Chapter 15
Teachers' Stories of Mathematical Subject Knowledge: Accounting for the Unexpected

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Introduction

In this chapter, we report an innovative assessment feedback tool – we call it a 'mathsmap' – and describe how two pre-service primary school teachers in England made sense of such a personalised diagnostic map to reflect on their subject knowledge in mathematics (Ryan & Williams, 2007a, 2007b). The mathsmap provides both a summative and a diagnostic profile of attainment and errors across a test of a constructed 'primary teacher mathematics curriculum' (Ryan & McCrae, 2005, 2005/2006).

The use of the mathsmap to reflect learning at a personal level is seen to provoke 'accounts' or 'stories' that might inform pre-service teachers' pedagogical content knowledge. In making their mathsmap comprehensible to themselves, the two pre-service teachers reported here, Lorna and Charlene,[1] were provoked to account for their own knowledge 'troubles', that is, to narrate their metacognition. We were interested, in particular, in their view of themselves as mathematical learners and how this would impact on their pedagogical content knowledge and teacher identity.

We offer a method for encouraging such reflection by having pre-service teachers personally confront their *patterns* of responses as indicated on their mathsmap. This tool is different from other feedback devices in drawing attention to non-normative responses of two kinds: unexpected correct and unexpected incorrect responses. Being told that responses are not 'expected' causes dissonance, or 'trouble' *to be explained*; such troubles generate 'accounts' or stories narrated to normalise them (Bruner, 1996). This also provides the researcher or teacher educator with some insight into pre-service teacher self-knowledge, indeed their metacognitive

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This chapter draws on our earlier work reported in Ryan and Williams (2007b) extending the analysis of Lorna and Charlene's accounts of their learning – in particular in 'accounting for the unexpected' as an opportunity for learning and development of pedagogical content knowledge.

[1]Lorna and Charlene are pseudonyms.

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knowledge, and perhaps their sense of self-efficacy or agency as learners. Such insights, we suggest, may also inform the design of teacher education courses.

In our earlier work on classifying the mathematical errors that children make on standardised tests, we concluded that most errors and misconceptions are the result of intelligent constructions (see for example, Ryan & Williams, 2007a). Similarly, it is such intelligent constructions that adults make that we sought to explore here with the pre-service teachers in our study, and to identify any turning points in their mathematical autobiography as they narrated or 'storied' their own learning (Bruner, 1996, pp. 144–149) around their unexpected troubling successes and errors. We think that such activity may play a significant part in the development of pedagogical content knowledge – that knowledge that Shulman (1986) referred to as including "the most useful forms of representation of . . . ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). See Chapter 2 by Petrou and Goulding, this volume, for further discussion of Shulman's work. We suggest that knowledge of one's own methods of making the subject comprehensible to oneself is a necessary first step for reflective teaching.

Teachers' Mathematical Knowledge

In the recent political climate of international league tables and government initiatives to 'drive up' standards in primary schools, the spotlight has been on the teacher and their teaching: a model of deficiency (Sanders & Morris, 2000, p. 398) in particular tempts a quick response of measure-and-fix. Yet what is a robust and useful 'measure' and what is the 'fix'? The interplay of subject matter knowledge and 'effective' teaching is complex: strong subject knowledge is arguably a central and necessary condition for more effective mathematics teaching, but it is not sufficient. We believe that a more productive approach would be to ask what sort of subject knowledge informs more effective teaching, and how might a novice teacher take control of their identity as a mathematics learner themselves and use this positively in their teaching?

In England, initial teacher education providers have been required since the late 1990s to 'audit' their pre-service teachers' mathematical knowledge and to support its development – 'gaps' in knowledge are to be filled, errors and misconceptions 'fixed' and 'connections' made between key mathematical concepts. Such political imperatives are set against the reality of the background of pre-service teachers' own school experiences – what they bring with them to their education training courses.

> Even if we were confident about trainees' knowledge and understanding of mathematics, we would need to recognise that the vast majority of trainees have tended to specialise in non-mathematical subjects after the age of 16, and may need to refresh and deepen their understanding of mathematics before entering the classroom (Goulding, Rowland, & Barber, 2002, p. 690).

Of course the affective dimension – beliefs about mathematics and attitudes to its learning – is also part of the complexity of what pre-service teachers bring to their teacher training. We think that reflecting on their own identity as a mathematics learner offers the pre-service teacher an opportunity to seize the power of metacognition; knowing how one learns and how one breaks through difficulties in understanding are perhaps potentially liberating. Co-ordinating subject matter knowledge and pedagogical content knowledge on initial teaching practice is also a balancing act, and the novice is most vulnerable to expectations – mostly their own – of classroom management in the first instance.

Goulding, Rowland, and Barber (2002) examined how mathematical subject knowledge of pre-service teachers has been conceptualised and they reported on how it has been audited in three institutions in England. The items they used in their audit instruments explored both the substantive and syntactic knowledge of the trainees, that is "knowledge *of* mathematics (meanings underlying procedures) and knowledge *about* mathematics (what makes something true or reasonable in mathematics)" (p. 692).

Their study reported trainees' difficulties, errors and misconceptions and initial findings on the complex relationship between subject matter knowledge as assessed by their audits and actual teaching performance. They hypothesised that subject knowledge "would influence both students' planning and their teaching, a cognitive dimension encompassing beliefs about mathematics, and their confidence in the classroom" (p. 694). They were "persuaded that the relationship involves both cognitive and affective dimensions" but one of their dilemmas was whether audit requirements were "creating anxiety and dislike of mathematics or acting as a useful lever for development" (p. 701).

