Chapter 14
Using Theories to Build Kindergarten Teachers' Mathematical Knowledge for Teaching

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Introduction

Around the world, there are moves to strengthen the mathematical development of children in preschool settings, and to strengthen the preparation of preschool teachers to support such development. A joint position paper published in the United States by the National Association for the Education of Young Children (NAEYC) and the National Council for Teachers of Mathematics (NCTM) stated that "high quality, challenging, and accessible mathematics education for 3- to 6-year-old children is a vital foundation for future mathematics learning" (NAEYC & NCTM, 2002, p. 1). As such, they recommend that "teachers of young children should learn the mathematics content that is directly relevant to their professional role" (NAEYC & NCTM, p. 14). Similarly, the Australian Association of Mathematics Teachers (AAMT) and Early Childhood Australia (ECA) published a joint position paper calling for the adoption of "pedagogical practices that encourage young children to see themselves as mathematicians" (AAMT/ECA, 2006, p. 2). They too recommended that early childhood staff be provided with "ongoing professional learning that develops their knowledge, skills and confidence in early childhood mathematics" (AAMT/ECA, 2006, p. 4). In England, the Practice Guidance for the Early Years Foundation Stage (2008) offers suggestions for practitioners in how to foster children's knowledge of counting, calculations, shapes and measures.

All too often, preschool teachers receive little or no preparation for teaching mathematics to young children (Ginsburg, Lee, & Boyd, 2008). Moreover, research on preschool teachers' mathematical knowledge is limited and investigating the ways in which tools may be used for building kindergarten teachers' mathematical knowledge for teaching is critical. Recently, Tsamir (2008) described how theories of mathematical knowledge may be used as tools in mathematics teacher education. We extend this idea and describe how combining theories of teachers' knowledge

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with theories of mathematical knowledge may be used as a tool to build kindergarten teachers' mathematical knowledge for teaching.

There is a wide range of theories relevant to the development of mathematical knowledge. In this chapter we focus on Tall and Vinner's (1981) concept image-concept definition (CICD) theory and describe how familiarizing kindergarten teachers with this theory may be used to build their mathematical knowledge for teaching. Much of kindergarten children's knowledge is based on their perceptions and manipulations of their surrounding. Left unchecked, intuitive interpretations created at this age often become rigid and difficult to undo at a later stage (Fischbein, 1987). It is therefore relevant to introduce this theory to kindergarten teachers so that they may plan activities that help young children assimilate concepts of higher complexity and abstraction during the early years, encouraging children to build concept images which are in line with concept definitions.

In framing the mathematical knowledge kindergarten teachers need for teaching, we draw on the works of Shulman (1986) and of Ball and her colleagues (Ball, Bass, & Hill, 2004; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008). Clearly, all teachers need to know the mathematics they are teaching. Kindergarten teachers, for example, need to be able to discriminate between triangles and non-triangles. Yet, this is not sufficient. Teachers must be able to explain why a figure is, or is not a triangle. They also need to know effective ways of presenting figures to their students so that they too will be able to differentiate between triangles and non-triangles.

In this chapter, we describe how combining theories embedded in the realm of teacher knowledge with theories embedded in the realm of mathematics knowledge and familiarizing practicing kindergarten teachers with this combination was used to build their geometrical knowledge for teaching. The chapter begins by describing the separate theories and how they may be combined to build a more comprehensive and refined tool for building and evaluating mathematical knowledge for teaching. It then illustrates how this tool was used to build kindergarten teachers' knowledge for teaching geometrical concepts. We also illustrate how kindergarten teachers used the combination of theories to inform their practice. Finally, we address how the combined theories tool described here may be further developed and used.

Combining Theories of Teacher Knowledge with Theories of Mathematics Knowledge

Dimensions of Knowledge for Teaching

In his seminal work, Shulman (1986) described and analyzed components of teachers' knowledge necessary for teaching. As already described in Chapter 2, two of the major components identified were subject-matter knowledge (SMK) and pedagogical content knowledge (PCK). Ball and her colleagues further developed Shulman's theory, focusing on mathematics, but retaining a basic framework that can be generalized to other subject areas. SMK was further divided into common content knowledge (CCK) and specialized content knowledge (SCK). CCK may be defined as "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399), whereas SCK is "mathematical knowledge not typically needed for purposes other than teaching" (Ball et al., 2008, p. 400). Pedagogical content knowledge may be further differentiated into knowledge of content and students (KCS) and knowledge of content and teaching (KCT). KCS is "knowledge that combines knowing about students and knowing about mathematics", whereas KCT "combines knowing about teaching and knowing about mathematics" (Ball et al., 2008, p. 401).

We illustrate the dimensions of knowledge for teaching within the context of geometry by offering a few examples taken from Ball et al. (2008). Knowing that the diagonals of a parallelogram are not necessarily perpendicular may be considered knowledge typical of anyone who knows mathematics (CCK). Knowing "how the mathematical meaning of *edge* is different from the everyday reference to the edge of a table" (p. 400) is an example of SCK. Knowing which shapes young students are likely to identify as triangles, and that confusion between area and perimeter may lead to erroneous answers, are examples of KCS. Knowing how to sequence the presentation of examples and which examples may deepen students' conceptual knowledge is KCT.

Shulman's and Ball's theories have been used to explore teachers' mathematical knowledge in several specific mathematical contexts such as division of fractions (Tirosh, 2000) and multiplication and subtraction of whole numbers (Ball et al., 2008). These theories have not been explicitly combined with the more general mathematics knowledge CICD theory suggested by David Tall and Shlomo Vinner in the 1980s. In the next section we review Tall and Vinner's theory taking into consideration Fischbein's theory of intuitive knowledge.

