Chapter 4
Changed Views on Mathematical Knowledge in the Course of Didactical Theory Development: Independent Corpus of Scientific Knowledge or Result of Social Constructions?

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Introduction

This contribution tries in an exemplary way to look at the case of the historical development of important tendencies in mathematics education (Mathematikdidaktik) in Germany in the last 40 years. This description can only follow one line of development; it cannot and will not summarize other research approaches existing in mathematics education in Germany. A major concern is to investigate the clarification process of the central objects of mathematics education research and to analyze the important role that the content matter 'mathematics' plays for teaching and learning processes. The main interest of this paper is to better understand the special German case of how theoretical considerations for mathematics education developed, changed and expanded. This development cannot and will not explain a universal, all-embracing theory of mathematics education, but it reflects one important German tradition (without looking here at other traditions) and is an example of a theoretical evolution of mathematics education. Within this historical development, there are to be found strategies of comparing, contrasting and of (locally) integrating theories and theoretical aspects.

> 'Mathematics learning' as an object of didactical considerations has consistently over time been regarded as the triad 'Learner – Teacher – Learning/Teaching-Content'. In pedagogics, these three elements are labeled as the 'didactical triangle' since Friedrich Herbart (1776–1848) (see Peterßen, 2001, p.140, and Künzli, 2000, pp. 48–49). According to Herbart "... education within instruction does not [take place] in the immediate relationship between educator and pupil, but educator and pupil [enter] into an indirect relationship to each other. Between them stand the instruction objects"[1] (Peterßen, 2001, p. 140, translated by H.S.).

In mathematics education (in Germany), the didactical triangle (see Fig. 4.1) has a long tradition. The vertices for mathematics education represent: (1) the mathematical knowledge, (2) the student, and (3) the teacher (cf. Steinbring, 1998a).

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[1] One can suppose that this is where the famous didactical triangle originates.

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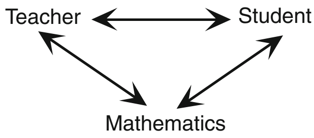

Fig. 4.1 The didactical triangle

The schema of the didactical triangle with its three elements will be used as a kind of 'test instrument' for the following considerations and analyses. By using this triangle, the following orientating questions will be asked for the elaboration of the changes and developments in mathematics education:

(A) Which explicit and implicit (unconscious) concepts and role descriptions exist about the three 'elements': mathematics, teacher and students?
(B) Which explicit and implicit (unconscious) concepts and role descriptions exist about the relationships or interactions between the three 'elements': mathematics, teacher and students?
(C) What is explicitly or implicitly (unconsciously) seen as the central and crucial means (among the three 'elements': mathematics, teacher and students) of positively influencing and improving the learning process?

These questions, combined with the resulting insights and answers, will help to provide a more fully differentiated picture of which research concepts and objects have been predominant in mathematics education in the course of its historical development, and how the role and nature of (school) mathematical knowledge has changed and been redesigned in the course of this development.

The 'Stoffdidaktik' Elaboration of Mathematical Knowledge as an Essential Factor Influencing Teaching and Learning Processes

Until the mid-1960s, the emphasis in Germany was on didactical works and analyses which concentrated on school-mathematical knowledge, its didactical elementarization and on subject matter aspects. These works were essentially linked to mathematics as a pre-given content for learning and instruction, and specific features of a genuinely mathematics education research approach had not yet become noticeable in them. Especially within the German-speaking countries, this didactical research paradigm developed as *stoffdidaktik*[2] (Content based Didactics).

[2] In this paper 'stoffdidaktik' is restricted to a certain fundamentalist form of content-related mathematical analysis based on ideas from the New Math era. Later, there were further developments and modifications of the *stoffdidaktik* approach – no longer explicitly linked to the New Math era – that relate the analysis of mathematical content knowledge to the learning processes of students. These kinds of *stoffdidaktik* still exist; there are also types of *stoffdidaktik* that emphasize the *epistemological* analysis of mathematical content matter.

> ... 'stoffdidaktik' is dominated by too simple a model to solve didactical questions and research problems. [It] acts on the assumption that mathematical knowledge – as researched and developed in the academic discipline – is essentially unchanged and absolute .... Though 'stoffdidaktik' in the meantime notices the problems of understanding that students have in learning, and accordingly it specifically proceeds to prepare the pre-given mathematical disciplinary knowledge for instruction as a mathematical content, to elementarise it and to arrange it methodically; yet the principle remains unchallenged that mathematical knowledge represents a finished product, and that the teaching-learning-process can be organised linearly, emanating from the content, over the teacher, into the students' heads, and can ultimately be controlled and influenced at every step by mathematics educators (Steinbring, 1997, p. 67, see also Steinbring, 1998a, and Steinbring, 1998b, pp. 161–162).

Under the abbreviated label *stoffdidaktik*, this direction was especially represented by 'didactically oriented content analysis'.

> The research complex of didactically oriented content analysis (Sachanalysen) has lately engaged mathematics education in the Federal Republic of Germany in a particular way .... The research methods of this area are identical with those of mathematics, so that outsiders have sometimes gained the impression that, here, mathematics (particularly elementary mathematics) and not mathematics education is being conducted .... The goal of 'didactically oriented content analysis' which essentially follows mathematical methods is to give a better foundation for the formulation of content-related learning goals and for the development, definition and use of a differentiated methodical set of instruments. (Griesel, 1974, p. 118, translated by H.S.).
>
> What does progress in mathematics education depend upon? 1. Upon the state of development of the analysis of the content, the methods and the application of mathematics. 2. Upon didactical ideas and insights, which make it possible to attend better, or at all, to a subject area within instruction. (Griesel, 1971, p. 7, translated by H.S.).

Griesel names four further influential factors (general experience and statistically based evidence about instruction, insights into the mathematical learning process, development-psychological and sociological conditions); yet the didactical work on the 'content' is the most important.

