Chapter 12
The Knowledge Quartet as an Organising Framework for Developing and Deepening Teachers' Mathematics Knowledge

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Introduction

In this chapter, we present some findings from a study which evaluated the effectiveness of one classroom-based approach to the development of elementary mathematics teaching. This approach drew on earlier research into teachers' mathematical content knowledge at the University of Cambridge, when a framework for the analysis of mathematics teaching – the Knowledge Quartet – was developed. In the work to be reported here, this framework was used to identify and develop a group of beginning teachers' mathematics content knowledge for teaching. First, we shall give a rationale for our focus on teachers' content knowledge in action in the classroom and a brief description of the study which led to the development of the Knowledge Quartet.

Rationale

Education researchers and government agencies have identified limitations in teachers' mathematical content knowledge (e.g. Ball, 1990; Ma, 1999; Ofsted, 2000). These limitations have been perceived as a factor in unsatisfactory pupil achievement (Williams, 2008). Difficulties associated with teachers' mathematical content knowledge are particularly apparent in the elementary sector where generalist teachers often lack confidence in their own mathematical ability (Brown, McNamara, Jones, & Hanley, 1999; Green & Ollerton, 1999). The 'reform' movement in mathematics teaching, and enquiry-based approaches to learning, which have been influential in curriculum reform in several countries, arguably require teachers to have a greater depth of mathematical content knowledge than was needed for teaching more 'traditional' mathematics (e.g. Borko et al., 1992; Goulding, Rowland, & Barber, 2002). Identifying, developing and deepening teachers' mathematical

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content knowledge, has therefore become a priority for policy makers and mathematics educators around the world.

There is not a simple relationship between teachers' formal qualifications in mathematics and the achievement of their pupils (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997; Begle, 1979). Several researchers have argued that mathematical content knowledge needed for teaching is not located in the minds of teachers, but rather is realised through the practice of teaching (Hegarty, 2000; Mason & Spence, 1999). From this perspective, knowledge for teaching is constructed in the context of teaching, and can therefore be observed only as 'in vivo' knowledge in this context. Teaching requires knowledge in several different domains, and a number of knowledge taxonomies reflect this multidimensional perspective (see Chapter 2 by Petrou and Goulding, this volume). Hegarty (2000) argued that the effects of these different kinds of teacher knowledge can only be understood within the contexts of dynamic teaching situations. He presented a model which represents the teacher as having a number of incomplete sets of relevant insights, elements of which come together in instances of teaching to form a new insight specific to that situation. This is resonant with the contention of Mason and Spence (1999) that *knowing-about* mathematics and mathematics teaching is only realised as *knowing-to* in the act of teaching. The perspective on teacher knowledge at the heart of this chapter – the Knowledge Quartet – provides a framework for analysis of the mathematics content knowledge that informs teacher insights when they are brought together in practice, so that the distinction between different kinds of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching. In the following section, we outline the fundamental, observational research that gave rise to the 'tool' that lies at the heart of this chapter.

Developing the Knowledge Quartet

Context and Purpose of the Research

In the UK, the majority of prospective, 'trainee' teachers are graduates who follow a one-year course leading to a Postgraduate Certificate in Education (PGCE) in a university education department. Over half of the PGCE year is spent teaching in schools under the guidance of a school-based mentor, or 'cooperating teacher'. Placement lesson observation is normally followed by a review meeting between the cooperating teacher and the student-teacher. On occasion, a university-based tutor will participate in the observation and the review. Thirty years ago, Tabachnick, Popkewitz, and Zeichner (1979) found that "cooperating teacher/student teacher interactions were almost always concerned with . . . procedural and management issues . . . There was little or no evidence of any discussion of substantive issues in these interactions" (p. 19). The situation has not changed, and more recent studies also find that mentor/trainee lesson review meetings typically focus heavily on organisational features of the lesson, with very little attention to the mathematical content of mathematics lessons (Borko & Mayfield, 1995; Strong & Baron, 2004).

The purpose of the research from which the Knowledge Quartet emerged was to develop an empirically-based conceptual framework for lesson review discussions *with a focus on the mathematics* content of the lesson and the role of the trainee's mathematics subject matter knowledge (SMK) and pedagogical content knowledge (PCK). In order to be a useful tool for those who would use it in the context of *practicum* placements, such a framework would need to capture a number of important ideas and factors about mathematics content knowledge in relation to teaching, within a small number of conceptual categories, with a set of easily-remembered labels for those categories.