Concept Image-Concept Definition (CICD)

Having precise definitions for mathematical concepts ensures mathematical coherence and provides the foundation for building mathematical theories. However, these same mathematical concepts may have been encountered by the individual in other forms prior to being formally defined. Even after they are defined, mathematical concepts often invoke images both at the personal as well as the collective level. The term concept image is used to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). The concept definition refers to "a form of words used to specify that concept" (p. 152). A formal concept definition is a definition accepted by the mathematical community whereas a personal concept definition may be formed by the individual and change with time and circumstance. Because the concept image actually contains a conglomerate of ideas, some of these ideas may coincide with the definition while others may not. For example, a function may be formally defined as a correspondence between two sets which assigns to each element in the first set exactly one element in the second set. Some students claim that a function is a rule of correspondence (Vinner, 1991). This image does not contradict the definition. However, it is limited and eliminates the possibility of an arbitrary correspondence.

When a problem is posed to an individual, there are several cognitive paths that may be taken which take into account both the concept image and concept definition. At times, although the individual may have been presented with the definition, this particular path may be bypassed. According to Vinner (1991), an intuitive response is one where "everyday life thought habits take over and the respondent is unaware of the need to consult the formal definition" (p. 73). Intuitive knowledge is both self-evident and immediate and is often derived from experience (Fischbein, 1987). As such, it does not always promote the logical and deductive reasoning necessary for developing formal mathematical concepts. "Sometimes, the intuitive background manipulates and hinders the formal interpretation" (Fischbein, 1993, p. 14). Fischbein (1993) considered the figural concepts an especially interesting situation where intuitive and formal aspects interact. The image of the figure promotes an immediate intuitive response. Yet, geometrical concepts are abstract ideas derived from formal definitions. Thus, as we consider the notions of concept image and concept definition, we take into account aspects of Fischbein's theory related to intuitive and formal knowledge.

Although Tall and Vinner (1991) introduced their theory within the context of advanced mathematical thinking, the interplay between concept definition and concept image is part of the process of concept formation at any age. Young children learn about and develop concepts, including geometrical concepts, before they begin kindergarten. As such, their concept image is often limited to their immediate surroundings and experiences and is based on perceptual similarities of examples, also known as characteristic features (in line with Smith, Shoben, & Rips, 1974). This initial discrimination may lead to only partial concept acquisition in that children may consider some non-examples to be examples and yet may consider some examples to be non-examples of the concept. Regarding geometrical concept formation, van Hiele (1958) theorized that students' geometrical thinking progresses through a hierarchy of five levels, eventually leading up to formal deductive reasoning. At the most basic level, students use visual reasoning taking in the whole shape without considering that the shape is made up of separate components. At the second level, students begin to notice the different attributes of different shapes but the attributes are not perceived as being related. At the third level, relationships between attributes are perceived and definitions are meaningful. Kindergarten children begin to perceive attributes, but need guidance in order to assess which attributes are critical for identifying a figure and which are not. Familiarizing kindergarten teachers with the CICD theory may enlighten teachers to the tension which may exist between the concept image and concept definition and inform their instruction in ways that will promote children's advancement along the van Hiele levels of thinking.

The Combined Framework

Ball's notions of CCK, SCK, KCS and KCT allow us to differentiate between types of knowledge necessary for teaching. We suggest that the four dimensions of teachers' knowledge be combined with theories of mathematics knowledge in order to provide a finer grain and more focused lens with which to study mathematics teachers' knowledge for teaching. Such a framework would allow us to investigate, for example, teachers' knowledge of the psychological aspects of student's mathematical errors. Here we suggest how these four dimensions may be combined with Tall and Vinner's CICD theory and illustrate this framework within the context of geometry focusing on teachers' knowledge for teaching triangles.

	Domains of teachers' knowledge			
Domains of mathematical thinking	CCK	SCK	KCS	KCT
Concept image	Cell 1	Cell 2	Cell 3	Cell 4
Concept definition	Cell 5	Cell 6	Cell 7	Cell 8

Cell 1: CCK-Image. Here we address the common knowledge of a concept's image. This includes knowing to draw examples and non-examples of triangles.

Cell 2: SCK-Image. Here we address the specialized knowledge of a concept's image necessary for teaching. This includes a rich concept image of triangles which incorporates scalene and obtuse triangles with different orientations and not just equilateral and isosceles triangles. It may also include a broad image of non-examples for triangles beyond circles and squares (Tsamir, Tirosh, & Levenson, 2008).

Cell 3: KCS-Image. Here we address knowledge related to students and concept images. This includes knowing that the equilateral triangle is a prototypical triangle (Hershkowitz, 1990) and that young children may not identify as a triangle a long and narrow triangle such as the scalene triangle, even when admitting that it has three points and lines (Shaughnessy & Burger, 1985). We also include in this cell knowledge of the van Hiele model (e.g., van Hiele & van Hiele, 1958) for students' geometrical thinking and being able to recognize, for example, that a student's concept image at the most basic level takes in the whole shape without considering its components. As such, this cell includes knowing that a rounded 'triangle' is often identified as a triangle (Hasegawa, 1997) because children take in the likeness of the whole shape, ignoring that the shape is missing vertices.

Cell 4: KCT-Image. Here we address knowledge related to teaching and concept images. This includes knowing which examples and non-examples to present to a student which will broaden his or her concept image of a triangle to include, for example, triangles with different orientations.

Cell 5: CCK-Definition. Here we address common knowledge related to a concept's definition. It includes knowing that a triangle may be defined as a polygon with three straight sides.