In a critical comparison between (German) 'didactically oriented content analysis' and (French) 'ingénierie didactique', Strässer (1994) states that *stoffdidaktik* ultimately pursued the goal of elaborating school-mathematical subject areas – similar to mathematical areas in Bourbakism – in a logically consistent way and built upon unambiguous foundations. As an example, Strässer quotes from the foreword to the two-volume book by G. Holland *Geometry for Teachers and Students (1974/1977)*:

> This book arguably offers the reader a complete axiomatic composition of the Euclidian geometry of the plane, which in its system of concepts as well as in the choice and organisation of the geometrical contents orientates itself as much as possible to contemporary geometry instruction in school (Holland, 1974, p. 7, translated by H.S.).

An archetype for *stoffdidaktik* was uniform mathematics, as it was exemplarily given by Bourbaki and then by the so-called New Mathematics. Connected with this archetype of uniform, axiomatic mathematics, the illusion for work in *stoffdidaktik* was that mathematics for teachers, students and pupils (i.e., school-mathematics) could also ultimately be elaborated in a logically correct and consistent manner, definite and absolute for all teaching and learning processes.

> The whole of mathematical knowledge, ordered in this way, is, in principle, describable with a single, universal language. This uniformity . . . means essentially that the elementary concept of the number '5' and the more abstract concept of the 'expectation of a binomially distributed random variable' are objects at the same level of description by mathematical set-language. This product of the mathematical knowledge corpus reflects the preoccupations of the historical period during which it originated; its logical clarity, the construction from the simple to the complex and abstract, as well as its uniform language, are together imagined to provide the ideal preparation of knowledge for its acquisition and its understanding – as was also for example the maxim of the movement of so-called 'New Mathematics' (Steinbring, 1998b, p. 161).

The *stoffdidaktik* work undertaken focused initially on the school mathematics of higher school grades (especially grammar school, the German Gymnasium covers the grades 5–13, age 10–19); then, at the end of the sixties, with didactical works in the frame of the movement of 'New Mathematics' (especially the works of Z. P. Dienes), it was extended to mathematics instruction in primary school (grades 1–4, ages 6–10).

> The modernisation of mathematics instruction in primary school only started much later, about the year 1966, when the inventive ideas of Z. P. Dienes became familiar . . .. We can speak of a modernisation of mathematics instruction in primary school and in grades 5 and 6 . . . (Griesel, 1971, p. 8, translated by H.S.).

For a summary, characterizing the position of *stoffdidaktik* as described in this paragraph, the three aforementioned questions (A, B and C) shall now be consulted and answered in a general way. About the mathematical content, there clearly is the conception that ultimately a uniform, objective and unchangeable content of teaching and learning is to be elaborated in didactics according to the paradigm of scientific mathematics. The teaching, learning and understanding processes of the participating persons (teacher and students) are orientated around the rigid subject matter structures: the teacher is the 'conveyor' of the didactically prepared content to the student(s) who are seen as passive receivers. The relations between the three elements of the didactical triangle are of an essentially linear nature: the mathematical knowledge arrives by means of the preparation and transfers from the teacher to the students. In the research paradigm of *stoffdidaktik*, the scientific elaboration of mathematical knowledge is the central and crucial means practiced for steering and optimizing mathematical instruction, learning and understanding processes.

The Synchronization Between the Dynamics of Knowledge Development and the Processes of Teaching and Learning

The international criticism of New Mathematics (Kline, 1973) led also in Germany to a long-term critical altercation with New Mathematics. Furthermore, the scientific debate about the status and the objects of a science genuinely concerned with mathematics education took place over a longer period (Steiner in ZDM, 1974; Winter, 1985; Wittmann, 1992). One prominent voice, Winter (1985, pp. 80–81), states:

> So-called Sachanalysen ('didactically oriented content analyses') can have a downright calamitous effect on the school reality, if they refer reductionistically solely to mathematics (perhaps even to assumed mathematics) and fade out other essential constituents of learning mathematics . . .. [One] inevitably encounters problems of the goals and forms of learning itself, which are not, or hardly, explained in the Sachanalysen . . .. In general: Sachanalysen are in danger of losing focus on the outer-mathematical reality and thus on the students' experience of the world, and this is only one pedagogical sin of such reductions (Winter, 1985, p. 80/81, translated by H.S.).

The relation between mathematical learning content and teaching and learning processes did not work in the way imagined from the perspective of *stoffdidaktik*. A new perspective on the subject matter content needed to be developed which took the sequential development and dynamics of teaching and learning processes into account. Freudenthal (1973, p. 114) emphasized the process character of mathematics for learning in a paradigmatic way:

> It is true that words such as mathematics, language, and art have a double meaning. In the case of art it is obvious. There is a finished art studied by the historian of art, and there is an art exercised by the artist. It seems to be less obvious that it is the same with language; in fact linguists stress it and call it a discovery of de Saussure's. Every mathematician knows at least unconsciously that besides ready-made mathematics there exists mathematics as an activity. But this fact is almost never stressed, and non-mathematicians are not at all aware of it.

Mathematics, as an activity, implies that learning becomes an active process in the construction of knowledge.

> The opposite of ready-made mathematics is mathematics in statu nascendi. This is what Socrates taught. Today we urge that it be a real birth rather than a stylized one; the pupil himself should re-invent mathematics . . .. The learning process has to include phases of directed invention, that is, of invention not in the objective but in the subjective sense, seen from the perspective of the student (Freudenthal, 1973, p. 118).

Development processes are not uniform, universal or homogeneous. Subjective characteristics of those people keeping the process going, as well as situated representations, notations and interpretations of mathematical knowledge, are manifold, divergent and partly heterogeneous. Further, cultural contexts, subjective influences and situated dependencies are both active and inevitable; such are the reasons for an observable diversity and non-uniformity of the emerging knowledge.

