Tim Rowland
Kenneth Ruthven
*Editors*

# Mathematical Knowledge in Teaching

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# Mathematical Knowledge in Teaching

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# Chapter 1
# Introduction: Mathematical Knowledge in Teaching

**Tim Rowland and Kenneth Ruthven**

## Background: The Topic and the Book

This book examines the issue of mathematical subject knowledge in teaching. There is now widespread agreement that the quality of primary and secondary school mathematics teaching depends crucially on the subject-related knowledge that teachers are able to bring to bear on their work. However, when discussion starts to focus in on the specific forms and functions of mathematical knowledge for teaching, there is much less concurrence. There is now, however, a prevalent suggestion that effective teaching calls for distinctive forms of subject-related knowledge and thinking. These are particularly live issues for policy and practice because of the longstanding difficulties in recruiting teachers who are confident and conventionally well qualified in mathematics, and because of rising concern that teaching of the subject has not adapted sufficiently to the changing circumstances of schools and their students. The issues to be examined in this book are, then, of considerable significance in addressing world-wide aspirations to raise standards of teaching and learning in mathematics through developing more effective approaches to characterising, assessing and developing mathematical knowledge for teaching within the professional workforce.

Public discussion often proceeds on the basis either that teachers need only have such mathematical competence as they are expected to develop in their students, or that it may also be beneficial for them to have somewhat 'more advanced' knowledge of mathematics than the subject matter they are teaching. However, it is now clear that such perspectives fail to do justice to the situation. Rather, a number of Anglo-American, Continental-European and East-Asian traditions point to particular forms of subject-related knowledge that underpin effective teaching of mathematics and to distinctive mechanisms for developing such knowledge. An important objective of this book, then, is to develop a critical synthesis of different perspectives on mathematical knowledge for and in teaching and to establish

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their professional implications. This synthesis seeks to bring key conceptual and theoretical problems to the fore and to find cogent ways of addressing them. For example, much work in this field has taken for granted the conceptions of mathematical knowledge, successful learning and effective teaching reflected in the customary educational practices familiar to the researchers. We seek to adopt a more detached perspective which seeks explicitly to take account of differing and changing perspectives on such matters. Equally, much work in this field has treated mathematical knowledge for teaching as residing solely with the classroom teacher. We aim to follow an approach which recognises the part played by textbooks and other tools and resources in classroom teaching and learning (forming a larger distributed system through which mathematical knowledge comes into play). We intend also to consider the possibility that some students help to bring such knowledge into play in the classroom (in particular through systematic reflection and thoughtful interaction with teacher and peers); indeed, that some students may themselves develop personal mathematical knowledge which includes aspects of what has been taken to be knowledge distinctively for teaching.

Inevitably, however, the different contributions to this book reflect the current flux in thinking about this area. Certain well-established ideas, notably that of 'pedagogical content knowledge', provide shared but contested points of reference, while alternative ways of thinking, drawn from a variety of sources, offer contrasting accounts and suggest neglected aspects. The authors of each chapter bring their special expertise to bear on the phenomenon of mathematical knowledge in teaching, reflecting different perspectives about the knowledge of mathematics teachers and different ways of 'knowing' within teaching. The coherence of the book comes less from consensus on the issues and more from a collective understanding and appreciation of the different perspectives and convictions of the contributors as a whole, developed within a series of seminars on *Mathematical Knowledge in Teaching* held over a 2-year period and sponsored by the Nuffield Foundation. Nevertheless, the contributions all broadly proceed from a view that attempts to assess and develop mathematical knowledge for teaching are unlikely to be meaningful or successful unless they take the classroom context of teachers' professional work into account, so that the focus of fundamental and applied research in the field becomes teachers' mathematical knowledge *in* teaching. This belief in the social and institutional situatedness of the domain – if the research is to be of any practical use – justified the title of the seminar series that brought the authors together in the first place, and thus explains the title of this book.

## Introduction to Section 1

Every profession – every 'job', for that matter – has its own knowledge base. However, discussion of the relation between knowledge and the profession of teaching must take into account the way in which knowledge itself lies at the heart of education. Thus, Kelly (1995, p. 103) writes that "... knowledge is the very stuff of education; it is impossible to conceive of, or to plan, any educational activity without recognising the central role that some knowledge-based transaction must play in it."

The relationship between teacher knowledge as the means of education and student knowledge as its goal is explored in the first section of the book, which focuses on some of the approaches which have been developed to conceptualise mathematical knowledge in teaching. One debate running through this first section is the extent to which mathematical knowledge in teaching is wholly derivable from considerations of a mathematical kind, or whether pedagogical knowledge rooted in other disciplines, such as psychology and the human and social sciences more broadly, needs to be incorporated into the model. Another crucial question is whether mathematical knowledge in teaching is located 'in the head' of the individual teacher or is somehow a social asset, meaningful only in the context of its application.

Each of the main contributions introduces and examines a particular type of approach to theorising mathematical knowledge in teaching. In each case, the related conceptual framework provides significant insight into key forms of mathematical knowledge which play a part in successful teaching. Each of these frameworks is then brought to bear on the analysis of the part played by mathematical knowledge in successful teaching. Beginning with Shulman's essentially individualist conception of teacher knowledge, and moving through overtly mathematical conceptions to social practice-based reconceptualisations, the authors explain how and why each conceptual framework can be seen to represent a significant advance on, or alternative to, earlier conceptions. The value of these theories is demonstrated in their practical application: our contributors were also presented with the challenge of identifying significant implications of bringing these conceptual frameworks to bear on policy and practice in school teaching, teacher education and professional development. The final contribution to this first section offers a critical appreciation of these approaches and develops a more overarching framework for synthesising their differing contributions to the analysis of key issues of policy and practice.

## Introduction to Section 2

Socio-cultural considerations of the nature of mathematical knowledge in teaching lead naturally to the subject matter in the second section of the book. Here, the various authors bring to light the different ways in which conceptions of mathematical knowledge in teaching are determined by cultural assumptions at local, institutional and national levels. These, for the most part tacit, assumptions are the invisible substrate from which theory and practice in the characterisation, assessment and development of teachers' mathematical content knowledge proceed. Such cultural 'givens', typically unseen and unquestioned, have the potential to limit what can be imagined and, therefore, what can be achieved in the practice of education. On a more positive note, these embedded starting points can also foster attitudes, habits and practices in one culture that would be difficult to sustain in another. The ability to see possibilities beyond familiar ways of being and doing can be achieved not only by a more anthropological perspective on our own practices, but also by looking to practice in cultures other than our own. In this latter respect, the study of educational theory and practice beyond our own shores (wherever they might be) has proved to be especially illuminating in the recent decades.

Thus, the focus of the chapters in the second section of the book is on different ways in which the cultural context shapes the development of mathematical knowledge in teaching. Each of the main contributions examines some aspect of cultural variation and shaping, and each introduces a conceptual framework by which the cultural embedding of mathematical knowledge in teaching can be understood. The relevant chapter then illustrates the use of this conceptual framework to analyse a particular aspect of the cultural variation and shaping of mathematical knowledge in teaching, identifying significant implications of bringing this conceptual framework to bear on policy and practice in school teaching, teacher education and professional development. The final contribution to this section offers a critical appreciation of these frameworks and illustrations and provides a synthesis of their implications for issues of policy and practice.

## Introduction to Section 3

The focus in the third section is on the various means by which progress is being made in enhancing mathematical knowledge in teaching, within teacher education programmes and professional development initiatives. The different approaches featured in this section are explicit in identifying the 'tools' that are brought to bear on the development of teacher knowledge. In this case, 'development' includes identification and assessment of the mathematical knowledge (and the various kinds of such knowledge) that teachers bring to their work. The approach in each case is *principled*, in that the relevant tools have themselves been developed in research or are firmly rooted in research in mathematics education and teacher education. The authors report firsthand the development and application of different theorised tools for the development of mathematical knowledge in teaching. While these tools differ considerably, each creates favourable conditions for teachers to reflect deeply on what they know and understand about mathematics and/or mathematics teaching, so that this knowledge becomes available to them for regulation and control.

Each of the main contributions in this third section introduces the relevant tool and explains its guiding principles and underlying rationale. In doing so, the authors relate their accounts to the conceptual frameworks discussed in earlier chapters, especially those in the first section. In this way, a 'theoretical loop' (Skott, 2005) can be seen to be activated, in which practice feeds on theory and feeds back to theory. In each case, the authors illustrate the use of the relevant tool to develop and/or assess mathematical knowledge in and for teaching and explain how and why such a tool represents a significant advance on, or alternative to, existing ones. The authors identify significant issues which might arise in developing further and wider use of these tools and ways in which these issues might be addressed. The final contribution to this section offers a critical appreciation of these tools and examines how they might contribute to a more overarching system aimed at developing and assessing mathematical knowledge in and for teaching.

## Critical Discussion and Synthesis

Each section of the book concludes with a chapter that offers critical discussion and synthesis of the preceding chapters. Likewise, the book concludes with a chapter that reviews how each section has contributed to a comprehensive and systematic understanding of mathematical knowledge in teaching, and how the book as a whole is intended to constitute a foundation for a systematic and reflexive research programme. The purpose of such a programme, and of this book in the first place, must be to enhance theory and practice in mathematics teacher education and professional development and, ultimately, the mathematics learning experiences of students in all phases of education.

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# Part I
# Conceptualising Mathematical Knowledge in Teaching

# Chapter 2
# Conceptualising Teachers' Mathematical Knowledge in Teaching

**Marilena Petrou and Maria Goulding**

## Introduction

It seems obvious that a teacher's mathematical knowledge is an important ingredient for teaching, and while a teacher needs to be able to do the mathematics required for the curricular level being taught, this may not be sufficient to ensure pupil progress. Indeed, research on effective primary teachers indicates that those who produce the highest numeracy gains in pupils do not necessarily hold advanced qualifications in mathematics (Askew, Brown, Rhodes, Johnson, & William, 1997). We need to know what other factors come into play, and how these interact with each other in the teaching process. Internationally, however, there is no universal agreement on a widely-accepted framework for describing teachers' mathematical knowledge in teaching (Tirosh & Even, 2007). This is a concern not only for pre-service teacher education and professional development courses, but also for research, since without some common understanding of what subject knowledge means and what it looks like in practice, there can be no coherent approach to designing courses, or answering research questions about the role of teachers' mathematical knowledge in teaching. None of these issues is politically neutral. International comparisons of pupil performance in the 1990s fuelled widespread anxiety in both the United States and England and Wales about mathematical standards. Teachers were seen to be part of the problem of relatively poor performance and strengthening their subject knowledge as a contribution to the improvement of overall standards. In this chapter, the meaning, importance and limitations of several analytical models of teachers' mathematical knowledge will be discussed with some reference to their political context, and a synthesis will be proposed. Finally, the implications of using this synthesis in future teacher development programmes will be presented.

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## Shulman's Conceptualisation

In the United States in the 1980s, broad agreement on the inadequacy of the traditional curriculum for preparing students for the demands of the 21st century, coupled with disappointing results from international comparisons of mathematical achievement, led to calls for reform in the mathematics curriculum, and its teaching and assessment (Schoen, Fey, Hirsch, & Coxford, 1999). At the same time, but independently from the reform agenda, Shulman (1986, p. 6) identified a blind spot with respect to content knowledge in the teacher effectiveness research for as much attention to be paid to what the teacher is teaching as is paid to generic pedagogic factors such as wait time and time on task: "How do teachers decide what to teach, how to represent it, how to question students about it and how to deal with problems of misunderstanding?" His work with secondary teachers of English, biology, mathematics and social studies allowed him and his colleagues to develop a more coherent theoretical framework of teacher knowledge which has since become widely influential (Grossman, Wilson, & Shulman, 1989; Shulman, 1986, 1987). There was an immediate interest in his ideas, and Shulman's (1986) new conceptualisation led to a new phase, continuing to the present, of research on teacher knowledge. In these studies, the focus is on teaching itself, and on providing rich descriptions of teachers' actions while teaching. Ball, Thames, and Phelps (2008) point out that Shulman's work has been cited in more that 1,200 refereed journal articles in a wide variety of disciplines. Given its importance, a review of the model will be presented, interwoven and supplemented with a discussion of how the ideas have been modified and expanded by others.

Shulman and his colleagues proposed different categories of teacher knowledge that are needed for effective teaching. Although the specific boundaries and the names of the categories varied across publications, one of the most detailed descriptions of their model is given in Shulman (1987) publication. Here Shulman proposed seven different categories of teacher knowledge:

– general pedagogical knowledge;
– knowledge of learners' characteristics;
– knowledge of educational context;
– knowledge of educational purposes and values;
– content knowledge;
– curriculum knowledge;
– pedagogical content knowledge.

The first four categories listed above refer to general aspects of teacher knowledge and were not the focus of Shulman's work (Ball et al., 2008), which focused on the missing-content dimension of teacher knowledge. In his account, however, Shulman (1987) made it clear that an emphasis placed on this was not intended to limit the importance of general categories of teacher knowledge in teaching.

The last three categories – content knowledge, curriculum knowledge and pedagogical content knowledge – describe the content dimensions of teacher knowledge and together make up, what Shulman referred to as, the missing paradigm in research on teaching. These three content dimensions of teacher knowledge are discussed in detail in Shulman (1986) publication. Content knowledge includes knowledge of the subject and its organising structures, and is what Shulman called Subject Matter Knowledge (SMK). SMK refers to "the amount and organization of knowledge per se in the mind of the teacher" (Shulman, 1986, p. 9). Shulman went on to suggest that, "to think properly about content knowledge requires going beyond knowledge of the facts or the concepts of a domain" (p. 9). Thus, understanding subject matter not only includes awareness of its facts, but also goes beyond the facts to include understanding of its structure. SMK consists of what Schwab (1978) named 'substantive' and 'syntactic' knowledge. Substantive knowledge concerns the organisation of key facts, theories, models and concepts, while syntactic knowledge concerns the processes by which theories and models are generated and established as valid. For example, in mathematics, syntactic knowledge consists of activities such as formulating and testing generalisations and constructing proofs. Ball (1991) echoes Schwab's concepts of substantive and syntactic knowledge discipline makes a distinction between knowledge of mathematics and knowledge about mathematics.

The second content-related category is curriculum knowledge, that is:

> Represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and the contraindications for the use of particular curriculum or program materials in particular circumstances (Shulman, 1986, p. 10).

Thus, curriculum knowledge is knowledge of the available instructional materials, such as the curriculum and textbooks (what Shulman calls lateral curriculum knowledge), as well as knowledge of the topics and the ways in which these were addressed during the previous and subsequent years in schools (what Shulman calls vertical curriculum knowledge). The United States does not have a national curriculum, and Shulman's early work coincided with a period in the US when a variety of reform programs were being funded (Schoen et al., 1999). Shulman's description of content knowledge conjures up a view of a loose curriculum frame with a degree of choice about materials and approaches, which may not be applicable in different contexts. Contemporary practice in the UK, for instance, is strongly constrained by official guidance and assessment systems; so the teachers' curriculum knowledge not only includes the materials and resources from which they can draw, but also the frame in which they are working. Thus teachers may not draw on the full range of what is available to them, or even know about what is available to them, because they are limited by the testing regime.

The last and most influential of the three content-related categories is the new concept of Pedagogical Content Knowledge (PCK). PCK is:

> That special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding [. . .] It goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching (Shulman, 1986, p. 9).

It includes the content specific representations, examples and applications that teachers use in order to make subject matter comprehensible to students together with the strategies that teachers use in order to overcome their students' difficulties. PCK suggests that it is not just knowledge of the subject, or knowledge of pedagogy that is needed in mathematics teaching, but rather a combination of both.

The power of PCK is illustrated by a number of researchers who have assimilated, criticised and reformulated the concept. For example, Meredith (1995) suggests a reformulation of the concept of PCK and claims that a wider framework, extending PCK to incorporate alternative forms of teaching, is needed.

In particular, Meredith (1995, p.176) argues that PCK, as defined by Shulman:

> seems to imply one type of pedagogy rooted in particular representations of prior knowledge. Most of the research posits a teacher-directed, didactic model of teaching. PCK does not seem to encompass alternative views of teaching which, for instance, conceive of learners as autonomous agents constructing their own understanding of subject matter.

Meredith's stance is clearly influenced by the reform agenda mentioned earlier, which included a much greater emphasis on problem solving, in contrast to the predominant teaching method where teachers explain and illustrate procedures, and pupils practice the procedures using examples. Her argument is that PCK, as defined by Shulman, seems to see the teacher's role as transmitting mathematical knowledge and helping learners to acquire understanding. In addition, Meredith claims that Shulman's conceptualisation does not acknowledge that teachers develop different forms of PCK depending on the knowledge and beliefs they bring to learning. Shulman's concept of PCK:

> is perfectly adequate if mathematical knowledge is seen as absolute, incontestable, unidimensional and static. On the other hand, teachers who conceive of subject knowledge as multidimensional, dynamic and generated through problem solving may require and develop very different knowledge for teaching (Meredith, 1995, p. 184).

It is debatable, however, whether Shulman's conceptualisation necessarily entails a view of knowledge as incontestable or of teachers as transmissive. It seems just as important for teachers working in a constructivist way to have a range of alternative ways of representing mathematics and responding to pupils' ideas and to have a rich knowledge base on which to draw.

Although Shulman's work was ground-breaking and his ideas continue to influence the majority of research in the field, later researchers in the same tradition argue that it is not sufficiently developed to be operationalised in research on teacher knowledge and teacher education. According to Ball et al. (2008), the distinction between Shulman's notions of content knowledge and pedagogical content knowledge is often unclear. In addition, the conceptualisation does not acknowledge the interactions between the different knowledge categories (Hashweh, 2005) and can be criticised for presenting what seems to be a static view of teaching knowledge. It ignores the dynamic nature of knowledge, and that teacher knowledge often develops through classroom interactions with the students on the subject matter (Fennema & Franke, 1992).

## Fennema and Franke's Conceptualisation

Fennema and Franke (1992) work is discussed here because it responds to the last criticism and focuses specifically the case of mathematics teaching. Their model builds on and modifies Shulman's framework by suggesting that the knowledge needed in teaching is interactive and dynamic in nature. They propose a model of teacher knowledge that can be used to describe what teachers need in mathematics teaching. They argue that mathematical knowledge for teaching includes four components: knowledge of the content, knowledge of pedagogy, knowledge of students' cognition and teachers' beliefs (Fig. 2.1). Their model centres on teacher knowledge as it occurs in the context of the classroom. Central to their conceptualisation is the claim that knowledge is interactive in nature, and that, in a given context, teachers' knowledge of content is related to knowledge of pedagogy and students' cognition and combines with beliefs to create a knowledge set that determines teaching practices and teachers' behaviour in the classroom. Moreover, they suggest that knowledge is of a dynamic nature and claim that teaching is a process in which teachers can change their existing knowledge and create new knowledge.

The content of the mathematics component includes:

> teachers' knowledge of the concepts, procedures, and problem-solving processes within the domain in which they teach [. . .] It includes knowledge of the concepts underlying the procedures, the interrelatedness of these concepts, and how these concepts and procedures are used in various types of problem solving (Fennema & Franke, 1992, p. 162).

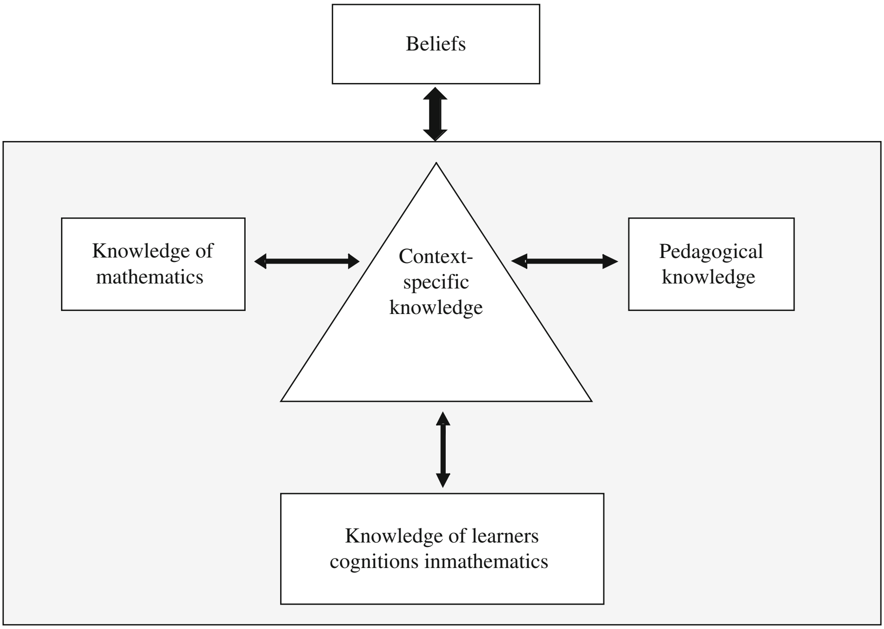


**Fig. 2.1** Teacher knowledge: developing in context (Fennema & Franke, 1992, p. 162)

The parallels between the definition of content of mathematics domain and the definition of SMK as conceptualised by Shulman (1986) are clearly visible. Central to both accounts is the idea that teachers need not just to know the procedures, but also to understand the concepts underlying them. They need to know that something is so, and also why it is so.

The pedagogical knowledge component refers to "teachers' knowledge of teaching procedures such as effective strategies for planning, classroom routines, behavior management techniques, classroom organisation procedures, and motivational techniques" (Fennema & Franke, 1992, p. 162). From Fennema and Franke's conceptualisation, it can be said that the pedagogical knowledge component is related to Shulman's category of general pedagogical knowledge which includes broad principles and strategies of classroom management. Furthermore, in considering teachers' knowledge of pedagogy, they discuss teachers' knowledge of representation in a manner similar to Shulman's conceptualisation, according to which the use of representations is central in teaching.

The learner's cognition component includes:

> knowledge of how students think and learn and, in particular, how this occurs within specific mathematics content [. . .] as well as understanding the processes that students will use and the difficulties and successes that are likely to occur (Fennema & Franke, 1992, p. 162).

In Shulman's conceptualisation of knowledge, students' conceptions are considered as part of teachers' pedagogical knowledge. In Fennema and Franke (1992) model, this kind of knowledge is considered as a category on its own, not as a subcategory of teachers' pedagogical knowledge. What is common to both accounts is the idea that knowledge of how students think and learn is central to effective mathematics teaching. This idea is consistent with later publications from Shulman's colleagues (Grossman, 1990). The foremost claim in Grossman's approach is the recognition that teachers must also know their learners. According to Marks (1990), this includes knowing learners' cognitive processes, typical patterns of understanding, common errors, things frequently found to be difficult or easy, and interpreting students' understanding in the midst of a lesson.

Fennema and Franke (1992) see teacher knowledge as both interactive and dynamic in nature. Knowledge is developed in a specific context and often develops through interactions with the subject matter and the students in the classroom. In their model, all aspects of teacher knowledge and beliefs are related to each other, and all must be considered to understand mathematics teaching. They suggest that no one domain of teacher knowledge has a singular role in 'effective' mathematics teaching.

Therefore, for Fennema and Franke, the challenge for research in the field of teacher knowledge is to develop methodology that can encompass all of these, with the aim of understanding the interaction between different categories of teacher knowledge, the roles they play in mathematics teaching and how these roles differ as teachers' knowledge changes when they interact with their students. Adding to this, they claim that the key to understanding this kind of relationship requires researchers to carefully take into account the context in which teachers work, as central to the knowledge and beliefs that are evidenced in mathematics teaching.

## The Mathematics Teaching and Learning to Teach Project (MTLT) and the Learning Mathematics for Teaching Project (LMT): A Practice-Based Framework of Teachers' Mathematical Knowledge for Teaching

The third US model to be described, whilst influenced by critiques of Shulman, falls broadly within his cognitive tradition. Like Fennema and Franke, the team at University of Michigan examines ways in which Shuman's ideas can be operationalised in mathematics education. For the past 15 years, the work of Mathematics Teaching and Learning to Teach Project (MTLT) and Learning Mathematics for Teaching Project (LMT) have focused both on the teaching of mathematics and on the mathematics used in teaching. The aim was to develop a practice-based theory of content knowledge needed for mathematics teaching. The first project focused on what teachers do while teaching. By teaching was meant:

> everything that teachers do to support the instruction of their students [. . .] the interactive work of teaching lessons in classrooms, and all the tasks that arise in the course of that [. . .] Each of these tasks involves knowledge of mathematical ideas, skills of mathematical reasoning [. . .] fluency with examples, and thoughtfulness about the nature of mathematical proficiency (Ball, Hill, & Bass, 2005, p. 17).

The team used qualitative methods to collect and analyse data in order to investigate what teachers do as they teach mathematics and what mathematical knowledge and skills teachers need to hold in order to be able to teach mathematics effectively. Data analysis led to the conceptualisation of a model of mathematical knowledge for teaching as illustrated in Fig. 2.2.

This model builds on Shulman's work by clarifying the distinction between SMK and PCK. The team's work attempts to validate Shulman's conceptualisation by developing reliable and valid measures of mathematical knowledge for teaching. The model suggests that Shulman's SMK can be divided into three categories: common content knowledge, specialised content knowledge and horizon knowledge. Common content knowledge refers to mathematical knowledge and skills that are used in any setting, not necessarily that of teaching, and includes an individual's ability to calculate an answer and to solve mathematical problems correctly. Specialised content knowledge, a central idea in the model proposed, is the knowledge that is used in classroom settings and is needed by teachers in order to teach effectively (Ball et al., 2008). Finally, horizon knowledge includes teachers' awareness of how the mathematical topics covered in previous years in schools are related to curriculum topics addressed in the subsequent years in schools.

In addition to this, they suggest that PCK, as conceptualised by Shulman, can be divided into Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Curriculum (KCC) (Ball et al.,

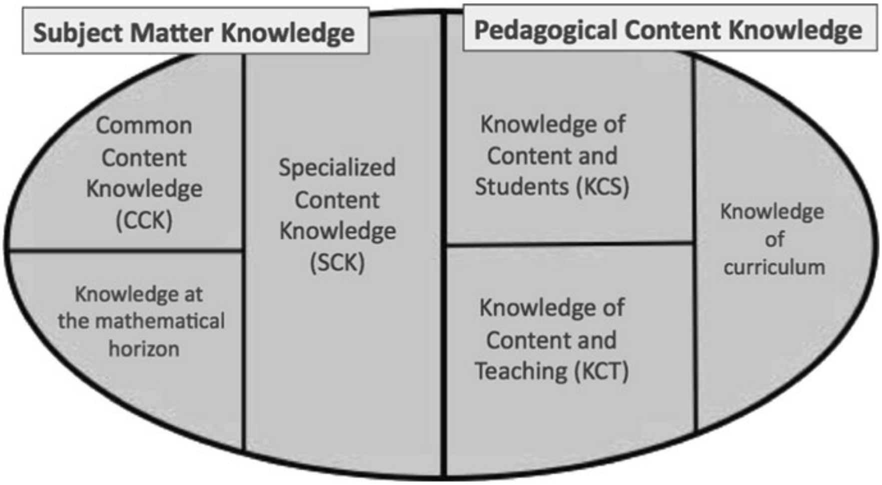


**Fig. 2.2** Mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008, p. 403)

2008). KCS is ‘knowledge that combines knowledge about students and knowing about mathematics’ (Ball et al., 2008, p. 36). This means that teachers must be able to anticipate students’ difficulties and obstacles, hear and respond appropriately to students’ thinking and choose appropriate examples and representations while teaching. Both in planning and teaching, teachers must show awareness of students’ conceptions and misconceptions about a mathematics topic.

Finally, KCT is ‘knowledge that combines knowledge about mathematics and knowledge about teaching’ (Ball et al., 2008, p. 401). It refers to teachers’ decisions on the sequencing of activities and exercises, their awareness of the possible advantages and disadvantages of representations used while they teach and to their decisions to pause a classroom discussion for more clarifications, or to use a student’s opinion to make a mathematical remark.

The framework presented in Fig. 2.2 supports Shulman (1986) idea that knowledge for teaching includes a specialised knowledge of content which elaborates the constructs of SMK and PCK. For example, the two central dimensions of PCK as defined by Shulman (1986) are included in the constructs of KCS and KCT. These are, firstly, teachers’ awareness of their students’ conceptions and misconceptions and, secondly, the representations and examples that teachers use in order to make subject matter comprehensible to students. Furthermore, this framework develops the concept of SMK in more detail by proposing its sub-domains and developing measures of these sub-domains. For instance, the way horizon knowledge is defined is clearly related to Shulman’s notion of vertical curriculum knowledge.

However, Ball et al.’s (2008) conceptualisation of mathematical knowledge for teaching does not acknowledge the importance of teachers’ beliefs in their teaching. Research has suggested that teachers’ “beliefs about the nature of mathematics may be tied up with subject-matter knowledge in the way in which teachers approach mathematical situations" (Goulding, Rowland, & Barber, 2002, p. 691). If teachers believe that mathematics is principally a subject of rules and routines which have to be remembered, then their own approach to unfamiliar problems will be constrained, and this may impact on their teaching. Beliefs may be particularly salient in the development of syntactic knowledge, where conjecturing, finding evidence and seeking explanations is quite different from finding rules and routines in recognizable contexts.

Furthermore, Ball et al.'s new framework can be criticised for its account of the concept of SCK. This concept is central in the conceptualisation of teachers' mathematical knowledge and is defined as the mathematical knowledge that is used in classroom settings and needed by teachers in order to teach effectively. The definition of SCK, as it stands, does not clearly distinguish between SCK and PCK. After all, PCK is also uniquely needed by teachers and is used in classroom settings. As Shulman (1987, p. 8) noted, PCK is "a special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding."

What is valuable in the development of this framework of teacher knowledge is that the Michigan team made some progress in identifying the relationship between teacher knowledge and students' achievement in mathematics. Hill, Rowan, and Ball (2005) argue that teachers' mathematical knowledge is related to students' achievement in mathematics and they provide evidence that teachers with weak knowledge transmit this to their students. Another central contribution of the MTLT and LMT projects was the development of measures of teachers' mathematical knowledge. In his account, Shulman (1986) hopes that researchers working in the field of teacher knowledge will manage to develop instruments that could be used to test aspects of teacher knowledge. Within the work of these two projects, the research teams managed to develop a series of multiple choice items that can be used to measure mathematical knowledge for teaching. These kinds of items could reasonably be used to inform the content and structure of different courses within teachers' initial training.

Ball's work was of interest to both sides of the 'maths wars' that followed what had seemed like a broad consensus on the need for major curriculum, teaching and assessment reform in the United States. Into this acrimonious debate, the work of Liping Ma, a Ball's student at Michigan, brought surprising agreement between the reformers and the traditionalists. Her comparative study of Chinese and US teachers' knowledge of fundamental mathematics found a knowledge gap between the US and Chinese teachers (who had significantly fewer years of formal mathematics education at school) which mirrored the gap between US and Chinese students found in other studies. She argued that in the US, the lack of attention to mathematical content knowledge in teacher education programmes reinforced low quality school mathematics education and that this was an important impediment to reform. Her conclusions found favour with both the reformers advocating a focus on problem solving and teachers as facilitators of children's thinking, and the traditionalists with a concern for underlying structure, formal reasoning and deductive proof. (Shulman in Ma, 1999).

## The Knowledge Quartet

Also, in a broadly cognitive tradition, but in the different political circumstances of England and Wales, is the theoretical model reported in the findings of the SKIMA (Subject Knowledge in Mathematics) project by members of the Faculty of Education at the University of Cambridge (Rowland, 2005; Rowland, 2007; Rowland, Huckstep, & Thwaites, 2003). This arose out of earlier work on primary pre-service mathematical subject knowledge which was a response to the increased prescription of the initial teacher training curriculum and its assessment by the government, over a decade after the introduction of the National Curriculum in 1989.

The Knowledge Quartet is a theoretical framework which arose from the investigation of the mathematical content knowledge of pre-service elementary school teachers in England and Wales. The project is set within the theoretical framework set out by Shulman (1986) but responds to Fennema and Franke (1992) by categorising situations from classrooms where mathematical knowledge surfaces in teaching. The team's approach to investigate the relationship between pre-service teachers' SMK and PCK of mathematics was to observe and videotape mathematics lessons taught by pre-service teachers in a 1-year Postgraduate Certificate of Education course.

The detailed analysis of the lessons observed resulted in the identification of a framework called 'The Knowledge Quartet'. This framework can be used as a tool for classifying ways in which the pre-service teachers' SMK and PCK come into play in the classroom. The Knowledge Quartet consists of four dimensions, namely, Foundation, Transformation, Connection and Contingency.

> (Foundation) consists of trainees' knowledge, beliefs and understanding acquired in the academy, in preparation for their role in the classroom. Such knowledge and beliefs inform pedagogical choices and strategies in a fundamental way [. . .] the second category (Transformation) concerns knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself [. . .] Connection binds together certain choices and decisions that are made for the more or less discrete parts of mathematical content [. . .] contingency concerns classroom events that are almost impossible to plan for (Rowland et al., 2003, pp. 97–98).

Each component of the Knowledge Quartet is composed of a number of codes. The key components of the Foundation category are teachers' knowledge and understanding of mathematics pedagogy, as well as their beliefs about it. Transformation includes the kind of representation and examples used by teachers, as well as teachers' explanations and questions asked from students. The third category, Connection, includes the links made between different lessons, between different mathematical ideas and between the different parts of a lesson. It also includes the sequencing of activities for instruction and an awareness of possible difficulties and obstacles that students may have with different mathematical topics and tasks. Finally, the fourth category, Contingency, concerns teachers' readiness to respond to students' questions, to respond appropriately to students' wrong answers and to deviate from their lesson plan. In other words, it concerns teachers' readiness to react to situations that are almost impossible to plan for.

This framework elaborates the constructs of SMK and PCK as these were defined by Shulman, and takes up Hashweh (2005) suggestion that what is missing from Shulman's account is the identification of interactions among the different categories of teachers' knowledge. The Knowledge Quartet can be used in understanding the ways in which SMK and PCK are related and come into play in the classroom. In this framework, all aspects of teachers' knowledge and beliefs come together as resources from which to draw both in planning and in the act of teaching. The Knowledge Quartet can be seen as a response to Fennema and Franke (1992) call to develop studies that focus on the identification of a framework for thinking about the ways in which different components of teachers' knowledge are integrated and come into play in the classroom. In addition, the Foundation dimension of the framework can be understood as a response to Meredith (1995) call for a model that acknowledges that pre-service teachers may develop different forms of PCK depending on the knowledge and views they bring to their training.

The Knowledge Quartet is currently used as a framework for lesson observation and for mathematics learning development within the primary PGCE programme at Cambridge University (Rowland, 2007). The framework is also being applied to support teaching development in early career teachers in England (Turner, 2006) and structuring initial teacher education in Ireland (Corcoran, 2007). Finally, the framework was used with the aim of understanding what relationship can be observed between Cypriot pre-service teachers' mathematical knowledge and their teaching (Petrou, 2009). Petrou (2009) argued that, in general, the Knowledge Quartet was comprehensive in the classification of teaching situations in which participants' mathematical knowledge surfaces in teaching. Issues related to the interpretation and use of textbooks in mathematics teaching were not addressed by the framework; however, they proved important in analysing the mathematics lessons taught in Cypriot classrooms. This suggested that when adapting the Knowledge Quartet for observing lessons in Cyprus, and indeed, in many other countries, there is a need to take careful account of possible differences between the context in which the framework was originally developed and the context in which it is being applied.

The neglect of textbooks in the English conception of the Knowledge Quartet may be partly because they were less visible than they are in Cyprus. But it may be because the original study was very focused on SMK and PCK and not so much on curriculum knowledge. According to Shulman (1986), curriculum knowledge includes the 'knowledge of instructional materials', such as textbooks. So, the enlargement of the Knowledge Quartet to include use of textbooks brings curriculum knowledge within the orbit of the Knowledge Quartet in a way that it was not before. Petrou (2009) argues that the original analysis of the video data in the English study may have been limited by the neglect of curriculum knowledge. Indeed, although texts were less visible, practice in the UK is strongly framed by the National Curriculum and its associated assessment systems. Therefore, since the curriculum is the key to mathematics teaching in the UK, and we imagine this is to be true in many countries, it seems reasonable to argue that 'curriculum knowledge' is important in any attempt to understand what teachers need to know in order to teach mathematics effectively.

## Synthesis

The models about teacher knowledge described above can be understood as elaborating, and not replacing, Shulman's (1986) conceptualisation of content-related categories of teacher knowledge, and in particular, the notions of SMK and PCK. The conceptualisations of teacher knowledge proposed are not inconsistent; rather, they build on each other. Even though the researchers have stressed different domains of teacher knowledge, all focus on the importance of seeing the content to be taught as an important part of teaching.

However, the conceptualisations raise questions about whether the distinction between SMK and PCK could and should be made. Aubrey (1997) and McEwan and Bull (1991) agree with McNamara (1991) in arguing that there is no distinction between SMK and PCK, and that for teachers, all knowledge is pedagogic. Similarly, An, Kulm, and Wu (2004, p. 146) define PCK as "the knowledge of effective teaching, which includes three components, knowledge of content, knowledge of curriculum, and knowledge of teaching."

The advantage of these perspectives is that they acknowledge the importance of teaching and are very appropriate when we are seeking to understand what is going on in the classroom. After all, "teachers use mathematical knowledge not so much for the doing of mathematics but rather for the teaching of mathematics" (Hodgen, 2003, p. 106).

However, considering all mathematical knowledge in teaching as pedagogic may not be helpful in teacher preparation and development programmes. There is an argument for some specific attention to be paid to SMK. For instance, those who have had limited experience of reasoning and proof, may need opportunities to work on this at their own level as learners of mathematics. The important thing here would be that the examples and activities chosen for this work would not be far removed from the school curriculum. For instance, reasoning associated with odd and even numbers and operations upon them, or algebraically generalizing patterns in number squares, could be appropriate for primary pre-service teachers. Secondary teachers may not be called upon to prove the rules for operations on directed numbers in the classroom, but being exposed to this exercise could enlarge their own understanding and help them to see that understanding is a continual process. In geometry, secondary teachers may know the conditions for congruency of the triangles, but may not fully understand why these conditions hold and what can be deduced, once congruency has been established. The unpacking and deepening of SMK can be seen as part of the process of transformation required for robust PCK to be developed.

For this reason, the synthesis proposed here (Fig. 2.3) maintains a distinction between SMK and PCK, but recognizes the interplay between the two categories. We position ourselves with Fennema and Franke (1992) in believing that teacher knowledge can only be understood in the context in which they work. We take the

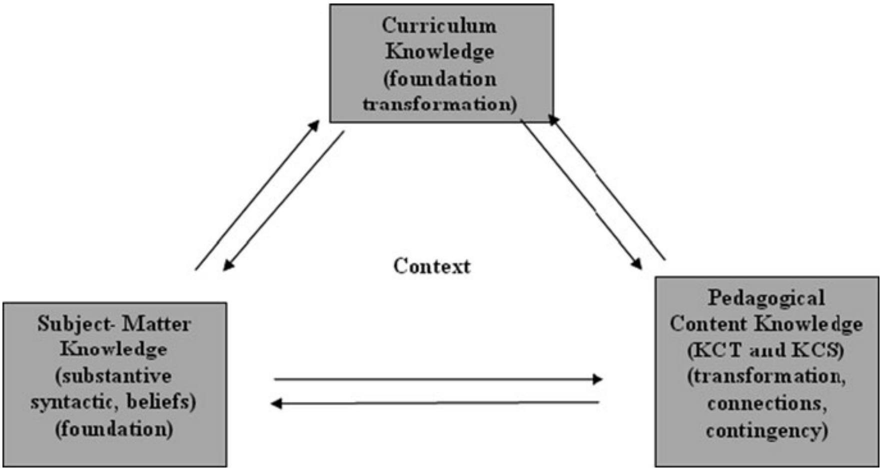


**Fig. 2.3** Synthesis of models on teacher mathematical knowledge

view that the context in which teachers work is the structure that defines the components of knowledge central to mathematics teaching. Included in this 'context' are the educational system, the aims of mathematics education, the curriculum and its associated materials (such as textbooks) and the assessment system. Within a particular country, therefore, the curriculum and its associated materials provide the frame within which teachers work. They reflect beliefs about what mathematics is, what students need to know about mathematics, and in what ways mathematics needs to be taught. However, we acknowledge that context is also local. This would include the resources, both material and human, that teachers have in their school or locality, as well as the practices and ethos of the workplace.

In focusing on investigating the nature, role and importance of SMK, PCK and the related categories in the Knowledge Quartet, the earlier models neglect the importance of curriculum knowledge in conceptualising mathematical knowledge for teaching. In Ball et al., (2008, p. 403), Curriculum Knowledge is provisionally placed within pedagogical content knowledge:

> We are not yet sure whether this may be a part of our category of knowledge of content and teaching or whether it may run across the several categories or be a category in its own right. We also provisionally include a third category within subject matter knowledge, what we call "horizon knowledge" [. . .] Again we are not sure whether this category is part of subject matter knowledge or whether it may run across the other categories. We hope to explore these ideas theoretically, empirically, and also pragmatically as the ideas are used in teacher education or in the development of curriculum materials for use in professional development.

Ball et al. seem to recognise that there is need in their model to further refine and investigate the concept of curriculum knowledge.

Our model suggests that Curriculum Knowledge as defined by Shulman (1986) is central in understanding what teachers need to know in order to teach mathematics effectively. In addition, the model implies that teachers' SMK and PCK can determine the ways in which teachers understand, interpret and use the mathematics curriculum and its associated materials. Indeed, research suggests that teachers interpret the curriculum materials, such as textbooks, in different ways and that their interpretation determines the ways they use these in their teaching (Ball & Cohen, 1996; Petrou, 2009). In a study that focuses on investigating how pre-service teachers interpret and use mathematics textbooks, Nicol and Crespo (2006) show that teachers have various approaches to using textbooks, ranging from adherence (the textbook is seen as authority), elaboration (the textbook is seen as the main resource, but teachers elaborate it with other resources), and creation (the teacher examines the textbook with a critical eye for its potential and limitations in deciding what to teach). Clearly, SMK and PCK are factors influencing these three approaches.

Elsewhere, Ball and Feiman-Nemser (1988) show how pre-service teachers' beliefs and knowledge of mathematics influence the way they use textbooks. They describe how one teacher had problems in understanding the suggestions in the textbook about a lesson on measurement, because of her insufficient knowledge of the content and of how students make sense of measurement. Thus, curriculum materials may provide ways of organising content, activities and tasks that help teachers in their planning and teaching. However, in order to implement a curriculum that was designed to promote students' deep understanding of mathematics effectively, teachers themselves need the understanding to exploit the potential of texts and other resources.

## Implications and Limitations

The synthesis proposed above demonstrates the multidimensional nature of teacher knowledge and the connections between different categories, but like all models, there are inevitably some over-simplifications. We have already pointed out the ambiguous boundary between SMK and PCK, but feel that the two categories are useful organizing devices in describing teacher knowledge for research purposes, and particularly in devising pre-service and professional development programmes. For instance, an appropriate level of mathematical knowledge will be one of the criteria for entry into a course of training. This may simply be a question of providing evidence of achieving a specific examination qualification, but it could also be probed in an interview. We would also argue for the provision of opportunities for all teachers to work on mathematics relevant to the school curriculum, but at their own level. Later chapters in this book will cover those areas of primary teachers' knowledge, such as fractions, division, and proof, where common weaknesses have been identified. Work on these areas and others, about which less is known, would not only strengthen the resource of SMK from which teachers can draw in teaching, but would also give them opportunities to reflect upon the experience of doing mathematics themselves. Moreover, they may experience different ways of learning mathematics which could impact on their beliefs of how it can be learned.

In many of the projects described earlier, this kind of approach to strengthen SMK has already been adopted. One of the dilemmas here, however, is how the approach is implemented. If implemented through an audit, the dimensions of fear and control may inhibit pre-service teachers and encourage compliance rather than engagement. If the approach is developmental, there is a danger that it receives lower priority than other elements, and is not taken very seriously. Moreover, a developmental approach may be time consuming, expensive and difficult to track as teachers move through their professional careers.

One of the common features of the different models of teacher knowledge discussed here is the largely individualistic assumption which underpins them. Despite the acknowledgement of context, the focus tends to be on the knowledge that an individual teacher brings to a course of teacher education and then into the classroom. This can result in a deficit view of the individual teacher, who at worst needs remediating and at best developing, rather than seeing teacher knowledge as a product of the educational system in which she is located. We cannot assume that the frameworks discussed here are universal. Even if there are some commonalities, there may be great differences in emphasis in various cultural contexts and different priorities for research and development. Switching attention to the system would mean paying more attention to the prior mathematical experiences of teachers and to the resources available to teachers for their own use. In situations where the assessment of subject knowledge is seen as a means of weeding out prospective teachers, more communal ways in which teachers' mathematical knowledge in teaching could be encouraged, whether in training or professional development. This could include paying more attention to the mathematical knowledge which teachers want help with, as well as the elements deemed important by teacher educators. It could also mean more use of peer teaching and development.

Switching the focus from the individual teacher to the system brings in the third element – curriculum knowledge. The curriculum and its associated materials can act as both a resource and a constraint on the teacher. Critical analysis can expose the intentions of both the policy makers and the writers of curriculum materials, whether these materials are for pupils (e.g. textbooks), or for teachers (e.g. teachers' guides). These may be quite different from the knowledge that teachers use and reveal through their practice, either because of inadequacies in the teacher's knowledge base, or because of informed rational choices. We need to know more about how teachers use curriculum materials to improve their teaching, and which curriculum materials are most effective in doing this. For instance, building on the work of Davis and Krajcik (2005) and Stylianides (2007) provides a theoretical framework showing how features of educative curriculum materials may promote teacher learning by enabling them to engage their students effectively in tasks related to proof. These materials may be strengthening teachers' own knowledge of proof through the development of their teaching.

We would also argue that the Knowledge Quartet is a framework that can be used for developing educative curriculum materials. For instance, in terms of the Transformation dimension, curriculum materials can provide several appropriate representations of different ideas. Also, a discussion of the possible advantages or disadvantages of using a certain representation can support teachers in adapting and using the suggested representation in their teaching. This can help make teachers better prepared not only to explain concepts to their students, but also to understand the ways their students make sense of different ideas in mathematics. However, such guidance cannot help every teacher. Educative curriculum materials, like any innovation, are not a solution in themselves. Their effectiveness might be limited by teacher knowledge and beliefs. Nevertheless, Ball and Cohen (1996) showed that when educative elements were included in the guidance provided to teachers about using textbooks, and when these textbooks were used as an important part of teacher training, this resulted in favourable gains in both teachers' SMK and PCK.

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# Chapter 3
# Knowing and Identity: A Situated Theory of Mathematics Knowledge in Teaching

**Jeremy Hodgen**

A group of prospective secondary teachers are engaged in a school mathematics problem involving fractions: *why can you multiply to multiply, but not add to add?* All the prospective teachers are well qualified. In fact, several have doctorates in mathematics. All, of course, can add, subtract, multiply and divide fractions with ease. Yet, they are finding the problem of explanation exceedingly difficult.

How is it that such an apparently elementary problem can cause a group of mathematical experts such problems? Mathematically, the problem involves the algorithms for arithmetic involving fractions. One multiplies the numerators and the denominators to multiply fractions: $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$, but one does not add them to add: *in general*, $\frac{a}{b} + \frac{c}{d} \neq \frac{a+c}{b+d}$. Of course, the problem is that these prospective teachers have never been asked this question before. But the problem becomes more complex when posed in the context of teaching in that it is no longer simply a *mathematical* question (how to show the statement is true), but also a *pedagogical* question (how to enable others to see the statement is true). At the heart of these mathematical and pedagogical questions lie some of the "big ideas" of school mathematics: the notion of rational numbers as division of integers, the relationship between multiplication and addition and the ways in which rational number may be represented. The notion of pedagogical content knowledge (PCK) as developed by Shulman, Ball and others is one response to this complexity: mathematics teaching requires a specialist knowledge of mathematics for teaching that integrates a knowledge of mathematics and pedagogy. These approaches have been discussed in depth in Chapter 1 by Goulding and Petrou in this book. Yet, as Goulding and Petrou indicate, these approaches have downplayed the importance of context. In this chapter, I take this critique further. I examine this issue of context and argue that mathematics teacher knowledge is not simply *applied* within the context of teaching mathematics but is rather *situated* within the complex and social world of mathematics classrooms. In

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other words, to simply focus on application and context is to underplay the ways in which social structures support (or hinder) teacher knowledge and its use.

My analysis draws on what Lerman (2000) terms the 'social turn' in mathematics education. A key work in this social turn is Lave and Wenger's (1991) monograph examining the nature of learning as apprenticeship and re-casting knowledge in terms of situated cognition. Whilst this original work largely considered learning in informal settings outside formal education, it has nevertheless been influential in the formal context of mathematics education (Boaler, 2002; Greeno, 1998), particularly in relation to the perennial issue of how students use or transfer the mathematics learnt in school, into real world contexts. This notion of transfer – how knowledge learnt in one context can be used or applied in a different context – has in turn been the subject of much contentious debate (e.g., Anderson, Greeno, Reder, & Simon, 2000), although it is arguable that this debate has often been characterised more by misunderstandings than by genuine disagreement. As Putnam and Borko (2000, p. 12) argue,

> It is easy to misinterpret scholars in the situative camp as arguing that transfer is impossible—that meaningful learning takes place only in the very contexts in which the new ideas will be used. The situative perspective is not an argument against transfer, however, but an attempt to recast the relationship between what people know and the settings in which they know—between the knower and the known.

From this perspective, knowledge is social and contextualised rather than individual and general, whilst knowledge about mathematics teaching is less about general principles and more about 'intertwined collections of more specific patterns that hold across a variety of situations' (Putnam & Borko, 2000, p.13). It is a recognition of the similarities and differences between these patterns that enables the growth of a more abstract mathematical knowledge.

## The Problem of Mathematics Teacher Knowledge

It appears self-evident that teachers should know about mathematics in order to teach it effectively. But teacher knowledge in mathematics is an area of some controversy. There is evidence that poor subject knowledge in mathematics has a negative impact on teaching (e.g., McDiarmid, Ball, & Anderson, 1989; Rowland, Martyn, Barber, & Heal, 2000). There is considerably less consensus on what constitutes the mathematical knowledge necessary for teaching. Some have argued that improving teachers' knowledge of mathematics per se will lead to better teaching (e.g., Alexander, Rose, & Woodhead, 1992). However, the evidence base in this area suggests otherwise. Several studies, for example, have found no link between teachers' mathematical knowledge, as measured in terms of academic mathematical qualifications, and effective teaching (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997; Begle, 1968). What is clear is that the connection between teacher knowledge and teaching outcomes is neither simple nor straightforward.

To deal with this problem, research has focused on exploring the nature of teacher knowledge in mathematics. One strand of this research has been to link mathematical knowledge for teaching to ways of knowing in the discipline of mathematics. Lampert (1986), for example, distinguishes between *procedural* and *principled* knowledge of mathematics. Procedural knowledge is a rule-guided 'knowing that' and concerns mathematical procedures and their use to compute correct answers. Principled knowledge, on the other hand, is a wider and more conceptual 'knowing how' and includes the knowledge of mathematical concepts that enable the construction of procedures for solving mathematical problems. Lampert's distinction has similarities to Skemp's (1976) distinction between instrumental and relational understandings, Prestage and Perks (2001) learner-knowledge and teacher-knowledge, and Thompson, Philipp, Thompson, and Boyd's (1994) calculational and conceptual orientations.

Increasingly, researchers have argued that mathematical knowledge for teaching is distinct and different to the knowledge necessary to practice mathematics. As I have already noted, a key starting point for much of this work is Shulman's (1986) notion of pedagogical content knowledge which 'goes beyond the subject per se to the dimension of subject knowledge *for teaching* . . . the particular form of content knowledge that embodies the aspects of content most germane to its teachability' (p. 9, original emphasis). The nature of pedagogical content knowledge is itself, however, something of a contested idea within the education research community. McNamara (1991), for example, argues that there is no clear distinction between subject knowledge and pedagogical content knowledge. Indeed, Corbin and Campbell (2001) argue that pedagogical content knowledge is most useful as a metaphor that locates teacher knowledge as embedded within the complex and unpredictable practice of teaching. Another critique is epitomised by Brown and McIntyre (1993), who argue that much of teachers' knowledge is tacit, craft knowledge that cannot be codified as theoretical abstract knowledge. For Brown and McIntyre, the knowledge of an expert teacher is more intuitive and, in a very real sense, less explicit than that of a novice.

Taking this notion of tacit knowledge further, situated theorists problematise the very nature of knowledge, arguing that teachers' mathematical knowledge, like any other form of knowledge, is located in social practice (Greeno, 1998; Putnam & Borko, 2000). Hence, in a development of Lave and Wenger's (1991) work, Adler (1998) refers to a dynamic, contextualized and active process of 'knowing' rather than the more static, abstract and passive notion of 'knowledge'. Thus, teacher knowledge is *embedded* in the practices of teaching and any attempt to describe this knowledge abstractly is likely to fail to capture its dynamic nature.

## A Case Study from Primary Mathematics: Alexandra's Knowledge of the Multiplication and Division of Fractions

In this section, I discuss the case of Alexandra,[1] a primary teacher, and her knowledge of proportional reasoning. I contrast Alexandra's knowledge of

[1] Alexandra is a pseudonym.

proportional reasoning in the context of developing lessons and leading professional development sessions with her knowledge in the context of a structured mathematics interview.

This case study is drawn from a 4-year longitudinal study into the professional change of six teachers involved as teacher-researchers in the Primary Cognitive Acceleration in Mathematics Education (CAME) Project research team (Johnson, Hodgen, & Adhami, 2004). This team consisted of four researchers, four teacher-researchers and the Local Education Authority mathematics advisor. During the school year 1997/1998, the team met fortnightly to develop Thinking Maths lessons specifically for primary children aged 9–11. During the second phase of the project, over the school years 1998/1999 and 1999/2000, a further cohort of teachers joined the project to implement the Thinking Maths lessons more widely. In Phase 2, the teacher-researchers led professional development sessions aimed at enabling this new cohort of teachers to teach the Thinking Maths lessons in their own classes.

In the research study, data collection was qualitative using multiple methods, including observations of seminars, lessons and professional development sessions, interviews with individuals and groups, and structured mathematical interviews (adapted from Millett, Askew, & Simon, 2004).[2] Here, my focus is on the mathematics interview, which took place in December 2000. During this interview, Alexandra was asked to solve several problems and to suggest models, stories, or diagrams to use when teaching the ideas to children. The questions themselves largely related to two aspects underlying the elementary mathematics curriculum: rational number and multiplicative reasoning. I focus on three related questions from this interview:

> How would you solve these problems? What would be a good story, diagram or model for them?

$$0.5 \times 0.2 \quad 3 \div 0.75 \quad 1\frac{3}{4} \div \frac{1}{2}$$

I was particularly interested in the extent to which the teachers could generate a variety of appropriate and pedagogically useful illustrations, and in the range of different meanings of multiplication and division that they drew upon. Ma (1999), for example, describes three models of division: measurement, partitive, and factors and product. These broadly relate the understandings of multiplication in terms of repeated addition, scaling and arrays (and areas). There is extensive research evidence to suggest that the area model is used in only limited ways in UK primary (and secondary) mathematics classrooms (Nunes, 2001).

As a Primary CAME teacher-researcher, Alexandra was involved in the development of a number of lessons addressing students' misconceptions in collaboration with other teacher-researchers and academics. Specifically, she, together with another teacher, developed two lessons focusing on fractions: 'Share an Apple', and

[2] This drew on previous work at King's (Askew et al., 1997), which in turn drew on a range of sources. An item on division of fractions, $1\frac{3}{4} \div \frac{1}{2}$, for example, was drawn from Ball (1990) work and is also discussed in Ma (1999).

'Halving and Thirding' (Johnson et al., 2003). In Share an Apple, the focus is on representations and comparisons of fractions. So, for example, children are asked to consider various ways of representing and comparing the magnitude of simple fractions of everyday objects. In Halving and Thirding, the focus is on developing and connecting different representations for the multiplication of fractions, including repeated multiplication by $^1/_2$ and $^1/_3$, with a particular focus on developing the area model for multiplication and linking this to other representations. Alexandra herself suggested this focus on the area model based on her experiences of team-teaching the lesson. She also led the professional development sessions introducing these lessons and had contributed to an academic paper on their development.

The research reported here took place in the context of the National Numeracy Strategy (NNS) in England, a national initiative focused on primary mathematics pedagogy (Brown, Millett, Bibby, & Johnson, 2000). One feature of the NNS was the appointment of several hundred Numeracy Consultants. The role of these local primary mathematics specialists was to support teachers and to deliver professional development to them. Throughout much of her participation in Primary CAME, Alexandra was also a Numeracy Consultant, whose responsibilities included delivering training aimed at enhancing primary teachers' subject knowledge of mathematics. In this mathematics educator role, I observed her teach several National Numeracy training sessions on both fractions and multiplication, during which she appeared to be fluent with a variety of both techniques and representations. In addition, in response to her perceptions of weaknesses in these training materials and in collaboration with another Numeracy Consultant, she developed a further session for teachers in which she focused on the use of the area model of multiplication in relation to fractions together with the concept of equivalence.

Given these experiences, I had expected Alexandra to demonstrate a sophisticated understanding of multiplication in the mathematics interview. Yet, her knowledge appeared to be very significantly weaker in this setting: in the interview, she appeared to know 'less' and to know it less securely. Alexandra could successfully answer all the questions performing most of the necessary calculational procedures correctly, although on several questions this took a considerable amount of time and whilst solving the problems, she made several mistakes which she corrected during the interview. At one point, she indicated some awareness of her limited understanding referring to division by fractions [$1^3/_4 \div {}^1/_2$] as follows: "If I was doing that the way I was taught to do it, I would just turn that all upside down. And I have real problems with this idea of division by fractions." However, she was unable to carry out this procedure and solved the question by converting to decimals mentally, and then using a calculator. To solve 0.5 × 0.2, she used a standard multiplication algorithm, as in Fig. 3.1.

$$
\begin{array}{r}
0.5 \\
\times\ 0.2 \\
\hline
1\ 0 \\
0\ 0\ 0 \\
\hline
0.1\ 0 \\
\hline
\end{array}
$$

**Fig. 3.1** Alexandra's procedure for solving 0.5 × 0.2

As she carried out the algorithm, she commented on how she knew where to place the decimal point in the product: “There are two decimal places in the question, so there must be two decimal places in the answer.” This, together with her inclusion of the multiplication by zero, strongly suggests that her understanding of this method is certainly heavily reliant on procedural knowledge.

Although Alexandra read the answer correctly as 0.1 and used the same form as in the question, she did not notice that this could be read as a tenth or that the calculation was equivalent to either of the relatively simple ‘half of two tenths’ or ‘half of a fifth’. Hence, she appeared to have no strategy to check, or make sense of the result of this calculation procedure. Indeed, she could not generate an illustration of this problem. Whilst she did not get this problem ‘wrong’, her knowledge did appear to be partial and limited.

Alexandra found the generation of any models extremely difficult and required considerable support and prompting to tackle these questions. Indeed, she asked me, with apparent disbelief, if I could do it. She provided a single story for just two of the three problems. Reflecting her preference for decimal fractions, she found $3 \div 0.75$ relatively straightforward, after I had suggested thinking about contexts involving measures: “how many lots of 75 pence can you get from three pounds.” However, she had considerable difficulty with $1^3/_4 \div {}^1/_2$, eventually producing the following story:

> If you said that was one, and that was three quarters you’d get three halves and half a half out of it. But that’s not very helpful is it? . . . One, OK, that’s one and three quarters, so you can get one, two, three. Three halves out of it. And half of a half.

The example is more ‘helpful’ than Alexandra suggests. Repeated addition does provide one satisfactory explanation for the answer. The ‘pure’ mathematics context of numbers is, in this case, rather more helpful than the commonplace use of pizzas or cakes to illustrate problems involving fractions. Yet, whilst Alexandra’s subject knowledge here appears stronger than any of the US teachers in Ma’s (1999) study, this example does highlight a problematic issue. Alexandra had developed the two fractions lessons with the specific aim of enabling children to develop a range of models for the representation of fractions. The Halving and Thirding lesson had used both measurement and area representations for the multiplication of fractions, an aspect of the lesson which she herself had highlighted several times during the lesson simulation to Phase 2 teachers. It is somewhat surprising that, given these fairly intense lesson development experiences together with her experiences as a mathematics educator, she was not able in the interview to draw on the area model to division by fractions, or more significantly, to the multiplication of decimal fractions. Indeed, she was unable to provide an illustration of $0.5 \times 0.2$. More surprising still is her reaction to being asked to think of models, given that I had observed her emphasise different meanings of multiplication and division (including repeated subtraction/addition and the area/array models) and the need to understand children’s different ways of seeing mathematical relationships when leading training sessions. Of course, this does not mean that she did *not know* other models. However, the difficulty that she encountered generating these stories does suggest that she lacked an intuitive familiarity with these and different models of multiplication/division. Alexandra's failure to draw on her experiences of developing the fractions lessons suggests that her knowledge was highly contextualised.

This case presents a dilemma. This is not a case where transfer has failed. Faced with these interview problems in other situations, Alexandra 'knew' more and performed 'better', and thus, in these different contexts, her mathematics appeared 'good enough'. Certainly, her knowledge *in context* appeared stronger than her knowledge *out of context.* In order to make sense of this, Lave and Wenger's insights about situated cognition are helpful. Alexandra's mathematics knowledge for teaching had developed in large part within the context of teaching, teacher education and curriculum development. Contrary to common wisdom, this knowledge was *situated*; it was 'known' in the context of teaching.

In the context of teaching, Alexandra 'knew', for example, about different models for the multiplication of fractions in the context of lesson development and, as a tutor during INSET sessions, when such knowledge was explicitly part of her role. Significantly, these were the settings where she was working in collaboration with others, and she had access to lesson or course guidance. She was not simply a passive participant in these contexts, nor was she simply 'delivering' the pre-prepared course materials. In fact, Alexandra's knowledge appeared to be relatively strong in both settings, and at least as strong as that of Numeracy Consultants in general[3]: it allowed her not only to participate in the discussions within the research team, but also to respond authoritatively to teachers' questions. However, Alexandra's knowledge in teaching did not simply derive from a more general individual mathematics knowledge. Rather, her knowledge in teaching was supported by the social communities and relationships in which she acted as an expert. These communities provided the cognitive and discursive tools with which Alexandra could be knowledgeable mathematically. It was distributed in the sense that it was 'stretched over' (Lave, 1988) and supported by other individuals and artefacts, in particular, lesson materials and structures. In other words, in being situated, her knowledge was both *social* and *distributed* (Putnam & Borko, 2000).

It is important to recognise that the interview was something of a 'testing' and artificial situation. The problems posed were deliberately 'tricky', and the situation raised issues of mathematics anxiety. Alexandra certainly seemed to perceive the interview as something of a threat to her professional identity. It is quite possible that in normal classroom contexts, Alexandra would be less unsettled. However, an important aspect of teacher knowledge is that it can act as a resource to enable a teacher to act in an unpredicted or unexpected situation. Thus, the situated, social

[3]My evidence here is partly based on my own observations and partly based on evidence gathered for the Leverhulme Numeracy Research Programme, an extensive 5-year longitudinal study of primary mathematics covering the period of the introduction of the NNS and the appointment of Numeracy Consultants (Millett, Brown, & Askew, 2004b).

and distributed nature of a teacher's mathematical knowledge for teaching may hinder the teacher's ability to respond appropriately in novel contexts, for which the teacher does not have an instant recourse to support her knowledge.

## Is this just an Issue for Primary Teaching?

The literature on teacher knowledge is dominated by research in primary/elementary education, and one could be forgiven for concluding that the problem of teacher knowledge is primarily an issue in this sector. In a sense, this emphasis is unsurprising since the problem of teacher knowledge is brought into sharp focus in a sector where the majority of teachers are generalists[4] and primary teachers generally have considerably less formal education in mathematics. As a result, their mathematical knowledge is likely to be weaker and more influenced by contextual factors. Certainly, most *mathematically trained* secondary teachers' mathematical knowledge is likely to be rather more secure than that of most primary teachers, particularly when it comes to solving school mathematical problems of the sort Alexandra was asked to solve.[5]

The evidence, whilst less extensive, suggests that secondary teachers' knowledge is no less situated. Thompson and Thompson (1994), for example, describe a middle-school specialist teacher whose knowledge of rate and speed was strong and fluent: he himself could solve classroom problems with ease. Yet this very fluency was a barrier to teaching. When observed teaching a student one-to-one, the teacher conceived, albeit implicitly, of speed in terms of the covariance of distance and time, whilst the student's understanding was additive and discrete. The student did not have an image of motion as the simultaneous accumulation of distance and time (i.e. direct proportion). The teacher's own connections between representational structures and 'calculational' procedures for solving the problem were so strong that, when working with a student, he "*saw* (i.e. imputed) appropriate reasoning any time [the student] employed an appropriate calculation" (p. 299, emphasis in original). In a later analysis, they argue that the teacher's understandings of division and proportionality were so "packed" that they were "insensitive to conceptual subtleties in the situations" (Thompson & Thompson, 1996, p. 4).

One aspect of the power of mathematics lies in this "packed" and abbreviated nature. A fluent mathematician can choose the most appropriate representation for solving a problem irrespective of whether this representation is actually appropriate for modelling this particular problem. The essence of teacher knowledge involves an *explicit* recognition of this – "unpacking" the mathematical ideas (Ball & Bass,

[4]I recognize that there are a number of educational systems internationally (e.g. in Israel) in which there are specialists teachers of elementary mathematics. Nevertheless, the generalist remains the norm.

[5]Not all secondary teachers of mathematics are mathematically trained, of course. In England, for example, a significant proportion of them have weak mathematics qualifications, particularly those teaching lower secondary mathematics (Johnston-Wilder et al., 2003).

2000). On the other hand, doing mathematics only requires an *implicit* recognition of this. Indeed, fluency in mathematics arguably involves developing such implicit understandings. To a competent mathematician, the nuances of meaning inherent in these different pedagogical representations of mathematics can seem trivial and unimportant.[6] Hence, it may be that there is a tension for many secondary teachers of mathematics in that some aspects of mathematics knowledge for teaching run counter to the habits and norms of mathematics as a discipline.

This is not to argue that mathematics knowledge does not matter, but rather that mathematical knowledge is not sufficient in isolation. Lloyd and Wilson (1998) discuss how a teacher's sophisticated understanding of functions enabled him to implement an innovatory reform-focused curriculum. Lloyd and Wilson's teacher had previously taught a traditional curriculum for 14 years. They argue that the teacher's rich and well-articulated mathematical knowledge enabled innovation, but only in the context of curriculum materials and a related professional development programme that supported the innovation. Like Alexandra, Lloyd and Wilson's teacher's mathematical knowledge for teaching was supported by artefacts and social structures. Unlike Alexandra, his knowledge was also supported by a rich understanding of mathematics.