Cell 6: SCK-Definition. Here we address the specialized knowledge of a concept's definition. In mathematics, definitions are apt to contain only necessary and sufficient conditions required to identify an example of the concept. Other critical attributes may be reasoned out from the definition. Thus, this cell includes knowing that defining the triangle as a three-sided polygon implies that it must be a closed figure with three vertices. It includes knowing that the triangle may be defined as a three-sided polygon, or a polygon with three angles, or a polygon with three vertices and that all three definitions are equivalent.

Cell 7: KCS-Definition. Here we address knowledge related to students and concept definitions. It includes knowing that a minimalist definition may not be appropriate for young students at the first or second van Hiele level because they do not necessarily perceive that a polygon with three sides must have three vertices. For example, research has suggested that for young children, the association between a triangle and the attribute of 'threeness' may be stronger than the necessity for it to be closed or for its vertices to be pointed (Tsamir et al., 2008).

Cell 8: KCT-Definition. Here we address knowledge related to teaching and concept definitions. It includes speaking to children with precise language, calling the vertices of a triangle by their proper name as opposed to referring to them as corners. It also includes knowing which examples and non-examples of a triangle to present to children which may encourage children's use of concept definitions and promote their advancement along the van Hiele levels of geometrical thinking. For example, presenting non-examples of a triangle which are not intuitively recognized as such, may encourage children to refer back to the concept definition when identifying the figure as a non-example of a triangle (Tsamir et al., 2008).

The combined theory suggested above may be used to build teachers' knowledge in at least two ways. First, it serves as a tool for teacher educators by allowing the teacher educator to focus on the specific knowledge being promoted. In much the same way, when explicitly presented to teachers it may also serve to focus the teachers on the knowledge they are building and its use in teaching.

Setting

For the past 2 years, we have been providing professional development for groups of kindergarten teachers. Our program, *Starting Right: Mathematics in Kindergartens*, carried out in collaboration with the Rashi Foundation and the Israel Ministry of Education, is an integrated program where both kindergarten teachers and the children learning in the kindergartens are participants. Teachers participate in a professional development course. Children participate in a mathematically-enriched environment created by their kindergarten teachers. A major aim of the professional development course is to increase the children's mathematical knowledge by increasing their teachers' mathematical and pedagogical content knowledge.

The segments described in this study relate to two different groups of teachers who participated at different times in our program. The first and second authors were co-instructors of the course. Teachers met with the instructors on a weekly basis (4 h per week), either at a local educational center or in one of the kindergartens. The first two segments report on one group of teachers whereas the third segment reports on the second group. Each of the segments illustrates how the combined theory suggested above may be used to build and assess kindergarten teachers' knowledge for teaching geometry.

Research Segments

Building Kindergarten Teachers' SCK Regarding Concept Images and Concept Definitions of Triangles

What image comes to mind when one thinks about a triangle and what specialized knowledge must teachers know regarding this concept image? The concept image of a layperson (i.e. not a teacher) may differ from that of a teacher. Recall that according to Tall and Vinner (1981), the concept image consists of mental images, properties, and processes associated with the concept. A concept image may also change with time and experience. Studies have shown that when asked to draw a triangle, most students will draw either an equilateral triangle or isosceles triangle with a horizontal base (Hershkowitz, 1990). On the other hand, we would expect teachers to have a more elaborate and richer image of triangles which would include triangles whose sides and angles are unequal and triangles with different orientations.

During our first meeting with the kindergarten teachers, we asked the teachers to draw three examples of triangles and three non-examples of triangles. We began with triangles for several reasons. First, the preschool curriculum in Israel specifies that kindergarten children should be able to recognize many different examples of triangles. Second, we hypothesized that kindergarten teachers would be somewhat familiar with the definition of a triangle and perhaps less familiar with definitions for other geometrical figures. For example, although squares and rectangles may be familiar to many, the hierarchical nature of quadrilaterals makes the square a complex figure to define (De Villiers, 1994).

First, we note that all of the examples teachers drew were indeed triangles and all of the non-examples teachers drew were indeed not triangles. In other words, the teachers demonstrated CCK of the concept image for a triangle. Eight of the nine teachers drew at least one example of an equilateral triangle with a horizontal base. The ninth teacher drew a triangle with unequal sides but with a horizontal base. Five teachers drew only equilateral triangles with horizontal bases for all three examples of triangles. Two teachers drew one right triangle each, with horizontal bases. Only three teachers drew examples of triangles with a different orientation. Regarding the non-examples, all of the teachers drew geometrical figures such as circles, squares, and trapezoids. This introductory task afforded us a glimpse into the teachers' concept image of a triangle and indicated that their concept image of a triangle was more closely aligned with common knowledge rather than the specialized knowledge of a teacher.

Teachers were then asked to write down on a piece of paper the definition of a triangle. Although no two definitions were exactly alike, each teacher was able to give a valid definition of a triangle. In other words, the teachers demonstrated what may be considered CCK of the concept definition. Building teachers' SCK was done gradually and began by comparing the different definitions teachers had given for a triangle. The instructor gave the following instructions:

> Look at the definitions (now written on the board) and try to think which are correct and which are incorrect . . . if there are definitions which are unacceptable, explain why. If there are definitions which you approve of more than others, explain why. If there are definitions for which a slight revision may improve the acceptability of that definition, then write it. Perhaps there is more than one correct definition.

The teachers engaged in the task and then discussed the results together. Pointing to the first definition, "A triangle is a shape with three sides and three vertices", the instructor requested the teachers to raise their hands if they agreed that it was a valid definition. The following discussion ensued:

I: The question is very simple. Is this a definition of a triangle or not? That means that you can only vote yes or no. There is nothing in between and everyone has to vote.
H: How many times can I vote (yes)?
I: For each of the definitions you can either vote yes or no.
E: Is the question then if it's (the definition written on the board) closer to yes being a definition or closer to not being a definition?
I: There is no approximation. In mathematics it either is or is not (a valid definition).