The focus of this particular research was therefore to identify ways that teachers' mathematics content knowledge – both SMK and PCK – can be observed to 'play out' in practical teaching. The teacher-participants in this study were novice, trainee elementary school teachers, and the observations were made during their school-based placements. Whilst we believe certain kinds of knowledge to be *desirable* for elementary mathematics teaching, we are convinced of the futility of asserting what a beginning teacher, or a more experienced one for that matter, *ought* to know. Our interest is in what a teacher *does* know and believe, and how opportunities to enhance knowledge can be identified. We have found that the Knowledge Quartet, the framework that arose from this research, provides a means of reflecting on teaching and teacher knowledge, with a view to developing both.

Method

The participants in the study were enrolled on a 1-year PGCE course in which each of the 149 trainees specialised either on the Early Years (pupil ages 3–8) or the Primary Years (ages 7–11). Six trainees from each of these groups were chosen for observation during their final school placement. The six were chosen to reflect a range of outcomes of a subject-knowledge audit administered 3 months earlier. Two mathematics lessons taught by each of these trainees were observed and videotaped, i.e. 24 lessons in total. The trainees were asked to provide a copy of their planning for the observed lesson. As soon as possible after the lesson, the observer/researcher wrote a succinct account of what had happened in the lesson, so that a reader might immediately be able to contextualise subsequent discussion of any events within it. These 'descriptive synopses' were typically written from memory and field notes, with occasional reference to the videotape if necessary.

From that point, we took a grounded approach to the data for the purpose of generating theory (Glaser & Strauss, 1967). In particular, we identified in the videotaped lessons aspects of trainees' actions in the classroom that seemed to be significant in the limited sense that it could be construed to be informed by a trainee's mathematics subject matter knowledge or their mathematical pedagogical knowledge. We realised later that most of these significant actions related to choices made by the trainee, in their planning or more spontaneously. Each was provisionally assigned an 'invented' code. These were grounded in particular moments or episodes in the tapes. This provisional set of codes was rationalised and reduced (e.g. eliminating duplicate codes and marginal events) by negotiation and agreement in the research team. The 18 codes generated by this inductive process are itemised later in this chapter. The name assigned to each code is intended to be indicative of the type of issue identified by it: for example, the code *adheres to textbook* (AT) was applied when a lesson followed a textbook script with little or no deviation, or when a set of exercises was 'lifted' from a textbook, or other published resource, sometimes with problematic consequences. By way of illustration of the coding process, we give here a brief account of an episode that we labelled with the code *responding to children's ideas* (RCI). It will be seen that the contribution of a child was unexpected. Within the research team, this code name was understood to be potentially ironic, since the observed response of the teacher to a child's insight or suggestion was often to put it to one side rather than to deviate from the planned lesson script, even when the child offered further insight on the topic at hand.

Code RCI*: an illustrative episode*. Jason was teaching elementary fraction concepts to a Year 3 class (pupil age 7–8). Each pupil held a small oblong whiteboard and a dry-wipe pen. Jason asked them to "split" their individual whiteboards into two. Most of the children predictably drew a line through the centre of the oblong, parallel to one of the sides, but one boy, Elliot, drew a diagonal line. Jason praised him for his originality, and then asked the class to split their boards "into four". Again, most children drew two lines parallel to the sides, but Elliot drew the two diagonals. Jason's response was to bring Elliot's solution to the attention of the class, but to leave them to decide whether it was correct. He asked them:

Jason: What has Elliot done that is different to what Rebecca has done?
Sophie: Because he's done the lines diagonally.
Jason: Which one of these two has been split equally? [...] Sam, has Elliot split his board into quarters?
Sam: Um... yes... no...
Jason: Your challenge for this lesson is to think about what Elliot's done, and think if Elliot has split this into equal quarters. There you go Elliot.

At that point, Jason returned the whiteboard to Elliot, and the question of whether it had been partitioned into quarters was not mentioned again. What makes this interesting mathematically is the fact that (i) the four parts of Elliot's board are not congruent, but (ii) they have equal areas; and (iii) this is not at all obvious. Furthermore, (iv) an elementary demonstration of (ii) is arguably even less obvious. This seemed to us a situation that posed very direct demands on Jason's SMK and arguably his PCK too. It is not possible to infer whether Jason's "challenge" is motivated by a strategic decision to give the children some thinking time, or because he needs some himself.

Equipped with this set of codes, we revisited each lesson in turn and, after further intensive study of the tapes, elaborated each descriptive synopsis into an analytical account of the lesson. In these accounts, the agreed codes were associated with relevant moments and episodes, with appropriate justification and analysis concerning the role of the trainee's content knowledge in the identified passages, with links to relevant literature.