## The Contribution of Situated Theories: What Does This Mean for Teacher Knowledge?

There is no doubt that Shulman's (1986) pedagogical content knowledge and the work of Ball and others provide a very significant contribution to understanding teacher knowledge. However, the analysis that I have presented here strongly suggests that mathematics teacher knowledge is very much more deeply embedded in practice than the PCK literature generally acknowledges. Whilst subsequent work has emphasised the aspects of Shulman's work that attempt to codify teacher knowledge, it is often overlooked that he did examine the forms of knowledge. This neglected area of Shulman's work relates to the way teacher knowledge is 'held' and used in teaching. Shulman conceives of knowledge as involving propositional, case and strategic aspects. These are discussed in some depth by Goulding and Petrou in Chapter 2. The case and strategic aspects of knowledge do certainly go some way towards recognizing the interrelationship between knowledge and its use. Shulman conceives of teaching "*theory* through cases" (p. 11). Further, he suggests that the strategic may be better captured as a process of "knowing" rather than the more static "knowledge" (p. 14) and argues that this "comes into play as the teacher confronts particular situations or problems whether theoretical, practical or moral, where principles collide and no simple solution is possible" (p. 13). This aspect

[6] See Saunders (1999) for an example in which a professional mathematician rejects the pedagogical distinction between fractions as operators and quantities as "playing tricks" (p. 3) and indicative of the de-professionalisation of teachers.

of Shulman's work provides many insights, particularly regarding the application and use of teacher knowledge. Nevertheless, it is a largely individual conception of knowledge. One consequence is a negative focus on the problem of teacher knowledge in terms of finding and fixing individual deficits (Askew, 2008). However, some aspects of mathematics teachers' subject knowledge are more difficult than others to pin down and codify. Almost inevitably, the focus on knowledge is concentrated on the more easily describable ideas (e.g. number facts) with much less emphasis placed on the more ephemeral but equally important ideas that Yackel and Cobb (1996) term socio-mathematical norms such as symmetry.

In viewing knowledge as situated, social and distributed, the situated perspective presents a significant advance. A major contribution is that this approach places much greater emphasis on the communities in which mathematics teachers are engaged rather than on individual knowledge. In principle, it is certainly desirable for teachers to 'possess' a sophisticated knowledge of mathematics for teaching that is evident in a variety of contexts, both inside and outside the classroom. I have little doubt, for example, that, were the gaps in Alexandra's mathematics knowledge to be addressed, her knowledge of mathematics in teaching – and her teaching of mathematics – would also improve. However, it is also important to bear in mind that the key setting in which teachers 'use' and 'apply' their mathematics knowledge is in the classroom. In Alexandra's case, there certainly were significant gaps in her knowledge of rational number, as evident in the mathematics interview. But ultimately, the quality of a teacher's mathematical knowledge in interview situations does not matter in itself, except possibly for research purposes. What does matter is that a teacher's mathematical knowledge as situated in teaching contexts is sufficient for successful learning to occur. The evidence presented here suggests that classroom knowledge is not a straightforward contextualisation or application of a more abstract and general a priori mathematical knowledge.

A second contribution relates to the nature of learning. Adler (1998) argues that becoming a mathematics teacher involves learning to talk both *within* and *about* mathematics teaching and learning, rather than simply learning new knowledge. In their study involving a group of mathematics teachers from one middle school, Stein, Silver, and Smith (1998) similarly highlight the importance of story and narrative in restructuring and reworking knowledge about mathematics teaching. They see this restructuring of existing knowledge and experience as more important than the acquisition of new knowledge – echoing Askew et al.'s (1997) findings about the importance of teachers' beliefs about mathematics in the teaching of numeracy. Stein et al. (1998) place these notions of story and narrative in the context of teachers' professional identities, arguing that teacher learning is best conceived of as a process of identity change.

One criticism of the situated learning literature, and in particular the work of Lave and Wenger, is that the context is conceived of as relatively static and fixed. Hence, individual learning can appear as following fixed and predictable trajectories of learning. Holland, Lachicotte, Skinner, and Cain's (1998) conceptualisation of identity in terms of agency and social structure provides a way of understanding the unexpected and surprising nature of learning. In an analysis of students' mathematical identities, Boaler and Greeno (2000) relate Holland et al.'s (1998) conception of identity to Belenky, Clinchy, Goldberger, and Tarule's (1986) notions of authority and knowing. They link procedural knowing to an acceptance of external authority in mathematics; and conceptual or principled knowing, to a more questioning and critical stance – the need to 'know why'. Similarly, Povey, Burton, Angier, and Boylan (1999) discuss how developing an authorial stance towards mathematics enables teachers to develop such a critical stance. Hodgen and Johnson (2004) examine teacher motivation and the reasons why teachers participate (or do not participate) in learning about mathematics education, arguing that the motivation to change is inextricably linked to teachers' identities and the social context in which they are located. Focusing on the aforementioned case of Alexandra, they discuss how the context of a school mathematics lesson prompted her to make an explicit connection between spatial and numerical representations (seeing the Cartesian system as "like a 2D number line", p. 236) and to 'see' the mathematical nature of diagrams and representations of fractions. Clearly, these are key components of mathematical knowledge for teaching, but Hodgen and Johnson conceive of her learning as an *authorial* choice in response to the particular demands of circumstance. This focus on identity highlights part of the difficulty of teacher learning. Bartholomew (2006), for example, uses the notion of the 'defended self' to highlight how mathematics teachers may resist learning because they perceive it as a threat to their being. Hodgen and Askew (2007) suggest that imagination plays a key role in overcoming such threats, thus developing and transforming teachers' relationships with and knowledge of (school) mathematics.

A third contribution relates to the analysis of learning settings. The situative perspective is often seen as providing a critique of current practices in schooling rather than offering an alternative vision (Lerman, 2000).[7] Greeno's (1998) work, however, provides a useful method of analysing learning situations. He highlights the importance of understanding the constraints and affordances: constraints that enable participants (teachers and learners) to predict and anticipate activities and outcomes; affordances that provide opportunities for participants to draw on practices from elsewhere. Boaler (2000) highlights the importance of the social context of learning. In a re-analysis of her study of open-ended and traditional approaches to school mathematics (2002), she describes how the students, who experienced the open-ended approach, more easily related school mathematics to out-of-school contexts in part because of the similarities in the way mathematics was practiced. In a similar vein, Lave (1992) argues that much problem-solving in schools is not authentic: in contrast to the messy and complex problems of the real world, school mathematics problems tend to be straightforward and routine. But Putnam and Borko (2000, pp. 4–5) argue that the problem of authenticity is related to the authenticity of learning rather than necessarily to the authenticity of problems themselves: "Authentic activities foster the kinds of thinking and problem-solving skills that are important in out-of-school settings, whether or not the activities themselves mirror what practitioners do". This highlights the two-fold problem of authenticity in mathematics.

[7] See, for example, Lave and Wenger's (1991) rather brief and simplistic critique of school education.

Mathematics teaching involves two stages of re-contextualisation of mathematics knowledge: a re-contextualisation of teachers' own mathematics learner knowledge for the classroom to enable students to re-contextualise this classroom mathematics for out-of-school contexts.

## Implications for the Practices of Teaching, Teacher Education and Development

Recognising the situated nature of mathematics knowledge suggests that focusing exclusively on mathematics knowledge in isolation from the classroom context is unlikely to be effective. In developing strategies directed at improving teacher knowledge, there is a need to examine the contextual constraints and affordances which help or hinder teachers to act knowledgably in the classroom (Greeno, 1998). A crucial issue is to examine how collective knowledge can be harnessed to support an individual teacher's mathematical knowledge in the classroom.

One productive strategy is to provide tools that focus on the use of teacher knowledge in the practice of teaching mathematics. One constructive tool of this kind, the Knowledge Quartet, is described in some depth elsewhere in this book, particularly in the Chapter 12 by Turner and Rowland. To date, like much of the literature on mathematics teacher knowledge, this approach has focused on primary or elementary mathematics teaching, although there is every reason to suggest that this approach could be useful in secondary teaching and teacher education. In particular, key aspects of the Knowledge Quartet resonate with active research topics in secondary mathematics, including the choice and construction of mathematical examples.

A second implication relates explicitly to the social aspect of teacher knowledge. If teacher knowledge is supported by social structures and relationships, then it is likely to be productive to focus on developing shared expertise rather than individual 'knowledge'. The efficacy of collaborative approaches to mathematics teacher education is well-established (e.g. Clarke, 1994) and the situated perspective lends further theoretical weight to such approaches. Millett, Brown and Askew (2004a) highlight the importance of the professional community of teachers in a school and find that some primary schools appear to be able to successfully 'share' mathematics knowledge and expertise amongst a group of teachers through a mathematics co-ordination team.

A third implication concerns lesson materials, textbooks and, more broadly, the distributed aspects of teacher knowledge. There is certainly an urgent need to examine how textbooks and other materials can best support teacher knowledge in the practice of teaching. However, there is a great deal of evidence that materials on their own are insufficient (e.g. Askew, 1996). Spillane (1999) argues that for professional change of any significance, mathematics teachers need social spaces in which they have access to "rich deliberations about the substance . . . a practising of reform ideas with other teachers and reform experts includ[ing] material resources or artefacts that support [these] deliberations" (p. 171). Looking at Alexandra's subject knowledge development, one of the significant features was her engagement in lesson development (Hodgen & Johnson, 2004; Johnson et al., 2004). There has been a great deal of focus on translating the Japanese practice of lesson study to a Western context. But actually, this may be a misguided attempt to transfer a very contextualised cultural practice. What made a difference for Alexandra was not lesson study per se, but rather the more general practice of lesson development carried out in collaboration with others: constructing pedagogic strategies, examples, tasks, etc. that enable students to do and learn mathematics. Key to this is that lesson development is not merely a pedagogic exercise; it necessitates the investigation and exploration of topics from school mathematics, as described in Chapter 5 by Watson and Barton. That such apparently simple and elementary topics can challenge mathematical experts, is clear from the example of prospective secondary teachers cited at the beginning of this chapter.

A fourth implication relates to identity, care and relationships. For many primary teachers, the problem of maths anxiety is well-documented (Bibby, 1999). However, simply reducing anxiety and enabling teachers to 'feel better' about mathematics can lead to complacency (Askew, 1996). Askew and I have argued that teachers' knowledge of mathematics is both intellectual and emotional (Hodgen & Askew, 2007). The motivation to do mathematics – or to teach mathematics – is both individual and social. This is as true for well-qualified and knowledgeable secondary teachers, as it is for primary teachers. However, interventions related to teachers' knowledge of mathematics have generally focused on cognitive and pedagogic issues: teachers' mathematics subject knowledge, how children learn and teaching approaches. These issues are, of course, important, but the importance of identity in coming to know as suggested by the situated perspective, implies that such an approach is doomed to failure unless placed within an affective frame in which teachers have space to question and enjoy mathematics and mathematics teaching. In analysing mathematics subject knowledge, for example, Askew (2008) presents a convincing case for a focus on the big ideas – or socio-mathematical norms – of precision and generalization, as well as the romance of the subject.

Finally, there are implications for research into mathematics teacher knowledge. There is an increasing interest in the measurement of teachers' mathematics knowledge and the relationship with student learning (Hill, Rowan, & Ball, 2005). However, the situated perspective suggests that problem goes beyond this issue of codification in that teachers' knowledge is not only situated but also social and distributed. The testing of individual teachers is likely to focus on de-contextualised mathematics knowledge which, as in the case of Alexandra above, may be very different from their classroom knowledge. Nevertheless, the issue of how mathematics teacher knowledge is enacted and the relationship with classroom practice remains poorly understood, and research in this area, like the research in mathematics teacher education generally (Adler, Ball, Krainer, Lin, & Novotna, 2005), is largely limited to small scale studies. Given the analysis above, approaches that focus on the notion of re-contextualisation (Adler & Davis, 2006) may offer insights in this area.

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# Chapter 4
# Changed Views on Mathematical Knowledge in the Course of Didactical Theory Development: Independent Corpus of Scientific Knowledge or Result of Social Constructions?

**Heinz Steinbring**

## Introduction

This contribution tries in an exemplary way to look at the case of the historical development of important tendencies in mathematics education (Mathematikdidaktik) in Germany in the last 40 years. This description can only follow one line of development; it cannot and will not summarize other research approaches existing in mathematics education in Germany. A major concern is to investigate the clarification process of the central objects of mathematics education research and to analyze the important role that the content matter 'mathematics' plays for teaching and learning processes. The main interest of this paper is to better understand the special German case of how theoretical considerations for mathematics education developed, changed and expanded. This development cannot and will not explain a universal, all-embracing theory of mathematics education, but it reflects one important German tradition (without looking here at other traditions) and is an example of a theoretical evolution of mathematics education. Within this historical development, there are to be found strategies of comparing, contrasting and of (locally) integrating theories and theoretical aspects.

> 'Mathematics learning' as an object of didactical considerations has consistently over time been regarded as the triad 'Learner – Teacher – Learning/Teaching-Content'. In pedagogics, these three elements are labeled as the 'didactical triangle' since Friedrich Herbart (1776–1848) (see Peterßen, 2001, p.140, and Künzli, 2000, pp. 48–49). According to Herbart "... education within instruction does not [take place] in the immediate relationship between educator and pupil, but educator and pupil [enter] into an indirect relationship to each other. Between them stand the instruction objects"[1] (Peterßen, 2001, p. 140, translated by H.S.).

In mathematics education (in Germany), the didactical triangle (see Fig. 4.1) has a long tradition. The vertices for mathematics education represent: (1) the mathematical knowledge, (2) the student, and (3) the teacher (cf. Steinbring, 1998a).

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[1] One can suppose that this is where the famous didactical triangle originates.

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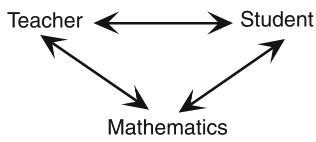
**Fig. 4.1** The didactical triangle




The schema of the didactical triangle with its three elements will be used as a kind of 'test instrument' for the following considerations and analyses. By using this triangle, the following orientating questions will be asked for the elaboration of the changes and developments in mathematics education:

(A) Which explicit and implicit (unconscious) concepts and role descriptions exist about the three 'elements': mathematics, teacher and students?
(B) Which explicit and implicit (unconscious) concepts and role descriptions exist about the relationships or interactions between the three 'elements': mathematics, teacher and students?
(C) What is explicitly or implicitly (unconsciously) seen as the central and crucial means (among the three 'elements': mathematics, teacher and students) of positively influencing and improving the learning process?

These questions, combined with the resulting insights and answers, will help to provide a more fully differentiated picture of which research concepts and objects have been predominant in mathematics education in the course of its historical development, and how the role and nature of (school) mathematical knowledge has changed and been redesigned in the course of this development.

## The 'Stoffdidaktik' Elaboration of Mathematical Knowledge as an Essential Factor Influencing Teaching and Learning Processes

Until the mid-1960s, the emphasis in Germany was on didactical works and analyses which concentrated on school-mathematical knowledge, its didactical elementarization and on subject matter aspects. These works were essentially linked to mathematics as a pre-given content for learning and instruction, and specific features of a genuinely mathematics education research approach had not yet become noticeable in them. Especially within the German-speaking countries, this didactical research paradigm developed as *stoffdidaktik*[2] (Content based Didactics).

[2] In this paper 'stoffdidaktik' is restricted to a certain fundamentalist form of content-related mathematical analysis based on ideas from the New Math era. Later, there were further developments and modifications of the *stoffdidaktik* approach – no longer explicitly linked to the New Math era – that relate the analysis of mathematical content knowledge to the learning processes of students. These kinds of *stoffdidaktik* still exist; there are also types of *stoffdidaktik* that emphasize the *epistemological* analysis of mathematical content matter.

> ... 'stoffdidaktik' is dominated by too simple a model to solve didactical questions and research problems. [It] acts on the assumption that mathematical knowledge – as researched and developed in the academic discipline – is essentially unchanged and absolute .... Though 'stoffdidaktik' in the meantime notices the problems of understanding that students have in learning, and accordingly it specifically proceeds to prepare the pre-given mathematical disciplinary knowledge for instruction as a mathematical content, to elementarise it and to arrange it methodically; yet the principle remains unchallenged that mathematical knowledge represents a finished product, and that the teaching-learning-process can be organised linearly, emanating from the content, over the teacher, into the students' heads, and can ultimately be controlled and influenced at every step by mathematics educators (Steinbring, 1997, p. 67, see also Steinbring, 1998a, and Steinbring, 1998b, pp. 161–162).

Under the abbreviated label *stoffdidaktik*, this direction was especially represented by 'didactically oriented content analysis'.

> The research complex of didactically oriented content analysis (Sachanalysen) has lately engaged mathematics education in the Federal Republic of Germany in a particular way .... The research methods of this area are identical with those of mathematics, so that outsiders have sometimes gained the impression that, here, mathematics (particularly elementary mathematics) and not mathematics education is being conducted .... The goal of 'didactically oriented content analysis' which essentially follows mathematical methods is to give a better foundation for the formulation of content-related learning goals and for the development, definition and use of a differentiated methodical set of instruments. (Griesel, 1974, p. 118, translated by H.S.).
>
> What does progress in mathematics education depend upon? 1. Upon the state of development of the analysis of the content, the methods and the application of mathematics. 2. Upon didactical ideas and insights, which make it possible to attend better, or at all, to a subject area within instruction. (Griesel, 1971, p. 7, translated by H.S.).

Griesel names four further influential factors (general experience and statistically based evidence about instruction, insights into the mathematical learning process, development-psychological and sociological conditions); yet the didactical work on the 'content' is the most important.

In a critical comparison between (German) 'didactically oriented content analysis' and (French) 'ingénierie didactique', Strässer (1994) states that *stoffdidaktik* ultimately pursued the goal of elaborating school-mathematical subject areas – similar to mathematical areas in Bourbakism – in a logically consistent way and built upon unambiguous foundations. As an example, Strässer quotes from the foreword to the two-volume book by G. Holland *Geometry for Teachers and Students (1974/1977)*:

> This book arguably offers the reader a complete axiomatic composition of the Euclidian geometry of the plane, which in its system of concepts as well as in the choice and organisation of the geometrical contents orientates itself as much as possible to contemporary geometry instruction in school (Holland, 1974, p. 7, translated by H.S.).

An archetype for *stoffdidaktik* was uniform mathematics, as it was exemplarily given by Bourbaki and then by the so-called New Mathematics. Connected with this archetype of uniform, axiomatic mathematics, the illusion for work in *stoffdidaktik* was that mathematics for teachers, students and pupils (i.e., school-mathematics) could also ultimately be elaborated in a logically correct and consistent manner, definite and absolute for all teaching and learning processes.

> The whole of mathematical knowledge, ordered in this way, is, in principle, describable with a single, universal language. This uniformity ... means essentially that the elementary concept of the number '5' and the more abstract concept of the 'expectation of a binomially distributed random variable' are objects at the same level of description by mathematical set-language. This product of the mathematical knowledge corpus reflects the preoccupations of the historical period during which it originated; its logical clarity, the construction from the simple to the complex and abstract, as well as its uniform language, are together imagined to provide the ideal preparation of knowledge for its acquisition and its understanding – as was also for example the maxim of the movement of so-called 'New Mathematics' (Steinbring, 1998b, p. 161).

The *stoffdidaktik* work undertaken focused initially on the school mathematics of higher school grades (especially grammar school, the German Gymnasium covers the grades 5–13, age 10–19); then, at the end of the sixties, with didactical works in the frame of the movement of 'New Mathematics' (especially the works of Z. P. Dienes), it was extended to mathematics instruction in primary school (grades 1–4, ages 6–10).

> The modernisation of mathematics instruction in primary school only started much later, about the year 1966, when the inventive ideas of Z. P. Dienes became familiar .... We can speak of a modernisation of mathematics instruction in primary school and in grades 5 and 6 ... (Griesel, 1971, p. 8, translated by H.S.).

For a summary, characterizing the position of *stoffdidaktik* as described in this paragraph, the three aforementioned questions (A, B and C) shall now be consulted and answered in a general way. About the mathematical content, there clearly is the conception that ultimately a uniform, objective and unchangeable content of teaching and learning is to be elaborated in didactics according to the paradigm of scientific mathematics. The teaching, learning and understanding processes of the participating persons (teacher and students) are orientated around the rigid subject matter structures: the teacher is the 'conveyor' of the didactically prepared content to the student(s) who are seen as passive receivers. The relations between the three elements of the didactical triangle are of an essentially linear nature: the mathematical knowledge arrives by means of the preparation and transfers from the teacher to the students. In the research paradigm of *stoffdidaktik*, the scientific elaboration of mathematical knowledge is the central and crucial means practiced for steering and optimizing mathematical instruction, learning and understanding processes.

## The Synchronization Between the Dynamics of Knowledge Development and the Processes of Teaching and Learning

The international criticism of New Mathematics (Kline, 1973) led also in Germany to a long-term critical altercation with New Mathematics. Furthermore, the scientific debate about the status and the objects of a science genuinely concerned with mathematics education took place over a longer period (Steiner in ZDM, 1974; Winter, 1985; Wittmann, 1992). One prominent voice, Winter (1985, pp. 80–81), states:

> So-called Sachanalysen ('didactically oriented content analyses') can have a downright calamitous effect on the school reality, if they refer reductionistically solely to mathematics (perhaps even to assumed mathematics) and fade out other essential constituents of learning mathematics .... [One] inevitably encounters problems of the goals and forms of learning itself, which are not, or hardly, explained in the Sachanalysen .... In general: Sachanalysen are in danger of losing focus on the outer-mathematical reality and thus on the students' experience of the world, and this is only one pedagogical sin of such reductions (Winter, 1985, p. 80/81, translated by H.S.).

The relation between mathematical learning content and teaching and learning processes did not work in the way imagined from the perspective of *stoffdidaktik*. A new perspective on the subject matter content needed to be developed which took the sequential development and dynamics of teaching and learning processes into account. Freudenthal (1973, p. 114) emphasized the process character of mathematics for learning in a paradigmatic way:

> It is true that words such as mathematics, language, and art have a double meaning. In the case of art it is obvious. There is a finished art studied by the historian of art, and there is an art exercised by the artist. It seems to be less obvious that it is the same with language; in fact linguists stress it and call it a discovery of de Saussure's. Every mathematician knows at least unconsciously that besides ready-made mathematics there exists mathematics as an activity. But this fact is almost never stressed, and non-mathematicians are not at all aware of it.

Mathematics, as an activity, implies that learning becomes an active process in the construction of knowledge.

> The opposite of ready-made mathematics is mathematics in statu nascendi. This is what Socrates taught. Today we urge that it be a real birth rather than a stylized one; the pupil himself should re-invent mathematics .... The learning process has to include phases of directed invention, that is, of invention not in the objective but in the subjective sense, seen from the perspective of the student (Freudenthal, 1973, p. 118).

Development processes are not uniform, universal or homogeneous. Subjective characteristics of those people keeping the process going, as well as situated representations, notations and interpretations of mathematical knowledge, are manifold, divergent and partly heterogeneous. Further, cultural contexts, subjective influences and situated dependencies are both active and inevitable; such are the reasons for an observable diversity and non-uniformity of the emerging knowledge.

The contrast between uniform scientific mathematics (oriented towards a generally valid (research) product) and the different perspectives and interpretations of mathematics produced in social environments for different application domains (tied up in situatedly-framed development processes) becomes extremely apparent against the background of the different cultures in which mathematical knowledge is used and experienced. The culture of the researching and teaching mathematician and the culture of mathematics teaching face one another in an obviously distinct, and sometimes opposing, way. The role the Bourbakist mother structures play for the unity of mathematics cannot be understood by mere appropriation of the principles given by these structures. The culture of mathematical science and the historical development of mathematics form the necessary background for an understanding.

These principles of the unity of mathematical knowledge cannot easily be transferred to school mathematics. With such an endeavor, school mathematics would lose its cultural background and become mere formalistic signs and formulas. In order to understand these signs and formulae, the formation of a new, distinct culture, a kind of mathematical re-invention, would again be necessary. From the point of view that mathematical knowledge has to be seen as a newly-emerging culture, one has to question the unity of mathematics in learning and teaching processes. If mathematical knowledge can only be meaningfully interpreted in the frame of a specific cultural environment, then there is not simply one single, but many different forms of practicing mathematics.

Wittmann (1995, pp. 358–359) distinguishes between specialized, scientific mathematics and the general social 'phenomenon' of mathematics.

> [One] . . . must conceive of 'mathematics' as a broad societal phenomenon whose diversity of uses and modes of expression is only a part reflected by specialized mathematics as typically found in university departments of mathematics. I suggest a use of capital letters to describe MATHEMATICS as mathematical work in the broadest sense; this includes mathematics developed and used in science, engineering, economics, computer science, statistics, industry, commerce, craft, art, daily life, and so forth according to the customs and requirements specific to these contexts . . .. It should go without saying that MATHEMATICS, not specialized mathematics, forms the appropriate field of reference for mathematics education. In particular, the design of teaching units, coherent sets of teaching units and curricula has to be rooted in MATHEMATICS.

On the basis of this position about the role of mathematical knowledge in instruction processes, Wittmann characterizes didactics of mathematics as a 'design science' (1998, 2001). In German-speaking mathematics education, especially concerning teacher education at universities and teachers' further professional education, Wittmann is a protagonist for a new perception on the role and the meaning of mathematical knowledge for teaching and learning processes, which critically distances itself from New Mathematics.

At the Institute for Didactics of Mathematics (IDM), founded at the University of Bielefeld in 1973, fundamental studies about mathematics education positions, problems and research questions were carried out in three working groups of scientists. In the "Mathematics Teacher Education" working group (Arbeitsgruppe Mathematiklehrerbildung, 1981), two central research approaches in mathematics education were mainly pursued: (1) the particular epistemological nature of mathematical knowledge, and (2) the central role of the teacher within mathematical teaching and learning processes.

Historical, philosophical and epistemological analyses were elaborated as a basis for characterizing mathematical knowledge ultimately as theoretical knowledge. A central criterion of theoretical mathematical knowledge – also observable in the course of its historical development – lies in the transition from pure object or substance thinking to relation or function thinking.

The transition from a substance concept to a relational concept is a central part of Ernst Cassirer's epistemological philosophy.

> . . . the theoretical concept in the strict sense of the word does not content itself with surveying the world of objects and simply reflecting its order. Here the comprehension, the 'synopsis' of the manifold is not simply imposed upon thought by objects, but must be created by independent activities of thought, in accordance with its own norms and criteria (Cassirer, 1957, p. 284).

And in another passage, Cassirer (1923, p. 20) writes:

It is evident anew that the characteristic feature of the concept is not the 'universality' of a presentation, but the universal validity of a principle of serial order. We do not isolate any abstract part whatever from the manifold before us, but we create for its members a definite relation by thinking of them as bound together by an inclusive law.

This understanding of theoretical mathematical concepts as referring to relations, rather than to objects or to the empirical properties of objects, constitutes the basic step towards developing mathematics education into a scientific discipline.

> For didactics, for instance, it is obvious that the didactic problem in its deeper sense, that is in the sense that it is necessary to work on it scientifically, is constituted by the very fact that concepts will reflect relationships, and not things. Analogously, we may state for the problem of the application of science that it will become a real problem only where the relationship between concept and application is no longer quasi self-evident, but where to establish such a relationship requires independent effort (Jahnke & Otte, 1981, pp. 77–78).

A perception that mathematical knowledge does not reflect things, but relations, implies a differentiated view of teaching and learning mathematics as independent activities of the participating persons. Thus, the role of the teacher comes to the fore.

> A description of the requirements on the teacher and the teaching activity has been attempted in the debate about the relation between teaching and learning. From this debate, one can record as a consequence that 'teaching' cannot be derived from descriptions of 'learning' – and that according to the opinion of many authors the developmental status of learning theories is more advanced than that of teaching theories. After all, the conception that the contents of teacher education should essentially consist of insights about the student's learning process is very common.
>
> What is the specificity of teaching? The specificity of teaching lies within the content of the activity, which aims at effectuating learning. Every theory of academically institutionalised education thus presupposes a concept of teaching and cognition, but also requires perceptions about the questions by which mechanisms the teaching/learning process leads or shall lead to an interactionally imparted forming of the learner." (AG Mathematiklehrerbildung, 1981, p. 57, translated by H.S.).

According to this perspective, theoretical and empirical works about the particularity of the teacher's activity have been carried out in the aforementioned working group of scientists at IDM Bielefeld (see for instance: Bromme,1981, 1992; Bromme & Seeger, 1979). These concepts and works about the activity of the mathematics teacher reveal, in particular, that within the didactical triangle, the teacher and his role are determined neither by the mathematical knowledge nor by the learning students. For instance, Bromme (1981, 1992) analyses central aspects of the teacher's activity (e.g., the preparation of mathematics instruction) under the perspective that teachers are to be regarded as experts in their professional field of work. In addition to the two essential fields of professional teacher knowledge: ‘content knowledge’ and ‘pedagogical content knowledge’ (according to Shulman, 1986), Steinbring (1998c) elaborates the particularity of ‘epistemological knowledge for mathematics teachers’ with a view to the theoretical and dynamic character of mathematics. This knowledge concerns insights about the particular epistemological nature of mathematical knowledge for teaching and learning processes, which are not contained in the ‘pedagogical content knowledge’, which Shulman (1986, p. 9) briefly describes as “. . . the ways of representing and formulating the subject that make it comprehensible to others”.

Furthermore, the independent role of the learning child with his or her cognitive predispositions moved to the centre of didactical research, a position which it had already taken for a longer time in primary school didactics. In a summarizing main lecture at the Federal Congress for ‘Didaktik der Mathematik in Osnabrück’ in 1991, Peter Sorger sums up a view taken in German mathematics education:

> Today, we know so much more, especially about the individual primary school child, about his cognitive activities, about his thinking, about the initiation and course of mathematical learning processes, about the influences of the individual learning history onto new learning situations, about the variety of possible thinking and solution strategies, which the adults’ perceptions are always in danger of cutting too short. The diagnosis, analysis and therapy of learning difficulties have also been thoroughly researched (Sorger, 1991, p. 39, translated by H.S.).

The research on this topic in particular uses methods from reference disciplines and they are not reducible to mathematical works (i.e., they essentially contribute to an independent research profile for mathematics education).

Again, the three questions (A, B and C) shall be asked and answered in a general way, in order to characteristically sum up the positions about mathematical instruction (respectively teaching and learning processes) described in this section. The mathematical content is interpreted more diversely and its dynamic and procedural character is particularly emphasized. (School-) Mathematical knowledge is not identical with scientific mathematical research knowledge, but, at the same time, it is theoretical knowledge which means that it is subject to a particular epistemology (also in the frame of the activities of teaching and learning). It is this *developmental* aspect of mathematical knowledge that makes possible to coordinate the ongoing students’ learning activities with the teachers’ teaching activities. These more differentiated perceptions of mathematics education negate an immediate dependence of the teacher on mathematical knowledge and of the student on the instructing teacher. From this perspective, learning mathematics is autonomous: “socially and actively discovering, independent learning by the students”; teaching is likewise viewed as an independent activity (AG Mathematiklehrerbildung of the IDM Bielefeld).

The three elements of the didactical triangle, (1) the mathematical knowledge, (2) the student and (3) the teacher stand ‘apart’ and gain independence as well as their own dynamics with new didactical research questions. The relations between these elements are of a rather indirect nature. For instance, the teacher is now regarded rather like a moderator or initiator of learning processes, while the student is conceded his own responsibility for his mathematical understanding and learning processes. The developing mathematical knowledge becomes manifest in different ways in different using and teaching/learning practices; it is no longer consistently and universally given, for example, on the basis of the Bourbakian structure types (Bourbaki, 1971).

The 'steering' of the students' learning processes by the teacher can no longer be perceived as mechanical conducting. The 'functioning' of the didactical triangle now rather represents a reciprocal process between its three elements and not a linear or circular movement of mathematics via the teacher to the students, or vice-versa. The mathematical knowledge (now in its new interpretation as theoretical knowledge within a development process) remains important, but shows itself in different characteristics in learning and teaching activities; however, the student's learning activities and the teacher's teaching activities also have an essential influence on the whole process.

At first, didactical research concentrated rather on the three relatively autonomous elements of the didactical triangle, (1) the mathematical knowledge, (2) the students and (3) the teacher; only with the beginning of mathematics education interaction research was the co-action of the three elements taken seriously and treated explicitly as the central object of didactical research.

## Mathematics Education Research and Mathematical Teaching-Learning-Practice as Independent Institutional Systems

For a long time, researchers in mathematics education research took the standpoint that mathematics teaching practice had to strictly follow the insights constructed by mathematics education. This point of view is also found in those didactical works which emphasize the procedural character of mathematical knowledge and of mathematical teaching and learning situations. There still exist perceptions according to which instruction practice could be directly improved by educational research.

Mathematics education is faced with the tension between scientific research and constructive development work. This problematique has been discussed intensively for a long time, for instance in the scientific debates about the so-called 'Theory-practice-problem' (Bazzini, 1994; Even & Loewenberg Ball, 2003; Seeger & Steinbring, 1992; Steinbring, 1994; Verstappen, 1988).

Facing this complementary task of research and constructive development, mathematics education is confronted with the fundamental question: "What is the particular nature of the relation between theory and practice?" One traditional solution to this question that educational research exclusively provides the necessary knowledge and prescriptions for school practice has been decidedly criticized and replaced by other conceptions.

An essential criticism has been developed by means of the work of the research group around Heinrich Bauersfeld at the IDM (Bielefeld). Since the beginning of the eighties, research started in which everyday mathematics teaching as autonomous social events was taken seriously and analyzed under an interactionistic perspective (e.g., Bauersfeld, 1978, 1988; Cobb & Bauersfeld, 1995; Krummheuer, 1984, 1988; Maier & Voigt, 1991, 1994; Voigt, 1984, 1994). Everyday mathematics instruction is seen as a peculiar culture, which is neither completely nor directly determined by the scientific discipline 'mathematics', nor can it be directly guided and improved by mathematics education research results.

Voigt (1996, p. 384, translated by H.S.) calls this the 'turn to everyday life' of the authentic classroom in mathematics education:

> . . . the 'turn to everyday life' . . . with its criticism of 'holiday didactics' . . . contained the claim of assigning a greater meaning than before to the features of everyday instruction. In ethnographic observations of instruction and interpretative studies, one saw a corrective for conceptions of instruction which emerge at the didactical desk; one was disillusioned by the effects of the school reforms (see among others the 'New Mathematics') and wanted to understand better the surprising stability of everyday instruction, its own progress and its traditions. At the same time, there was the hope of being able to better connect with the experience and the problem awareness of the practitioners through softer methods of empirical research".

(School) Practice and (content-related educational) science need to be seen as two relatively autonomous institutions and fields of work between which there are no direct possibilities of influence or change (see Bartolini-Bussi & Bazzini, 2003; Krainer, 2003; Scherer & Steinbring, 2006; Steinbring, 1994, 1998c). Each of the two fields is subject to its own expectations and aims, as well as to system-internal requirements and norms which cannot be externally invalidated in order to apparently be able to directly interfere in and to purposefully regulate from within the other field.

The relative separation and autonomy of (content-related educational) theory and (school) practice, however, does not mean that there are no reciprocal actions between the two at all. Rather, in the relation between theory and practice, the respective other field can be seen as a necessary environment in which irritations and stimulations occur, which indirectly animates the first field in order to implement changes, alternative ways of proceeding and further developments. What is important here is to notice that not only such changes within (school) practice, but also within content-related educational theory, must ultimately occur and establish themselves from the inside and 'out of themselves'. In order for this to happen, irritations and stimulations from the outside are helpful and necessary, yet they are not deterministic influencing instruments.

Under this fundamentally changed perspective on the 'theory-practice-problem', the didactical triangle takes on a different orientation function for mathematics education research. It no longer represents an ideal paradigmatic schema against which everyday instruction must be measured, but instead becomes an instrument for the analysis of actual mathematics instruction in which the reciprocal interconnectedness between the three relevant elements participating in the instruction process are systematically captured.

In works of interpretative classroom research, social interactions and their patterns and mechanisms were the centre of research interest; the mathematical teaching and learning content was, in principle, faded out. Thus, the particular relation between two elements of the didactical triangle (2) the student, and (3) the teacher within the frame of everyday instruction events was prioritized.

The interactionist perspective relies mainly on two (previously neglected) basic aspects: the learning child (in the classroom) and the interaction between the learner and the teacher. In this research context, one has to distinguish between two theoretical perspectives:

> The one is an individual-psychological perspective which emphasizes the learner's autonomy and his cognitive development and which leads to the concept of student-oriented, 'constructivistic' mathematics instruction. The other is a collectivistic perspective which criticizes the 'child-centered ideology' of the first perspective and understands learning mathematics as the socialization of the child into a given classroom culture . . . (Voigt, 1994, p. 78).

These two research perspectives are based on reference to different scientific disciplines. The individual-psychological perspective relies, for example, on cognitive psychology as well as on radical constructivism (von von Glasersfeld, 1991) and the collectivistic perspective uses sociological and ethnographic theories. In the analyses of mathematical interactions, one or the other of these two theoretical orientations is often emphasized.[3] An over-emphasis on either the individual-psychological or the collectivistic perspective was a major critique and a starting point for the working group around H. Bauersfeld to develop a theoretical concept which explicitly brings together the individual cognitive perspective and the collective social perspective, as a basis for qualitative analyses of interaction.

On the one hand, it asserted that a single student cannot discover all school knowledge by himself. "Culture, we can say, is not discovered; it is traded or falls into oblivion. All this indicates for me that we should rather be more careful when talking about the discovery method or about the conception that discovery is the basic vehicle of instruction and education" (Bruner, 1972, p. 85). On the other hand, it is considered doubtful that effective participation in social interaction patterns can lead to successful mathematics learning.

> In everyday lessons, interaction patterns often can be reconstructed in which the teachers influence every step of the students' activities without creating favourable conditions for the student to make desirable learning processes in problem solving and developing concepts . . .. We should resist the temptation of identifying learning mathematics with the student's successful participation in interaction patterns (Voigt, 1994, p. 82).

Consequently, an interaction theory was developed in which both perspectives were connected to each other:

> [A]n interaction theory of teaching and learning mathematics [offers] a possibility of regarding social aspects of learning mathematics and at the same time of avoiding the danger of overdoing the cultural and social dimensions. For the interaction theory emphasizes the processes of sense making of individuals that interactively constitute mathematical meanings.

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[3] Concerning the individual-psychological perspective, see e.g. Cobb, Yackel, and Wood (1991); and for the collectivistic perspective, see Solomon (1989).

> The interaction theory of teaching and learning mathematics uses findings and methods of micro sociology, particularly of symbolic interactionism and ethnomethodology …. Of course the interaction-theoretical point of view does not suffice if one wants to understand classroom processes holistically (Voigt, 1994, p. 83).

The interaction-research approach of the social epistemology of mathematical knowledge (Steinbring, 2005) understands itself as an important, independent and complete model inasmuch as the particularity of the social existence of mathematical knowledge is an essential component of this theoretical approach of interaction analysis. In this theoretical conception of the social epistemology of mathematical knowledge, the epistemological particularity of the subject matter ‘mathematical knowledge’ dealt with in the interaction, constitutes a basis for its theoretical examination.

Epistemology-based interaction research in mathematics education accentuates the assumption that a specific social epistemology of mathematical knowledge is constituted in classroom interaction and this assumption influences the possibilities of how to analyze and interpret mathematical communication. This assumption includes the following view of mathematics: mathematical knowledge is not conceived as a ready-made product, characterized by correct notations, clear cut definitions and proven theorems. If mathematical knowledge in learning processes could be reduced to this description, the interpretation of mathematical communication would become a direct and simple concern. When observing and analyzing mathematical interaction, one would only have to diagnose whether a participant in the discussion has used the ‘correct’ mathematical word, whether he or she has applied a learned rule in the appropriate way, and then has gained the correct result of calculation.

Mathematical concepts are constructed in interaction processes as symbolic relational structures and are coded by means of signs and symbols that can be combined logically in mathematical operations. This interpretation does not require a fixed, pre-given description for the mathematical knowledge (the symbolic relations have to be actively constructed and controlled by the subject in interactions). Further, certain epistemological characteristics of this knowledge are required and explicitly used in the analysis process (i.e., mathematical knowledge is characterized in a consistent way as a structure of relations between (new) symbols and reference contexts).

The intended construction of meaning for the unfamiliar new mathematical signs, by trying to build up reasonable relations between signs and possible contexts of reference and interpretation, is a fundamental feature of an epistemological perspective on mathematical classroom interaction. This intended process of constructing meaning for mathematical signs is an essential element of every mathematical activity, whether this construction process is performed by the mathematician in a very advanced research problem, or whether it is undertaken by a young child when trying to understand elementary arithmetical symbols with the help of the place value table. The focus on this construction process allows mathematics teaching and learning at different school levels to be viewed as an authentic mathematical endeavour.

In epistemologically-oriented mathematical classroom research, the subject of teaching and learning mathematical knowledge is taken into account as an important element within the didactical triangle. For empirical, interpretative research, the didactical triangle takes a descriptive function – and it has no prescriptive function – with which guidelines for instruction practice are provided. As a descriptive schema, the didactical triangle serves to characterize an essential and complex (i.e., not further dissectible) fundamental object of mathematics education research: namely, (everyday) mathematical interactions and communications within teaching and learning processes.

To sum up, one can ascertain the following alongside the three questions (A, B and C). The three elements of the didactical triangle, (1) the mathematical knowledge, (2) the student and (3) the teacher, are seen in the institutional context of the joint interaction as relatively independent 'systems', which are engaged in reciprocal actions with each other. The mathematical interactions between teacher and students take place between autonomous subjects, who are aware of each other during the reciprocal communication, but who cannot directly influence the psyche or the consciousness of the other. The communicated and negotiated mathematical knowledge is interactively constructed within this social context on the basis of its epistemological basic conditions of consistence and structure.

Accordingly, the teacher continues to take the role of a moderator or a facilitator of learning occasions for the students, who continue to be responsible for their own understanding processes and participate by means of socially and actively discovering mathematics learning. The instructional communication process emerges and constitutes itself within the actual course of teaching and learning; it cannot be planned and prepared in detail beforehand. Mathematics as a teaching-learning-object develops within the social interaction, and is in different ways the 'subjective property' of the persons taking part in the interaction.

The question about decisive means for positively changing and affecting the teaching, learning and understanding processes (C) gains a more differentiated background. Changes and improvements cannot take place from the outside, or by means of a direct intervention. Changes can only be encouraged in the participating autonomous systems and then need to be continued and realized within the systems themselves. This concerns the learning student to whom the teacher can ultimately only offer opportunities to learn for himself. But this is also true for the teacher and the development of his professional teaching activity in connection with mathematics education research.

## Mathematical Knowledge in Teaching: A Case Illustrating the Epistemology-based Interaction View on Teaching Learning Processes

In what follows, a short teaching episode is used to illustrate exemplarily how, from the perspective of epistemologically-oriented empirical instruction research, the three elements of the didactical triangle, (1) the mathematical knowledge, (2) the

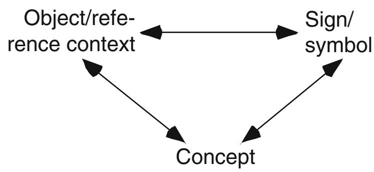


**Fig. 4.2** The epistemological triangle

student, and (3) the teacher, autonomously and interactively generate mathematical knowledge within this social situation (based on Steinbring, 2005).

Mathematics teaching and learning deal with the use and interpretation of mathematical signs, symbols and symbol systems. The mediation between mathematical signs or symbols and structured reference contexts can be described with the help of the epistemological triangle (see Fig. 4.2) (see Steinbring, 2005, 2006). This triangle serves as a theoretical instrument for analyzing the connection of yet unfamiliar mathematical signs/symbols, of partly familiar reference contexts for the signs/symbols and of fundamental mathematical concept principles, which regulate the mediation between signs and reference contexts.

This epistemological triangle is a theoretical schema, in which the corners reciprocally 'determine' each other; thus, none of the three elements can be explicitly or unequivocally given in order to then deductively determine the other elements. A fundamental concept is necessary to regulate the mediation between sign and reference context, and in the further development of mathematical knowledge, the fundamental conceptual knowledge is enhanced and differentiated.

The following classroom scene is taken from a third grade class working on the topic of 'figurate numbers'. All the children are sitting in a circle facing the board. Displayed on the board are dot patterns for the first five rectangular numbers (divided into two triangular configurations by means of different colours) together with the values of the respective triangular or rectangular numbers (see Fig. 4.3). The discussion now focuses on determining the amounts and the configuration for the 6th position.

88 T Yes. So what can we do to find out if this is always true?
89 S Nothing.
90 T Christopher.
91 Ch I notice something.
92 T Yes, tell us.
93 Ch Up there it goes four. Then it goes six. Then it goes eight. And then it goes ten. [*At this moment, T points at the number 20 and then at the number 30 on the left hand side of the table*]. Then it goes twelve [*T now points at the empty field below the number 20*]. Therefore, there should be thirty-two on the other seventeen [*2 sec pause*] um, forty-two should be on that and on the other one twenty-seven
94 T I see. You mean . . ., that's quite an interesting idea, Christopher. You mean, here there should be forty-two? [*points at the empty field below the number 30*]
95 Ch Yes. [*T writes the number 42 in the table*]

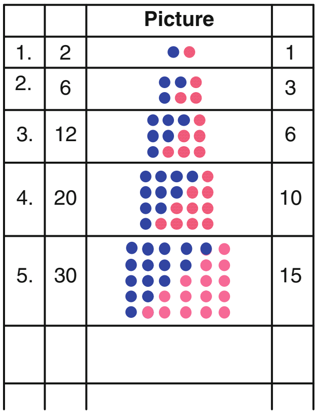


**Fig. 4.3** The connection between triangular and rectangular numbers

96 T Yes. And there? [*points at the empty field below the number 15 on the right side of the table*]
97 Ch Twenty-seven.
98 T Why do you think there should be twenty-seven? . . . Can you give a reason for that?
99 Ch No.
100 T No? . . . Nico.
101 N Twenty-one.
102 T Why do you think [it's] twenty-one?
103 N Because twenty and twenty are forty [*points at the ten's decimal place of the number 42*] and one and one are two [*points at the unit's place of the number 42*].
104 T Mhm [*writing the number 21 in the table*]. Oh yes, then we already know the next thing. But we ought to check whether it is indeed correct from the picture, whether it is really always like this.

This classroom scene will first be structured and summarized. First the teacher asks again his question as to whether it is always true, and after that he asks: "How can we find out whether this is always true?"

Phase 1 (90–97): Christopher continues the numbers in the column of the rectangular numbers and derives a new triangular number.

Christopher notices something. He names the sequence of numbers one after another: ". . . there it goes 4, then it goes 6, then it goes 8, then it goes 10, then it goes 12". With this, he seems to refer to the second column, and the teacher points at this column, at the numbers 20 and 30. Christopher names the respective difference or increase between the numbers in his sequence. Then he infers: "Therefore there should be thirty-two and on the other seventeen". He has (mistakenly?) constructed a number bigger by 2 in the left number column and he does the same in the right number column: from 15 to 17. Christopher corrects his statement: "Forty-two should be on that and on the other one twenty-seven". Here he has raised the two numbers by 12. The teacher confirms the first number with the question whether "... here there should be forty-two?" and he writes this number down after Christopher has agreed. Christopher repeats once again that, in the other position, there should be '27'.

Phase 2 (98–100): Christopher cannot justify his procedure.

The teacher asks Christopher to justify his claims. But Christopher cannot justify why '27' is supposed to be here.

Phase 3 (100–104): Nico corrects Christopher's triangular number and gives a justification for his claim.

Nico says '21' and means that this number is correct. He justifies this with the following 'calculation': "Because twenty and twenty are forty [*points at the tens decimal place of the number 42*] and one and one are two [*points at the units place of the number 42*]." The teacher agrees with him and writes down the new numbers (see Fig. 4.4). The teacher formulates a new 'research mission': "But we ought to check whether it is indeed correct from the picture, whether it is really always like this".

This detailed description of the short mathematical interaction between the two boys and their teacher clearly shows that the mathematical knowledge and understanding of this knowledge emerges, and is not completely fixed and clear-cut (as, for instance, *stoffdidaktik* (see part 2) would assume). The learning process is not simply a procedure of acquiring step-by-step the correct and undisputed mathematical rules and expressions.

The reactions and the following proposals in this mathematical interaction – contributed by the boys as well as by the teacher – develop and evolve according to the ongoing intention to commonly clarify and gain an understanding of, and a meaning for, the mathematical knowledge in question. This is what is meant by a parallel development of mathematical knowledge and the teaching and learning processes (see Part III).

The following (limited) epistemological analysis will show how central ideas of Part IV become relevant: the mathematical knowledge that develops in this communication process is open; it has to be constructed and interpreted by

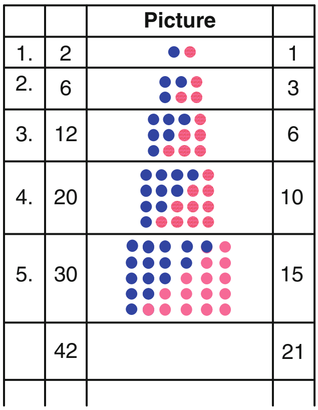


**Fig. 4.4** New triangular and rectangular numbers

the participants, but it is subject to epistemological constraints of coherence and consistency. The knowledge is not a priori given and fixed, but develops within its epistemological frames by subjective constructions and alternating social interactions.

First, Christopher's contributions, together with the teacher's pointing gestures, can be understood in the following way. Christopher names the differences between the rectangular numbers that have been written down. The teacher points at the respective number column. Christopher seems to have in view the additive continuation of the number sequence. He continues this characterization: "Then it goes 12"; this increase by 2 is supposed to lead to the new rectangular number in the sixth row.

Christopher now uses his arithmetical progression as a justification for the new numbers. He infers first, the two numbers 32 and 17, numbers which differ by 2 from their antecedent numbers; perhaps he transfers the increase of the differences directly to the new situation and corrects himself immediately. Now he seems to add 12 in both cases, and he names the numbers 42 and 27.

In his argumentation, Christopher referred to the arithmetical continuation pattern of the rectangular numbers without the geometrical situation. In a rather typical manner, the teacher takes the 'correct part' out of Christopher's argument. He says: "You mean . . ., that's quite an interesting idea, Christopher. You mean, here there should be forty-two? [*points at the empty field below the number 30*]" (94). And then: [*T writes the number 42 in the table*] (95). In this way, there is an implicit agreement in this social interaction that one part of the expected answer is correct, namely '42'. After the teacher's question, Christopher confirms that he believes that 27 belongs to the empty field. However, he cannot justify his claim.

Nico continues the knowledge construction. His justification: "Because 20 + 20 are 40 and 1 + 1 are 2" results in halving 42 into 20 + 20 and 1 + 1. If this is put in connection with the relation "Always half", which has been thoroughly discussed before, Nico intends a justification by using this relation. Again the teacher confirms this correct number as before by writing down 21 (104).

The knowledge constructions of the two boys can be characterized epistemologically in the following way. The mediation between sign/symbol and reference context carried out in this situation can be shown by the following epistemological triangle (Fig. 4.5).

For Christopher's knowledge construction, the analysis shows that he developed a continuation principle for the sixth rectangular number from the given arithmetical pattern. His counting by twos – 4, 6, 8, 10, 12 – is meant to suggest that the difference between the rectangular numbers is always an increase by '2' and that, therefore, '12' must now be added to the value of the fifth number. This addition of '12' is transferred to the fifth triangular number, and '27' is determined as the sixth triangular number. Christopher constructs a general arithmetical relation between the rectangular numbers in a verbal way and transfers it directly to the triangular numbers. This connection is inferred only from the arithmetical structure. No justification is given, for instance, using the geometric pattern of the rectangular numbers.

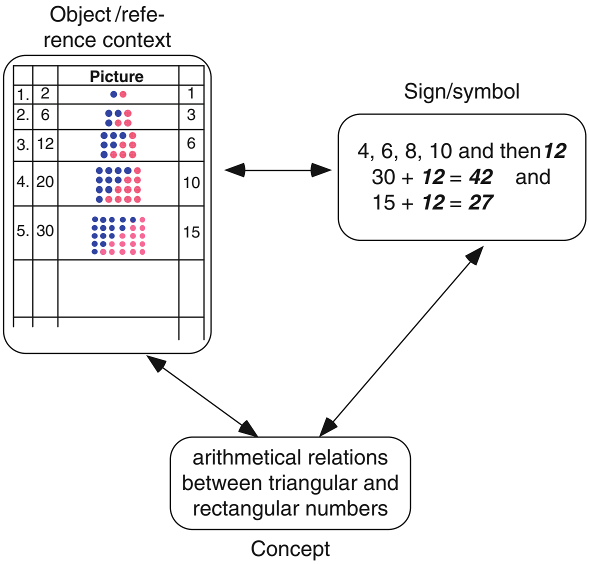


**Fig. 4.5** The epistemological triangle: Christopher considers arithmetical distances of number sequences

In Nico's knowledge construction, the rectangular number '42' is halved in a particular way. The intention connected with this proposal is not articulated directly. The brief argument is restricted to the procedure of the arithmetic bisection or doubling only. Nico constructs a brief verbally-formulated sign "20 + 20 = 40 and 1 + 1 = 2" with reference to the number 42 which was noted on the poster. This mediation between sign/symbol and reference context is represented as above in the epistemological triangle (Fig. 4.6).

In their contributions, both students constructed new knowledge relations which could not be directly inferred from knowledge that was already there. These knowledge relations were restricted to the arithmetical number symbols and structures with no reference to the geometrical configurations.

The question why the structure, that was observed locally in the numbers, is really generally valid needs, for example, the reference to the geometrical, general patterns of the triangular and rectangular configurations.

Based on this analysis, it can be stated that Christopher and Nico constructed mathematical signs that are not connected to the presupposed problem knowledge, but signs that use the visible arithmetical structure of the numbers on the poster.

In this short episode, the teacher intervened at certain moments to confirm correct answers or correct parts of answers or arguments developed by the two students. To give an example:

94 T I see. You mean …, that's quite an interesting idea, Christopher. You mean, here there should be forty-two? [*points at the empty field below the number 30*].

The teacher also writes the number in question, 42, in the table (he gives similar feedback to the student Nico when writing the proposed number 21 in the

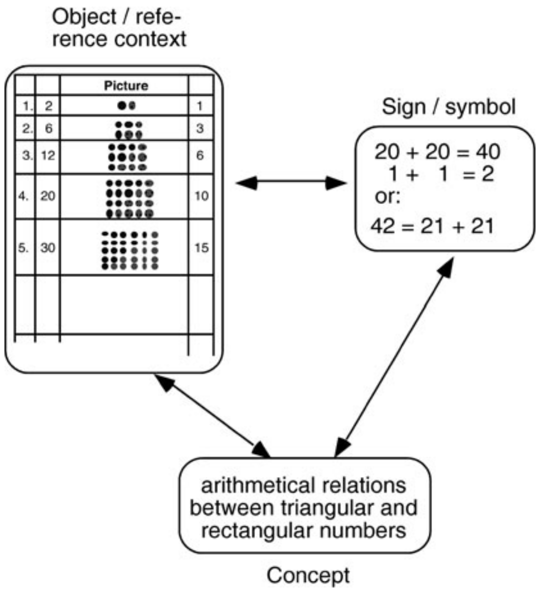


**Fig. 4.6** The epistemological triangle: Nico dissects 42 into 20 + 20 and 1 + 1

table). In this way, the teacher comments and moderates the contributions and arguments developed by the students. He does not simply follow an anticipated 'correct' solution procedure strictly, but he accepts, at least in great part, the activities and proposals by the students and he guides them. Surely the students also have learned how to participate in a question-answer game in mathematics teaching and they are certainly conscious of the teacher's feedback as questioning some proposed numbers (this cannot be the right one), or as writing down other numbers (these are the expected right numbers). This exemplifies how, through common interaction, mathematical knowledge develops along the epistemological constraints (see Part IV).

Looking back to the earlier sections (2, 3 and 4), again a further summarizing interpretation can be given concerning the three elements of the didactical triangle, (1) the mathematical knowledge, (2) the student, and (3) the teacher. The mathematical knowledge, essentially the important mathematical relations and structures, are in a way interactively constructed between the boys and the teacher. Thus, the theoretical (school-) mathematical knowledge (1) evolves here in a communication process between these three persons.

Christopher (2) argues in a situation-bound relational justification context. He constructs new arithmetical relations in the given structure that are also transferred to the arithmetical continuation of the triangular numbers without an additional underlying justification.

Nico (2) argues within an algorithmic justification context. He communicates factual knowledge. He seems to have in mind a relation between rectangular and triangular numbers by saying 21 + 21 = 42. Also, Nico does not produce true new mathematical knowledge as his argumentation refers exclusively to arithmetical relations, and does not take the geometrical knowledge problem into consideration.

The teacher (3) participates in this interaction as a moderator and he comments on students' proposals in a way of pointing at acceptable and unacceptable suggestions, thus guiding the process of negotiating the evolving mathematical relations of theoretical knowledge.

Looking at the didactical triangle as a descriptive instrument (see end of Part IV) in order to label the essential elements and their reciprocal actions within mathematical teaching and learning processes, the new interpretation from an epistemological mathematics education research perspective becomes clear: mathematical knowledge is interactively constructed by the participants on the basis of specific epistemological conditions thereof, which are effective also within instructional learning processes and which, in this teaching learning context, lead to a socially-developed epistemology of (school) mathematical knowledge.

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# Chapter 5
# Teaching Mathematics as the Contextual Application of Mathematical Modes of Enquiry

**Anne Watson and Bill Barton**

## One of Bill's Experiences

The syllabus I am using requires five lessons on 2×2 matrices for my class of 14-year-olds. We have looked at arrays and gone through the operations +, −, ×, ÷ with other matrices and 1×2 vectors. The final section is on matrices as transformations of the unit square: reflections, stretches, shears and rotations. We do not quite finish, so I use a little of the next lesson in a tight syllabus. In response to an invitation to the students to give me a random matrix so we can look at its effect, I get a 3×3 matrix suggested. Smart kid. The class appear to have understood the 2-dimensional concept, so I extend, draw a unit cube and watch as they quickly pick up the idea and stretch and reflect it in a plane. No problem – until the same child, flushed with her success, asks about a 4×4 matrix with a smile, knowing that there are only three dimensions. I seize the moment to demonstrate the power of mathematics to go beyond our experience and soon hypercubes are being reflected through 3-D space using the patterns of 2×2 and 3×3 reflections. The keen students take home work on problems in 5 or 6 dimensions. But that lesson has been used up, and half the next one, and I am dreadfully behind my schedule. After the lesson, why did I not feel concerned? And why, 30 years later, do I remember that lesson as one of my best?

## One of Anne's Experiences

One student, a good mathematics graduate training to be a teacher, told me that he had expected to shut down his intellectual engagement with mathematics in order to teach at a lower level than had been normal for him. Instead, he had found thinking about mathematics as a teacher every bit as mathematical and challenging as his first degree. An example occurred when thinking about preparing a lesson on straight line graphs, when he suddenly became aware that the schoolbook use of the

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term ‘linear’ for functions of the form $y = mx + c$ did not equate with his university use of ‘linear’ to refer to functions for which $f(\lambda x) = \lambda f(x)$ and $f(x + y) = f(x) + f(y)$. That is, keeping within the school syllabus, $f(x) = 3x - 4$ is not linear in the sense of transformations of vector spaces. He used this realisation to springboard a brief discussion with his school students of other mathematical meanings that appear to vary as you learn more mathematics, such as: multiplication not always making things bigger; translation being a different kind of symmetry from reflection, rotation and enlargement because it is not a matrix transformation; division not always being represented by sharing, and so on.

## Introduction

As teachers and educators, we have spent many hours in mathematics classrooms observing and participating. Knowing mathematics means being able to use mathematical concepts mathematically: the two cannot be separated. For us, lists of required mathematics content knowledge are hypothetical until mathematical modes of seeking, using, and exemplifying understanding are understood. We have explored the teacher task of preparation of resources in order to investigate our hypothesis that mathematical modes of enquiry are an important component of mathematical knowledge for teachers. We set up an artificial resource preparation exercise amongst a group of knowledgeable mathematics educators and recorded their collaboration to develop a description of how mathematical enquiry affected the process. This chapter is a contribution to making fluency in mathematical modes of enquiry an integral part of the conceptualisation of mathematical knowledge in teaching.

## The Roles of Mathematical Modes of Enquiry in Teaching

Research about how teachers’ mathematical knowledge relates to student achievement is of varied quality and gives varied results. Wilson, Floden, and Ferrini-Mundy (2001) undertook a systematic review of the literature and found that, while many studies find that personal knowledge does make a difference to teaching as one would expect, one study suggested that between four and six ‘courses’ did make a difference, but further courses did not make a further statistical difference (see Monk, 1994). However, they point out that different studies do not necessarily look at similar kinds of subject matter. In the Oxford Internship course, we find that undergraduate courses in real analysis, linear algebra, number theory, and abstract algebra enable teachers to use connections and overarching similarity within mathematics in their work, history of mathematics courses enable them to portray mathematics as a human and cultural endeavour, while engineering and physics courses enable them to provide interesting contexts and motivations for mathematics. This does not mean that new teachers with these backgrounds definitely use them in these ways, nor that those without cannot provide these experiences.

The question of connection between knowledge and teaching is still open and another way to approach it is to ask how mathematical knowledge influences the tasks of teaching in practice. The work of the SKIMA project team (Chapter 12 by Turner and Rowland, this volume) goes some way towards this in the primary phase by analysing epistemologically the way that teachers structure mathematical conceptual understanding in lessons.

Watson and DeGeest (2008) analysed 40 lesson videos from 18 teachers, teaching 11–14 year-old students on a range of subjects. The teachers worked in teams to plan and review lessons. They had shared aims, sometimes observed each other's lessons, and all taught in an interactive style coordinating students' responses to a range of varied tasks. Most of them believed they were teaching similarly to each other. After students had completed tasks involving mathematical techniques and processes, the teachers initiated further interactions which focused on discussing mathematical implications and connecting, integrating and affirming mathematical ideas. Teachers with little university-level mathematics in their past qualifications either ignored these aspects, or undertook them in everyday ways, such as: *implications* would be in terms of mentioning real-world applications, or saying what was important for an upcoming test; *connections* to easier mathematics already used in the work would be made, and no *integration* within mathematics was offered; new knowledge would be *affirmed* through its usefulness in getting answers, or students were affirmed by being praised for effort. By contrast, teachers with more mathematical past qualifications were more likely to discuss mathematical implications by comparing methods or exploring more complex cases; to connect and compare new experiences to other areas of mathematics that were structurally similar or conceptually related; and to suggest proof, or experiments, or explanatory usefulness to affirm what had been done.

Since Shulman (1986, 1987), mathematical knowledge for teaching has often been theorised using the idea of acquisition of types of content knowledge for teaching. For example, Kennedy (1999) claims that teachers need to understand the ways students hold mathematical conceptions, to know what representations and analogies will be useful in teaching, and to understand developmental stages. While such models might be useful for adding nuance to a continuum of pedagogical content knowledge and subject matter knowledge, in our view they risk missing out a crucial aspect of what a mathematics teacher does in relation to mathematics: *teachers enact mathematics*. In discussing mathematical knowledge for teaching, we can easily be drawn into a curriculum that claims a need for knowledge about quadratic equations, differentiation, the history of negative numbers, stages in development of number awareness, common misconceptions and so on. What is often missed is the teacher's mathematical thinking and awareness which Mason described as 'knowing-to' act in the moment (Mason & Johnston-Wilder, 2004, p. 289). It is not just a question of what teachers know, but how they know it, how they are aware of it, how they use it and how they exemplify it. As Chapter 3 by Hodgen (this volume) shows, the knowledge that is overtly apparent in teaching does not necessarily correlate with the knowledge that is displayed in audits. However, we believe from our own experience that there is more to these apparent mismatches than a general argument about situatedness can provide, so our question is: how do people who are fluent in mathematics bring this fluency to bear on teaching tasks?

Experiences like those at the start of this chapter have led us to consider mathematical modes of enquiry: a teacher 'sparking off' a student comment to make a wider point about mathematics and extend the students' thinking when the moment was ripe, at the later cost of having to squeeze the syllabus; a new teacher challenging himself with elementary material by thinking about mathematical definitions, assumptions and implications; a lecturer becoming carried away with making connections and using new perspectives to re-view familiar material; new teachers treating a curriculum topic as an arena for comparing examples, definitions, assumptions and implications. Again and again we observe, in ourselves and in others, that some of the best teaching and learning moments occur when mathematical modes of enquiry are invoked. We have come to believe that they are central to what a teacher does.

In this chapter we investigate this belief by enacting the selection and preparation of resources appropriate for mathematics teaching in secondary school. This grounded investigation of a mathematical perspective into teacher thinking confirms our belief, but leaves many questions untouched. Without understanding more about *how* mathematical knowledge is brought to bear on the tasks of teaching, descriptions and audits of necessary knowledge are hypothetical.

## Mathematical Modes of Enquiry

In March, 2008, the ICMI Centennial conference had a Working Group on *Disciplinary Mathematics*[1] *and School Mathematics*, in which questions were asked about the relationship between research mathematics and what happens in secondary classrooms. Initially, it appeared that strongly differing orientations were being expressed; on the one hand, it was argued that school mathematics had to be a 'shadow' of the discipline and, on the other, that it was fundamentally different in context (Watson, 2008). However, a consensus did emerge that "students learn through reasoning that resembles mathematical thought" (Barton & Gordeau, 2008, p. 255). It was noted that a significant difference between disciplinary and school mathematical experiences was the mediation of the teacher as provider of tasks, language and authority for validation. This begs the question of how the teacher can best undertake the mediation. 'Working as a mathematician' was one answer to this question and yet, as has been observed by many, the teacher is not usually or solely acting as a mathematician while teaching. While this leads some authors to look at social issues of identity, agency, language, power and so on, for us, any discussion of the mathematics involved in teaching has to start from an understanding of what doing mathematics entails and then seeing how this acts out in teaching. Without

[1] In this Group, 'Disciplinary Mathematics' was intended to mean, and was taken to mean, mathematics as a research discipline.

this focus, there is a temptation to see teaching mathematics as to do with exercising power versus constructing identity, rather than as an arena for acting mathematically.

What do we mean by 'acting mathematically'? Krutetskii's (1976) seminal study of gifted Soviet mathematics students identified several common features. These students all had a tendency to:

- grasp formal structure;
- think logically in spatial, numerical and symbolic relationships;
- generalise rapidly and broadly;
- curtail mental processes;
- be flexible with mental processes;
- appreciate clarity and rationality;
- switch from direct to reverse trains of thought;
- memorise mathematical objects.