In the above segment, teachers come to realize that definitions must be precise. On the other hand, different definitions may be equivalent and thus there may be more than one definition for a particular concept. Although the instructor's approach may be considered quite direct, it became the norm with these kindergarten teachers that the instructor gave the closing argument of each discussion. Discussing the merits of each of the definitions led to a more general discussion of definitions:

E: Maybe we first need to know what a definition is.
R: A definition must be clear.
Y: That you don't argue with.
R: In a dictionary.

The teachers have begun to realize that it is important to first ascertain what is meant by a definition in mathematics before they can discuss if what is written may be considered a valid definition of a triangle. Differentiating between everyday dictionary definitions and mathematical definitions is another aspect of SCK related to concept definitions and was discussed further in the following lesson as teachers reviewed various definitions for a triangle found in dictionaries and mathematics textbooks.

During the next lesson, Tall and Vinner's CICD theory was presented explicitly to the teachers. The teachers had been discussing which of the dictionary definitions would be unacceptable and for what reasons. One teacher quoted the following definition for a triangle, "a closed figure made up of straight lines." Another teacher responds, "But that can be like . . . a crown that you make. It doesn't say how many sides." This exchange prompted the instructor to introduce the notions of concept image and concept definition:

> Notice the connection between your thoughts and your knowledge, between your imagination and your knowledge . . . Vinner investigated mathematical concepts that also have a visual presentation. However, he also said that in mathematics we have definitions and we must work according to these definitions. This is the concept definition. The concept image is what we imagine in our thoughts when we close our eyes and think of the concept.

In the elementary school, concept definitions may be used to differentiate between critical and non-critical attributes of a concept, in order to identify examples and non-examples of that concept. After introducing the notion of a concept definition, the instructor adds, "to define is to simply characterize a group of mathematical entities . . . to say what can be called by this (concept) name and what cannot." The instructor then refers to the examples and non-examples of triangles the teachers drew during the first lesson pointing out that these illustrate each teacher's concept image whereas currently, the discussion at hand has revolved around the concept definition of a triangle.

In the above segment, the combined framework was essentially used by the instructor to assess current knowledge and then to direct and focus the knowledge being built. "From a cognitive point of view, prior knowledge has to be considered as a possibly influential characteristic" (Blömeke, Felbrich, Müller, Kaiser, & Lehmann, 2008). Assessing current knowledge is an essential first step to building new knowledge. The combined framework served to differentiate between CCK of the concept image, which the teachers exhibited, and SCK of the concept image, which the teachers seemed to lack. This was true as well for the teachers' CCK and SCK of the concept definition. After assessing current knowledge, the instructor began by focusing on SCK related to concept definitions leading eventually to an explicit discussion of the CICD theory.

Differentiating Between SCK and KCT

Throughout the program a clear differentiation was made between mathematical knowledge for the teachers and mathematical knowledge as it is applied in the classroom. Initially, teachers found it difficult to separate these two domains of knowledge.

M: This is very confusing. You started off by talking about kindergarten children (in the beginning of the lesson) and now you decided to talk about mathematical thinking.

I: Let's put things in order. First, we must talk about the mathematics as is. First we (the teachers) need to know what a triangle is. The kindergarten will wait. Tomorrow morning we are not going to talk with the children about triangles.

A: I see us as kindergarten teachers, sitting with the students with the classic square, the classic rectangle, and the classic triangle and then we say, "What is this?" The child should say it's a triangle but according to what does he decide if it's a triangle or not?

I: Just a second. We'll get there. We'll definitely talk about it but for now it's just us. Differentiating between the children and us is very important. Part of what we will learn will be important mathematical knowledge that we will know but that we won't necessarily tell it as such to the children because it may not be appropriate.

The instructor is stressing the difference between KCT and SCK and that they are two different ways of knowing mathematics. She further explains the necessity for this differentiation, "My strong belief is that first you need to know what you are dealing with mathematically because otherwise there will be no basis for how you answer the child."

During the second lesson, as teachers discuss various definitions for a triangle, the difference between SCK and KCT is again brought up:

I: What is the source of this definition?

C: A geometry text book.

I: For which grade?

M: Junior high school.

Y: And you also need to know for what (mathematics) level the textbook is geared to.

I: Ok. I want to make something clear. In the end, we will bring to the kindergarten a definition which we feel is appropriate for the kindergarten. But, now we are talking about definitions which would be acceptable to mathematicians . . . Now, you need to decide which definition is valid and which is not.

H: Wait a minute. Are you talking about for us or for the kindergarten?

I: For you.

The teachers are beginning to realize that a formal concept definition must be accepted by the mathematical community. This is part of the SCK being developed during these first two lessons related to concept definitions. On the other hand, knowing how to adapt a formal concept definition to the age and level of the students is an aspect of KCT. Although a triangle may be defined as a three-sided polygon, the teachers agreed that this definition would be unsuitable for young children for two reasons. First, it is quite unlikely that young children would comprehend the meaning of the term polygon. Second, a minimalist definition, although mathematically acceptable, does not stress all of the critical attributes that all examples share. As the instructor summarized:

> On the one hand, a definition in the kindergarten should take into considerations all of the critical attributes that are derived from the mathematical definition. On the other hand, it should take into consideration psychological aspects. We created a definition that includes closure, pointed vertices, straight sides, and the number three. Children should work according to this definition.