The contrast between uniform scientific mathematics (oriented towards a generally valid (research) product) and the different perspectives and interpretations of mathematics produced in social environments for different application domains (tied up in situatedly-framed development processes) becomes extremely apparent against the background of the different cultures in which mathematical knowledge is used and experienced. The culture of the researching and teaching mathematician and the culture of mathematics teaching face one another in an obviously distinct, and sometimes opposing, way. The role the Bourbakist mother structures play for the unity of mathematics cannot be understood by mere appropriation of the principles given by these structures. The culture of mathematical science and the historical development of mathematics form the necessary background for an understanding.

These principles of the unity of mathematical knowledge cannot easily be transferred to school mathematics. With such an endeavor, school mathematics would lose its cultural background and become mere formalistic signs and formulas. In order to understand these signs and formulae, the formation of a new, distinct culture, a kind of mathematical re-invention, would again be necessary. From the point of view that mathematical knowledge has to be seen as a newly-emerging culture, one has to question the unity of mathematics in learning and teaching processes. If mathematical knowledge can only be meaningfully interpreted in the frame of a specific cultural environment, then there is not simply one single, but many different forms of practicing mathematics.

Wittmann (1995, pp. 358–359) distinguishes between specialized, scientific mathematics and the general social 'phenomenon' of mathematics.

> [One] . . . must conceive of 'mathematics' as a broad societal phenomenon whose diversity of uses and modes of expression is only a part reflected by specialized mathematics as typically found in university departments of mathematics. I suggest a use of capital letters to describe MATHEMATICS as mathematical work in the broadest sense; this includes mathematics developed and used in science, engineering, economics, computer science, statistics, industry, commerce, craft, art, daily life, and so forth according to the customs and requirements specific to these contexts . . .. It should go without saying that MATHEMATICS, not specialized mathematics, forms the appropriate field of reference for mathematics education. In particular, the design of teaching units, coherent sets of teaching units and curricula has to be rooted in MATHEMATICS.

On the basis of this position about the role of mathematical knowledge in instruction processes, Wittmann characterizes didactics of mathematics as a 'design science' (1998, 2001). In German-speaking mathematics education, especially concerning teacher education at universities and teachers' further professional education, Wittmann is a protagonist for a new perception on the role and the meaning of mathematical knowledge for teaching and learning processes, which critically distances itself from New Mathematics.

At the Institute for Didactics of Mathematics (IDM), founded at the University of Bielefeld in 1973, fundamental studies about mathematics education positions, problems and research questions were carried out in three working groups of scientists. In the "Mathematics Teacher Education" working group (Arbeitsgruppe Mathematiklehrerbildung, 1981), two central research approaches in mathematics education were mainly pursued: (1) the particular epistemological nature of mathematical knowledge, and (2) the central role of the teacher within mathematical teaching and learning processes.

Historical, philosophical and epistemological analyses were elaborated as a basis for characterizing mathematical knowledge ultimately as theoretical knowledge. A central criterion of theoretical mathematical knowledge – also observable in the course of its historical development – lies in the transition from pure object or substance thinking to relation or function thinking.

The transition from a substance concept to a relational concept is a central part of Ernst Cassirer's epistemological philosophy.

> . . . the theoretical concept in the strict sense of the word does not content itself with surveying the world of objects and simply reflecting its order. Here the comprehension, the 'synopsis' of the manifold is not simply imposed upon thought by objects, but must be created by independent activities of thought, in accordance with its own norms and criteria (Cassirer, 1957, p. 284).

And in another passage, Cassirer (1923, p. 20) writes:

It is evident anew that the characteristic feature of the concept is not the 'universality' of a presentation, but the universal validity of a principle of serial order. We do not isolate any abstract part whatever from the manifold before us, but we create for its members a definite relation by thinking of them as bound together by an inclusive law.

This understanding of theoretical mathematical concepts as referring to relations, rather than to objects or to the empirical properties of objects, constitutes the basic step towards developing mathematics education into a scientific discipline.

> For didactics, for instance, it is obvious that the didactic problem in its deeper sense, that is in the sense that it is necessary to work on it scientifically, is constituted by the very fact that concepts will reflect relationships, and not things. Analogously, we may state for the problem of the application of science that it will become a real problem only where the relationship between concept and application is no longer quasi self-evident, but where to establish such a relationship requires independent effort (Jahnke & Otte, 1981, pp. 77–78).

A perception that mathematical knowledge does not reflect things, but relations, implies a differentiated view of teaching and learning mathematics as independent activities of the participating persons. Thus, the role of the teacher comes to the fore.

> A description of the requirements on the teacher and the teaching activity has been attempted in the debate about the relation between teaching and learning. From this debate, one can record as a consequence that 'teaching' cannot be derived from descriptions of 'learning' – and that according to the opinion of many authors the developmental status of learning theories is more advanced than that of teaching theories. After all, the conception that the contents of teacher education should essentially consist of insights about the student's learning process is very common.
>
> What is the specificity of teaching? The specificity of teaching lies within the content of the activity, which aims at effectuating learning. Every theory of academically institutionalised education thus presupposes a concept of teaching and cognition, but also requires perceptions about the questions by which mechanisms the teaching/learning process leads or shall lead to an interactionally imparted forming of the learner." (AG Mathematiklehrerbildung, 1981, p. 57, translated by H.S.).

According to this perspective, theoretical and empirical works about the particularity of the teacher's activity have been carried out in the aforementioned working group of scientists at IDM Bielefeld (see for instance: Bromme,1981, 1992; Bromme & Seeger, 1979). These concepts and works about the activity of the mathematics teacher reveal, in particular, that within the didactical triangle, the teacher and his role are determined neither by the mathematical knowledge nor by the learning students. For instance, Bromme (1981, 1992) analyses central aspects of the teacher's activity (e.g., the preparation of mathematics instruction) under the perspective that teachers are to be regarded as experts in their professional field of work. In addition to the two essential fields of professional teacher knowledge: ‘content knowledge’ and ‘pedagogical content knowledge’ (according to Shulman, 1986), Steinbring (1998c) elaborates the particularity of ‘epistemological knowledge for mathematics teachers’ with a view to the theoretical and dynamic character of mathematics. This knowledge concerns insights about the particular epistemological nature of mathematical knowledge for teaching and learning processes, which are not contained in the ‘pedagogical content knowledge’, which Shulman (1986, p. 9) briefly describes as “. . . the ways of representing and formulating the subject that make it comprehensible to others”.