The identification of these fine categories was a stepping stone with regard to our intention to offer a practical framework for use by ourselves, our colleagues and teacher-mentors, for reviewing mathematics teaching with trainees following lesson observation. An 18-point tick-list (like an annual car safety check) was not quite what was needed. Rather, the intended purpose demanded a more compact, readily understood scheme, which would serve to frame a coherent, content-focused discussion between teacher and observer. The key to the solution of our dilemma was the recognition of an association between elements of subsets of the 18 codes, enabling us to group them (again by negotiation in the team) into four broad, superordinate categories, which we have named (I) foundation (II) transformation (III) connection (IV) contingency. These four units are the dimensions of what we call the 'Knowledge Quartet'.

Each of the four units is composed of a small number of subcategories that we judged, after extended discussions, to be of the same or a similar nature. An extended account to the research pathway described above is given in Rowland (2008). Naturally, we are immersed in the process from which the codes emerged. We believe, however, that our names for the codes are less important to other users of the 'quartet' than a broad sense of the general character and distinguishing features of each of broad units, which we shall outline in a moment. The Knowledge Quartet has now been extensively 'road tested' as a descriptive and analytical tool. As well as being re-applied to analytical accounts of the original data (the 24 lessons), it has been exposed to extensive 'theoretical sampling' (Glaser & Strauss, 1967) in the analysis of other mathematics lessons, in England and beyond. As a consequence, two additional codes[1] have been added to the original 18, but in its broad conception, we have found the quartet to be comprehensive as a tool for thinking about the ways that content knowledge comes into play in the classroom. We have found that many moments or episodes within a lesson can be understood in terms of two or more of the four units; for example, a *contingent* response to a pupil's suggestion might helpfully *connect* with ideas considered earlier. Furthermore, the application of content knowledge in the classroom always rests on *foundational* knowledge.

Conceptualising the Knowledge Quartet

The concise conceptualisation of the Knowledge Quartet which now follows draws on the extensive range of data referred to above. As we observed earlier, the practical application of the Knowledge Quartet depends more on teachers and teacher educators understanding the broad characteristics of each of the four dimensions than on their recall of the contributory codes.

[1] These new codes, derived from applications of the KQ to classrooms in Ireland and Cyprus, are *teacher insight* (Contingency) and use of *instructional materials* (Transformation) respectively.

Foundation

Contributory codes: awareness of purpose; identifying errors; overt subject knowledge; theoretical underpinning of pedagogy; use of terminology; use of textbook; reliance on procedures.

The first member of the quartet is rooted in the foundation of the teacher's theoretical background and beliefs. It concerns their knowledge, understanding and ready recourse to what was learned at school, and at college/university, including initial teacher preparation, in preparation (intentionally or otherwise) for their role in the classroom. It differs from the other three units in the sense that it is about knowledge 'possessed',[2] irrespective of whether it is being put to purposeful use. For example, we could claim to have knowledge about division by zero, or about some probability misconceptions – or indeed to know where we could seek advice on these topics – irrespective of whether we had had to call upon them in our work as teachers. Both empirical and theoretical considerations have led us to the view that the other three units flow from a foundational underpinning.

A key feature of this category is its *propositional* form (Shulman, 1986). It is what teachers learn in their 'personal' education and in their 'training' (pre-service in this instance). We take the view that the possession of such knowledge has the potential to inform pedagogical choices and strategies in a fundamental way. By 'fundamental' we have in mind a rational, reasoned approach to decision-making that rests on something other than imitation or habit. The key components of this theoretical background are: knowledge and understanding of mathematics per se; knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt.

In summary, this category that we call 'foundation' coincides to a significant degree with what Shulman (1987) calls 'comprehension', being the first stage of his six-point cycle of pedagogical reasoning.

Transformation

Contributory codes: teacher demonstration; use of instructional materials; choice of representation; choice of examples.

The remaining three categories, unlike the first, refer to ways and contexts in which knowledge is brought to bear on the preparation and conduct of teaching. They focus on knowledge-in-action as *demonstrated* both in planning to teach and in the act of teaching itself. At the heart of the second member of the quartet, and acknowledged in the particular way that we name it, is Shulman's observation that the knowledge base for teaching is distinguished by " . . . the capacity of a

[2]The use of this acquisition metaphor for knowing suggests an individualist perspective on Foundation knowledge, but we suggest that this 'fount' of knowledge can also be envisaged and accommodated within more distributed accounts of knowledge resources (see Chapter 3 by Hodgen, this volume).

teacher to *transform* the content knowledge he or she possesses into forms that are pedagogically powerful" (1987, p. 15, emphasis added). This characterisation has been echoed in the writing of Ball (1988), for example, who distinguishes between knowing some mathematics 'for yourself' and knowing in order to be able to help someone else learn it. As Shulman indicates, the presentation of ideas to learners entails their re-presentation (our hyphen) in the form of analogies, illustrations, examples, explanations and demonstrations (Shulman, 1986, p. 9). Our second category, unlike the first, picks out behaviour that is directed towards a pupil (or a group of pupils), and which follows from deliberation and judgement informed by foundation knowledge. This category, as well as the first, is informed by particular kinds of literature, such as the teachers' handbooks of textbook series or in the articles and 'resources' pages of professional journals. Increasingly, in the UK, teachers look to the internet for bright ideas, and even for readymade lesson plans. The trainees' choice and *use of examples* has emerged as a rich vein for reflection and critique. This includes the use of examples to assist concept formation, to demonstrate procedures, and the selection of exercise examples for student activity.