It was agreed that in the kindergarten children would be presented with the following definition: A triangle is a closed figure with three straight lines and three pointed vertices.[1]

The above segment illustrates how the combined framework was used to focus teachers' attention on the types of knowledge being built. In our program we found that teachers were eager to implement their newly acquired knowledge in the classroom. While this is, of course, commendable, the teachers needed to sort out the difference between the mathematical knowledge needed for teaching and the pedagogical knowledge needed to convey the mathematics to their students. By making this difference explicit, teachers were first able to focus on their knowledge of concept definitions and then focus on the teaching of concept definitions.

In the following section, we describe how the combined theories tool was used to develop another aspect of kindergarten teachers' KCT related to the concept image and concept definition of triangles.

Building Kindergarten Teachers' KCT Regarding Concept Definitions and Concept Images of Triangles

The formation of geometrical concepts, as with many mathematical concepts, is a complex process in which examples play an important role (Watson & Mason, 2005). Initially, the mental construct of a concept includes mostly visual images based on perceptual similarities of examples, also known as characteristic features (Smith et al., 1974). This initial discrimination may lead to only partial concept acquisition. Later on, examples serve as a basis for both perceptible and non-perceptible attributes, ultimately leading to a concept based on its defining features. Visual representations, impressions and experiences make up the initial concept image. Formal mathematical definitions are usually added at a later stage. According to the Principles and Standards for School Mathematics (NCTM, 2000), young children "need to see many examples of shapes that correspond to the same geometrical concept as well as a variety of shapes that are non-examples of the concept" (p. 98). Thus, another important aspect of KCT is knowing which examples and non-examples to present to children that will promote the development of an appropriate concept image as well as encourage children to refer to the concept definition.

[1] It is important to note that precise language was used with the teachers as well as with the children. Terms such as corners and turns were not used. As such, 'vertices' is the appropriate translation from Hebrew.

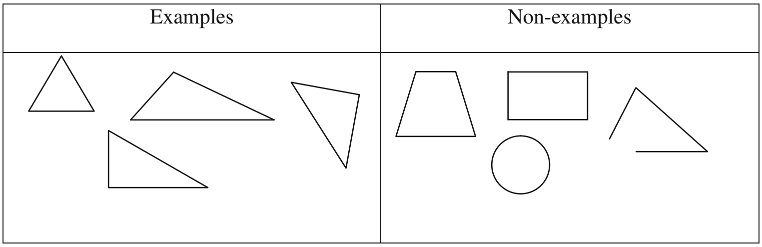


Fig. 14.1 A sample of examples and non-examples of triangles used by teachers

In this section, we describe a segment which took place with the second group of kindergarten teachers during the fifth lesson of their course. The teachers had been instructed to assess their children's knowledge regarding the identification of examples and non-examples of triangles and are now discussing the results. It soon becomes obvious that the results were largely dependent on the choice of examples and non-examples the teachers had chosen to use for this assessment. (See Fig. 14.1 for a sample of some of these examples and non-examples.) The instructor explains:

> The results do not give us a complete picture of what the children know and what they are capable of knowing. We have found in our work with children that almost all of the children correctly identify this (pointing to an equilateral triangle with a horizontal base) as a triangle and only a third of the children will correctly identify the same triangle if it is turned upside down. The typical concept image of the triangle is this (pointing to an equilateral triangle with a horizontal base).

In other words, in order to properly assess children's knowledge, the teacher should include examples that are not necessarily part of the child's intuitive concept image.

Choosing examples that are not necessarily part of the child's concept image may also encourage the child to refer back to the concept definition (Tsamir et al., 2008). As the instructor claims, "it is important to work with many examples and non-examples . . . going over the critical attributes and at the same time creating a world of images." The teachers are then instructed to think about the figures along two dimensions: a mathematical dimension and a psycho-didactical dimension. The mathematical dimension divides the figures into examples and non-examples of triangles according to the concept definition. The psycho-didactical dimension divides the figures into what is and is not intuitively identified as examples and non-examples according to the child's current concept image.

Knowledge of how to choose appropriate examples and non-examples was evident later on during the course as teachers discussed pentagons. In order to create the examples, teachers discussed the concept definition of a pentagon:

S: I want to know if there is an exact definition for a pentagon.
R: A closed figure with five sides.
O: A five-sided polygon.
S: Ok.
I: And what about a definition for the children?

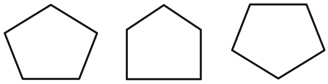
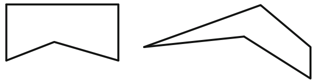
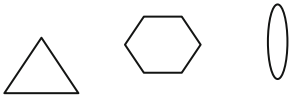
Dimensions	Psycho-didactical	
Mathematical	Intuitive	Non-intuitive
Examples		
Non-examples		

Fig. 14.2 Teachers' suggestions of examples and non-examples of pentagons

S: For the children I would say five sides, five vertices, and closed.
O: A closed figure . . . like we did before . . . with five sides and five vertices.

Working together in groups, the teachers came up with the following suggestion of examples and non-examples to use in various activities (see Fig. 14.2).

It may be surprising that the teachers placed the upside down pentagon in the section for intuitive pentagons. After all, the teachers had previously experienced that upside down triangles are not necessarily part of the child's concept image of a triangle. However, at this point, the teachers felt that the upside-down pentagon may be considered intuitive. The children had already been presented with triangles of various orientations and could successfully identify an upside down triangle as an example of a triangle. In other words, the children's concept image of triangles had changed and the teachers were choosing examples based on the children's current concept image of geometrical figures. On the other hand, triangles cannot be concave and so concave figures, such as the concave pentagon, may not currently be part of the child's concept image. The teachers had gained knowledge of their students (KCS) and used this knowledge in their teaching (KCT).