Furthermore, the independent role of the learning child with his or her cognitive predispositions moved to the centre of didactical research, a position which it had already taken for a longer time in primary school didactics. In a summarizing main lecture at the Federal Congress for ‘Didaktik der Mathematik in Osnabrück’ in 1991, Peter Sorger sums up a view taken in German mathematics education:

> Today, we know so much more, especially about the individual primary school child, about his cognitive activities, about his thinking, about the initiation and course of mathematical learning processes, about the influences of the individual learning history onto new learning situations, about the variety of possible thinking and solution strategies, which the adults’ perceptions are always in danger of cutting too short. The diagnosis, analysis and therapy of learning difficulties have also been thoroughly researched (Sorger, 1991, p. 39, translated by H.S.).

The research on this topic in particular uses methods from reference disciplines and they are not reducible to mathematical works (i.e., they essentially contribute to an independent research profile for mathematics education).

Again, the three questions (A, B and C) shall be asked and answered in a general way, in order to characteristically sum up the positions about mathematical instruction (respectively teaching and learning processes) described in this section. The mathematical content is interpreted more diversely and its dynamic and procedural character is particularly emphasized. (School-) Mathematical knowledge is not identical with scientific mathematical research knowledge, but, at the same time, it is theoretical knowledge which means that it is subject to a particular epistemology (also in the frame of the activities of teaching and learning). It is this *developmental* aspect of mathematical knowledge that makes possible to coordinate the ongoing students’ learning activities with the teachers’ teaching activities. These more differentiated perceptions of mathematics education negate an immediate dependence of the teacher on mathematical knowledge and of the student on the instructing teacher. From this perspective, learning mathematics is autonomous: “socially and actively discovering, independent learning by the students”; teaching is likewise viewed as an independent activity (AG Mathematiklehrerbildung of the IDM Bielefeld).

The three elements of the didactical triangle, (1) the mathematical knowledge, (2) the student and (3) the teacher stand ‘apart’ and gain independence as well as their own dynamics with new didactical research questions. The relations between these elements are of a rather indirect nature. For instance, the teacher is now regarded rather like a moderator or initiator of learning processes, while the student is conceded his own responsibility for his mathematical understanding and learning processes. The developing mathematical knowledge becomes manifest in different ways in different using and teaching/learning practices; it is no longer consistently and universally given, for example, on the basis of the Bourbakian structure types (Bourbaki, 1971).

The ‘steering’ of the students’ learning processes by the teacher can no longer be perceived as mechanical conducting. The ‘functioning’ of the didactical triangle now rather represents a reciprocal process between its three elements and not a linear or circular movement of mathematics via the teacher to the students, or vice-versa. The mathematical knowledge (now in its new interpretation as theoretical knowledge within a development process) remains important, but shows itself in different characteristics in learning and teaching activities; however, the student’s learning activities and the teacher’s teaching activities also have an essential influence on the whole process.

At first, didactical research concentrated rather on the three relatively autonomous elements of the didactical triangle, (1) the mathematical knowledge, (2) the students and (3) the teacher; only with the beginning of mathematics education interaction research was the co-action of the three elements taken seriously and treated explicitly as the central object of didactical research.

Mathematics Education Research and Mathematical Teaching-Learning-Practice as Independent Institutional Systems

For a long time, researchers in mathematics education research took the standpoint that mathematics teaching practice had to strictly follow the insights constructed by mathematics education. This point of view is also found in those didactical works which emphasize the procedural character of mathematical knowledge and of mathematical teaching and learning situations. There still exist perceptions according to which instruction practice could be directly improved by educational research.

Mathematics education is faced with the tension between scientific research and constructive development work. This problematique has been discussed intensively for a long time, for instance in the scientific debates about the so-called ‘Theory-practice-problem’ (Bazzini, 1994; Even & Loewenberg Ball, 2003; Seeger & Steinbring, 1992; Steinbring, 1994; Verstappen, 1988).

Facing this complementary task of research and constructive development, mathematics education is confronted with the fundamental question: “What is the particular nature of the relation between theory and practice?” One traditional solution to this question that educational research exclusively provides the necessary knowledge and prescriptions for school practice has been decidedly criticized and replaced by other conceptions.

An essential criticism has been developed by means of the work of the research group around Heinrich Bauersfeld at the IDM (Bielefeld). Since the beginning of the eighties, research started in which everyday mathematics teaching as autonomous social events was taken seriously and analyzed under an interactionistic perspective (e.g., Bauersfeld, 1978, 1988; Cobb & Bauersfeld, 1995; Krummheuer, 1984, 1988; Maier & Voigt, 1991, 1994; Voigt, 1984, 1994). Everyday mathematics instruction is seen as a peculiar culture, which is neither completely nor directly determined by the scientific discipline 'mathematics', nor can it be directly guided and improved by mathematics education research results.

Voigt (1996, p. 384, translated by H.S.) calls this the 'turn to everyday life' of the authentic classroom in mathematics education:

> . . . the 'turn to everyday life' . . . with its criticism of 'holiday didactics' . . . contained the claim of assigning a greater meaning than before to the features of everyday instruction. In ethnographic observations of instruction and interpretative studies, one saw a corrective for conceptions of instruction which emerge at the didactical desk; one was disillusioned by the effects of the school reforms (see among others the 'New Mathematics') and wanted to understand better the surprising stability of everyday instruction, its own progress and its traditions. At the same time, there was the hope of being able to better connect with the experience and the problem awareness of the practitioners through softer methods of empirical research".