This study also drew on Ball's (1990) call to encourage pre-service teachers to 'revisit' and perhaps 'unlearn' their own school experiences of mathematics. It is such unlearning that Ball believed provoked a deeper self-awareness and articulation of beliefs about mathematics (as cited in Goulding et al., 2002, p. 692). In exploring the relationship between audited subject matter knowledge and confidence, Goulding et al. (2002) cautioned that a simple "emphasis on the audit and the remediation process [may have] had a demotivating effect" on some pre-service teachers in their study (p. 700), but not for most:

> There is stiff competition to gain a place on these courses: most students are resilient, well-motivated and goal-oriented. Indeed the majority of those who required some remediation, including some very weak students, appeared to respond positively to the opportunity and reported in evaluations that they were pleased to address some of their weaknesses before the main teaching practice (p. 700).

Sanders and Morris (2000) tested the "factual knowledge and central concepts" of their pre-service teachers but not their "understanding or knowledge of the organizing principles and ideas of mathematics" (p. 399). Their students were encouraged to take responsibility for improving their own learning but it was found that "self-directed study inevitably had a low priority", and in the first year of Sanders and Morris's study this was found not to be a satisfactory approach in improving performance (p. 400). With another cohort they also examined the effects on confidence when their pre-service teachers were expected to re-examine their own mathematical knowledge and skills. Following an initial test, remediation was provided and re-sits undertaken on a voluntary basis. The authors were disappointed that only 40% of the students ‘grasped the nettle’ and took advantage of the support offered. Some students were “empowered by poor test results to tackle their knowledge deficits” but others “found ways of ‘excusing’ poor results” as involving technical terms or non-coverage at school (p. 407). Some students focussed only on the topics they would be teaching on their upcoming assessed placement (p. 406).

Murphy (2003) suggests that pre-service teachers may not be clear about how audited subject knowledge relates to the teaching of primary mathematics. Her study sought to examine trainee teachers’ perceptions of the value of an auditing process. She found that “only about half of the trainee teachers felt that their improved confidence had come from [the audit process] and only about one third of the trainee teachers saw that it had made a difference to their ability to teach primary mathematics” (p. 86). She found that one group of *less confident* pre-service teachers viewed the auditing process as ‘filling in gaps’ and gained confidence in their subject knowledge and their own teaching as a result. However, a second group of *confident* trainees did not see the value of the audit process and may have regarded the process as ‘jumping hoops’. Murphy suggests that differing views of the audit process may reflect “differing beliefs in mathematics as a discipline” (p. 89) and hoped that a larger proportion of trainee teachers would “see the relevance of subject knowledge to their teaching of primary mathematics” (p. 90) in response to an improved content and process of audit.

Barber and Heal’s (2003) study focussed on the role of social interaction and collaboration in learning and the effectiveness of peer tutoring in enhancing primary trainee mathematical subject knowledge. Peer interaction was used as a teaching strategy – providing opportunities for both the tutor *and* the tutee to ‘explain’ their mathematics. The pairings of high scoring and low scoring trainees were made on the basis of an initial audit. Generally low-scoring trainees had reported low levels of confidence and many of them also reported panic when required to ‘do maths’ and needing time to ‘recall’ knowledge. The peer tutors were trained in the art of explanation and reported “how enlightening it was to hear so many alternative ways of approaching each problem [on the audit] and how instructive to realise that their own perspective on the problem was not the only one” (p. 69).

The feedback from the peer tutoring sessions was positive and “pointed to the mediating influence of emotional factors” and improved confidence (Barber & Heal, 2003, p. 69). The authors suggest further development of peer tutoring with attention to the nature of ‘ideal’ pairing, tutor training, ‘quality control’ and the different needs of different ‘bands’ of trainees. The authors cautioned leaving trainees to organise self-study – they found that “half of those who identified themselves as having poor subject knowledge at the beginning of the course achieved the lowest scores in the formal audit” (p. 70).

Rowland, Barber, Heal, and Martyn (2001) are also wary of guided self-study as an adequate ‘treatment’ for poor subject knowledge. Some of the pre-service teachers in their study had difficulty in communicating what they could ‘see’ in mathematical situations and thus faced considerable cognitive obstacles in working alone (p. 93). Goulding (2003) reported that the SKIMA (Subject Knowledge in Mathematics) group – a collaboration of researchers in four UK universities investigating weaknesses in knowledge, self-assessment and the link between subject knowledge and teacher competence – had found that peer support groups and peer tutors "seemed to be successful in boosting the confidence of weak trainees and also that of the stronger trainees who acted as peer tutors" (p. 76).

Thus far, we have a deficit model of pre-service primary teacher subject matter knowledge – 'gaps' to be filled, errors and misconceptions to be 'fixed' and new connections to be made. Tests and audits have traditionally reconstituted the knowledge base of secondary school as the expected base of subject knowledge for teaching which then provokes different 'fixes' including notions of relearning, 'unlearning' and remediation.

Some of the research above prompts further attention to pre-service teacher awareness of their 'problems', the effects of anxiety, motivation for change and 'tools' for exploring these. We attempt to go a little further then highlighting also notions of identity and agency in the personal professional development of the pre-service teacher. In particular, we look to the pre-service teacher 'storying' their mathematical autobiography by accounting for the unexpected: exploring their learning identity and perhaps bearing fruitful pedagogical content knowledge that will be played out in their ongoing story of being a teacher of mathematics. See also Chapter 13 by Corcoran and Pepperell, this volume, for further discussion on identity and narratives shared by participants in Lesson Study.

Testing Subject Knowledge

We now provide a brief outline of the audit 'tool' we have used with pre-service teachers who were interested in exploring their patterns of response on a written test. We too have taken the traditional route by starting with the school curriculum. The test we used, the *Teacher Education Mathematics Test* [*TEMT*] (Australian Council for Educational Research, 2004), had been developed by first constructing a 'primary teacher curriculum' using documents based on Australian and United Kingdom secondary school curricula. Similar tests can be developed using a reasonable sample of the targeted population (see Ryan & McCrae, 2005, 2005/2006 for detail of methodology). The level of attainment targeted Australia's school level 5/6 (understood to be 'functional numeracy level') – this is the equivalent of GCSE[2] grade C, the minimum mathematics requirement for entry to initial teacher education courses in England.