These tendencies have been elaborated by Cuoco, Goldenberg, and Mark (1996) to attach their specific manifestations in various branches of mathematics as a taught subject in schools and undergraduate courses. They have also extended the list to include the qualities of 'sustained niggling' that bother mathematicians. Their characterisation of 'habits of mind' includes: pattern-sniffing, experimenting, visualising, forming conjectures, reasoning proportionally, loving systems, embracing unifying theories, looking at variance and invariance, extending meanings, thinking generally from examples and exemplifying from generalities.

Sustained niggling is also described by Hadamard (1945) and extended to include moments when insight occurs unexpectedly after being totally engaged with a problem for a period of time, then relaxing to do something else. This common experience reminds us that the natural ways in which the mind works includes reflection, organising, and seeking ways to compare and generalise experience. Mason (1988) puts some structure on 'sustained niggling' by focusing on stages and states of mathematical thinking. His inspiration came from Polya's (1962) classic work on problem-solving, encapsulating Polya's extensive list of the many strategies on which mathematicians can call. Mason sloganised these as 'specialise, generalise, conjecture, convince', but the common use of these strategies in the curriculum is as instructions rather than as descriptions of behaviour. This sometimes leads to an assumption that these actions should happen in a given order. It is more common for mathematical thinkers to roam between and within these approaches. It is also worth noting that 'specialise' implies a special choice of examples, rather than using examples as data for inductive purposes. We mention this here because purposeful generation and use of examples is also a major feature of being mathematical and also one that characterises good planning and teaching. For example, in a lesson about probability that we observed, the teacher offered examples in which $P(r) + P(\text{not } r) = 1$ emerged as a conjecture that was obvious to many learners, followed by an example in which $P(r) = 1$ and a following example in which $P(\text{not } r) = 1$. Creating and using examples to structure generality requires that teachers see what they are teaching in terms of generalities rather than techniques. Using extreme and special examples, as this teacher did, is a commonly used mode of enquiry to see how far a conjecture holds up, but one which is rarely explicit in textbooks. This teacher clearly understood this and tried to communicate it to her students: ‘Look’, she said ‘the mathematics is telling you something’.

Of course, one cannot be mathematical without the specific intellectual toolkit and repertoire of mathematics. The ‘habits of mind’ model includes many of the intellectual tools and skills of mathematics such as the use of representation and generalisation in the example above. Simon (2006) described key developmental understandings of mathematics not as first order knowledge, but as foundations for learning other ideas. We see these key understandings as threads that run throughout mathematics, so that the ways in which we read mathematical situations are profoundly influenced by them. For example, understanding number multiplicatively as a first resort, not as something to be used if additive models fail, is key to understanding much secondary and tertiary mathematics; understanding functions as mathematical objects, rather than as algebraic representations of data sets, is key to understanding much higher mathematics. While conceptual understanding is an aspect of mathematical knowledge and pedagogic design based on these understandings is important, it is only part of what constitutes mathematical knowledge. To become a fluent mathematician, a student must learn to act in certain ways in mathematical and other situations, to develop mathematical habits of mind, to enact mathematical modes of enquiry, and to think in terms of these underpinning key understandings. The *stoffdidaktik* approach to pedagogy described in Chapter 4 by Steinbring (this volume) offers scientific design which, as he describes, ignores the dynamic co-construction of knowledge which takes place through interaction in classrooms. But rather than take a position in a debate between external design and live interaction, we take a different viewpoint: a teacher has both to design and plan, and also to respond in dynamic situations, and hence needs to know what it means to act mathematically while both preparing and teaching.

Silverman and Thompson (2008) show that merely being offered situations in which conceptual development of key understandings are made apparent is not guaranteed to lead to improved mathematics teaching. Nor, as we know from experience, is experience of exploratory and interactive teaching necessarily going to lead to better teaching. Why does this transfer not take place? To contribute to this puzzle, we want to develop the reverse story: how do teachers who have over time developed significant ways of thinking about and interpreting mathematics bring that experience and knowledge to bear on pedagogic tasks?

## An Artificial Teacher Activity

In order to explore our belief that mathematical modes of enquiry are central to effective teacher activity, we set up an artificial teacher activity, that of planning to use resources, and asked two other experienced mathematics educators to join us. All four of us regard ourselves as mathematicians in our habits of mind.

The behaviour of mathematics education experts demonstrates what is possible when there is a background of fluency and experience. For example, Carlson and Bloom (2005) examined the behaviour of 12 mathematicians when solving problems and used their observations to devise a framework for describing successful problem-solving heuristics. Wild and Pfannkuch (1999) investigated models of statistical thinking by examining statisticians and graduate students at work. The argument for this approach is that knowing what expert behaviour can be like provides a frame for thinking about development. Our investigation is much smaller in scale and involves ourselves as subjects; however, the principle of examining expert behaviour to inform the possibilities of more general behaviour remains valid. Like those other researchers, we do not claim that our behaviour is what novices would do, nor that ours was the best, or the only possible behaviour, nor that what we did was all that is required for planning. Rather, we are using this situation to expose possible roles for mathematical expertise in planning to use resources. The aim is to become more articulate about these possibilities and thus contribute to discussions about *how* mathematical expertise can contribute to planning and teaching. The study draws its methodological base from participant observation; we constitute in part the group being observed. Nevertheless, the ethnographic nature of our account is subject to concerns about subjectivity, observation choices and generalisability. We acknowledge these while arguing that this case study both supports and illuminates roles for mathematical modes of enquiry in teaching.

We took two starting points and gave ourselves the artificial task of using them to devise teaching situations, role-playing teachers embarking on a shared planning exercise.

The first stimulus was a page of exercises from a school textbook chosen because it exemplified the kind of exercises in pure mathematics that are very familiar to any secondary mathematics teacher. The second stimulus was that day's newspaper and was chosen because many new teachers talk of 'everyday relevance' as a motivation for learning mathematics. For this activity, we assumed the content aims of the author of the textbook page, and also assumed that exploring the mathematics of an 'everyday' issue from the newspaper could be a lesson aim in itself. The task for the team of four was to describe the possibilities they could see in these stimuli for a teacher faced with creating lessons based on them. Of course, it is very rare that a teacher would have the luxury of 2 hours with three interested colleagues to create two lessons. Lessons are more usually created within a continuous curriculum, and with certain aims.

Our approach was first to identify, according to our mathematical knowledge, the range of potential mathematics afforded by these artefacts. In addition we agreed to discuss, after the event, what mathematical knowledge, experience and modes of enquiry we had used in this identification.

We audio-recorded the discussion and then analysed it to identify the mathematical practices and repertoire implicit and explicit in our responses. Each of the two authors wrote field notes soon after the event, and later separately analysed the recorded discussion, tracking mathematical actions and comments. These summaries were returned to the other participants for verification and combined to construct the descriptions below.

## Stimulus 1: Problems About Inverse Proportion

The textbook page concerned inverse proportionality, expressing this relationship in a variety of ways, including two which offered interplay between data sets and algebraic representations (see Appendix). A range of letters as variables was used, and the independent variable in the proportional relationship was itself a function in some of the questions. For example, in one question the independent variable was expressed as a positive square root.

Our initial responses may be described as alerts: Pedro[2] noted that the symbol $\propto$ is not universal and, for example, is not used in Portugal; Bill noted that the term 'inverse' has multiple mathematical meanings that students of this level would know (for example, 'inverse operation' and 'invert and multiply'); Anne noted that particular letters used on the page, such as $t$, $L$ and $v$, have common contextual meanings in mathematics and science that were not preserved in these examples; and John noted that the two ways of expressing inverse proportion algebraically $x = \frac{k}{y}$ (implied in Qn.1a) and $\frac{x_1}{x_2} = \frac{y_2}{y_1}$ (implied in Qn.2) might not be obviously equivalent because of the difference in notation. All of us saw these expressions of relationship as fundamental to a fluent understanding and recognition of inverse proportionality, too important to be offered only as implications, as in Qn.1 and Qn.2.

The group then began to respond in more detail to features of the set of exercises. For example, Qn.4 essentially asked for the factors of 12, and could be completed without thinking about inverse proportion. Another feature noted was that all the numbers on the page were simple integers or simple fractions and have simple multiplicative relationships. The discussion quickly moved into a suggestion by Pedro that Qns. 7 and 8 would be more interesting if the table did not include any matched pair. For example, instead of

| | | | | |
|---|---|---|---|---|
| $y$ | 2 | 4 | | 1/4 |
| $z$ | 8 | | 16 | |

one could offer

| | | | | |
|---|---|---|---|---|
| $y$ | | 4 | | 1/4 |
| $z$ | 8 | | 16 | |

[2]We acknowledge the help of Pedro Palhares and John Mason in preparing this chapter.

Such an exercise would not only make the question more open, but students working with it would be led naturally into the concept of the constant of proportionality, $k$. We wondered if the necessity to understand an expression such as $t^2$ to be a variable in itself, as is offered in Qn.8, detracted from the main idea of inverse proportion or encouraged a focus on ‘inverseness’. The extensive discussion ended up with agreement that a whole series of lessons could be constructed so that students would develop for themselves the concept of inverse proportionality by engaging with suitably constructed and sequenced tasks that either avoided completion by merely templating numbers, or that challenged such completion by affording cognitive conflict. However, we agreed that the page as a whole attempted to avoid the possibility that learners would get locked into simplistic assumptions about the relationship and did offer the potential for complex engagement with the concept approached from several different perspectives, using different symbolisations and also within composite functions.

The point we are making here is that it was our mathematical engagement with the page that problematised the content, and our pedagogical experience that suggested what can be done about it in the classroom. From our initial alerts about ambiguities of language, symbolism, and possible confusion of equivalent expressions, we deduced that this page could not be used effectively without considerable discussion with students about the purpose and meaning of inverse proportionality. If such discussion was effective, then it is unclear what purpose the exercise would serve. The questions are so different that fluency would be unlikely to be achieved, and the conceptual understanding afforded by connecting and relating different questions would, for most students, need to be developed through discussion, or through the teacher orientating the class towards it. While we recognised the potential of comparing answers and approaches between questions, our pedagogical experience tells us that, on their own, learners are unlikely to make these comparisons. We decided that, in a classroom, we would probably pick out particular questions and discuss the implications of them for the meaning of proportionality and the ways that proportional relationships might appear in symbolic form.

From here, the group moved away from the exercises themselves to consider what further mathematics could be developed through using such examples. Graphical plots, the effects of using different types of graph paper, and a general study of hyperbolae were the first suggestions; we all regularly use electronic graph-plotters as a mathematical tool alongside data sets and/or symbolic representations to increase our understanding of relationships. Next we talked about the hidden inverse proportionality in functions of two variables such as $y^3 x^2 = 8$, which can be decomposed as $g(y)f(x) = k$. By plotting $g(y)$ against $1/f(x)$ ($f(x) \neq 0$), a level of complexity following that offered by the exercise, we noted that such functions go through the origin and the constant of proportionality is always the gradient. Finally, we discussed how one might go about developing an algebra of ‘varies in relation to’, such as ‘if $x$ varies with $y$ this way, and $y$ varies with $z$ that way, how does $x$ vary with $z$?’ We agreed that this question could be explored by students at this mathematical level with appropriate technology.

Further discussion focused on contexts outside pure mathematics: where does inverse proportion occur and how could the concept emerge from student experiences? The Gas Law example (Qn.10) was related to the difficulty of opening and closing a fridge door, and the inverse proportion arising from a fixed amount of a resource being used by differing numbers of people were suggested. We noted that many examples exist in physics and wondered whether it was wise to use letters that have conventional physical meanings in equations that do not have those meanings. At a more general level, it was realised that proportionality could be regarded as the invariant relationship between a pair of variables independent of scaling. Familiar examples are density, expressing the invariant relationship between volume and mass for a particular substance, and the gravitational constant, expressing the invariant relationship between vertical force and mass.

Finally we turned to the meta-question of where our ideas had come from. What mathematical learning in our own histories had led to us expressing these thoughts? There were some particular answers: Bill is interested in language and mathematics so he focused on words; Anne had a current preoccupation with notation because, in her experience, some notations lead too easily to manipulation without meaning (Watson, 2009); some of John's responses were triggered by his rejection of all 'copy and complete' exercises because he has found such tasks are likely to lead to over-attention to answers, and failure to grasp underlying meanings; Pedro has a habit of using 'make something different' to extend a mathematical situation when he is working on his own mathematics. John's habit leads him to reject textbook pages, whereas Pedro adapts a textbook task and uses it to scaffold engagement in something challenging (see also Prestage & Perks, 2001).

We all agreed that much of how we reacted was already rehearsed in other situations. We have habits of analysing the variation, relationships and constraints implied in collections of mathematical objects. We had all looked at the questions as a sequence of mathematical objects which were exemplifying inverse proportionality. For example, we all had the habit of treating functions as variables, so a shift to thinking about $g(y) = kf(x)$ rather than $y = kx$ was a fluent way to raise more questions. We asked what meanings and understandings can be inferred from this collection. We brought prior experiences of many different kinds and we realised that, during the course of the discussion, we had each done some new thinking, for example, thinking about the relationship between equal ratios and constants of proportionality was not something we had explicitly done before in a general way, although it was, perhaps tacitly, embedded in mathematical experiences such as the Sine Rule. What made us alert to that?

On reflection, it was clear that we had responded on several mathematical levels: we thought about symbolic representation; equivalence of notations; relevant and irrelevant features of examples; the nature and representation of relevant functions; applications and meanings in other contexts; domain of applicability as represented in these questions; the affordances of other exemplifications; and the extension of the concept of proportionality into an algebra of relations. We had pedagogic responses (what would these exercises afford for students? how could they be changed to afford more?), as well as mathematical ones (what mathematics can we do in this situation? what potential mathematical meanings are suggested in the given presentation?), and we were attuned to ‘reading’ the mathematical text offered by each other in addition to that in the textbook.

## Stimulus 2: The Day’s Newspaper

Bill brought a newspaper and had already identified three potential articles to use as stimuli for mathematical activity. The first was a map of UK divided into different electoral regions. Three of us immediately started thinking about the Four Colour Theorem and its variants (for example, using regular-shaped areas or on different surfaces); the fourth member of the group thought of statistical questions. We asked ourselves what it was that made the stimulus different for each of us, but got only vague answers such as prior experience, familiarity with similar stimuli, disposition towards classical mathematical questions or social applications. A second article was about car finances and prompted a brief conversation about sequences and series, economic indices, and the volume of crushed scrap cars.

The item we decided to discuss in depth concerned the possibility of a new outbreak of smallpox and a discussion of plagues and epidemics. We started talking about modelling spread of disease. What was interesting, however, was that two of us began sifting our memories for the appropriate differential equations and the best variables to use (how do we write down the probability of two people meeting and one infecting the other?), while another looked at the situation more globally as ‘some form of exponential growth’ and thought about the reasons why or when exponential functions are the appropriate models. Both approaches ultimately lead to similar ideas, but it generated a discussion of how these two approaches can be balanced, which (if either) is useful in the school curriculum and the importance of the interaction between the approaches.

In this discussion, we all acknowledged the accessibility of our intellectual resources for modelling populations, such as knowledge of appropriate functions and prior experience of working on similar issues, and that there are many different modelling tools that can be used (graph theory, statistical mechanics, statistical data analysis, and so on). None of us considered plotting given data to explore the situation empirically in the hope that a generality would emerge from the data, but as teachers we felt we would have to consider such an approach since some students are oriented to ‘pattern-spotting’ by their algebra curriculum.

We became aware of several mathematical issues that could arise in the classroom. One issue was the way models shift between discrete and continuous formulations, often without this being made explicit. Our facility comes from multiple situated experiences of using functions in many contexts, and this situatedness probably contributes to difficulties in being meaningfully explicit with students. Any invented rule about what sort of situations require discrete or continuous treatment is likely to be over-simplistic and prey to counter-examples. Another is the idea of big numbers and how we develop our appreciation of what big numbers mean in a practical sense. Our own appreciation appears to have developed during adulthood through being genuinely interested in ‘big number’ situations, such as those that arise in natural disasters.

Our discussion then turned to other links that could be made from this material. Issues of disease prevention, the experience of false positive tests, and other medical science issues that could be regarded as part of general education and for which mathematics teachers (alongside other subject teachers) have some responsibility. We reflected on our uncertainty about some of this: what are the dangers that we (or teachers at large) might not be informed sufficiently and thereby give bad practical advice? What are the problems for mathematics teachers entering these areas? This led to considering the wider role of teachers in grappling with big issues, and the extent to which the skills of mathematical analysis are important in nearly every case. We discussed the Game of Life (Gardner, 1970), John Conway’s original idea behind the mathematics of cellular automata, and related simulation “games” that model the rise and fall of populations as contexts in which the setting up of rules and examining the consequences could be raised with students. We moved on to talk about iterative models, convergence, cycling and divergence. This led us to ideas of chaos.

Our meta-reflection on the knowledge we had brought to our discussion focussed on our prior experience. We were strongly aware that the main resource we brought to the newspaper article discussion was our experience of using mathematics to deal with social and economic questions. This kind of knowledge differs from the kinds we used for the proportionality discussion. It hinges on a sustained mathematical outlook on big social and political questions rather than being a form of pedagogic knowledge. In our newspaper discussion, we did not get close to the task of designing classroom tasks that would expose learners to the mathematical affordances of the issues of disease. Rather, we talked about mathematics being used as a tool to illuminate social and educational issues. The question we failed to address was whether the outcomes of engaging with such material should be the understanding of mathematical concepts, proficient use of mathematical tools, deeper insight into the social issues, or some combination of the three. We agreed that, whatever the outcome, engaging in such mathematical teaching requires knowledge and experience of using mathematics as an applied tool. Being mathematical in and with the world is both implicit and explicit in our practice. The consequence of this is the importance of mathematics teachers continuing to think about new mathematics, new applications and new areas of social relevance, not because that is what the four of us do, but because it contributes to mathematical richness and complexity.

## Discussion

In our responses to these two stimuli, the dominant knowledge brought to bear on the pedagogic tasks of planning and teaching was the personal mathematical past experience of the protagonists. In the work on proportionality, the pedagogic task was to discuss the affordances of the exercise and suggest better design, but the knowledge we used to critique the design was about representation, equivalence, exemplification, classes of functions, associated meanings, applications, tendency to treat relations as objects, and understanding of how to extend mathematical concepts beyond their obvious domains without having to think consciously about these strategies.

In the smallpox example, the protagonists brought past experience of modelling, suitable functions, differential equations, and an over-arching judgement about newspaper situations that might afford useful kinds of mathematical engagement. This process is the opposite of the application of mathematics to real world contexts, through which one gains the experience necessary to make these kinds of judgements.

Some mathematical modes of enquiry which arose in the situations above are:

- interpreting mathematical statements (e.g., identifying variables, identifying relations, or constructing particular senses of structure);
- flexibility with representations (e.g., changing them, manipulating them, setting up alternatives, comparing them, identifying different potentials that arise from their use);
- flexibility with approaches to the mathematics (e.g., starting with equation detail or the overall type of function) and making the links between these approaches;
- purposeful playing with an idea (e.g., instantiating it, finding particularly special or extreme examples, simplifying, asking 'what if', noticing what changes and what stays invariant);
- conjecturing, testing, deductive reasoning, and other forms of justifying;
- summing up by organising mathematical ideas; saying what is known or not known;
- linking; finding similarities or isomorphisms through comparing formal structures.

Comparing these modes of enquiry to the characteristics identified by Krutetskii above suggests that we were able to 'unpack' curtailed and generalised processes and concepts using generic mathematical exploration strategies, in order to lay out the ideas to be used in teaching. This is a different meaning of 'unpack' than is often used in relation to mathematical knowledge in teaching, because ours draws on the mathematical knowledge and experience that we have, and on the ways we have individually encapsulated it, rather than elaborating surface procedural knowledge.

One aspect of what a teacher provides, that a text cannot, is a range of mathematical options offered with judgement. While a text or website may contain several options, they are not offered at a fine level in response to students' reactions. To 'know-to' respond, teachers need: a repertoire of critical examples, a means to construct more, and a sense of when it is appropriate to present them; to understand when and why different representations are useful in a student's learning trajectory; to make purposeful connections between mathematical ideas; and to focus attention on just that aspect of a relationship that is important for the task at hand. In our activity, connections were illustrated in the smallpox example; thinking about appropriate use of representations was illustrated in the proportion example, as was our construction of new examples (or modified ones) to focus attention on particular relationships. Our argument is that a teacher for whom these are ways of being mathematical is more likely to be able to act fluently in all classroom mathematical contexts, compared with one who has learned a repertoire of pedagogical strategies without personal mathematical involvement. For example, a teacher might encourage students to use a given sequence of problem-solving strategies without having used them in tricky cases themselves. Those who undertake mathematical exploration develop a pedagogic repertoire through dynamic engagement in classroom mathematics.

Another aspect that a teacher provides is prediction. It is a peculiarly educational task to predict the consequences of a particular experience for other people, but in order to do this in mathematics, a teacher has to think about the mathematical consequences of particular conceptual instances. An example at a complex level would be the introduction of a concept within a limited frame which may later generalise to wider frames. A further kind of prediction is to look at an example or set of examples and see with new eyes what generalisations might be inferred from them. Often the cause of a so-called 'misconception' is inappropriate generalisation, or paying attention to the wrong variables. Unless teachers consciously generalise from examples themselves, they may not understand how this process can go wrong even when students are thinking very carefully.

Mathematics teachers have to react mathematically in the moment. Reacting in the moment means understanding mathematics deeply enough to be aware of affordances and opportunities. An important mode of enquiry in mathematics is an awareness of all the possible connections and directions of mathematical development from any particular situation. An equation can be read this way or that, the expressions that comprise it can be factorised or expanded or 'divided through' or several other options. It may conform to this pattern or that, it might be better to see it as a polynomial of a particular order, or make a substitution to transform it into another type of equation. It may be better to graph it or put it in matrix form. Knowing all of these (and their consequences) and choosing between them is part of both mathematics teaching and mathematical activity.

Finally, mathematics teachers must be able to "see behind". They need to be able to interpret students' mathematical texts and verbal explanations to understand what the student might be meaning, or is trying to express, and then work in such a way that the mathematical thought becomes clearly expressed. Again, this is a mathematical mode of working, it is what a mathematician might do when interpreting a text, or when thinking through a problem: having an idea, struggling to express it exactly, and it is through this process that justification and, ultimately, proving are born. A particular manifestation of this mode is in the similarity between reading a mathematics textbook and reading students' mathematical attempts. Reading text is a mathematical mode of enquiry. Most of the available typographies of mathematical knowledge in teaching claim that knowledge of misconceptions is important, but we would say that personal experience of how misconceptions come to be constructed is a more powerful source of pedagogic knowledge. What enables us to work with these alternative constructions is our analysis of how they could have been arrived at, in terms of inferred relationships between variables, or inferred connections between representations. The analytical process is generally more useful than trying to accumulate a list of possible curriculum misconceptions to be remembered and anticipated.

## Moving Forward

What might teacher educators do with this understanding of the importance of mathematical modes of enquiry? Let us assume for a moment that our teachers have already developed a set of key modes. A classic situation is that teachers cannot see how to use these modes in their work because school mathematics is strongly framed by the curriculum and assessment regimes. We suggest that it is not so much a matter of learning new or more modes; it is more a matter of maintaining mathematical activity so that these modes of enquiry are active and remain part of the teacher's way of being a mathematician in all pedagogic situations.

A recent experience working with teachers underscores the value of teachers maintaining their mathematical modes. A year-long research study (Barton & Paterson, 2009) centred on practising teachers who were supported in their quest for new mathematics learning in areas in which they had uncertainties. Each teacher chose their own topic and worked with the support of mathematicians and mathematics educators from a university department of mathematics. They met together as a group to share their insights and experiences. Topics included proof, the history of the exponential function, mathematical modelling, trigonometry, and the concept of probability. In the light of what we have said in this chapter, it is interesting that some of their 'topics', proof and modelling, could be described as mathematical modes of enquiry.

Teacher A found that she was restructuring her teaching to engage with 'big ideas' in the classroom because that was what had been useful and interesting for her in her own study. Teacher B saw a distinction between how she studied herself and the methods she had used before to pass tests. Teacher C drew distinctions between what you could do for yourself and why you sometimes need a teacher to explain. In their meta-discussions, the research team found that studying for themselves had led teachers to give more attention to learners' voices and less to their own voices in the classroom. The team also found that the 'deepest learners' were the better listeners. There were several ways in which the experience of studying mathematics impacted on the teachers' practice:

- they taught their own new knowledge directly, or encouraged students to replicate their learning;
- they explicitly used in class the new mathematical connections they had learned;
- they passed on to students their renewed knowledge of how to study;
- they expressed new insights and approaches more in terms of base structure and less on procedure.

All these teachers, in a short period of time, adapted their teaching, not only as a result of their new learning, but as a result of using mathematical modes of enquiry. Examples were: attention to ‘big ideas’; reproducing learning experiences; being aware of the assistance that might support understanding a mathematical concept; looking for and using connections between different mathematical ideas; and seeking mathematical structure. Another mode was the way the teachers used each other, both as support in their mathematical explorations and also as mathematical sounding boards for ideas and sources of linked knowledge.

Not only did the teachers attempt to reproduce such modes of enquiry in their classrooms, but they all reported talking to their students about their own learning experiences – a meta-level discussion of mathematical modes. Two of these teachers pursued this approach in further research.

## Conclusion

Mathematical knowledge includes its own modes of enquiry, but these are not necessarily explicit, nor are they always drawn into action when planning to teach. Mathematical modes of enquiry are learned by engaging in authentic mathematical experiences, although such learning can be enhanced both by having these modes modelled, and also by having the modes explicitly discussed at a meta-level. This applies to both teachers and students. We know from observations and from the literature (e.g., Kane, 2002; Lortie, 1975) that teachers appear much more likely to draw on their past experiences of being taught, and on the norms around them in school, than on being mathematical. Only rarely do we observe teachers applying their full repertoire of mathematical modes in a planned and explicit way.

It is with these experiences in mind that we argue that teaching mathematics is the contextual application of modes of mathematical enquiry, and that, too often, the modes of enquiry used in planning and teaching are drawn from a set limited both by teachers’ own mathematical experiences and by the ways they were taught. We have also shown in this chapter how knowledgeable mathematical enquiry can act in a shared planning process. We report some evidence that teachers will engage in personal mathematical experiences using mathematical modes of enquiry, and finding ways to organise situations involving those modes in the classroom. We hypothesise that such activity will result in better mathematics learning for their students. Teachers’ own mathematical knowledge and enquiry can provide an integrating context for all aspects of mathematics teaching in which the separate actions of doing, planning, teaching and learning mathematics connect and inform each other.

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## Appendix

From Rayner, D. (2000). *Higher GCSE mathematics: Revision and practice* (p. 188). Oxford, UK: Oxford University Press. Reproduced by kind permission of Oxford University Press.

## Exercise 12

1. Rewrite the statements connecting the variables using a constant of variation, $k$.
   (a) $x \propto \frac{1}{y}$ (b) $s \propto \frac{1}{t^2}$(c) $t \propto \frac{1}{\sqrt{q}}$
   (d) $m$ varies inversely as $w$ (e) $z$ is inversely proportional to $t^2$
2. $T$ is inversely proportional to $m$. If $T = 12$ when $m = 1$, find:
   (a) T when m = 2 (b) $T$ when $m = 24$.
3. $L$ is inversely proportional to $x$. If $L = 24$ when $x = 2$, find:
   (a) L when x = 8 (b) $L$ when $x = 32$.
4. $b$ varies inversely as $e$. If $b = 6$ when $e = 2$, calculate:
   (a) the value of $b$ when $e = 12$ (b) the value of $e$ when $b = 3$.
5. $x$ is inversely proportional to $y^2$. If $x = 4$ when $y = 3$, calculate:
   (a) the value of $x$ when $y = 1$ (b) the value of $y$ when $x = 2\frac{1}{4}$.
6. $p$ is inversely proportional to $\sqrt{y}$. If $p = 1.2$ when $y = 100$, calculate:
   (a) the value of $p$ when $y = 4$ (b) the value of $y$ when $p = 3$.
7. Given that $z \propto \frac{1}{y}$, copy and complete the table:

| $y$ | 2 | 4 | | $\frac{1}{4}$ |
|---|---|---|---|---|
| $z$ | 8 | | 16 | |

8. Given that $v \propto \frac{1}{t^2}$, copy and complete the table:

| $t$ | 2 | 5 | | 10 |
|---|---|---|---|---|
| $z$ | 25 | | $\frac{1}{4}$ | |

9. $e$ varies inversely as ($y$ – 2). If $e = 12$ when $y = 4$, find:
   (a) e when y = 6 (b) $y$ when $e = 1$.
10. The volume $V$ of a given mass of gas varies inversely as the pressure $P$. When $V = 2$ m$^3$, $P = 500$ N/m$^2$. Find the volume when the pressure is 400 N/m$^2$. Find the pressure when the volume is 5 m$^3$.

# Chapter 6
# Conceptualising Mathematical Knowledge in Teaching

**Kenneth Ruthven**

Each of the preceding four chapters in this section of the book has examined the development of a particular line of thinking about mathematical knowledge in teaching. My task in this chapter is to offer a critical appreciation of these approaches, and to create a more overarching framework for synthesising their differing contributions to the analysis of key issues of policy and practice.

## Subject Knowledge Differentiated

The first line of thinking can be described as *Subject knowledge differentiated* (Petrou and Goulding, Chapter 2). Its fundamental thrust is that expert teaching requires more than what would ordinarily constitute expert knowledge of a subject. Thus, its central concern is to identify types of subject-related knowledge that are distinctive to teaching so as to develop a taxonomy of such knowledge. The goal is to provide an overarching heuristic framework that can guide the analysis, assessment and development of professional knowledge. This project has its roots in Shulman's pioneering sketch of a taxonomy of knowledge for teaching. Chapter 2 discusses subsequent work that has sought to refine Shulman's model in the light of more direct practical experience of assessing and developing mathematical knowledge in teaching.

Petrou and Goulding focus first on the way in which the range of subject-related aspects identified in the Shulman knowledge taxonomy was extended by the model of professional knowledge proposed by Fennema and Franke (1992). Because of the significance of *knowledge of learner cognitions* within the *Cognitively Guided Instruction* [CGI] approach with which Fennema and Franke had been involved, it is not surprising that they singled this out as a primary category of professional knowledge within their model. This of course reflected the much wider trend (as discussed more fully by Steinbring in Chapter 4) to conceive teaching less in terms of direct instruction and more in terms of indirect (radical constructivist) facilitation, or

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(social constructivist) mediation, of students' construction of knowledge. Likewise, sensitised to the crucial interaction between knowledge and beliefs through their experience of working with elementary-school teachers to develop the CGI instructional approach, Fennema and Franke incorporated teacher beliefs as well as teacher knowledge into their model. Finally, perhaps the most important feature of Fennema and Franke's model was their insistence on the need to acknowledge "the interactive and dynamic nature of teacher knowledge", and to examine it "as it occurs in the context of the classroom" (p. 162) (as discussed more fully by Hodgen in Chapter 3). Both the other (and later) programmes of research that Petrou and Goulding then go on to discuss in depth in Chapter 2 share this concern to attend to knowledge in classroom (inter)action; however, these programmes position themselves differently as regards the continuing centrality of the Shulman taxonomy.

From an extensive programme of research and development work conducted at the University of Michigan, Ball, Thames, and Phelps (2008) have proposed a refinement of the Shulman taxonomy in which some of its core categories are further subdivided, and others reassigned. This refinement was informed by study of the way in which mathematical knowledge plays out in classroom practice, conducted with a view to developing operational measures of teacher knowledge. In the Michigan model, Shulman's *pedagogical content knowledge* [PCK] is conserved, but subdivided into *knowledge of content and students* [KCS], typically "an amalgam, involving a particular mathematical idea or procedure and familiarity with what students often think or do" (p. 401); and *knowledge of content and teaching* [KCT], typically "an amalgam, involving a particular mathematical idea or procedure and familiarity with pedagogical principles for teaching that particular content" (p. 402). Moreover, in this model, Shulman's *curricular knowledge* becomes a further subcategory of pedagogical content knowledge in the form of *knowledge of content and curriculum* [KCC]. Shulman's *content knowledge*, too, is conserved, as an overarching category of *subject matter knowledge* [SMK], but differentiated into *common content knowledge* [CCK] used in settings other than teaching (and so, of course, by non-teachers), and *specialised content knowledge* [SCK] which uniquely enables "teachers ... to do a kind of mathematical work that others do not" (Ball et al., 2008, p. 400); for example, when they respond to students' questions and find a telling example to make a specific mathematical point; or when they modify tasks to make them either easier or harder by anticipating the effects of changing particular didactical variables that affect students' approaches and responses (which also implies a potential involvement of aspects of pedagogical content knowledge). Finally, the more tentative subcategory of *horizon content knowledge* [HCK] is proposed, which concerns teacher awareness of how mathematical topics are related across the span of mathematics, and of how their development unfolds, as when teachers connect a topic being taught to topics from prior or future years, or explain how it will contribute to longer-term mathematical goals and purposes (although presumably such knowledge develops largely through encounters, as both student and teacher, with particular curriculum schemes and materials, raising the question of its relation to *knowledge of content and curriculum*).

Appraising this refinement of Shulman's taxonomy, Petrou and Goulding question the viability of demarcating specialised content knowledge from pedagogical content knowledge. Nevertheless, the taxonomic urge encourages fine distinctions:

> [S]izing up the nature of an error, especially an unfamiliar error, typically requires . . . flexible thinking about meaning in ways that are distinctive of specialized content knowledge. In contrast, familiarity with common errors and deciding which of several errors students are most likely to make are examples of knowledge of content and students. (Ball, Thames, & Phelps, 2008, p. 401).

Such examples also highlight the plurality of routes through which knowledge-in-use can emerge in teaching situations. Likewise, the use of 'amalgam' signals that many teaching problems cannot be adequately framed in 'pure' terms drawn from a single knowledge domain, or even by drawing on several domains independently. Put simply, satisfactory resolution of teaching problems must take account of, and often trade off between, interacting considerations of quite different types, framed in correspondingly different terms. This gives rise to solutions that often involve an irreducible fusion of such considerations, not reducible to the practice, or even logic, of any single pure knowledge domain. Moreover, for reasons both of ecological adaptation and cognitive economy, much professional knowledge comes to organise itself around paradigmatic problems and solutions that involve this type of fusion since these are closer to experienced teaching situations.

Thus, the contrasting approach taken at the University of Cambridge by Rowland, Huckstep, and Thwaites (2003, 2005) has been to develop a taxonomy more directly grounded in analysis of teacher knowledge-in-use in the course of actual classroom teaching episodes. While the *Knowledge Quartet* acknowledges parallels to the Shulman knowledge taxonomy, it does not seek to refine that model. Rather, it is designed to provide a guide to mathematical knowledge-in-use that is well suited to supporting teachers' professional reflection and learning. This Cambridge taxonomy, like the Michigan one, establishes prototypical systems of classification rather than logical ones, evoked through paradigmatic examples more than formulated through tight definitions. Essentially, the *Knowledge Quartet* provides a repertoire of ideal types that provide a heuristic to guide attention to, and analysis of, mathematical knowledge-in-use within teaching. However, whereas the basic codes of the taxonomy are clearly grounded in prototypical teaching actions, their grouping to form a more discursive set of superordinate categories – Foundation, Transformation, Connection and Contingency – appears to risk introducing too great an interpretative flexibility unless these categories remain firmly anchored in grounded exemplars of the subordinate codes.

Petrou and Goulding conclude Chapter 2 by proposing a synthesis of the different taxonomies of teacher mathematical knowledge that they have reviewed. In my view, while attempting this task is a valuable exercise in comparing and clarifying the models, it is not one that can be completed satisfactorily. The comparability of categories from the different taxonomies is no less problematic than the distinction between categories within any one. Influenced by experience of transposing the *Knowledge Quartet* from an English to a Cypriot context, Petrou and Goulding's synthesis gives more priority to curriculum knowledge than does either the Michigan or the Cambridge model. In effect, Petrou and Goulding revert to something close to core elements of the original Shulman taxonomy, wherein curriculum knowledge sits alongside content (or subject matter) knowledge and pedagogical content knowledge. This provides a reasonable match to Ball et al.'s refinement of the Shulman taxonomy, but its fit to the *Knowledge Quartet* is more problematic.

The Shulman knowledge taxonomy and its subsequent variants have mesmerised the field rather at the expense of the model of pedagogical reasoning that accompanied early accounts of the taxonomy (Wilson, Shulman, & Richert, 1987). In particular, this model incorporates a crucial process of *transformation*, which focuses on the *interpretation* and *representation* of disciplinary concepts, and on their *adaptation* to some general schooling situation and their *tailoring* to a particular group of students. If we view transformation as a process of problem solving, we see that it is subject to a range of constraints, both mathematical and pedagogical, which often cannot be considered in isolation from one another. Thus, what might appear to be simply a solution to a mathematical problem may have also been conditioned by pedagogical constraints and vice versa. Equally, where a teacher's solution to a problem of classroom teaching is also conditioned by curricular constraints, this can disguise a further interplay of mathematical and pedagogical considerations behind the institutionalised curriculum. Moreover, while solutions to such teaching problems may become crystallised as stable knowledge, they may equally be subject to continuing adaptation and refinement, and they will vary between teachers and across teaching settings. From this viewpoint, it becomes clearer why it has been so difficult to make demonstrable progress in establishing persuasive and productive knowledge taxonomies.

Petrou and Goulding's synthesis also incorporates the setting or context of teaching as an explicit, though relatively underdeveloped, element of their model. On the structural side, "the context in which teachers work is the structure that defines the components of knowledge central to mathematics teaching", and "this 'context' [includes] the educational system, ... the curriculum and its associated materials, such as textbooks and the assessment system". On the agentic side, "teachers' [knowledge] can determine the ways in which [they] understand, interpret and use the mathematics curriculum and its associated materials". Petrou and Goulding also draw attention to the "largely individualistic assumption which underpins" the models of teacher knowledge that they have discussed and suggest that attention needs to be given to teacher knowledge in relation to the wider systems within which it functions and develops.

## Subject Knowledge Contextualised

Analysing teacher (and teaching) knowledge from this wider perspective is the focus of the second line of thinking to be reviewed here; what can be described as *Subject knowledge contextualised* (Hodgen, Chapter 3). The fundamental thrust of this approach is that the (collective as well as individual) use and development of subject-related knowledge in teaching is strongly influenced by material and social context: the central concern of this approach is to identify and analyse significant facets of this contextual shaping. The goal is to acknowledge the embeddedness of knowledge in professional activity mediated by teaching tools and social organisation, so providing a model better adapted to guide the analysis, assessment and development of mathematical knowledge in teaching. This project as a whole draws on more general socio-cultural models, and Chapter 3 pursues this line of argument by using these models to characterise examples from studies of mathematical knowledge in teaching.

In Chapter 3, Hodgen seeks to illustrate the embeddedness of mathematical knowledge in professional practice, principally through the case of an experienced advisory teacher for primary mathematics who is actively involved in leading professional and resource development that includes intensive work on particular mathematical topics. Observed in her ordinary professional work, the teacher is able to function competently and confidently in tasks involving conceptual as well as computational mathematical activity. Observed in a more pressured interview situation involving apparently similar tasks, she does not display the same competence and confidence. While Hodgen notes that the teacher's capacity to access relevant knowledge may have been disrupted by anxiety triggered by the interview setting, the main explanation that he proposes is that the teacher's normal competence and confidence depends on the support ordinarily available to her through working collaboratively, using lesson materials, and drawing on curricular guidance.

Indeed, as Hodgen later points out, another important issue, already recognised by Shulman, is the form in which professional knowledge is held and the way in which it is organised and accessed. Here, cognitive studies of expert mathematics teaching at school level (Leinhardt, Putnam, Stein, & Baxter, 1991) have found teachers' knowledge and reasoning about a particular topic to be organised in terms of 'curriculum scripts' closely tailored to the actual work of teaching; these memory structures provide loosely ordered repertoires of action and argumentation, including relevant representations and explanations as well as markers for anticipated student difficulties. Likewise, Hodgen suggests that restructuring existing knowledge and experience may play a more important part in learning to teach and developing as a teacher than acquiring wholly new knowledge. As noted earlier by Fennema and Franke, such restructuring is often likely to extend to belief as well as knowledge, so that teacher learning also involves a degree of reconstruction of identity.

While the approach developed in Chapter 3 offers a plausible broadening of perspective, it appears to rest as yet on a relatively slender and fragmentary evidential base. The other cases invoked are all treated much more briefly. One of them provides a counterpoint to the main case in showing the distribution of professional expertise across personal knowledge, teaching tools and social organisation. In it, the capacity of a high-school mathematics teacher to implement an innovatory reform-oriented curriculum appears to rest not just on his rich and well-articulated mathematical knowledge, exercised and perhaps further developed through teaching a traditional curriculum for many years, but also on the support provided by the curriculum materials and the professional development associated with the innovation. The other cases represent a particular type of dysfunction where the personal knowledge of teachers is not well-adapted to the particular context of school teaching. In the first of these, proficiency in academic mathematical practice does not provide prospective secondary teachers with ready resources to respond to a naïve question about what, for them, are taken-for-granted techniques, a type of question often posed to teachers in school mathematical practice. In the second of these cases, compacted knowledge of a mathematical topic accompanied by automated recognition of mathematical connections appear to impair the sensitivity of a middle-school mathematics teacher to his students' thinking, preventing him from connecting with, and making sense of, this thinking. In effect, these cases show not only that knowledge of more advanced mathematics does not, of itself, help a teacher to function effectively (for the reasons reviewed by Petrou and Goulding in Chapter 2), but that more elementary knowledge may have taken on a curtailed and automated character that stands in the way of effective functioning in many teaching situations. It seems that shifts in the preferred modalities of mathematical thinking associated with more advanced study can create 'expert blind spots' for teachers (Nathan & Petrosino, 2003; van Dooren, Verschaffel, & Onghena, 2002).

The approach to thinking about mathematical knowledge in teaching developed in Chapter 2 focuses on different facets of individual knowledge and understanding of mathematics, which enable teachers to deploy the ideas and methods of the subject flexibly across teaching situations without reliance on contextual supports. While the approach developed in Chapter 3 accepts the desirability of teachers having these types of individual knowledge and understanding of mathematics, it acknowledges the reality that many school systems are obliged to operate with teachers who lack independent personal competence and confidence in the subject. Consequently, from a broader perspective which examines teaching in context, this approach suggests that teaching tools and social organisation provide potentially important mechanisms to help the teaching force function more effectively, and that these are also potentially capable of contributing to the development of teachers' individual knowledge and understanding of the mathematics that they are teaching.

## Subject Knowledge Interactivated

The third line of thinking in this section on conceptualising mathematical knowledge in teaching can be described as *Subject knowledge interactivated* (Steinbring, Chapter 4). Unlike the other chapters, this one does not specifically examine teacher knowledge and learning, but focuses rather on an evolution of thinking about the character of mathematical knowledge and how it is mediated through teaching and learning. The fundamental thrust of the evolution that Steinbring describes is towards a view in which mathematical knowledge is taken to be only indirectly communicable and locally constructible through social interaction. Hence, the central concern of this approach is with the epistemic and interactional processes through which mathematical knowledge is (re)contextualised and (re)constructed in the classroom.

For Steinbring, this forms part of a more general view that emphasises reciprocal interaction between relatively autonomous systems, rather than direct action of one on another; a view applicable not just to relations between teacher and student, but also to those between teacher educator and classroom teacher, educational researcher and teacher developer. Nevertheless, by examining *Stoffdidaktik*, Steinbring is focusing on a tradition, which plays a central part in the subject-related components of teacher education in the German-speaking world (Keitel, 1992). Thus, I will add to my précis of Steinbring's argument some observations of my own, based on my understanding of the form and function of *Stoffdidaktik* in teacher education, as gleaned from working with colleagues in German-speaking countries and from examining mathematics texts used in teacher education there.

Steinbring employs the widely-recognised 'didactical triangle', consisting of Mathematics/Content, Teacher/Teaching and Student/Learning (and the relations between them), as a device to compare three stages in the evolution of this central component of the German tradition of *Mathematikdidaktik*. In order to trace the evolutionary process, he describes changes at each stage in terms of the way in which these elements, and the relations between them, are conceived. At the first stage, that of classical *Stoffdidaktik*, attention was focused on Mathematics/Content, in particular on mathematically systematic analysis of curricular content to find an optimal presentation and sequencing for teaching purposes. Fundamental assumptions, then, were that achieving such a presentation is largely a problem of mathematical analysis, and that such a presentation then provides an unproblematic basis for effective teaching, and so for effective learning. Consequently, the other elements of the triangle, Teacher/Teaching and Student/Learning, received little attention at this stage, and the relationship between the elements was assumed to be a linear one in which well-analysed Content is relayed by the Teacher to the Student. One can see how, within teacher education, this view gave classical *Stoffdidaktik* a central place: a good teacher must be well-versed in the analysis of subject content for teaching purposes.

The second stage was precipitated by new views of the Student as a sense-making agent in the classroom, and of Learning as a process of knowledge construction. This was linked, in turn, to a new view of Mathematics/Content in which the processes and products of academic mathematical practice were less unquestioningly accorded a privileged place. Under these circumstances, the focus of attention within *Mathematikdidaktik* enlarged to include Student/Learning as well as Mathematics/Content, and the style in which Mathematics/Content was treated shifted (as exemplified by Freudenthal's influential work on didactical phenomenology and progressive mathematisation). These new views rather neglected the element of Teacher/Teaching, indeed sometimes treated it with a degree of suspicion. Within the institution of teacher education, reformed *Stoffdidaktik* responded by taking a broader perspective on Mathematics/Content and incorporating greater attention to Student/Learning, often in the form of the results of psychological studies of student errors and misconceptions (e.g., Padberg, 1989; Vollrath, 1994). In the third stage of the evolution described by Steinbring, attention has turned to direct analysis of knowledge construction in the classroom, particularly to its epistemic and interactional aspects. Although Mathematics/Content, Teacher/Teaching and Student/Learning are regarded as relatively independent systems, attention now focuses on their operation and particularly their reciprocal interaction. The type of analysis that results from this approach is illustrated in the final section of Chapter 4. Steinbring argues, of course, that such analyses carry no direct implications for teacher education or for teaching. Nevertheless, one can see how exposure to such examples might stimulate teacher educators to incorporate this type of fine-grained analysis of the construction of mathematical knowledge through classroom interaction into their work with prospective and serving teachers.

At one point, Steinbring introduces a comparison with the Anglo-American tradition. He suggests that there is a further type of mathematical knowledge relevant to teaching that goes beyond content knowledge and pedagogical content knowledge as characterised by Shulman: what he terms *epistemological knowledge for mathematics teachers*. We can perhaps best understand this by first observing how Steinbring's use of the epistemological triangle draws attention to what otherwise might be taken as unproblematic mathematical entities. This approach highlights the way in which such entities are constituted by virtue of relationships between Concept, Sign/Symbol and Object/Reference Context, and how these relationships provide a key focus for the unfolding construction and negotiation of knowledge in classroom interaction. In effect, the epistemological triangle might be seen as a necessary expansion of the Mathematics/Content element within the didactical triangle to enable it to preserve its heuristic function given this new view which emphasises "the theoretical and dynamic character of mathematics". It is necessary, then, to treat with some caution apparent parallels between relations implied by the didactical triangle and the subcategories of pedagogical content knowledge in the refined (Michigan) version of the Shulman knowledge taxonomy. Superficially at least, Knowledge of Content and Students focuses on the relation (or, in this view, the interaction) between Mathematics/Content and Students/Learning; Knowledge of Content and Teaching focuses on the relation (or interaction) between Mathematics/Content and Teacher/Teaching. Knowledge of Content and Curriculum presumably resides within the Mathematics/Content element within the didactical triangle, but parallels are problematic given the way that the analysis presented by Steinbring has shown how views of this element have changed markedly within *Mathematikdidaktik*, as previously unacknowledged complexities have been recognised.

## Subject Knowledge Mathematised

The final line of thinking developed in this section of the book can be described as *Subject knowledge mathematised* (Watson and Barton, Chapter 5). Its fundamental thrust is that teachers must act mathematically in order to enact mathematics with their students, and that doing so calls for a kind of knowledge rather different from that which normally receives emphasis in discussions of mathematical knowledge in teaching. The central concern of the chapter, then, is to characterise those *mathematical modes of enquiry,* which underpin any authentic form of mathematical activity, and to show how teachers employ them to foster such activity in their classrooms.

While also embracing Krutetskii's 'mathematical abilities', the chapter locates its approach principally within a tradition leading from Polya's 'mathematical heuristic' to Cuoco, Goldenberg and Mark's 'mathematical habits of mind'. The chapter draws on each of these sources to exemplify the range of intellectual dispositions and strategies which can be thought of as mathematical modes of enquiry. The chapter then uses a simulated exercise in the planning of teaching situations from contrasting types of resource to provide more fully-developed examples of mathematical modes of enquiry in action within the work of teaching. Retrospective analysis of the thinking stimulated by this exercise also leads to more mathematical modes being identified. In an authentic piece of planning, of course, contextual aspects of the teaching situation would also figure, and might indeed shape key aspects of the process. Nevertheless, the artificiality of the exercise does help to focus attention on the mathematical modes of enquiry and their significance. In effect, what Watson and Barton are doing is "organizing the reality with mathematical means" (Freudenthal, 1973, p. 44), bringing out how both the enaction of teaching and its planning can be treated as processes of mathematising. This underpins their broader critique of predominant perspectives on mathematical knowledge in teaching, challenging an apparent focus on frozen mathematical content at the expense of fluid mathematical process (so taking further some of the critiques of Shulman's analysis reviewed in Chapter 2). Where teacher knowledge is concerned, the crux for Watson and Barton is that "a teacher for whom these [modes] are ways of being mathematical is more likely to be able to act fluently in all classroom mathematical contexts, compared with one who has learned a repertoire of pedagogical strategies without personal mathematical involvement".

Freudenthal (1991, p. 30) has described mathematisation as "the process by which reality is trimmed to the mathematician's needs and preferences" and this seems a very apt description of the approach set out in Chapter 5. A recurring feature of the argument is an emphasis on personal mathematical experience as a source of insight. Reviewing the planning exercise, Watson and Barton conclude that "the dominant knowledge brought to bear on the pedagogic tasks of planning and teaching was the personal mathematical past experience of the protagonists". But potential limitations of the personal mathematical experience and thinking of teachers as a guide to the mathematical experience and thinking of their students are not explored. Challenging the value of teachers' learning about common mathematical misconceptions amongst students, Watson and Barton suggest that "personal experience of how misconceptions come to be constructed is a more powerful source of pedagogic knowledge". Arguing that mathematics teachers need to be able to "see behind" students' productions, they propose that this is exactly "what a mathematician might do when interpreting a text, or when thinking through a problem".

This approach comes close to a mode of thought that has been found to be prevalent amongst subject-specialist teachers, in which an explicit mathematical narrative provides the organising structure for a tacit pedagogical one.

> In mathematics teachers, the subject-matter-specific pedagogical content knowledge is to a large part tied to mathematical problems. In a way, it is “crystallized” in these problems, as research in everyday lesson planning has shown. In their lesson preparation, experienced mathematics teachers concentrate widely on the selection and sequence of mathematics problems . . . Nevertheless, pedagogical questions of shaping the lessons are also considered by teachers in their lesson planning, as these questions codetermine the decision about tasks. By choosing tasks with regard to their difficulty, their value for motivating students, or to illustrate difficult facts, and so forth, the logic of the subject matter is linked to teachers’ assumptions about the logic of how the lesson will run, and how the students will learn. . . . Teachers often do not even realize the integration they effect by linking subject-matter knowledge to pedagogical knowledge. One example of this is their (factually incorrect) assumption that the subject matter (mathematics) already determines the sequence, the order, and the emphasis given to teaching topics. The pedagogical knowledge that flows in remains, in a way, unobserved. To teachers who see themselves more as mathematicians than as pedagogues, their teaching decisions appear to be founded “in the subject matter” (Bromme, 1994, p. 76).

Too mathematically purist a stance risks isolating discussion of mathematical knowledge in teaching from productive perspectives that are very much in sympathy with the idea of mathematical modes of enquiry, but which frame such ideas in different terms. For example, one of the cases which Collins, Brown, and Newman (1989) use to illustrate their model of ‘cognitive apprenticeship’ is Schoenfeld’s (1985) approach to teaching mathematical problem solving which lies in the same tradition of critical refinement of Polya’s mathematical heuristic as does the mathematical modes of enquiry approach. Schoenfeld, however, is not uncomfortable with the language and concepts of the cognitive sciences; indeed, he uses that apparatus to generate fresh insights into teaching for mathematical problem solving. Others characterise his teaching in similar terms:

> To students, learning mathematics had meant learning a set of operational methods, what Schoenfeld calls *resources*. Schoenfeld’s method [involves] teaching students that doing mathematics consists not only in applying problem-solving procedures, but in reasoning about and managing problems using heuristics, control strategies and beliefs. Schoenfeld’s teaching employs the elements of modelling, coaching, scaffolding and fading in a variety of activities designed to highlight different aspects of the cognitive processes and knowledge structures required for expertise (Collins, Brown, & Newman, 1989, p. 470).

The language employed by Watson and Barton in the Conclusion to Chapter 5 suggests that theirs is not a purist stance. For example, the references to “authentic mathematical experiences”, “having modes modelled”, and “having the modes explicitly discussed at a meta-level” appear to borrow from the same forms of language as cognitive apprenticeship. Equally, Collins et al. (1989, p. 474) acknowledge how mathematical and pedagogical considerations (of exactly the type identified by Bromme in the quotation above) interact in Schoenfeld’s thinking about his teaching: “Schoenfeld places a unique emphasis on the careful sequencing of problems. He has designed problem sequences to achieve four pedagogical goals: motivation, exemplification, practice, and integration.”

## Reconceptualising Subject Knowledge in Teaching

Originating in response to perceived inadequacies in received views of mathematical knowledge in teaching, each of the lines of thinking presented in Chapters 2–5 has given rise to productive reconceptualisations.

The first pair of approaches focus on how mathematical knowledge can be functionally adapted and developed in ways that specifically support the teaching role. *Subject knowledge differentiated* challenges the received idea that the mathematical knowledge required in teaching is simply that developed through studying the subject to a level which provides adequate facility in (and perspective on) the material to be taught. This approach has reconceived the issue productively by identifying types of subject knowledge that are closely linked to teacher activity in promoting effective learning, which are not normally developed as a student of the subject, and that appear to be distinctive to teaching (or, at least, developed much more fully than in other professions). This also introduces a challenge to the received idea that teaching simply involves the direct application of mathematical knowledge that is universal in its character and organisation. *Subject knowledge contextualised* further reconceives this issue by showing how teachers' knowledge undergoes a professionally-specific adaptation, in which organising structures are developed that support coordinated attention to the mathematical and pedagogical facets of teaching, and that involve an important degree of (sometimes unacknowledged) fusion between mathematical and pedagogical concepts. *Subject knowledge contextualised* has also challenged what appears to be an unexamined emphasis in received views of mathematical knowledge in teaching on such knowledge as individually and independently held. In the face of widespread difficulties in securing and developing a mathematically proficient workforce within teaching, this issue has been reconceived productively by drawing attention to the distribution of subject knowledge across teaching tools and professional communities, and so to the contribution that (when appropriately developed and organised) these resources can make to supporting knowledgeable subject teaching and the development of teachers' subject knowledge.

The second pair of approaches share an emphasis on a teaching role that centres on supporting student knowledge construction. *Subject knowledge interactivated* challenges the received idea that mathematical knowledge can be pre-formulated in a way that enables it to be simply relayed by teachers to students (or indeed by teacher educators to prospective teachers). This approach has reconceived the issue productively by showing the types of interactional process through which mathematical knowledge undergoes a process of (re)construction by means of active negotiation between the participants in classroom mathematical activity, mediated by appropriately designed tasks. In particular, this highlights the types of subject-specific epistemic and interactional competence that are required in effective teaching of this style. Closely related is the challenge which *Subject knowledge mathematised* offers to the received idea that it is sufficient for teachers to know their subject in the sense simply of being familiar with the finished mathematical material to be taught and with associated difficulties that students may encounter.

This approach reconceives the issue productively by highlighting the way in which teachers are responsible for leading classroom enactment of mathematical activity through which knowledge can be (re)constructed, and by illustrating how, at its best, such joint activity provides a means through which students become conversant with the mathematical modes of enquiry that underpin such (re)construction (i.e., how they can be led to develop syntactic as well as substantive competence (Schwab, 1978)).

In the face of the widespread difficulties noted earlier in securing and developing a mathematically proficient workforce within teaching, this latter pair of approaches might be seen as setting a somewhat utopian (and overly challenging) standard for mathematical knowledge in teaching. The counter to this is that more modest strategies may only be capable of effecting marginal improvement within received practices of mathematics teaching which are fundamentally flawed. Indeed, from the more radical perspective, the problem of subject expertise in teaching is just one component of a much larger issue of the social reproduction of mathematical knowledge. In this view, inadequate mathematical knowledge on the part of individual teachers is a subsidiary phenomenon that ultimately resides in the inadequacies of received practices, not just of mathematics teaching but of mathematical communication more broadly, because these lack mechanisms through which the thinking processes and learning strategies that underpin the development of mathematical knowledge are made accessible to students and significant to their teachers.

Within the practice of academic mathematics, for example, Thurston (1994, p. 8) has argued that established protocols for communication fail to provide effective means of revealing underlying thinking processes:

> We mathematicians need to put far greater effort into communicating mathematical *ideas*. To accomplish this, we need to pay much more attention to communicating not just our definitions, theorems, and proofs, but also our ways of thinking. We need to appreciate the value of different ways of thinking about the same mathematical structure. We need to focus far more energy on understanding and explaining the basic mental infrastructure of mathematics . . . This entails developing mathematical language that is effective for the radical purpose of conveying ideas to people who don't already know them.

Thurston is arguing that effective mathematical communication involves some degree of interactivation and mathematisation of the knowledge at stake. In particular, he criticises approaches to teaching mathematics that neglect such interactivation in favour of a reductive focus on demathematised knowledge.

> In classrooms . . . we go through the motions of saying for the record what we think the students "ought" to learn, while the students are trying to grapple with the more fundamental issues of learning our language and guessing at our mental models. Books compensate by giving samples of how to solve every type of homework problem. Professors compensate by giving homework and tests that are much easier than the material "covered" in the course, and then grading the homework and tests on a scale that requires little understanding. We assume that the problem is with the students rather than with communication: that the students either just don't have what it takes, or else just don't care. Outsiders are amazed at this phenomenon, but within the mathematical community, we dismiss it with shrugs (p. 6).

This highlights how (presumably excellent) conventional mathematical knowledge on the part of (university) teachers is unable to compensate for what Thurston characterises as 'often dysfunctional' cultural practice. Translated into the language of cognitive apprenticeship, the type of teaching practice sketched above models mathematical activity in restricted terms, employs excessive scaffolding without progressive fading, and neglects cognitive and metacognitive articulation and reflection. The result is impoverished joint activity, weak interaction between teacher and students, and a corresponding polarisation of their classroom roles that accentuates the shortfall of (particularly tacit) knowledge to which students are given access.

Under such conditions, few students will develop powerful and flexible strategies of mathematical thinking and learning. A fundamental problem of mathematical knowledge in teaching is that the school and university experience of many prospective and practising teachers has been of this limited type, creating reflexes that are difficult to change. The models of subject thinking and learning that prospective teachers have developed as students are well known to constitute an important base for the forms of teaching practice that they go on to develop. Arguably indeed, taking on a teaching role involves a recasting of intrapersonal metacognition into interpersonal activity and dialogue. Interestingly too, this argument suggests that the degree to which (what have been presumed to be) distinctive elements of mathematical knowledge for teaching can be differentiated from other mathematical knowledge may actually be mediated by cultural practices of teaching. Specifically, the degree of differentiation between what is regarded as teaching-specific, rather than as more generic, mathematical expertise may be directly related to the asymmetry of teacher and student roles in classroom mathematical activity, and inversely related to the attention accorded there to cognitive and metacognitive articulation and reflection. That provides one of the reasons why we have been keen in this book, particularly in the next section, to extend our consideration of issues of mathematical knowledge in teaching beyond the English-speaking world in which the predominant conceptualisations reviewed in Chapters 2 and 3 have been developed.

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# Part II
# Understanding the Cultural Context of Mathematical Knowledge in Teaching

# Chapter 7
# The Cultural Location of Teachers' Mathematical Knowledge: Another Hidden Variable in Mathematics Education Research?

**Paul Andrews**

## Introduction

Much work on mathematics teacher knowledge has drawn on the earlier conceptualisations of Shulman (1986). In brief, his model of teacher knowledge, prompted more by concerns about inadequate teacher education programmes rather than individual teacher competence, comprised three components: subject matter content knowledge, pedagogical content knowledge and curricular knowledge. More recently, a number of researchers have developed frameworks that essentially present mathematics teacher knowledge in generalised forms that acknowledge not only the role of subject matter knowledge in successful teaching – an understanding of the substantive and syntactic properties of mathematics – but also transformative pedagogic knowledge, whereby subject matter knowledge is made amenable to multiple presentations. Such models include, for example, Sherin's (2002) discussion of *content knowledge complexes*, reflecting the automated and simultaneous application of both content and pedagogic content knowledge, Rowland, Huckstep, and Thwaites' (2005) four-dimensional model of primary preservice teacher knowledge, and the University of Michigan team's model incorporating *common content knowledge, specialized content knowledge, horizon knowledge*, *knowledge of content and students* and *knowledge of content and teaching* (Ball, Thames, & Phelps, 2008). All such models can be construed as representations or developments of Shulman's (1986) content and pedagogical content knowledge. However, such categorisations, in their presentations of essentially personal attributes, seem to locate teacher knowledge within the individual. While this is not of itself problematic, it is difficult to understand, particularly in the light of the arguments below, how such frameworks are not consequences of particular cultural contexts drawing on both systemic imperatives and didactic folklore.

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As colleagues will know, this is an allusion to the 2002 book, *Beliefs: A hidden variable in mathematics education*, edited by Gilah Leder, Erkki Pehkonen and Günter Törner

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As can be seen from the above, with few exceptions, most mathematics teacher knowledge research has been undertaken by US-based researchers, many of whom seem to have overlooked the possibility that teachers' mathematical knowledge, as manifested in their observable behaviour, is a cultural construction, as evidenced by other US-based researchers (Hiebert et al., 2003; Schmidt et al., 1996; Stigler, Gallimore, & Hiebert, 2000). In this paper, a framework for analysing teachers' mathematical knowledge that explicitly acknowledges the cultural discourse in which mathematics teaching and learning occur is proposed. It does not seek to replace existing models, but to complement them. Admittedly, some teacher knowledge researchers have acknowledged the limitations of work undertaken in single cultural contexts (Ball et al., 2008), but few have considered whether frameworks developed in one cultural context are applicable in another (although the studies of An, Kulm, and Wu (2004) and Delaney, Ball, Hill, Schilling, and Zopf (2008) are exceptions). This is an issue of some salience. For example, discussing Spanish teachers' subject knowledge, Escudero and Sánchez (2007, p. 314) examine several textbook representations of Thales' theorem, one of which reads "if several parallel straight lines (AA,´ BB,´ CC)´ are cut by two transversal lines (AC, AĆ)´, the ratio of any two segments of one of these transversals is equal to the ratio of the corresponding segments of the other transversal". Such presentations, largely unknown in current English texts, not only highlight differences in curricular expectations, but also provoke the reaction that whether they are categorised as common or specialised content knowledge is probably of less importance than whether they should be required knowledge of teachers working within curriculum frameworks that do not privilege them. Thus, in an attempt to shift teacher knowledge from the personal construct embedded in much recent research, this paper considers teacher knowledge as a social construction located within particular cultural contexts.

## Mathematical Knowledge in Teaching: A Culturally-Located Model

It is probably not an unreasonable conjecture that not only are teachers' actions reflections of their goals, but also that their goals reflect an idealised view of what it is they want to achieve for their students. In this respect, Reeve and Jang (2006) describe two forms of long term goals focused respectively on learner autonomy and learner conformity, while others, in accordance with psychological research, have discussed goals as focused on learner mastery or learner performance (Wolters & Daugherty, 2007). In respect of this study, the professional goals and ambitions teachers have for their students are construed as idealised learning outcomes relating to knowledge, skills and dispositions that are addressed by an equally idealised set of didactics that may or may not be linked with learner autonomy or conformity, mastery or performance orientation. In this manner, teachers' long term goals reflect an *idealised* curriculum. For example, an *idealised* curriculum may relate to learner acquisition of adaptive expertise, or the flexible application of an integrated and connected set of concepts and procedures (Baroody, 2003; Kilpatrick, Swafford, &

Findell, 2001). Another may reflect Dutch expectations in which learner experiences and teachers' structuring of mathematics are located in problems that are imaginably real to the learner (Van den Heuvel-Panhuizen, 2003). Importantly, Andrews' (2007a) investigation of English and Hungarian mathematics teachers' professional goals found most English teachers articulating goals concerning mathematics as applicable number and the means by which learners are prepared for a world beyond school, while their Hungarian colleagues privileged mathematics as problem-solving and logical thinking. Such differences highlight two characteristics of the idealised curriculum: it is located in individual experience and it is articulable.

According to Hufton and Elliott (2000, p. 117) teachers' practices are so "deep in the background of the schooling process ... so taken-for-granted ... as to be beneath mention". In this regard, a number of researchers have attempted to "reveal taken-for-granted and hidden aspects of teaching" (Hiebert et al., 2003, p. 3) and have unveiled unnoticed but culturally-located practices characteristic of the systems under scrutiny (Schmidt et al., 1996; Stigler et al., 2000). The consensus seems to be that teachers employ pedagogical strategies which, through repeated enactment, are not only typical of a country's lessons, but also beneath their consciousness (Cogan & Schmidt, 1999). This sense of typicality has found confirmation in Andrews' research in which mathematics teachers from four European countries have been observed to behave, at least as far as seven generic learning outcomes and ten generic didactic strategies are concerned, in ways that align them closely with their national colleagues and distinguish them from their overseas colleagues (Andrews, 2007b, 2009a, 2009b). Explanations suggest that cultures "shape the classroom processes and teaching practices within countries, as well as how students, parents and teachers perceive them" (Knipping, 2003, p. 282). Thus, it seems that teachers' actions, in addition to being informed by individuals' *idealised* curricula, are informed by culturally-located and beneath articulation *received* curricula. The *received* curriculum, characterised by its hidden, inarticulable, properties is amenable only to inference and can be construed as a set of collective practices and goals.