(School) Practice and (content-related educational) science need to be seen as two relatively autonomous institutions and fields of work between which there are no direct possibilities of influence or change (see Bartolini-Bussi & Bazzini, 2003; Krainer, 2003; Scherer & Steinbring, 2006; Steinbring, 1994, 1998c). Each of the two fields is subject to its own expectations and aims, as well as to system-internal requirements and norms which cannot be externally invalidated in order to apparently be able to directly interfere in and to purposefully regulate from within the other field.

The relative separation and autonomy of (content-related educational) theory and (school) practice, however, does not mean that there are no reciprocal actions between the two at all. Rather, in the relation between theory and practice, the respective other field can be seen as a necessary environment in which irritations and stimulations occur, which indirectly animates the first field in order to implement changes, alternative ways of proceeding and further developments. What is important here is to notice that not only such changes within (school) practice, but also within content-related educational theory, must ultimately occur and establish themselves from the inside and 'out of themselves'. In order for this to happen, irritations and stimulations from the outside are helpful and necessary, yet they are not deterministic influencing instruments.

Under this fundamentally changed perspective on the 'theory-practice-problem', the didactical triangle takes on a different orientation function for mathematics education research. It no longer represents an ideal paradigmatic schema against which everyday instruction must be measured, but instead becomes an instrument for the analysis of actual mathematics instruction in which the reciprocal interconnectedness between the three relevant elements participating in the instruction process are systematically captured.

In works of interpretative classroom research, social interactions and their patterns and mechanisms were the centre of research interest; the mathematical teaching and learning content was, in principle, faded out. Thus, the particular relation between two elements of the didactical triangle (2) the student, and (3) the teacher within the frame of everyday instruction events was prioritized.

The interactionist perspective relies mainly on two (previously neglected) basic aspects: the learning child (in the classroom) and the interaction between the learner and the teacher. In this research context, one has to distinguish between two theoretical perspectives:

> The one is an individual-psychological perspective which emphasizes the learner's autonomy and his cognitive development and which leads to the concept of student-oriented, 'constructivistic' mathematics instruction. The other is a collectivistic perspective which criticizes the 'child-centered ideology' of the first perspective and understands learning mathematics as the socialization of the child into a given classroom culture . . . (Voigt, 1994, p. 78).

These two research perspectives are based on reference to different scientific disciplines. The individual-psychological perspective relies, for example, on cognitive psychology as well as on radical constructivism (von von Glasersfeld, 1991) and the collectivistic perspective uses sociological and ethnographic theories. In the analyses of mathematical interactions, one or the other of these two theoretical orientations is often emphasized.[3] An over-emphasis on either the individual-psychological or the collectivistic perspective was a major critique and a starting point for the working group around H. Bauersfeld to develop a theoretical concept which explicitly brings together the individual cognitive perspective and the collective social perspective, as a basis for qualitative analyses of interaction.

On the one hand, it asserted that a single student cannot discover all school knowledge by himself. "Culture, we can say, is not discovered; it is traded or falls into oblivion. All this indicates for me that we should rather be more careful when talking about the discovery method or about the conception that discovery is the basic vehicle of instruction and education" (Bruner, 1972, p. 85). On the other hand, it is considered doubtful that effective participation in social interaction patterns can lead to successful mathematics learning.

> In everyday lessons, interaction patterns often can be reconstructed in which the teachers influence every step of the students' activities without creating favourable conditions for the student to make desirable learning processes in problem solving and developing concepts . . .. We should resist the temptation of identifying learning mathematics with the student's successful participation in interaction patterns (Voigt, 1994, p. 82).

Consequently, an interaction theory was developed in which both perspectives were connected to each other:

> [A]n interaction theory of teaching and learning mathematics [offers] a possibility of regarding social aspects of learning mathematics and at the same time of avoiding the danger of overdoing the cultural and social dimensions. For the interaction theory emphasizes the processes of sense making of individuals that interactively constitute mathematical meanings.

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[3]Concerning the individual-psychological perspective, see e.g. Cobb, Yackel, and Wood (1991); and for the collectivistic perspective, see Solomon (1989).

> The interaction theory of teaching and learning mathematics uses findings and methods of micro sociology, particularly of symbolic interactionism and ethnomethodology .... Of course the interaction-theoretical point of view does not suffice if one wants to understand classroom processes holistically (Voigt, 1994, p. 83).

The interaction-research approach of the social epistemology of mathematical knowledge (Steinbring, 2005) understands itself as an important, independent and complete model inasmuch as the particularity of the social existence of mathematical knowledge is an essential component of this theoretical approach of interaction analysis. In this theoretical conception of the social epistemology of mathematical knowledge, the epistemological particularity of the subject matter 'mathematical knowledge' dealt with in the interaction, constitutes a basis for its theoretical examination.

Epistemology-based interaction research in mathematics education accentuates the assumption that a specific social epistemology of mathematical knowledge is constituted in classroom interaction and this assumption influences the possibilities of how to analyze and interpret mathematical communication. This assumption includes the following view of mathematics: mathematical knowledge is not conceived as a ready-made product, characterized by correct notations, clear cut definitions and proven theorems. If mathematical knowledge in learning processes could be reduced to this description, the interpretation of mathematical communication would become a direct and simple concern. When observing and analyzing mathematical interaction, one would only have to diagnose whether a participant in the discussion has used the 'correct' mathematical word, whether he or she has applied a learned rule in the appropriate way, and then has gained the correct result of calculation.

Mathematical concepts are constructed in interaction processes as symbolic relational structures and are coded by means of signs and symbols that can be combined logically in mathematical operations. This interpretation does not require a fixed, pre-given description for the mathematical knowledge (the symbolic relations have to be actively constructed and controlled by the subject in interactions). Further, certain epistemological characteristics of this knowledge are required and explicitly used in the analysis process (i.e., mathematical knowledge is characterized in a consistent way as a structure of relations between (new) symbols and reference contexts).