Test versions (each of 45 items) were constructed across the six strands of the constructed 'primary teacher curriculum' involving Number (16 items in each test), Measurement (8), Space and Shape (8), Chance and Data (6), Algebra (5), and

[2]GCSE is the *General Certificate of Secondary Education* in England which assesses children's attainment at the end of current compulsory schooling, usually at 16 years of age.

Reasoning and Proof (2). The tests were in a pen-and-paper multiple choice format and timed for a 45-min period. A Rasch analysis (Bond & Fox, 2001; Rasch, 1980; Wright & Stone, 1979) of the responses of a large sample of students was undertaken using *Quest* software (Adams & Khoo, 1996).

Rasch scaling uses one version of item-response modelling: a one-parameter stochastic model of persons' responses to items. Here responses are modelled by a probability function characterised by one parameter – the item 'difficulty'. The model "can help transform raw data from the human sciences into abstract, equal-interval scales. Equality of intervals is achieved through log transformations of raw data odds, and abstraction is accomplished through probabilistic equations" (Bond & Fox, 2001, p. 7).

The Rasch model assigns a *difficulty* parameter to each test item, estimated by its facility, and a so-called *ability* parameter to each person, estimated by their raw score on the test. These parameters are calculated as 'log-odds' units called *logits*. The logit scale is an interval scale, and the Rasch model "routinely sets at 50% the probability of success for any person on an item located at the same level on the item-person logit scale" (Bond & Fox, 2001, p. 29). That is, a person located at an 'ability' of x has a 50% probability of correctly answering an item of 'difficulty' x, an increasing probability of answering items below that difficulty and a decreasing probability of answering items above that difficulty. The sample data builds the measure by assigning parameters to items and persons (just one each) that minimise data-model residuals.

The *Quest* program automatically calculates these item parameters (item difficulty estimates with the default mean difficulty set at zero), and person estimates (student ability estimates) and the model-fit statistics (how well items and persons fit the model) from the data. *Quest* also provides classical statistics. For further discussion of Rasch modelling and analysis see Williams and Ryan (2000).

The *TEMT* test items were scaled in terms of their difficulty and each person was located on the *same* scale in terms of their ability as measured by the test. The data were found to be compatible with the Rasch model, and test reliability and goodness of fit were strong (Ryan & McCrae, 2005, 2005/2006).

We follow the psychometric tradition and use the term 'ability' in this chapter purely in a technical sense, as a measure of the underlying construct that the test is measuring (called the latent trait in the psychometric literature). In our context, the measure is of performance or attainment on the items in this test. There is no imputation of meaning attached to the term 'ability' other than what one can construe from the face value of the items themselves.

In the construction of the multiple choice test, distracters were purposefully chosen from known or suspected errors drawn from research on children's understanding but also from research on teacher knowledge. They were used to mitigate against guessing in the multiple choice format but also, more interestingly, to provide a finer-grained detail of pre-service teachers' knowledge. Guessing/errors were not specifically penalised in the estimation of student ability.

In a second study, another cohort of pre-service teachers in England ($N = 87$) also took a *TEMT* assessment in the second year of their initial teacher training.

Their patterns of response were very similar to the larger Australian sample ($N = 426$) used to validate the test originally (Ryan & McCrae, 2005/2006). The pre-service teachers in the England sample included pre-service primary trainees, non-mathematics specialist secondary trainees and a small group of mathematics primary/secondary specialist trainees. Participation was on a voluntary basis with the promise of personalised diagnostic feedback from the test to assist their subject knowledge development.

The 87 trainees in England were given an individual map of their responses as diagnostic feedback. A questionnaire gathered information on what sense they made of their map; in addition, two pre-service teachers from this cohort volunteered to be interviewed to see what sense they made of this feedback and how they intended to address their indicated mathematical needs (Ryan & Williams, 2007b).

Personalised Diagnostic Maps of Subject Knowledge

Quest software also produces an output for each individual, called a *kidmap* (here called a mathsmap), highlighting their correct *and* incorrect response patterns. The map summarises an individual's performance according to the Rasch model's expectations. All test items are scaled on a vertical axis from lowest to highest in terms of the difficulty of each item (from easy to hard). Each individual then is mapped left or right of the axis in terms of achievement of the item or not (achieved or not achieved). The overall ability score locates each student on the axis and a fit statistic ('fit') indicates how well the student fits the Rasch model. We show Lorna's mathsmap in Fig. 15.1.

In the mathsmap the 45 items of the test are located along the vertical *scale* according to their overall difficulty. It can be seen that item 33 was the easiest and item 30 the hardest. Those items that Lorna answered correctly are located on the left of the diagram, and those that she answered incorrectly appear on the right. Lorna answered item 33 correctly and item 30 incorrectly. The mathsmap also *locates* Lorna's 'ability' on the same vertical logit scale (centrally marked by 3Xs): her ability measure estimate was 0.91[3] and she answered 64.44% (29/45) of the items correctly (see the statistics in the top margin). The dotted lines around the estimate of Lorna's ability represent ± 1 standard error for the estimate. Additionally Lorna's *actual* option choices (1, 2, 3, 4, 5 or 0), made for each incorrect item on the right-hand side, are indicated in parentheses; thus 30(4) indicates that Lorna incorrectly chose option 4 for the hardest item 30. This gives further diagnostic information (Ryan & McCrae, 2005/2006).

The individual would be *expected* to achieve all the items at and below their ability estimate with an increasing probability for those items further below. Lorna

[3]Lorna's 'ability' is located at 0.91 logits on the scale which indicates that she is nearly one standard deviation above the mean of the item difficulties. We can therefore compute the probability of her correctly answering an item of difficulty 'd' as being approximately $\exp(0.91 - d)/[1 + \exp(0.91 - d)]$; thus, for the average item with $d = 0$, this is approximately 70% for Lorna.

