERCISE 1.6.14. Does Q_n contain any copies of $K_{2,3}$?

EXERCISE 1.6.15. Show that every graph X contains a bipartite subgraph with at least half the number of edges of X.

EXERCISE 1.6.16. Let X be a graph in which every vertex has even degree. Show that it is possible to orient the edges of X such that the indegree equals the outdegree for each vertex.

EXERCISE 1.6.17. Show that a graph X is connected if and only if for any partition of its vertex set into two non-empty sets, there exists at least one edge between the two sets.

EXERCISE 1.6.18. Show that in a connected graph any two paths of maximum length have at least one common vertex.

EXERCISE 1.6.19. Let X be a graph with n vertices and e edges. Show that there exists at least one edge uv such that

$$d(u) + d(v) \geq \frac{4e}{n}.$$

EXERCISE 1.6.20. If X is a graph on n vertices containing no K_3's, then the number of edges of X is less than or equal to $\lfloor \frac{n^2}{4} \rfloor$ edges. Give an example of a graph on n vertices containing no K_3's with $\lfloor \frac{n^2}{4} \rfloor$ edges.