Connection

Contributory codes: making connections between procedures; making connections between concepts; anticipation of complexity; decisions about sequencing; recognition of conceptual appropriateness.

The next category binds together certain choices and decisions that are made for the more or less discrete parts of mathematical content – the learning, perhaps, of a concept or procedure. It concerns the *coherence* of the planning or teaching displayed across an episode, lesson or series of lessons. Mathematics is notable for its coherence as a body of knowledge and as a field of enquiry. Indeed, a great deal of mathematics is held together by deductive reasoning. The pursuit of coherence and mathematical connections in mathematics pedagogy has been stimulated recently by the work of Askew et al. (1997): of six case study teachers found to be highly effective, all but one gave evidence of a 'connectionist' orientation. The association between teaching effectiveness and a set of articulated beliefs of this kind lends a different perspective to the work of Ball (1990) who also strenuously argued for the importance of connected knowledge for teaching.

Related to the integrity of mathematical content in the mind of the teacher and his/her management of mathematical discourse in the classroom, our conception of coherence includes the *sequencing* of topics of instruction within and between lessons, including the ordering of tasks and exercises. To a significant extent, these reflect deliberations and choices entailing not only knowledge of structural connections within mathematics itself, but also awareness of the relative cognitive demands of different topics and tasks.

Contingency

Contributory codes: responding to children's ideas; use of opportunities; deviation from agenda; teacher insight.

Our final category concerns the teacher's response to classroom events that were not anticipated in the planning. In some cases, it is difficult to see how they could have been planned for, although that is a matter for debate. In commonplace language this dimension of the quartet is about the ability to 'think on one's feet': it is about *contingent action.* Shulman (1987) proposes that most teaching begins from some form of 'text' – a textbook, a syllabus, ultimately a sequence of planned, intended actions to be carried out by the teacher and/or the students within a lesson or unit of some kind. Whilst the stimulus – the teacher's intended actions – can be planned, the students' responses can not.

Brown and Wragg (1993) suggest that 'responding' moves are the lynch pins of a lesson, important in the sequencing and structuring of a lesson, and observe that such interventions are some of the most difficult tactics for novice teachers to master. The quality of such responses is undoubtedly determined, at least in part, by the knowledge resource available to the teacher, as the earlier illustrative episode with Jason demonstrates.

Having now set out the conceptual apparatus underpinning the tool in focus in this chapter, we now proceed to an account of its use by a group of teachers in their professional development over a 4-year period.

The Knowledge Quartet and Mathematics Teaching Development

The Knowledge Quartet was developed to identify, describe and analyse mathematics content knowledge revealed in teaching, in order to provide a framework for reflection and discussion of lessons. We were then motivated to investigate whether, and in what ways, the framework could be used to develop and deepen mathematical content knowledge. In a study begun in 2004, the first author evaluated the Knowledge Quartet as a tool for the identification and development of teachers' SMK and PCK (see e.g. Turner, 2008).

This longitudinal study took place over 4 years, during which the participants could be regarded as 'beginning teachers'. Each of these years was considered a different phase of the study. It began with 12 participants in their PGCE graduate teacher preparation year. As expected, this cohort reduced to nine in the second year, to six in the third year and finally to four in the fourth and final year of the study. This attrition was predicted, and a consequence of the participants' relocation, changes in commitment to and participation in the project. Four case studies, of Amy, Jess, Kate and Lisa, were built using data from lesson observations, post-lesson reflective interviews, participants' reflective written accounts, group interviews and individual interviews over the 4 years.

The study was based on a model of teacher professional development through reflection both *in* and *on* teaching action (Schön, 1983). The Knowledge Quartet was used to focus the teachers' reflections on the mathematics content knowledge realised in their teaching. The teachers used the framework as a tool to support their reflections on and discussions about their mathematics teaching over the course of the study. Videotapes of the participants' lessons were used to aid recall and to allow in-depth analysis and reflection on, their teaching.