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------------------------------ PERSONAL MAP ----------------------------------
 Lorna                                                     ability:    0.91
 group:     all                                             fit:       1.14
 scale:     numeracy                                        % score:  64.44

------------Harder Achieved---------------------Harder Not Achieved----------
                                     truncated
                                      |   |   30(4)
                                      |   |
                                      |   |   23(5)
                                      |   |   36(3)
                                      |   |   40(4)
                                      |   |
                                      |   |   37(4)
                                      |   |   20(2)
                        38     8      |   |
                              19      |   |   14(2)
.........................................
                                      |   |
                                      |   |
                              16      |XXX|   13(4)
                              42      |   |   39(2)
                  28    18     3      ........32(1)............................
                                      |   |   29(3)
                         7     5      |   |   11(3)
                                      |   |
                                      |   |
 45   43    34    26    15     9      |   |   10(1)
                               4      |   |
                        44    25      |   |
                                      |   |
                               2      |   |   41(1)
                        31     6      |   |   12(3)
                                      |   |
                              17      |   |
                               1      |   |
                                      |   |
                              22      |   |   21(5)
                                      |   |
                                      |   |
                                      |   |
                                      |   |
                                      |   |
                              24      |   |
                                      |   |
                                      |   |
                        35    27      |   |
                                      |   |
                                      |   |
                                      |   |
                              33      |   |
------------ Easier Achieved--------------------- Easier Not Achieved----------
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Fig. 15.1 Mathsmap for Lorna

has a 50% probability of answering items *at* her ability estimate (note that item 16 is correct and item 13 is incorrect – both are located at her ability level). She would have been expected, with an increasing probability, to have correctly answered items 39, 32, 29, 11, 10 and so on, but she did not. Lorna would not have been expected to correctly answer items 38, 8 and 19 located *above* her estimate (on the left), but she *did* respond correctly to these items. In a perfect 'goodness of fit' to the Rasch model, the top left and bottom right quadrants would be empty, so items in these quadrants are particularly engaging for discussion in the first instance.

Each individual mathsmap indicates the secure and non-secure curriculum areas of a pre-service teacher: the non-secure items may indicate 'gaps' in knowledge, 'rusty' or long-forgotten knowledge or faulty conceptions. We found that discussion of them compelled a 'storying' of their mathematical knowledge and history by our two interviewees.

Bruner (1996) suggests that:

> Stories pivot on breached norms. That much is already clear. That places 'trouble' at the hub of narrative realities. Stories worth telling and worth construing are typically born in trouble. (p. 142)

Thus, the two shadowed quadrants (top left and bottom right) of the mathsmap in Fig. 15.2 list breached norms, and therefore 'trouble' to be explained, perhaps normalised, at least to be explored and brought to some narrative reality. Narrative interpretations may be idiosyncratic, but perhaps there are universals in the realities they construct (Bruner, 1996, p. 131). Bruner suggests also that 'turning points' are crucial to the narratives – "pivotal events in time when the 'new' replaces the 'old'" (p. 144), so we think that the stories of unexpected item mapping in the mathsmap may provide both the interviewee and the interviewer with insights into the subject matter knowledge and personal histories of mathematical learning that the pre-service teachers bring with them.

The pre-service teacher trainees were given their own mathsmap and guidance on how to read it. They were also given a list of the descriptors of the test items rather than the actual test items in order that the curriculum area indicated by the descriptor was targeted for study by the trainee, in a broad sense, rather than in terms

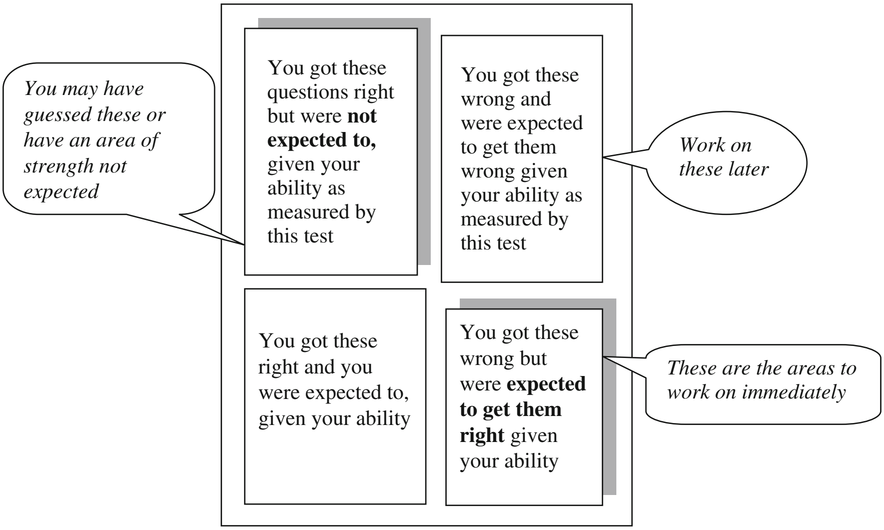

Fig. 15.2 Interpreting the mathsmap quadrants

Table 15.1 Descriptors for Lorna's unexpected *incorrect* items in the bottom right quadrant of her mathsmap

Item	Curriculum description of item
29	Algebra: multiplying simple algebraic expressions by a number
11	Chance: likelihood/probability of everyday events (numerical)
10	Shape and Space: identifying Cartesian co-ordinates
41	Algebra: from tables of values to algebraic rule
12	Chance: recognising dependent events (reduced sample)
21	Measures: finding perimeter of a rectangle – words

Table 15.2 Descriptors for Lorna's unexpected *correct* items in the top left quadrant of her mathsmap

Item	Curriculum description of item
38	Shape & Space: rotation of a shape about an internal point
8	Shape & Space: interpreting drawings on a grid
19	Shape & Space: finding one missing length for similar shapes

of item-specificity. See for example Table 15.1 for Lorna's 'easier not achieved' item descriptors – she was expected to get these items correct but did not.

Lorna's unexpectedly correct items are shown in Table 15.2. They are all Shape and Space test items.