The intended construction of meaning for the unfamiliar new mathematical signs, by trying to build up reasonable relations between signs and possible contexts of reference and interpretation, is a fundamental feature of an epistemological perspective on mathematical classroom interaction. This intended process of constructing meaning for mathematical signs is an essential element of every mathematical activity, whether this construction process is performed by the mathematician in a very advanced research problem, or whether it is undertaken by a young child when trying to understand elementary arithmetical symbols with the help of the place value table. The focus on this construction process allows mathematics teaching and learning at different school levels to be viewed as an authentic mathematical endeavour.

In epistemologically-oriented mathematical classroom research, the subject of teaching and learning mathematical knowledge is taken into account as an important element within the didactical triangle. For empirical, interpretative research, the didactical triangle takes a descriptive function – and it has no prescriptive function – with which guidelines for instruction practice are provided. As a descriptive schema, the didactical triangle serves to characterize an essential and complex (i.e., not further dissectible) fundamental object of mathematics education research: namely, (everyday) mathematical interactions and communications within teaching and learning processes.

To sum up, one can ascertain the following alongside the three questions (A, B and C). The three elements of the didactical triangle, (1) the mathematical knowledge, (2) the student and (3) the teacher, are seen in the institutional context of the joint interaction as relatively independent 'systems', which are engaged in reciprocal actions with each other. The mathematical interactions between teacher and students take place between autonomous subjects, who are aware of each other during the reciprocal communication, but who cannot directly influence the psyche or the consciousness of the other. The communicated and negotiated mathematical knowledge is interactively constructed within this social context on the basis of its epistemological basic conditions of consistence and structure.

Accordingly, the teacher continues to take the role of a moderator or a facilitator of learning occasions for the students, who continue to be responsible for their own understanding processes and participate by means of socially and actively discovering mathematics learning. The instructional communication process emerges and constitutes itself within the actual course of teaching and learning; it cannot be planned and prepared in detail beforehand. Mathematics as a teaching-learning-object develops within the social interaction, and is in different ways the 'subjective property' of the persons taking part in the interaction.

The question about decisive means for positively changing and affecting the teaching, learning and understanding processes (C) gains a more differentiated background. Changes and improvements cannot take place from the outside, or by means of a direct intervention. Changes can only be encouraged in the participating autonomous systems and then need to be continued and realized within the systems themselves. This concerns the learning student to whom the teacher can ultimately only offer opportunities to learn for himself. But this is also true for the teacher and the development of his professional teaching activity in connection with mathematics education research.

Mathematical Knowledge in Teaching: A Case Illustrating the Epistemology-based Interaction View on Teaching Learning Processes

In what follows, a short teaching episode is used to illustrate exemplarily how, from the perspective of epistemologically-oriented empirical instruction research, the three elements of the didactical triangle, (1) the mathematical knowledge, (2) the

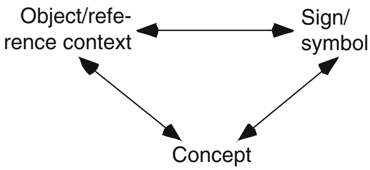


Fig. 4.2 The epistemological triangle

student, and (3) the teacher, autonomously and interactively generate mathematical knowledge within this social situation (based on Steinbring, 2005).

Mathematics teaching and learning deal with the use and interpretation of mathematical signs, symbols and symbol systems. The mediation between mathematical signs or symbols and structured reference contexts can be described with the help of the epistemological triangle (see Fig. 4.2) (see Steinbring, 2005, 2006). This triangle serves as a theoretical instrument for analyzing the connection of yet unfamiliar mathematical signs/symbols, of partly familiar reference contexts for the signs/symbols and of fundamental mathematical concept principles, which regulate the mediation between signs and reference contexts.

This epistemological triangle is a theoretical schema, in which the corners reciprocally 'determine' each other; thus, none of the three elements can be explicitly or unequivocally given in order to then deductively determine the other elements. A fundamental concept is necessary to regulate the mediation between sign and reference context, and in the further development of mathematical knowledge, the fundamental conceptual knowledge is enhanced and differentiated.

The following classroom scene is taken from a third grade class working on the topic of 'figurate numbers'. All the children are sitting in a circle facing the board. Displayed on the board are dot patterns for the first five rectangular numbers (divided into two triangular configurations by means of different colours) together with the values of the respective triangular or rectangular numbers (see Fig. 4.3). The discussion now focuses on determining the amounts and the configuration for the 6th position.

88 T Yes. So what can we do to find out if this is always true?
89 S Nothing.
90 T Christopher.
91 Ch I notice something.
92 T Yes, tell us.
93 Ch Up there it goes four. Then it goes six. Then it goes eight. And then it goes ten. [*At this moment, T points at the number 20 and then at the number 30 on the left hand side of the table*]. Then it goes twelve [*T now points at the empty field below the number 20*]. Therefore, there should be thirty-two on the other seventeen [*2 sec pause*] um, forty-two should be on that and on the other one twenty-seven
94 T I see. You mean . . ., that's quite an interesting idea, Christopher. You mean, here there should be forty-two? [*points at the empty field below the number 30*]
95 Ch Yes. [*T writes the number 42 in the table*]

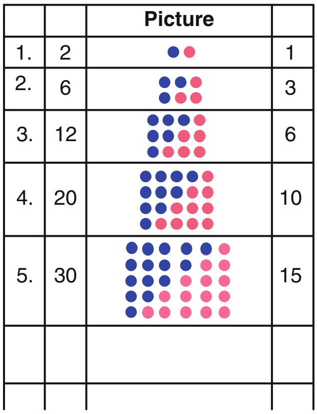


Fig. 4.3 The connection between triangular and rectangular numbers

96 T Yes. And there? [*points at the empty field below the number 15 on the right side of the table*]
97 Ch Twenty-seven.
98 T Why do you think there should be twenty-seven? . . . Can you give a reason for that?
99 Ch No.
100 T No? . . . Nico.
101 N Twenty-one.
102 T Why do you think [it‘s] twenty-one?
103 N Because twenty and twenty are forty [*points at the ten’s decimal place of the number 42*] and one and one are two [*points at the unit’s place of the number 42*].
104 T Mhm [*writing the number 21 in the table*]. Oh yes, then we already know the next thing. But we ought to check whether it is indeed correct from the picture, whether it is really always like this.