Of course, irrespective of the *idealised* and *received* curricular determinants of their actions, most teachers work within systemically defined curricular frameworks, which the second international mathematics study described as *intended* curricula – systemically located expectations of learner outcomes which frequently reflect historical values and imperatives. An assumption too frequently made in mathematics education circles is that mathematics curricula, particularly in culturally similar countries, share many similarities and few substantial differences. In this regard, a brief analysis of the mathematics curricula of Flanders and Hungary and their expectations with regard to linear equations, the countries and topic represented in this chapter, is salient.

The Flemish curriculum for mathematics[1] in the first grade of secondary education is located in three domains: number theory, algebra and geometry. Within

[1] See http://www.ond.vlaanderen.be/dvo/english/

each strand are three core objectives concerning (1) concept formation and knowledge of facts; (2) procedures; and (3) cohesion between concepts. The particular expectations for simple linear equations are:

- First grade of secondary education: Use letters to represent generalisations and unknowns, solve equations of the first degree with one unknown and simple problems, which can be converted to such an equations.
- Second grade of secondary education: Solve equations of the first and second degree in one unknown, and problems which can be converted into such equations.

The generic expectations focus on issues concerning language, problem solving as both a mathematical activity and as a means modelling the real world, 'the importance and the need for providing proof, which is inherent in mathematics', and a need for students to 'develop self-regulation by focusing on the problem, planning, executing and monitoring the solution process; develop self-confidence as a result of successfully solving mathematical problems; develop a sense of independence and determination in tackling problems'.

The Hungarian curriculum[2] for grades 5–8 (upper primary) includes an introduction locating the learning of mathematics within a developmental framework, acknowledging explicitly the transition from concrete and inductive to abstract and deductive. It addresses the affective domain while making no concession to ensure that children are presented with intellectually challenging mathematics. These aims are supported by five broad themes concerning the application of acquired mathematical concepts: the development of a mathematical approach; problem-solving skills and logical thinking; the application of acquired learning methods and thinking; and developing the right attitude towards learning. Curriculum content is also characterised according to five broad themes, namely: methods of thinking, algebra and arithmetic, mathematical relations, functions and sequences, geometry, and probability and statistics. First degree equations with a single unknown are covered in each of the 4 years. Students should

- Year 5: Solve simple equations of the first degree by deduction, breaking down, checking by substitution along with simple problems expressed verbally.
- Year 6: Solve simple equations of the first degree and one variable with freely selected method.
- Year 7: Solve simple equations of the first degree by deduction and the balance principle. Interpret texts and solve verbally expressed problems. Solve equations of the first degree and one variable by the graphical method.
- Year 8: Solve deductively equations of the first degree in relation to the base set and solution set. Analyse texts and translate them into the language of mathematics. Solve verbally expressed mathematical problems.

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[2]See http://www.okm.gov.hu/letolt/nemzet/kerettanterv36.doc

Such details highlight differences in the underlying systemic conceptions of mathematics and its teaching and, it is argued, confirm that teachers' expected subject knowledge is clearly a function of the system in which they work. For example, the Hungarian curriculum's developmental framework, reflected in the annual visitation and incremental conceptual growth of linear equations, differs markedly from the espoused expectations of the Flemish.

In sum, teachers' mathematical knowledge in teaching is a social construction drawing, inter alia, on the culturally-located *idealised*, *received* and *intended* curricula. It is conjectured that the closer the three are aligned, the more coherent both subject knowledge and its didactic manifestation are likely to be; although if systemic expectations are limited, then even closely aligned curricula may result in limited opportunities for learning. Also, of course, if the three curricula are unaligned, then the result may be didactic anarchy, as in the case of Mrs. Oublier whereby reform-oriented practices were incorporated alongside unconsciously held traditional beliefs about mathematics teaching and normative classroom behaviours (Cohen, 1990). In this chapter, the tripartite framework of *idealised*, *received* and *intended* curricula is examined from the perspective of its explaining variation in the ways in which teachers present mathematics to their students. To achieve this objective, two sequences of lessons taught on linear equations to students in grade 8 in Flanders and Hungary are examined. Each sequence was taught by a teacher defined locally as effective in the manner of the learner's perspective study (Clarke, 2006).

## The Project

Funded by the European Union, the Mathematics Education Traditions of Europe (METE) project examined aspects of mathematics teaching in Belgium (Flanders), England, Finland, Hungary and Spain. The main dataset comprised video recordings of four sequences of lessons taught in each country on agreed topics by teachers defined locally, in the manner of the learner's perspective study (Clarke, 2006) as effective. After recording, videotapes were downloaded and compressed for ease of sharing and coded against a generic schedule developed in a bottom-up and iterative manner during the first year of the project. Full details of this process can be seen in Andrews (2007c), although it is probably sufficient to say that the final coding schedule comprised seven generic learning outcomes and ten generic didactic strategies which project colleagues thought, on the basis of a year's live observations, reflected well their perceptions of the mathematics teaching of the five countries.[3] Codes were applied to the episodes of a lesson, where an episode was defined "as that part of a lesson in which the teacher's observable didactic intention remained constant" (Andrews, 2007c, p. 499), with no limit to the number of codes that could be applied. The only criterion was the presence of a learning

[3] Working definitions of the codes can be seen below in Table 7.2.

outcome or didactic strategy at some point during the episode. Several quantitative analyses have been undertaken and these have proved effective in highlighting similarities and differences in the emphases found in the respective countries' episodes (Andrews, 2007b, 2009a, 2009b). However, much work has still to be done in respect of qualitative analyses, and this chapter represents a first pass at that process.

Two sequences of four lessons on the topic of linear equations, taught to grade 8 students in Flanders and Hungary, are reported below. The topic was chosen because it reflects an important transition as mathematics passes from concrete and inductive to abstract and deductive. Particularly pertinent to the analysis presented in this chapter is research highlighting a distinction between arithmetical and algebraic equations. On one hand, arithmetic equations, with the unknown on one side only, are generally assumed to be susceptible to undoing (Filloy & Rojano, 1989). On the other hand, algebraic (non-arithmetic) equations, with unknowns on both sides, cannot be solved by arithmetic-based approaches and require not only that the learner "understand that the expressions on both sides of the equals sign are of the same nature (or structure)" (Filloy & Rojano, 1989, p. 19), but also that they are able to operate on the unknown as an entity and not a number. Thus, arithmetic equations are procedural, while algebraic or non-arithmetic equations are structural (Kieran, 1992). The choice of these two sequences was based on the availability of English language mathematics curricula, and because they provided interesting similarities and differences in the ways the two teachers concerned – Pauline in Flanders and Eva in Hungary – conceptualised and presented this iconic topic to their students. These similarities and differences, drawing on the generic learning outcomes and didactic strategies exploited in the METE project, can be seen in Table 7.1. In respect of similarities of learning outcomes, for example, both teachers emphasised student acquisition of conceptual knowledge and procedural knowledge, while

**Table 7.1** Percentage of all Flemish and Hungarian episodes coded for each of the generic learning outcomes and didactic strategies alongside the same for Pauline and Eva

| | Flanders | Pauline | Hungary | Eva |
|---|---|---|---|---|
| Conceptual knowledge | 71 | 44 | 64 | 40 |
| Derived knowledge | 5 | 4 | 6 | 0 |
| Structural knowledge | 17 | 8 | 40 | 20 |
| Procedural knowledge | 57 | 80 | 51 | 87 |
| Mathematical efficiency | 13 | 12 | 36 | 33 |
| Problem-solving | 7 | 4 | 31 | 33 |
| Reasoning | 35 | 20 | 45 | 13 |
| Total episodes | 111 | 25 | 78 | 15 |

| | Flanders | Pauline | Hungary | Eva |
|---|---|---|---|---|
| Activating | 23 | 20 | 35 | 47 |
| Exercising | 3 | 4 | 5 | 0 |
| Explaining | 52 | 44 | 59 | 67 |
| Sharing | 61 | 40 | 97 | 100 |
| Exploring | 6 | 4 | 0 | 0 |
| Coaching | 39 | 48 | 45 | 40 |
| Assessing | 20 | 32 | 36 | 27 |
| Motivating | 10 | 12 | 46 | 20 |
| Questioning | 49 | 28 | 87 | 100 |
| Differentiating | 6 | 12 | 0 | 0 |

neither attended to derived knowledge. However, there were substantial differences in the emphases placed on structural knowledge, mathematical efficiency and problem solving. Thus, the two teachers offer interesting and culturally different contexts for discussing the proposed framework.

Both teachers were in their late twenties with between 6 and 7 years' experience. The Flemish lessons were drawn from a middle track class in an unremarkable comprehensive in a provincial university city while the Hungarian derived from a Budapest gimnazium. Each sequence is reported as accurately as data permit. Videographers were instructed to focus on the teachers whenever they were speaking. Teachers wore radio-microphones while a static microphone was placed strategically to capture as much student talk as possible. Lessons were transcribed and subtitles constructed so that colleagues from each country could watch any lesson from another. Thus, any description of a lesson will be informed by several layers of interpretation and choice – videographer, transcriber, interpreter and writer. The descriptions below are also informed by a decision to focus attention on the activities and tasks teachers present and the manner in which they implemented them. As Thompson, Carlson, and Silverman (2007, p. 416) observe, "... tasks do not have agency. Tasks do not elicit behavior any more than a hammer elicits hammering". Thus, the manner in which the two teachers presented their tasks and the means by which this presentation facilitated student agency informed the interpretation and reporting of the two sequences. The order of the two sequences is determined by nothing other than alphabetical order.

## Pauline

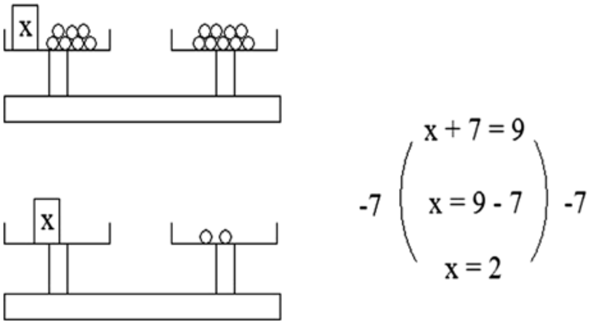
Pauline's class comprised 25 grade 8 students of average attainment. It was clear from conversations between teacher and students that simple linear equations had been covered earlier in the students' learning of mathematics; an experience, drawing on a vocabulary of task and outcome, where the unknown was always located on the right hand side of an equation and solved by a process of intuitive undoing.

The *first* lesson began with Pauline posing a problem involving characters from the cartoon series The Simpsons: if Bart, Lisa and Maggie, are 7, 5 and 0 years old respectively and their mother, Marge, is 34 years old, in how many years would the sum of the children's ages equal their mother's? Pauline drew a table of values before completing, collaboratively, the first three columns.

| | | | | |
|---|---|---|---|---|
| Marge's age | 34 | 35 | 36 | ... |
| Children's total age | 12 | 15 | 18 | ... |

Individual completion of the remaining columns was followed by a discussion during which the solution of 11 years and the fact that with every year's increase in Marge's age the sum of the children's increased by three were agreed. Pauline then introduced an unknown, $x$, to represent the number of years to pass before the two sums would be equal. This led, after several closed questions, to her writing that Marge would acquire $34 + x$ years, while the children would reach $7 + x$, $5 + x$ and $0 + x$ respectively. Lastly, she wrote $34 + x = 7 + x + 5 + x + 0 + x$, which she simplified to $34 + x = 12 + 3x$.

After this Pauline demonstrated how each of the two rows of the table of values could be represented graphically to show an intersection after 11 years. This brief exposure to an alternative approach was followed by her modelling, by means of questions and the introduction of the balance, the solution to $x + 7 = 9$. She sketched the left-hand image of the two below and asked how the unknown, $x$, could be found. A student suggested that removing seven from both sides would not only maintain the balance but also provide a unique value for $x$, as reflected in the second image. The process was then summarised symbolically as shown. This was followed by Pauline modelling, by means of the bracketed process, solutions to $x - 2 = 10$, $3x = 8$ and $\frac{x}{3} = 7$.




Finally, in this particular episode, Pauline drew the table below on an OHT and explained, with no reference to the balance or any questions posed to her class, the relationship between each of her four exemplars to their respective formalisations. During this time, she included aspects of the history of equations on the same slide and their significance in the work of, for example, Descartes.

| | | |
|---|---|---|
| a = b | ⇒ | a + c = b + c<br>a – c = b – c<br>a. c = b. c<br>a: c = b: c |

The lesson ended with an exercise involving problems of the form, $x - 3 = 10$, $200 - x = 20$ and so on. The solutions to these were to be placed in a crossword-like grid.

The *second* lesson began with students continuing the exercise started the previous lesson. After several minutes Pauline initiated a class discussion focused on particular difficulties. For example, she spent some time discussing the solution to $\frac{2x}{3} = 30$ and how division by 3/2 was equivalent to multiplying by its inverse. The class then returned to the exercise before answers were presented on the overhead projector and students corrected their own work. During both periods of seatwork Pauline circulated the room helping individuals. In this early part of the lesson, a student asked whether they should continue to use the balance method and was told that, for the time being, she should continue to use it.

In the next phase of the lesson, Pauline wrote $6(x - 5) - 8 = x - 3$ on the board and indicated to her students that this was a substantially more difficult equation than those previously experienced but that, if they concentrated on what she was about to show them, they would soon be able to solve it confidently. When asked about the differences between this new equation and those solved earlier, one student commented that there was an $x$ in both the *task* and the *outcome*. Pauline commented that they were no longer to think about equations as *task* and *outcome* but to talk about *left terms* and *right terms*. Following this, Pauline began a formal treatment in which the algebra, including actions, was written on the left side of the board and justificatory annotations on the right. Throughout the process Pauline questioned continuously. The following is what she wrote.

$6(x - 5) - 8 = x - 3$ (1) Eliminate brackets

A discussion followed in which Pauline drew from her students notions of associativity and commutativity before settling on distributivity as the warrant for what she was about to do. This included an aside, written on a different board, during which she discussed which rule would be applied to $6.(5x)$. A student initially proposed the brackets rule (*de haakjes regel*) before Pauline steered them to associativity.

$6x - 30 - 8 = x - 3$ (2) Calculate if possible

$6x - 38 = x - 3$

$-x(\ ) - x$

$6x - 38 - x = -3$

Although this was not annotated, Pauline asserted the need to collect like terms and, in particular, get the unknowns to one side and the numbers to the other.

$5x - 38 = -3$ (3) Get x in one term and the rest in the other

$+\,38(\quad) + 38$

$5x = -3 + 38$ (4) Calculate if possible

$5x = 35$

:5( ):5 (5) Divide both terms by the coefficient of x (Pauline encouraged her students to use the expression *factor* of $x$)

$$x = \frac{35}{5}$$

$$x = 7$$

Having obtained a solution and discussed the uniqueness of the value obtained, Pauline undertook a check. This took the following form.

$$\begin{aligned} 6 \cdot (7 - 5) - 8 &= 7 - 3 \\ 6 \cdot 2 - 8 &= 4 \\ 12 - 8 &= 4 \\ 4 &= 4 \end{aligned}$$

The *third* and *fourth* lessons followed similar forms. They began with Pauline revisiting the equations posed at the end of the previous lesson by means of closed questions supplemented by overhead transparencies on which she had prepared solutions. Subsequently, students were invited to work on selected problems from their text while she circulated the room helping individuals before further discussion of solutions. One point of interest arose during the public solution to an equation in which all coefficients were fractions; Pauline explained to her students that by rewriting all fractions over a common denominator and then multiplying the whole equation by the common denominator reduced the equation to one with integer coefficients that could be solved as all those that had gone before.

## Eva

This sequence of lessons was taught to a class of 20 grade 7 students in a Budapest gimnazium. The *first* lesson began with Eva presenting several open sentences, some of which were mathematical, to revisit notions of truth and the role of the basic set in determining the validity of a statement. This was followed by a brief discussion in which she defined an equation as comprising two expressions connected by an equals sign and that such expressions may or may not contain variables or unknowns depending on circumstances. Throughout, Eva wrote much on the board, although it is not clear whether or not students were expected to make notes. Next, Eva posed an exercise in which three open sentences were to be solved in relation to a basic set defined as integers in the range $-3 \leq \square \leq 3$. The open sentences were $5 - \square = 8$, $5 - \square > 6$ and $\square .2 = 7$.

Once solutions or ideas had been shared, Eva posed several oral problems like, “Kala is twice as old as her sister. The sum of their ages is 24, how old are they?” Each was solved individually before students shared their solution strategies. At the close of this episode, Eva commented that such problems could be solved mentally with the application of logic.

Eva now split the class into four groups. Each was given a different word problem for translating into an equation. Eva made it clear that she did not expect the equation to be solved, just constructed. The first group’s problem was:

> Some friends went on a trip. The first day they covered just 2 km. The second day they covered 2/10 of the remaining journey. If they covered 6 km on the second day, how long was their journey?

Each group worked for several minutes before a representative demonstrated how its equation had been derived from the text. Eventually, all four stories had been heard and four identical equations, $0{,}2.(x - 2) = 6$, had been written on the board. This process lasted many minutes with Eva offering many prompts before all the equations had emerged correctly. She then recounted the story for each equation before inviting approaches to their solution. A volunteer proposed a *thinking backwards* strategy and, showing no anxiety concerning division by 0,2, obtained a value of $x = 32$. Eva then checked this solution against the text of each problem.

This was followed by a new word problem:

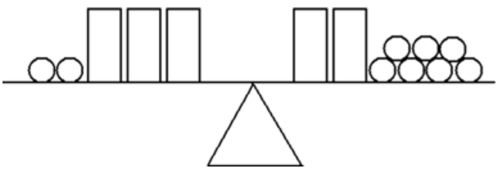
> On two consecutive days, the same weight of potatoes was delivered to the school's kitchen. On the first day, 3 large bags and 2 bags of 10 kg were delivered. On the second day, 2 large bags and 7 bags of 10 kg were delivered. If the weight of each large bag was the same, what weight of potatoes was in the large bag?

As before, Eva asked her students to construct an equation and soon, despite some hesitation concerning units, a girl wrote on the board $3x + 20 = 2x + 70$. Eva asked whether their intuitive strategies would be sufficient and suggested that the balance principle would be able to help them. The class was questioned as to what this meant and several contributed suggestions indicating their understanding of its function in relation to equations. With help from her students, Eva wrote

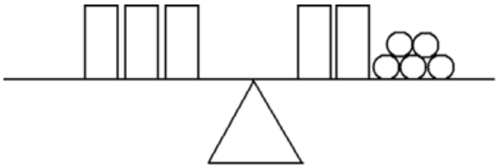
$$\begin{aligned} 3x + 20 &= 2x + 70| - 20 \\ 3x &= 2x + 50| - 2x \\ x &= 50 \text{ kg} \end{aligned}$$

She reminded the class of the need to check and did so, substituting 50 back into each expression separately before comparing for equality. Lastly, she posed some algebraic equations for homework.

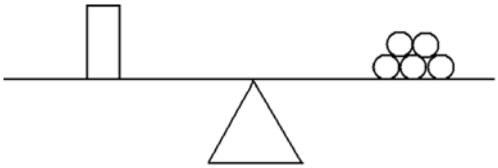
The *second* lesson began with Eva, by means of much questioning, revisiting the previous lesson. When her students reached the potato problem, she reminded them of the scale principle and, having reminded them of the importance of specifying the meaning of $x$, drew the following.



Drawing on what her students suggested, Eva then rubbed out two small bags on each side, leaving.



Finally, she erased two large bags from each side to show that one large bag balanced 5 bags of 10 kg.



On completion of the drawings, Eva asked students how the diagrams could be represented symbolically. The students offered sufficient for her to repeat what she had written on the board the previous lesson. At this point, she reminded her class of the importance of checking and did so.

Eva then set about checking the previous lesson's homework. With contributions from various students, some oral and others written on the board, the class worked publicly through solutions to several equations with the unknown on both sides. Throughout, Eva focused her students on the balance principle and, in relation to one particular problem, highlighted the difficulty of expressing negative terms on the scales. On several occasions, more than one solution was reported for an equation with all being compared and points concerning elegance and efficiency made. In every case, Eva insisted that solutions were checked.

After this, an exercise was set. The first, $10a - 4 + 3a - 11a = 2 + 4a + 4 - 7a$, was solved collectively with Eva orchestrating the process. Then students worked quietly on their own before Eva paused to consider the solution to an apparently problematic equation; $7.(2 - c) + 5 - 4.(c - 8) = -4 - 3.(c + 3)$. A student went to the board and, after prompting, eliminated the brackets before completing the rest unproblematically. Finally, two similar equations were set for homework.

The *third* and *fourth* lessons, following the sharing of homework solutions, comprised sequentially posed and publicly solved equations. Some were located entirely within a world of mathematics while others were first derived from word problems. One word problem, by way of example, was "A stake is driven through a pond into the ground. If 1/4 of the stake's length is in the ground, 3/5 in water and 2.8 m above the water, how long is the stake?" Thus, in this and other equations, coefficients were frequently fractions.

## Discussion

In the following, the two sequences of lessons are examined for their similarities and key differences. This provides a basis for evaluating the effectiveness of the tripartite curriculum as a complementary framework to the Shulman-related model discussed above. To facilitate this process, the percentages, drawn from Andrews (2009a, 2009b), of all Flemish and Hungarian episodes coded for each of the generic learning outcomes and didactic strategies are presented in Table 7.1, alongside the same summary statistics for the episodes of each of the two teachers. Details of the codes are beyond this chapter, although working definitions can be seen in Table 7.2. Each teacher is discussed separately with the *intended* curriculum being considered before the *received* and *idealised*, due to the former being a systemic bench mark. As above, Pauline is considered first.

### *Pauline*

Firstly, in respect of the *intended* curriculum and the particular expectations relating to equations, Pauline unambiguously and consistently addressed letters as unknowns and solutions to first degree equations. There were occasional episodes in which concept formation was addressed, as in her use of the Simpsons' problem and the invocation of the balance when solving $x + 7 = 9$. She placed considerable emphasis on the development of students' procedural knowledge. Indeed, it could be argued that the key objective of all four lessons was the development of procedural fluency. Also, despite her use of terminology relating to arithmetical structures as well as the extended and formalised discussion of the use of the denominator during the fourth lesson, there were few attempts to address cohesion between topics. In respect of generic expectations, problem solving was limited to challenging context-independent equations with no attempt to derive equations and then solve them. In a similar vein, Pauline made no attempt to engage her students in real world modelling. Lastly, there were several occasions when Pauline seemed to have been caught off-guard by her students' suggestions. In each case, she promised to address students' concerns or suggestions once she had completed her planned activity but never did, indicating rare opportunities for students to acquire self-regulatory competence. Overall, the resonance between the systemic expectations and Pauline's observed behaviour is strong in some areas and weak in others. Her competence with the substantive and syntactic knowledge pertaining to equations seemed secure, although her application of that knowledge to contexts other than those of mathematics itself seemed limited. In short, her mathematical knowledge in teaching appeared less resonant with the *intended* curriculum than Flemish authorities might have wished.

Secondly, there are aspects of Pauline's practice that resonate with what the literature indicates is a Flemish mathematics education tradition, although, as will be discussed below, there is also variation from it. For example, Pauline's failure to address in any significant manner mathematical problem solving and unambiguous

**Table 7.2** Working definitions of the codes used in Andrews (2009a, 2009b)

| | |
|---|---|
| *Observable didactic strategies (Andrews, 2009b)* | |
| Activating prior knowledge | The teacher focuses learners' attention on mathematics covered earlier in their careers as preparation for activities to follow |
| Exercising prior knowledge | The teacher focuses learners' attention on mathematical content covered earlier in their careers in the form of a period of revision unrelated to any activities that follow |
| Explaining | The teacher explains an idea or solution. This may include demonstration, explicit telling or the pedagogic modelling of higher-level thinking or procedures. In such instances, the teacher is the informer with little or no student input |
| Sharing | The teacher engages learners in a process of public sharing of ideas, solutions or answers. The teacher's role is one of manager rather than explicit informer |
| Exploring | The teacher explicitly engages learners in an activity which is not teacher-directed, from which a new mathematical idea is explicitly intended to emerge |
| Coaching | The teacher offers hints, prompts or feedback to facilitate their understanding of or abilities to undertake tasks or to correct errors or misunderstandings |
| Assessing | The teacher explicitly assesses or evaluates learners' responses to determine the overall attainment of the class |
| Motivating | The teacher, through actions beyond those of mere personality, explicitly addresses learners' attitudes, beliefs or emotional responses towards mathematics |
| Questioning | The teacher explicitly uses a sequence of questions, perhaps Socratic, which lead pupils to build up new mathematical ideas or clarify or refine existing ones |
| Differentiation | The teacher treats students differently in terms of the kind of tasks or activities, the kind of materials provided and/or the kind of expected outcome in order to make instruction optimally adapted to the learners' characteristics and needs |
| *Observable learning outcomes (Andrews, 2009a)* | |
| Conceptual K. | The teacher encourages the conceptual development of his or her students |
| Derived K. | The teacher encourages the development of new mathematical knowledge or entities from existing knowledge or entities |
| Structural K. | The teacher emphasises connections between different mathematical entities |
| Procedural K. | The teacher encourages the acquisition of skills, procedures, techniques or algorithms |
| Efficiency | The teacher encourages learners' development of procedural flexibility, awareness of elegance or critical comparison of working |
| Problem solving | The teacher encourages learners' engagement with the solution of non-trivial or non-routine tasks |
| Reasoning | The teacher encourages learners' development and articulation of justification and argumentation |

emphasis on procedural knowledge accords with earlier findings that Flemish mathematics teaching is largely transmissive (Waeytens, Lens, & Vandenberghe, 1997) and privileges declarative knowledge and lower-order procedural skills (Janssen, De Corte, Verschaffel, Knoors, & Colemont, 2002) above those of problem solving and adaptive expertise (Verschaffel, De Corte, & Borghart, 1997). When the quantitative data derived from Pauline's lessons are considered, it would seem that she placed greater emphasis on procedural knowledge and lower emphases on conceptual knowledge, structural knowledge and reasoning than her Flemish colleagues (Andrews, 2009a). Moreover, in respect of her didactic emphases, she questioned and shared more rarely than her colleagues and this, coupled with a tendency to coach and assess more frequently (Andrews, 2009b), could be construed not only as distinguishing her practice from the Flemish collective, but further evidence of a teacher who, through frequent checks that students were succeeding with the work set, places great value in procedural competence. In summary, the evidence highlights elements of Pauline's *received* curriculum that identify her with her colleagues and elements that suggest divergence from it.

Thirdly, in respect of the *idealised* curriculum, some conjectures are possible. Pauline attempted neither to engage her students in translating word problems into manipulable symbolic forms nor to offer them opportunities to model real world situations. Moreover, on the single occasion she derived an equation from a word problem – in this case an algebraic equation – she abandoned it and introduced the balance as a strategy for solving arithmetic equations. However, introducing analytical approaches to solve equations amenable to a process of reversal is didactically unproductive (Pirie & Martin, 1997; Nogueira de Lima & Tall, 2008). Additionally, when she derived the equation, little input was sought from her students, reflecting a practice whereby the problems posed, despite being mathematically challenging, were never resolved in a genuinely collaborative manner, with collectively undertaken activity always leading to a predetermined outcome. These latter observations, further supported by her frequent use of pre-prepared OHTs comprising model solutions, allude to a conception of teaching in which the role of the teacher is to structure learners' opportunities so tightly that not only is her students' learning trajectory entirely predetermined but also any potential deviations are thwarted. Thus, in sum, Pauline's *idealised* curriculum, despite the complexity of the problems posed, seemed to reflect a teacher-centred and procedurally-focused perspective on mathematics teaching and learning. The totality of the above suggests that her *idealised* curriculum was at odds with elements of the *intended* curriculum and, in the characteristics of her observed practice, that her *received* curriculum also diverged from that of the Flemish collective.

## *Eva*

In terms of the *intended* curriculum, Eva's lessons show adherence to systemic developmental expectations, with the explicit curriculum content of the 4 years of equations being observed at various times in the first two lessons. For example, the year 5 and year 6 objectives were observed subsequent to the activity in which each of four groups translated a word problem into the same equation. The emphasis on the balance was not only introduced at the end of the first lesson, but also repeated systematically at the start of the second, where clear links were made between the concrete and abstract. The early emphasis on the translation of word problems, an objective of the year 7 curriculum, was observed in all four lessons, while relating the solving of equations to the base and solution sets was explicitly addressed at the start of the first lesson in a manner that indicated that this was not an unfamiliar aspect of their work. Problem solving skills were regularly addressed in both the translation of complex word problems into equations and the expectation that the students would solve non-routine equations involving unknowns on both sides along with negative numbers, fractions and brackets in various manifestations. Also, the constant sharing of solutions provided ample opportunity for students to engage with and explain their mathematical reasoning. In sum, the evidence suggests that Eva adhered closely to systemic expectations in respect of her presentation of both linear equations and generic learning outcomes.

In respect of the *received* curriculum, some interesting insights emerged. Firstly, Eva's use of the balance in solving the potato problem reflected very closely a lesson observed by Andrews (2003) in which the teacher, László, not only presented pictures of the balance alongside a symbolic representation but also located his entire exposition around a physical balance and small bags containing an unknown number of glass marbles. Thus, it is not inconceivable that such presentations form part of a received didactic culture. Eva's lessons not only adhered closely to a previously observed cycle of problem posing, solving and sharing (Andrews, 2003; Szendrei & Torok, 2007) but also reflected a tradition in which concrete materials and drawings are used to scaffold students' learning of mathematics (Depaepe, De Corte, Op't Eynde, & Verschaffel, 2005). The problems posed were frequently difficult. For example, the translation from text to symbols of some of the word problems leading to the shared equations of lesson one, and the pole driven through water into the ground of lesson four were challenging, and presented not inconsiderable difficulty for some students. Additionally, Eva's constant invocation of brackets, negatives and fractions imparted a different sense of difficulty in accordance with earlier findings that Hungarian teachers operate with the general rather than the particular (Andrews, 2003), while her consistently high expectations in respect of procedural competence resonated closely with earlier observations (Andrews, 2003, 2007b, 2009a; Depaepe et al., 2005). Thus, there is considerable evidence that Eva's classroom behaviours, as manifestations of her *received* curriculum, resonate closely with what the literature says of Hungarian teachers generally. However, when compared with her colleagues, the data presented in Table 7.1 show, as with Pauline above, lower emphases on conceptual knowledge, structural knowledge and reasoning and a substantially increased emphasis on procedural knowledge. In respect of her didactic practices, the same table shows little substantial variation between Eva's practice and that of her colleagues other than a lower emphasis on explicit motivational strategies.

In relation to her *idealised* curriculum, several inferences can be made. The ways in which Eva facilitated the collective construction of both procedural and conceptual knowledge through the use of non-routine problems reflects, it is argued, social constructivist principles. Moreover, her frequent use of realistic word problems, as construed by the Dutch realistic mathematics education tradition as imaginably real (Van den Heuvel-Panhuizen, 2003), presents an atypical perspective from the Hungarian norm in which teachers rarely pose problems related to any context other than mathematics itself (Andrews, 2003). Also, the manner in which Eva constructed her students' engagement with both conceptual and procedural elements indicates an implicit emphasis on their acquisition of adaptive expertise. Thus, it seems that Eva's *idealised* curriculum is located in beliefs about collectively constructed knowledge, which, supported by the systematic use of realistic problems, facilitate her ambitions concerning learner acquisition of adaptive expertise.

In sum, despite some apparent discrepancies, the evidence indicates a close resonance between the three curricula: the *intended* is reflected closely in Eva's observed practice. Eva's *received* curriculum resonates well with both the *intended* and the collective Hungarian, which seems equally resonant with her *idealised*. There are differences, but it could be argued that Eva's use of realistic problems allows her to not only address the *intended* curriculum with authority but also raise her practice above that of the collective *received.*

## Conclusion

Quantitative analyses indicated that Pauline's and Eva's observable learning objectives differed from the collectives of their respective countries. For example, both were observed to privilege procedural knowledge while simultaneously placing lower emphases on conceptual knowledge, structural knowledge and reasoning. This highlights, I propose, the significance not only of acknowledging the cultural context in which teaching and learning occur, but also the topic under scrutiny. The lessons above were both on linear equations, a topic with limited opportunities for teachers to focus on, say, conceptual and structural knowledge, particularly when compared with many topics in, say, geometry. In similar vein, linear equations present fewer opportunities for high-level reasoning but many more opportunities for procedural work than would the angle properties of polygons; both Pauline and Eva offered considerable variation in respect of the exercises they posed, drawing on a various forms of coefficients, exploiting brackets in different ways and so on. This would be in contrast to the limited procedural opportunities embedded in, say, the angle sum of a triangle. Thus, their observable behaviours may not have deviated quite as far from the collective *received* curriculum as initially suggested. Other deviations from the *received* curricula, as in Pauline's lower emphasis on questioning and sharing, are conjectured to reflect individual *idealised* curricula and are not necessarily topic related.

Of course, it would have been possible to analyse the subject knowledge manifestations of both Pauline and Eva against existing frameworks. For example, Pauline's apparent reluctance to deviate from her planned sequences of activity could be construed as low-level contingency (Rowland et al., 2005). However, whether or not this reluctance is a reflection of, essentially, a deficit in her pedagogic repertoire or a desire not to deviate from a well-defined and articulated procedure is difficult to determine. What is clear, although possible explanations can be found in her *idealised* curriculum, is that her observed behaviours set her apart from both the *intended* expectations of her educational system and the *received* practices of her colleagues. In similar vein, would the content knowledge observed in Eva's lessons be manifestations of Ball et al.'s (2008) *common* or *specialised* content knowledge? The answer to this, I propose, requires acknowledgement of both *intended* and *received* curricula. For example, the expectations of mathematics for all learners in Hungary, at least as far as the *intended* and *received* curricula indicate, are high. Consequently, common content knowledge in that country would be qualitatively different from countries where systemic expectations are low. In summary, the above shows that mathematical knowledge in teaching is a relative and not an absolute construct and confirms that the proposed tripartite curriculum model provides a worthwhile, but complementary, alternative to existing frameworks.

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# Chapter 8
# How Educational Systems and Cultures Mediate Teacher Knowledge: 'Listening' in English, French and German Classrooms

**Birgit Pepin**

## Introduction

In recent years, the question of teacher knowledge has received an increasing amount of attention from researchers who have investigated the professional knowledge of teachers from different angles. It is accepted that what teachers know is one of the most important influences on what happens in classrooms. It has also been established that the conceptual tools that teachers possess, in order to deal with their work, depend to a large extent on the cultural and systemic traditions of the educational environment in which they work (Andrews & Hatch, 2000; Hiebert et al., 2003; Pepin, 1999).

From my own research (e.g. Pepin, 1999; Pepin & Haggarty, 2003), that of colleagues, and larger-scale studies such as TIMSS (e.g. Hiebert et al., 2003), it is clear that the work of teaching differs from country to country (e.g. Cogan & Schmidt, 1999). Whilst the quantity and quality of teachers' mathematical knowledge has been an area of great concern (e.g. Ma, 1999), it is, however, less clear how to measure teacher knowledge, what it consists of and how it is comparable across countries. Comparisons of, or simply 'looking into', different knowledges may develop deeper understandings of what we mean by 'knowledge in/for teaching'.

For more than 10 years, I have studied mathematics teachers and their curricular practices in mathematics classrooms in different countries, particularly in England, France and Germany. The goal of these studies has been to develop a deeper understanding of what is going on in mathematics classrooms at the lower secondary level, especially with respect to teaching and learning mathematics with understanding, and to the influence and nature of curricular materials, such as texts, used in classrooms. The comparative perspective has helped to highlight the particular features of teachers' pedagogic practice, to discover alternatives and, in turn, to develop a deeper understanding of those features and practices, so as to stimulate discussion about choices within teachers' immediate environments and countries.

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Teachers have to make decisions and to assess whether students understand the tasks and activities provided; support pupil learning and perhaps adapt the pace of instruction accordingly; initiate and maintain discussions when appropriate; challenge and further student thinking; perhaps adapt the materials, or indeed choose purposefully among materials, and so on (e.g. Ball, 1997; Even & Tirosh, 1995, 2002). Different classroom environments and cultures, constraints and affordances provided by different settings and opportunities for developing particular mathematical practices are likely to influence teachers' perceptions of what it means to teach and learn mathematics with understanding, as well as what kinds of knowledge are needed to do that. Teaching mathematics successfully means identifying with and applying the norms of the classroom community which is likely to be different in different contexts, whether they vary from school to school (e.g. Eisenman & Evan, 2007), or from country to country (e.g. Stigler & Hiebert, 1999). Teachers need knowledge of, and perhaps 'internalised' these norms.

In terms of what knowledge is brought into play in any classroom, Sherin (2002) found that in classroom exchanges of ideas, teachers typically negotiate between three areas of knowledge: their understanding of subject matter, their perception of curriculum materials and their personal theories of learning. Effective teachers, she argues, 'weave' between these areas of knowledge (and deepen their own understanding of them) so as to increase student understanding. This reflection-in-action can either take place in the classroom (e.g. Sherin, 2002), or beyond. These studies argue that it involves, in the first instance, listening carefully to students' expressions of mathematical content (Davis, 1997), and that, by doing so, teachers notice significant mathematical moments and respond appropriately (e.g. Sherin, 2002).

Moreover, a large number of studies (e.g. Boaler, Wiliam, & Brown, 2000; Clark, 1997) have reported the cognitive advantages to students from being able to participate in mathematical discussion. However, pedagogic practice that is able to move students' thinking forward involves more than developing a respectful and trusting environment for discussion and problem solving (O'Connor & Michaels, 1996). 'Effective' pedagogy demands careful attention to what students have to say and what they do; 'noticing' and listening carefully is needed (Yackel, Cobb, and Wood, 1998), as well as interacting 'knowledgeably' at critical moments (Jaworski, 1994). Researchers have provided evidence of the critical role of the teacher in listening to students and orchestrating mathematical discourse (e.g. Manoucheri & Enderson, 1999). The literature reveals that teachers' sensibility for redirecting discussion to ensure that important mathematical ideas are being developed is dependent on a range of pedagogical content knowledge skills (e.g. Turner et al., 1998). Broad guidelines have been provided, for example, by the National Council of Teachers of Mathematics (2000, p. 19), and these include what teachers might do to enhance effective classroom discourse: "Effective teaching involves observing students (and) listening carefully to their ideas and explanations". They argue that effective pedagogy demands careful attention to students' articulation of ideas. Franke and Kazemi (2001) claim that effective teachers try to 'delve into the minds' of students by noticing and listening carefully to what they have to say.

Thus, here the notion of ‘listening’ takes on a form different to how it may be perceived in everyday language. Typically, there is a notion that teachers ‘tell’ and students listen; teachers teach and students (have to) learn. I argue that teachers (have to) ‘listen’ in order to teach, and to listen to student talk amongst other sources. Listening becomes teacher learning to teach and requires action: it is an active rather than a passive process. The notion of listening to teach implies what to listen *for* as well as *how* to listen (Schultz, 2003), under particular circumstances and in particular contexts, as part of the knowledge in and for teaching.

‘Knowledge in/for teaching’ can take different shapes. Placing ‘listening’ at the centre of teaching is not a common thing to do and stands in contrast to many countries’ prescriptive texts from which mathematics teachers are expected to teach. Ball (1993, p. 388) emphasises that adopting a listening approach to teaching mathematics is complex, and that it is hard: “The ability to *hear* what children are saying transcends disposition, aural acuity, and knowledge, although it also depends on all of these.” In this chapter, I explore teacher knowledge with respect to ‘listening’ and ‘hearing’ students in English, French and German classrooms.

## Listening to and ‘Hearing’ Students

Trying to find out what and how students are learning is central to teaching, although it seems impossible to know with certainty what students learn. Teachers make sense of their students’ work and understanding nearly all the time when working with them in classrooms; it is not a separate activity but an integral part of instruction. Listening to students ‘carefully’ seems to be an important factor in this and, indeed, is assumed in terms of constructivist teaching: “. . . researchers and teachers must learn to listen and to hear the sense and alternative meanings in these [students’] approaches.” (Confrey, 1991, p. 111)

However, this is a difficult task and many studies have identified the difficulties in attending to pupil understanding and strategies (e.g. Even & Wallach, 2004). Mason (2002) illustrates that noticing what students say and do is complex; he proposes the *discipline of noticing* which enhances awareness and sensitivities to student experiences. Moreover, there are various ways in which teachers can listen to their students’ mathematical ideas. Davis (1997) outlines three different orientations:

(1) an evaluative orientation – where teachers listen to students’ ideas in order to diagnose and correct their mathematical understanding;
(2) an interpretative orientation – where the purpose is to access pupil thinking, rather than to assess;
(3) a hermeneutic orientation – where teachers listen by engaging pupils in the process of negotiation of meaning and understanding.

Despite reform efforts, the first appears, interestingly if expectedly, to be the most common where teachers tend to ‘tell’ and explain rather than ‘listen’ (Crespo, 2000).

Further, Ball (1997) coined the term 'to hear students' when trying to understand what students know: to understand what they say, do and show. She identified three main "challenges":

(1) the challenge to 'listen across divides' – teachers facing the difficulty to understand what students (who are usually younger than they are) think, say and mean, within structures perhaps not familiar to them;
(2) the challenge to listen through the multiple influences of contexts – teachers facing the 'variability in students' thinking shaped by context and moment' (Ball, 1997, p. 800); and
(3) the challenge to listen with and through desire – teachers facing the disappointment when their students are confused, and conquering the 'fragility of understanding' when probing for clarity and evidence of student understanding.

These three ways of 'hearing students' potentially provide a useful tool to analyse teacher action and pedagogic practice.

Moreover, Arcavi and Isoda (2007) described and analysed an approach to develop teachers' 'productive listening capabilities'. They define 'listening' to students as "giving careful attention to hearing what students say (and to see what they do), trying to understand it and its possible sources and entailments." (p. 112) According to their view, it is not a passive attitude, but should include the following components:

- Detecting, taking up, and creating opportunities in which students are likely to engage freely, expressing their mathematical ideas;
- Questioning students in order to uncover the essence and sources of their ideas;
- Analysing what one hears (sometimes in consultation with peers) and making the enormous intellectual effort to take the 'other's perspective' in order to understand it on its own merits; and
- Deciding in which ways the teaching can productively integrate students' ideas (p. 112).

They see this approach not only as beneficial for implementing a constructivist approach, but also in terms of affective value and role model provision: students feel that their voice is heard, they feel valued, and they may listen to others (peers or the teacher) more carefully. In turn, this may provide opportunities to reflect on their and others' thinking, which is likely to develop deeper mathematical understandings.

Summarising these three positions, it seems that there is no 'common use' of the term 'listening' in teaching: 'to listen' and 'to hear' students seem to be used nearly interchangeably. However, there is a common notion of listening as paying careful attention to what students say and do, and this is often linked to student experiences in the classroom and, further, to their developing understandings. Conceptualising teaching as listening suggests that the teacher is always learning and that this shapes his/her decision making in the classroom (as reflection in action) and contributes to teachers' professional growth.

Teaching is often perceived as 'telling': teachers talk and students listen. Instead, and leaning on previous education research (e.g. Schultz, 2003), I put listening at the centre of teaching: to listen to teach mathematics. Rather than a passive stance, listening to teach necessitates action, and the act of listening is based on student-teacher interaction and it focuses on meaning-making in the classroom. This potentially includes listening to what students say, what they write and do, perhaps their gestures and, moreover, the immediate classroom environment and wider context in which the lesson takes place. Listening to teach mathematics involves knowledge of the learners (and their development), knowledge of the immediate classroom situation and the wider socio-cultural context and, of course, knowledge of the subject matter. This provides the framework for my analysis.

## Teacher Knowledge, Pedagogic Practice and Classroom Environments

In terms of comparative education research in mathematics classrooms, there is a variety of studies linking teacher knowledge and pedagogic practice. Stigler and Hiebert (1999), for example, argue that different beliefs about the nature of mathematics, the nature of learning, the role of the teacher, the structure of the lesson, and teacher responses to individual student differences lead to different modes of instruction in the US and Japan. Others argue that there is a 'globalisation effect' in terms of moderation across models of schooling in national education systems (Meyer, Ramirez, & Soyson, 1992) and, still others argue that these interact with national educational traditions in ways that influence mathematics lessons and classrooms (LeTendre et al., 2001). As an example, Delaney, Loewenberg Ball, Hill, Schilling, and Zopf (2008) argue that international comparisons of teachers' mathematical knowledge need to be considered in the light of differences that may exist in the knowledge that teachers use in each country. These studies highlight the complexity of, and the difficulty in, determining correspondence between constructs in the different contexts.

However, there is an agreement that knowledge in and for teaching is 'situated', that is 'knowledge of and adapted to particular contexts' (Putnam & Borko, 2000). Putnam and Borko (2000, p. 13) argue that "this professional knowledge is developed in context, stored together with characteristic features of classrooms and activities, organised around the tasks that teachers accomplish in classroom settings, and accessed for use in similar situations." Similarly, in her work on pupil knowledge and identity construction, Boaler (2002, p. 1) views knowledge "not as an individual attribute, but as something that is distributed between people and activities and systems of their environment". She contends that:

> What is fundamental to the situated perspective is an idea that knowledge is co-produced in settings, and is not the preserve of individual minds. Situated perspectives suggest that when people develop and use knowledge, they do so through their interactions with broader social systems. This may mean that they are learning from a book (written by others) or

> teacher, or engaging in individual reflection of some socially produced ideas. But the different activities in which learners engage co-produce their knowledge, so that when students learn algorithms through the manipulation of abstract procedures, they do not only learn the algorithms, they learn a particular set of practices and associated beliefs (Boaler, 2000, p. 3).

Thus, these authors propose a shift from a focus only upon knowledge, to one that attends to the inter-relationships of knowledge, practice and identity. Moreover, there is a shift from teacher or pupil knowledge to knowledge that is constituted through the course of mutual engagement and interactions. It is a proposal that is fundamental to Lave and Wenger's (1991) social practice theory, where the notions of 'community of practice' and 'connectedness of knowing' are central features (Walshaw & Anthony, 2008).

## The Study

In a previous study (Pepin, 1997, 1999), I have developed an understanding of varying practices in mathematics classrooms in England, France and Germany, using an ethnographic framework. It emerged that national educational traditions are a large determinant and influence on teachers' pedagogies in the 3 countries. More recently, I have investigated with Linda Haggarty, the ways in which mathematics textbooks are used by teachers in English, French and German lower secondary classrooms (Pepin & Haggarty, 2001; Haggarty & Pepin, 2002). This work suggested that the use of curricular materials (such as textbooks), together with the selection of (mathematical) tasks, impacts to a large extent the mathematical 'diet' offered to students.

For this chapter, I have re-analysed some of the data collected over the years in terms of teacher knowledge for 'listening'. The selected data (for this study) consisted of extended lesson observations and interviews with 12 teachers, four in each country, plus shorter observations and interviews with an additional ten teachers in each country. I re-analysed the data on the basis of my understandings of 'listening' and using a socio-cultural approach to gain new understandings about teacher knowledge and listening to pupils. The main questions addressed are:

(1) How do teachers perceive 'knowledge for/in teaching'?
(2) How do teachers 'listen' to students, making use of their own knowledge for/in teaching? What are the characteristically different ways of 'listening'? What are the similarities and differences?
(3) How is this knowledge influenced by/embedded in the educational environment?

A procedure involving the analysis of themes similar to that described by Woods (1996) and by Burgess (1984) was adopted. Moreover, I tried at one level to maintain the coherence of the teacher cases through a holistic story of the case that is respondent-validated by participant teachers and anchored on their own interviews and my observations. At another level, I analysed across teacher cases using my conceptual framework of ‘listening’, testing the hypotheses offered by the literature, and building explanations and theorisations grounded in the data. On a third level, I looked for similarities and differences of teacher ‘listening’ across country cases. However, due to the additional cross-cultural dimension, it was important to address the potential difficulties with cross-national research, in particular, issues related to conceptual equivalence, equivalence of measurement, and linguistic equivalence (Warwick & Osherson, 1973; Pepin, 2002). Particularly important were the findings of Delaney et al. (2008), who compared teacher ‘mathematical knowledge for teaching’ across the US and Ireland, highlighting the value of validity checks of constructs in both contexts. In this respect, it was important to locate and understand teacher pedagogic practices and the classroom cultures in England, France and Germany, and it was useful to draw on knowledge gained from earlier research which highlighted the complex nature of teachers’ work and classroom environments in the 3 countries, in addition to potential influences (e.g. systemic developments and educational traditions).

## Mathematics Classroom Environment

Teachers (and learners) of mathematics at secondary level work in different environments in England, France and Germany. Whereas in England and France, most pupils go to comprehensive schools, in Germany, pupils are divided into those going to the local grammar school *Gymnasium* (about 40%), to the technical school, *Realschule*, or the secondary modern school, *Hauptschule*. Furthermore, pupils in the 3 countries experience different organisations of schooling, which in turn have implications for the ‘mathematical diet’ they are provided with, and experience. In England, most schools apply a ‘setting system’ to teach mathematics in perceived ability sets, using differentiated texts which provide sets at different ‘levels’ with different mathematical ‘diets’. In France, most pupils are taught in mixed-ability groups (and provided with the same textbook): pupils are said to be ‘entitled’ to the same curriculum. In Germany, pupils are effectively streamed by the three school types, but within those streams they are taught in mixed-ability groups. The three school types also have their own mathematics curricula and textbooks and, notably for teacher knowledge, different types of teacher education.

In previous studies (e.g. Pepin 1999), I identified characteristic ‘profiles’ of classroom situations in England, France and Germany. Teachers assigned significance and value to particular practices which are commonly concerned with pupil engagement and assessment of understanding. For example, in the English classroom, the main aim was to (relatively briefly) explain a particular mathematical notion and let pupils get as much practice as possible. Of particular importance was that pupils were attentive during teacher explanations and subsequently worked on their own whilst teachers attended to individual pupils’ needs. The French teachers regarded their main aim as facilitating mathematical thinking by initiating tasks and helping pupils to think around a particular concept, in whole-class conversation, as individuals or in groups, followed by practice. Thus, of particular importance was that pupils would discover the concept with the help of selected cognitive activities. The main objective in the German mathematics classrooms was to discuss mathematical content. Teachers initiated tasks or discussed exercises from the homework in a conversational style before giving pupils exercises to practice on their own. They particularly valued that most pupils would be involved in a teacher-led discussion about the mathematical content.

Moreover, there appeared to be particular ‘customary ways’ that all teachers used in their teaching. For example, teachers in all three countries asked pupils to work on exercises from textbooks for a considerable amount of time so that pupils could practice what has been explained and teachers could monitor understanding. However, in England, many pupils at Key Stage[1] 4 and almost all at Key Stage 3 had not been issued with a textbook to use in school and at home; they only worked from textbooks during lessons under teacher guidance. Thus, it is likely that the majority of these pupils only ever had access to the textbook in class, and consequently, had to rely entirely on teacher-guided input. In France, the situation was quite different: every pupil had a textbook provided by the school to be used in school and at home. In Germany, pupils had to buy their own textbooks which were selected by schools/teachers from a ministry-approved range. Thus, already at the outset, there are differences in the roles and importance assigned to textbooks in the classroom environment, and for students, in terms of access to texts (Pepin, 2009a).

In recent analyses (Pepin, 2009a, 2009b), I have investigated the role of textbooks and particular constructs such as ‘negativity’ (Vlassis, 2004) in the ‘figured worlds’ (Holland, Lachicotte, Skinner, & Cain, 1998) of English, French and German classrooms. I have argued that the ‘figured worlds’, in which pupils work, influence the development of their identities as learners of mathematics and are different across the 3 countries. These were linked to the mathematical tasks provided and mediated by teachers, the practices that pupils are engaged in when doing those tasks, and the environment they work in and experience in class. Analysing mathematical tasks (in selected textbooks), it was found that the majority of textbook tasks were procedural in nature, and while they allegedly help pupils to become ‘procedurally fluent’ (Kilpatrick, Swafford, & Findell, 2001), they also portray this as necessary to becoming a competent learner, whilst at the same time, it can be argued, obscuring the meaning and concept of mathematics. In terms of teacher mediation of those tasks chosen mostly from textbooks, providing students with a series of ‘coherent’ and appropriate tasks seemed important, and these depended to a large extent on teachers’ beliefs and the environment in which they worked. Within the limits of the system, whether students were taught in mixed classes (France), setted (England), or streamed (Germany), teachers had the freedom to select tasks that could potentially guide their instruction, and to mediate those tasks in ways they thought best. These classrooms set the context of activity and provide the frames whereby meanings of actions are mediated and conveyed (Pepin, 2009a).

[1] In England, compulsory schooling is divided into four key stages. The teachers in this study taught pupils in Key Stage 3 (age 11–14) and Key Stage 4 (14–16).

## Teacher Knowledge and Listening to Pupils

### *Content Knowledge for Teaching*

When talking about mathematical knowledge for teaching, most people would probably argue that content knowledge matters for teaching. Much research has gone into this, and concepts such as pedagogical content knowledge (Shulman, 1986) have been further developed and refined (e.g. Ball, Thames, & Phelps, 2008). When asked explicitly about what knowledge is necessary for teaching mathematics, most teachers in the study emphasized mathematical content knowledge (see Table 8.1). However, how they 'defined' this was different. English teachers claimed that it means "maths-wise to be confident and competent", which includes having sufficient knowledge "if kids go off on a little bit of a tangent which has relevance". They also talked about knowing how to make the mathematics 'digestible' for the pupils/group they teach, to 'adapt any topic in a hundred different ways according to what children (one is) teaching'.

French teachers also pointed to subject knowledge and interestingly linked it to the ability to 'step back' from the mathematics content. Teachers emphasised the 'distance' (*recul*) that a teacher needs to have, with respect to his/her subject.

> . . . to have enough knowledge of the subject, of the mathematics, in order to have enough distance in terms of what one teaches (Teacher 3, France- my translation)
>
> At the pedagogic level one has to accept to step back, in terms of mathematical knowledge . . . (Teacher 1, France- my translation)

This is an interesting notion which certainly involves a process of reflection. This reflection is likely to involve consciously thinking about one's experiences with the mathematics, turning ideas over in one's head, looking at things from a different perspective, stepping back to review things, and consciously deciding what one is doing and why. This process is likely to increase knowledge of the subject.

In Germany, the *Hauptschule* teachers emphasized in the study the importance of subject knowledge and elaborated on it in terms of 'conveying the content correctly', whereas *Gymnasium* teachers highlighted aspects of 'logical thinking' in connection with it. Logic was seen as the basis for their mathematics teaching and learning and teachers worried that pupils often had problems with logic and reasoning. The second most important knowledge aspect was knowledge about the children, in the sense that *all* students must be heard (not only those with their hands up). In particular, the *Hauptschule* teachers stressed the 'background knowledge of the children' in order to "be able to act educationally soundly in problem situations, not only through negative sanctions." (Teacher 1, Germany- my translation)

Thus, it appears that even similar kinds of knowledge, commonly referred to as mathematics 'subject' or 'content' knowledge, are perceived differently in different educational environments. This is most 'visibly' illustrated by the German case teachers who worked within one country and *Land*, but in different school types within that *Land*. This also implies that they have gone through different

**Table 8.1** 'Knowledge' in/for teaching mathematics

| | | 'Listening knowledge' in/for teaching | | |
|---|---|---|---|---|
| | Knowledge in/for teaching | Individual child/learner | Group/class/set | Environment/context |
| English teachers | Subject knowledge: Mathematical content knowledge<br>How to make the mathematics 'digestible'/Adaptation of mathematics according to ind./group | Important to get to know pupils (mathematical background)<br>Practices: Circulating room; individual attention/help; ind. misconceptions | "There is no fixed formula" but always related to group/set/level of group<br>Practices: "hands-up" Encourage discussion | Comprehensive schools & setting<br>Never enough time: "we are under the most relentless pressure" |
| French teachers | Subject knowledge<br>'Stepping back' from the content (*recule*)<br>Multiplicity of ways of solving a problem and teaching a topic | "Test pupils" – "at the border line of having understood or not" | Several pupils at the board<br>Practices: Identification of common mistakes/misconceptions<br>Sharing methods/making solutions available for everyone<br>Discussion | Comprehensive schools & mixed ability (*collège*)<br>Difficulty of keeping pupils/class at the 'same speed', 'same rhythm'<br>Practices: to 'keep the class together' (e.g. dictation) |
| German teachers | Subject knowledge:<br>HS: to be 'correct'<br>GS: to think logically<br>To know the children:<br>HS: pers. background<br>GS: common problems connected to logic and reas. | HS: pupil background (pastoral)<br>GS: Practices:<br>Test pupil on board | GS: Pupil mistakes – as sites for learning<br>GS: Practices:<br>Whole class discussions to 'involve everybody in thinking process' | Tripartite system & streaming<br>HS: pastoral care 'heavy'<br>GS: importance of mathematical reasoning, logic and proof |

teacher education. Teacher education for *Hauptschule* teachers shares the patterns of primary school teacher education; it may be argued that it also shares its ‘philosophy’, that is the ‘education of the child’, which would explain teachers’ discourse and emphasis of pastoral responsibilities in terms of teacher knowledge. *Gymnasium* teacher education focuses on the subject matter (and its teaching), which may explain the emphasis on logic and reasoning in their explanations of subject knowledge.

## *‘Listening Knowledge’ in/for Teaching*

Listening to pupils includes ways of figuring out what students are learning, and also includes ways of monitoring students in terms of involvement and with respect to difficulties with the mathematical content (Wallach & Even, 2007). Teachers constantly ‘read’ their students; they make judgments about how things are going, for the group as a whole, as well as for particular students (Ball, 1997). But listening is also an essential feature of the classroom culture in terms of equitable learning environments.

What it means to listen can be illustrated by describing instances from classroom episodes, teacher actions and ways of ‘hearing’ students. In the classrooms studied, there were characteristically different teacher practices: differences in the ways they approached ‘listening’ to pupils. Teachers talked and reasoned about their practices differently, in the sense that they used differently figured ‘knowledges’ to ‘hear’ students (see Table 8.1).

From what teachers said and did in their classrooms, teacher listening can be divided into broadly three categories: those that relate to the individual learner, to the group as a whole, and to the educational environment. In *England*, the mathematics teachers studied geared their teaching to the group/set of children they had in front of them, “. . . [and] that’s why it is important to get to know them as well as you can . . .” (Teacher 1, England). For them, there was no ‘fixed formula’ or ‘best way of teaching a topic’. Instead of considering the situation in mathematical terms, the emphasis was on knowledge about (1) the group and set in order to pitch the lesson at the right level; (2) about the individual children, so that the teacher would know about their background in terms of mathematical experiences; and (3) how the children feel on that particular day.

“I think every time I teach something I do it differently, according to how they are, how they feel, what their experiences [are] and so on . . .” (Teacher 1, England). Pupil engagement was an important aspect of these teachers’ pedagogic practices, and they had particular ways of ‘hearing’ and finding out whether individual pupils were truly involved or not, and how students could be re-engaged.

> “I try and get them involved as much as I can . . . I look to who is concentrating, who is may be not . . . try and involve them with eye contact if they are drifting a bit . . . I try and get kids put their hands up . . . and work out if I have lost them or whether they are just not concentrating . . . as much pupil involvement as possible . . . I also try to get them to explain things . . .” (Teacher 2, England)

When pupils were working on tasks, teachers typically circulated round the room and attended to individual pupils or small groups, which gave them additional information. It appears that there were a number of indications for teachers, that children had or had not understood, and these needed careful 'listening', which included knowing about and watching out for particular signs. For example, 'the number of hands up', or 'how confident [pupils] look', were the usual cues, or the 'weaker ones tend[ed] to pull faces'.

One teacher pointed out that this is something that needed to be learnt over the years and she provided strategies what to look out for:

> "A lot comes with watching the children do maths ... if possible getting them to say what they are doing as they are doing it, to watch them do it so that one can see the processes they go through ... learn from them what the processes involve ... mathematical knowledge and then this ability to break down the procedures and to understand what the child is thinking ... if they have done some wired method, to work out what they have done, why they have done it, whether it is a genuine method." (Teacher 2, England)

Thus, for her, listening to children discuss mathematics, watching them do it and considering their ideas carefully, was likely to develop teacher knowledge in teaching. She was very clear that the source of this teacher knowledge lay with the pupils, and that what is needed from the teacher was an 'open eye' and an 'attentive ear'.

In summary, a characteristic of *English* mathematics teachers' listening practice was that they attended to individual pupils more than their 'continental' colleagues, in the sense that relatively little time was spent with the group as a whole and more time on talking and listening to individual pupils. 'Individualistic listening', a characteristic of English teacher listening practices, is supported by their educational environment including the setting system, in the sense that each child is expected to get the individual support she/he needs to progress. Teachers generally dismissed any 'best practice' for teaching, as it did not reflect the personal nature of learning; thus they felt that they needed to know a 'hundred different ways' of teaching and learning a topic, to be able to attend to their pupils appropriately. This attention to the individual, in turn, meant not only that each pupil was likely to get a little time, on average, from the teacher, but also that teachers themselves felt pressurised. There was never enough time to 'do it all': "we are under the most relentless pressure". More generally, teachers appeared to feel responsible for the learning of their pupils and there was the expectation that the teacher would identify those who did not understand and help them along.

In contrast, *French* teachers attended and listened to the group as a whole, in particular, to pupils' mathematical misconceptions and common errors. In terms of pedagogic practice, teachers typically got up to four pupils to the board and asked them to do different exercises and explain their work to the whole class. They also identified 'test' pupils: those who would be representative of a group of pupils who had not understood.

> Whilst they are at the board it allows me to see what they have done ... very often I ask several to the board ... that allows me to correct relatively quickly ... I try to take those

> pupils who are just at the border line of having understood and not . . . (Teacher 1, France- my translation)
>
> . . . either I check in their books, or I have one or two 'test pupils' . . . when I realise that there is one who has not understood, that means that there are five or ten [who have not understood] depending on the pupil. (Teacher 1, France- my translation)

For monitoring learning of a whole group, teachers needed in-depth knowledge about their pupils' developing understandings (and whom to take to the board) about common errors and the ways these can be used as sites for learning in a whole class context. In fact, misconceptions and mistakes appeared to be accepted as learning sites. The conversations in class often centered on the nature of the method that was used and mistakes were viewed as methods that could be improved.

> The most important [things] are the mistakes . . . for the same multiplication there were about 15 different answers and we try to understand the different mistakes they have made . . . what interests me, I told them, is that they tell me what mistakes they have done, that we look at them, that we try to understand them and that they subsequently don't do them any more . . . (Teacher 3, France- my translation)

Thus, errors were viewed as useful points of discussion; pupils were very willing and were used to share their understandings with peers and with the teacher. More importantly, mistakes were regarded as "natural and constructive consequences of building improved methods of solving problems" (Hiebert et al., 1997, p. 168). This not only developed a classroom culture based on 'listening' to each other's conceptions, but also an atmosphere where students understood that they were responsible for the correctness of their own work. It appeared to be the teacher's role, dependent to a large extent on his/her knowledge and experience, to make these solutions available for everyone, to share their knowledge, and allow pupils to use the analysis of methods and the mathematical reasoning related to it to determine correctness. Students appeared to accept this responsibility.

Interestingly, a characteristic of French teachers was that they talked about keeping pupils working more or less at the 'same speed'. Considering that they were teaching in mixed-ability classes, this needed considerable skill and careful 'listening' to the pace of the class. Their strategies to 'keep the class together' varied; one of them was dictation.

> Rather than distributing a worksheet, I also like to dictate the text (problem), specially in geometry . . . that allows everybody to work more or less at the same speed, and from time to time that obliges them to listen, to be attentive [rather than doing the things immediately] . . . they are forced to follow a bit . . . (Teacher 3, France- my translation)

In terms of exploration and open problems, it appeared to be more difficult to 'walk the same rhythm'. "I like to make them explore a bit, but to have everybody search at the same speed, the same rhythm, is difficult to manage . . ." (Teacher 1, France- my translation). However hard they tried, teachers realised that this aim was not manageable with the 'heterogeneity' of their classes, and it concerned them.

In summary, a characteristic of the French mathematics teachers' listening practices was that they tried to figure out where most of their pupils were (in terms of understanding the mathematical concept), calling several pupils to the board and modelling the problem-solving processes on the board, working with the whole group in a collaborative activity. The focus of classroom interactions was problem-solving methods; it required careful listening to pupils' explanations (also those at the board) and suitably guiding the whole group to act on the information in some way, such as using it to help solve another problem. Mistakes and common errors were seen as a natural part of the process of improving methods of solution: they were sites for learning. Working in mixed-ability classes, teachers developed skills to listen to the group as a whole. Working at the 'same speed' and 'keeping the class together' were central concerns in this mixed-ability group environment.

Amongst *German* teachers, a distinction has to be made between *Gymnasium* and *Hauptschule*. As explained earlier, their working conditions in terms of pupil intake and school culture as well as their teacher education, were quite different. Accordingly, the *Hauptschule* teachers had more pastoral care responsibilities than their *Gymnasium* colleagues, which meant that they 'listened' differently and to different things. In more concrete terms, they lamented that, typically, at the beginning of each lesson they were "bombarded with things that have nothing to do with the subject instruction . . . [as a form tutor he had to deal with] problems of pupils, also private (home) problems . . . [problems with] friendships . . . (Teacher 1, Germany-my translation). Listening to pupils' problems became a necessary part of the start of the lesson, where pupils appeared to 'pour' their problems onto the teacher. Teachers often felt 'angry' that 'too much and too often' things were fore-grounded that had nothing to do with their (mathematics) lessons. Thus, balancing the pastoral side (including listening to pupil problems) and the subject-oriented tasks seemed to be a major dilemma for those teachers and they felt that they could not do justice to either side.

Similar to their French colleagues (albeit in a mixed-ability teaching environment), the German *Gymnasium* teachers typically listened to the group by calling a pupil (not several) to the board to explain a particular task which was subsequently discussed with the whole group, and teachers used reasoning and proof to guide their explanations. When a task was presented to a whole class by the teacher, a considerable amount of time was spent on discussing how to address that problem, on questions and comments, potential solutions, comparison of answers, and so on. Doing mathematics meant spending time to reflect, listening, explaining, restructuring and trying out methods; in short, thinking about and reasoning within mathematics. This involved skilful guidance by the teacher, listening to pupil explanations and 'hearing' their problems and developing understandings. Interestingly, teachers mentioned the 'time aspect': they "waited for a relatively long time for an answer, so that as many as possible get thinking." In fact, they appeared to be committed not to rush their lessons.

Indeed, one *Gymnasium* teacher explicitly referred to "the ability to listen [to pupils]" as an essential quality of a teacher. She believed that many of her colleagues had an 'underdeveloped' understanding of pupils' problems (with mathematics) because they did not listen and understand what pupils were saying and meaning.