Narrative Accounts – The Impetus of 'Troubles'

Our two interviewees had quite different mathsmap profiles. Lorna had an ability estimate of 0.91 (29 items correct, and located at the 56th percentile) and Charlene had an ability estimate of 2.00 (36 items correct, and located at the 86th percentile). Both volunteered to be interviewed on how they interpreted their mathsmap. They had quite different profiles in terms of mathematical confidence, life experience and school teaching practice.

Lorna narrated her unexpected correct responses with a story of her growth in competence and confidence in her capacity to learn. She was very animated and excited by her ability to have overcome a recent school teaching experience which had shown up her lack of knowledge in the Shape and Space area of the mathematics curriculum, and said she now felt confident about tackling her problem area of algebra as a consequence of her success. On the other hand, although (or just possibly because) Charlene was a higher scorer, her accounts for her unexpected errors told a story of 'slips', tending to marginalise explanations that might invoke her need to learn or fill knowledge gaps. She said that she often got "carried away" and made silly errors, but she also thought that she needed to improve her mental maths skills (Ryan & Williams, 2007b).

We discuss Lorna's and Charlene's interviews and point to the way 'accounting for the unexpected' in both cases impelled a story of themselves as learners or mathematicians. The resources these two pre-service teachers drew on in their story outlined and 'coloured' (or perhaps constructed) their metacognitive knowledge of learning. This leads us to propose the mathsmap as a tool for provoking pre-service teachers to 'story' their own learning and knowledge, and hence evoke cultural models of 'learning' in general.

Lorna

Lorna was a 'mature' trainee studying on a 4-year BA Primary (Hons) education degree course, qualifying her to teach in primary schools. She was not confident about her mathematics ability and said that she had achieved a C grade in mathematics in school O-levels[4] some 20 years before.

Lorna: ... I always think I am near the bottom ten percent (laughs).

However she had answered 64% of the test items correctly and was highly motivated to address areas of weakness in subject knowledge. She was energised by her unexpected responses.

Lorna: [The mathsmap] identified areas I thought I was weak in and some I didn't .... Yeah, there *were* some surprises! In both what I thought I knew and in some areas I thought I was rusty. Some areas I didn't think I was quite so wonderful on and I got them right, which surprised me. I thought, 'Oh, well not too bad at all!' 'Cos I was thinking I was sort of, virtually *way* down and had *mountains* to climb and now it shows, 'No I don't, I'm sort of in the middle with having just over half, with 64 percent.' So I've not got as much climbing to do. I thought maybe with just a few small steps and I'll be there.

Lorna was surprised that she had achieved some of the items above her ability estimate as indicated in the top left quadrant of her mathsmap (these were all Shape and Space items – see Fig. 15.1 and Table 15.2). Once this curriculum area was identified she explained her unexpected success by *recent targeting* while on teaching practice, because she already knew that this was an area of weakness – she had not guessed here.

Lorna: Well that's interesting, that! Because on my teaching practice last year with year 6, I did a unit of work in term 1 for Shape and Space and it was all about quadrilaterals and rotating shapes and the size of angles (and) symmetry. So maybe that is where that has come from, that not only I have taught them but I have learnt as well ... So I have ... as well as teaching children I have learned myself, so I know I have learnt more from

[4] O-level was the pre-1988 forerunner of the national GCSE examination.

what I have taught, as well as teaching at the same time . . . (Excited) so that tells me that maybe with time and practice that this area here [bottom right quadrant] will come, up . . . over [into the 'correct' quadrant].

She told a story of low confidence with a belief of "mountains yet to climb" in addressing subject knowledge. She had confronted a setback on her recent teaching practice which had highlighted lack of knowledge and this then became a pivotal moment, a 'turning point' – it is not only a story of directed self-study but one of a deep connection made with learning *as she taught*. Her teaching practice had been the motivating factor and it is clear here that she had been determined not to 'put it away' because it confronted her own professional identity as a teacher of mathematics.

Lorna: I've not got as much hard work to do as I thought I did. 'Cos I was dreading it. I tend to hide things and put them away and think if I put it away and can't see it, it doesn't matter and won't bother me but sometimes you've just got to . . . After my teaching practice what I did, I did flounder with Shape and Space, I did. I had some really bad lessons. The first lesson I did, the teacher she just said right we'll just put that to one side and I think we'll start again. And she gave me some help. And I went home and I studied and studied and studied. And it did, it shows it does help. I wasn't expected to get them right and I did.

After her unsuccessful lesson, Lorna's school mentor had given her time to study and prepare the lessons for this area again, so Lorna had collected textbooks and had used the internet to study Shape and Space extensively on her own in order to feel more confident.

Lorna: (I used the book on) subject knowledge, it's the one we have here in the library. And I went out and bought it and I just sat and read and read and read on Shape and Space . . .. I think it's by Suggate . . .. It was in the directed reading notes we were given to do every week. I went to that one because I'd done the (chapter) on algebra, because I was rusty on algebra. So I read up on algebra and found it really useful. It worked for me. The vocabulary was good for me. So I thought, right, I'll go for it and use it for Shape and Space. And obviously it did, it helped, it worked. I thought, now I know what to do and I went out and bought it.

She then referred to the items in the bottom right quadrant – 'easier not achieved' which she now felt she could be successful with using the same study strategy.

Lorna: It shows me that there are a lot of concepts there that are quite rusty because I am 39 – (that's) 20 years after [my own schooling] . . . so that tells me that maybe through teaching that I, (with) just a little bit of homework and practice, that I could move those quite easily up . . . over, to there [left]. . . . Because I do *fear* maths, I see maths as a bully. It is *my* bully.

> And this has shown me that I can overcome this, and become an effective maths teacher.

Lorna also identified algebra as one of her "rusty" areas and was becoming confident that she could move it 'over the line'. She asked to discuss an actual test item. Her discussion of item 41 (see Table 15.1) showed that she could now talk her way through the item on matching a table to an algebraic rule (see Fig. 15.3) after having done some personal study on algebra.

Which of the following tables represent the function $y = x^2 + 3$?