The relationship between knowledge of students and knowledge of teaching was observed several times during the year. Towards the end of the year, the teachers discussed how to help children who still had difficulties identifying various examples and non-examples of geometrical figures. Referring to triangles, one teacher stressed the need to help children recall the concept definition. She suggested the following:

> E: First we need to strengthen the critical attributes. So, we start with the triangle they are used to (referring to the equilateral triangle) and we put it down in different directions and ask the child what has changed and what has not and to check again the critical attributes. Regarding the hostile triangles (referring to those which do not coincide with the child's concept image) I would greatly enlarge the triangle so it would be much clearer to the child and ask him again to check the critical attributes, the sides and vertices.

Notice that this teacher has identified two possible stumbling blocks for the children. The first is children's difficulty with orientation. She isolates this difficulty using the triangle most likely to coincide with children's concept image and focusing only on the changing orientation. The second is children's difficulty in identifying non-intuitive triangles. Her suggestion of enlarging the triangles directs the child to notice the straight sides and pointed vertices of the triangle. Similarly, when discussing children's difficulties in identifying concave pentagons, a different teacher suggests enlarging the figure and cutting it out so that children can feel the hidden vertex. During the next lesson, this teacher described how she carried out this suggestion and that the enlarged concave pentagon was indeed helpful.

In this segment, we see the results of explicitly introducing kindergarten teachers to the CICD theory and explicitly discussing with them the difference between the mathematical knowledge they as teachers need to know and applying this knowledge in the kindergarten. We can see a clear difference between the examples and non-examples teachers chose for triangles in the beginning of the year to those they chose for pentagons in the middle of the year. Teachers are cognizant of the need to present a suitable definition of a pentagon for their children. They are aware of the tension between the concept image and concept definition and devise activities that will enrich the children's concept image while strengthening their awareness of the concept definition. Using the combined theory as a lens, we may say that the teachers are accessing their KCS related to concept images and concept definitions in order to build their KCT related to concept images and concept definitions.

Kindergarten Children's Knowledge of Pentagons

At the end of the year, 166 kindergarten children were presented with six figures and asked to identify examples and non-examples of pentagons (see Fig. 14.3). Of these children, 81 learned in eight kindergartens taught by teachers who participated in the program and 85 children were from six other kindergartens. All 14 kindergartens were located in the same low-socioeconomic neighbourhood. Children were interviewed individually, in a quiet corner of the kindergarten classroom.

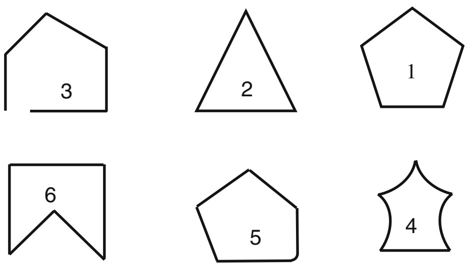
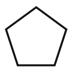
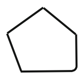


Fig. 14.3 Figures presented to kindergarten children

Each of the six figures was printed on a separate card. After presenting each card in the same order to each child, two interview questions were asked: Is this a pentagon? Why? The first question ascertained if the child identified examples and non-examples of a pentagon. The second question allowed us to study the child's reasoning about identification of a figure.

Results indicated that more program children than non-program children correctly identified the figures as examples or non-examples of a pentagon (*see* Table 14.1). It should be noted that according to the Israeli national preschool mathematics curriculum, kindergarten children should be able to identify examples and non-examples of pentagons. Noticeably, nearly all the program children correctly identified both pentagons as examples of a pentagon. Although most non-program children correctly identified the convex pentagon as a pentagon, only 13% correctly identified the concave pentagon as a pentagon. As one child claimed, "It looks like a different figure." Perhaps, the concave pentagon did not coincide with their concept image of a pentagon. Furthermore, when asked to explain their identifications, only 25% of non-program children gave reasons which pointed to the critical attributes of a pentagon. This coincides with Clements, Swaminathan, Hannibal, and Sarama (1999) who found that young children rarely refer to the properties of a figure, relying more on a holistic, visual approach to identification. On the other hand, 85% of program children pointed to critical attributes in their reasoning. For example, one program child correctly identified the triangle as not being a pentagon and added, "it only has three vertices and three sides". This child's reasoning coincides with the third van Hiele level of geometric reasoning where the definition is meaningful and the child takes notice of critical attributes. On the other hand, a non-program child who also correctly identified the triangle as not being a pentagon claimed, "It's not like a pentagon. It's a triangle". This child first stipulates that the figure at hand does not coincide with his image of a pentagon. He then correctly claims that this figure is a triangle. Referring to the name of a shape implies that the child has taken into consideration the whole shape without regard for its components (Markman,

Table 14.1 Frequency (in %) of correct identifications per figure

	Figures					
	Examples		Non-examples			
	1	6	2	3	4	5
						
Program children	95	94	99	100	93	96
Non-program children	76	13	93	82	73	56

1989). This type of reasoning coincides with the first van Hiele level of geometric reasoning.

Not all program children immediately identified the concave pentagon as a pentagon. For example, one child examined this pentagon and thought out loud, "It's not a pentagon. Let's check. (The child counts the vertices.) It is a pentagon because it has five sides and five vertices and it's closed". Perhaps, this pentagon did not yet coincide with his concept image. Yet, he was able to correct his identification by referring to the critical attributes mentioned in the concept definition. This example illustrates how focusing on promoting different aspects of teachers' knowledge may eventually filter down to promoting student's knowledge.