This classroom scene will first be structured and summarized. First the teacher asks again his question as to whether it is always true, and after that he asks: “How can we find out whether this is always true?”

Phase 1 (90–97): Christopher continues the numbers in the column of the rectangular numbers and derives a new triangular number.

Christopher notices something. He names the sequence of numbers one after another: “. . . there it goes 4, then it goes 6, then it goes 8, then it goes 10, then it goes 12”. With this, he seems to refer to the second column, and the teacher points at this column, at the numbers 20 and 30. Christopher names the respective difference or increase between the numbers in his sequence. Then he infers: “Therefore there should be thirty-two and on the other seventeen”. He has (mistakenly?) constructed a number bigger by 2 in the left number column and he does the same in the right number column: from 15 to 17. Christopher corrects his statement: “Forty-two should be on that and on the other one twenty-seven”. Here he has raised the two numbers by 12. The teacher confirms the first number with the question whether "… here there should be forty-two?" and he writes this number down after Christopher has agreed. Christopher repeats once again that, in the other position, there should be '27'.

Phase 2 (98–100): Christopher cannot justify his procedure.

The teacher asks Christopher to justify his claims. But Christopher cannot justify why '27' is supposed to be here.

Phase 3 (100–104): Nico corrects Christopher's triangular number and gives a justification for his claim.

Nico says '21' and means that this number is correct. He justifies this with the following 'calculation': "Because twenty and twenty are forty [*points at the tens decimal place of the number 42*] and one and one are two [*points at the units place of the number 42*]." The teacher agrees with him and writes down the new numbers (see Fig. 4.4). The teacher formulates a new 'research mission': "But we ought to check whether it is indeed correct from the picture, whether it is really always like this".

This detailed description of the short mathematical interaction between the two boys and their teacher clearly shows that the mathematical knowledge and understanding of this knowledge emerges, and is not completely fixed and clear-cut (as, for instance, *stoffdidaktik* (see part 2) would assume). The learning process is not simply a procedure of acquiring step-by-step the correct and undisputed mathematical rules and expressions.

The reactions and the following proposals in this mathematical interaction – contributed by the boys as well as by the teacher – develop and evolve according to the ongoing intention to commonly clarify and gain an understanding of, and a meaning for, the mathematical knowledge in question. This is what is meant by a parallel development of mathematical knowledge and the teaching and learning processes (see Part III).

The following (limited) epistemological analysis will show how central ideas of Part IV become relevant: the mathematical knowledge that develops in this communication process is open; it has to be constructed and interpreted by

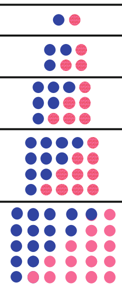
		Picture	
1.	2		1
2.	6		3
3.	12		6
4.	20		10
5.	30		15
	42		21

Fig. 4.4 New triangular and rectangular numbers

the participants, but it is subject to epistemological constraints of coherence and consistency. The knowledge is not a priori given and fixed, but develops within its epistemological frames by subjective constructions and alternating social interactions.

First, Christopher's contributions, together with the teacher's pointing gestures, can be understood in the following way. Christopher names the differences between the rectangular numbers that have been written down. The teacher points at the respective number column. Christopher seems to have in view the additive continuation of the number sequence. He continues this characterization: "Then it goes 12"; this increase by 2 is supposed to lead to the new rectangular number in the sixth row.

Christopher now uses his arithmetical progression as a justification for the new numbers. He infers first, the two numbers 32 and 17, numbers which differ by 2 from their antecedent numbers; perhaps he transfers the increase of the differences directly to the new situation and corrects himself immediately. Now he seems to add 12 in both cases, and he names the numbers 42 and 27.

In his argumentation, Christopher referred to the arithmetical continuation pattern of the rectangular numbers without the geometrical situation. In a rather typical manner, the teacher takes the 'correct part' out of Christopher's argument. He says: "You mean . . ., that's quite an interesting idea, Christopher. You mean, here there should be forty-two? [*points at the empty field below the number 30*]" (94). And then: [*T writes the number 42 in the table*] (95). In this way, there is an implicit agreement in this social interaction that one part of the expected answer is correct, namely '42'. After the teacher's question, Christopher confirms that he believes that 27 belongs to the empty field. However, he cannot justify his claim.

Nico continues the knowledge construction. His justification: "Because 20 + 20 are 40 and 1 + 1 are 2" results in halving 42 into 20 + 20 and 1 + 1. If this is put in connection with the relation "Always half", which has been thoroughly discussed before, Nico intends a justification by using this relation. Again the teacher confirms this correct number as before by writing down 21 (104).

The knowledge constructions of the two boys can be characterized epistemologically in the following way. The mediation between sign/symbol and reference context carried out in this situation can be shown by the following epistemological triangle (Fig. 4.5).

For Christopher's knowledge construction, the analysis shows that he developed a continuation principle for the sixth rectangular number from the given arithmetical pattern. His counting by twos – 4, 6, 8, 10, 12 – is meant to suggest that the difference between the rectangular numbers is always an increase by '2' and that, therefore, '12' must now be added to the value of the fifth number. This addition of '12' is transferred to the fifth triangular number, and '27' is determined as the sixth triangular number. Christopher constructs a general arithmetical relation between the rectangular numbers in a verbal way and transfers it directly to the triangular numbers. This connection is inferred only from the arithmetical structure. No justification is given, for instance, using the geometric pattern of the rectangular numbers.