Table 1

x	0	1	5	10
y	3	4	8	13

Table 2

x	0	1	2	5
y	3	4	7	28

Table 3

x	0	1	2	3
y	3	5	7	9

A. Table 1 only B. Table 2 only C. Table 3 only D. Tables 1 and 3 E. Tables 2 and 3

Fig. 15.3 Item 41: 'Algebra: from tables of values to algebraic rule'

Lorna: Question 41. (Looking at her test script) I wrote at the side 'guessed, no idea!'

Interviewer: Do you want to talk through now what you are thinking perhaps?

Lorna: First thought, ooh, algebra! Right! So, you've got to work out – I can graph this scale, if *x* is squared plus 3, you are going to have a plus – you're going to have it going plus 3 every time *but* it's got to be squared as well. So you're going to have to take 3 off, and then you've got to have a number that you can get a square root from. This is *after* now reading about algebra. *Before* I would have just thought, oh, well it must start with a 3. And then I've thought, no, hang on, how am I going to do this? I just didn't know. And then I thought, oh *x,* in the top row in table 1, you've got 1, then I felt, well '*x* squared', 1 times 1 is 1 plus 3 is 4 (pointing to it) . . . And then the next number along in table 1, *x*. I've thought if *x* is 5, I've not squared it, I've just added 3. And the next one along in table 1 is *x* is 10, and then the answer below is 13. I've just added 3, I've just guessed, *panicked* and just gone for number 1 [option A] which was table 1.

Here Lorna constructs an account of her mistake of 'adding 3', which she had originally thought was because she "guessed". Now 'after reading about algebra', she can see "*x* is squared plus 3 . . . you're going to have it going plus 3 every time *but* it's got to be squared as well." She reinforces this formulation of the function by inverting it and emphasising the need for a square root.

We note that in talking about her own thinking 'before', she switches tenses as in "I would just have thought" and "I just didn't know". Here she constructs her old thinking to include a squaring of the *x*, re-working the first *x*-value in Table 15.1, getting the right value of 4; but then "I've thought, if x is 5, I've not squared it, I've just added 3 . . . I've just added 3, I've just guessed."

This is a pivotal event where Lorna replaces the 'old' with the 'new' (Bruner, 1996. p. 144) in her story of her algebraic understanding. What began with a powerless statement, "I guessed, no idea" becomes, by the end of her story, a new guess, "I've just added 3" which we pedagogues would conceptualise as a self-diagnosis. This is an important storying of her self 'before' and 'after' her learning about algebra, and we think offers insight into her potential metacognitive learning about her own learning.

Charlene

Charlene was a science specialist trainee on a 3-year BSc (Hons) in primary and secondary education degree course, qualifying her to teach as a generalist in England's Key Stage 2 (middle and upper primary school) or as a science specialist in Key Stage 3 (lower secondary school) and perhaps Key Stage 4 (upper secondary school). She was confident with the mathematics in the test – she had answered 80% of the items correctly and was interested in seeing where she had made mistakes. She had achieved a B grade on her AS-level[5] mathematics two years previously. She reported that her mathsmap (see Fig. 15.4) was initially a puzzle but once she had read the detailed instructions it made sense.

Charlene: When I first looked at it, I was like 'what is this!" I was looking at it thinking 'how do you read that?' But then, once I'd . . . actually looked at it properly, and then read a few of the instructions, I was like 'that's easy!', it made sense, and it seemed the best way, probably, to present the information.

Charlene confirmed that the items in the bottom right quadrant (easier, not achieved items) (see Table 15.3) made sense as items she should have answered correctly and seemed to have an understanding of the type of errors she would have made: silly mistakes rather than knowledge problems.

Charlene: I mean, they looked like the sort of things that I . . . probably would have had problems with or made a silly mistake on, like the decimal point (question 16) . . . and also probably (question) 5 because it's 'measuring, in lengths, mm, cm and metres' so that will be converting, which is easy for me to make a mistake in. . . . I just, I don't know, I just get carried away. I jump one step ahead, and it all goes pear-shaped . . . 'Cos sometimes I try and think too advanced for the questions, 'cos I did AS [A- level year 1], not very well, maths, but I do sometimes think there's more to it than what's there.

[5] AS is the first year of the Advanced level which constitutes the final 2 years (called AS and A2) of post-compulsory schooling in England.

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------------------------------ PERSONAL MAP ----------------------------------
 Charlene                                                     ability:   2.00
 group:     all                                               fit:       0.87
 scale:     numeracy                                          % score:  80.00

------------Harder Achieved ---------------------Harder Not Achieved ----------
                                       truncated
                                         |   |   30(4)
                                         |   |
.........................................        23(5)
                              36         |   |
                                         |   |   40(4)
                                         |   |
                                         |XXX|   37(1)
                              20         |   |
                          38   8         |   |
                                         ........14(1)...19(2)....................
                                         |   |
                                         |   |
                                         |   |
                              13         |   |   16(3)
                          42  39         |   |
                     32   18   3         |   |   28(3)
                              29         |   |
                          11   7         |   |    5(2)
                                         |   |
                                         |   |
  45   43   34   26   15   10  9         |   |
                               4         |   |
                          44  25         |   |
                                         |   |
                          41   2         |   |
                     31   12   6         |   |
                                         |   |
                              17         |   |
                               1         |   |
                                         |   |
                          22  21         |   |
                                         |   |
                                         |   |
                                         |   |
                                         |   |
                                         |   |
                              24         |   |
                                         |   |
                                         |   |
                          35  27         |   |
                                         |   |
                                         |   |
                                         |   |
                              33         |   |
                                         |   |
------------ Easier Achieved -------------------- --Easier Not Achieved ----------
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Fig. 15.4 Mathsmap for Charlene

Charlene suggests that she "get(s) carried away", or thinks in a "too advanced" manner rather than having missing knowledge, that may explain her errors.