> I have often experienced that colleagues . . . don't really listen, because they are not able to leave their own train of thoughts and to follow another person's thinking processes . . . not able to pick up what [pupils] are saying . . . although teachers should be able to. (Teacher 3, Germany- my translation)

In summary, a characteristic of the German mathematics teachers' listening practices was that they were different according to the environments in which they were working. *Hauptschule* teachers would have needed a large repertoire of strategies and knowledge to teach the mathematics in a 'difficult' environment and allow for access and equity in their classrooms. This was hard, often impossible, for teachers to establish because a great part of their time was spent on pastoral responsibilities and 'listening' to pupils' private problems. The *Gymnasium* teachers' practices involved collaborative working in a whole-class context, in whole-class discussions, and concentrating on mathematics (rather than pastoral aspects). Teacher listening to pupils and pupils listening to each other were important aspects of teachers' strategies for developing pupil understanding, and for involving every pupil in the thinking processes (e.g. reasoning and proof) that are typical for mathematics.

## Discussion and Conclusions

Exploring teacher knowledge with respect to listening to pupils is not common, perhaps even less in terms of teacher listening (to learn) to teach mathematics. Teacher teaching is often equated with 'telling' and pupil learning with listening. I argue here that it is worth considering 'listening' in terms of 'listening to teach mathematics', which implies that the knowledge that both bring to the situation (e.g. about the learner, about the situation and about mathematics) constitute the starting place for the teaching (and learning) of mathematics (Schultz, 2003).

Teachers do not often mention their 'listening' to pupils. However, seen in a different light, listening is an essential feature of teaching and learning mathematics and of the mathematics classroom culture. If the role of the teacher is to create a classroom in which "all students can reflect on mathematics and communicate their thoughts and actions" (Hiebert et al., 1997, p. 29), then listening must be an important part of it. All of our teachers in England, France and Germany 'listened' to their pupils in different ways and under different 'conditions' afforded by diverse environments in which teachers (and pupils) worked and learnt. It appears that the three different educational systems and environments provide and 'create' varying contexts for teaching and learning mathematics and that different knowledge is needed to be effective in each.

In summary, 'teacher listening to pupils' was different in the English, French and German classrooms studied in the sense that teachers needed different kinds of knowledge to listen 'appropriately' within their respective environments. Even similar kinds of knowledge (e.g. subject knowledge) appeared to be differently situated in these culturally figured worlds. Whereas in one context (England), content knowledge was seen to serve the adaptation of the mathematics to become 'digestible' for the students (a practical consideration), in another (France), the emphasis was on the development of the knowledge residing within the teacher and reflection (stepping back) was necessary for that. The German cases also illustrated the influence of context and environment on the knowledge perceived to be appropriate for teaching mathematics: in one context (*Hauptschule*) subject knowledge was about the 'correctness' of the mathematics; in another (*Gymnasium*) about 'thinking logically' – different types of subject knowledge.

The crucial point for listening is the 'quality of listening' and this is differently perceived in different educational environments. Whereas in one environment (England), it appeared that quality of listening means 'listening' to individual pupils talking about and developing their understandings of mathematics, in others (France and German *Gymnasiums*), teachers develop listening skills for the group as a whole in order to involve everybody in mathematical discussion and thinking processes. Keeping pupils at the 'same speed' is a concern, in particular, in a mixed-ability learning environment. These differences in the 'quality of listening' help us to better understand the extent to which mathematical knowledge for and in teaching is culturally specific.

Moreover, knowledge in and for teaching can be applied as well as developed through 'listening to' and 'hearing' pupils: applied in terms of knowledge of mathematics, for example, to analyse an error mathematically with a group of pupils; developed in terms of, for example, developing a repertoire of common problems students have with a particular task/topic.

Considering 'listening' seriously may help us to redefine, further theoretically develop and analytically clarify teacher knowledge for/in teaching, and drawing on international comparisons may help us to sharpen and deepen our constructs. A clearer sense of the categories of content knowledge for teaching (see Ball et al., 2008), for example, might inform the design of support materials for teachers. Locating 'listening to teach' at the centre of teaching mathematics means taking a learning stance: learning from (and 'listening' to) the spoken and written words, what was left unsaid and gestures. Listening is an active process that may allow for 'quiet change'.

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# Chapter 9
# Modelling Teaching in Mathematics Teacher Education and the Constitution of Mathematics for Teaching

**Jill Adler and Zain Davis**

## Introduction

The QUANTUM[1] research project in South Africa has as its central concern answering the question of *what* is constituted as *mathematics in and for teaching* in formalised in-service teacher education in South Africa and *how* it is constituted. Entailed in the question is an understanding that, in practice, selections of content in mathematics teacher education are varyingly drawn from mathematics and the arena of education (including mathematics education, teacher education and teaching experience). Debate continues as to whether and how mathematics teacher education programmes should integrate or separate out opportunities to learn *mathematics* and *teaching*. Programmes range across a spectrum of integration and separation of mathematics and teaching, including variations in the degree to which opportunities for teachers to learn both mathematics and teaching are presented as embedded in problems of practice. Hence our concern with what, how and with what possible effects mathematical knowledge and related practices are constituted in and across a range of programmes, across diverse teacher training institutions in South Africa.

Our study has included three cases from three different teacher education sites where teachers were enrolled in in-service 'upgrading' programmes: two cases specialising in a fourth and final year of accredited mathematics teacher education, and the other specialising at the honours level.[2] In our analysis we were struck by the

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[1]QUANTUM is the name given to a Research and Development project on quality mathematical education for teachers in South Africa. The development arm of QUANTUM focused on *qu*alifications for *t*eachers *u*nderqualified in *m*athematics (hence the name) and completed its tasks in 2003. QUANTUM continues as a research project.

[2]In South Africa, teachers are required to obtain a 4-year post-school qualification in education to practice. Those teachers who obtained only 3 (or fewer) year qualifications under previous dispensations are now required to enrol for further study on in-service programmes to 'upgrade' their teaching qualifications.

T. Rowland, K. Ruthven (eds.), *Mathematical Knowledge in Teaching*, Mathematics Education Library 50, DOI 10.1007/978-90-481-9766-8_9, 

observation that in each case teachers were presented with strong, though different, images of the mathematics teacher and, thereby, of mathematics teaching. This is no surprise. As a professional practice, we expect aspects of practice to be modelled and further that such modelling will vary across programmes and contexts. Our primary interest was, however, not in modelling per se, but in how the modelling of mathematics teaching related to the constitution of mathematics in each case. In this chapter, we describe our observations and the analytic resources recruited to that end, building on previous work reported in Adler and Davis (2006), Davis, Adler, and Parker (2007), Adler and Huillet (2008). We will argue that three different orientations to learning mathematics for teaching are exhibited across our cases – referred to here as 'look at my practice', 'look at your practice' and 'look at (mathematics teaching) practice' – and present different opportunities for learning mathematics in and for teaching.

We begin with a discussion of teacher education in South Africa, and a location of the chapter in debates on mathematics for teaching.

## Mathematics Teacher Education in Post Apartheid South Africa

Fifteen years into the new democratic dispensation in South Africa, school mathematics remains an area of national concern, a critical element of which is the preparation and development of mathematics teachers. Shortages of secondary school teachers persist, as do concerns with the quality of mathematics teaching and poor learner performance across grade levels (Carnoy et al., 2008). As is well known, the majority of black secondary teachers who trained under apartheid had access to only a 3-year College of Education diploma. The quality of that training in general and in mathematics in particular was, by and large, poor (see Welch (2002) for a more detailed discussion). Consequently many current secondary mathematics teachers have not had adequate opportunities to learn further mathematics and/or study school mathematics from a teaching perspective. Formal upgrading programmes for teachers – specifically, an Advanced Certificate in Education (with Mathematics specialisation) – continue to be offered. In initial teacher education, in addition to the usual degree plus Post-Graduate Certificate in Education, secondary mathematics teachers can qualify by obtaining a Bachelor of Education (B.Ed.) programme currently being implemented in some Higher Education Institutions, including that of one of the authors. A specialization for teaching mathematics in secondary schools is possible within the degree, with the mathematics courses being designed and taught in the School of Education. Admission criteria for gaining access to a B.Ed. degree with a specialization in mathematics are less demanding than those for entry into mathematics courses offered in a B.Sc. or B.A. degree programme. Typically, many of the students entering the B.Ed. programme are not strong performers in mathematics in school. Degrees in science, engineering and business science attract the mathematically strong students. Thus, and as has been argued (Adler, 2002), both pre- and in-service mathematics teacher education programmes need to deal simultaneously with redress (past inequality), repair (apartheid education did damage) and reform (orient teachers to the bias and focus of the new school curriculum).

Most teacher educators would agree that it is important for secondary mathematics teachers to learn substantial mathematics in their undergraduate degrees; many would simultaneously agree with the contention that novice teachers (including those who enjoyed tertiary level studies in mathematics) come into the profession with superficial understandings of the mathematics they learnt (Parker, 2009). From her survey of research on mathematics teacher education policy and practice, Parker concludes: "What these studies point to is that a strong mathematics subject identity is important for successful secondary school mathematics teaching, where success is measured by school learner success", and further that while the claim that teachers need to know the subject matter they teach has strong intuitive appeal, ". . . exactly what they need to know to teach at various levels, and how they need to know this are still debated and remain topics for further research" (Parker, 2009, pp. 35–36). There are two critical points here. The first is that in both pre- and in-service secondary mathematics teacher education programs in South Africa, mathematical dispositions and know-how need to be produced, and in ways that will enable teachers to project mathematical identities in their teaching; however, the what and how of such programmes remain contentious. Secondly, programmes are presented with both opportunity (for innovation towards such productions) and challenge (having to do so in conditions of inequality, poor quality and, relatively speaking, limited resources). Hence the focus in the QUANTUM research project: the what and how of such programmes and their potential effects.

Precisely because socio-economic inequality persists and is pervasive in South Africa, vigilance is required with respect to who has opportunity to learn what in the context of teacher education as much as in school itself. The cases described in this chapter open up such discussion and in doing so contribute to the discussion of culture and the notion of mathematics in and for teaching in this book. In the first instance, the South African context itself gives rise to questions and insights specific to prevailing local conditions. A consideration of the context throws a spotlight on the particular challenges in teacher education, which are nevertheless not unique to South Africa. In their similarities and differences, the cases we discuss here may be treated as windows into cultural practices within and across mathematics teacher education itself, and mathematics in and for teaching within it.

Over the past two decades, a range of studies has developed out of Shulman's seminal study of teachers' professional knowledge (Shulman, 1987), a considerable number of which have been located in mathematics teaching contexts (Ball, Bass, & Hill, 2004; Ball, Thames, & Phelps, 2008; Even, 1990; Even, 1993; Krauss, Neubrand, Blum, & Baumert, 2008; Ma, 1999; Marks, 1992; Rowland, Huckstep, & Thwaites, 2005; Adler & Huillet, 2008). A number of the studies have sought to elaborate SMK (e.g. Even, 1990, 1993) or to unpack PCK, and the boundary between PCK and SMK (e.g. Adler & Huillet, 2008; Marks, 1992). Others have appropriated the notions of PCK and SMK, sharpened them with respect to mathematics and then explored the relationship between, for example, teachers' SMK and

PCK (e.g., Krauss et al., 2008), or, more broadly, the relationship between recently constructed measures of teachers mathematical knowledge for teaching, the quality of their instruction and student learning (e.g. Ball et al., 2008; Hill et al., 2008). In what could be understood as a move to manage the tension between audit and evaluation (Williams, this volume), Ball, Hill and their colleagues argue that their measures are indeed derived from and validated in observations of practice. This strand of their research has identified tasks of teaching and their specific mathematical entailments (Hill et al., 2008; Rowland et al., 2005). Together these studies have contributed significantly to a developing discourse on mathematical knowledge for teaching.

Shulman's work, and Ball's elaboration and development of that work in studies of primary mathematics teaching in the USA, is discussed in many of the chapters in this volume and in detail in that of Goulding and Petrou. Ball et al. are aware of the cultural location of their work, and there are studies that have examined their measures of mathematical knowledge for teaching in different cultural contexts, such as Ireland (see Delaney, Ball, Hill, Schilling, & Zopf, 2008); and we are aware of a similar study underway in Ghana. However, how their measures are shaped and in what ways, by both curriculum in use and reform discourses in the USA is not elaborated. As Andrews argues (Chapter 7, this volume), there is a cultural specificity of mathematics in use in teaching, that is, of forms and functions of PCK across contexts. A particular contribution of this chapter then, is its description of how mathematics in and for teaching comes to 'live' in mathematics teacher education in a range of South African institutions.

## Studying Mathematics and Teaching in Mathematics Teacher Education

Our observations are, of course, a function of how we have read teacher education practice. We have developed a methodology[3] that enables us to describe what and how mathematics is constituted in teacher education practice. We accept as axiomatic that pedagogic practice entails continuous evaluation (Bernstein, 2000), the function of which is the constitution of criteria for the production of legitimate texts. Further, any evaluative act, implicitly or explicitly, has to appeal to some or the other ground in order to authorise the selection of criteria. Our unit of analysis is what we call an *evaluative event*, that is, a teaching-learning sequence that can be recognised as focused on the pedagogising of particular mathematics and/or teaching content, the latter being the 'object' of the event. In other words, an evaluative event is an evaluative sequence aimed at the constitution of a particular mathematics/teaching object. The shift from one event to the next is taken as marked by a change in the object of attention. Evaluative events therefore vary in temporal extent

[3]The methodology is detailed in a range of publications from the QUANTUM study already mentioned. It draws substantially from Davis' (2001, 2005) Hegelian elaboration on Bernstein's proposition asserting that pedagogic discourse is necessarily evaluative.

and can also be thought of as made up of a series of two or more sub-events when it is productive to do so, as in cases where the content that is elaborated is itself a cluster of distinct but related contents. The evaluative activity that inheres in an event can be thought of as a series of pedagogic judgements, as defined in Davis (2001). By describing observed pedagogic practice in terms of evaluative event series we produce units for the analysis of pedagogy.

## *Reading 'What' in the Constitution of Mathematics in and for Teaching*

Each course, all its contact sessions and related materials were analysed and partitioned into evaluative events. After identifying starting and endpoints of each event or sub-event, we first noted whether the object of attention was mathematical and/or pedagogic (i.e. about teaching), and coded this *M* or *T* respectively. We added codes of *m* and *t* where some assumed background knowledge either of mathematics or of teaching was also in focus. For example, a focus on misconceptions in mathematics learning was coded as *T*, as a teaching object. The code *Tm* was used when the discussion of misconceptions, for example, included assumed mathematical knowledge.

We worked with the idea that in pedagogic practice, in order for some content to be learned, it has to be represented as an object available for semiotic mediation in pedagogic interactions between teacher and learner. An initial orientation to the object, then, is one of immediacy: The object exists in some initial (re)presented form. Subsequent to the moment of immediacy, pedagogic interaction generates a field of possibilities for predicating the object through related judgements made on what is and is not the object, which might be thought of as a moment of pedagogic reflection in which criteria are constituted. All judgement, hence all evaluation, necessarily appeals to some or other locus of legitimation to ground itself, even if only implicitly. Legitimating appeals can be thought of as qualifying reflection in attempts to fix meaning. We therefore examine *what* is appealed to and *how* appeals are made in order to deliver up insights into the constitution of mathematics for teaching (MfT) in mathematics teacher education. Given that mathematics teacher education draws varyingly from the domains of mathematics, mathematics education and mathematics teaching, what come to be taken as the grounds for evaluation are likely to vary substantially within and across sites of pedagogic practice in teacher education. We eventually described the grounds appealed to across the three courses in terms of six ideal-typical categories: (1) mathematics, (2) mathematics education, (3) metaphor, (4) experience of teaching (adept or neophyte), (5) curriculum, and (6) the authority of the adept.

By way of example, we present the analysis of three evaluative events in one session in one of our cases, numbered Case 1 here, where the first event was divided into seven sub-events. This was the fourth 3-h session in a course: *Teaching and Learning Mathematical Reasoning*. The course comprised seven such sessions in total. The focus of the particular session discussed here was 'misconceptions'.

**Assignment 2**

Consider the following problem given to grade 8, 9 or 10 learners:

**Someone makes a conjecture that $x^2 + 1$ can never equal 0 if x is a real number.**

**Is this person correct or not? Justify your answer.**

Your task is to:

1. Predict the misconceptions that might arise when Grade 8, 9 or 10 learners attempt this problem.
2. Discuss the importance of these misconceptions for you as a teacher, drawing on the paper by Smith et al.
3. Discuss how you would work with these misconceptions in a Grade 8, 9 or 10 classroom.

You should write about 4–5 pages in total (1200–1500 words).

All teachers have experiences of learners' misconceptions in mathematics. How we think about and work with learners' misconceptions might differ from teacher to teacher, depending on how we view learning and the role of the teacher. In Hatano's paper, he argued that misconceptions give us evidence that learners are in fact constructing their own knowledge and so they are important for teachers. Thus from a constructivist perspective, misconceptions are seen as an important part of learning. In this week's paper, Smith *et al*. argue very strongly that misconceptions are a normal part of learning and are to be expected on the difficult road to mathematical understanding. Sasman *et al* argue that we should try to counter misconceptions with cognitive conflict although they argue that this is very difficult. In the session, we will critically discuss these papers. Our guiding questions will be: Can we consider misconceptions to be an important part of learning? How might teachers best work with misconceptions in the classroom?

**Required reading**

1.Smith, J.P., DiSessa, A.A. and Roschelle, J. (1993) Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences*, 3(2), 115–163.

2. Sasman, M., Linchevski, L., Olivier, A., and Liebenberg, R. (1998) Probing children's thinking in the process of generalization. Paper presented at the fourth annual congress of the Association for Mathematics Education of South Africa (AMESA), Pietersburg, July 1998.

**Fig. 9.1** Assessment task case 1

Students had been provided an assessment task marked "Assignment 2", shown in Fig. 9.1 below, which was accompanied by an introductory paragraph and two papers. Students (most of whom were practicing secondary teachers) were expected to read the introduction and study the papers as preparation for the lecture.

We use parts of this session to show how events/sub-events begin and end and how they were analysed, specifically their categorisation as either *T* or *M*, as well as *t* or *m*; and then what was recorded as legitimating appeals. We show here that appeals over this session varied across mathematical principles, mathematics education, practical experience of teaching and curriculum knowledge (i.e., ideal-typical categories 1, 2, 4 and 5), with mathematics education dominant. As will become evident, an idea of what a misconception is in mathematics teaching and learning was constituted in this session in interaction between the lecturer, the students and the range of discursive and practice-based resources (research papers, a video record and a transcript) made available for the session by the lecturer.

The lecture began with a viewing of a video extract of a typical secondary township school Grade 10 class, where the learners had worked on a problem and were discussing it as a class with the teacher. In addition to the video extract, students had a transcript of the classroom discourse. After the video had played and the

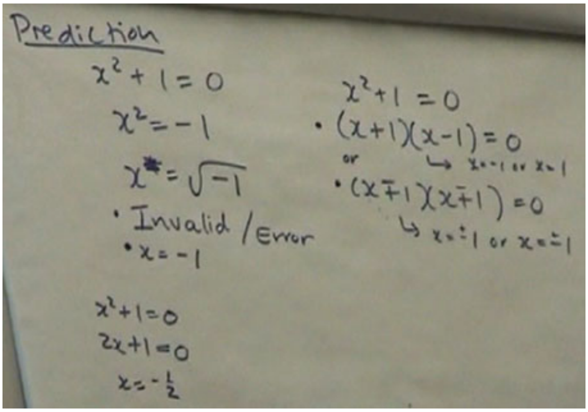


**Fig. 9.2** Students' anticipations of school learners' misconceptions

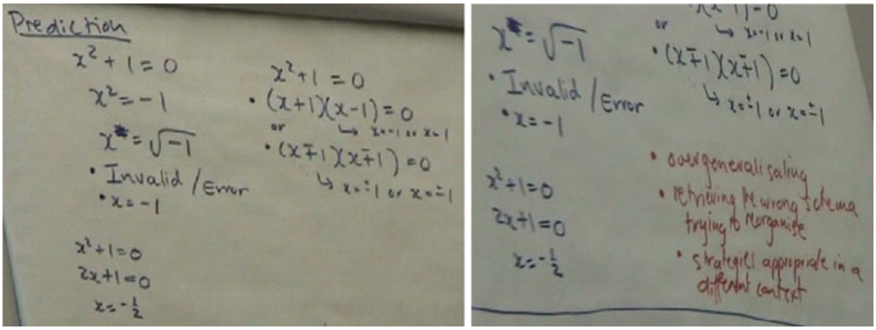


**Fig. 9.3** Students' ideas rephrased by the lecturer

lecturer had discussed the ethics of observing and respecting data from a colleague's classroom, she directed attention to the students' anticipations of school learners' misconceptions, as required by task 1 of Assignment 2 (see Fig. 9.1). This was marked as the beginning of event 1 of session 4. The resulting series of lecturer-student interactions was recorded as sub-event 4.11.

Ideas offered by students were recorded on a flip chart (Figs. 9.2 and 9.3) and rephrased by the lecturer (L = lecturer; Sn = student n).

*L:* *(After recording the students' suggestions shown in* Fig. 9.2.) *So you are telling me here the one misconception you predicted that didn't come up on the tape is that learners will try to solve the expression, and learners in the tape didn't do that . . . Did any other prediction you had come up that didn't involve solving?*

*S1:* *They will take any real number for x. Say, try x is equal to 2.*

*L:* *Why would you see this as a misconception?*

*Ss:* *They will try a few numbers.*

*L:* *What kind of numbers at grade 10?*

We captured and categorised this sub-event (4.11) as having a teaching object in focus (specific misconceptions) in the context of mathematics, i.e., *Tm*. What students were to grasp was a notion of misconceptions in mathematics learning (*T*), and the mathematics in discussion was incidental and presumed known (*m*). The immediate representation was the task from a Grade 10 class, recontextualised as the focus of their assignment and focus of this session. Reflection in this event was on student predictions. Criteria legitimating student suggestions (i.e., the grounds functioning as to whether and how this was a misconception) were located in students' practical experience.

Table 9.1 shows how we recorded and categorised each of the events and sub-events in this session. All sub-events 4.11–4.17 of event 4.1 were directed at the notion of misconceptions. Before we present the table, we describe sub-events 4.12 and 4.17 in some detail in order to illuminate further our rules for recognition of the notion and legitimating appeals.

Following the recording of predicted misconceptions, the session moved on to categorising the misconceptions listed and evident in the video extract students had watched. The announcement by the lecturer below marked the beginning of sub-event 4.12:

*L: I think there are different kinds of misconceptions here that we can see . . . three different ones.*

As in the previous sub-event, discussion between the lecturer and students followed. The lecturer probed student offerings with the following questions: "... where is it [the misconception] coming from?", "Why might it make sense to the learner?", "How would Smith [or DiSessa] say that?", thus directing students to the published texts on misconceptions that they had read in preparation for the session. The types of misconceptions identified and discussed were again recorded on the flip chart. Over-generalising, using wrong schema or strategies from a different set of problems (none of which are sensible here) are indicated in Fig. 9.3.

Substitution using examples was noted separately as "testing the conjecture". The lecturer returns to this in sub-event 4.15 (below), with the question: are some misconceptions "more correct" than others? Sub-event 4.12 was categorised as *Tm*: again, the notion of misconceptions was in focus, and specifically the identification of types of misconceptions as described in the mathematics education research texts students were required to read. Appeals were consistently to the field of mathematics education. Misconceptions named and recognised in the field of mathematics education (e.g., over-generalising, retrieving wrong schema, strategies appropriate in a different context) were to be found in the texts read by the students. The beginning of sub-event 4.13 was marked by the lecturer bringing into focus students' view that misconceptions originate in teaching, and ends with reference to the texts where over-generalising is described as something that learners will do as they learn something new. The example from the video discussed is where learners want to find a value for $x$, and suggest $x = 1$, equating the value of $x$ with the coefficient of $x^2$.

Sub-event 4.14 was marked by the lecturer effecting a shift in focus to other contributions from learners in the video, and she posed the question of whether some

**Table 9.1** Categorisation of three evaluative events and sub-events in case 1

| Event | Specific object | Object | Appeals to . . . | Image of teaching |
|---|---|---|---|---|
| 4.11 | Learner misconceptions: Specific misconceptions articulated by students and experienced in their own teaching | *Tm* | Experience of teaching | The students are asked to think about their own teaching. Thus, the student (as teacher) as he/she sees him or herself is the image of teaching. We describe this as: *look at your teaching practice* |
| 4.12 | Learner misconceptions: Distinguishing types of misconception, specifically overgeneralization, using wrong schema, strategies from a different problem, testing the conjecture | *Tm* | Mathematics education | Students here are looking at a video of another teacher, together with a transcript of the videoed episode. In addition, examples of teaching related to misconceptions are present in the research papers they have read and are referring to. We describe the collection of images here, all of which are external to the lecturer and the students as: *look at (mathematics teaching) practice* |
| 4.13 | Learner misconceptions: specifically over-generalising | | Mathematics education | As in 4.12: *look at (mathematics teaching) practice* |
| 4.14 | Learner misconceptions: Some are more 'correct' i.e., mathematical, than others | *Tm* | Mathematics, mathematics education | As in 4.12: *look at (mathematics teaching) practice* |
| 4.15 | Justification is mathematical (*M*), misconceptions more/less mathematical (*Tm*) | *M Tm* | Mathematics | As in 4.12: *look at (mathematics teaching) practice* |
| 4.16 | Value/meaning of $\sqrt{-1}$; when $\sqrt{-1}$ declared invalid (*Mt*) is or is not a misconception (*Tm*) | *Mt Tm* | Mathematics, experience of teaching, curriculum | As in 4.12: *look at (mathematics teaching) practice* |
| 4.17 | Reasoning theoretically or empirically | *M* | Mathematics, mathematics education | As in 4.12: *look at (mathematics teaching) practice* |
| 4.2 | Classifying mathematics tasks | *Tm* | Mathematics education | Video of another teacher; own teaching; texts including texts from previous sessions: *look at (mathematics teaching) practice; look at your teaching* |
| 4.3 | Using misconceptions in teaching | *Tm* | Mathematics education, mathematics, experience of teaching | Video of another teacher; own teaching; texts: *look at (mathematics teaching) practice, and look at your teaching* |

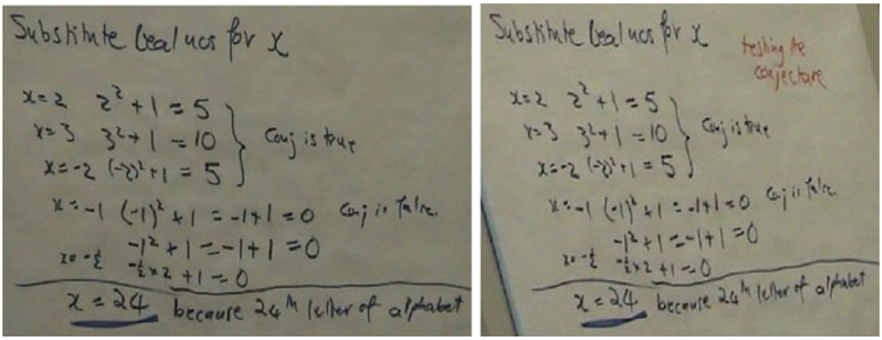


**Fig. 9.4** Additional solutions offered by learners in the video

misconceptions were ‘more correct’ than others. The lecturer focused attention on the suggestion by one learner that $x^2 + 1 = x^2 + 1$ (and thus not 0), and asked if the statement was more or less ‘correct’ than the suggestion, $x = 1$. As with sub-event 4.12, the object of subsequent two sub-events was categorised as *Tm*. Appeals were made to the field of mathematics education, specifically to the types of misconceptions identified in the texts the students had read.

In sub-event 4.15, the recognition and marking of misconceptions continued. Focus shifted from strategies that were not productive to two additional solutions offered by learners in the video: (1) the ‘numerical’ solution (where students substituted 0, then 1, then –1 and then agreed with the conjecture (see Fig. 9.4); and (2) the reasoning that if $x^2 + 1$ is equal to 0, then $x^2$ must be equal to -1.

The lecturer asked students “which response would you prefer?” And, after some interaction between the lecturer and students, and students themselves, the lecturer stated that the learners (in the video):

*L:* *. . . are trying to falsify this [referring to the conjecture], to prove the opposite. If they can’t, then they will prove it is true. The teacher [in the video] thought they are trying to get to zero . . .. It is a systematic approach, trying to test the conjecture.*

As indicated in Table 9.1, we categorised this as *Tm*, with appeals located in mathematics, rather than mathematics education as previously. The criteria for judging what is more or less correct are mathematical principles.

The categorisation of the remaining sub-events making up event 4.1 and then events 4.2 and 4.3 are summarised below, with an interesting appeal in event 4.16 to curriculum knowledge. In event 4.16, there was discussion of whether learners’ conception of the square root of –1 as not valid was a misconception. There is a suggestion in the video that not “valid” and “error” as responses derive from the displays of calculators when students/learners attempt to perform a calculation like finding the square root of –1. In the end, in a context where complex numbers are not part of the curriculum and learners’ experience (indeed the problem was explicitly restricted to real numbers), declaring the square root of –1 “not valid” could not be classified as incorrect, and consequently, not as a misconception either.

### *Reading 'How' in the Constitution of Mathematics in and for Teaching*

Our data suggests that the image of teaching is a significant element of pedagogic practice in teacher education and so of the constitution of teaching and/or mathematical objects in this practice. The last column in Table 9.1 describes the location of the image of teaching in each of the events. As discussed in the introduction to this chapter, across the cases students were presented, both implicitly and explicitly, with images of the mathematics teacher and mathematics teaching. In the events summarised above, the most visible image of mathematics teaching is in the video students watch and consider in the session. While the most visible, it was not the only image. The initial image of teaching in this session, however, is that of the students (as practising teachers) themselves. Additional implicit images of mathematics teaching are contained in discussion in the research texts. Students are thus presented with a range of images of teaching. While this includes their own teaching practice, the dominant images are located in recognisable situations, distant from the course itself, and in the broader practices of mathematics teaching. We refer to this imaging of teaching and the teacher as "look at (mathematics teaching) practice".

There were similarities and differences in the way mathematics teaching was modelled across the cases, and it is our contention that images of mathematics teaching are instrumental in the way in which appeals emerge, and thus how mathematics in and for teaching comes to be constituted. We elaborate on this claim through the case discussions following. It is evident in Table 9.1 that the notion of 'misconception' is filled out in time and over time and the recognition and realisation criteria (Bernstein, 2000) for discerning and marking misconceptions are exhibited through appeals.

In addition, there were similarities and differences in the strength of the lecturer's control over criteria for what is and is not legitimate in the practice (Bernstein, 2000). Varying strengths become evident through the consistency and spread of appeals within and across cases, as we elaborate below. In Case three, as illustrated in Session 4, the lecturer has strong control over criteria, selecting what is to be focused on, and directing students to linking learner contributions in the video and its transcript to descriptions of misconceptions in the readings for the session.

The illustrations of the three events in Session 4, with elaboration of some of the sub-events within event 4.1, reveal the methodology employed in the project and specifically how events were recognised and described. We now move on to discuss the three cases we studied.

## Three Cases of Mathematics Teacher Education

The discussion of each case begins with a general statement of the approach to learning mathematics for teaching, and so a reading of the practice to be acquired. This is then supported by extracts from events, including those that illustrate appeals different in kind from those described earlier. The extracts are selected for illustrative purposes and to discuss the way mathematics teaching is modelled and the mathematical knowledge that is in focus, and thus our interpretation of what and how MfT came to be constituted in each of the cases. We begin with Case 1.

## *Case 1: Teaching and Learning Mathematical Reasoning*

The practice to be acquired in this course was the interrogation of records of practice with mathematics education as a resource. The image of teaching was presented in a range of records of practice including video of other teachers. We referred to this as: *Look at (mathematics teaching) practice.* The structure of each of the sessions of the course was similar to that of Session 4, as illustrated and described above. The image of the school learner and the teacher were continually subjected to interrogation from discursive resources constituted by mathematics education. The principles structuring the activity in the course were explicit and distanced from the teacher educator herself. The teachers were required to describe, justify and explain their thinking in relation to both what they brought to the discussion or observed and what they had read. The records of practice were the images of practice constituted as objects for interrogation by the field of mathematics education. The pattern of interaction between the lecturer and students was similar throughout the course, where the academic text was emphasised and made to frame criteria for what was and was not legitimate. Within the focus on mathematics teaching as object in Case 1, mathematics itself came into focus and mathematical principles functioned to ground notions of teaching.

Table 9.2 summarises the appeals made for legitimating the texts within this pedagogic practice. Evidence for our description of the practice to be acquired lies in the table. In the total of 34 events across the course, 31 (91%) direct appeals are made to mathematics education texts. We also note from Table 9.2 that there is a spread of appeals across possible domains, reflecting the complex resources that constitute knowledge for teaching mathematics within the practice.

We note that appeals to the metaphorical and to the authority of the lecturer (which we elaborate and exemplify in discussion of Case 2 following) are low, suggesting that mathematics is presented as a reasoned activity and interrogation of practice is through the field of mathematics education. Secondly, the relatively high percentage of appeals to experience, together with appeals to mathematics education shows a particular kind of evaluation at work. We noticed with interest that in this course, there are 95 appeals across 34 events. We suggest that this density of appeals reflects strong pedagogic framing (control of the criteria by the lecturer), a key feature that marks out the different practices across cases.

## *Case 2: Algebra Content and Pedagogy*

In Case 2, the practice to be acquired was a particular pedagogy modelled by the lecturer who presented the activity as a specific practical accomplishment. We refer to this as: *Look at my practice. Look at me and you will see and experience what*

**Table 9.2** Distribution of appeals in Case 1

| | Mathematics | Mathematics education | Metaphorical | Experience of either adept or neophyte | Curriculum | Authority of the adept |
|---|---|---|---|---|---|---|
| Mathematics | 5 | 6 | 5 | 0 | 0 | 0 |
| Proportion of appeals ($N = 16$) (%) | 31.3 | 37.5 | 31.3 | 0 | 0 | 0 |
| Teaching | 15 | 25 | 0 | 23 | 10 | 6 |
| Proportion of appeals ($N = 79$) (%) | 19 | 31.7 | 0 | 29.1 | 13.7 | 7.5 |
| Mathematics and teaching | 20 | 31 | 5 | 23 | 10 | 6 |
| Proportion of appeals ($N = 95$) (%) | 21.1 | 32.6 | 5.3 | 24.2 | 10.5 | 6.3 |
| Proportion of events ($N = 34$) (%) | 58.8 | 91.2 | 14.7 | 67.7 | 29.4 | 17.7 |

*it means to teach algebra. Do what I do, and the way I do it*. The lecturer worked with her students (the teachers) in ways similar to that which she advocated they work with their learners. That this is set up as a practical accomplishment is clearly recognised in and across the course sessions. The lecturer also stated on a number of occasions: "I am not teaching you content, that you must do on your own …. I am teaching you how to teach [algebra]". She further emphasised that it was not enough to know how to carry out a calculation, but that teachers "also need to understand why it works". Lectures were structured around and supported by a booklet of activities and exercises that dealt with "different methods of introducing and teaching algebra in the Senior Phase". In other words, teachers on the course were to (re)learn how to teach grades 7–9 algebra.[4] The teaching sequence below captures this central feature of Case 2 and illustrates how the modelling of mathematics teaching – 'look at me and see how to teach' – functioned, together with the mathematics that came into focus.

In the first few sessions of the course, the focus was on learning to teach some of the general properties of operations on numbers and rules of algebra, for example, rules for operating on exponential expressions. The lecturer frequently employed everyday and visual metaphors, sometimes combined them. For example, the distribution of food and the act of commuting between towns were used to illustrate the distributive and commutative laws, respectively.[5] With respect to the distributive law, its introduction in class (i.e. the beginning of an evaluative event) was through a descriptive metaphor of distributing food. The distributive law was then elaborated through a visual metaphor represented on the lecturer's board, as shown in Fig. 9.5.

Students on the course were thus offered metaphorical and visual representations of the distributive law, which were intended, at once, to enable them to understand the distributive law and have ways of presenting it to their learners so that they too might achieve understanding: look at me, and you will see what and how to teach.

In this case, and we are not suggesting a necessary relationship here, mathematics comes to be constituted as sensible in the strict sense of the term (it is what we see/experience) and not as reasoned activity. Let us elaborate: Fig. 9.5 shows that the lecturer used areas of squares and rectangles to establish further grounds for accepting the distributive law, grounds that brought in mathematical features, but

[4]Most of the teachers on this programme were initially trained to teach in primary schools and were upgrading a 3-year qualification and improving their level of teaching. A design principle of the course was that by learning to teach algebra, the teachers would themselves have opportunities to (re)learn algebra.

[5]More generally, it is interesting to note that in instances such as these there is a question of the integrity of the metaphor with respect to the mathematical idea being 'exemplified'. This specific point is a general concern in mathematics education where the everyday is frequently recruited to invest mathematical objects and notions with meaning. Given the intelligible nature of mathematical ideas, this presents teachers with difficulties of finding useful and meaningful metaphors.

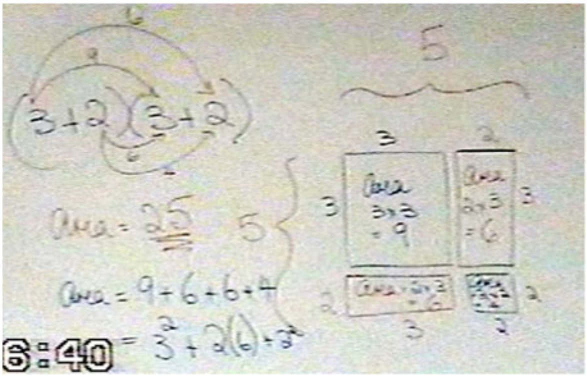


**Fig. 9.5** Area and the distributive law

nevertheless remain at the level of the sensible. A geometrical metaphor is employed to generate a representation of binomial–binomial multiplication as an exemplification of the distributive law. The idea seems to be that since the learner can recognise that $5 \times 5 = 25$, and that $5 = 3 + 2$, and also that $(3 + 2)(3 + 2)$ must therefore be 25, she/he will be convinced that binomial–binomial multiplication must function as described by the lecturer. The products corresponding to the areas of the four rectangles produced by the partitioning of 5 into $(3 + 2)$ are identified with the products produced during the calculation of $(3 + 2)(3 + 2)$. The validity of the calculations performed in both representations of binomial–binomial multiplication depicted (arithmetic and geometric) relies on the distributive law, so that neither is a direct demonstration of the validity of the other.

What is of great importance in this practice, however, is that a *visual demonstration* of the procedure for (binomial–binomial) multiplication is presented to teachers. In terms of our analytic tools, the legitimating appeal here (qualifying reflection on the notion of the distributive law in mathematics) *is metaphorical*. The appeals to Mathematics in Case 2, where the focus was on learning to teach rules of algebra, were, for the most part, of the form of using numbers to test and assert the validity of mathematical statements, or, of actually asserting a procedure or rule (as with the distributive law), which was then redescribed metaphorically.

In Case 2, we find the distribution of appeals shown in Table 9.3. We see that only four of 36 events explicitly appealed to teaching; three of those appeals were to the localised experiences of the teachers and one to the official curriculum. No appeals were made to the arena of mathematics education. This observation supports the point made earlier that the teaching of mathematics is presented as a practical accomplishment modelled by the lecturer, where its principles are to be tacitly acquired. The framing of criteria with respect to mathematics teaching is weak. Moreover, as Table 9.3 shows, the meaning of mathematics was strongly grounded in metaphor. What we find provocative here is that in this practice, neither mathematics nor teaching is underpinned by principles – the ground functioning here is at the level of the sensible and metaphorical.

**Table 9.3** Distribution of appeals in Case 2

| | Mathematics | Mathematics education | Metaphorical | Experience of either adept or neophyte | Curriculum | Authority of the adept |
|---|---|---|---|---|---|---|
| Mathematics | 15 | 0 | 25 | 1 | 0 | 0 |
| Proportion of appeals ($N = 41$) (%) | 36.6 | 0 | 61 | 2.4 | 0 | 0 |
| Teaching | 0 | 0 | 0 | 3 | 1 | 0 |
| Proportion of appeals ($N = 4$) (%) | 0 | 0 | 0 | 75 | 25 | 0 |
| Mathematics and teaching | 15 | 0 | 25 | 4 | 1 | 0 |
| Proportion of appeals ($N = 45$) (%) | 33.3 | 0 | 55.6 | 8.9 | 2.2 | 0 |
| Proportion of events ($N = 36$) (%) | 41.7 | 0 | 69.4 | 11.1 | 2.8 | 0 |

## *Case 3: Reflecting on Mathematics Teaching*

In Case 3, the practice to be acquired was that of reflecting on practice, understood as the conscious examination and systematisation of one's own mathematics teaching practice. In the terms we have used for other cases, the students here are to learn by *looking at your own practice.* The Reflecting on Mathematics Teaching (RMT) course that is in focus in this section was one of two specialist mathematics education courses; the remaining four specialist courses were mathematics courses. RMT was delivered through seven 3-hour fortnightly Saturday sessions and a week long vacation school. RMT students were supplied with the learning materials and expected to work through them independently in preparation for the contact sessions. In the materials and in the contact sessions the lecturer explicitly positioned teachers as already experienced and knowledgeable. The course notes suggest that teachers would acquire the 'tools and the space' to think about and improve their teaching through action research. It would help them to 'systematise what they already do', namely, reflect on their practice to improve mathematics teaching and learning. Teachers were expected to use their existing mathematical and professional competence to engage independently at home with the course materials to identify a problem in their teaching and then plan and implement an intervention. In preparation for the contact sessions, they were thus expected to work through the activities to produce resources from their own practice for reflection and further elaboration.

However, by the second contact session it was clear that the presumed mathematical and professional competences[6] for teaching that were to be used as the main resource for the course were absent. Whatever the reasons, the teachers did not bring expected examples from their own practice to the sessions. That reality presented major obstacles to progress in the course and in response the lecturer inserted an example of what was required. She did so by modelling the 'expert practice' required. The image was elaborated through examples of how the lecturer (as expert teacher) would go about planning for and engaging in mathematics classroom teaching. The focus fell on the practices themselves, while the principles of the practice that she herself used were rendered implicit. Indeed, starting from an orientation to learning mathematics for teaching by reflecting on students' own practices, the orientation that emerged in this Case (see Table 9.4) resembled that exhibited in Case 2: look at my practice.

Unexpected obstacles to the planned arrangements for teaching are not unique to the course, though, in this instance, there were sustained and substantial difficulties the lecturer had to confront. We include it for illustration here for two reasons. Firstly, it points to a well-established orientation in teacher education (self-reflection), or what we have called 'look at yourself'. Secondly, it highlights for us the hidden assumptions in such an orientation – that students (teachers) can recognise in their own practice that which is intended to be interrogated in the programme

[6]For example, a deep knowledge of the school mathematics required by the new curriculum, or professional competence such as an ability to produce a year plan based on a curriculum document.

**Table 9.4** Distribution of appeals in Case 3

| | Mathematics | Mathematics education | Metaphorical | Experience of either adept or neophyte | Curriculum | Authority of the adept |
|---|---|---|---|---|---|---|
| Mathematics | 3 | 0 | 1 | 0 | 0 | 1 |
| Proportion of appeals ($N = 5$) (%) | 60 | 0 | 20 | 0 | 0 | 20 |
| Teaching | 3 | 10 | 0 | 28 | 5 | 23 |
| Proportion of appeals ($N = 69$) (%) | 4.4 | 14.5 | 0 | 40.6 | 7.3 | 33.3 |
| Mathematics and teaching | 6 | 10 | 1 | 28 | 5 | 24 |
| Proportion of appeals ($N = 74$) (%) | 8.1 | 13.5 | 1.4 | 37.8 | 6.8 | 32.4 |
| Proportion of events ($N = 36$) (%) | 15.4 | 25.6 | 2.6 | 71.8 | 12.8 | 61.5 |

and reveals unintended consequences of such. Here, the majority of students did not follow the expected practice (suggestions) with the result that the resources required in the contact sessions for enabling progress in the module were absent. The lecturer tried to overcome the problem by modelling an example of the required expert practice. The lecturer drew on principled knowledge to produce the examples she used; however, as noted earlier, the principles that structured her activity remained implicit. The image (of the teacher and of teaching) that came to be presented, though unintended, was (as in Case 2) the lecturer herself, and the dominant ground and criteria for interpreting practice was the experience she demonstrated with respect to both mathematics and teaching.

## Mathematics for Teaching Across Cases of Mathematics Teacher Education

In each of the three cases, we have discussed criteria for what was to count as either mathematics or mathematics teaching. The appeals and grounds that illuminated the criteria ranged across mathematics, mathematics education, metaphorical recruitments of the everyday teaching experience and curriculum, evidencing our earlier point that mathematics teacher education does indeed draw from a range of domains. Significantly, however, the spread of appeals differed across the cases in nature, extent and density.

While we do not and cannot claim any necessary causal relations here, two observations are pertinent. The first is that there was a dominance of particular appeals in each case, illuminating different orientations to practice. In Case 1, the dominant appeals were to mathematics education in the main (91.2% of all events included appeals to mathematics education), together with appeals to mathematics itself (58.8%) and to the students' experience as practicing teachers (67.7%). In Case 2, appeals were strongly grounded in metaphor (69.4%) together with mathematics (41.7%). In Case 3, as a result of the lecturer having to shift orientation from reflection on examples of practice brought by students themselves to examples she provided on the spot, dominant appeals were to experiences of teaching (71.8%) and to her authority (61.5%).

Second, and co-incident with types and spread of appeals was their relative density. Of the three cases, the distribution of appeals was least dense in Case 2: 45 appeals across 36 events in the course overall; and most dense in Case 1: 95 across 34 events, with Case 3 somewhere between: 74 appeals across 36 events. The constitution of mathematics for teaching in these three cases as reflected in the operation of pedagogic judgement and criteria in use, was different. Consequently, while students in each of these sites of teacher education were offered opportunities for learning *mathematics for teaching*, the opportunities were of different kinds and at different levels of sophistication.

The density and nature of appeals correlated further with the way in which teaching was modelled in each of the cases. Modelling the practice is, we may wish to argue, a necessary feature of all teacher education: there needs to be some demonstration/experience (real or virtual) of the valued practice. That is, it seems necessary for students to encounter some image of what mathematics teaching performances should look like (cf. Ensor, 2004). In the Algebra course of Case 2, the image of teaching was located in the performance of the lecturer whose concern (stated repeatedly through the course) was that the teachers themselves experience particular ways of learning mathematics. Such an experiential base was believed to be necessary, if they were to enable others to learn in the same way. The mathematical examples and activities in the course thus mirrored those that the teachers were to use in their Grades 7–9 algebra class. However, the teaching perspective on the school mathematics content remained at the level of practical demonstration, presenting students with instances they could imitate and hence no principled ways in which to engage with Grade 7–9 algebra, nor with how it could/should be taught. In Case 1, the model of teaching mathematical reasoning was externalised and distanced from both the lecturer and the teacher-students themselves, and located in images and records of the practice of teaching, specifically in video records of local teachers teaching mathematical reasoning and related transcripts and copies of learners' work. Teaching practices were objects to be described and analysed by drawing on discursive resources (texts, explaining, arguing, describing practice in systematic ways) situated within the field of mathematics education.

We have been struck in our presentation of this work how the identification of the different orientations to modelling teaching across our cases resonates deeply with colleagues in the field. The pedagogic forms in Cases 2 and 3, in particular, are very familiar in South Africa. We see these as a function of ideologies and discourses in teacher education practice that assert the importance of teacher educators practicing what they preach (the need to 'walk the talk'). Such pressure is particularly strong when new practices (reforms) are being advocated and so a significant feature of in-service teacher education. More generally, the modelling forms also reflect well-known theory-practice discourses, in particular, that theories without investment in practice are empty.

## In Conclusion

In this chapter we have presented our in-depth analyses of selected courses in mathematics teacher education and what and how practice (in this instance, mathematics for teaching) was differently constituted. Our findings thus need to be understood as a result of a particular lens, a lens that we believe has enabled a systematic description of what is going on 'inside' teacher education practice, and in particular, 'what' comes to be the content of mathematics for teaching; that is, the mathematical content and practices offered in these courses and 'how' this occurs. We are calling this 'mathematics for teaching'. It is not an idealised or advocated set of contents or practices, but rather a description of what is recognised as content through our gaze. This content is structured by a particular pedagogic discourse, a component of which is the projection and modelling of the activity of teaching itself. In Bernstein's terms, we have seen through an examination of evaluation at work and of how images of teaching are projected; that different opportunities for learning mathematics in and for teaching are offered to teachers by different programmes. The research we have done suggests that developing descriptions of what does or should constitute mathematics for teaching outside of a conception of how teaching is modelled is only half the story.

Returning to the introduction to this chapter and the South African context where concerns with quality are accompanied by concerns to address inequality, important questions arise for further research. Do particular orientations necessarily give rise to a particular kind of mathematics in and for teaching? How do the ranging forms we have described relate to teachers' learning from and experiences of mathematics for teaching and, ultimately, the quality of their teaching? What possible consequences follow for social justice in and through teacher education itself? These questions have their basis in our empirical work. The orientation "look at my practice" in Case 2 was part of a course for teachers coming from rural schools and where it is fair to say historical disadvantage is at its most acute. Further research needs to pursue: for which teachers, in what contexts, there are opportunities for learning mathematics for teaching and with what effects.

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# Chapter 10
# Audit and Evaluation of Pedagogy: Towards a Cultural-Historical Perspective

**Julian Williams**

## Introduction

In this chapter, I first outline a theory of audit and evaluation building on Cultural-Historical Activity Theory (CHAT) and Power's (1999) critical perspective and present an analysis of audit and evaluation in education in general. Second, I draw on some recent empirical case studies – mainly conducted with my doctoral students – of teachers' knowledge, in particular, teachers' understanding of their students' mathematical knowledge. These studies showed that the teachers we studied sometimes mis-judged their students knowledge; that their judgments were influenced by their own mathematical knowledge and by their teaching experience; and that their knowledge of their students was task-situated and tool-mediated rather than 'in the head'. Shulman's notion of pedagogical content knowledge is reconceptualised via CHAT as a boundary object between reflection on teaching and the practice of teaching. Third, I argue the need to examine our methodology for tapping teacher knowledge with due recognition of the danger, or opportunity, presented by teacher knowledge auditing. I finally develop some further implications of CHAT perspectives on pedagogy and its audit or evaluation: The distribution of knowledge across activity systems involves contradictions between 'de-coupling' and 'colonisation', which I attribute to exchange-value and use-value contradictions in the knowledge economy.

I argue that the problem of auditing and evaluating teachers' knowledge for teaching requires us to answer some basic questions:

- What is the purpose of the practices of audit and evaluation, in general and of teachers' knowledge, in particular?
- What kinds of knowledge do mathematics teachers need in order to teach or to produce evidence for auditors?

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- How can teacher knowledge be audited/assessed: what tools/technologies are available, or needed?

For the analysis of activities and their purposes, I turn to CHAT for theoretical perspectives. The main citations to the original corpus of literature on CHAT are usually to Vygotsky, Luria and Leont'ev, but also sometimes to Bakhtin and Voloshinov, and to the Western developers and disseminators, Cole (1996) and Engestrom (1987, 1991); see also Roth and Lee (2007) for a general review, and Williams and Wake (2007a, 2007b) or Ryan and Williams (2007) in the specific mathematics education context.

It is important to mention that this tradition has influenced another significant socio-cultural current well-known to mathematics educators: that of situated learning in communities of practice (Lave & Wenger, 1991; Lave, 1996; Wenger, 1998). However, CHAT theory has an extensive history and more extended repertoire of social-psychological, Marxist/ian concepts such as 'subject', 'object', 'system', 'tool-mediation' and 'division of labour' in 'activity' that will prove useful. Thus, auditing and evaluation refer to different social practices, arise in distinct activities, defined by different motives and engage the subjects and subjectivities of those involved in contradictory ways. I ultimately attribute the tensions between audit and evaluation and, what Power calls 'de-coupling' and 'colonising', to contradictions between the use-value and exchange-value of mathematics in a knowledge economy (see Williams et al., 2009; Williams, in press). I will be drawing on all these concepts here and argue that these perspectives raise new horizons with regard to audit and evaluation of teachers and teacher knowledge.

## Accounting for the Dialectic of Audit

Let me begin with the function of audit in society in general, as the recent introduction of audit to teacher knowledge and education has a historical context and Power's work, inter alia, will prove useful (Power, 1999). The critical sociological literature on 'audit' suggests that we face an audit explosion in all public sectors of the economy from health and education to policing. It is widely recognised how dysfunctional this can be and the literature is not without passionate, strongly politically-positioned critiques of its often deleterious, often unintended impact on practice (See several chapters in Strathern, 2000).

In a recent case in the UK, a hospital is discovered to have killed approximately 400 patients as the unintended consequence of its efforts to meet the audit requirements for becoming a 'Trust', giving it certain advantages in terms of funding and autonomy over non-Trust hospitals. On the other hand, the consequences of NOT auditing can have unintended consequences too: As I write this chapter, I hear that one consequence of cancelling the national tests for 14-year-olds in England is that many Shakespeare theatre companies have experienced a sudden burst of cancellations by schools (Shakespeare used to be on the test syllabus and so it was worth motivating students even at the cost of a school trip). In both cases the diagnosis is quite simple: management is mediated by proxy measurements for ‘use’ that do not faithfully measure use-value – in the one case the impact on ‘health’ is measured by Accident and Emergency department wait-times, in the other the educational value is measured by the national test scores.

On the other hand, auditing practices are no doubt here to stay and seem to go from strength to strength in the UK: in education, despite powerful critiques and the cancellation of some of the national tests, one may argue it is stronger than ever before. The managerial elites need audit to protect themselves from their own lack of accountability and potential accusations of bad judgment, indeed of having made any personal judgment at all, as ‘personal judgment’ is the only one that can thereby become critical personally (Power, 1999).

With international audits such as TIMSS and PISA, one sees league tables going global: it is not difficult to imagine potentially homogenising effects on education internationally to suit the international labour market and much EU policy seems directed along these lines. Indeed, the nation-states in these circumstances may come to have less room for manoeuvre themselves (Williams, 2005, 2009).

Yet in the education literature, it seems, our theoretical understanding of auditing practices is slim. Power (1999) made an important contribution with the conceptions of ‘de-coupling and colonising’ in this context. I introduce these notions here and their relation to ‘audit’ versus ‘evaluation’ practices:

> Audit focuses on verification . . . Audit is a normative check whereas evaluation . . . addresses cause and effect issues; audit is orientated to compliance whereas evaluation seeks to explain the relationship between the changes observed and the programme (Power, 1999, p. 118).

Furthermore:

> Although forms of self-evaluation are viewed as a necessary component threshold for any spending to be taken seriously, cost effectiveness auditing sits *above* them . . .. (p. 118)

Audit is driven by a degree, perhaps a healthy degree, of mistrust and by the need for accountability and some degree of transparency of procedure: thus the audit holds the auditee to ‘account’ to the auditor who, as a result, may influence a flow of resource – essentially it is economic and about the power to control.

> (T)he development of auditable performance measures is much more than a technical issue: it concerns the power to define the dominant language of evaluation (within a hierarchy of economy, efficiency, and effectiveness) . . .. (Power, 1999, p. 117)

Notice here that the hierarchy places economy and efficiency over effectiveness, which implies a colonisation of practice by audit, or the authority of cost-benefits, or of exchange value over the use-value of the outcomes of a practice. Yet evaluation is at heart a self-valuation process, an attempt by practitioner(s) to reflect and to understand and improve what they do: this use is – in the professions, at least – a use-value.

In particular, Power has shown how the tensions involved in audit arise from contradictions between audit from the bottom-up, reflecting evaluations by practitioners and professionals on the ground and audit from the top-down, based on performance objectives set by managers under the regime of ‘New Public Management’. He has used these notions to ground insights into contrasting, empirical case studies, in finance, in health and in Higher Education. My purpose here is to use his approach to re-conceptualise ‘formative’ assessments (broadly corresponding to evaluation) and ‘summative’ assessments (broadly corresponding to audit) and thus reinforce the controversial insight that both are necessary components of a functional assessment process (see Williams and Ryan (2000), which builds on Black and Wiliam (1998)).

Power shows that the entire history of audit involves a problematic: the purpose of audit is said to be to reduce the necessity of relying on the validity of local custom and practice, e.g. of professional subjective judgment. And yet the audit practice is – or claims to be – *itself* unauditable, i.e. it relies in the end on the professional judgment of experienced auditors and this judgment essentially includes their subjective evaluation (based on finite, even quite small sets of data and impressions) of the people they are auditing. In practice, there is a ‘gap’ between what audit promises and what can realistically be ‘known’ (with limited resources a significant part of the problem).

In addition, the survival of evaluation in a regime of audit creates the need for new measures, i.e. for measures of primary products of practices that professionals believe to be valuable. In some areas of education this can be problematic and I will argue this is the case with teacher knowledge. The problem of measurement technologies has led in some spheres to second-order constructs, whereby the *processes* of management are measured instead of their products (what we have come to know as Quality Assurance, or what Power refers to as control of control). Thus, it seems auditors can be persuaded to use second order measures as long as they are credible and can be counted. What auditors need is a politically acceptable system that can be credibly said to hold the system being audited to ‘account’ for its outcomes, increasingly against costs on a ‘value for money’ basis.

It emerges from Power’s account that credible auditing *in practice* always needs to engage with its auditees and their practices. Increasingly, the auditors expect (and on grounds of efficiency this is inevitable) the auditees to actively ‘comply’ with the audit and this provides room for manoeuvre if auditees are to collect data, or maybe even construct their own measures. Indeed, in persuading doctors to collect measures, Power recalls that one of the first moves of audit was to use the evaluation data that doctors already used to monitor practice for formative purposes. Similarly, academics have been brought to engage with research assessment exercises as a means of accounting for their research practices and the associated flow of resource.

Of course there is a huge tension in the purposes of audit and evaluation, as Black and Wiliam (1998) and others have pointed out (an account of this is in Williams and Ryan (2000)). When the ‘bottom-up’ evaluation process breaks free from top-down audit pressures, Power calls this ‘de-coupling’. In the extreme, if decoupled from the practice it is supposed to record, audit may thereby be rendered totally ineffective in holding local practice accountable: typically organisations de-couple by setting up specialist departments to isolate the productive parts of the organisation from its effects. In a number of assessment projects in Manchester, we tried to develop formative and diagnostic work in connection with summative assessment as a means of offering some de-coupling possibilities: by encouraging teachers to focus on the formative aspects of their work with national tests, we sought to counter the most offensive effects of 'teaching to the test' – where summative testing effectively colonises teaching practice (see Williams & Ryan, 2000; Williams, Wo, & Lewis, 2007). The use of an 'audit' instrument by Ryan and Williams in the service of teachers' metacognitive evaluation (see Chapter 15 by Ryan and Williams, this volume) is another pertinent example: a device designed principally to measure students' mathematics knowledge can sometimes be 'turned' into a tool for self-evaluation.

When evaluators on the ground find themselves using instruments devised by Ofsted (the national inspection agency for schools in England) to observe each others' lessons (see e.g. Williams, Corbin, & MacNamara, 2007b; Corbin, MacNamara, & Williams, 2003) then the auditing practice 'takes over' the evaluation practice on the ground. Power refers to this as 'colonisation' and this neatly describes what happens when teaching becomes dominated by preparation for the tests that were introduced as audit measures. The account of this in Williams et al. (2007a), however, revealed that colonisation can sometimes be contested: teachers can develop surprising resources for turning accountability systems to their own purposes, e.g. turning audit into evaluation. Thus, when teachers – who were also 'managers' required to audit their colleagues' compliance with the so-called three part lesson – saw a 'great lesson' that did *not* conform, they went straight out and told everyone about it.

An important conclusion for understanding of auditing practices is that the tension referred to is actually caused by a 'contradiction' between opposite purposes of assessment for audit (usually summative) and evaluation (usually formative). These purposes are part of the contradictory 'objects' of two distinct 'activities' ('audit' and 'evaluation') that engage with distinct Activity Systems. The audit system collects data for managers and ultimately the state to count the 'success' of their expenditures in practice. Ultimately, the audit justifies a flow of further resource to the primary practice. We say that what is audited is thereby shown to have 'exchange value'.

On the other hand, the primary professional practice being audited in general self-evaluates as part of its own system in terms of the usefulness of its outcomes, with a view to improving practice in utilitarian terms. In this context, what is evaluated normally is supposed to have 'use-value'. (For a full exposition of a theory of value in education, see Williams (in press) and Williams et al. (2009).) The trouble arises when the two activities engage together, share common instruments and objects, even subjectivities, as they inevitably do. The hospital manager responsible for the bid for Trust status leans on the Accident and Emergency staff to cut wait-times, patients get insufficient or inappropriate care, patients die. The school bursar no longer has the funds for the trip to the Shakespeare play, because the argument formerly applied by the English staff no longer holds; the funds are distributed elsewhere, the school trip to see Shakespeare is cancelled.

Audit and evaluation in practice always mutually engage and feed off each other: audit MUST engage with local practice to be credible and inevitably WILL try to colonise local practices even to the point of endangering their use. On the other hand, local practice demands to be resourced and professional practitioners will feed the audit system with data accordingly. Indeed, this engagement with audit generally offers opportunities for subversion and the local effects of audit can generally be, to some extent, de-coupled and made useful in evaluation precisely because of auditors' need for credibility.

Thus, there is always a political struggle over audit and evaluation, their distinct values, systems, objects, sources of credibility and power bases. To understand this is essential to understanding the education system today. For instance, to attempt to 'deny' audit may be to try to refuse to engage with powerful social forces and so leave the field open to their colonisation. Rather, we may criticise existing audits, subvert them and devise better technologies that reassert the use of professional evaluation and reflection.

In contrast to the effects of audit on learners in schools, professional auditing of teachers' knowledge has so far made quite limited inroads into professional practice. In the UK, there has been the introduction of a requirement for teacher educators to audit elementary aspects of teachers' mathematical knowledge for primary school teaching. Summing up the recent literature, I conclude that we do not know much about the effects of this on trainees or on practice in general.

We know that many of the auditing instruments used are essentially crude tests of mathematics not much different from school mathematics tests, assessing mainly substantive elementary mathematics, though a few attempt to touch on syntactical knowledge (e.g. Rowland, Barber, Heal, & Martyn, 2003). In general, these audits repeat the school assessments that the trainee teachers would have completed some years earlier in school and the same deficit model applied: as Murphy (2003) points out, this can lead to a kind of complacency ('jumping through hoops') among those who pass and desperation (or worse, denial) among those who do not. Almost nothing in this work has been done to actually audit 'teaching knowledge', i.e. knowledge distributed in the act of teaching.

Let us take an exemplary audit item from this literature: "Some children have measured their desk to be 53 cm by 62 cm. State the possible limits to the lengths of their sides" (Goulding, Rowland, & Barber, 2003).

It is quite evident to me that any teacher in the flow of teaching is going to think: "Well, 53 cm could be anywhere between 50 and 53, as these tape measures tend to stretch, and the desk may have been a true rectangle many years ago, . . . but why do you ask?" But of course, we are not auditing knowledge in (or even for) teaching, we are auditing schooled knowledge with its arcane conventions, language and values ("State the limits . . ."), stripped of any practical or pedagogical or even mathematical sense of purpose. In this context, it is surprising that educators reflecting on these audits see them as being broadly positive for teachers in training, i.e. an improvement on what went before. But then, they did write these audits/tests and they do know what went before.

What of the prospect of auditing knowledge for, or distributed in, teaching then? Ball, Thames, and Phelps (2008) report that they have developed some proxies and I will report some potential instruments later in this paper that similarly bear on teacher knowledge of their students' learning. (In addition, for an account of such assessment of teachers' knowledge and their own learning, see Chapter 15 by Ryan and Williams, this volume) .

But first, we proceed with the analysis of audit-versus-evaluation by considering: (i) the contradictory purposes of audit and evaluation (exchange versus use); (ii) what kinds of knowledge teachers need to teach (for use) and what they need to display (for exchange); and (iii) what technologies of assessment we need to prevent audit from colonising evaluation.

## What Is the Purpose of Audit and Evaluation of Teachers' Knowledge?

The CHAT analysis of the contradiction referred to above has its roots in two distinct activities and activity systems: those of the auditors (teachers' managers, certifiers, accreditors and maybe even teacher educators) and those of the auditees (the teachers and student teachers themselves, but maybe also the teachers and their educators, too). Teachers' knowledge, for the purpose of the activity of teaching, has a different 'meaning' and 'purpose' from that of teacher knowledge for audit and for their auditors: it can be considered a boundary concept, and when reified in audit practices becomes a boundary object at the interface between the two and so its meaning is contested.

In socio-cultural theory, boundary objects are considered to be interesting theoretically and methodologically: they serve contradictory purposes, being involved in distinct activities, and as such they can provide insights into system dynamics (see e.g. Star & Griesemer, 1989). Thus, by exposing the audit item above to the (incorrect, improper and subversive) point of view of the practising classroom teacher engaged in the (imaginary) flow of teaching, I reveal that it assesses for audit, but not for teaching practice.

Who are the auditors or more significantly the commissioners of audit here, and what does 'teacher knowledge' mean for them? I mention a number of groups that may each have distinct interests and between whom there are potentially yet more contradictions and tensions. Politicians, their officers and teacher educators may have need of data to record and monitor the success of their work, and hence to account to their own public audiences – and hence ensuring their own flow of resource.

For these groups, some measure of 'their' teachers' knowledge may provide essential exchange value in meeting their social need for accountability. But in addition, in order for these measures to be credible, there is a need for auditors and their own audiences to believe that the measure does represent something real, some use-value in teaching: this can only be determined by an articulation of a relation to the practice of teaching. Thus, the measure of 'percentage of teachers who are graduates in mathematics', say, is only a viable audit measure if there is a credible relation between this measure and teaching or potential teaching quality. This all offers much disputable terrain, but the contest over credibility is not only, or even essentially, an academic one. It is in everyday political discourses and discourses of common sense that the battle is fought by operators on the political stage.

Who are the auditees here and what is knowledge for them? It may be the teachers, for whom their knowledge is both use-value (knowledge needed for them to be able 'to teach') and exchange value (the means for them to stake a claim to professional status, possibly accreditation). This signals another contradiction in the commodification of knowledge. The student teachers may unhelpfully become engaged subjectively in this audit process: they may 'pass' and therefore their knowledge is credibly 'assured', and perhaps then they may have less motivation to learn more. Or, they may 'fail' and believe that they *are* failures, and as tried and well-practised failures they may proceed to learn instrumentally to try to pass and even teach this expertise and knowledge to others.

In conclusion: the essential primary tensions and contradictions of audit and evaluation reside in the contradictions 'within' the objects of the activities of teaching and auditing, between the exchange and use-value of the knowledge being audited for the different, contradictory social groups with their different interests. Resolving these contradictions involves auditing and giving exchange value to 'useful' knowledge: the introduction of syntactical knowledge to audit is a good start, albeit that this has proved somewhat problematic to teacher education so far.