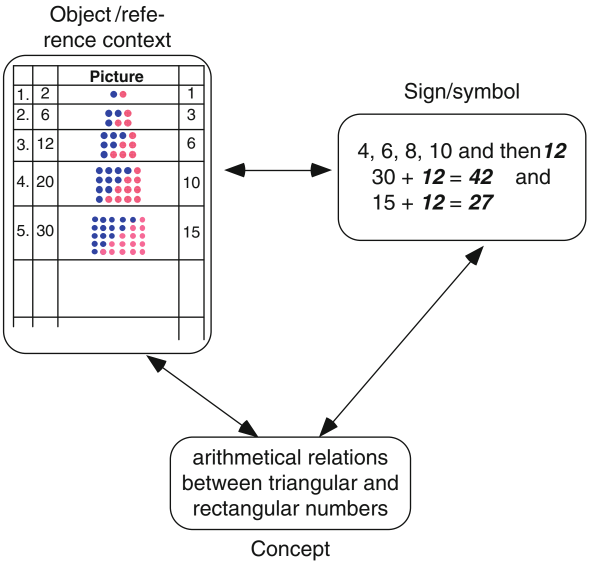


Fig. 4.5 The epistemological triangle: Christopher considers arithmetical distances of number sequences

In Nico's knowledge construction, the rectangular number '42' is halved in a particular way. The intention connected with this proposal is not articulated directly. The brief argument is restricted to the procedure of the arithmetic bisection or doubling only. Nico constructs a brief verbally-formulated sign "20 + 20 = 40 and 1 + 1 = 2" with reference to the number 42 which was noted on the poster. This mediation between sign/symbol and reference context is represented as above in the epistemological triangle (Fig. 4.6).

In their contributions, both students constructed new knowledge relations which could not be directly inferred from knowledge that was already there. These knowledge relations were restricted to the arithmetical number symbols and structures with no reference to the geometrical configurations.

The question why the structure, that was observed locally in the numbers, is really generally valid needs, for example, the reference to the geometrical, general patterns of the triangular and rectangular configurations.

Based on this analysis, it can be stated that Christopher and Nico constructed mathematical signs that are not connected to the presupposed problem knowledge, but signs that use the visible arithmetical structure of the numbers on the poster.

In this short episode, the teacher intervened at certain moments to confirm correct answers or correct parts of answers or arguments developed by the two students. To give an example:

94 T I see. You mean …, that's quite an interesting idea, Christopher. You mean, here there should be forty-two? [*points at the empty field below the number 30*].

The teacher also writes the number in question, 42, in the table (he gives similar feedback to the student Nico when writing the proposed number 21 in the

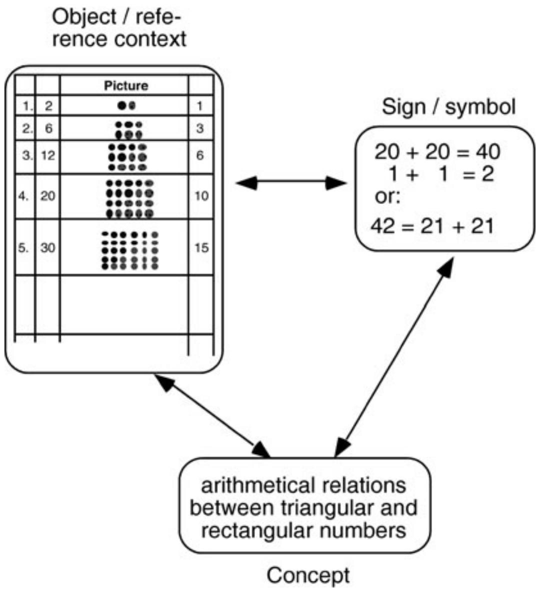


Fig. 4.6 The epistemological triangle: Nico dissects 42 into 20 + 20 and 1 + 1

table). In this way, the teacher comments and moderates the contributions and arguments developed by the students. He does not simply follow an anticipated 'correct' solution procedure strictly, but he accepts, at least in great part, the activities and proposals by the students and he guides them. Surely the students also have learned how to participate in a question-answer game in mathematics teaching and they are certainly conscious of the teacher's feedback as questioning some proposed numbers (this cannot be the right one), or as writing down other numbers (these are the expected right numbers). This exemplifies how, through common interaction, mathematical knowledge develops along the epistemological constraints (see Part IV).

Looking back to the earlier sections (2, 3 and 4), again a further summarizing interpretation can be given concerning the three elements of the didactical triangle, (1) the mathematical knowledge, (2) the student, and (3) the teacher. The mathematical knowledge, essentially the important mathematical relations and structures, are in a way interactively constructed between the boys and the teacher. Thus, the theoretical (school-) mathematical knowledge (1) evolves here in a communication process between these three persons.

Christopher (2) argues in a situation-bound relational justification context. He constructs new arithmetical relations in the given structure that are also transferred to the arithmetical continuation of the triangular numbers without an additional underlying justification.

Nico (2) argues within an algorithmic justification context. He communicates factual knowledge. He seems to have in mind a relation between rectangular and triangular numbers by saying 21 + 21 = 42. Also, Nico does not produce true new mathematical knowledge as his argumentation refers exclusively to arithmetical relations, and does not take the geometrical knowledge problem into consideration.

The teacher (3) participates in this interaction as a moderator and he comments on students' proposals in a way of pointing at acceptable and unacceptable suggestions, thus guiding the process of negotiating the evolving mathematical relations of theoretical knowledge.

Looking at the didactical triangle as a descriptive instrument (see end of Part IV) in order to label the essential elements and their reciprocal actions within mathematical teaching and learning processes, the new interpretation from an epistemological mathematics education research perspective becomes clear: mathematical knowledge is interactively constructed by the participants on the basis of specific epistemological conditions thereof, which are effective also within instructional learning processes and which, in this teaching learning context, lead to a socially-developed epistemology of (school) mathematical knowledge.

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