She said that in converting 0.125 to a fraction on the test (item 16, 'Number: decimal to fraction conversion') she probably 'misread' one of the answer options (C: 125/100) which she had selected thinking it was 125/1,000 (not one of the options). But she also said that her mental mathematics skills needed improvement and her processing on this item showed that she was using repeated addition to find how many 125s in 1,000.

Table 15.3 Descriptors for Charlene's unexpected *incorrect* items in the bottom right quadrant of her mathsmap

Item	Curriculum description of item
16	Number: decimal to fraction conversion
28	Data: graphs – generating rules of the form $y = mx+c$ from graph points
5	Measures: ordering metric lengths stated in mm, cm, m

Charlene: (Reading the question) '0.125 is the same as' (Pause) It's . . . not sure how to do . . . it's 1, 2, 5 over a thousand. I think I probably went for C originally. (Checks) Yes . . . Because I just must have missed out, misread one of the noughts, seeing there was an extra nought on it, because that was an automatic . . .

Interviewer: What would you go for now?

Charlene: (Long pause) I need to improve my mental maths. I can't. (Pause) I'll have to do it the long way . . .

Interviewer: What's the long way?

Charlene: (Laughs) I'm doing, how many, I'm working out the multiples of 125, to work out whereabouts (writing) a thousand . . .

Interviewer: You've got 125, 250.

Charlene: 375, 500. OK, so 4 is 500, so, 8 would be a thousand. So it's '1 over 8', which is B.

Interviewer: You've gone for B. So why do you think you went for C originally, again, can you express that?

Charlene: Because I misread the 100 as 1,000, so I just assumed it was 125 over 1,000 when it was 125 over 100. And I think even when we came out, somebody mentioned that, and I thought, oops, maybe I did pick the wrong one then.

This account matches Charlene's first explanation for her 'mistakes' as getting "carried away" or "jumping ahead" so that things go "pear-shaped". She says she "misread" and 'saw' an extra nought in the denominator of the option C fraction and processed quickly here as a one-step item. Here for item 16, her thinking does not appear to be "too advanced" or anticipate a two-step item, but rather suggests a seldom-used mental fact which took her a little time to re-construct.

One of Charlene's items located *at* her ability level (see Fig. 15.4, item 37) was answered incorrectly. The curriculum description was 'dissection and tessellation: understanding Pythagoras' theorem' and involved interpreting a classic proof by area dissection (see Fig. 15.5). It was the fifth hardest item on the test but discriminated well at the top end of the ability range. Charlene said it was an unusual question because it was asking for a proof.

An internet animation demonstrates the theorem of Pythagoras by dissection and *drag-and-drop* transformations of the shapes shown on the diagram.

What will the transformations show to demonstrate the theorem?

A. That D and C will fit exactly into E
B. That A, B, C, D and E will fit exactly into F
C. That A, B, C, D and G will fit exactly into F
D. That A, B and C will fit exactly into G

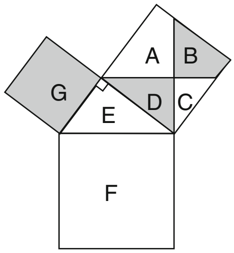

Fig. 15.5 Pythagoras' Theorem

Charlene: (Laughs while reading the question) No, it's just, yes, what's this on about? I think it could just be the question itself as well, (if) you've not really experienced that sort of thing ... It's something that's got to prove Pythagoras' theorem and that ... Is that '*a* squared plus *b* squared equals *c* squared'? Is that Pythagoras?

Interviewer: Is it?

Charlene: (Pause) I don't ..., or is it *sohcah* ... No, *sohcahtoa* is different. It is '*a* squared plus *b* squared equals *c* squared'. (Pauses)

Interviewer: What would that mean in relation to this picture?

Charlene: (Pauses and laughs) I haven't got a clue! (Pauses) I don't know what it means, the diagram ... '*a* squared plus *b* squared equals ...'

Interviewer: What does that mean?

Charlene: It means the length of the two short sides, both squared, and added together, is the same as the length of the longest side, the hypotenuse, squared ... (pauses)

Charlene juggled good-naturedly with the item here saying "what's this on about?" and recognised that 'previous experience' of something like this would help as it was an unexpected type of test question. She 'knew' the Pythagorean theorem but appeared not to have a geometrical image of it and did not make any connection with 'square' shapes in this or further discussion – this is not surprising of course if the theorem is simply represented as a numerical/algebraic formula without visualisation. But the point is that she does not consider this as an instance of a missing conception of 'square'.

Comparison, Contrast and Limitations

Lorna and Charlene had very different mathematical backgrounds, levels of confidence and motivation to improve their subject knowledge. As a mature student, Lorna was highly motivated and aware of her "rusty" knowledge and particular areas of weakness. She had in fact underestimated her mathematical ability as measured on this test, which was above average for her cohort, whereas she had thought she had "mountains to climb". As a result of uncomfortable exposure of poor subject knowledge on her own school teaching practice, she had already targeted Shape and Space for study and was very pleased that her mathsmap indicated that she had achieved beyond her current expected ability level. It appeared she was also very motivated by her school mentor who had given her the opportunity to “start again”. She was very independent and willing to put in a lot of extra time – she commented that the younger students wanted it all done for them. Lorna had targeted algebra from her mathsmap for personal study already and demonstrated in discussion that her confidence in articulating algebraic structure was growing. She seemed to be very positive about the sort of feedback the mathsmap gave her and considered her subject knowledge as a ‘work in progress’.

Charlene was a high achieving science student who had recently completed AS-level. She was very confident about her mathematics ability and had quickly made sense of her mathsmap. She did not identify any areas of subject knowledge weakness, generally explaining most of her errors as simple processing errors due to her tendency to rush or to anticipate questions as more complex than they were. This seemed to be generally the case from discussion of her errors, though she exhibited some fundamental scale misconceptions related to linear graphing (in item 28, for example) with prototypical misreading of the scale. She did not appear to be alert to multi-step questions though she could identify them in discussion afterwards. Charlene did note that her mental mathematics skills needed further work, but predominantly diagnosed her errors as ‘slips’, and her narrative leaves little space for knowledge gaps or misconceptions. However she said she would prefer to have the actual test questions back to review to see whether she had just made a silly mistake or whether she did not actually understand something.