The participants were initially introduced to, and familiarised with, the Knowledge Quartet in their training year. One lesson taught by each of them was videotaped and analysed during their final *practicum* placement. These videotapes were used in one-to-one stimulated recall interviews with the participants, using the Knowledge Quartet to focus on the mathematical content of each lesson. During their first year of teaching, the participants were given focused feedback, structured by the Knowledge Quartet framework, on three videotaped lessons. This was intended to support and develop their own use of the framework. They then watched the videotapes and wrote reflective accounts of these lessons. In the second year of their teaching, the most intensive period of data collection, participants used the framework more independently, supported by discussions with the researcher and by group meetings. Interviews and observations in the third year of their teaching gave final indications of the development in participants' mathematical content knowledge as it was evidenced in their teaching.

Lesson observations were analysed by the first author using the dimensions and constituent codes of the Knowledge Quartet. Transcripts of one–one interviews and group meetings, and the participants' written reflective accounts, were analysed using the computer-aided qualitative analysis software NVivo. This gave rise to a hierarchy of emergent codes and themes, which informed the final analysis of the data. As we will demonstrate below, there was evidence from the study that use of the Knowledge Quartet as a framework for reflection had a positive influence on the development of the participants' content knowledge for teaching by focusing reflection on the mathematical content of their teaching, as opposed to the more managerial and generic aspects that tend, as we remarked earlier, to dominate lesson review.

The analysis brought out two overarching aspects in the development of the participants. The first of these related to their conceptions of mathematics teaching – an aspect of Foundation knowledge coded in the longitudinal study under the themes of 'beliefs' and 'confidence'. The second related more broadly to developments in mathematical content knowledge in relation to the four dimensions of the Knowledge Quartet.

Development in Conceptions of Mathematics Teaching

From a comprehensive review of literature across several disciplines, Kuhs and Ball (1986) identified four dominant views of the way mathematics should be taught:

- a classroom-focused view;
- a content-focused with an emphasis on performance view;
- a content-focused with an emphasis on conceptual understanding view;
- a learner-focused view.

Kuhs and Ball (1986) give detailed accounts of these four views, making them accessible as a framework for analysis. These models were not seen as exclusive: teachers would be expected to hold elements of more than one view simultaneously, and over time. Ernest (1989) subsequently included two additional models of conceptions which combined characteristics of these views. However, these combinations could be accounted for within Kuhs and Ball's simpler four-model framework.

Evidence from the different data sources combined to reveal that each of the participants held complex views of mathematics teaching incorporating elements from all four of Kuhs and Ball's dominant views. Though the initial balance of these elements varied, the NVivo analysis indicated a pattern in the direction of change in the four case studies. Jess, Lisa and Kate began the study with predominantly content-focused views of teaching and Amy with a predominantly learner-focused view. There was evidence that the teachers moved towards views with greater emphasis on developing conceptual understanding in pupils rather than on developing procedural performance. There was also a pattern of change in conceptions of mathematics teaching towards a learner-focused view. In the following section, selected data from the study are used to substantiate the claims made above. Inevitably, only a small selection of representative data can be presented here.

Jess began her career with a content-focused view of mathematics teaching which emphasised performance. In her first year of teaching, she commented:

> The thing is, if it [using written algorithms] works for them what's the problem? I have found some of the less able children have been shown how to do carrying, and they've got it and they use that all the time. (Jess, group interview, Phase 2)

Over the course of the study, she moved towards a view which emphasised conceptual understanding though she continued to consider performance to be important.

> I think teaching procedures are important, especially for low ability children who need to have a strategy to rely on. However, I have made a conscious effort to make my teaching more conceptual so it becomes much more real than just practicing something and it probably means it is much easier to apply in a new situation as it means something. (Jess, individual interview, Phase 4)

Like Jess, Lisa focused on procedures at the beginning of her career.

> The children can often do what I want them to do when it is like that because it is in small steps. (Lisa, post-lesson interview, Phase 1)

By her third year of teaching, there was evidence that Lisa had become more concerned with conceptual understanding. Commenting on a lesson she had taught at the beginning of the year, she wrote;

> It might have helped if I had been more encouraging of them to use jottings as well as to write the number sentence. They didn't need them for the numbers involved but it probably would have helped them with what the concept was. (Lisa, post-lesson reflective interview, Phase 3)

In her first year of teaching, Amy focused on both the performance (ability to count accurately) and conceptual understanding of individual children:

> I had planned which children I would ask to count which box of items so that I could differentiate the counting task or assess individual skills. I deliberately chose Katie so I could assess whether she had the cardinal principle. (Amy, post-lesson reflective interview, Phase 2)

In addition to being interested in whether the children could count accurately, Amy used their 'performance' to assess whether they understood the concept of cardinality. By her second year of teaching, Amy appeared to move further towards a learner-focused view:

> Teachers often talk too much, including me; more focus should be given to the children rather than the teacher. I have learned to really watch children. It is great to be able to see from the other side and see how they are responding. (Amy, group interview, Phase 3)

There was some evidence in Jess' second year of teaching that she was trying to understand the thinking of individuals as well as of groups of children and use this to inform her teaching.