## What Kinds of Knowledge 'Should' (Mathematics) Teachers 'Have' for – or 'Display' in – Teaching?

This is the favourite territory of dispute for the mathematics teacher educator and many a researcher: 'we' all like to say that we want more than for teachers to 'just' have/display mathematical knowledge, facts and skills they can 'pass on' to children/students, while for the public, common sense suggests that this is just what teachers should know and do. This disjuncture between the teacher-educator discourse and that of the general public (to whom government accounts) is ultimately what gives audit so much room for colonisation in the practice of teacher education.

We must also ask, what do teachers need 'before' and 'when' they teach? Note that in this question the acquisition metaphor (to 'have' knowledge) is implicit and the process of 'display' appears somewhat strange or at least non-normative (Sfard, 1998). Note also the 'before' and 'when' that signify distinct audit/evaluation purposes again at the boundary between training and teaching institutions. This is another boundary that signals a contradiction between the exchange value in the feeder institution (e.g. the teacher-training institution) and the use-value in the receiver system such as the school where the teacher will practise. In general, audit and evaluation at the transition or boundary between institutions becomes problematic (see Williams et al., 2009).

Ultimately, it must be argued that the 'acquisition' of certain objects of knowledge (concepts, etc.) in pre-service training practices become mediating tools in the subsequent practice of teaching. But the third generation Western version of CHAT due to Engestrom and Cole asks us to attend to contradictions arising from just this kind of linkage between the two. What is reified in one system tends to need a lot of work to become a useful tool in another. When assessment in training becomes a tool of audit, this strengthens the links between training systems with a third system, i.e. the audit system. It becomes more difficult to structure it to the purpose of 'use' in teaching, as each system has its own objectives, its own technologies and discourses. Thus, according to Ball et al. (2008) teacher knowledge 'in mathematics teaching' is multi-dimensional: this implies that audit instruments that credibly measure this will yield multiple measures and make auditing very complicated and perhaps even impossible. In such cases audit tends to reduce multiple measures to one, thereby constraining the evaluation of use.

In part, this becomes a matter of technology: can we devise assessment tools that bring the training practices in line with 'use', but still satisfy the demands of the audit culture for some measure of knowledge that is credible?

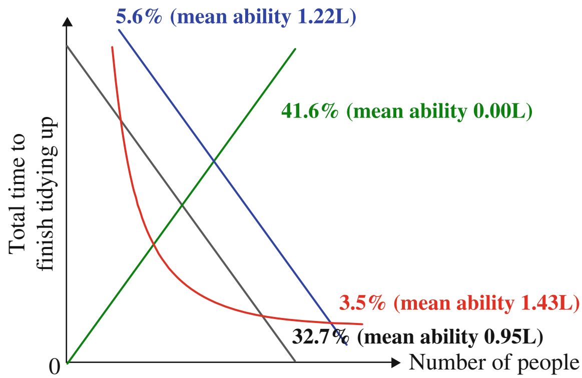
## How Can We Audit/Assess Teacher Knowledge: What Tools/Technologies Do We Have?

The need for the development of appropriate technologies is by now apparent: the demands of audit require a credible measure, but credibility and de-coupling demand a sense of authenticity in relation to the primary products of teaching. One very simple technology in the field of formative assessment is an apt case to discuss. It is one of a number of studies conducted in which diagnostic assessment instruments were designed for students, but were adapted to assess their teachers' knowledge too. In addition to the case described here below, we found in a study of primary and secondary school teachers' knowledge about probability that the effect of teaching experience is distinct to that of prior subject matter knowledge (with the more experienced teachers better predicting learners' errors, but the less experienced teachers showing better schooled knowledge of the topic; see Afantiti-Lamprianou and Williams (2003)).

I now describe one example in some detail, following the account given in Hadjidemetriou and Williams (2002, 2003). A diagnostic assessment tool, developed from items from the research literature (Bell & Janvier, 1981; Bell, Brekke, & Swan, 1987a, 1987b; Hart, 1981) was constructed, (a) to elicit pupils' graphical conceptions and misconceptions, and (b) to function as a questionnaire for assessing (and measuring) teachers' perception of the difficulty of the items for their learners, based on a test instrument of items already calibrated for 14-year-old students learning about graphs. The test instrument was given to a sample of pupils and their teachers in order to establish a link between these two groups and to compare results. (Pupils' group interviews and teachers' semi-structured interviews also helped us to validate responses and to gain an insight into the thinking of learners and teachers.)

The items of the diagnostic instrument were deliberately posed in such a way as to 'surface' known 'everyday' graphical conceptions. It developed from an analysis

**Fig. 10.1** Showing the main responses (*line-graphs*), frequencies (%) and mean ability parameters (measured in Logits[1]) to an item from Hadjidemetriou and Williams (2002)




of the key literature in the field of children's thinking and involved misconceptions such as 'slope-height confusion' (e.g. Bell & Janvier, 1981; Clement, 1985), the tendency towards linear, smooth and other 'prototypical' graphs (Leinhardt, Zaslavsky, & Stein, 1990), the 'graph as picture' misconception, pupils' tendency towards reversing the x- and y- co-ordinates, misreading the scale and so on (Williams & Ryan, 2000). One example is given in Fig. 10.1: the four interesting responses included linear and inverse correlation and lines that either crossed (32.7%) or failed to cross (5.6%) the x- and y-axes.

The pupils' test was scaled using a Rasch methodology resulting in a five-level hierarchy of responses, each level of which was described as a characteristic performance including errors which diagnose significant misconceptions (Hadjidemetriou & Williams, 2002). The key point about Rasch methodology is that it helps develop a unidimensional interval scale for an underlying 'attainment' construct (which, in this case, we take to be 'graphical understanding': note that in the psychometric literature, the term 'ability' is always used as the technical term for this dimension, but for obvious reasons we try to avoid this). Because the Rasch model observes the principle of conjoint additivity, it is the most parsimonious one-dimensional model that meets the essential audit requirement, i.e. that scores can be legitimately added, subtracted or averaged.

However, group interviews also gave us the opportunity to validate the test responses, in particular, that the interpretation of the errors found in the test are symptomatic of the misconceptions discussed in the literature. In general, we found such interpretations to be valid, with just one problematic case of a misconception concerning children's slope-height confusion (Hadjidemetriou & Williams, 2002).

Twelve experienced teachers also participated in the study. They were asked to answer all the items and: (a) to predict how difficult their children would find

[1]The average ability [see comment above] of those on the test that made these responses is measured in logits: one logit corresponds approximately to one standard deviation of a normal distribution. Thus, those that drew straight lines with negative slopes would be about one standard deviation above the average of those that drew a positive slope if the sample were normal.

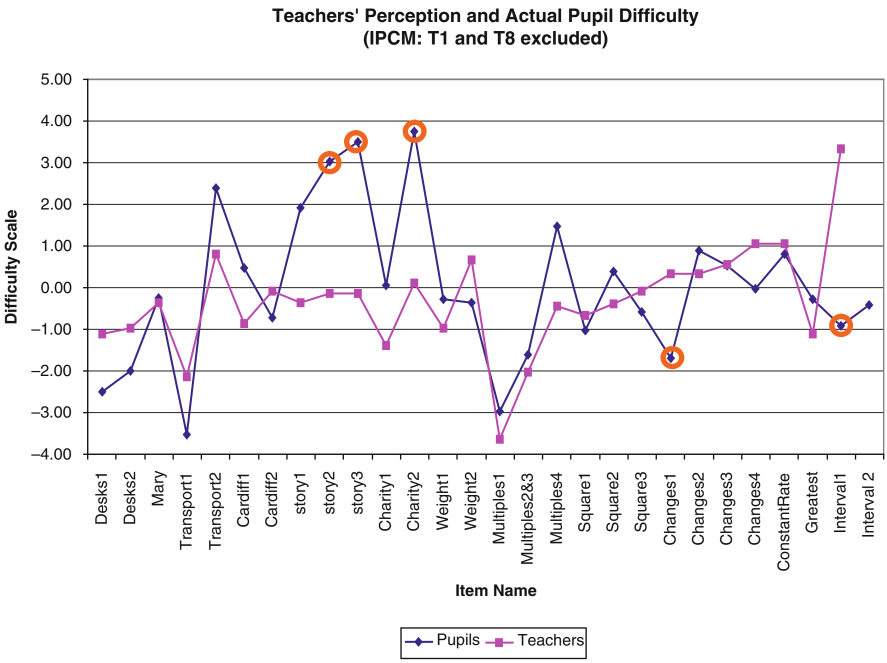



**Fig. 10.2** Teachers' perception and actual pupil difficulty: from Hadjidemetriou and Williams (2002)

the items (on a five-point scale starting from Very Difficult, Difficult, Moderate, Easy, Very Easy); (b) to suggest likely errors and misconceptions the children would make; and (c) to suggest methods/ideas they would use to help pupils overcome these difficulties. Teachers' knowledge was further explored through semi-structured interviews. From the teachers' rating scale data using Rasch models, we scaled the teachers' perception of difficulty and contrasted it with the learners' difficulty hierarchy (see Fig. 10.2). It was shown that some teachers over- or under-estimated the difficulties of some items. In Fig. 10.3, the circled items are the items that the teachers 'most mis-estimated' in terms of their difficulty. Data from questionnaires and interviews suggested that these mis-estimations were due either to: (a) the teachers having the misconception the item was designed to elicit (i.e. a failure of content knowledge), or (b) the teachers incorrectly assuming that pupils required formal understanding of mathematical concepts to answer questions correctly, i.e. a failure of pedagogical content knowledge.

The teachers' interviews, on the other hand, confirmed that the majority of them follow similar instructional sequences and that these are aligned with the prescribed National Curriculum. They also revealed that teachers' judgement of what is difficult is structured by this curriculum sequence: i.e. they sometimes incorrectly think that topics being more 'advanced' in the curriculum implies they are more difficult.

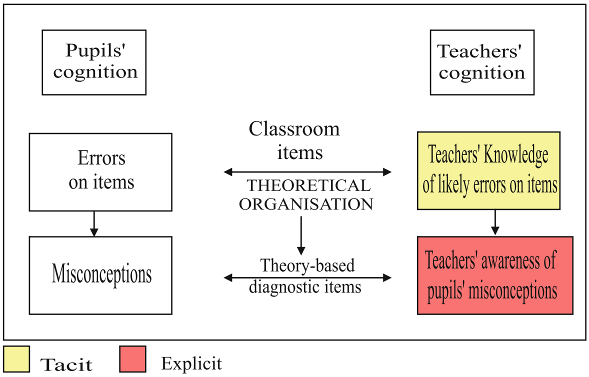



**Fig. 10.3** The emergence of knowledge about misconceptions: from Hadjidemetriou and Williams (2002)

Finally, we were struck by these teachers' apparent lack of awareness, in general, of their children's conceptions and misconceptions (see Table 10.1). When asked what misconceptions they might anticipate in their planning of teaching, few had much to say; yet when asked to predict errors in response to the test instrument, they were better able to predict what their pupils would do. Thus, these teachers' (who might generally be described as 'leading teachers' in the sense that they were all experienced, promoted to leading positions, or active in education in their region) audited knowledge was highly sensitive to the methodology adopted to collect it (Hadjidemetriou & Williams, 2002). We concluded that their knowledge is 'distributed' and that well-researched tools might make all the difference in what they are able to articulate, or to show in practice (see also Chapter 3 by Hodgen, this volume). This suggests consequences for their planning of teaching perhaps, but also for the results of audit.

Shulman (1986) proposes that pedagogical content knowledge appears in three different forms: propositional knowledge (e.g. knowledge of students' errors and misconceptions drawn from the literature), case knowledge (e.g. a personal, vivid

**Table 10.1** Misconceptions identified by 12 teachers: from Hadjidemetriou and Williams (2002)

| Teacher *Misconception* | 1* | 2* | 3* | 4* | 5* | 6* | 7* | 8* | 9 I | 10 I | 11 I | 12 Q |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Slope height* | | | | | Q | Q | Q | Q | | | | Q |
| *Linearity* | | | | | | | | | | | I | |
| *Y=X prototype* | | | | | | | Q | Q | | | | |
| *Origin prototype* | | | | | | | Q | Q | | | | |
| *Picture as graph* | | | Q | Q | Q | Q | Q | Q | I | I | | |
| *Co-ordinates* | | | IQ | I | | I | IQ | Q | I | I | | |
| *Scale* | I | I | I | I | I | IQ | IQ | IQ | | | I | Q |

'Q', 'I', and 'QI' indicate whether the misconception/error was mentioned by the teacher in the Questionnaire (Q), Interview (I) or both (IQ) while ∗ indicates the teachers who were both interviewed and answered the questionnaire.

classroom experience of an error that a teacher was surprised by) and strategic knowledge (i.e. the art of acting in the moment, in particular, to act in situations of information overload, openness, or lack of knowledge relevant to the situation). Much knowledge is presented by teacher educators in the form of declarative statements or propositions, possibly framed around a theory, in a logical form. But these often lack richness of context and are, therefore, hard for practitioners to recall or use in practice. According to Shulman, these limitations make propositional knowledge hard to apply. Case knowledge, on the other hand, may bring these propositions to life and embed them in context:

> Case knowledge is knowledge of the specific, well-documented and richly described events. Whereas cases themselves are reports of events, the knowledge they represent is what makes them cases. The cases may be examples of specific instances of practice- detailed descriptions of how an instructional event occurred- complete with particulars of contexts, thought and feelings. (Shulman, 1986, p. 11)

By providing teachers with the appropriate tools that will 'surface' errors and misconceptions, we hoped to enrich this kind of well-organised but well-contextualised and usable knowledge. Thus, such pedagogical tools might help mediate research knowledge, which might thereby be transformed aptly for teaching practice. All that is then needed is the strategic judgment to use the knowledge effectively in practice.

This link between 'case knowledge' and 'propositional knowledge' is, in our view, generally best conceptualised not just as a cognitive one (i.e. it is not only based on what teachers know and keep in their mind), but one which is socio-culturally structured, i.e. mediated by well-researched tools in practice. Figure 10.3 illustrates the relationship proposed.

This suggests that teachers acquire (maybe largely through classroom practice) knowledge about their pupils' errors. This knowledge is tacit, based on the tasks and items used in the classroom. This also relates to teachers' propositional knowledge. However, if these propositions and pupils' errors and misconceptions are theoretically organised around tasks that aim to diagnose them, then, firstly, deeper cognitive problems such as misconceptions come to the surface, and secondly, teachers are made aware of them. We concluded that a well-designed diagnostic tool that includes items which will elicit errors that reveal theoretically-based errors (i.e. misconceptions), might help to transform teachers' tacit knowledge into explicit knowledge that could be used in planning.

In terms of CHAT, the propositional knowledge relates most clearly to research and perhaps teacher education practices of 'reflection' on teaching; it is mediated by scientific language, and facilitates reflection and planning, and discourses about teaching generally. But case knowledge is mediated much more obviously by the everyday language of the context of teaching in classroom action, or generally in interaction with learners. Strategic knowledge is wholly embedded in the practice of teaching in the flow of the moment.

Thus, in CHAT terms, I argue that Shulman's three components of pedagogical content knowledge reveal the way such knowledge sits at the boundary between two different practices. On the one hand, we have the reflection, discussion and theorisation of teaching of the kind found in an inquiry group, or perhaps a staff room, or privately when a teacher is engaged in evaluating, planning, problem solving or reviewing strategies. On the other hand, we have the teaching practice itself. I argue that the different perspectives on knowledge revealed by the two practices explain the difference and the relation between the forms of knowledge proposed by Shulman (propositional, case, and strategic knowledge).

## Conclusion

In summary, previous studies have shown that (a) the teachers we studied sometimes mis-judged their students' knowledge, and their judgments are influenced not only by their own mathematical knowledge, but also by their teaching experience and the intended curriculum; and (b) their knowledge of their students can be strongly 'task-situated' and 'tool-mediated' rather than 'in the head'. In fact teacher knowledge is distributed.

All this is suggestive of the observation that audit and evaluation are tool-mediated, and that these tools shape cognition in practice. But the triple objects of audit, training and teaching practices are at stake here: the tools we use are at the boundary of all three activity systems and need somehow to satisfy the needs of the three systems if they are to become stable. On the contrary, the contradictory demands of the three practices may create instability, political contestation and tend towards colonisation or de-coupling. The triumvirate involves an interesting set of power relationships.

Audit tools can be critical in shaping the backwash effects of audit and need to be thought through in terms of their affordances for the colonising or decoupling of practices. It seems to be important that the tools we designed potentially coordinated training and teaching practice and also the propositional and case knowledge implicated. It also seems to be important that they can be used to construct summative measures and hence offer tools for audit. In this sense they might provide affordances for three systems and practices.

There will always be this uneasy struggle over the use of assessment tools. If any one community gains the upper hand such as that implied by decoupling or colonisation, it can lead to dysfunctional practices that may serve no-one. Even Prime Minister Tony Blair was discomforted when confronted on live television by a patient who observed that, since his government had introduced an audit measure of waiting times for medical appointments that punished centres where times went over 2 days, doctors' surgeries had started refusing to make appointments (de-coupling) more than 2 days in advance, leaving the patient angry and frustrated. Why don't auditors see this coming?

Let us imagine, then, the unintended consequences of audit in advance. If audit tools become a means to control a flow of resource, one should ask how this will distort their use in evaluation. In general, it becomes more important for the subject to get the right answer (a measure has to have right answers) than to learn. We must anticipate that if a measurement becomes high stakes in the assignment of exchange value, then the use of the tool for knowledge creation purposes in the primary practice at stake may be compromised.

In conclusion, I have argued that educational researchers need to understand audit as a practice and the contradictions inherent in it that might be politically exploitable. I have given an example from our own development work of how tools that audit knowledge-for-teaching provoke the realisation that knowledge is socio-culturally distributed. A credible audit of knowledge-for-teaching requires engagement with useful evaluation of learning and the development of case and propositional knowledge that might be productive for teaching in practice. Hence credibility of audit tends to produce a de-coupling effect – perhaps a necessary corrective in these colonised times. On the other hand, the engagement of 'evaluation and use' value with 'audit and exchange' value imposes some of the constraints of audit on practice, e.g. audit abhors multidimensionality and complexity. This contradiction fuels tendencies towards colonisation.

## Discussion: Towards a Collective Subject

I have appealed to social, cultural analyses of audit, evaluation and assessment practices in the foregoing argument, and especially to the way that tension and political conflict arises from contradictory practices and their objects (e.g. use and exchange values). In reflecting on empirical work in assessment, I have focussed on how particular tools may mediate audit and evaluation in significant ways.

It is increasingly obvious and widely recognised, I believe, that practice is mediated by tools and that, therefore, audit is sensitive to the technologies of surveillance available. Less obvious or less well known is our analysis of the social forces at work in audit and the consequences for understanding what is possible, and how and why productive or unproductive coupling, de-coupling or colonisation might be designed. Finally, I suggest some new directions where the CHAT perspective might lead.

First, CHAT recognises tool-mediation in object-orientated activity as only one mediating factor among many that may be the source of significant contradictions and therefore, dynamics. In addition, CHAT recognises the division of labour, governed by social, cultural, historically-formed 'norms' that position differently disposed subjects in collective activity. Furthermore, in particular, CHAT recognises the inner contradictions within the subject and within the object of activity, and between activities and their bounding activities through boundary objects and boundary crossers (Cole, 1996). Finally, CHAT recognises the possibility of 'expansion', for instance, via the re-formulation of the object of activity, or the formation of the collective subject (Engestrom, 1987, 1991).

Where might these notions lead in the case of the audit of teacher knowledge? First, it is significant that pedagogical knowledge is distributed in assessment tools, but this is only one cultural reification of a more general social distribution of knowledge.

To credibly formulate the problem of auditing pedagogic knowledge at the level of the individual teacher or student-teacher logically requires the presumption of a 'normative' level playing field consisting of (a) a scheme of work, the departments'/schools' plans, etc., (b) a standardised textbook and teaching resources, and (c) a common assessment and professional development system, inter alia. However, each institutional, classroom and pedagogical context is different. Appealing for a normative uniformity of affordances in school context is not only unrealistic, it is equitable to the point of being revolutionary.

If pedagogic knowledge then includes the assembly of knowledge distributed across the learning-teaching environment, then one must evaluate this in its social context. The result may be to question, not why a teacher is unaware of the learners' needs, say, but why the scheme of work, the departments' or schools' plans, the text book and the assessment and professional development system as a whole, are systemically unaware of the learners' needs. In this view, a teacher's knowledge can only be evaluated at the level of the system and the remediation of the system is at issue: the blame for weaknesses becomes distributed. But so then is the remedy, which demands a collective organisation of the many agents involved: this raises the possibility of a community of teachers as a collective subject (see Williams et al., 2007b). It may make sense to think increasingly of teacher knowledge in this way as a collective property of a collectively cognising subject: perhaps the department or the school, or even 'the mathematics teaching profession' as a whole, though it is necessary to work out the appropriate levels at which evaluation becomes useful.

Then there is the question of the 'double bind' (Engestrom, 1987). The central contradiction of schooling, the principal source of alienation of learners, is that between exchange value (the learning of knowledge for accreditation, i.e. for advantage in the future distribution wars over resources, capitals etc.) and use-value (learning useful knowledge that enhances the capacity of the individual/social subject to act usefully). The teacher may experience the same contradiction in relation to their own pedagogic knowledge. The thrust of the argument for the formation of a collective subject rests in finding allies that share an interest in escaping this double bind and in rewriting the rules. What might this mean for teacher knowledge?

The contest over values, surely, will not be decided within the teaching profession alone and is manifest and of interest throughout society. But I suggest that understanding audit and evaluation at least requires us to see how values are critically at issue for our profession as well.

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# Chapter 11
# The Cultural Dimension of Teachers' Mathematical Knowledge

**Andreas J. Stylianides and Seán Delaney**

## A Case for Considering Culture in Research on Teachers' Mathematical Knowledge

Much of the teacher knowledge literature has emerged from research programmes in North America. In one review of research on the topic, the educational philosopher Gary Fenstermacher identified four categories of teacher knowledge. Each category was linked to researchers based in the United States or Canada (Fenstermacher, 1994). The categories he identified were personal practical knowledge (associated with Jean Clandinin and Michael Connolly), knowledge developed from reflective practice (associated with Donald Schön), types of knowledge about teaching (associated with Lee Shulman), and knowledge generated by teacher-researchers (associated with Marilyn Cochran-Smith and Susan Lytle). Although this research originated in North America, it has influenced research on teacher knowledge elsewhere.

Take for example Shulman's work, which has inspired much research on teacher knowledge over the last two decades. In 1986, Shulman drew attention to the fact that researchers at the time were attending to generic aspects of teaching, such as classroom management and student reinforcement, whereas subject matter knowledge was being relatively neglected. Shulman's work inspired researchers to look more closely at the content preparation of teachers in all school subject areas and at all levels from primary school to college. In particular, his idea of pedagogical content knowledge (e.g., Shulman, 1986) captured the attention of many educators so that by now the term is taken for granted (Bullough, 2001). Although a huge amount of teacher knowledge research has taken place in the United States (e.g., Ball & Bass, 2003; Simon, 1993), researchers throughout the world have responded to the call to look at teacher content knowledge in several school subjects (e.g., Padilla, Ponce-de-León, Rembado, & Garritz, 2008; Rowland, Huckstep, &

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Thwaites, 2005). Whether the studies originated inside or outside the United States, a substantial number of them cite the work of Shulman.

The timing of Shulman's article was good, coming at a time when teachers and teacher educators in the United States were being criticised by several reports (see Bullough, 2001, for a discussion of these reports). But the popularity of the construct of pedagogical content knowledge inside and outside the United States is probably due to the way in which it brought together content knowledge and the practice of teaching. By combining content knowledge and the practice of teaching, the construct implied that a special kind of subject matter knowledge is unique to teaching. Although the idea of pedagogical content knowledge appealed to numerous researchers, many of them used the term in different ways and the construct needed additional specification, even in the United States (Ball, Thames, & Phelps, 2008). But even if a construct is well specified in a given country, it can be problematic if that construct is applied in a new setting where it may be interpreted differently. When a construct has different meanings in different settings, it is considered to lack conceptual equivalence (e.g., Harachi, Choi, Abbott, Catalano, & Bliesner, 2006).

The example of pedagogical content knowledge is an illustration of how a teacher knowledge construct developed in one country is assumed to apply universally. But assumptions of universality need to be treated with caution. In relation to intelligence tests, Straus (1969) made the following point:

> The items used in most standard intelligence tests contain many references to objects and events which would be outside the range of experience of a village child in Africa or India. Of course, some children would get the correct answer to these "culturally biased" items, but these are likely to be children who have had exposure to modern urban settings. Thus, children getting the highest scores will not necessarily be the brightest children, but rather the more "Westernized" (p. 234).

Although the construct of intelligence had been developed to the satisfaction of researchers in "westernized" settings, when tests based on the construct were transferred to another setting, the construct was different and students' scores on the tests, which were based on the construct, had little meaning in relation to the construct as originally conceived. Straus acknowledged that remedying such problems in research poses practical difficulties, including those of time and cost.

It is possible to understand why the cultural dimension of pedagogical content knowledge was not acknowledged when the construct was introduced. As Shulman himself noted about the teaching effectiveness studies which were popular when he proposed the idea of pedagogical content knowledge, "to conduct a piece of research, scholars must necessarily narrow their scope, focus their view, and formulate a question far less complex than the form in which the world presents itself in practice" (Shulman, 1986, p. 6). Shulman focused on specific school subjects at secondary school level and studied teachers in California. But can observations of teachers in one US state produce a construct that has the same meaning throughout the world? For example, Shulman asked about the knowledge needed by a teacher when presented with "flawed or muddled textbook chapters," and what "analogies, metaphors, examples, demonstrations, and rephrasings" the teacher can use to explain, represent or clarify ideas (Shulman, 1986, p. 8). Yet in some countries, flawed or muddled textbook chapters may be rare, reducing the necessity for teachers to possess such knowledge; and metaphors can be sensitive to national and organisational cultures (Gibson & Zellmer-Bruhn, 2001) so that knowing a useful metaphor in one setting may be unhelpful in another.

We have used the example of pedagogical content knowledge to urge caution in assuming that ideas about teacher knowledge which apply in one setting have universal application. Variations in how teacher knowledge is conceived matter because conceptions may be expressed similarly but understood differently in various countries. That is problematic for policy makers, researchers or educators who need to be explicit about the meaning of terms they use. Furthermore, some researchers are currently studying mathematical knowledge held by student teachers in several countries (Tatto et al., 2008). In order to interpret the findings of the mathematics known by student teachers, it is important to know what *kind* of mathematical knowledge they hold and why that knowledge is important in the particular countries in which they will teach.

Acknowledging the cultural dimension of teachers' mathematical knowledge is a relatively recent phenomenon (Ball et al., 2008; Delaney, 2008; Delaney, Ball, Hill, Schilling, & Zopf, 2008). One reason for the increased attention to the role of culture in teacher knowledge may be due to our growing understanding of the influence culture has in many aspects of life, from homicide rates (Nisbett, 1993) to safety on aeroplanes (Gladwell, 2008; Merritt, 2000). Although the work pilots do is similar from country to country, cultural attributes, such as taboos against questioning a more senior colleague, interact with their training and other factors to shape how they do their work. In her study of 9,400 pilots in 19 countries, Merritt (2000) concluded that "the effects of national culture can be seen over and above the professional pilot culture, and that one-size-fits all training is not appropriate" (p. 299).

A major reason for our interest in teacher knowledge is to inform the professional formation and development of teachers so that they in turn can help to raise the mathematical achievement of their students. If Stigler and Hiebert (1999) and others are correct that teaching is a cultural activity, then the knowledge teachers possess or need may depend on the culture in which they are working. Alternatively, if, like flying planes, teaching is largely the same from country to country[1] and teachers require the same knowledge wherever they teach, cultural attributes are likely to interact differently with teachers' acquisition of that knowledge from one country to another. In both cases, the cultural dimension of teacher knowledge needs to be considered. The four chapters in this section of the book add considerably to this discussion in relation to teachers' mathematical knowledge, and illustrate some of the avenues currently being pursued within the rapidly growing body of research that acknowledges and studies the cultural dimension of teachers' mathematical knowledge.

[1] For an overview of this argument, see Dale (2000).

In what follows, we review the four chapters with a focus on the interplay between the cultural context and mathematical knowledge for/in teaching. We make a distinction between "mathematical knowledge *for* teaching" and "mathematical knowledge *in* teaching." We use the former term to describe knowledge that can enable teachers to effectively support student learning of mathematics. In a sense, this kind of knowledge can be understood as being essential, or necessary, for successful teaching (as defined within a particular cultural setting). We use the latter term to describe knowledge that teachers use as they teach mathematics, i.e., teachers' knowledge as manifested in their practice. There is no suggestion about the capacity of this kind of knowledge to necessarily support a particular form of teaching (successful or not).

The review illuminates three different, but complementary, aspects of the cultural embedding of mathematical knowledge for/in teaching. The first aspect, which is represented by the chapters of Andrews and Pepin, situates mathematical knowledge *in* teaching in the context of different national educational systems. The second aspect, which is represented by the chapter of Adler and Davis, situates mathematical knowledge *for* teaching in the context of diverse teacher education programmes. The final aspect, which is represented by the chapter of Williams, situates mathematical knowledge *for* teaching in the culture of a "knowledge economy". In all cases, the identified context of mathematical knowledge for/in teaching denotes the main (rather than the exact or only) cultural locus of this knowledge as reflected in the chapters.

We acknowledge that the focus of our review on the cultural embedding of mathematical knowledge for/in teaching inevitably downplays some important contributions made by the chapters that did not fit directly within the scope of our review. We will allude to some of these contributions in the final section of our chapter where we will consider implications of the four chapters for teacher education research and practice.

## The Interplay Between the Cultural Context and Mathematical Knowledge for/in Teaching

### *The Cultural Embedding of Mathematical Knowledge in Teaching in the Context of National Educational Systems*

Andrews and Pepin both located teachers' mathematical knowledge in the national cultural discourse in which mathematics teaching and learning occur, and considered teacher knowledge as a social construction that is shaped by the particular national educational system wherein it functions. They argued that consideration of the characteristics of different educational systems (curricular expectations, typical teaching practices, etc.) can offer useful insight into explaining the variation observed in the ways teachers' mathematical knowledge is manifested in teaching practices of these systems. Accordingly, the two chapters acknowledged the importance, and examined the role, of the cultural embedding of mathematical knowledge *in* teaching in the study of mathematics teachers' practices in different countries. The cross-national comparative aspect of the chapters became, then, a means by which the authors understood and described mathematical knowledge in teaching in their selected countries.

Having outlined in general terms the chapters' common position on the cultural embedding of mathematical knowledge in teaching, we will now consider the development of this position in each chapter separately.

Andrews criticised existing frameworks on teachers' mathematical knowledge in that they tend to consider this knowledge as a personal construct, paying insufficient attention to its cultural embedding in the context of national educational systems that have their own systemic imperatives and didactic folklore. In an attempt to contribute to the development of existing frameworks, Andrews proposed a complement to these frameworks, a tripartite classification of what he called "idealised", "received", and "intended" curricula. This classification considers teacher knowledge as a social construction that is located in the classification's constituent and culturally dependent curricula: one that describes teachers' personal and articulable perspectives on mathematics teaching and learning (idealised curriculum), one that describes hidden and inarticulable aspects of teachers' practices that are taken for granted within an educational system (received curriculum), and a third one that describes systemically defined expectations of learning outcomes that often reflect societal or historical values (intended curriculum).

Andrews used this classification as an analytic tool to examine mathematical knowledge *in* teaching as manifested in two lesson sequences on linear equations taught by a Flemish teacher and a Hungarian teacher to grade 8 students in their respective countries. The findings of the examination, which had a comparative cross-national nature, suggested the utility of the classification in revealing culturally relevant aspects of mathematical knowledge in teaching. The importance of the findings lies in that the revealed aspects could remain tacit, or defy explanation, under alternative examinations that would use existing frameworks on teachers' mathematical knowledge. For example, Andrews discussed the case of the Flemish teacher who seemed reluctant to deviate from her planned lesson activities, an attribute of her practice that could be construed as a low level "contingency" (see Rowland et al., 2005). However, Andrews observed that it is difficult for one to determine whether this teacher's reluctance suggests a deficit in her pedagogical practice or whether it actually reflects a conscious decision on the part of the teacher not to deviate from well-articulated procedures. Andrews noted, then, that the terms "intended" and "received" curricula offer a useful language for one to describe the teacher's observed behaviours: these behaviours set the teacher apart both from systemic expectations of the Flemish educational system (intended curriculum) and from practices shared among her colleagues (received curriculum).

Pepin began from the premise that the practice of "listening" is central to mathematics teaching (and thus an important element of mathematical knowledge *for* teaching) and examined how mathematical knowledge with respect to listening is manifested in the teaching practices of English, French, and German teachers. Specifically, she used a socio-cultural approach to examine mathematical knowledge *in* teaching from the point of view of listening, using data from interviews and lesson observations with 42 teachers (14 in each country). The cross-national nature of Pepin's examination illuminated, like Andrews' study did, culturally relevant aspects of mathematical knowledge in teaching that seemed to be shaped by the national educational contexts in which the teaching practices were embedded.

In particular, Pepin's examination showed that teachers' listening (and, by implication, teachers' knowledge with respect to listening) took different forms in the three countries and that this variation might be explained in terms of different aims, values, or school types in place in each country's educational system. In England, an aspect of teachers' listening was its individualistic nature, which might be explained with reference to one of the aims of the English educational system to provide students with the individual support they need to make progress in their studies. Contrary to what was observed in England, teachers' listening in France tended to attend to the group as a whole; this aspect of French teachers' listening might be attributed to the fact that the French educational system values whole-class discussions of mathematical problems. Finally, the considerable variation that was observed among German teachers' listening might be explained in terms of the different school types where the teachers worked. For example, German teachers who worked in secondary modern schools (Hauptschulen), which are considered to be educationally challenging working environments for teachers, tended to listen more for the correctness of students' contributions and less for the logical underpinnings of these contributions, which was one of the characteristics of the listening practices of their colleagues who worked in the local grammar schools (Gymnasien).

To conclude, the two chapters reinforced, extended, and further exemplified an important point made by prior comparative research: the cultural aspects of national educational systems not only influence what mathematics teaching looks like in these systems and students' learning outcomes (e.g., Cogan & Schmidt, 1999; Hiebert et al., 2003), but also the nature and manifestation of teachers' mathematical knowledge in teaching practice.

## *The Cultural Embedding of Mathematical Knowledge for Teaching in the Context of Diverse Teacher Education Programmes*

Adler and Davis examined the constitution of mathematical knowledge *for* teaching in various teacher education cultures, which were shaped by the broader, socio-economically diverse South African context. Adler and Davis argued that descriptions of the constitution of mathematical knowledge for teaching in teacher education would be incomplete without serious consideration of how mathematics teaching was *modelled* in it, i.e., the images of the mathematics teacher and, by implication, of mathematics teaching, presented to pre- or in-service teachers in teacher education programmes. Accordingly, Adler and Davis studied how mathematics teaching was modelled in teacher education programmes as a means of describing the kinds of learning opportunities teachers are afforded in these programmes to develop mathematical knowledge for teaching.

In analysing how mathematics teaching is modelled, and thus in interpreting the mathematical knowledge for teaching constituted, in teacher education programmes, Adler and Davis used a methodology that built on the works of Bernstein (1996) and Davis (2001). The methodology was premised on the assumption that pedagogic practice entails continuous evaluation, with every evaluative act, a form of pedagogic judgement, appealing to an authorising ground (being mathematics, mathematics education, teaching experience, etc.) in order to legitimise the pedagogic judgement. Adler and Davis applied this methodology in case studies of three teacher education programmes for in-service teachers in South Africa, and derived three different models of mathematics teaching (one for each programme).

The models of mathematics teaching, and the corresponding kinds of learning opportunities for teachers to develop mathematical knowledge for teaching that these support or imply, are referred to in the chapter as (1) "look at my practice", (2) "look at your own practice", and (3) "look at mathematics teaching practice". In the first model, developing mathematical knowledge for teaching is by emulation of the practice (performance) of the teacher educator who aims to provide teachers with an experiential base of the (reform-oriented) practice teachers are expected to enact in their classrooms. In the second model, developing mathematical knowledge for teaching is by systematic reflection on teachers' own practices as part of an action-research paradigm to teacher professional development. In the third model, developing mathematical knowledge for teaching is by interrogation of records of classroom practice, using analytic tools derived from the field of mathematics (teacher) education.

In light of their findings of how mathematical knowledge is constituted in teacher education, Adler and Davis raised questions about, and set a foundation for future investigations of, the role of teacher education in redressing or reproducing socio-economic inequality in South Africa and elsewhere. *Who* has access to *what* learning opportunities in teacher education for developing mathematical knowledge for teaching? How does teachers' acquired knowledge in teacher education shape teachers' capacity to teach mathematics and, by implication, the learning opportunities that the teachers ultimately offer to students in schools in different areas and of different socio-economic status? For example, Adler and Davis observed that the first model of mathematics teaching (look at my practice) was promoted in a teacher education programme for teachers from rural and socio-economically disadvantaged schools. To what extent, then, does this particular teacher education orientation to mathematics teaching account for, or contribute to, the generally low student learning outcomes in these schools?

## *The Embedding of Mathematical Knowledge for Teaching in a "Knowledge Economy" Culture*

Williams situates his chapter in the culture of the "knowledge economy". One feature of a knowledge economy is a political requirement to audit services and service providers in order to establish a cost-benefit analysis for expenditure in particular areas. Teachers' mathematical knowledge *for* teaching has not been immune from this societal preoccupation with auditing. The goal of auditing teacher knowledge is to establish an "exchange value" so that if it is found to be satisfactorily present, resources will continue to flow towards teachers who possess such knowledge or towards the teacher educators who help teachers develop such knowledge. In contrast to auditing, evaluating mathematical knowledge for teaching is concerned with determining its "use-value", or how it can be used in the practice of teaching.

Although audit and evaluation may often be in conflict, common to both endeavours is the need for tools to audit and evaluate teacher knowledge. According to Williams, developers of such tools face two major challenges. One is that knowledge is sensitive to the tool that is used to audit or evaluate it. Propositional knowledge is removed from practice and often found in research on teaching, whereas case knowledge and strategic knowledge are more directly connected to classroom practices; a tool designed to evaluate one type of knowledge may be ineffective in evaluating another type. The second challenge identified by Williams is that teacher knowledge is distributed; rather than being held "in the head" of any individual teacher, such knowledge is held by teachers collectively – in a school or in the profession. Given these challenges, any tool designed to evaluate or audit teacher knowledge involves compromise. Furthermore, Williams contends, the tools that are shaped will ultimately shape our conception of teacher knowledge.[2]

An analysis of culture is central to Williams's thesis. When people are immersed in a culture, they can be unaware of how it shapes their thoughts and actions. But when features of a culture are highlighted or contrasted with other cultures, biases and orientations become apparent. By attending to contemporary Western society's bias towards audit, and the potential for evaluation to be conflated with audit, it is possible to consider how such an orientation affects our understanding of teacher knowledge.

## Implications for Teacher Education

Implications for teacher education of the cultural dimension of *mathematics* and *mathematics education* (the values inherent in them, their historical and social bases, etc.) have been considered elsewhere (see, e.g., Bishop, 1988; Gerdes, 1998). The four chapters in this section add to these implications via another route: that of acknowledging the cultural dimension of *teachers' mathematical knowledge.* In this section, we discuss implications that derive from this route and concern how teacher education is, or might be, influenced by the cultural dimension of teachers' mathematical knowledge.[3]

[2]In interpreting and describing Williams's view, we paraphrased an expression attributed to McLuhan (1964, 1994) in Lewis H. Lapham's introduction to the 1994 edition of *Understanding Media*: "we shape our tools and then our tools shape us" (p. xi). McLuhan's actual quotation seems to be that "the *beholding* of idols, or the use of technology conforms men to them" (p. 45).

[3]We use "teacher education" broadly to include both the initial training of pre-service teachers and the continued professional development of in-service teachers.

An important concern of mathematics teacher education is to articulate a curriculum that will help teachers develop mathematical knowledge that is useful in teaching. Some of the research that is available to inform such a curriculum has evaluated the knowledge – often the absence of knowledge – held by teachers and student teachers (e.g., Ball, 1990; Borko et al., 1992; Ma, 1999; Stein, Baxter, & Leinhardt, 1990). The methods used in this research included teacher observation, asking teachers to respond to mathematics teaching scenarios, and categorising mathematical objects. But Williams would advocate adopting a sceptical approach to the findings of such research because in at least one case described by him the knowledge teachers were deemed to hold was sensitive to the methodology used to audit it. Although such methods have yielded compelling data to inform, by implication, teacher education curricula, Williams's chapter cautions against complacency with existing tools for evaluating teacher knowledge and advocates the need to be mindful of the cultural-boundedness of any tools used. Future research into teachers' mathematical knowledge, therefore, would benefit from using multiple and innovative means to study mathematical knowledge in and for teaching. This would ensure that mathematics teacher educators have rich and diverse data about teacher knowledge to draw on when designing and delivering teacher education curricula.

Another area influenced by the cultural dimension of teachers' mathematical knowledge concerns the role of teacher educators. The chapters by Andrews and Pepin both made the point that the manifestation of teachers' mathematical knowledge in teaching practice is shaped by cultural aspects of the national educational systems wherein teachers work. In cases where these cultural aspects are aligned with visions of effective teaching and learning of mathematics in the respective educational systems, a potentially important element of the role of teacher educators would be to facilitate teachers' acculturation to the existing systems. Part of this process of acculturation would happen naturally anyway, assuming that prospective teachers were themselves educated in those systems in which they will be employed.

Yet, in several educational systems nowadays, new visions of effective teaching and learning are introduced in the context of curricular reforms. These new visions deviate from some previous systemically defined expectations of learning outcomes or how to achieve those outcomes, thereby creating a need for teacher educators to acculturate teachers to novel (from the point of view of a given educational system) conceptions of teaching and learning mathematics. The "apprenticeship-of-observation" (Lortie, 1975) would be a major obstacle to the process of acculturation to novel conceptions, as "it is an ally of continuity [of existing practices] rather than of change" (p. 67).[4] Accordingly, a different potentially important element of teacher educators' role would be to help teachers become more aware of, and reflect critically on, their "cultural scripts for teaching" (see Stigler & Hiebert, 1999) with

[4]The apprenticeship-of-observation is a process through which students internalise (in the most part unconsciously) the practices of their own teachers. Lortie (1975) commented on the apprenticeship-of-observation: "[T]he apprenticeship-of-observation undergone by all who enter teaching begins the process of socialization in a particular way; it acquaints students with the tasks of the teacher and fosters the development of identifications with teachers" (p. 67).

an eye towards developing new conceptions of teaching and learning that better meet the goals of curricular reforms in the respective educational systems.[5] The method, however, by which teacher educators can acculturate teachers to new forms of teaching and learning remains unclear and is a fertile direction for future research.

The chapter by Adler and Davis informs this issue with its discussion of how mathematics teaching is modelled in teacher education. It would seem that traditional discourses in teacher education would advocate the importance of the "look at my practice" model of mathematics teaching on the basis that teachers might not be expected to enact reform-oriented teaching without having first experienced for themselves this kind of teaching from the learners' point of view. Notwithstanding this argument in favour of the "look at my practice" model when acculturating teachers to new forms of teaching and learning, Adler and Davis's chapter suggests that an effective teacher education practice would incorporate a variety of models of mathematics teaching, for each model would underpin different kinds of learning opportunities for teachers to develop mathematical knowledge for teaching.

Another question to be asked by teacher educators is the extent to which the unit to be concerned with is the individual teacher. Williams argues that the knowledge that matters for teaching is distributed across the system and that strengths and shortcomings are located not in individuals but across textbooks, school plans, assessment, professional development and so on. Take, for example, the issue of textbooks: "educative curriculum materials" (Davis & Krajcik, 2005), which are concerned not only with student learning but also with teacher learning, are more likely to complement teachers' knowledge in ways that will have a positive impact on their teaching than other kinds of materials. Yet, existing curriculum materials, even those which are reform-oriented, fall short of meeting key expectations for being considered "educative" (Stylianides, 2007). But even the availability of educative curriculum materials does not imply by itself that teachers use these materials productively to enhance their knowledge for teaching (Castro, 2006). Consequently, teacher educators need to study the wider context in which individual teachers' knowledge interacts with other aspects of the system. This study can inform teacher educators' understanding of the mathematical knowledge required of individual teachers, of specialists within a school, of teachers collectively in a school, and of the entire teaching profession.

Williams argues further that the systems across which teachers use their mathematical knowledge differ to such an extent that auditing the knowledge of individual teachers is futile. If Williams is correct about the distributed nature of teacher knowledge and its cultural specificity, how should teacher educators plan and assess their courses? What are the distinctive features of different pedagogical contexts

[5] According to Stigler and Hiebert (1999), people within an educational system share a mental picture of what teaching is like, that is, they share a "cultural script for teaching." A major factor involved in the development of teachers' cultural scripts for teaching is the apprenticeship-of-observation. Indeed, Stigler and Hiebert (1999) argued that "we learn how to teach indirectly, through years of participation in classroom life, and that we are largely unaware of some of the most widespread attributes of teaching in our own culture" (p. 11).

that affect the mathematical knowledge required by teaching? Is it feasible to offer prospective teachers individualised programmes to prepare them specifically for the diverse contexts in which they might teach and to evaluate every teacher's knowledge in the teacher's unique teaching environment? Implementing such a system in many contemporary models of teacher education would pose practical and economic difficulties. But, is it possible to adapt existing models of mathematics teacher education to take on board the idea of distributed mathematical knowledge of mathematics for teaching? Research that would directly address such questions may yield fruitful answers to inform teacher education.

For example, one way in which teacher educators can incorporate the distributed knowledge assumption of mathematical knowledge for teaching is to review how they plan for and organise learning in their courses. Hewitt and Scardamalia (1998) identified six strategies for distributed learning processes which could be modified for and used in the teacher education context. These strategies are to:

1. support educationally effective peer interactions,
2. integrate different forms of discourse,
3. focus students on communal problems of understanding,
4. promote awareness of participants' contributions,
5. encourage students to build on each others' work, and
6. emphasise the work of the community.

Such strategies for developing teacher knowledge recognise the distributed nature of knowledge for teaching mathematics at the level of teacher education. The central goal in using the strategies could be to create a "Knowledge-Building Community" where the focus would be on advancing knowledge through "reading relevant resource materials, posing questions, offering theories, conducting experiments, and generally working with peers to make sense of new ideas" (Hewitt & Scardamalia, 1998, p. 82). By applying the work of Hewitt and Scardamalia to teachers' acquisition of mathematical knowledge, and being mindful of the cultural dimension of that knowledge, the possibility is opened for the chapters in this section to enhance our understanding of how teachers acquire, or can be supported in acquiring, mathematical knowledge for teaching.

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# Part III
# Building Mathematical Knowledge in Teaching by Means of Theorised Tools

# Chapter 12
# The Knowledge Quartet as an Organising Framework for Developing and Deepening Teachers' Mathematics Knowledge

**Fay Turner and Tim Rowland**

## Introduction

In this chapter, we present some findings from a study which evaluated the effectiveness of one classroom-based approach to the development of elementary mathematics teaching. This approach drew on earlier research into teachers' mathematical content knowledge at the University of Cambridge, when a framework for the analysis of mathematics teaching – the Knowledge Quartet – was developed. In the work to be reported here, this framework was used to identify and develop a group of beginning teachers' mathematics content knowledge for teaching. First, we shall give a rationale for our focus on teachers' content knowledge in action in the classroom and a brief description of the study which led to the development of the Knowledge Quartet.

### *Rationale*

Education researchers and government agencies have identified limitations in teachers' mathematical content knowledge (e.g. Ball, 1990; Ma, 1999; Ofsted, 2000). These limitations have been perceived as a factor in unsatisfactory pupil achievement (Williams, 2008). Difficulties associated with teachers' mathematical content knowledge are particularly apparent in the elementary sector where generalist teachers often lack confidence in their own mathematical ability (Brown, McNamara, Jones, & Hanley, 1999; Green & Ollerton, 1999). The 'reform' movement in mathematics teaching, and enquiry-based approaches to learning, which have been influential in curriculum reform in several countries, arguably require teachers to have a greater depth of mathematical content knowledge than was needed for teaching more 'traditional' mathematics (e.g. Borko et al., 1992; Goulding, Rowland, & Barber, 2002). Identifying, developing and deepening teachers' mathematical

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content knowledge, has therefore become a priority for policy makers and mathematics educators around the world.

There is not a simple relationship between teachers' formal qualifications in mathematics and the achievement of their pupils (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997; Begle, 1979). Several researchers have argued that mathematical content knowledge needed for teaching is not located in the minds of teachers, but rather is realised through the practice of teaching (Hegarty, 2000; Mason & Spence, 1999). From this perspective, knowledge for teaching is constructed in the context of teaching, and can therefore be observed only as 'in vivo' knowledge in this context. Teaching requires knowledge in several different domains, and a number of knowledge taxonomies reflect this multidimensional perspective (see Chapter 2 by Petrou and Goulding, this volume). Hegarty (2000) argued that the effects of these different kinds of teacher knowledge can only be understood within the contexts of dynamic teaching situations. He presented a model which represents the teacher as having a number of incomplete sets of relevant insights, elements of which come together in instances of teaching to form a new insight specific to that situation. This is resonant with the contention of Mason and Spence (1999) that *knowing-about* mathematics and mathematics teaching is only realised as *knowing-to* in the act of teaching. The perspective on teacher knowledge at the heart of this chapter – the Knowledge Quartet – provides a framework for analysis of the mathematics content knowledge that informs teacher insights when they are brought together in practice, so that the distinction between different kinds of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching. In the following section, we outline the fundamental, observational research that gave rise to the 'tool' that lies at the heart of this chapter.

## Developing the Knowledge Quartet

### *Context and Purpose of the Research*

In the UK, the majority of prospective, 'trainee' teachers are graduates who follow a one-year course leading to a Postgraduate Certificate in Education (PGCE) in a university education department. Over half of the PGCE year is spent teaching in schools under the guidance of a school-based mentor, or 'cooperating teacher'. Placement lesson observation is normally followed by a review meeting between the cooperating teacher and the student-teacher. On occasion, a university-based tutor will participate in the observation and the review. Thirty years ago, Tabachnick, Popkewitz, and Zeichner (1979) found that "cooperating teacher/student teacher interactions were almost always concerned with . . . procedural and management issues . . . There was little or no evidence of any discussion of substantive issues in these interactions" (p. 19). The situation has not changed, and more recent studies also find that mentor/trainee lesson review meetings typically focus heavily on organisational features of the lesson, with very little attention to the mathematical content of mathematics lessons (Borko & Mayfield, 1995; Strong & Baron, 2004).

The purpose of the research from which the Knowledge Quartet emerged was to develop an empirically-based conceptual framework for lesson review discussions *with a focus on the mathematics* content of the lesson and the role of the trainee's mathematics subject matter knowledge (SMK) and pedagogical content knowledge (PCK). In order to be a useful tool for those who would use it in the context of *practicum* placements, such a framework would need to capture a number of important ideas and factors about mathematics content knowledge in relation to teaching, within a small number of conceptual categories, with a set of easily-remembered labels for those categories.

The focus of this particular research was therefore to identify ways that teachers' mathematics content knowledge – both SMK and PCK – can be observed to 'play out' in practical teaching. The teacher-participants in this study were novice, trainee elementary school teachers, and the observations were made during their school-based placements. Whilst we believe certain kinds of knowledge to be *desirable* for elementary mathematics teaching, we are convinced of the futility of asserting what a beginning teacher, or a more experienced one for that matter, *ought* to know. Our interest is in what a teacher *does* know and believe, and how opportunities to enhance knowledge can be identified. We have found that the Knowledge Quartet, the framework that arose from this research, provides a means of reflecting on teaching and teacher knowledge, with a view to developing both.

## *Method*

The participants in the study were enrolled on a 1-year PGCE course in which each of the 149 trainees specialised either on the Early Years (pupil ages 3–8) or the Primary Years (ages 7–11). Six trainees from each of these groups were chosen for observation during their final school placement. The six were chosen to reflect a range of outcomes of a subject-knowledge audit administered 3 months earlier. Two mathematics lessons taught by each of these trainees were observed and videotaped, i.e. 24 lessons in total. The trainees were asked to provide a copy of their planning for the observed lesson. As soon as possible after the lesson, the observer/researcher wrote a succinct account of what had happened in the lesson, so that a reader might immediately be able to contextualise subsequent discussion of any events within it. These 'descriptive synopses' were typically written from memory and field notes, with occasional reference to the videotape if necessary.

From that point, we took a grounded approach to the data for the purpose of generating theory (Glaser & Strauss, 1967). In particular, we identified in the videotaped lessons aspects of trainees' actions in the classroom that seemed to be significant in the limited sense that it could be construed to be informed by a trainee's mathematics subject matter knowledge or their mathematical pedagogical knowledge. We realised later that most of these significant actions related to choices made by the trainee, in their planning or more spontaneously. Each was provisionally assigned an ‘invented’ code. These were grounded in particular moments or episodes in the tapes. This provisional set of codes was rationalised and reduced (e.g. eliminating duplicate codes and marginal events) by negotiation and agreement in the research team. The 18 codes generated by this inductive process are itemised later in this chapter. The name assigned to each code is intended to be indicative of the type of issue identified by it: for example, the code *adheres to textbook* (AT) was applied when a lesson followed a textbook script with little or no deviation, or when a set of exercises was ‘lifted’ from a textbook, or other published resource, sometimes with problematic consequences. By way of illustration of the coding process, we give here a brief account of an episode that we labelled with the code *responding to children’s ideas* (RCI). It will be seen that the contribution of a child was unexpected. Within the research team, this code name was understood to be potentially ironic, since the observed response of the teacher to a child’s insight or suggestion was often to put it to one side rather than to deviate from the planned lesson script, even when the child offered further insight on the topic at hand.

Code RCI*: an illustrative episode*. Jason was teaching elementary fraction concepts to a Year 3 class (pupil age 7–8). Each pupil held a small oblong whiteboard and a dry-wipe pen. Jason asked them to “split” their individual whiteboards into two. Most of the children predictably drew a line through the centre of the oblong, parallel to one of the sides, but one boy, Elliot, drew a diagonal line. Jason praised him for his originality, and then asked the class to split their boards “into four”. Again, most children drew two lines parallel to the sides, but Elliot drew the two diagonals. Jason’s response was to bring Elliot’s solution to the attention of the class, but to leave them to decide whether it was correct. He asked them:

Jason: What has Elliot done that is different to what Rebecca has done?
Sophie: Because he’s done the lines diagonally.
Jason: Which one of these two has been split equally? [...] Sam, has Elliot split his board into quarters?
Sam: Um. . . yes. . . no. . .
Jason: Your challenge for this lesson is to think about what Elliot’s done, and think if Elliot has split this into equal quarters. There you go Elliot.

At that point, Jason returned the whiteboard to Elliot, and the question of whether it had been partitioned into quarters was not mentioned again. What makes this interesting mathematically is the fact that (i) the four parts of Elliot’s board are not congruent, but (ii) they have equal areas; and (iii) this is not at all obvious. Furthermore, (iv) an elementary demonstration of (ii) is arguably even less obvious. This seemed to us a situation that posed very direct demands on Jason’s SMK and arguably his PCK too. It is not possible to infer whether Jason’s “challenge” is motivated by a strategic decision to give the children some thinking time, or because he needs some himself.

Equipped with this set of codes, we revisited each lesson in turn and, after further intensive study of the tapes, elaborated each descriptive synopsis into an analytical account of the lesson. In these accounts, the agreed codes were associated with relevant moments and episodes, with appropriate justification and analysis concerning the role of the trainee's content knowledge in the identified passages, with links to relevant literature.

The identification of these fine categories was a stepping stone with regard to our intention to offer a practical framework for use by ourselves, our colleagues and teacher-mentors, for reviewing mathematics teaching with trainees following lesson observation. An 18-point tick-list (like an annual car safety check) was not quite what was needed. Rather, the intended purpose demanded a more compact, readily understood scheme, which would serve to frame a coherent, content-focused discussion between teacher and observer. The key to the solution of our dilemma was the recognition of an association between elements of subsets of the 18 codes, enabling us to group them (again by negotiation in the team) into four broad, superordinate categories, which we have named (I) foundation (II) transformation (III) connection (IV) contingency. These four units are the dimensions of what we call the 'Knowledge Quartet'.

Each of the four units is composed of a small number of subcategories that we judged, after extended discussions, to be of the same or a similar nature. An extended account to the research pathway described above is given in Rowland (2008). Naturally, we are immersed in the process from which the codes emerged. We believe, however, that our names for the codes are less important to other users of the 'quartet' than a broad sense of the general character and distinguishing features of each of broad units, which we shall outline in a moment. The Knowledge Quartet has now been extensively 'road tested' as a descriptive and analytical tool. As well as being re-applied to analytical accounts of the original data (the 24 lessons), it has been exposed to extensive 'theoretical sampling' (Glaser & Strauss, 1967) in the analysis of other mathematics lessons, in England and beyond. As a consequence, two additional codes[1] have been added to the original 18, but in its broad conception, we have found the quartet to be comprehensive as a tool for thinking about the ways that content knowledge comes into play in the classroom. We have found that many moments or episodes within a lesson can be understood in terms of two or more of the four units; for example, a *contingent* response to a pupil's suggestion might helpfully *connect* with ideas considered earlier. Furthermore, the application of content knowledge in the classroom always rests on *foundational* knowledge.

## *Conceptualising the Knowledge Quartet*

The concise conceptualisation of the Knowledge Quartet which now follows draws on the extensive range of data referred to above. As we observed earlier, the practical application of the Knowledge Quartet depends more on teachers and teacher educators understanding the broad characteristics of each of the four dimensions than on their recall of the contributory codes.

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[1] These new codes, derived from applications of the KQ to classrooms in Ireland and Cyprus, are *teacher insight* (Contingency) and use of *instructional materials* (Transformation) respectively.

## Foundation

Contributory codes: awareness of purpose; identifying errors; overt subject knowledge; theoretical underpinning of pedagogy; use of terminology; use of textbook; reliance on procedures.

The first member of the quartet is rooted in the foundation of the teacher's theoretical background and beliefs. It concerns their knowledge, understanding and ready recourse to what was learned at school, and at college/university, including initial teacher preparation, in preparation (intentionally or otherwise) for their role in the classroom. It differs from the other three units in the sense that it is about knowledge 'possessed',[2] irrespective of whether it is being put to purposeful use. For example, we could claim to have knowledge about division by zero, or about some probability misconceptions – or indeed to know where we could seek advice on these topics – irrespective of whether we had had to call upon them in our work as teachers. Both empirical and theoretical considerations have led us to the view that the other three units flow from a foundational underpinning.

A key feature of this category is its *propositional* form (Shulman, 1986). It is what teachers learn in their 'personal' education and in their 'training' (pre-service in this instance). We take the view that the possession of such knowledge has the potential to inform pedagogical choices and strategies in a fundamental way. By 'fundamental' we have in mind a rational, reasoned approach to decision-making that rests on something other than imitation or habit. The key components of this theoretical background are: knowledge and understanding of mathematics per se; knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt.

In summary, this category that we call 'foundation' coincides to a significant degree with what Shulman (1987) calls 'comprehension', being the first stage of his six-point cycle of pedagogical reasoning.

## Transformation

Contributory codes: teacher demonstration; use of instructional materials; choice of representation; choice of examples.

The remaining three categories, unlike the first, refer to ways and contexts in which knowledge is brought to bear on the preparation and conduct of teaching. They focus on knowledge-in-action as *demonstrated* both in planning to teach and in the act of teaching itself. At the heart of the second member of the quartet, and acknowledged in the particular way that we name it, is Shulman's observation that the knowledge base for teaching is distinguished by " . . . the capacity of a

[2]The use of this acquisition metaphor for knowing suggests an individualist perspective on Foundation knowledge, but we suggest that this 'fount' of knowledge can also be envisaged and accommodated within more distributed accounts of knowledge resources (see Chapter 3 by Hodgen, this volume).

teacher to *transform* the content knowledge he or she possesses into forms that are pedagogically powerful" (1987, p. 15, emphasis added). This characterisation has been echoed in the writing of Ball (1988), for example, who distinguishes between knowing some mathematics 'for yourself' and knowing in order to be able to help someone else learn it. As Shulman indicates, the presentation of ideas to learners entails their re-presentation (our hyphen) in the form of analogies, illustrations, examples, explanations and demonstrations (Shulman, 1986, p. 9). Our second category, unlike the first, picks out behaviour that is directed towards a pupil (or a group of pupils), and which follows from deliberation and judgement informed by foundation knowledge. This category, as well as the first, is informed by particular kinds of literature, such as the teachers' handbooks of textbook series or in the articles and 'resources' pages of professional journals. Increasingly, in the UK, teachers look to the internet for bright ideas, and even for readymade lesson plans. The trainees' choice and *use of examples* has emerged as a rich vein for reflection and critique. This includes the use of examples to assist concept formation, to demonstrate procedures, and the selection of exercise examples for student activity.

## Connection

Contributory codes: making connections between procedures; making connections between concepts; anticipation of complexity; decisions about sequencing; recognition of conceptual appropriateness.

The next category binds together certain choices and decisions that are made for the more or less discrete parts of mathematical content – the learning, perhaps, of a concept or procedure. It concerns the *coherence* of the planning or teaching displayed across an episode, lesson or series of lessons. Mathematics is notable for its coherence as a body of knowledge and as a field of enquiry. Indeed, a great deal of mathematics is held together by deductive reasoning. The pursuit of coherence and mathematical connections in mathematics pedagogy has been stimulated recently by the work of Askew et al. (1997): of six case study teachers found to be highly effective, all but one gave evidence of a 'connectionist' orientation. The association between teaching effectiveness and a set of articulated beliefs of this kind lends a different perspective to the work of Ball (1990) who also strenuously argued for the importance of connected knowledge for teaching.

Related to the integrity of mathematical content in the mind of the teacher and his/her management of mathematical discourse in the classroom, our conception of coherence includes the *sequencing* of topics of instruction within and between lessons, including the ordering of tasks and exercises. To a significant extent, these reflect deliberations and choices entailing not only knowledge of structural connections within mathematics itself, but also awareness of the relative cognitive demands of different topics and tasks.

## Contingency

Contributory codes: responding to children's ideas; use of opportunities; deviation from agenda; teacher insight.

Our final category concerns the teacher's response to classroom events that were not anticipated in the planning. In some cases, it is difficult to see how they could have been planned for, although that is a matter for debate. In commonplace language this dimension of the quartet is about the ability to 'think on one's feet': it is about *contingent action*. Shulman (1987) proposes that most teaching begins from some form of 'text' – a textbook, a syllabus, ultimately a sequence of planned, intended actions to be carried out by the teacher and/or the students within a lesson or unit of some kind. Whilst the stimulus – the teacher's intended actions – can be planned, the students' responses can not.

Brown and Wragg (1993) suggest that 'responding' moves are the lynch pins of a lesson, important in the sequencing and structuring of a lesson, and observe that such interventions are some of the most difficult tactics for novice teachers to master. The quality of such responses is undoubtedly determined, at least in part, by the knowledge resource available to the teacher, as the earlier illustrative episode with Jason demonstrates.

Having now set out the conceptual apparatus underpinning the tool in focus in this chapter, we now proceed to an account of its use by a group of teachers in their professional development over a 4-year period.

## The Knowledge Quartet and Mathematics Teaching Development

The Knowledge Quartet was developed to identify, describe and analyse mathematics content knowledge revealed in teaching, in order to provide a framework for reflection and discussion of lessons. We were then motivated to investigate whether, and in what ways, the framework could be used to develop and deepen mathematical content knowledge. In a study begun in 2004, the first author evaluated the Knowledge Quartet as a tool for the identification and development of teachers' SMK and PCK (see e.g. Turner, 2008).

This longitudinal study took place over 4 years, during which the participants could be regarded as 'beginning teachers'. Each of these years was considered a different phase of the study. It began with 12 participants in their PGCE graduate teacher preparation year. As expected, this cohort reduced to nine in the second year, to six in the third year and finally to four in the fourth and final year of the study. This attrition was predicted, and a consequence of the participants' relocation, changes in commitment to and participation in the project. Four case studies, of Amy, Jess, Kate and Lisa, were built using data from lesson observations, post-lesson reflective interviews, participants' reflective written accounts, group interviews and individual interviews over the 4 years.

The study was based on a model of teacher professional development through reflection both *in* and *on* teaching action (Schön, 1983). The Knowledge Quartet was used to focus the teachers' reflections on the mathematics content knowledge realised in their teaching. The teachers used the framework as a tool to support their reflections on and discussions about their mathematics teaching over the course of the study. Videotapes of the participants' lessons were used to aid recall and to allow in-depth analysis and reflection on, their teaching.

The participants were initially introduced to, and familiarised with, the Knowledge Quartet in their training year. One lesson taught by each of them was videotaped and analysed during their final *practicum* placement. These videotapes were used in one-to-one stimulated recall interviews with the participants, using the Knowledge Quartet to focus on the mathematical content of each lesson. During their first year of teaching, the participants were given focused feedback, structured by the Knowledge Quartet framework, on three videotaped lessons. This was intended to support and develop their own use of the framework. They then watched the videotapes and wrote reflective accounts of these lessons. In the second year of their teaching, the most intensive period of data collection, participants used the framework more independently, supported by discussions with the researcher and by group meetings. Interviews and observations in the third year of their teaching gave final indications of the development in participants' mathematical content knowledge as it was evidenced in their teaching.

Lesson observations were analysed by the first author using the dimensions and constituent codes of the Knowledge Quartet. Transcripts of one–one interviews and group meetings, and the participants' written reflective accounts, were analysed using the computer-aided qualitative analysis software NVivo. This gave rise to a hierarchy of emergent codes and themes, which informed the final analysis of the data. As we will demonstrate below, there was evidence from the study that use of the Knowledge Quartet as a framework for reflection had a positive influence on the development of the participants' content knowledge for teaching by focusing reflection on the mathematical content of their teaching, as opposed to the more managerial and generic aspects that tend, as we remarked earlier, to dominate lesson review.

The analysis brought out two overarching aspects in the development of the participants. The first of these related to their conceptions of mathematics teaching – an aspect of Foundation knowledge coded in the longitudinal study under the themes of 'beliefs' and 'confidence'. The second related more broadly to developments in mathematical content knowledge in relation to the four dimensions of the Knowledge Quartet.