Summing Up and Looking Ahead

In this chapter we proposed a theoretical framework which combined a theory of teachers' knowledge with a theory of mathematical knowledge and illustrated how it may be used as a tool in promoting teachers' mathematical knowledge for teaching. "A crucial trait of a valuable framework of teacher knowledge is the extent to which it identifies that knowledge needed for student learning and understanding" (Graeber & Tirosh, 2008, p. 124). Other tools conceptualize teachers' knowledge based solely on the work teachers do. We add that it is equally important to frame teachers' knowledge based on the knowledge we wish our students to gain. Viewing teachers' mathematics knowledge through two lenses – one of teachers' knowledge and one of mathematical knowledge – allows us to pinpoint more precisely what teachers need to know for teaching mathematics. Of course, teachers need to know which examples of triangles to present and in what order to present them (KCT). However, if we recognize that some examples will enhance students' concept image of a triangle and others will encourage students' use of the definition, we may accordingly develop teachers' mathematical knowledge for teaching each of these aspects.

The theorized tool we described, combined Ball and her colleagues' conceptualization of teachers' knowledge for teaching with Tall and Vinner's CICD theory in order to promote kindergarten teachers' knowledge for teaching geometry. There are several variables in this proposal. There is the mathematical context used to illustrate the use of this tool, the grade-level at which the teachers taught, the action taken with the tool, and the theories we chose to combine. Each of these variables represents possible directions for further development and wider use of the tool.

Regarding the mathematical context, we found that for kindergarten teachers, the context of geometry provided a natural venue for discussing images and definitions. Beginning with triangles and other two-dimensional polygons, the teachers could easily discuss the figures they saw and drew and began to understand the need for concept definitions. They also came to acknowledge that not every concept definition may be adapted for the young children in kindergarten. This came up when discussing circles and the concept image and concept definition of a circle.

It was decided that for the circle, a child's concept image may currently be enough. These discussions carried on as the teachers discussed three-dimensional solids such as pyramids, spheres and cylinders. Although this chapter specifically used the context of geometry to illustrate the combined framework, we believe that the generality of the CICD theory allows it to be applied to building teacher's knowledge of additional mathematical contexts. In the kindergarten, for example, we used the combined framework for building teachers' knowledge of equivalent sets (Tirosh, Tsamir, Levenson, & Tabach, submitted). As with geometry, we used the combined theory to build teachers' SCK of the concept image of equivalent sets and differentiated between this knowledge and KCT regarding this concept image. The same was done for the concept definition. If the use of this tool is to be expanded to other preschool mathematical contexts (such as patterns and measurement), then perhaps prior research will be necessary in order to first investigate children's concept image and concept definition in these contexts.

Regarding the grade-level at which the teachers taught, this chapter illustrated promoting knowledge for teaching in kindergarten. We believe that the combined theory has potential to be used as a tool for promoting teachers' knowledge for teaching in other grades as well. In both elementary and secondary schools, studies have shown that tension exists between students' concept images and concept definitions within various mathematical contexts (Bingolbali & Monaghan, 2008; Even & Tirosh, 1995; Gray, Pinto, Pitta, & Tall, 1999; Levenson, Tsamir, & Tirosh, 2007; Schwarz & Hershkowitz, 1999; Vinner & Dreyfus, 1989). Perhaps at the high school level, teachers are more cognizant of the necessity for definitions than preschool teachers are. On the other hand, they may pay less attention to concept images. This issue will need to be addressed by perhaps placing extra emphasis on these cells during professional development.

In this chapter, we illustrated some ways in which the combined framework could be used to promote teachers' SCK and KCT related to concept images and concept definitions in the kindergarten. The next step would be to demonstrate how the combined framework may be used to promote KCS related to concept images and concept definitions. Another issue that arises from pondering the use of the tool is the degree of explicitness when presenting the tool to teachers. Upon reflection, the four dimensions of teachers' knowledge were not made as explicit to the teachers as was the concept image-concept definition theory. We believe that it is important to make both theories equally explicit to teachers. This issue is being addressed in our current courses where the four dimensions of teachers' knowledge are explicitly presented and discussed.

Choosing which theories to combine is a significant issue which needs to be addressed. Regarding our goals for professional development, it is too simplistic to say that we aimed to enhance teachers' knowledge. As the first section of this book indicates, conceptualizing mathematical knowledge for teaching is complex. Our choice of using the four domains of knowledge described by Ball and her colleagues arose mostly from our necessity to use a finer grain than provided by Shulman's (1986) often used notions of SMK and PCK. In retrospect, we found that this choice was well suited for conceptualizing the knowledge needed for teaching some mathematical concepts in kindergarten. As noted in the beginning of this paper, most kindergarten teachers have little training in teaching mathematics. As such, each of the four domains needed attention.

Tsamir (2008) recognized the complexity of choosing which mathematical knowledge theories to present to teachers from the vast offering of theories relevant to mathematics teaching. Regarding mathematics teaching, the CICD theory is a widely recognized mathematics education theory which spans students of all ages and is relevant to many different mathematical contexts (Hershkowitz, 1989; Schwarz & Hershkowitz, 1999; Tall & Vinner, 1981; Vinner & Dreyfus, 1989). It informs our understanding of mathematical concept formation. It allows us to predict and analyze students' errors. Another direction for widening the use of the tool would be to consider combining other theories of mathematical knowledge with theories of teachers' knowledge. Tsamir (2008) described how familiarizing secondary school teachers with Fischbein's (1993) theory of the three components of knowledge and Stavy and Tirosh's (1996) theory of the intuitive rules may promote secondary teachers' mathematical and pedagogical knowledge. The choice of theories may depend on the mathematical context as well as the activities or tasks which take place in the classroom. For example, parts of the intuitive rules theory are especially appropriate when engaging in comparison tasks. Combining this theory with the four dimensions of teachers' knowledge may then focus us, for example, on developing teachers' knowledge of how and when students use these rules (KCS). Another direction for addressing this issue might be to pool mathematical education theories that investigate students' mathematical learning and possible sources of errors. For example, Fischbein's (1993) theory mentioned above, the intuitive rules theory (Stavy & Tirosh, 1996), and Tall and Vinner's (1981) CICD theory all have elements of intuitive thinking. The mathematics education research community should consider how to combine these theories in order to provide a more comprehensive theory for investigating students' mathematical thinking as well as teachers' mathematics knowledge for teaching.