In both cases, the limitations of the mathsmap as a tool become apparent. Firstly, it was fortunate that Lorna was able to identify Shape and Space as an area of strength but it is not particularly well-designed to profile topic strength as it is a short, item-focussed diagnostic tool. Secondly, Charlene being a high-scorer receives less diagnostic feedback than Lorna. In a computer adaptive test format where items target the ability, Charlene would be presented with more challenging test items and would thus receive more diagnostic feedback from her mathsmap. Finally, for the same reason (that the items are generally distant from the ability), we might expect a particularly weak student to get less value out of the mapping tool as currently designed.

Subject knowledge is one component – but an important one – in building mathematical knowledge in teaching. We have shown here how two pre-service teachers made use of one subject knowledge audit tool to narrate their metacognition. We think that such opportunities for personal narration may provoke agency and provide a basis for further development of pedagogical knowledge.

Conclusion and Discussion

In previous work we and others have shown how teacher errors can provide opportunities for pre-service teachers to examine the basis for their own understandings, as well as identifying areas for attention by teacher educators (for example, Rowland et al., 2001; Ryan & Williams, 2007b). We have offered here one method for encouraging teacher reflection by having pre-service teachers personally explore their responses, errors and misconceptions with a mathsmap. We are aware, however, that with such a focus, the deficit model can be predominant. The mathsmap is different from other feedback devices in drawing attention to non-normative responses of the two kinds. The unexpected correct and incorrect responses can be productively cast as 'trouble' to be explained, thus compelling stories to account for them (Bruner, 1996). Such accounts – it seems to us – provide opportunities to explore students' metacognitive knowledge, and even the sense of agency in the students' own learning.

Thus, Lorna narrated her unexpectedly correct responses with a story of her growth in competence and confidence in her capacity to learn. It is difficult not to interpret this as a very positive indicator. Charlene was a higher scorer and she narrated her unexpected errors with a story of 'slips' rather than considering a need to fill gaps in her subject knowledge.

We do not want to over-interpret these two limited cases, but rather point to the way 'accounting for the unexpected' in both cases impelled a story of themselves as learners or mathematicians. The resources they used – for example, whether they invoked 'misconceptions' or not – reflected their metacognitive knowledge of learning and hence tapped their pedagogical content knowledge. Interestingly, other work asking primary teachers to account for the unexpected errors of their children (as produced on the children's mathsmaps) have similarly provoked accounts from their teachers, which draw on explanations such as 'slips' or 'we've done a lot of that recently' (Petridou & Williams, 2007). This leads us to propose the mathsmap can be a tool for provoking students to 'story' their own learning and knowledge, and hence becomes a diagnostic of the cultural models available to them for narrating stories of 'learning' in general, which we argue is an important component of pedagogical knowledge and might be critical in the formation of professional identity.

Bruner (1996) suggests that cultural norms are constructed through canonical stories. One cultural norm identified above draws on the notion of learning as 'filling gaps' in knowledge – a norm that some have argued is dangerously reminiscent of the 'empty vessel' notion. Yet it 'works' for Lorna because she is able to see how her own efforts have 'filled the gap', and so the story reinforces her identity as an agentive, active self-improver. Perhaps we can identify other models in the data, or in others' stories of learning. If not, we suspect, these students will enter teaching with a very limited repertoire of models for learning and hence for their role in teaching.

Reconceptualising the stories in the literature about trainees' learning as 'canons' may make other narrative options available. For example, when students tell of the importance of multiple methods in their own learning of problem solving, might this be part of another canonical narrative of teaching, one which is more connectionist, and one which negates traditional teaching of procedures? If this connection can be firmly established, then we will understand why it is so important for teachers to experience such problem solving themselves as learners, and how these experiences can provide the resources for professional development of the connectionist teacher.

Of course, such stories need to be evaluated and sifted. The cultural model of 'magical influences' which is common in many cultures – including our own – whereby errors are 'just slips' may not be a helpful model with which to narrate pedagogy. But a critical approach to identity formation would ask that even these intuitive stories need to be written and examined by educational criteria.

The research issue which might arise is how teachers' identities, dispositions towards, and knowledge of, learning and teaching may benefit from such reflections on their own experiences of learning. We argue that the task is to study narratives of learner identities, and even *professional identities*, and narratives of learners-becoming-teachers, and to understand the critical events that mediate this long term professional identity work.

Chapter 10 by Williams (this volume) argues that Shulman's 'propositional knowledge' arises from scientific reflection on practice and is largely mediated by academic, formal conceptual language, but that 'case knowledge' is embedded in practical teaching situations and is typically mediated by conceptual language from the classroom and staffroom. By extension from this we can argue that 'case knowledge' for teaching can be generated from learning experiences too, and that this is the most obvious and appropriate source of this for pre-service teachers. It is pertinent and compelling to note that 'case study' knowledge is traditionally told through narrative accounts, sometimes even biographies, and is as close to a 'story telling' genre of research reportage as one finds in social and educational research.

We conclude with some research questions that should help frame the next steps in this line of research:

- What are the canonical stories of learning and teaching we want our students/teachers to be able to tell?
- What experiences do our pre-service teachers need to reflect on to generate this cultural knowledge?
- There may be many ways of narrating learning and teaching. How might professional identities result from teachers positioning themselves in relation to these canons? By what criteria might we best evaluate the canonical stories of learning and teaching?

These questions reformulate old questions in the conceptual framework of cultural narrative – old wine in new bottles. But it might be helpful to think of teacher education in this way. Stories, narratives, parables, folk tales and the like have been the chief means by which cultural knowledge has been shared for many thousands of years and arguably continues to be so; how better can we explain that the ancient common sense views of pedagogy continue to thrive in schools despite the volumes of book-knowledge 'available' to educators?

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