## *Development in Conceptions of Mathematics Teaching*

From a comprehensive review of literature across several disciplines, Kuhs and Ball (1986) identified four dominant views of the way mathematics should be taught:

- a classroom-focused view;
- a content-focused with an emphasis on performance view;
- a content-focused with an emphasis on conceptual understanding view;
- a learner-focused view.

Kuhs and Ball (1986) give detailed accounts of these four views, making them accessible as a framework for analysis. These models were not seen as exclusive: teachers would be expected to hold elements of more than one view simultaneously, and over time. Ernest (1989) subsequently included two additional models of conceptions which combined characteristics of these views. However, these combinations could be accounted for within Kuhs and Ball's simpler four-model framework.

Evidence from the different data sources combined to reveal that each of the participants held complex views of mathematics teaching incorporating elements from all four of Kuhs and Ball's dominant views. Though the initial balance of these elements varied, the NVivo analysis indicated a pattern in the direction of change in the four case studies. Jess, Lisa and Kate began the study with predominantly content-focused views of teaching and Amy with a predominantly learner-focused view. There was evidence that the teachers moved towards views with greater emphasis on developing conceptual understanding in pupils rather than on developing procedural performance. There was also a pattern of change in conceptions of mathematics teaching towards a learner-focused view. In the following section, selected data from the study are used to substantiate the claims made above. Inevitably, only a small selection of representative data can be presented here.

Jess began her career with a content-focused view of mathematics teaching which emphasised performance. In her first year of teaching, she commented:

> The thing is, if it [using written algorithms] works for them what's the problem? I have found some of the less able children have been shown how to do carrying, and they've got it and they use that all the time. (Jess, group interview, Phase 2)

Over the course of the study, she moved towards a view which emphasised conceptual understanding though she continued to consider performance to be important.

> I think teaching procedures are important, especially for low ability children who need to have a strategy to rely on. However, I have made a conscious effort to make my teaching more conceptual so it becomes much more real than just practicing something and it probably means it is much easier to apply in a new situation as it means something. (Jess, individual interview, Phase 4)

Like Jess, Lisa focused on procedures at the beginning of her career.

> The children can often do what I want them to do when it is like that because it is in small steps. (Lisa, post-lesson interview, Phase 1)

By her third year of teaching, there was evidence that Lisa had become more concerned with conceptual understanding. Commenting on a lesson she had taught at the beginning of the year, she wrote;

> It might have helped if I had been more encouraging of them to use jottings as well as to write the number sentence. They didn't need them for the numbers involved but it probably would have helped them with what the concept was. (Lisa, post-lesson reflective interview, Phase 3)

In her first year of teaching, Amy focused on both the performance (ability to count accurately) and conceptual understanding of individual children:

> I had planned which children I would ask to count which box of items so that I could differentiate the counting task or assess individual skills. I deliberately chose Katie so I could assess whether she had the cardinal principle. (Amy, post-lesson reflective interview, Phase 2)

In addition to being interested in whether the children could count accurately, Amy used their 'performance' to assess whether they understood the concept of cardinality. By her second year of teaching, Amy appeared to move further towards a learner-focused view:

> Teachers often talk too much, including me; more focus should be given to the children rather than the teacher. I have learned to really watch children. It is great to be able to see from the other side and see how they are responding. (Amy, group interview, Phase 3)

There was some evidence in Jess' second year of teaching that she was trying to understand the thinking of individuals as well as of groups of children and use this to inform her teaching.

> I have started to get children to explain in more detail what they have said so I understand where they are coming from, and also so some of the other children start to realise some of these things too. (Jess, reflective account, Phase 3)

Kuhs and Ball (1986) suggest that teachers with a learner-focused view of mathematics teaching would adopt a problem solving or enquiry approach in their teaching. There was some evidence in Jess' third year of teaching that she was seeing the advantages of such an approach for understanding and developing children's mathematical thinking.

> When it gets around to working out what they know, it proves more if they have done problem solving. Like really, like hands on, like thinking and trying to think about the calculations they are doing really helps, rather than paper methods. (Jess, group interview, Phase 4)

Lisa also moved towards a problem solving approach to her mathematics teaching. This was apparent when she taught a similar lesson in her third year of teaching to one she had taught in her first year, about the complements in ten. In the earlier lesson, Lisa systematically demonstrated finding each of the complements in ten by dividing ten objects between two sets. In the later lesson, Lisa asked the children to *investigate* how many different ways the ten objects might be divided between the two sets.

There was evidence that the conceptions of the four teachers moved in similar directions, although from different starting points and to different degrees. Research shows that such movement does not occur through teaching experience alone (e.g. Wilson and Cooney, 2002). It was evident that Lisa's use of the Knowledge Quartet influenced the move towards focusing on conceptual understanding and towards a learner-focused view of mathematics teaching.

> It [the Knowledge Quartet] certainly gets me thinking a lot more about what I know and how I am going to teach them, like watching how they've learned. (Lisa, Group interview, Phase 2)

There was also evidence that Amy's use of the Knowledge Quartet both supported and developed her learner-focused view of mathematics teaching.

> I think the Knowledge Quartet has pushed me to think from the other side and see more clearly how the children see and what they need. It makes me try to put myself in their heads. (Amy, group interview, Phase 2)

A comment made by Kate in her second year of teaching suggested that use of the framework helped her to focus less on organisational matters, and more on conceptual understanding and on the learner.

> The first few things I would be thinking of are the organisational things, and then I try to think 'did they learn anything' and 'was the learning alright' even if the organisation wasn't kind of thing. So, I think it is useful to have some kind of structure to help you know what you need to know and what they need to know and how to learn it. I think what I have said and how I have explained things, I am more aware than I would be if I didn't have such a clear idea of what I was looking for. (Kate, interview, Phase 2)

There is less clear evidence from Jess's data that her use of the Knowledge Quartet was instrumental in moving her conceptions in a specific direction. However, she clearly saw the Knowledge Quartet as instrumental in improving her mathematics teaching.

> I think it is the only subject we have feedback on our teaching really . . . it is the only thing that actually comes close to constructive. What you've really thought about and tried to improve things and get in the right order . . . I think it has probably increased our maths teaching a lot more. (Jess, group interview, Phase 4)

In focusing and framing reflection on the mathematical content of teaching, the Knowledge Quartet appears to have been influential in confronting the conceptions of the teachers in the study. These conceptions generally shifted towards a view of mathematics teaching that was concerned with conceptual understanding and which focused on the learner. An increasingly learner-focused view was reflected in the adoption of more problem solving and enquiry approached to teaching mathematics.

### *Development of Content Knowledge*

Use of the Knowledge Quartet was also found to be instrumental in developing the participants' mathematical content knowledge for teaching. There was evidence that reflection, focused by the Knowledge Quartet on the mathematical content of their mathematics teaching, enhanced the development of SMK, and particularly of PCK, in the teachers over the 4 years of the study. Developments in mathematical content knowledge were particularly evident in observations of teaching when two lessons taught by the same participant on similar topics were observed. For example, Amy was observed teaching lessons on counting in her training year and again in her first year of teaching. In the first lesson, Amy made use of a number of counting activities which involved the understanding of one or more of the principles of counting (Gelman and Gallistel, 1978) to which she had been introduced in her pre-service training. However, post-lesson discussions revealed that Amy was not aware of how these activities might help develop children's understanding of the principles. Amy's knowledge and use of the principles of counting was much more explicit in the second lesson.

> When I was planning this lesson, I drew on my knowledge of the pre-requisites for counting: knowing the number names in order, one to one correspondence, the cardinal principle, being able to count objects that cannot be moved/touched and counting objects that cannot be seen e.g. sounds or beats. (Amy, post-lesson interview, Phase 1)

This pedagogical content knowledge informed Amy's teaching in a way that it had not in the earlier lesson on counting. Her reflections on the previous lesson, mediated by the Knowledge Quartet, had prompted her to recall the classic Gelman and Gellistel work and seemed to have influenced Amy towards making the pre-requisites for counting more explicit in a similar lesson the following year.

Participants' written reflective accounts also suggested developments in their content knowledge, in relation to all four dimensions of the Knowledge Quartet. In relation to the *foundation* dimension, reflecting on their teaching using the Knowledge Quartet helped participants to recognise limitations in their SMK, which they then attempted to rectify. For example, Jess recognised the difficulty she had in distinguishing between the partition and quotition structures of division.

> Explaining dividing in terms of grouping and sharing still gets me mixed up. It is something I need to work on myself. The aim was to explain in terms of grouping. In future I am going to sort this out before the lesson so my physical representations don't get mixed up. (Jess, reflective account, Phase 2)

Through reflecting on her teaching, another of the participants, Kate, realised that she had not understood the difference between two subtraction structures (Rowland, 2006) and that this had affected her teaching.

> Because I had not really thought of 'find the difference' as a different sort of subtraction operation, but had thought of it just as different vocabulary for asking the question, I didn't really think about my choice of example in terms of looking for examples for which it would be sensible to do a 'difference' operation rather than a take away. (Kate, reflective account, Phase 2)

There was also evidence of developments in the teachers' content knowledge in relation to the *transformation* dimension of the Knowledge Quartet. All the participants were critical of their own teaching and the Knowledge Quartet framework channelled these criticisms in a constructive way onto the mathematical content of their lessons and onto how their pedagogy might be improved in relation to this content, e.g.

> When they were counting sounds it would have been helpful to match each sound to a held up finger . . . When I asked are there more frogs or more snakes I could have asked a child to come up and show these on the number line. (Amy, reflective account, Phase 1)

> I chose some quite big numbers to illustrate that drawing cubes and crossing them off may not always be reliable. It might have been better if I had chosen large numbers but with a small difference between them. (Kate, reflective accounts, Phase 2)

Amy's reflection led her to make suggestions for improvements to her teaching which focused on the use of demonstrations and representations. Kate suggested improvements to her teaching which related to her use of examples. These are all key aspects of the *transformation* dimension of the Knowledge Quartet.

There was also evidence that when reflecting on their teaching the participants were guided by the *connection* dimension of the Knowledge Quartet. The participants' reflection on their teaching focused on the connections they made, or had missed, and on how these might be further developed to enhance learning. Amy considered ways in which she could have made further connections in her lesson and clearly recognized the importance of making connections to aid children's learning.

> I could have linked the lesson to earlier work on counting or the OMS[3] (on counting and sharing fruit) earlier in the morning. I could have made reference to the good counting strategies one of the children used earlier when counting the fruit which would have enabled the children to make a connection and see their learning in context. (Amy, reflective account, Phase 2)

The sequencing of teaching is one aspect of the connection dimension of the Knowledge Quartet and in reflecting on her teaching Kate considered the appropriateness of the sequence she had used.

> Most of the children appeared to find measuring much easier than estimating making me think I should have done the activities in the opposite order. (Kate, reflective account, Phase 2)

Finally, there was evidence that the participants' reflection on their teaching focused on aspects of their content knowledge from within the *contingency* dimension of the Knowledge Quartet. For instance, Kate reflected on a teaching episode in which she had acted contingently.

> When estimating how many cubes long a book was Harriet-Mae said "eighty" and then corrected herself to say "eighteen". I used this as an example to question the children about which of these was a sensible estimate and we discussed why 80 was not. (Kate, reflective account, Phase 3)

Amy clearly felt that she became more able to act contingently over the course of the study.

> I am [more] aware of children's common misconceptions, and can therefore adapt in response contingently, or plan for these. Generally I think there is more contingent teaching going on and I am more confident to be flexible. I can respond quickly to a child by setting up an activity I know will extend from what they are doing. (Amy, group interview, Phase 3)

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[3] Oral and Mental Starter (OMS) was the term used in government guidance in the early 2000s for the beginning part of a mathematics lesson, in which children were expected to rehearse their knowledge of number bonds, calculation facts, etc.

It is likely that these teachers would have developed their practice in any case through systematic reflection. However, the instances discussed above suggest that the participants' reflection was focused on the mathematical content of their teaching by their use of the Knowledge Quartet. Our claim is that the Knowledge Quartet is an effective tool in this crucial respect. The teachers were alerted to issues relating to their mathematical content knowledge and they thought about ways to improve their teaching by addressing these issues. Kate explained how the Knowledge Quartet framework directed her reflection.

> If I think about my teaching in the car on the way home and I think, if it wasn't very good, why wasn't it very good? Was it the concept behind what I told them to do or was it the resources they had to do it with? So, that would be the Transformation and the first one would be Foundation. What would have enabled them to understand that better than they did? . . . I try and think, did they learn anything and, was the learning alright, even if the organisation wasn't. So, I think it is useful to have some kind of a framework. (Kate, interview, Phase 3)

Jess was convinced that her use of the Knowledge Quartet had been a positive influence on her teaching.

> I think the KQ has definitely improved my teaching. When I am planning I draw on the four areas unconsciously criticising what I plan to do, often asking myself questions – does that show what I want it to etc. (Jess, interview, Phase 3)

Amy explained why she found the framework useful and suggested that she saw the Foundation dimension as having 'overriding' importance in her teaching.

> I think it is good to be able to think about how you are putting different elements of your lesson into the different parts of the Quartet and also seeing how they link up. You feel like the Foundation theme is a kind of overriding one that comes into everything. (Amy, interview, Phase 3)

There was considerable evidence from observations of teaching, interviews and written reflective accounts that the participants' content knowledge for teaching developed over the course of the study and that this development was catalysed by reflection on their teaching supported by the Knowledge Quartet framework. Much evidence for the developments in mathematical content knowledge for teaching related to the PCK of the participants. However, there was also some evidence that the Knowledge Quartet supported development of the participants' SMK.

## Conclusion

This study shows that the Knowledge Quartet can be an effective tool in developing teachers' mathematical content knowledge through focused reflections on their mathematics teaching. All of the teachers who participated in the study reported above testified that they had found the Knowledge Quartet helpful when planning and evaluating their teaching and intended to continue using it after their participation in the project ceased. Participants particularly valued feedback on their teaching which focused on mathematical content, and found that the framework helped them to focus more effectively on mathematical content themselves. Analysis of the four case study participants suggested that the framework was influential both in developing their conceptions of mathematics teaching and in developing their mathematical content knowledge. Use of the Knowledge Quartet helped the participants move from a view of teaching which focused on children being able to carry out procedures, to one in which conceptual understanding was more important. There was also evidence that the case study participants developed more learner-focused views of mathematics teaching through their use of the framework. In focusing reflections on mathematical content, the framework was seen to be an effective tool to support development of the teachers' PCK and to identify and strengthen aspects of SMK.

Participants in the study found the four dimensions both helpful and easy to use. Those who worked with the framework for 4 years suggested that it had become part of their way of thinking, so that they automatically referred to the four dimensions when planning and evaluating their teaching.

> I think the KQ has definitely improved my teaching. When I am planning, I draw on the four areas unconsciously criticising what I plan to do, often asking myself questions – does that show what I want it to etc. (Jess, interview, Phase 3)

Evidence from this study strongly suggests that in addition to being a useful tool for analysis of mathematical content knowledge revealed in the practice of teaching, the Knowledge Quartet can support beginning teachers in developing their mathematical content knowledge.

Modes of initial teacher education in England are now very diverse, and include workplace 'apprenticeship' versions located in schools, such as School-Centred Initial Teacher Training (SCITT). Given the widespread concerns about the resource of mathematics knowledge in primary school staff (Williams, 2008), the development of trainees' content knowledge is a challenging issue for such programmes. The Knowledge Quartet therefore has particular relevance to these modes of teacher education, and we have taken up opportunities to promote it as a tool for content-focused lesson observation in these contexts. SCITT programme leaders have indicated that the framework is being found to be relevant and useful in such ITT schemes. One of them commented:

> It is the single most powerful tool I have come across that has enabled me to give effective feedback on trainees' subject knowledge for teaching in a focused way.

Mentors of trainee teachers at UK universities (our own, and others) have also found the framework helpful in identifying issues of content knowledge and in giving focused feedback to their mentees. Introducing the framework to student teachers during initial teacher education courses, and use of the framework by mentors and university tutors during *practicum* placements, has supported a focus on mathematical content knowledge during training. Familiarisation with the framework has helped teachers to continue to develop their conceptions of mathematics teaching and their mathematical content knowledge after beginning their teaching careers.

There remains the question of whether the Knowledge Quartet can be effectively used to develop mathematical content knowledge for teaching without the support of a 'more knowledgeable other', not necessarily a mathematics educator. A programme of mentor training, involving developing mentors' understanding of the Knowledge Quartet, might begin to establish a panel of 'knowledgeable others' in schools who could support colleagues. There might also be a 'cascade effect' as beginning teachers who have been supported in using the Knowledge Quartet by mathematics educators become the 'more knowledgeable others' within their schools. We recognise that this might lead to 'dilution' in the efficacy of the Knowledge Quartet. It seems likely that the conceptualisations of the four dimensions developed from the original empirical research would be interpreted in a number of alternative ways by unsupported teachers or mentors, and a book (Rowland, Turner, Thwaites, & Huckstep, 2009) has been written to assist in such a situation. However, there is evidence to show that the framework, even without expert support, would at least encourage teachers and mentors to focus on the mathematical content of teaching rather than on more managerial issues.

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# Chapter 13
# Learning to Teach Mathematics Using Lesson Study

**Dolores Corcoran and Sandy Pepperell**

## Introduction

In this chapter, we consider the use of Japanese lesson study in developing teaching practices and, in particular, the ways in which it is claimed to enhance mathematical knowledge for teaching. First, the general lesson study approach will be described with its key features outlined. The findings of a study carried out by Corcoran (2008) in Dublin will be reported and discussed, in order to examine the contribution of such an approach in the particular context of pre-service teacher education. In that study, engagement with peers in the lesson study enterprise transmuted students' negative attitudes to mathematics into a more positive, patient willingness to learn, and an optimism that they can go on learning mathematics in teaching. The report on the Dublin study will follow an overview of some of the published work reflecting claims made for the role of lesson study in focusing teachers on the knowledge for, and in, mathematics teaching.

## Enhancement of Teaching Through Lesson Study

The lesson study approach is built on the collective development of teaching effectiveness through collaborative work and reflection on practice and thus appears to offer a great deal to enhance mathematics teaching. For example, in a National Research Council report, Kilpatrick, Swafford, and Findell (2001) suggest that through the lesson study approach to professional development, "… teachers engage in very detailed analyses of mathematics, of students' mathematical thinking and skill, of teaching and learning" (p. 395), thus bringing together subject and pedagogy in reflecting on and refining practice. Engagement in these analyses is firmly rooted in group responsibility and in particular classroom contexts and draws on a range of resources both internal and external to the particular context.

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Lesson study is said to be premised on the Confucian saying that, "seeing something once is better than hearing about it one hundred times" (Yoshida, 2005). Its ultimate purpose is to gain new ideas about teaching and learning based on a better understanding of children's thinking so the observation of actual research lessons is at the core of the lesson study process. Yet, the lesson study cycle encompasses much more than studying children's responses while observing a lesson. It requires time dedicated to intensive *kyozai kenkyu* – a process in which teachers collaboratively investigate all aspects of the content to be taught and instructional materials available – and to *jyugyo kentuikai* – the post-lesson review session (Takahashi, Watanabe, Yoshida, & Wang-Iverson, 2005). Its main feature is collaborative planning and reflection that does not shy away from a critique of practice focused on the results of the group's work rather than on any individual. In these ways, it appears to offer teachers an opportunity to pool their collective teaching skills in situ as they adopt research goals appropriate to a particular school context for their lesson study. This approach, in general, addresses the situated and social view of teacher knowledge (see Chapter 3 by Hodgen, this volume), in that the focus for study is the lessons taught in particular schools with particular local concerns. As teachers plan and reflect together in groups, knowledge development is social. The 'research' lesson is planned collaboratively and teachers spend time clarifying the mathematics. The knowledge is drawing on a variety of sources including the teachers themselves, published curricula, research studies and 'experts' such as university teachers in the role of 'knowledgeable others' (Watanabe & Wang-Iverson, 2005).

In Japan, where it is integral to schools, lesson study is often credited with the success of Japanese students in international comparisons of mathematical achievement (Stigler & Hiebert, 1999). Internationally, there has been an increase in cross-cultural study of ways of teaching mathematics, and a growing interest in using lesson study as a basis for improving teaching in a variety of other contexts, most notably in the US. Increasingly, lesson study is being adopted in diverse school systems as a means of developing innovative classroom teaching and learning of mathematics (Asia-Pacific Economic Cooperation Education Network, 2008). For the purpose of this chapter, the discussion will now turn to a brief overview of how lesson study operates in Japan, the main elements of which have been used in projects in the US.

While there are various ways in which lesson study is carried out, the model that has been adopted in the US is mainly focused on individual schools, though reports of the work are often disseminated more widely. In Japan, this model begins with a group of teachers in a school identifying a particular teaching problem in their own school context. They then plan a lesson together where the strong focus is on the thinking and likely responses of pupils, but great care is taken over aspects of the teaching such as questions, resources and examples to be used. The focus in teaching, according to Tall (2008, p. 6), is on the mathematical knowledge of the teachers, but also on the need for "deep experience of how children think as they learn mathematics". Teachers also investigate possible teaching materials in the process. They then observe children's responses as one of the group teaches the (usually videotaped) lesson and they reflect together afterwards on the mathematical content of the lesson. The planning involves identifying the relevant mathematical knowledge and curriculum detail and although the focus is on one lesson, it is part of a sequence and progression in learning. The aim is not to produce perfect lessons to be offered as resources for others to use, but to be part of an ongoing process of deepening understanding of how teachers can bring about the meaningful learning of mathematics. This may include teaching specific methods (e.g. for calculation) as well as the solving of non-routine problems (Tall, 2008). The anticipated responses of pupils are an important aspect for discussion by teachers as are the potential difficulties that might be encountered. According to Fernandez (2005), the general approach is as follows. A group of teachers plans the detail of the lesson that one of the group will carry out in the classroom. The plan is written out in detail and, when the lesson is taught, other members of the study group (and sometimes invited visitors) observe what occurs. Feedback is then given after the lesson, usually starting with a reflection by the teacher who taught the lesson. After this, the lesson will be refined and other teachers may teach it again and follow this up with further analysis. The focus for reflection will be decided in advance, together with points for particular observation while the lesson is in progress.

## Lesson Study Appraised

In her work in the US, Fernandez (2005) was interested in the potential of the lesson study approach to support teachers in learning about mathematics for teaching, and also what the constraints might be for teachers whose own subject knowledge was limited. She emphasized that her interest was in "what lesson study has to offer, not on what teachers actually make of it" (p. 268). This is the central question being addressed in this chapter – what claims can be made for the contribution of lesson study to the development of mathematics for teaching? Elsewhere, Fernandez, Cannon, and Chokshi (2003) reviewed an initiative where Japanese teachers worked with teachers in the US to develop their work through lesson study. They suggest that through experience of lesson study teachers will draw on three critical lenses used by Japanese teachers to enhance their teaching of mathematics – the perspectives of researcher, of curriculum developer and of the pupils. The authors also report some of the difficulties that US teachers had in adopting these 'lenses' as ways of examining their teaching practice.

The *researcher* perspective requires teachers to observe the responses of pupils in a focused way and to gather specific and concrete evidence of those responses. US teachers' evaluations, according to Fernandez et al., were general and this might have resulted from the fact that the teachers appeared to find it difficult to be observers rather than teachers in this context. In the first lesson observation, the US teacher group tended to act as an extra pair of hands in the class while the Japanese teachers acted as an extra pair of eyes. In fact, had it not been for the presence and intervention of the Japanese teachers, much less would have been learned by the US teachers and, perhaps, it was their presence that had more influence on teacher learning than the participation in lesson study itself. There did, however, appear to be a potential for the addressing of mathematical teacher knowledge through the *curriculum developer* lens. Fernandez et al. suggest that the US teachers were not accustomed to discussing different ways of organising and sequencing elements of the curriculum, but rather tended to accept the authority of textbooks. Adopting this perspective, then, opens up possibilities for discussion and critical evaluation of how and why mathematical ideas might be arranged in order to maximize pupil understanding. It is claimed generally that the dynamic and interactive nature of the lesson study process offers participants multiple opportunities to deepen their knowledge of mathematics and of mathematics teaching. However, the work of Fernandez et al. suggests these will only be taken up in a context where teachers are enabled to position themselves as critical reflectors on their practice, who take ownership of their mathematical learning. In a later study of one school's lesson study work, Fernandez (2005) identifies the questions that arise for teachers in planning and implementing lessons which did result in discussion about mathematical knowledge for teaching, some resulting from children's difficulties and some from the unexpected ways children used to tackle activities, thereby giving an example of the 'opportunities' lesson study can offer. While she suggests that analysis of issues arising from the act of teaching can support the development of strategies for future teaching, at the same time she recognizes that it is not always possible to predict exactly what will occur next in teaching, so the matter of the development of mathematical knowledge is more vexed.

## The Role of Knowledgeable Other(s)

Fernandez (2005) observes that the US teachers were rather limited, at times, in their deliberations because of their own understanding about some of the connections in mathematics, in this case the relationship between fractions and division, and what 'whole' is referred to in fraction problems. However, she suggests positively that the cycle of planning and reflecting and the related analyses allowed space for teachers to expose and begin to address areas where they lacked confidence. Consequently, some teachers identified the need to develop their own understanding of mathematical ideas in order to discuss them fruitfully with pupils. According to Fernandez, it is the type of help and the manner in which it is given that will be crucial, that it "does not ask teachers to relinquish control of their work and [that it] does not overwhelm, alienate, or discourage teachers" (p. 285). Likewise, a study of Highlands school, also in the US, by Lewis, Perry, Hurd, and O'Connell (2006) found that, over time, teachers' observations became more focused and oriented to discussion of the detail of the mathematics they taught and the mathematical learning they analysed. Another change that they observed was the move towards using external sources of knowledge such as a wider range of texts, for comparison, and also research articles. Like the teachers in Fernandez's study, knowledge needs were identified through the study of local problems in their context.

While there are many positive commentaries on the potential for developing mathematical knowledge for teaching through lesson study, there have also been notes of caution. In particular, questions of the likely success of transferring an approach from one culture to another have been raised. Tall (2008, p. 1) suggests that it may be possible to learn from and use practices originating in another culture, if "we think reflectively about what it is we are trying to do in teaching mathematics." Teaching aims are central in decisions about approaches to practice and to teacher development and, without an examination of whether change in practice is required or possible, change may simply occur on the fringes of what happens in classrooms. One key feature of the way in which lesson study is described in the studies referred to here is that it is a 'bottom-up' rather than a 'top-down' model of teacher development. However, there are potential dangers in over-emphasising a localised, school-based approach. While knowledge can be seen as social and situated and, in the studies discussed, groups of teachers have been observed learning and developing confidence in recognizing what else they need to know, access to knowledge and expertise beyond the local context allows teachers to draw on a wider range of alternative views and to make informed, critical decisions to support the development of mathematical teaching in their own context. In relation to this, the role of knowledgeable other(s) who can provide such support is one which requires further exploration. In the next part of this chapter, the first author describes some of the findings from her research into a lesson study approach used with pre-service student teachers in Dublin.

## The Dublin Study

The Dublin study proposed to introduce Japanese lesson study to an Irish context and had as a primary goal the trialling of lesson study as a means of developing student teachers' mathematical content knowledge for primary teaching. Lewis et al. (2006, p. 5) offer two conjectures as to how lesson study might work to bring about the improvement of teaching. Conjecture 1 posits, "lesson study improves instruction through the refinement of lesson plans", while conjecture 2 proposes that "lesson study strengthens three pathways to instructional improvement; teachers' knowledge, teachers' commitment and community, and learning resources." This study was based on conjecture 2. The research project, therefore, is located in a theory of social practice, which conceptualises learning as *legitimate peripheral participation* (Lave & Wenger, 1991), and the student participants and I, as course facilitator/researcher, forming a *community of practice* (Wenger, 1998), where "membership [. . .] translates into an identity as a form of competence" (p. 153). The notion of identity formation as learning in practice and the possibility of mathematics knowledge for teaching arising from engagement in an enterprise dedicated to developing good mathematics teaching makes lesson study an attractive and potentially powerful tool for mathematics teacher development.

## Overview of the Lesson Study Elective Course

The lesson study research spanned the academic year 2006–2007 and took place in the context of a newly-offered elective module in education – Learning to Teach Mathematics Using Lesson Study – in an Irish college of education. Student teachers commonly pursue a concurrent model of teacher education there leading to an honours bachelor's degree in education (B. Ed) and including a single academic subject studied to degree level. Six third-year B. Ed student teachers participated. The lesson study protocols of collaborative lesson preparation and post-lesson collaborative reflection were adopted to further our goal of learning to teach the primary mathematics curriculum well. Each member of the elective group was involved in planning, teaching, analysing and revising mathematics lessons intended to promote children's mathematical reasoning. The lesson study elective course revolved around these mathematics lessons and extended over three cycles of lesson study. Research lessons were taught at two different school sites (see Table 13.1 for details). Because the student teachers came to the schools to teach the research lessons only, these are known as 'dive-in' lessons. As such, they lacked some of the rich potential for learning about their pupils available to class teachers working on lesson study within their own schools, yet the act of teaching and observing the research lessons for different age groups of children in widely different school settings constituted a valid and valuable lesson study experience for the student participants. A theme was chosen by the group for the research lessons in each cycle and each student teacher volunteered to teach particular lessons. On the research lesson days, the group divided into two with some members accompanying each 'teacher'.

**Table 13.1** Lessons taught during each lesson study cycle

| Lesson study cycle | School | Class/ages | Topic | Student teacher pseudonyms |
|---|---|---|---|---|
| Cycle 1 | St Peter's | 4th/9–10 years | Weight | Treasa |
| | St Paul's* | 4th/9–10 years | Weight | Finola |
| Cycle 2 | St Peter's* | 5th/10–11 years | Fractions | Bríd |
| | St Paul's | 3rd/8–9 years | Fractions | Ethna |
| Cycle 3 | St Peter's | 3rd/8–9 years | Division | Róisín |
| | St Paul's* | 5th/10–11 years | Fractions | Nóirín |

*Researcher present.

Three distinct aspects of the lesson study elective course emerged, and these were used to frame analysis. First, students participated in the course by engaging with the group in preparing, teaching and reflecting on lessons, i.e. by 'doing' lesson study. Secondly, participants also engaged with the elective course by 'doing' mathematics together, regularly. This aspect of engagement with interesting mathematics was for the students themselves and independent of mathematics to be taught in lessons. Thirdly, students participated in the elective by 'being' lesson study elective students, where engagement meant pursuing activities related to the elective enterprise but not essential to lesson study, for example, watching DVDs about lesson study and writing reflective journals.

## *Data Analysis*

Each of the six lessons was observed, recorded, transcribed and analysed using the Knowledge Quartet (KQ) framework (Rowland, Huckstep, & Thwaites, 2005). The KQ is a four-dimensional, practice-based framework for mathematics lesson observation and analysis developed inductively from analysis of videotaped lessons taught by novice teachers. The four dimensions are termed *Foundation, Transformation, Connection* and *Contingency*. Foundation includes teachers' knowledge, beliefs and understanding of mathematics and mathematics pedagogy, acquired before and during teacher preparation; this dimension is seen as underpinning the other three. Transformation encompasses the ways in which the teacher's own knowledge is transformed to make it accessible to the learner, especially through the use of representations and examples. Connection pertains to knowledge displayed when teachers make connections between and among mathematical ideas; it includes issues of sequencing and judgements about conceptual complexity. Finally, Contingency is manifested in the ways that a teacher responds to unanticipated events as they emerge during instruction. This could be described as 'thinking on your feet'. For further details of the KQ, see Chapter 12 by Turner and Rowland (this volume).

Lesson study community members were all encouraged to think of aspects of their mathematics lessons in terms of the four dimensions of the KQ, and the negotiation of the meaning of the framework as a language to describe mathematics teaching contributed to the *shared repertoire* (Wenger, 1998) of the community. As initially understood by the group, the KQ appeared linear in its exposition of the four dimensions of the mathematics knowledge required for teaching. Engagement in lesson study, however, brought about a reordering of the KQ components. By starting with a focus on children's learning of mathematics, a strong emphasis was placed on the Contingency dimension of teachers' mathematical knowledge in teaching, followed by the Connection dimension, which when contextualised by studying particular research lessons gave rise to revisiting the Transformation and Foundation dimensions. In lesson study cycle one, the lesson preparation and post-lesson reflection meetings were audio-recorded. In lesson study cycle two, these sessions were video-recorded. In lesson study cycle three, I was not present at the preparation meeting and the students' journals are the only record, although two post-lesson reflection meetings were audio-recorded and transcribed. For the purpose of coding these records, at first, I drew on concepts of *participation* and *identity* borrowed from Wenger (1998), but gradually the data analysis became more inductive, as various fresh indicators of mathematics teacher development were generated from the data rather than determined by reference to the literature or my own preconceptions. Since two research lessons were taught simultaneously on each occasion, I made personal observation notes for the one in which I was present. Students wrote a reflective journal for seven/eight of the lesson study sessions, and these were also important data. As well as describing some elements, which arose from analysis of the entire lesson study elective course, I shall illustrate here the crafting of a mathematics teacher identity by one student participant, Bríd, over the three cycles of lesson study.

## *'Doing' Lesson Study*

The dialectical nature of the learning of persons in activity presents a rich tapestry of interactions and interpretations of how that learning occurs. Rogoff, Matusov, and White contend, "learning involves transformation of participation in collaborative endeavour" (1996, p. 388). Learning to teach requires participation in and the collaboration of a group of people, and I sought evidence of this in my analysis of the lesson study community of practice. An agreed goal of each lesson study cycle was to establish what mathematical ideas or concepts the student teacher wanted pupils to engage with as a result of the particular lesson being planned, and to study children's responses to the mathematical task(s) during the lesson with a view to assessing the kind of mathematical thinking in evidence in the class. This became the shared enterprise of the community of practice.

### *Preparing the Lessons*: *Cycle One*

When the lesson study group turned to planning the lessons to be taught on weight, a tension emerged for some group members. What some students perceived as straying from the objective – "at times we could wander from that and begin including less relevant things" (Noirín's reflective journal entry 2) – was from another perspective, a process of exploration of the teaching resource materials available, discussion of the meanings of 'mass' and 'weight', leading to agreement to focus on the attribute of 'weight'. As a group, we engaged in study of resource materials which is meant to support the teacher of the lesson. I considered that my role as knowledgeable other (Watanabe & Wang-Iverson, 2005) was to collaborate with the team in order to enhance content knowledge, guide the thinking about pupil learning and support the team's work. A handbook of lesson study protocols advises that:

> Discussing the content, scope and sequence of curriculum helps teachers to be clear about where they are going with the lesson they are preparing and what outcomes they are looking for from the students. (Yoshida, 2005, p. 7)

As can be expected of student teachers, there was very little experience among them of 10-year-old children's current state of mathematics learning, or what they might be expected to know. Nor was there much evidence of theoretical underpinnings of pedagogy, which raised issues about these student teachers' Foundation knowledge for teaching primary mathematics. As a consequence, the effort expended in deciding which specific learning outcomes we were planning for extended the student teachers considerably. During this planning session and in later sessions, student teachers worked together to align content objectives from the curriculum with suitable contexts in which to base problems and activities designed to promote children's reasoning about the mathematical ideas underpinning each teaching objective chosen.

### Research Lessons: Cycle Two

In lesson study cycle two, the student teachers wanted to do something different and what they perceived as more difficult. They chose fractions as the topic to be taught. The main goal of these lessons was to provide children with an experience of exploring fractions in a realistic context while affording the student teachers an opportunity to study what the children already knew about fractions with a view to developing that understanding. In this instance, the school textbooks were put aside and the two lessons were planned as a fractions investigation activity.

### Learning Takes Time

Full participation in the community of practice was proving challenging, however, as this excerpt from one student teachers' journal attests:

> Initially I thought that by working together on a lesson we would work quicker but as we got more experience at lesson study, we began to spend longer discussing our intentions and really getting behind the mathematics and what we wanted the children to learn. It was now not a case of devising fun activities to enhance the lesson but a matter of questioning the mathematics and how best to teach it. (Nóirín's journal entry 4)

This spending of time on lesson study was proving to be personally demanding, yet inherently rewarding. We came back to 'doing' mathematics again with renewed interest and fresh eyes when, for the last hour of the preparation session, the agreed 'teachers' for lesson study cycle two took turns to practise their research lessons on the group.

## *'Doing' Lesson Study: Cycle Three*

Student participants conducted the planning for research lessons in cycle three in the absence of a knowledgeable other. Their success, as evidenced in the research lessons taught, shows that lesson study belongs to the participants, and that a knowledgeable other, while an integral part of the process, need not be centrally involved at every stage of the lesson study cycle. Using the Knowledge Quartet framework, there is strong evidence of learning along the Connection dimension in the data here, prompted by efforts to interpret the primary mathematics curriculum. Transformation issues were explored by Bríd, who reported:

> We also spent a lot of time debating whether or not to supply counters for the children to work with. Would they hinder or help them in their problem solving and would the distribution of them take time from the maths? Through our discussion we felt it best not to use them as they might distract the children from the actual problem. (Brid's journal entry 6)

The focus had shifted to actual mathematical details of the lesson and suitability of context, choice of example (3 as a divisor), whether or not to use counters, *et cetera* had become more central in the planning. All six student teachers' journal entries corroborate this engagement with the details of planning a successful lesson on division. However, the group also engaged in planning a lesson on percentages where the difficulties encountered by the group resulted in their abandoning the task until they could get 'expert' help. After outlining suggestions made to teach a lesson on percentages, which linked with fractions but was not aligned with what the community by now considered good practice, the embryonic lesson plan was shelved, in favour of a variation on Bríd's lesson. Nóirín balked at teaching the proposed lesson on percentages, because she did not know how it would relate to the class's current understandings of fractions. The student teachers' connection of the two mathematical topics raised is indicative of the presence of the second perspective characteristic of Japanese teachers – the curriculum developer lens (Fernandez et al., 2003). These students were becoming aware of complexity in the mathematical connections teachers are required to make in teaching the curriculum well.

## *'Doing' Mathematics*

Each of the lesson study sessions included some element of exploring mathematical ideas by the participants. The mathematical tasks presented in the first session were intended as an introduction to thinking about primary school mathematics in other than the traditional algorithmic terms. One illustrative example is offered:

> We then worked out some maths problems, in pairs. I was surprised at the simplicity some children (sic) worked out theirs in comparison to mine. The problem of the bus: 328 people to be transported in a forty-seater bus. I divided 40 into 328 directly to get my answer. My partner drew out circles of forty, until she had enough . . . that was how she realised she had enough buses. We both had the correct answer. This opened my mind and I realised that there is no right way of solving a maths problem. (Róisín's journal entry 1)

Reflective journals all referred in emotive terms to this element of the first session with one theme emerging strongly – the student teachers' differing relationships with mathematics were all "both complicated and powerful" (Mendick, 2006, p. 156). Each student reflected on her own responses to the problems in terms which ranged from comments on emotions like "fear . . . panicked and confused," through perceived personal deficiencies, "I always doubt my ability to do it," to the more measured "very interesting," and realisation of "how indoctrinated we are" (journal entry 1 of Ethna, Bríd, Nóirín and Treasa). Doing mathematics ourselves became an essential element of the lesson study elective and, while not explicitly part of the Japanese lesson study protocol, can be subsumed under the "purposeful learning" of the goal-driven pre-lesson planning phase of each cycle (Fernandez & Chokshi, 2005, p. 73). If the lesson study community of practice were to direct its research gaze on how children respond to mathematical tasks, then it had to direct its research gaze on members' own doing of mathematics also.

These student teachers accepted that to pose realistic problems and to focus on children's responses were aspects of good mathematics teaching practices that were challenging for them because of their own fragile relationships with mathematics.

Treasa: That's what I'm afraid of . . . cos I'm very . . .
I get very, I'm very insecure about maths. If they could say something and I'm standing there like an idiot saying 'God I don't know what to do next'. (Planning session lesson study cycle 1)

Making the lesson study elective sessions a safe place to question one's own and each other's mathematical ideas became an important element of the process. The role of the course tutor as knowledgeable other developed in the selection of interesting mathematics to engage the group, and in drawing pedagogical inferences from events in the group setting. I also sought to establish communication norms within the community of practice, which facilitated the expression of mathematical thinking. When working in community, all members had the responsibility to the enterprise and knowing when to use Contingent opportunities – by deviating from the planned agenda, or when to allow the discourse to continue uninterrupted – presented an occasional dilemma. The different approaches to mathematics were discussed and celebrated within the group and tended to mask the fact that some students were quicker and surer in proffering solutions than others. By focusing on improving pedagogy and constantly making connections between intentions and actions of the teacher and imagined and actual responses of the children, this aspect of difference within the group was minimised. Nonetheless, challenges in the communication of ideas between members emerged as a theme requiring further exploration.

## *'Being' in the Lesson Study Elective Community of Practice*

Each cycle of the lesson study elective course had a component that was not directly related to the preparation, teaching and reflecting on lessons taught. In cycle two, the student teachers engaged in watching two DVDs. These were representative of lesson study as practised by American teachers – *How Many Seats*? (Mills College Lesson Study Group, 2005), and as practised by a Japanese teacher – *To Open a Cube* (Mills College Lesson Study Group, 2003). The group members overtly identified with the lesson study process while owning their own practice. Reflections which followed this session were deeply insightful and focused on the student teachers' own learning about mathematics teaching from observing the two different settings for learning on the lesson study videos. Bríd's concluding remarks are salient:

> Overall, watching the DVDs gave me more insight into how kids think mathematically and ideas that I can use and take into account when planning. They also made me consider my own problem-solving abilities and realise that I am only as effective as my own level of thinking. This scares me a bit because my mathematical ability may prevent better learning by hindering rather than helping the pupils. I think that lesson study is vital to do with my colleagues so that I can both challenge my own thinking and receive support when planning for maths lessons. (Bríd's journal entry 6)

## Discussion

The research lessons on weight in the first cycle provided rich examples of the complexity of enacting the task of teaching for prospective teachers. The research lessons on fractions in the second cycle appeared to have taken on a research perspective, which arose from the goals of the lesson study enterprise. This was akin to the ‘researcher perspective’ used by Japanese teachers to enhance their teaching of mathematics (Fernandez et al., 2003). The two research lessons in lesson study cycle three were characterised by a more ‘improvisational’ approach to children’s learning of mathematics and were influenced by an attempt to use an interpretation of the critical lenses applied to student learning and curriculum development. We will now explore emergent themes suggested by the lesson study process relating to the development of mathematical knowledge for teaching.

### *Identity in Terms of Learning to Teach Mathematics*

By identity, we mean the learning that occurs while individuals are mutually engaged in a worthwhile enterprise (Lave, 1996). In this case, the student teachers pursued learning to teach mathematics together. By identity-work, we also mean the narratives people share while participating in such a community of practice. Identities are formed through participation and identification with the goals of the enterprise and as such are socially formed. These students’ “knowing and knowledgeability” (Roth & Lee, 2006) of and for good mathematics teaching was exhibited through their belonging to a community of practice dedicated to developing this very knowing and knowledgeability. Participation in the enterprise of studying mathematics teaching by engaging in actual teaching, and then reflecting critically on it as a group of individuals who are all similarly engaged, contributes to the identity of an individual engaged with learning to teach mathematics and to the community of practitioners building knowledge of and through the enterprise of lesson study dedicated to mathematics teaching. By putting the spotlight on the practice of the lesson study community as a whole, it is possible to illustrate, largely from the student teachers’ journals, how one participant re-positioned herself, through identity work as a pre-service primary teacher learning to teach mathematics well, thereby increasing her mathematical knowledge in teaching.

## The Case of Bríd

In lesson study cycle two, Bríd engaged with preparing a mathematically productive lesson on fractions, which emerged from practice of ‘doing’ fractions work and reading articles on good practice in teaching fractions. While acknowledging the contribution of the group, Bríd embraced her role as teacher, despite her feelings of inadequacy:

> While this session was of huge benefit to me, I am aware that I need to become more comfortable with fractions and I can only do this by immersing myself in them and engaging with them in a meaningful manner. I am hesitant about going into classrooms with them, but I have a funny feeling that it will be the children teaching me about fractions rather than me teaching them! Perhaps it is better to say it will be a joint sharing of learning. (Bríd's journal entry 4)

## *Descriptive Synopsis of Bríd's Lesson*

Bríd's lesson was focused on studying children's understanding of fractions as a designated number of equal parts of a whole and proposed using a pizza context. She set the scene by asking the class to imagine that it was one boy's birthday and that he had invited seven friends to share six pizzas with him. Children were given a teacher-made handout showing six identical circles on an A4 page, which they were invited to think of as pizzas. After some time working in pairs, the class was called to attend while some pairs were invited to the board to explain what they had done. Next, the children addressed a second similar problem, involving six pizzas divided between ten people, and later, other pairs of children were invited to explain their work, which Bríd illustrated on the board. A whole-class session concluded the lesson where the teacher talked the children through the process of adding a half and a quarter and finally elicited why the size of the unit was an important element in dealing with fractions (Fig. 13.1).

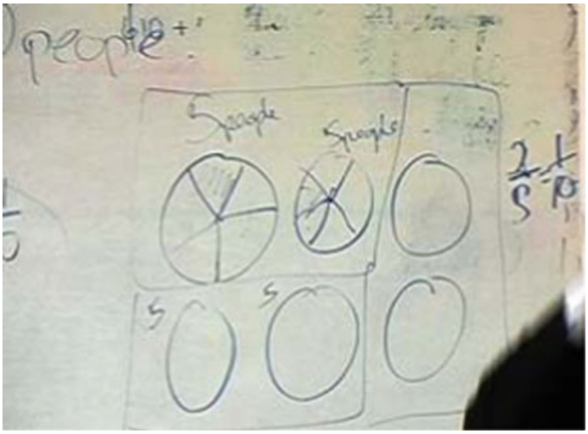


**Fig. 13.1** An innovative way of dividing six pizzas among 10 people?

## *Learning from Teaching*

Bríd's lesson plan had anticipated alternative answers and her strengthening mathematics teacher identity led to an engaging and challenging lesson. While she had planned the lesson to be about dividing pizzas, because this was a ‘dive-in’ lesson she did not know the children. Bríd invited a boy to tell her his name (Cathal) and then asked the class to imagine it was Cathal’s birthday and that he was having a pizza party. This tactic demonstrated strong pedagogical content knowledge on Bríd’s part. When Bríd’s lesson was analysed by the group, her opening scenario became one of the stories of the practice which was adopted in later lessons. It illustrates how learning about teaching occurs through engagement with the enterprise.

Despite the comprehensive planning, which focused on the mathematical content of the lesson, Bríd was confronted twice, in the act of teaching, with more sophisticated thinking on the part of some pupils than she had anticipated or was prepared for. In connection with the second problem, a child at the board divided half a pizza into five equal parts, which Bríd incorrectly called “fifths”. This was remedied on the spot, when one of her colleagues observing the lesson alerted Bríd to the error. In the second, two children suggested taking two pizzas together and giving each person a fifth of one pizza, resulting in 10 people getting a fifth each, from each of three sets of two pizzas. Bríd appeared confused by this innovative approach and while the children had articulated their thinking clearly, she wrote (incorrectly) on the board: $\frac{2}{5} = \frac{1}{10}$.

Aware of her own confused thinking, but unable to clarify it on the spot, Bríd quickly erased the ‘solution’ (without acknowledging that the children’s strategy led to $\frac{3}{5}$ of a pizza per person) and moved on with the rest of the lesson. Her knowledge of fractions was inadequate for the teaching activity she had set herself, and this mismatch between her plan to encourage children’s disparate ways of thinking and her ability to recognise the validity of all ideas, caused Bríd to consider the need to expand her facility with equivalence of fractions further. Her experience of enactment of the teaching role she had adopted as being less powerful than she had hoped and planned could have caused Bríd to be less adventurous in future, and stick to the textbook or to teaching by telling what she knew. However, such was Bríd’s engagement with the act of learning through teaching and the strength of her belonging to the collective enterprise that her lapse into mathematical misinformation on that occasion became a motivational force to learn more about fractions for teaching. This embarrassing episode also became a story for the community of practice and gave rise to considerable further mathematical work on fractions within the group. The meaning of the Knowledge Quartet dimension of Contingency was expanded for all participants in the community of practice through this challenge to Bríd’s knowledge of fractions and everyone’s Foundation mathematical understanding was expanded by her lesson. The case of Bríd is used here to exemplify how these students’ attitudes to the enterprise of teaching mathematics changed over the course. From feelings as individuals of inadequacy and fear, they had moved to a collective research orientation into how mathematics can be taught well, and into building the mathematical knowledge required to do so.

## Mathematics Teaching and Matters of Interpretation

Where mathematics is only one of many subjects to be taught by generalist primary teachers, learning how to teach mathematics can become largely a matter of interpretation of the language used and the meanings intended by teacher educators. Evidence from the lessons taught by the student teachers in this study indicates that they often experience difficulty in interpreting what is meant by contested terms like 'problem-based teaching' or 'realistic mathematics.' Mathematical process skills, such as 'communicating and expressing mathematical ideas', are widely interpreted in Ireland to mean the more generic notion of [teacher] 'talk and discussion'. Curricular guidelines on mathematics pedagogy, for example, the optimal use of materials and mathematical representations, are filtered in the light of past experiences. A community of practice by definition functions as an *economy of meaning*, which suggests that some meanings do achieve superior status (Wenger, 1998, p. 198). The role of a knowledgeable other is crucial in this economy. The lesson study community of practice became an important site where meanings of mathematical practices and mathematics teaching were negotiated through *engagement, imagination* and *alignment* (Wenger, 1998). Alignment with a reform interpretation of the mathematics curriculum, with good teaching practices, with recent research findings was critical to the lesson study enterprise, together with alignment of the lesson study community of practice with the other communities of practice with which it interacted. Accountability to the enterprise begets negotiations of meaning in a highly reflexive manner and participation in the lesson study community of practice involved negotiating and renegotiating meanings for an increasing number of mathematical ideas and practices.

## Findings

Learning to teach mathematics is hard work, and the six young women in the Dublin study faced the negative aspects of their different relationships with mathematics and worked collaboratively, with considerably energy to forge a new path for making sense of the primary mathematics curriculum and meaningful ways to teach it. By whole-hearted participation in the lesson study elective course, they validated the potential of lesson study as a means of learning to teach mathematics. But lesson study is not a panacea for mathematics teacher education or development. Rather, it is a process, which by design allows teachers to augment their mathematical and pedagogical skills for teaching the mathematics curriculum, by refining their goals and focusing on what and how children learn mathematics as a result of their practice. These outcomes do not result from individual effort, but from participation in practice. Japanese teachers have long realised that lesson study is a powerful means for teacher development and curricular change. Lesson study is, in essence, a road map for socio-cultural learning about mathematics in teaching. A road map is a useful, even necessary, tool with which to get from one place to another in unfamiliar territory. But, of itself, lesson study is not enough, no more than a road map is of use to the person who does not know how to read it, does not know where s/he is on the map, does not know the landmarks to look out for, and/or is not sure of the destination s/he is travelling towards. A second more specific set of directions is also required. The Knowledge Quartet is such a framework of mathematics knowledge in teaching that provides signposts to help answer the intermediate questions. Where are we now? In what direction are we heading? Why this representation or that example?

The student teachers in the Dublin study began by focusing on children's responses to the mathematics lessons they had planned and taught. From there, their attention moved in two directions: towards the Connection and Transformation dimensions of their mathematics teaching. The need for both dimensions – for example, the ability to make connections between mathematical ideas and procedures, to sequence material conducive to learning and to make optimal choices of representations and examples – becomes obvious when one studies Contingency opportunities which naturally arise in the course of any lesson. However, mathematics teaching is not a static activity. Rather, like the discipline of mathematics itself and the art of teaching, it is a dynamic cultural pursuit and the above three dimensions of the Knowledge Quartet arise from, are informed by and in turn transform the primary dimension which has been called Foundation knowledge. Findings from this study indicate that Foundation knowledge for mathematics teaching expands with participation in lesson study. There were marked changes in how these students approached planning for, and teaching of, mathematics. There was evidence in them, of a growing awareness of the depth and connectedness of mathematical ideas. They have developed a much more focused eye on how children build mathematical thinking and have expended considerable energy in designing opportunities for children to do so. The student participants in this study have all grown in self-confidence, a self-confidence that recognises personal agency and thrives on communal support. These student teachers and I have come to view mathematical knowledge for teaching primary mathematics in new ways, and in consequence, think differently about how mathematical knowledge can be developed or stifled by classroom experiences.

## *Lesson Study as a Tool for Developing Mathematical Knowledge in Teaching*

Viewing knowledge as situated in social contexts and constructed through social interaction requires a different view of how the development of mathematics teacher knowledge may or may not be fostered. The focus shifts from the individual to the communities in which mathematics teachers are engaged and the extent to which these communities support teacher learning and induct teachers into teaching practices. The scope for teachers to have ownership of, and to play an active part in developing their knowledge and expertise is also central to enabling the production of critically reflective practitioners who are better able to deal with the challenges faced in engaging with new knowledge, or knowledge constructed in the variety of teaching contexts they will experience. They will be able to deal with the discomfort that will inevitably be felt in having to revise their own mathematical knowledge in order to teach if the community in which they are learning to teach mathematics encourages a more collective responsibility where it is possible to be open about questions about mathematical subject knowledge as well as ways to teach it. Such communities of mathematics teachers are identified by Ma (1999) as contributing to what she calls the 'profound understanding of fundamental mathematics' characteristic of Chinese elementary school teachers. Recognising that knowledge is not simply located in the individual teacher but distributed over people and resources implies that the responsibility for development of mathematical knowledge for teaching is not an individual but a collective one, which participation in the practice of lesson study appears to meet.

Lesson study fosters the collective development of mathematical knowledge. Engagement with lesson study also enriches the personal knowledge base on which individual teachers draw in developing his or her own practice. Fernandez's work (2005) underlines the need for a knowledgeable other to act as a catalyst and to properly challenge accustomed ways of working. The Dublin study confirms that the practice of collectively studying teaching, in the immediacy of a research lesson, designed for a specific context, can then be extended and tested against an even wider knowledge and research base through working with a knowledgeable other. Lesson study, as a practice, accepts that it is always possible to improve one's teaching and to continually develop mathematical knowledge in the process. By recognising the importance of the knowledgeable other role, teachers are reminded of the importance of investigating multiple sources in preparing for teaching, while ownership of the practice remains firmly with the teachers themselves. The Dublin study leads us to conclude that engagement in a lesson study community with the purpose of learning about mathematics teaching also develops mathematical knowledge for teaching in the process.

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# Chapter 14
# Using Theories to Build Kindergarten Teachers' Mathematical Knowledge for Teaching

**Dina Tirosh, Pessia Tsamir, and Esther Levenson**

## Introduction

Around the world, there are moves to strengthen the mathematical development of children in preschool settings, and to strengthen the preparation of preschool teachers to support such development. A joint position paper published in the United States by the National Association for the Education of Young Children (NAEYC) and the National Council for Teachers of Mathematics (NCTM) stated that "high quality, challenging, and accessible mathematics education for 3- to 6-year-old children is a vital foundation for future mathematics learning" (NAEYC & NCTM, 2002, p. 1). As such, they recommend that "teachers of young children should learn the mathematics content that is directly relevant to their professional role" (NAEYC & NCTM, p. 14). Similarly, the Australian Association of Mathematics Teachers (AAMT) and Early Childhood Australia (ECA) published a joint position paper calling for the adoption of "pedagogical practices that encourage young children to see themselves as mathematicians" (AAMT/ECA, 2006, p. 2). They too recommended that early childhood staff be provided with "ongoing professional learning that develops their knowledge, skills and confidence in early childhood mathematics" (AAMT/ECA, 2006, p. 4). In England, the Practice Guidance for the Early Years Foundation Stage (2008) offers suggestions for practitioners in how to foster children's knowledge of counting, calculations, shapes and measures.

All too often, preschool teachers receive little or no preparation for teaching mathematics to young children (Ginsburg, Lee, & Boyd, 2008). Moreover, research on preschool teachers' mathematical knowledge is limited and investigating the ways in which tools may be used for building kindergarten teachers' mathematical knowledge for teaching is critical. Recently, Tsamir (2008) described how theories of mathematical knowledge may be used as tools in mathematics teacher education. We extend this idea and describe how combining theories of teachers' knowledge

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with theories of mathematical knowledge may be used as a tool to build kindergarten teachers' mathematical knowledge for teaching.

There is a wide range of theories relevant to the development of mathematical knowledge. In this chapter we focus on Tall and Vinner's (1981) concept image-concept definition (CICD) theory and describe how familiarizing kindergarten teachers with this theory may be used to build their mathematical knowledge for teaching. Much of kindergarten children's knowledge is based on their perceptions and manipulations of their surrounding. Left unchecked, intuitive interpretations created at this age often become rigid and difficult to undo at a later stage (Fischbein, 1987). It is therefore relevant to introduce this theory to kindergarten teachers so that they may plan activities that help young children assimilate concepts of higher complexity and abstraction during the early years, encouraging children to build concept images which are in line with concept definitions.

In framing the mathematical knowledge kindergarten teachers need for teaching, we draw on the works of Shulman (1986) and of Ball and her colleagues (Ball, Bass, & Hill, 2004; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008). Clearly, all teachers need to know the mathematics they are teaching. Kindergarten teachers, for example, need to be able to discriminate between triangles and non-triangles. Yet, this is not sufficient. Teachers must be able to explain why a figure is, or is not a triangle. They also need to know effective ways of presenting figures to their students so that they too will be able to differentiate between triangles and non-triangles.

In this chapter, we describe how combining theories embedded in the realm of teacher knowledge with theories embedded in the realm of mathematics knowledge and familiarizing practicing kindergarten teachers with this combination was used to build their geometrical knowledge for teaching. The chapter begins by describing the separate theories and how they may be combined to build a more comprehensive and refined tool for building and evaluating mathematical knowledge for teaching. It then illustrates how this tool was used to build kindergarten teachers' knowledge for teaching geometrical concepts. We also illustrate how kindergarten teachers used the combination of theories to inform their practice. Finally, we address how the combined theories tool described here may be further developed and used.

## Combining Theories of Teacher Knowledge with Theories of Mathematics Knowledge

### *Dimensions of Knowledge for Teaching*

In his seminal work, Shulman (1986) described and analyzed components of teachers' knowledge necessary for teaching. As already described in Chapter 2, two of the major components identified were subject-matter knowledge (SMK) and pedagogical content knowledge (PCK). Ball and her colleagues further developed Shulman's theory, focusing on mathematics, but retaining a basic framework that can be generalized to other subject areas. SMK was further divided into common content knowledge (CCK) and specialized content knowledge (SCK). CCK may be defined as "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399), whereas SCK is "mathematical knowledge not typically needed for purposes other than teaching" (Ball et al., 2008, p. 400). Pedagogical content knowledge may be further differentiated into knowledge of content and students (KCS) and knowledge of content and teaching (KCT). KCS is "knowledge that combines knowing about students and knowing about mathematics", whereas KCT "combines knowing about teaching and knowing about mathematics" (Ball et al., 2008, p. 401).

We illustrate the dimensions of knowledge for teaching within the context of geometry by offering a few examples taken from Ball et al. (2008). Knowing that the diagonals of a parallelogram are not necessarily perpendicular may be considered knowledge typical of anyone who knows mathematics (CCK). Knowing "how the mathematical meaning of *edge* is different from the everyday reference to the edge of a table" (p. 400) is an example of SCK. Knowing which shapes young students are likely to identify as triangles, and that confusion between area and perimeter may lead to erroneous answers, are examples of KCS. Knowing how to sequence the presentation of examples and which examples may deepen students' conceptual knowledge is KCT.

Shulman's and Ball's theories have been used to explore teachers' mathematical knowledge in several specific mathematical contexts such as division of fractions (Tirosh, 2000) and multiplication and subtraction of whole numbers (Ball et al., 2008). These theories have not been explicitly combined with the more general mathematics knowledge CICD theory suggested by David Tall and Shlomo Vinner in the 1980s. In the next section we review Tall and Vinner's theory taking into consideration Fischbein's theory of intuitive knowledge.

## *Concept Image-Concept Definition (CICD)*

Having precise definitions for mathematical concepts ensures mathematical coherence and provides the foundation for building mathematical theories. However, these same mathematical concepts may have been encountered by the individual in other forms prior to being formally defined. Even after they are defined, mathematical concepts often invoke images both at the personal as well as the collective level. The term concept image is used to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). The concept definition refers to "a form of words used to specify that concept" (p. 152). A formal concept definition is a definition accepted by the mathematical community whereas a personal concept definition may be formed by the individual and change with time and circumstance. Because the concept image actually contains a conglomerate of ideas, some of these ideas may coincide with the definition while others may not. For example, a function may be formally defined as a correspondence between two sets which assigns to each element in the first set exactly one element in the second set. Some students claim that a function is a rule of correspondence (Vinner, 1991). This image does not contradict the definition. However, it is limited and eliminates the possibility of an arbitrary correspondence.

When a problem is posed to an individual, there are several cognitive paths that may be taken which take into account both the concept image and concept definition. At times, although the individual may have been presented with the definition, this particular path may be bypassed. According to Vinner (1991), an intuitive response is one where "everyday life thought habits take over and the respondent is unaware of the need to consult the formal definition" (p. 73). Intuitive knowledge is both self-evident and immediate and is often derived from experience (Fischbein, 1987). As such, it does not always promote the logical and deductive reasoning necessary for developing formal mathematical concepts. "Sometimes, the intuitive background manipulates and hinders the formal interpretation" (Fischbein, 1993, p. 14). Fischbein (1993) considered the figural concepts an especially interesting situation where intuitive and formal aspects interact. The image of the figure promotes an immediate intuitive response. Yet, geometrical concepts are abstract ideas derived from formal definitions. Thus, as we consider the notions of concept image and concept definition, we take into account aspects of Fischbein's theory related to intuitive and formal knowledge.

Although Tall and Vinner (1991) introduced their theory within the context of advanced mathematical thinking, the interplay between concept definition and concept image is part of the process of concept formation at any age. Young children learn about and develop concepts, including geometrical concepts, before they begin kindergarten. As such, their concept image is often limited to their immediate surroundings and experiences and is based on perceptual similarities of examples, also known as characteristic features (in line with Smith, Shoben, & Rips, 1974). This initial discrimination may lead to only partial concept acquisition in that children may consider some non-examples to be examples and yet may consider some examples to be non-examples of the concept. Regarding geometrical concept formation, van Hiele (1958) theorized that students' geometrical thinking progresses through a hierarchy of five levels, eventually leading up to formal deductive reasoning. At the most basic level, students use visual reasoning taking in the whole shape without considering that the shape is made up of separate components. At the second level, students begin to notice the different attributes of different shapes but the attributes are not perceived as being related. At the third level, relationships between attributes are perceived and definitions are meaningful. Kindergarten children begin to perceive attributes, but need guidance in order to assess which attributes are critical for identifying a figure and which are not. Familiarizing kindergarten teachers with the CICD theory may enlighten teachers to the tension which may exist between the concept image and concept definition and inform their instruction in ways that will promote children's advancement along the van Hiele levels of thinking.

## *The Combined Framework*

Ball's notions of CCK, SCK, KCS and KCT allow us to differentiate between types of knowledge necessary for teaching. We suggest that the four dimensions of teachers' knowledge be combined with theories of mathematics knowledge in order to provide a finer grain and more focused lens with which to study mathematics teachers' knowledge for teaching. Such a framework would allow us to investigate, for example, teachers' knowledge of the psychological aspects of student's mathematical errors. Here we suggest how these four dimensions may be combined with Tall and Vinner's CICD theory and illustrate this framework within the context of geometry focusing on teachers' knowledge for teaching triangles.

| | Domains of teachers' knowledge | | | |
|---|---|---|---|---|
| Domains of mathematical thinking | CCK | SCK | KCS | KCT |
| Concept image | Cell 1 | Cell 2 | Cell 3 | Cell 4 |
| Concept definition | Cell 5 | Cell 6 | Cell 7 | Cell 8 |

Cell 1: CCK-Image. Here we address the common knowledge of a concept's image. This includes knowing to draw examples and non-examples of triangles.

Cell 2: SCK-Image. Here we address the specialized knowledge of a concept's image necessary for teaching. This includes a rich concept image of triangles which incorporates scalene and obtuse triangles with different orientations and not just equilateral and isosceles triangles. It may also include a broad image of non-examples for triangles beyond circles and squares (Tsamir, Tirosh, & Levenson, 2008).

Cell 3: KCS-Image. Here we address knowledge related to students and concept images. This includes knowing that the equilateral triangle is a prototypical triangle (Hershkowitz, 1990) and that young children may not identify as a triangle a long and narrow triangle such as the scalene triangle, even when admitting that it has three points and lines (Shaughnessy & Burger, 1985). We also include in this cell knowledge of the van Hiele model (e.g., van Hiele & van Hiele, 1958) for students' geometrical thinking and being able to recognize, for example, that a student's concept image at the most basic level takes in the whole shape without considering its components. As such, this cell includes knowing that a rounded 'triangle' is often identified as a triangle (Hasegawa, 1997) because children take in the likeness of the whole shape, ignoring that the shape is missing vertices.

Cell 4: KCT-Image. Here we address knowledge related to teaching and concept images. This includes knowing which examples and non-examples to present to a student which will broaden his or her concept image of a triangle to include, for example, triangles with different orientations.

Cell 5: CCK-Definition. Here we address common knowledge related to a concept's definition. It includes knowing that a triangle may be defined as a polygon with three straight sides.

Cell 6: SCK-Definition. Here we address the specialized knowledge of a concept's definition. In mathematics, definitions are apt to contain only necessary and sufficient conditions required to identify an example of the concept. Other critical attributes may be reasoned out from the definition. Thus, this cell includes knowing that defining the triangle as a three-sided polygon implies that it must be a closed figure with three vertices. It includes knowing that the triangle may be defined as a three-sided polygon, or a polygon with three angles, or a polygon with three vertices and that all three definitions are equivalent.

Cell 7: KCS-Definition. Here we address knowledge related to students and concept definitions. It includes knowing that a minimalist definition may not be appropriate for young students at the first or second van Hiele level because they do not necessarily perceive that a polygon with three sides must have three vertices. For example, research has suggested that for young children, the association between a triangle and the attribute of 'threeness' may be stronger than the necessity for it to be closed or for its vertices to be pointed (Tsamir et al., 2008).

Cell 8: KCT-Definition. Here we address knowledge related to teaching and concept definitions. It includes speaking to children with precise language, calling the vertices of a triangle by their proper name as opposed to referring to them as corners. It also includes knowing which examples and non-examples of a triangle to present to children which may encourage children's use of concept definitions and promote their advancement along the van Hiele levels of geometrical thinking. For example, presenting non-examples of a triangle which are not intuitively recognized as such, may encourage children to refer back to the concept definition when identifying the figure as a non-example of a triangle (Tsamir et al., 2008).