References

Australian Association of Mathematics Teachers and Early Childhood Australia (AAMT/ECA). (2006). *Position paper on early childhood mathematics*. Retrieved January 20, 2009, from http://www.aamt.edu.au/documentation/statements

Ball, D., Bass, H., & Hill, H. (2004). Knowing and using mathematical knowledge in teaching: Learning what matters. In A. Buffgler & R. Lausch (Eds.). *Proceedings of the 12th Annual Conference of the South African Association for Research in Mathematics, Science and Technology Education*. Durban: SAARMSTE.

Ball, D., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide?. *American Educator*, *29*, 14–22.

Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching. *Journal of Teacher Education*, *59*(5), 389–407.

Bingolbali, E., & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics*, *68*(1), 19–35.

Blömeke, S., Felbrich, A., Müller, C., Kaiser, G., & Lehmann, R. (2008). Effectiveness of teacher education. *ZDM – The International Journal on Mathematics Education, 29*(1), 719–734.

Clements, D. H., Swaminathan, S., Hannibal, M., & Sarama, J. (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education*, *30*(2), 192–212.

de Villiers, M.. (1994). The role and function of a hierarchical classification of quadrilaterals. *For the Learning of Mathematics*, *14*(1), 11–18.

Even, R., & Tirosh, D. (1995). Subject matter knowledge and knowledge about students as sources of teacher presentations of the subject matter. *Educational Studies in Mathematics*, *29*(1), 1–20.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht: Reidel Publishing Company.

Fischbein, E. (1993). The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity. In R. Biehler, R. Scholz, R. Straber, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 231–245). Dordrecht: Kluwer.

Ginsburg, H. P., Lee, J. S., & Boyd, J. S. (2008). Mathematics education for young children: What it is and how to promote it. *Social Policy Report*, *XXII*(I), 1–22.

Graeber, A., & Tirosh, D. (2008). Pedagogical content knowledge. In P. Sullivan & T. Wood (Eds.), *Knowledge and beliefs in mathematics teaching and teaching development* (pp. 117–132). Rotterdam: Sense Publishers.

Gray, E., Pinto, M., Pitta, D., & Tall, D. (1999). Knowledge construction and diverging thinking in elementary and advanced mathematics. *Educational Studies in Mathematics*, *38*, 111–133.

Hasegawa, J. (1997). Concept formation of triangles and quadrilaterals in the second grade. *Educational Studies in Mathematics*, *32*, 157–179.

Hershkowitz, R. (1989). Visualization in geometry – Two sides of the coin. *Focus on Learning Problems in Mathematics*, *11*(1), 61–76.

Hershkowitz, R. (1990). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition* (pp. 70–95). Cambridge, UK: Cambridge University Press.

Levenson, E., Tsamir, P., & Tirosh, D. (2007). Neither even nor odd: Sixth grade students' dilemmas regarding the parity of zero. *Journal of Mathematical Behavior*, *26*, 83–95.

Markman, E. (1989). *Categorization and naming in children*. Massachusetts: MIT Press.

National Association for the Education of Young Children & National Council of Teachers of Mathematics. (2002). *Position statement. Early childhood mathematics: Promoting good beginnings*. Available at www.naeyc.org/resources/position_statements/psmath.htm

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Practice Guidance for the Early Years Foundation Stage. (2008). Retrieved April 9, 2009, from www.standards.dfes.gov.uk/eyfs/resources/downloads/practice-guidance.pdf

Schwarz, B., & Hershkowitz, R. (1999). Prototypes: Brakes or levers in learning the function concept?. *Journal for Research in Mathematics Education*, *30*, 362–389.

Shaughnessy, J. M., & Burger, W. F. (1985). Spadework prior to deduction in geometry. *Mathematics Teacher*, *78*(6), 419–428.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14.

Smith, E., Shoben, E., & Rips, L. (1974). Structure and process in semantic memory: A featural model for semantic decisions. *Psychological Review*, *81*, 214–241.

Stavy, R., & Tirosh, D. (1996). Intuitive rules in mathematics and science: The case of "the more of A the more of B". *International Journal of Science Education*, *18*(6), 653–667.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.

Tirosh, D. (2000). Enhancing teachers' prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, *31*(1), 5–25.

Tirosh, D., Tsamir, P., Levenson, E., & Tabach, M. (submitted). From preschool teachers' professional development to children's knowledge: Comparing sets. The case of equivalence. *Journal of Mathematics Teacher Education*.

Tsamir, P. (2008). Using theories as tools in mathematics teacher education. In D. Tirosh & T. Wood (Eds.), *Tools and processes in mathematics teacher education* (pp. 211–234). Sense Publishers: Rotterdam The Netherlands.

Tsamir, P., Tirosh, D., & Levenson, E. (2008). Intuitive nonexamples: The case of triangles. *Educational Studies in Mathematics*, *69*(2), 81–95.

van Hiele, P. M., & van Hiele, D. (1958). A method of initiation into geometry. In H.Freudenthal (Ed.), *Report on methods of initiation into geometry*. Groningen: Walters.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking*. Dordrecht: Kluwer.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, *20*(4), 356–366.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: The role of learner generated examples*. Mahwah, NJ: Lawrence Erlbaum Associates.