> I have started to get children to explain in more detail what they have said so I understand where they are coming from, and also so some of the other children start to realise some of these things too. (Jess, reflective account, Phase 3)

Kuhs and Ball (1986) suggest that teachers with a learner-focused view of mathematics teaching would adopt a problem solving or enquiry approach in their teaching. There was some evidence in Jess' third year of teaching that she was seeing the advantages of such an approach for understanding and developing children's mathematical thinking.

> When it gets around to working out what they know, it proves more if they have done problem solving. Like really, like hands on, like thinking and trying to think about the calculations they are doing really helps, rather than paper methods. (Jess, group interview, Phase 4)

Lisa also moved towards a problem solving approach to her mathematics teaching. This was apparent when she taught a similar lesson in her third year of teaching to one she had taught in her first year, about the complements in ten. In the earlier lesson, Lisa systematically demonstrated finding each of the complements in ten by dividing ten objects between two sets. In the later lesson, Lisa asked the children to *investigate* how many different ways the ten objects might be divided between the two sets.

There was evidence that the conceptions of the four teachers moved in similar directions, although from different starting points and to different degrees. Research shows that such movement does not occur through teaching experience alone (e.g. Wilson and Cooney, 2002). It was evident that Lisa's use of the Knowledge Quartet influenced the move towards focusing on conceptual understanding and towards a learner-focused view of mathematics teaching.

> It [the Knowledge Quartet] certainly gets me thinking a lot more about what I know and how I am going to teach them, like watching how they've learned. (Lisa, Group interview, Phase 2)

There was also evidence that Amy's use of the Knowledge Quartet both supported and developed her learner-focused view of mathematics teaching.

> I think the Knowledge Quartet has pushed me to think from the other side and see more clearly how the children see and what they need. It makes me try to put myself in their heads. (Amy, group interview, Phase 2)

A comment made by Kate in her second year of teaching suggested that use of the framework helped her to focus less on organisational matters, and more on conceptual understanding and on the learner.

> The first few things I would be thinking of are the organisational things, and then I try to think 'did they learn anything' and 'was the learning alright' even if the organisation wasn't kind of thing. So, I think it is useful to have some kind of structure to help you know what you need to know and what they need to know and how to learn it. I think what I have said and how I have explained things, I am more aware than I would be if I didn't have such a clear idea of what I was looking for. (Kate, interview, Phase 2)

There is less clear evidence from Jess's data that her use of the Knowledge Quartet was instrumental in moving her conceptions in a specific direction. However, she clearly saw the Knowledge Quartet as instrumental in improving her mathematics teaching.

> I think it is the only subject we have feedback on our teaching really . . . it is the only thing that actually comes close to constructive. What you've really thought about and tried to improve things and get in the right order . . . I think it has probably increased our maths teaching a lot more. (Jess, group interview, Phase 4)

In focusing and framing reflection on the mathematical content of teaching, the Knowledge Quartet appears to have been influential in confronting the conceptions of the teachers in the study. These conceptions generally shifted towards a view of mathematics teaching that was concerned with conceptual understanding and which focused on the learner. An increasingly learner-focused view was reflected in the adoption of more problem solving and enquiry approached to teaching mathematics.

Development of Content Knowledge

Use of the Knowledge Quartet was also found to be instrumental in developing the participants' mathematical content knowledge for teaching. There was evidence that reflection, focused by the Knowledge Quartet on the mathematical content of their mathematics teaching, enhanced the development of SMK, and particularly of PCK, in the teachers over the 4 years of the study. Developments in mathematical content knowledge were particularly evident in observations of teaching when two lessons taught by the same participant on similar topics were observed. For example, Amy was observed teaching lessons on counting in her training year and again in her first year of teaching. In the first lesson, Amy made use of a number of counting activities which involved the understanding of one or more of the principles of counting (Gelman and Gallistel, 1978) to which she had been introduced in her pre-service training. However, post-lesson discussions revealed that Amy was not aware of how these activities might help develop children's understanding of the principles. Amy's knowledge and use of the principles of counting was much more explicit in the second lesson.

> When I was planning this lesson, I drew on my knowledge of the pre-requisites for counting: knowing the number names in order, one to one correspondence, the cardinal principle, being able to count objects that cannot be moved/touched and counting objects that cannot be seen e.g. sounds or beats. (Amy, post-lesson interview, Phase 1)

This pedagogical content knowledge informed Amy's teaching in a way that it had not in the earlier lesson on counting. Her reflections on the previous lesson, mediated by the Knowledge Quartet, had prompted her to recall the classic Gelman and Gellistel work and seemed to have influenced Amy towards making the pre-requisites for counting more explicit in a similar lesson the following year.

Participants' written reflective accounts also suggested developments in their content knowledge, in relation to all four dimensions of the Knowledge Quartet. In relation to the *foundation* dimension, reflecting on their teaching using the Knowledge Quartet helped participants to recognise limitations in their SMK, which they then attempted to rectify. For example, Jess recognised the difficulty she had in distinguishing between the partition and quotition structures of division.

> Explaining dividing in terms of grouping and sharing still gets me mixed up. It is something I need to work on myself. The aim was to explain in terms of grouping. In future I am going to sort this out before the lesson so my physical representations don't get mixed up. (Jess, reflective account, Phase 2)

Through reflecting on her teaching, another of the participants, Kate, realised that she had not understood the difference between two subtraction structures (Rowland, 2006) and that this had affected her teaching.

> Because I had not really thought of 'find the difference' as a different sort of subtraction operation, but had thought of it just as different vocabulary for asking the question, I didn't really think about my choice of example in terms of looking for examples for which it would be sensible to do a 'difference' operation rather than a take away. (Kate, reflective account, Phase 2)

There was also evidence of developments in the teachers' content knowledge in relation to the *transformation* dimension of the Knowledge Quartet. All the participants were critical of their own teaching and the Knowledge Quartet framework channelled these criticisms in a constructive way onto the mathematical content of their lessons and onto how their pedagogy might be improved in relation to this content, e.g.

> When they were counting sounds it would have been helpful to match each sound to a held up finger . . . When I asked are there more frogs or more snakes I could have asked a child to come up and show these on the number line. (Amy, reflective account, Phase 1)

> I chose some quite big numbers to illustrate that drawing cubes and crossing them off may not always be reliable. It might have been better if I had chosen large numbers but with a small difference between them. (Kate, reflective accounts, Phase 2)

Amy's reflection led her to make suggestions for improvements to her teaching which focused on the use of demonstrations and representations. Kate suggested improvements to her teaching which related to her use of examples. These are all key aspects of the *transformation* dimension of the Knowledge Quartet.

There was also evidence that when reflecting on their teaching the participants were guided by the *connection* dimension of the Knowledge Quartet. The participants' reflection on their teaching focused on the connections they made, or had missed, and on how these might be further developed to enhance learning. Amy considered ways in which she could have made further connections in her lesson and clearly recognized the importance of making connections to aid children's learning.

> I could have linked the lesson to earlier work on counting or the OMS[3] (on counting and sharing fruit) earlier in the morning. I could have made reference to the good counting strategies one of the children used earlier when counting the fruit which would have enabled the children to make a connection and see their learning in context. (Amy, reflective account, Phase 2)

The sequencing of teaching is one aspect of the connection dimension of the Knowledge Quartet and in reflecting on her teaching Kate considered the appropriateness of the sequence she had used.

> Most of the children appeared to find measuring much easier than estimating making me think I should have done the activities in the opposite order. (Kate, reflective account, Phase 2)

Finally, there was evidence that the participants' reflection on their teaching focused on aspects of their content knowledge from within the *contingency* dimension of the Knowledge Quartet. For instance, Kate reflected on a teaching episode in which she had acted contingently.

> When estimating how many cubes long a book was Harriet-Mae said "eighty" and then corrected herself to say "eighteen". I used this as an example to question the children about which of these was a sensible estimate and we discussed why 80 was not. (Kate, reflective account, Phase 3)

Amy clearly felt that she became more able to act contingently over the course of the study.

> I am [more] aware of children's common misconceptions, and can therefore adapt in response contingently, or plan for these. Generally I think there is more contingent teaching going on and I am more confident to be flexible. I can respond quickly to a child by setting up an activity I know will extend from what they are doing. (Amy, group interview, Phase 3)

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[3] Oral and Mental Starter (OMS) was the term used in government guidance in the early 2000s for the beginning part of a mathematics lesson, in which children were expected to rehearse their knowledge of number bonds, calculation facts, etc.

It is likely that these teachers would have developed their practice in any case through systematic reflection. However, the instances discussed above suggest that the participants' reflection was focused on the mathematical content of their teaching by their use of the Knowledge Quartet. Our claim is that the Knowledge Quartet is an effective tool in this crucial respect. The teachers were alerted to issues relating to their mathematical content knowledge and they thought about ways to improve their teaching by addressing these issues. Kate explained how the Knowledge Quartet framework directed her reflection.

> If I think about my teaching in the car on the way home and I think, if it wasn't very good, why wasn't it very good? Was it the concept behind what I told them to do or was it the resources they had to do it with? So, that would be the Transformation and the first one would be Foundation. What would have enabled them to understand that better than they did? . . . I try and think, did they learn anything and, was the learning alright, even if the organisation wasn't. So, I think it is useful to have some kind of a framework. (Kate, interview, Phase 3)

Jess was convinced that her use of the Knowledge Quartet had been a positive influence on her teaching.

> I think the KQ has definitely improved my teaching. When I am planning I draw on the four areas unconsciously criticising what I plan to do, often asking myself questions – does that show what I want it to etc. (Jess, interview, Phase 3)

Amy explained why she found the framework useful and suggested that she saw the Foundation dimension as having 'overriding' importance in her teaching.

> I think it is good to be able to think about how you are putting different elements of your lesson into the different parts of the Quartet and also seeing how they link up. You feel like the Foundation theme is a kind of overriding one that comes into everything. (Amy, interview, Phase 3)

There was considerable evidence from observations of teaching, interviews and written reflective accounts that the participants' content knowledge for teaching developed over the course of the study and that this development was catalysed by reflection on their teaching supported by the Knowledge Quartet framework. Much evidence for the developments in mathematical content knowledge for teaching related to the PCK of the participants. However, there was also some evidence that the Knowledge Quartet supported development of the participants' SMK.

Conclusion

This study shows that the Knowledge Quartet can be an effective tool in developing teachers' mathematical content knowledge through focused reflections on their mathematics teaching. All of the teachers who participated in the study reported above testified that they had found the Knowledge Quartet helpful when planning and evaluating their teaching and intended to continue using it after their participation in the project ceased. Participants particularly valued feedback on their teaching which focused on mathematical content, and found that the framework helped them to focus more effectively on mathematical content themselves. Analysis of the four case study participants suggested that the framework was influential both in developing their conceptions of mathematics teaching and in developing their mathematical content knowledge. Use of the Knowledge Quartet helped the participants move from a view of teaching which focused on children being able to carry out procedures, to one in which conceptual understanding was more important. There was also evidence that the case study participants developed more learner-focused views of mathematics teaching through their use of the framework. In focusing reflections on mathematical content, the framework was seen to be an effective tool to support development of the teachers' PCK and to identify and strengthen aspects of SMK.

Participants in the study found the four dimensions both helpful and easy to use. Those who worked with the framework for 4 years suggested that it had become part of their way of thinking, so that they automatically referred to the four dimensions when planning and evaluating their teaching.

> I think the KQ has definitely improved my teaching. When I am planning, I draw on the four areas unconsciously criticising what I plan to do, often asking myself questions – does that show what I want it to etc. (Jess, interview, Phase 3)

Evidence from this study strongly suggests that in addition to being a useful tool for analysis of mathematical content knowledge revealed in the practice of teaching, the Knowledge Quartet can support beginning teachers in developing their mathematical content knowledge.

Modes of initial teacher education in England are now very diverse, and include workplace 'apprenticeship' versions located in schools, such as School-Centred Initial Teacher Training (SCITT). Given the widespread concerns about the resource of mathematics knowledge in primary school staff (Williams, 2008), the development of trainees' content knowledge is a challenging issue for such programmes. The Knowledge Quartet therefore has particular relevance to these modes of teacher education, and we have taken up opportunities to promote it as a tool for content-focused lesson observation in these contexts. SCITT programme leaders have indicated that the framework is being found to be relevant and useful in such ITT schemes. One of them commented:

> It is the single most powerful tool I have come across that has enabled me to give effective feedback on trainees' subject knowledge for teaching in a focused way.

Mentors of trainee teachers at UK universities (our own, and others) have also found the framework helpful in identifying issues of content knowledge and in giving focused feedback to their mentees. Introducing the framework to student teachers during initial teacher education courses, and use of the framework by mentors and university tutors during *practicum* placements, has supported a focus on mathematical content knowledge during training. Familiarisation with the framework has helped teachers to continue to develop their conceptions of mathematics teaching and their mathematical content knowledge after beginning their teaching careers.

There remains the question of whether the Knowledge Quartet can be effectively used to develop mathematical content knowledge for teaching without the support of a 'more knowledgeable other', not necessarily a mathematics educator. A programme of mentor training, involving developing mentors' understanding of the Knowledge Quartet, might begin to establish a panel of 'knowledgeable others' in schools who could support colleagues. There might also be a 'cascade effect' as beginning teachers who have been supported in using the Knowledge Quartet by mathematics educators become the 'more knowledgeable others' within their schools. We recognise that this might lead to 'dilution' in the efficacy of the Knowledge Quartet. It seems likely that the conceptualisations of the four dimensions developed from the original empirical research would be interpreted in a number of alternative ways by unsupported teachers or mentors, and a book (Rowland, Turner, Thwaites, & Huckstep, 2009) has been written to assist in such a situation. However, there is evidence to show that the framework, even without expert support, would at least encourage teachers and mentors to focus on the mathematical content of teaching rather than on more managerial issues.

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