The combined theory suggested above may be used to build teachers' knowledge in at least two ways. First, it serves as a tool for teacher educators by allowing the teacher educator to focus on the specific knowledge being promoted. In much the same way, when explicitly presented to teachers it may also serve to focus the teachers on the knowledge they are building and its use in teaching.

## Setting

For the past 2 years, we have been providing professional development for groups of kindergarten teachers. Our program, *Starting Right: Mathematics in Kindergartens*, carried out in collaboration with the Rashi Foundation and the Israel Ministry of Education, is an integrated program where both kindergarten teachers and the children learning in the kindergartens are participants. Teachers participate in a professional development course. Children participate in a mathematically-enriched environment created by their kindergarten teachers. A major aim of the professional development course is to increase the children's mathematical knowledge by increasing their teachers' mathematical and pedagogical content knowledge.

The segments described in this study relate to two different groups of teachers who participated at different times in our program. The first and second authors were co-instructors of the course. Teachers met with the instructors on a weekly basis (4 h per week), either at a local educational center or in one of the kindergartens. The first two segments report on one group of teachers whereas the third segment reports on the second group. Each of the segments illustrates how the combined theory suggested above may be used to build and assess kindergarten teachers' knowledge for teaching geometry.

## Research Segments

### *Building Kindergarten Teachers' SCK Regarding Concept Images and Concept Definitions of Triangles*

What image comes to mind when one thinks about a triangle and what specialized knowledge must teachers know regarding this concept image? The concept image of a layperson (i.e. not a teacher) may differ from that of a teacher. Recall that according to Tall and Vinner (1981), the concept image consists of mental images, properties, and processes associated with the concept. A concept image may also change with time and experience. Studies have shown that when asked to draw a triangle, most students will draw either an equilateral triangle or isosceles triangle with a horizontal base (Hershkowitz, 1990). On the other hand, we would expect teachers to have a more elaborate and richer image of triangles which would include triangles whose sides and angles are unequal and triangles with different orientations.

During our first meeting with the kindergarten teachers, we asked the teachers to draw three examples of triangles and three non-examples of triangles. We began with triangles for several reasons. First, the preschool curriculum in Israel specifies that kindergarten children should be able to recognize many different examples of triangles. Second, we hypothesized that kindergarten teachers would be somewhat familiar with the definition of a triangle and perhaps less familiar with definitions for other geometrical figures. For example, although squares and rectangles may be familiar to many, the hierarchical nature of quadrilaterals makes the square a complex figure to define (De Villiers, 1994).

First, we note that all of the examples teachers drew were indeed triangles and all of the non-examples teachers drew were indeed not triangles. In other words, the teachers demonstrated CCK of the concept image for a triangle. Eight of the nine teachers drew at least one example of an equilateral triangle with a horizontal base. The ninth teacher drew a triangle with unequal sides but with a horizontal base. Five teachers drew only equilateral triangles with horizontal bases for all three examples of triangles. Two teachers drew one right triangle each, with horizontal bases. Only three teachers drew examples of triangles with a different orientation. Regarding the non-examples, all of the teachers drew geometrical figures such as circles, squares, and trapezoids. This introductory task afforded us a glimpse into the teachers' concept image of a triangle and indicated that their concept image of a triangle was more closely aligned with common knowledge rather than the specialized knowledge of a teacher.

Teachers were then asked to write down on a piece of paper the definition of a triangle. Although no two definitions were exactly alike, each teacher was able to give a valid definition of a triangle. In other words, the teachers demonstrated what may be considered CCK of the concept definition. Building teachers' SCK was done gradually and began by comparing the different definitions teachers had given for a triangle. The instructor gave the following instructions:

> Look at the definitions (now written on the board) and try to think which are correct and which are incorrect . . . if there are definitions which are unacceptable, explain why. If there are definitions which you approve of more than others, explain why. If there are definitions for which a slight revision may improve the acceptability of that definition, then write it. Perhaps there is more than one correct definition.

The teachers engaged in the task and then discussed the results together. Pointing to the first definition, "A triangle is a shape with three sides and three vertices", the instructor requested the teachers to raise their hands if they agreed that it was a valid definition. The following discussion ensued:

I: The question is very simple. Is this a definition of a triangle or not? That means that you can only vote yes or no. There is nothing in between and everyone has to vote.
H: How many times can I vote (yes)?
I: For each of the definitions you can either vote yes or no.
E: Is the question then if it's (the definition written on the board) closer to yes being a definition or closer to not being a definition?
I: There is no approximation. In mathematics it either is or is not (a valid definition).

In the above segment, teachers come to realize that definitions must be precise. On the other hand, different definitions may be equivalent and thus there may be more than one definition for a particular concept. Although the instructor's approach may be considered quite direct, it became the norm with these kindergarten teachers that the instructor gave the closing argument of each discussion. Discussing the merits of each of the definitions led to a more general discussion of definitions:

E: Maybe we first need to know what a definition is.
R: A definition must be clear.
Y: That you don't argue with.
R: In a dictionary.

The teachers have begun to realize that it is important to first ascertain what is meant by a definition in mathematics before they can discuss if what is written may be considered a valid definition of a triangle. Differentiating between everyday dictionary definitions and mathematical definitions is another aspect of SCK related to concept definitions and was discussed further in the following lesson as teachers reviewed various definitions for a triangle found in dictionaries and mathematics textbooks.

During the next lesson, Tall and Vinner's CICD theory was presented explicitly to the teachers. The teachers had been discussing which of the dictionary definitions would be unacceptable and for what reasons. One teacher quoted the following definition for a triangle, "a closed figure made up of straight lines." Another teacher responds, "But that can be like . . . a crown that you make. It doesn't say how many sides." This exchange prompted the instructor to introduce the notions of concept image and concept definition:

> Notice the connection between your thoughts and your knowledge, between your imagination and your knowledge . . . Vinner investigated mathematical concepts that also have a visual presentation. However, he also said that in mathematics we have definitions and we must work according to these definitions. This is the concept definition. The concept image is what we imagine in our thoughts when we close our eyes and think of the concept.

In the elementary school, concept definitions may be used to differentiate between critical and non-critical attributes of a concept, in order to identify examples and non-examples of that concept. After introducing the notion of a concept definition, the instructor adds, "to define is to simply characterize a group of mathematical entities . . . to say what can be called by this (concept) name and what cannot." The instructor then refers to the examples and non-examples of triangles the teachers drew during the first lesson pointing out that these illustrate each teacher's concept image whereas currently, the discussion at hand has revolved around the concept definition of a triangle.

In the above segment, the combined framework was essentially used by the instructor to assess current knowledge and then to direct and focus the knowledge being built. "From a cognitive point of view, prior knowledge has to be considered as a possibly influential characteristic" (Blömeke, Felbrich, Müller, Kaiser, & Lehmann, 2008). Assessing current knowledge is an essential first step to building new knowledge. The combined framework served to differentiate between CCK of the concept image, which the teachers exhibited, and SCK of the concept image, which the teachers seemed to lack. This was true as well for the teachers' CCK and SCK of the concept definition. After assessing current knowledge, the instructor began by focusing on SCK related to concept definitions leading eventually to an explicit discussion of the CICD theory.

## *Differentiating Between SCK and KCT*

Throughout the program a clear differentiation was made between mathematical knowledge for the teachers and mathematical knowledge as it is applied in the classroom. Initially, teachers found it difficult to separate these two domains of knowledge.

M: This is very confusing. You started off by talking about kindergarten children (in the beginning of the lesson) and now you decided to talk about mathematical thinking.

I: Let's put things in order. First, we must talk about the mathematics as is. First we (the teachers) need to know what a triangle is. The kindergarten will wait. Tomorrow morning we are not going to talk with the children about triangles.

A: I see us as kindergarten teachers, sitting with the students with the classic square, the classic rectangle, and the classic triangle and then we say, "What is this?" The child should say it's a triangle but according to what does he decide if it's a triangle or not?

I: Just a second. We'll get there. We'll definitely talk about it but for now it's just us. Differentiating between the children and us is very important. Part of what we will learn will be important mathematical knowledge that we will know but that we won't necessarily tell it as such to the children because it may not be appropriate.

The instructor is stressing the difference between KCT and SCK and that they are two different ways of knowing mathematics. She further explains the necessity for this differentiation, "My strong belief is that first you need to know what you are dealing with mathematically because otherwise there will be no basis for how you answer the child."

During the second lesson, as teachers discuss various definitions for a triangle, the difference between SCK and KCT is again brought up:

I: What is the source of this definition?

C: A geometry text book.

I: For which grade?

M: Junior high school.

Y: And you also need to know for what (mathematics) level the textbook is geared to.

I: Ok. I want to make something clear. In the end, we will bring to the kindergarten a definition which we feel is appropriate for the kindergarten. But, now we are talking about definitions which would be acceptable to mathematicians ... Now, you need to decide which definition is valid and which is not.

H: Wait a minute. Are you talking about for us or for the kindergarten?

I: For you.

The teachers are beginning to realize that a formal concept definition must be accepted by the mathematical community. This is part of the SCK being developed during these first two lessons related to concept definitions. On the other hand, knowing how to adapt a formal concept definition to the age and level of the students is an aspect of KCT. Although a triangle may be defined as a three-sided polygon, the teachers agreed that this definition would be unsuitable for young children for two reasons. First, it is quite unlikely that young children would comprehend the meaning of the term polygon. Second, a minimalist definition, although mathematically acceptable, does not stress all of the critical attributes that all examples share. As the instructor summarized:

> On the one hand, a definition in the kindergarten should take into considerations all of the critical attributes that are derived from the mathematical definition. On the other hand, it should take into consideration psychological aspects. We created a definition that includes closure, pointed vertices, straight sides, and the number three. Children should work according to this definition.

It was agreed that in the kindergarten children would be presented with the following definition: A triangle is a closed figure with three straight lines and three pointed vertices.[1]

The above segment illustrates how the combined framework was used to focus teachers' attention on the types of knowledge being built. In our program we found that teachers were eager to implement their newly acquired knowledge in the classroom. While this is, of course, commendable, the teachers needed to sort out the difference between the mathematical knowledge needed for teaching and the pedagogical knowledge needed to convey the mathematics to their students. By making this difference explicit, teachers were first able to focus on their knowledge of concept definitions and then focus on the teaching of concept definitions.

In the following section, we describe how the combined theories tool was used to develop another aspect of kindergarten teachers' KCT related to the concept image and concept definition of triangles.

## *Building Kindergarten Teachers' KCT Regarding Concept Definitions and Concept Images of Triangles*

The formation of geometrical concepts, as with many mathematical concepts, is a complex process in which examples play an important role (Watson & Mason, 2005). Initially, the mental construct of a concept includes mostly visual images based on perceptual similarities of examples, also known as characteristic features (Smith et al., 1974). This initial discrimination may lead to only partial concept acquisition. Later on, examples serve as a basis for both perceptible and non-perceptible attributes, ultimately leading to a concept based on its defining features. Visual representations, impressions and experiences make up the initial concept image. Formal mathematical definitions are usually added at a later stage. According to the Principles and Standards for School Mathematics (NCTM, 2000), young children "need to see many examples of shapes that correspond to the same geometrical concept as well as a variety of shapes that are non-examples of the concept" (p. 98). Thus, another important aspect of KCT is knowing which examples and non-examples to present to children that will promote the development of an appropriate concept image as well as encourage children to refer to the concept definition.

[1] It is important to note that precise language was used with the teachers as well as with the children. Terms such as corners and turns were not used. As such, 'vertices' is the appropriate translation from Hebrew.

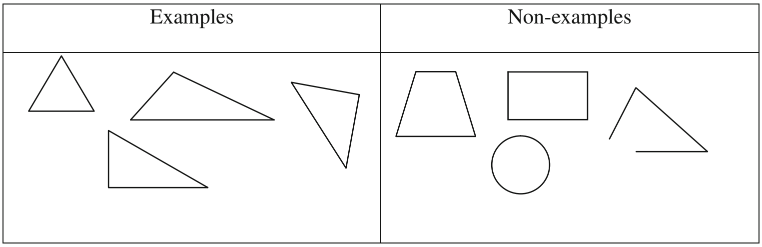



**Fig. 14.1** A sample of examples and non-examples of triangles used by teachers

In this section, we describe a segment which took place with the second group of kindergarten teachers during the fifth lesson of their course. The teachers had been instructed to assess their children's knowledge regarding the identification of examples and non-examples of triangles and are now discussing the results. It soon becomes obvious that the results were largely dependent on the choice of examples and non-examples the teachers had chosen to use for this assessment. (See Fig. 14.1 for a sample of some of these examples and non-examples.) The instructor explains:

> The results do not give us a complete picture of what the children know and what they are capable of knowing. We have found in our work with children that almost all of the children correctly identify this (pointing to an equilateral triangle with a horizontal base) as a triangle and only a third of the children will correctly identify the same triangle if it is turned upside down. The typical concept image of the triangle is this (pointing to an equilateral triangle with a horizontal base).

In other words, in order to properly assess children's knowledge, the teacher should include examples that are not necessarily part of the child's intuitive concept image.

Choosing examples that are not necessarily part of the child's concept image may also encourage the child to refer back to the concept definition (Tsamir et al., 2008). As the instructor claims, "it is important to work with many examples and non-examples . . . going over the critical attributes and at the same time creating a world of images." The teachers are then instructed to think about the figures along two dimensions: a mathematical dimension and a psycho-didactical dimension. The mathematical dimension divides the figures into examples and non-examples of triangles according to the concept definition. The psycho-didactical dimension divides the figures into what is and is not intuitively identified as examples and non-examples according to the child's current concept image.

Knowledge of how to choose appropriate examples and non-examples was evident later on during the course as teachers discussed pentagons. In order to create the examples, teachers discussed the concept definition of a pentagon:

S: I want to know if there is an exact definition for a pentagon.
R: A closed figure with five sides.
O: A five-sided polygon.
S: Ok.
I: And what about a definition for the children?

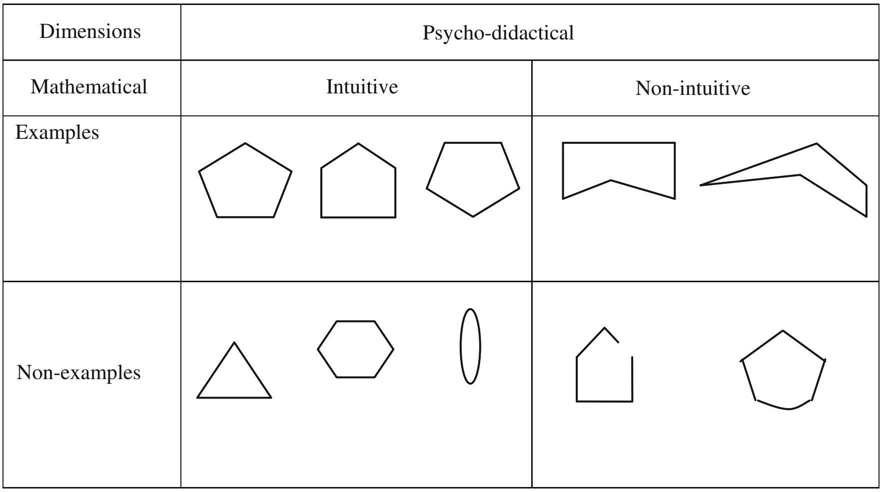



**Fig. 14.2** Teachers' suggestions of examples and non-examples of pentagons

S: For the children I would say five sides, five vertices, and closed.
O: A closed figure . . . like we did before . . . with five sides and five vertices.

Working together in groups, the teachers came up with the following suggestion of examples and non-examples to use in various activities (see Fig. 14.2).

It may be surprising that the teachers placed the upside down pentagon in the section for intuitive pentagons. After all, the teachers had previously experienced that upside down triangles are not necessarily part of the child's concept image of a triangle. However, at this point, the teachers felt that the upside-down pentagon may be considered intuitive. The children had already been presented with triangles of various orientations and could successfully identify an upside down triangle as an example of a triangle. In other words, the children's concept image of triangles had changed and the teachers were choosing examples based on the children's current concept image of geometrical figures. On the other hand, triangles cannot be concave and so concave figures, such as the concave pentagon, may not currently be part of the child's concept image. The teachers had gained knowledge of their students (KCS) and used this knowledge in their teaching (KCT).

The relationship between knowledge of students and knowledge of teaching was observed several times during the year. Towards the end of the year, the teachers discussed how to help children who still had difficulties identifying various examples and non-examples of geometrical figures. Referring to triangles, one teacher stressed the need to help children recall the concept definition. She suggested the following:

> E: First we need to strengthen the critical attributes. So, we start with the triangle they are used to (referring to the equilateral triangle) and we put it down in different directions and ask the child what has changed and what has not and to check again the critical attributes. Regarding the hostile triangles (referring to those which do not coincide with the child's concept image) I would greatly enlarge the triangle so it would be much clearer to the child and ask him again to check the critical attributes, the sides and vertices.

Notice that this teacher has identified two possible stumbling blocks for the children. The first is children's difficulty with orientation. She isolates this difficulty using the triangle most likely to coincide with children's concept image and focusing only on the changing orientation. The second is children's difficulty in identifying non-intuitive triangles. Her suggestion of enlarging the triangles directs the child to notice the straight sides and pointed vertices of the triangle. Similarly, when discussing children's difficulties in identifying concave pentagons, a different teacher suggests enlarging the figure and cutting it out so that children can feel the hidden vertex. During the next lesson, this teacher described how she carried out this suggestion and that the enlarged concave pentagon was indeed helpful.

In this segment, we see the results of explicitly introducing kindergarten teachers to the CICD theory and explicitly discussing with them the difference between the mathematical knowledge they as teachers need to know and applying this knowledge in the kindergarten. We can see a clear difference between the examples and non-examples teachers chose for triangles in the beginning of the year to those they chose for pentagons in the middle of the year. Teachers are cognizant of the need to present a suitable definition of a pentagon for their children. They are aware of the tension between the concept image and concept definition and devise activities that will enrich the children's concept image while strengthening their awareness of the concept definition. Using the combined theory as a lens, we may say that the teachers are accessing their KCS related to concept images and concept definitions in order to build their KCT related to concept images and concept definitions.

## Kindergarten Children's Knowledge of Pentagons

At the end of the year, 166 kindergarten children were presented with six figures and asked to identify examples and non-examples of pentagons (see Fig. 14.3). Of these children, 81 learned in eight kindergartens taught by teachers who participated in the program and 85 children were from six other kindergartens. All 14 kindergartens were located in the same low-socioeconomic neighbourhood. Children were interviewed individually, in a quiet corner of the kindergarten classroom.

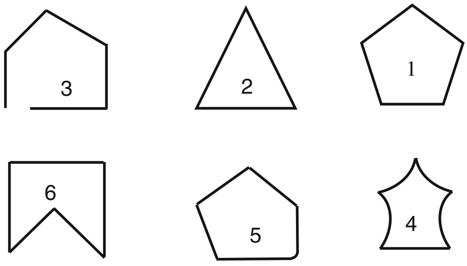
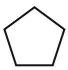
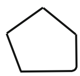



**Fig. 14.3** Figures presented to kindergarten children

Each of the six figures was printed on a separate card. After presenting each card in the same order to each child, two interview questions were asked: Is this a pentagon? Why? The first question ascertained if the child identified examples and non-examples of a pentagon. The second question allowed us to study the child's reasoning about identification of a figure.

Results indicated that more program children than non-program children correctly identified the figures as examples or non-examples of a pentagon (*see* Table 14.1). It should be noted that according to the Israeli national preschool mathematics curriculum, kindergarten children should be able to identify examples and non-examples of pentagons. Noticeably, nearly all the program children correctly identified both pentagons as examples of a pentagon. Although most non-program children correctly identified the convex pentagon as a pentagon, only 13% correctly identified the concave pentagon as a pentagon. As one child claimed, "It looks like a different figure." Perhaps, the concave pentagon did not coincide with their concept image of a pentagon. Furthermore, when asked to explain their identifications, only 25% of non-program children gave reasons which pointed to the critical attributes of a pentagon. This coincides with Clements, Swaminathan, Hannibal, and Sarama (1999) who found that young children rarely refer to the properties of a figure, relying more on a holistic, visual approach to identification. On the other hand, 85% of program children pointed to critical attributes in their reasoning. For example, one program child correctly identified the triangle as not being a pentagon and added, "it only has three vertices and three sides". This child's reasoning coincides with the third van Hiele level of geometric reasoning where the definition is meaningful and the child takes notice of critical attributes. On the other hand, a non-program child who also correctly identified the triangle as not being a pentagon claimed, "It's not like a pentagon. It's a triangle". This child first stipulates that the figure at hand does not coincide with his image of a pentagon. He then correctly claims that this figure is a triangle. Referring to the name of a shape implies that the child has taken into consideration the whole shape without regard for its components (Markman,

**Table 14.1** Frequency (in %) of correct identifications per figure

| | Figures | | | | | |
|---|---|---|---|---|---|---|
| | Examples | | Non-examples | | | |
| | 1 | 6 | 2 | 3 | 4 | 5 |
| |  |  |  |  |  |  |
| Program children | 95 | 94 | 99 | 100 | 93 | 96 |
| Non-program children | 76 | 13 | 93 | 82 | 73 | 56 |

1989). This type of reasoning coincides with the first van Hiele level of geometric reasoning.

Not all program children immediately identified the concave pentagon as a pentagon. For example, one child examined this pentagon and thought out loud, “It’s not a pentagon. Let’s check. (The child counts the vertices.) It is a pentagon because it has five sides and five vertices and it’s closed”. Perhaps, this pentagon did not yet coincide with his concept image. Yet, he was able to correct his identification by referring to the critical attributes mentioned in the concept definition. This example illustrates how focusing on promoting different aspects of teachers’ knowledge may eventually filter down to promoting student’s knowledge.

## Summing Up and Looking Ahead

In this chapter we proposed a theoretical framework which combined a theory of teachers’ knowledge with a theory of mathematical knowledge and illustrated how it may be used as a tool in promoting teachers’ mathematical knowledge for teaching. “A crucial trait of a valuable framework of teacher knowledge is the extent to which it identifies that knowledge needed for student learning and understanding” (Graeber & Tirosh, 2008, p. 124). Other tools conceptualize teachers’ knowledge based solely on the work teachers do. We add that it is equally important to frame teachers’ knowledge based on the knowledge we wish our students to gain. Viewing teachers’ mathematics knowledge through two lenses – one of teachers’ knowledge and one of mathematical knowledge – allows us to pinpoint more precisely what teachers need to know for teaching mathematics. Of course, teachers need to know which examples of triangles to present and in what order to present them (KCT). However, if we recognize that some examples will enhance students’ concept image of a triangle and others will encourage students’ use of the definition, we may accordingly develop teachers’ mathematical knowledge for teaching each of these aspects.

The theorized tool we described, combined Ball and her colleagues’ conceptualization of teachers’ knowledge for teaching with Tall and Vinner’s CICD theory in order to promote kindergarten teachers’ knowledge for teaching geometry. There are several variables in this proposal. There is the mathematical context used to illustrate the use of this tool, the grade-level at which the teachers taught, the action taken with the tool, and the theories we chose to combine. Each of these variables represents possible directions for further development and wider use of the tool.

Regarding the mathematical context, we found that for kindergarten teachers, the context of geometry provided a natural venue for discussing images and definitions. Beginning with triangles and other two-dimensional polygons, the teachers could easily discuss the figures they saw and drew and began to understand the need for concept definitions. They also came to acknowledge that not every concept definition may be adapted for the young children in kindergarten. This came up when discussing circles and the concept image and concept definition of a circle.

It was decided that for the circle, a child's concept image may currently be enough. These discussions carried on as the teachers discussed three-dimensional solids such as pyramids, spheres and cylinders. Although this chapter specifically used the context of geometry to illustrate the combined framework, we believe that the generality of the CICD theory allows it to be applied to building teacher's knowledge of additional mathematical contexts. In the kindergarten, for example, we used the combined framework for building teachers' knowledge of equivalent sets (Tirosh, Tsamir, Levenson, & Tabach, submitted). As with geometry, we used the combined theory to build teachers' SCK of the concept image of equivalent sets and differentiated between this knowledge and KCT regarding this concept image. The same was done for the concept definition. If the use of this tool is to be expanded to other preschool mathematical contexts (such as patterns and measurement), then perhaps prior research will be necessary in order to first investigate children's concept image and concept definition in these contexts.

Regarding the grade-level at which the teachers taught, this chapter illustrated promoting knowledge for teaching in kindergarten. We believe that the combined theory has potential to be used as a tool for promoting teachers' knowledge for teaching in other grades as well. In both elementary and secondary schools, studies have shown that tension exists between students' concept images and concept definitions within various mathematical contexts (Bingolbali & Monaghan, 2008; Even & Tirosh, 1995; Gray, Pinto, Pitta, & Tall, 1999; Levenson, Tsamir, & Tirosh, 2007; Schwarz & Hershkowitz, 1999; Vinner & Dreyfus, 1989). Perhaps at the high school level, teachers are more cognizant of the necessity for definitions than preschool teachers are. On the other hand, they may pay less attention to concept images. This issue will need to be addressed by perhaps placing extra emphasis on these cells during professional development.

In this chapter, we illustrated some ways in which the combined framework could be used to promote teachers' SCK and KCT related to concept images and concept definitions in the kindergarten. The next step would be to demonstrate how the combined framework may be used to promote KCS related to concept images and concept definitions. Another issue that arises from pondering the use of the tool is the degree of explicitness when presenting the tool to teachers. Upon reflection, the four dimensions of teachers' knowledge were not made as explicit to the teachers as was the concept image-concept definition theory. We believe that it is important to make both theories equally explicit to teachers. This issue is being addressed in our current courses where the four dimensions of teachers' knowledge are explicitly presented and discussed.

Choosing which theories to combine is a significant issue which needs to be addressed. Regarding our goals for professional development, it is too simplistic to say that we aimed to enhance teachers' knowledge. As the first section of this book indicates, conceptualizing mathematical knowledge for teaching is complex. Our choice of using the four domains of knowledge described by Ball and her colleagues arose mostly from our necessity to use a finer grain than provided by Shulman's (1986) often used notions of SMK and PCK. In retrospect, we found that this choice was well suited for conceptualizing the knowledge needed for teaching some mathematical concepts in kindergarten. As noted in the beginning of this paper, most kindergarten teachers have little training in teaching mathematics. As such, each of the four domains needed attention.

Tsamir (2008) recognized the complexity of choosing which mathematical knowledge theories to present to teachers from the vast offering of theories relevant to mathematics teaching. Regarding mathematics teaching, the CICD theory is a widely recognized mathematics education theory which spans students of all ages and is relevant to many different mathematical contexts (Hershkowitz, 1989; Schwarz & Hershkowitz, 1999; Tall & Vinner, 1981; Vinner & Dreyfus, 1989). It informs our understanding of mathematical concept formation. It allows us to predict and analyze students' errors. Another direction for widening the use of the tool would be to consider combining other theories of mathematical knowledge with theories of teachers' knowledge. Tsamir (2008) described how familiarizing secondary school teachers with Fischbein's (1993) theory of the three components of knowledge and Stavy and Tirosh's (1996) theory of the intuitive rules may promote secondary teachers' mathematical and pedagogical knowledge. The choice of theories may depend on the mathematical context as well as the activities or tasks which take place in the classroom. For example, parts of the intuitive rules theory are especially appropriate when engaging in comparison tasks. Combining this theory with the four dimensions of teachers' knowledge may then focus us, for example, on developing teachers' knowledge of how and when students use these rules (KCS). Another direction for addressing this issue might be to pool mathematical education theories that investigate students' mathematical learning and possible sources of errors. For example, Fischbein's (1993) theory mentioned above, the intuitive rules theory (Stavy & Tirosh, 1996), and Tall and Vinner's (1981) CICD theory all have elements of intuitive thinking. The mathematics education research community should consider how to combine these theories in order to provide a more comprehensive theory for investigating students' mathematical thinking as well as teachers' mathematics knowledge for teaching.

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# Chapter 15
# Teachers' Stories of Mathematical Subject Knowledge: Accounting for the Unexpected

**Julie Ryan and Julian Williams**

## Introduction

In this chapter, we report an innovative assessment feedback tool – we call it a 'mathsmap' – and describe how two pre-service primary school teachers in England made sense of such a personalised diagnostic map to reflect on their subject knowledge in mathematics (Ryan & Williams, 2007a, 2007b). The mathsmap provides both a summative and a diagnostic profile of attainment and errors across a test of a constructed 'primary teacher mathematics curriculum' (Ryan & McCrae, 2005, 2005/2006).

The use of the mathsmap to reflect learning at a personal level is seen to provoke 'accounts' or 'stories' that might inform pre-service teachers' pedagogical content knowledge. In making their mathsmap comprehensible to themselves, the two pre-service teachers reported here, Lorna and Charlene,[1] were provoked to account for their own knowledge 'troubles', that is, to narrate their metacognition. We were interested, in particular, in their view of themselves as mathematical learners and how this would impact on their pedagogical content knowledge and teacher identity.

We offer a method for encouraging such reflection by having pre-service teachers personally confront their *patterns* of responses as indicated on their mathsmap. This tool is different from other feedback devices in drawing attention to non-normative responses of two kinds: unexpected correct and unexpected incorrect responses. Being told that responses are not 'expected' causes dissonance, or 'trouble' *to be explained*; such troubles generate 'accounts' or stories narrated to normalise them (Bruner, 1996). This also provides the researcher or teacher educator with some insight into pre-service teacher self-knowledge, indeed their metacognitive

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This chapter draws on our earlier work reported in Ryan and Williams (2007b) extending the analysis of Lorna and Charlene's accounts of their learning – in particular in 'accounting for the unexpected' as an opportunity for learning and development of pedagogical content knowledge.

[1]Lorna and Charlene are pseudonyms.

T. Rowland, K. Ruthven (eds.), *Mathematical Knowledge in Teaching*, Mathematics Education Library 50, DOI 10.1007/978-90-481-9766-8_15,

knowledge, and perhaps their sense of self-efficacy or agency as learners. Such insights, we suggest, may also inform the design of teacher education courses.

In our earlier work on classifying the mathematical errors that children make on standardised tests, we concluded that most errors and misconceptions are the result of intelligent constructions (see for example, Ryan & Williams, 2007a). Similarly, it is such intelligent constructions that adults make that we sought to explore here with the pre-service teachers in our study, and to identify any turning points in their mathematical autobiography as they narrated or 'storied' their own learning (Bruner, 1996, pp. 144–149) around their unexpected troubling successes and errors. We think that such activity may play a significant part in the development of pedagogical content knowledge – that knowledge that Shulman (1986) referred to as including "the most useful forms of representation of . . . ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). See Chapter 2 by Petrou and Goulding, this volume, for further discussion of Shulman's work. We suggest that knowledge of one's own methods of making the subject comprehensible to oneself is a necessary first step for reflective teaching.

## Teachers' Mathematical Knowledge

In the recent political climate of international league tables and government initiatives to 'drive up' standards in primary schools, the spotlight has been on the teacher and their teaching: a model of deficiency (Sanders & Morris, 2000, p. 398) in particular tempts a quick response of measure-and-fix. Yet what is a robust and useful 'measure' and what is the 'fix'? The interplay of subject matter knowledge and 'effective' teaching is complex: strong subject knowledge is arguably a central and necessary condition for more effective mathematics teaching, but it is not sufficient. We believe that a more productive approach would be to ask what sort of subject knowledge informs more effective teaching, and how might a novice teacher take control of their identity as a mathematics learner themselves and use this positively in their teaching*?*

In England, initial teacher education providers have been required since the late 1990s to 'audit' their pre-service teachers' mathematical knowledge and to support its development – 'gaps' in knowledge are to be filled, errors and misconceptions 'fixed' and 'connections' made between key mathematical concepts. Such political imperatives are set against the reality of the background of pre-service teachers' own school experiences – what they bring with them to their education training courses.

> Even if we were confident about trainees' knowledge and understanding of mathematics, we would need to recognise that the vast majority of trainees have tended to specialise in non-mathematical subjects after the age of 16, and may need to refresh and deepen their understanding of mathematics before entering the classroom (Goulding, Rowland, & Barber, 2002, p. 690).

Of course the affective dimension – beliefs about mathematics and attitudes to its learning – is also part of the complexity of what pre-service teachers bring to their teacher training. We think that reflecting on their own identity as a mathematics learner offers the pre-service teacher an opportunity to seize the power of metacognition; knowing how one learns and how one breaks through difficulties in understanding are perhaps potentially liberating. Co-ordinating subject matter knowledge and pedagogical content knowledge on initial teaching practice is also a balancing act, and the novice is most vulnerable to expectations – mostly their own – of classroom management in the first instance.

Goulding, Rowland, and Barber (2002) examined how mathematical subject knowledge of pre-service teachers has been conceptualised and they reported on how it has been audited in three institutions in England. The items they used in their audit instruments explored both the substantive and syntactic knowledge of the trainees, that is "knowledge *of* mathematics (meanings underlying procedures) and knowledge *about* mathematics (what makes something true or reasonable in mathematics)" (p. 692).

Their study reported trainees' difficulties, errors and misconceptions and initial findings on the complex relationship between subject matter knowledge as assessed by their audits and actual teaching performance. They hypothesised that subject knowledge "would influence both students' planning and their teaching, a cognitive dimension encompassing beliefs about mathematics, and their confidence in the classroom" (p. 694). They were "persuaded that the relationship involves both cognitive and affective dimensions" but one of their dilemmas was whether audit requirements were "creating anxiety and dislike of mathematics or acting as a useful lever for development" (p. 701).

This study also drew on Ball's (1990) call to encourage pre-service teachers to 'revisit' and perhaps 'unlearn' their own school experiences of mathematics. It is such unlearning that Ball believed provoked a deeper self-awareness and articulation of beliefs about mathematics (as cited in Goulding et al., 2002, p. 692). In exploring the relationship between audited subject matter knowledge and confidence, Goulding et al. (2002) cautioned that a simple "emphasis on the audit and the remediation process [may have] had a demotivating effect" on some pre-service teachers in their study (p. 700), but not for most:

> There is stiff competition to gain a place on these courses: most students are resilient, well-motivated and goal-oriented. Indeed the majority of those who required some remediation, including some very weak students, appeared to respond positively to the opportunity and reported in evaluations that they were pleased to address some of their weaknesses before the main teaching practice (p. 700).

Sanders and Morris (2000) tested the "factual knowledge and central concepts" of their pre-service teachers but not their "understanding or knowledge of the organizing principles and ideas of mathematics" (p. 399). Their students were encouraged to take responsibility for improving their own learning but it was found that "self-directed study inevitably had a low priority", and in the first year of Sanders and Morris's study this was found not to be a satisfactory approach in improving performance (p. 400). With another cohort they also examined the effects on confidence when their pre-service teachers were expected to re-examine their own mathematical knowledge and skills. Following an initial test, remediation was provided and re-sits undertaken on a voluntary basis. The authors were disappointed that only 40% of the students 'grasped the nettle' and took advantage of the support offered. Some students were "empowered by poor test results to tackle their knowledge deficits" but others "found ways of 'excusing' poor results" as involving technical terms or non-coverage at school (p. 407). Some students focussed only on the topics they would be teaching on their upcoming assessed placement (p. 406).

Murphy (2003) suggests that pre-service teachers may not be clear about how audited subject knowledge relates to the teaching of primary mathematics. Her study sought to examine trainee teachers' perceptions of the value of an auditing process. She found that "only about half of the trainee teachers felt that their improved confidence had come from [the audit process] and only about one third of the trainee teachers saw that it had made a difference to their ability to teach primary mathematics" (p. 86). She found that one group of *less confident* pre-service teachers viewed the auditing process as 'filling in gaps' and gained confidence in their subject knowledge and their own teaching as a result. However, a second group of *confident* trainees did not see the value of the audit process and may have regarded the process as 'jumping hoops'. Murphy suggests that differing views of the audit process may reflect "differing beliefs in mathematics as a discipline" (p. 89) and hoped that a larger proportion of trainee teachers would "see the relevance of subject knowledge to their teaching of primary mathematics" (p. 90) in response to an improved content and process of audit.

Barber and Heal's (2003) study focussed on the role of social interaction and collaboration in learning and the effectiveness of peer tutoring in enhancing primary trainee mathematical subject knowledge. Peer interaction was used as a teaching strategy – providing opportunities for both the tutor *and* the tutee to 'explain' their mathematics. The pairings of high scoring and low scoring trainees were made on the basis of an initial audit. Generally low-scoring trainees had reported low levels of confidence and many of them also reported panic when required to 'do maths' and needing time to 'recall' knowledge. The peer tutors were trained in the art of explanation and reported "how enlightening it was to hear so many alternative ways of approaching each problem [on the audit] and how instructive to realise that their own perspective on the problem was not the only one" (p. 69).

The feedback from the peer tutoring sessions was positive and "pointed to the mediating influence of emotional factors" and improved confidence (Barber & Heal, 2003, p. 69). The authors suggest further development of peer tutoring with attention to the nature of 'ideal' pairing, tutor training, 'quality control' and the different needs of different 'bands' of trainees. The authors cautioned leaving trainees to organise self-study – they found that "half of those who identified themselves as having poor subject knowledge at the beginning of the course achieved the lowest scores in the formal audit" (p. 70).

Rowland, Barber, Heal, and Martyn (2001) are also wary of guided self-study as an adequate 'treatment' for poor subject knowledge. Some of the pre-service teachers in their study had difficulty in communicating what they could 'see' in mathematical situations and thus faced considerable cognitive obstacles in working alone (p. 93). Goulding (2003) reported that the SKIMA (Subject Knowledge in Mathematics) group – a collaboration of researchers in four UK universities investigating weaknesses in knowledge, self-assessment and the link between subject knowledge and teacher competence – had found that peer support groups and peer tutors "seemed to be successful in boosting the confidence of weak trainees and also that of the stronger trainees who acted as peer tutors" (p. 76).

Thus far, we have a deficit model of pre-service primary teacher subject matter knowledge – 'gaps' to be filled, errors and misconceptions to be 'fixed' and new connections to be made. Tests and audits have traditionally reconstituted the knowledge base of secondary school as the expected base of subject knowledge for teaching which then provokes different 'fixes' including notions of relearning, 'unlearning' and remediation.

Some of the research above prompts further attention to pre-service teacher awareness of their 'problems', the effects of anxiety, motivation for change and 'tools' for exploring these. We attempt to go a little further then highlighting also notions of identity and agency in the personal professional development of the pre-service teacher. In particular, we look to the pre-service teacher 'storying' their mathematical autobiography by accounting for the unexpected: exploring their learning identity and perhaps bearing fruitful pedagogical content knowledge that will be played out in their ongoing story of being a teacher of mathematics. See also Chapter 13 by Corcoran and Pepperell, this volume, for further discussion on identity and narratives shared by participants in Lesson Study.

## Testing Subject Knowledge

We now provide a brief outline of the audit 'tool' we have used with pre-service teachers who were interested in exploring their patterns of response on a written test. We too have taken the traditional route by starting with the school curriculum. The test we used, the *Teacher Education Mathematics Test* [*TEMT*] (Australian Council for Educational Research, 2004), had been developed by first constructing a 'primary teacher curriculum' using documents based on Australian and United Kingdom secondary school curricula. Similar tests can be developed using a reasonable sample of the targeted population (see Ryan & McCrae, 2005, 2005/2006 for detail of methodology). The level of attainment targeted Australia's school level 5/6 (understood to be 'functional numeracy level') – this is the equivalent of GCSE[2] grade C, the minimum mathematics requirement for entry to initial teacher education courses in England.

Test versions (each of 45 items) were constructed across the six strands of the constructed 'primary teacher curriculum' involving Number (16 items in each test), Measurement (8), Space and Shape (8), Chance and Data (6), Algebra (5), and

[2]GCSE is the *General Certificate of Secondary Education* in England which assesses children's attainment at the end of current compulsory schooling, usually at 16 years of age.

Reasoning and Proof (2). The tests were in a pen-and-paper multiple choice format and timed for a 45-min period. A Rasch analysis (Bond & Fox, 2001; Rasch, 1980; Wright & Stone, 1979) of the responses of a large sample of students was undertaken using *Quest* software (Adams & Khoo, 1996).

Rasch scaling uses one version of item-response modelling: a one-parameter stochastic model of persons' responses to items. Here responses are modelled by a probability function characterised by one parameter – the item 'difficulty'. The model "can help transform raw data from the human sciences into abstract, equal-interval scales. Equality of intervals is achieved through log transformations of raw data odds, and abstraction is accomplished through probabilistic equations" (Bond & Fox, 2001, p. 7).

The Rasch model assigns a *difficulty* parameter to each test item, estimated by its facility, and a so-called *ability* parameter to each person, estimated by their raw score on the test. These parameters are calculated as 'log-odds' units called *logits*. The logit scale is an interval scale, and the Rasch model "routinely sets at 50% the probability of success for any person on an item located at the same level on the item-person logit scale" (Bond & Fox, 2001, p. 29). That is, a person located at an 'ability' of $x$ has a 50% probability of correctly answering an item of 'difficulty' $x$, an increasing probability of answering items below that difficulty and a decreasing probability of answering items above that difficulty. The sample data builds the measure by assigning parameters to items and persons (just one each) that minimise data-model residuals.

The *Quest* program automatically calculates these item parameters (item difficulty estimates with the default mean difficulty set at zero), and person estimates (student ability estimates) and the model-fit statistics (how well items and persons fit the model) from the data. *Quest* also provides classical statistics. For further discussion of Rasch modelling and analysis see Williams and Ryan (2000).

The *TEMT* test items were scaled in terms of their difficulty and each person was located on the *same* scale in terms of their ability as measured by the test. The data were found to be compatible with the Rasch model, and test reliability and goodness of fit were strong (Ryan & McCrae, 2005, 2005/2006).

We follow the psychometric tradition and use the term 'ability' in this chapter purely in a technical sense, as a measure of the underlying construct that the test is measuring (called the latent trait in the psychometric literature). In our context, the measure is of performance or attainment on the items in this test. There is no imputation of meaning attached to the term 'ability' other than what one can construe from the face value of the items themselves.

In the construction of the multiple choice test, distracters were purposefully chosen from known or suspected errors drawn from research on children's understanding but also from research on teacher knowledge. They were used to mitigate against guessing in the multiple choice format but also, more interestingly, to provide a finer-grained detail of pre-service teachers' knowledge. Guessing/errors were not specifically penalised in the estimation of student ability.

In a second study, another cohort of pre-service teachers in England ($N = 87$) also took a *TEMT* assessment in the second year of their initial teacher training.

Their patterns of response were very similar to the larger Australian sample ($N = 426$) used to validate the test originally (Ryan & McCrae, 2005/2006). The pre-service teachers in the England sample included pre-service primary trainees, non-mathematics specialist secondary trainees and a small group of mathematics primary/secondary specialist trainees. Participation was on a voluntary basis with the promise of personalised diagnostic feedback from the test to assist their subject knowledge development.

The 87 trainees in England were given an individual map of their responses as diagnostic feedback. A questionnaire gathered information on what sense they made of their map; in addition, two pre-service teachers from this cohort volunteered to be interviewed to see what sense they made of this feedback and how they intended to address their indicated mathematical needs (Ryan & Williams, 2007b).

## Personalised Diagnostic Maps of Subject Knowledge

*Quest* software also produces an output for each individual, called a *kidmap* (here called a mathsmap), highlighting their correct *and* incorrect response patterns. The map summarises an individual's performance according to the Rasch model's expectations. All test items are scaled on a vertical axis from lowest to highest in terms of the difficulty of each item (from easy to hard). Each individual then is mapped left or right of the axis in terms of achievement of the item or not (achieved or not achieved). The overall ability score locates each student on the axis and a fit statistic ('fit') indicates how well the student fits the Rasch model. We show Lorna's mathsmap in Fig. 15.1.

In the mathsmap the 45 items of the test are located along the vertical *scale* according to their overall difficulty. It can be seen that item 33 was the easiest and item 30 the hardest. Those items that Lorna answered correctly are located on the left of the diagram, and those that she answered incorrectly appear on the right. Lorna answered item 33 correctly and item 30 incorrectly. The mathsmap also *locates* Lorna's 'ability' on the same vertical logit scale (centrally marked by 3Xs): her ability measure estimate was 0.91[3] and she answered 64.44% (29/45) of the items correctly (see the statistics in the top margin). The dotted lines around the estimate of Lorna's ability represent $\pm 1$ standard error for the estimate. Additionally Lorna's *actual* option choices (1, 2, 3, 4, 5 or 0), made for each incorrect item on the right-hand side, are indicated in parentheses; thus 30(4) indicates that Lorna incorrectly chose option 4 for the hardest item 30. This gives further diagnostic information (Ryan & McCrae, 2005/2006).

The individual would be *expected* to achieve all the items at and below their ability estimate with an increasing probability for those items further below. Lorna

[3] Lorna's 'ability' is located at 0.91 logits on the scale which indicates that she is nearly one standard deviation above the mean of the item difficulties. We can therefore compute the probability of her correctly answering an item of difficulty '$d$' as being approximately $\exp(0.91-d)/[1 + \exp(0.91-d)]$; thus, for the average item with $d = 0$, this is approximately 70% for Lorna.

```
------------------------------- PERSONAL MAP ----------------------------------
 Lorna                                                        ability:   0.91
 group:     all                                                fit:      1.14
 scale:     numeracy                                           % score:  64.44

------------ Harder Achieved ---------------------- Harder Not Achieved ----------
                                       truncated
                                         |   |   30(4)
                                         |   |
                                         |   |   23(5)
                                         |   |   36(3)
                                         |   |   40(4)
                                         |   |
                                         |   |   37(4)
                                         |   |   20(2)
                           38     8      |   |
                                 19      |   |   14(2)
..........................................
                                         |   |
                                         |   |
                                 16      |XXX|   13(4)
                                 42      |   |   39(2)
                     28    18     3      ........32(1)............................
                                         |   |   29(3)
                            7     5      |   |   11(3)
                                         |   |
                                         |   |
 45   43    34    26    15    9          |   |   10(1)
                                  4      |   |
                           44    25      |   |
                                         |   |
                                  2      |   |   41(1)
                           31     6      |   |   12(3)
                                         |   |
                                 17      |   |
                                  1      |   |
                                         |   |
                                 22      |   |   21(5)
                                         |   |
                                         |   |
                                         |   |
                                         |   |
                                         |   |
                                 24      |   |
                                         |   |
                                         |   |
                           35    27      |   |
                                         |   |
                                         |   |
                                         |   |
                                 33      |   |
------------ Easier Achieved ---------------------- Easier Not Achieved ----------
```

**Fig. 15.1** Mathsmap for Lorna

has a 50% probability of answering items *at* her ability estimate (note that item 16 is correct and item 13 is incorrect – both are located at her ability level). She would have been expected, with an increasing probability, to have correctly answered items 39, 32, 29, 11, 10 and so on, but she did not. Lorna would not have been expected to correctly answer items 38, 8 and 19 located *above* her estimate (on the left), but she *did* respond correctly to these items. In a perfect 'goodness of fit' to the Rasch model, the top left and bottom right quadrants would be empty, so items in these quadrants are particularly engaging for discussion in the first instance.

Each individual mathsmap indicates the secure and non-secure curriculum areas of a pre-service teacher: the non-secure items may indicate 'gaps' in knowledge, 'rusty' or long-forgotten knowledge or faulty conceptions. We found that discussion of them compelled a 'storying' of their mathematical knowledge and history by our two interviewees.

Bruner (1996) suggests that:

> Stories pivot on breached norms. That much is already clear. That places 'trouble' at the hub of narrative realities. Stories worth telling and worth construing are typically born in trouble. (p. 142)

Thus, the two shadowed quadrants (top left and bottom right) of the mathsmap in Fig. 15.2 list breached norms, and therefore 'trouble' to be explained, perhaps normalised, at least to be explored and brought to some narrative reality. Narrative interpretations may be idiosyncratic, but perhaps there are universals in the realities they construct (Bruner, 1996, p. 131). Bruner suggests also that 'turning points' are crucial to the narratives – "pivotal events in time when the 'new' replaces the 'old'" (p. 144), so we think that the stories of unexpected item mapping in the mathsmap may provide both the interviewee and the interviewer with insights into the subject matter knowledge and personal histories of mathematical learning that the pre-service teachers bring with them.

The pre-service teacher trainees were given their own mathsmap and guidance on how to read it. They were also given a list of the descriptors of the test items rather than the actual test items in order that the curriculum area indicated by the descriptor was targeted for study by the trainee, in a broad sense, rather than in terms

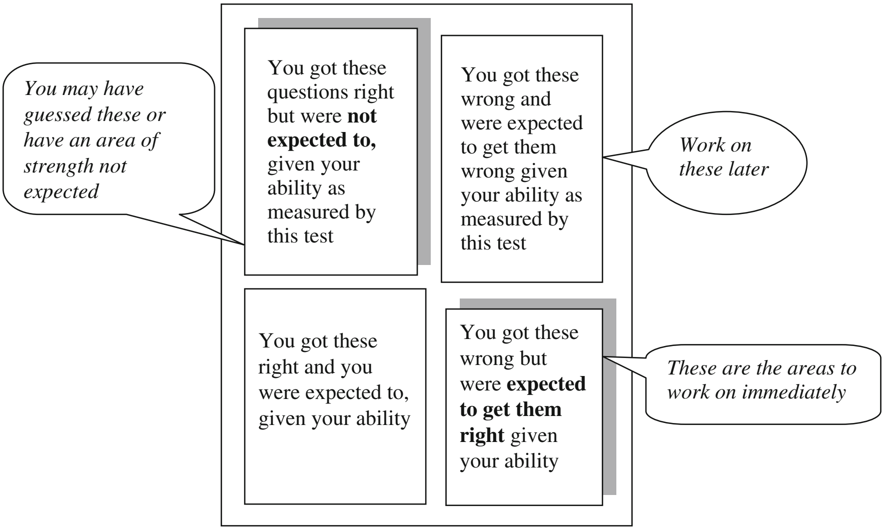


**Fig. 15.2** Interpreting the mathsmap quadrants

**Table 15.1** Descriptors for Lorna's unexpected *incorrect* items in the bottom right quadrant of her mathsmap

| Item | Curriculum description of item |
|---|---|
| 29 | Algebra: multiplying simple algebraic expressions by a number |
| 11 | Chance: likelihood/probability of everyday events (numerical) |
| 10 | Shape and Space: identifying Cartesian co-ordinates |
| 41 | Algebra: from tables of values to algebraic rule |
| 12 | Chance: recognising dependent events (reduced sample) |
| 21 | Measures: finding perimeter of a rectangle – words |

**Table 15.2** Descriptors for Lorna's unexpected *correct* items in the top left quadrant of her mathsmap

| Item | Curriculum description of item |
|---|---|
| 38 | Shape & Space: rotation of a shape about an internal point |
| 8 | Shape & Space: interpreting drawings on a grid |
| 19 | Shape & Space: finding one missing length for similar shapes |

of item-specificity. See for example Table 15.1 for Lorna's 'easier not achieved' item descriptors – she was expected to get these items correct but did not.

Lorna's unexpectedly correct items are shown in Table 15.2. They are all Shape and Space test items.

## Narrative Accounts – The Impetus of 'Troubles'

Our two interviewees had quite different mathsmap profiles. Lorna had an ability estimate of 0.91 (29 items correct, and located at the 56th percentile) and Charlene had an ability estimate of 2.00 (36 items correct, and located at the 86th percentile). Both volunteered to be interviewed on how they interpreted their mathsmap. They had quite different profiles in terms of mathematical confidence, life experience and school teaching practice.

Lorna narrated her unexpected correct responses with a story of her growth in competence and confidence in her capacity to learn. She was very animated and excited by her ability to have overcome a recent school teaching experience which had shown up her lack of knowledge in the Shape and Space area of the mathematics curriculum, and said she now felt confident about tackling her problem area of algebra as a consequence of her success. On the other hand, although (or just possibly because) Charlene was a higher scorer, her accounts for her unexpected errors told a story of 'slips', tending to marginalise explanations that might invoke her need to learn or fill knowledge gaps. She said that she often got "carried away" and made silly errors, but she also thought that she needed to improve her mental maths skills (Ryan & Williams, 2007b).

We discuss Lorna's and Charlene's interviews and point to the way 'accounting for the unexpected' in both cases impelled a story of themselves as learners or mathematicians. The resources these two pre-service teachers drew on in their story outlined and 'coloured' (or perhaps constructed) their metacognitive knowledge of learning. This leads us to propose the mathsmap as a tool for provoking pre-service teachers to 'story' their own learning and knowledge, and hence evoke cultural models of 'learning' in general.

## Lorna

Lorna was a 'mature' trainee studying on a 4-year BA Primary (Hons) education degree course, qualifying her to teach in primary schools. She was not confident about her mathematics ability and said that she had achieved a C grade in mathematics in school O-levels[4] some 20 years before.

Lorna: . . . I always think I am near the bottom ten percent (laughs).

However she had answered 64% of the test items correctly and was highly motivated to address areas of weakness in subject knowledge. She was energised by her unexpected responses.

Lorna: [The mathsmap] identified areas I thought I was weak in and some I didn't . . .. Yeah, there *were* some surprises! In both what I thought I knew and in some areas I thought I was rusty. Some areas I didn't think I was quite so wonderful on and I got them right, which surprised me. I thought, 'Oh, well not too bad at all!' 'Cos I was thinking I was sort of, virtually *way* down and had *mountains* to climb and now it shows, 'No I don't, I'm sort of in the middle with having just over half, with 64 percent.' So I've not got as much climbing to do. I thought maybe with just a few small steps and I'll be there.

Lorna was surprised that she had achieved some of the items above her ability estimate as indicated in the top left quadrant of her mathsmap (these were all Shape and Space items – see Fig. 15.1 and Table 15.2). Once this curriculum area was identified she explained her unexpected success by *recent targeting* while on teaching practice, because she already knew that this was an area of weakness – she had not guessed here.

Lorna: Well that's interesting, that! Because on my teaching practice last year with year 6, I did a unit of work in term 1 for Shape and Space and it was all about quadrilaterals and rotating shapes and the size of angles (and) symmetry. So maybe that is where that has come from, that not only I have taught them but I have learnt as well . . . So I have . . . as well as teaching children I have learned myself, so I know I have learnt more from

[4]O-level was the pre-1988 forerunner of the national GCSE examination.

what I have taught, as well as teaching at the same time . . . (Excited) so that tells me that maybe with time and practice that this area here [bottom right quadrant] will come, up . . . over [into the 'correct' quadrant].

She told a story of low confidence with a belief of "mountains yet to climb" in addressing subject knowledge. She had confronted a setback on her recent teaching practice which had highlighted lack of knowledge and this then became a pivotal moment, a 'turning point' – it is not only a story of directed self-study but one of a deep connection made with learning *as she taught*. Her teaching practice had been the motivating factor and it is clear here that she had been determined not to 'put it away' because it confronted her own professional identity as a teacher of mathematics.

Lorna: I've not got as much hard work to do as I thought I did. 'Cos I was dreading it. I tend to hide things and put them away and think if I put it away and can't see it, it doesn't matter and won't bother me but sometimes you've just got to . . . After my teaching practice what I did, I did flounder with Shape and Space, I did. I had some really bad lessons. The first lesson I did, the teacher she just said right we'll just put that to one side and I think we'll start again. And she gave me some help. And I went home and I studied and studied and studied. And it did, it shows it does help. I wasn't expected to get them right and I did.

After her unsuccessful lesson, Lorna's school mentor had given her time to study and prepare the lessons for this area again, so Lorna had collected textbooks and had used the internet to study Shape and Space extensively on her own in order to feel more confident.

Lorna: (I used the book on) subject knowledge, it's the one we have here in the library. And I went out and bought it and I just sat and read and read and read on Shape and Space . . .. I think it's by Suggate . . .. It was in the directed reading notes we were given to do every week. I went to that one because I'd done the (chapter) on algebra, because I was rusty on algebra. So I read up on algebra and found it really useful. It worked for me. The vocabulary was good for me. So I thought, right, I'll go for it and use it for Shape and Space. And obviously it did, it helped, it worked. I thought, now I know what to do and I went out and bought it.

She then referred to the items in the bottom right quadrant – 'easier not achieved' which she now felt she could be successful with using the same study strategy.

Lorna: It shows me that there are a lot of concepts there that are quite rusty because I am 39 – (that's) 20 years after [my own schooling] . . . so that tells me that maybe through teaching that I, (with) just a little bit of homework and practice, that I could move those quite easily up . . . over, to there [left]. . . . Because I do *fear* maths, I see maths as a bully. It is *my* bully.

> And this has shown me that I can overcome this, and become an effective maths teacher.

Lorna also identified algebra as one of her "rusty" areas and was becoming confident that she could move it 'over the line'. She asked to discuss an actual test item. Her discussion of item 41 (see Table 15.1) showed that she could now talk her way through the item on matching a table to an algebraic rule (see Fig. 15.3) after having done some personal study on algebra.

Which of the following tables represent the function $y = x^2 + 3$?

Table 1

| | | | | |
|---|---|---|---|---|
| $x$ | 0 | 1 | 5 | 10 |
| $y$ | 3 | 4 | 8 | 13 |

Table 2

| | | | | |
|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 5 |
| $y$ | 3 | 4 | 7 | 28 |

Table 3

| | | | | |
|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 |
| $y$ | 3 | 5 | 7 | 9 |

A. Table 1 only B. Table 2 only C. Table 3 only D. Tables 1 and 3 E. Tables 2 and 3

**Fig. 15.3** Item 41: 'Algebra: from tables of values to algebraic rule'

Lorna: Question 41. (Looking at her test script) I wrote at the side 'guessed, no idea!'

Interviewer: Do you want to talk through now what you are thinking perhaps?

Lorna: First thought, ooh, algebra! Right! So, you've got to work out – I can graph this scale, if $x$ is squared plus 3, you are going to have a plus – you're going to have it going plus 3 every time *but* it's got to be squared as well. So you're going to have to take 3 off, and then you've got to have a number that you can get a square root from. This is *after* now reading about algebra. *Before* I would have just thought, oh, well it must start with a 3. And then I've thought, no, hang on, how am I going to do this? I just didn't know. And then I thought, oh $x$, in the top row in table 1, you've got 1, then I felt, well '$x$ squared', 1 times 1 is 1 plus 3 is 4 (pointing to it) . . . And then the next number along in table 1, $x$. I've thought if $x$ is 5, I've not squared it, I've just added 3. And the next one along in table 1 is $x$ is 10, and then the answer below is 13. I've just added 3, I've just guessed, *panicked* and just gone for number 1 [option A] which was table 1.

Here Lorna constructs an account of her mistake of 'adding 3', which she had originally thought was because she "guessed". Now 'after reading about algebra', she can see "$x$ is squared plus 3 . . . you're going to have it going plus 3 every time *but* it's got to be squared as well." She reinforces this formulation of the function by inverting it and emphasising the need for a square root.

We note that in talking about her own thinking 'before', she switches tenses as in "I would just have thought" and "I just didn't know". Here she constructs her old thinking to include a squaring of the $x$, re-working the first $x$-value in Table 15.1, getting the right value of 4; but then "I've thought, if *x* is 5, I've not squared it, I've just added 3 . . . I've just added 3, I've just guessed."

This is a pivotal event where Lorna replaces the 'old' with the 'new' (Bruner, 1996. p. 144) in her story of her algebraic understanding. What began with a powerless statement, "I guessed, no idea" becomes, by the end of her story, a new guess, "I've just added 3" which we pedagogues would conceptualise as a self-diagnosis. This is an important storying of her self 'before' and 'after' her learning about algebra, and we think offers insight into her potential metacognitive learning about her own learning.

## Charlene

Charlene was a science specialist trainee on a 3-year BSc (Hons) in primary and secondary education degree course, qualifying her to teach as a generalist in England's Key Stage 2 (middle and upper primary school) or as a science specialist in Key Stage 3 (lower secondary school) and perhaps Key Stage 4 (upper secondary school). She was confident with the mathematics in the test – she had answered 80% of the items correctly and was interested in seeing where she had made mistakes. She had achieved a B grade on her AS-level[5] mathematics two years previously. She reported that her mathsmap (see Fig. 15.4) was initially a puzzle but once she had read the detailed instructions it made sense.

Charlene: When I first looked at it, I was like 'what is this!" I was looking at it thinking 'how do you read that?' But then, once I'd . . . actually looked at it properly, and then read a few of the instructions, I was like 'that's easy!', it made sense, and it seemed the best way, probably, to present the information.

Charlene confirmed that the items in the bottom right quadrant (easier, not achieved items) (see Table 15.3) made sense as items she should have answered correctly and seemed to have an understanding of the type of errors she would have made: silly mistakes rather than knowledge problems.

Charlene: I mean, they looked like the sort of things that I . . . probably would have had problems with or made a silly mistake on, like the decimal point (question 16) . . . and also probably (question) 5 because it's 'measuring, in lengths, mm, cm and metres' so that will be converting, which is easy for me to make a mistake in. . . . I just, I don't know, I just get carried away. I jump one step ahead, and it all goes pear-shaped . . . 'Cos sometimes I try and think too advanced for the questions, 'cos I did AS [A- level year 1], not very well, maths, but I do sometimes think there's more to it than what's there.

[5]AS is the first year of the Advanced level which constitutes the final 2 years (called AS and A2) of post-compulsory schooling in England.

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------------------------------ PERSONAL MAP ---------------------------------
 Charlene                                                      ability:   2.00
 group:     all                                                fit:       0.87
 scale:     numeracy                                           % score:  80.00

------------Harder Achieved ---------------------Harder Not Achieved ----------
                                              truncated
                                                |   |   30(4)
                                                |   |
.........................................   23(5)
                             36                 |   |
                                                |   |   40(4)
                                                |   |
                                                |XXX|   37(1)
                             20                 |   |
                         38   8                 |   |
                                                ........14(1)...19(2)....................
                                                |   |
                                                |   |
                                                |   |
                             13                 |   |   16(3)
                         42  39                 |   |
                    32   18   3                 |   |   28(3)
                             29                 |   |
                         11   7                 |   |    5(2)
                                                |   |
                                                |   |
  45   43   34   26   15   10   9               |   |
                              4                 |   |
                         44  25                 |   |
                                                |   |
                         41   2                 |   |
                    31   12   6                 |   |
                                                |   |
                             17                 |   |
                              1                 |   |
                                                |   |
                         22  21                 |   |
                                                |   |
                                                |   |
                                                |   |
                                                |   |
                                                |   |
                             24                 |   |
                                                |   |
                                                |   |
                         35  27                 |   |
                                                |   |
                                                |   |
                                                |   |
                             33                 |   |
                                                |   |
------------ Easier Achieved -------------------- --Easier Not Achieved ----------
```

**Fig. 15.4** Mathsmap for Charlene

Charlene suggests that she "get(s) carried away", or thinks in a "too advanced" manner rather than having missing knowledge, that may explain her errors.

She said that in converting 0.125 to a fraction on the test (item 16, 'Number: decimal to fraction conversion') she probably 'misread' one of the answer options (C: 125/100) which she had selected thinking it was 125/1,000 (not one of the options). But she also said that her mental mathematics skills needed improvement and her processing on this item showed that she was using repeated addition to find how many 125s in 1,000.

**Table 15.3** Descriptors for Charlene's unexpected *incorrect* items in the bottom right quadrant of her mathsmap

| Item | Curriculum description of item |
|---|---|
| 16 | Number: decimal to fraction conversion |
| 28 | Data: graphs – generating rules of the form $y = mx+c$ from graph points |
| 5 | Measures: ordering metric lengths stated in mm, cm, m |

Charlene: (Reading the question) '0.125 is the same as' (Pause) It's . . . not sure how to do . . . it's 1, 2, 5 over a thousand. I think I probably went for C originally. (Checks) Yes . . . Because I just must have missed out, misread one of the noughts, seeing there was an extra nought on it, because that was an automatic . . .

Interviewer: What would you go for now?

Charlene: (Long pause) I need to improve my mental maths. I can't. (Pause) I'll have to do it the long way . . .

Interviewer: What's the long way?

Charlene: (Laughs) I'm doing, how many, I'm working out the multiples of 125, to work out whereabouts (writing) a thousand . . .

Interviewer: You've got 125, 250.

Charlene: 375, 500. OK, so 4 is 500, so, 8 would be a thousand. So it's '1 over 8', which is B.

Interviewer: You've gone for B. So why do you think you went for C originally, again, can you express that?

Charlene: Because I misread the 100 as 1,000, so I just assumed it was 125 over 1,000 when it was 125 over 100. And I think even when we came out, somebody mentioned that, and I thought, oops, maybe I did pick the wrong one then.

This account matches Charlene's first explanation for her 'mistakes' as getting "carried away" or "jumping ahead" so that things go "pear-shaped". She says she "misread" and 'saw' an extra nought in the denominator of the option C fraction and processed quickly here as a one-step item. Here for item 16, her thinking does not appear to be "too advanced" or anticipate a two-step item, but rather suggests a seldom-used mental fact which took her a little time to re-construct.

One of Charlene's items located *at* her ability level (see Fig. 15.4, item 37) was answered incorrectly. The curriculum description was 'dissection and tessellation: understanding Pythagoras' theorem' and involved interpreting a classic proof by area dissection (see Fig. 15.5). It was the fifth hardest item on the test but discriminated well at the top end of the ability range. Charlene said it was an unusual question because it was asking for a proof.

An internet animation demonstrates the theorem of Pythagoras by dissection and *drag-and-drop* transformations of the shapes shown on the diagram.

What will the transformations show to demonstrate the theorem?

A. That D and C will fit exactly into E
B. That A, B, C, D and E will fit exactly into F
C. That A, B, C, D and G will fit exactly into F
D. That A, B and C will fit exactly into G

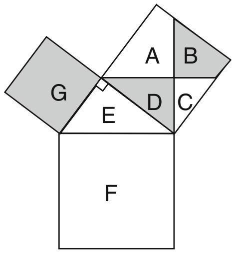


**Fig. 15.5** Pythagoras' Theorem

Charlene: (Laughs while reading the question) No, it's just, yes, what's this on about? I think it could just be the question itself as well, (if) you've not really experienced that sort of thing . . . It's something that's got to prove Pythagoras' theorem and that . . . Is that '*a* squared plus *b* squared equals *c* squared'? Is that Pythagoras?

Interviewer: Is it?

Charlene: (Pause) I don't . . ., or is it *sohcah* . . . No, *sohcahtoa* is different. It is '*a* squared plus *b* squared equals *c* squared'. (Pauses)

Interviewer: What would that mean in relation to this picture?

Charlene: (Pauses and laughs) I haven't got a clue! (Pauses) I don't know what it means, the diagram . . . '*a* squared plus *b* squared equals . . .'

Interviewer: What does that mean?

Charlene: It means the length of the two short sides, both squared, and added together, is the same as the length of the longest side, the hypotenuse, squared . . . (pauses)

Charlene juggled good-naturedly with the item here saying "what's this on about?" and recognised that 'previous experience' of something like this would help as it was an unexpected type of test question. She 'knew' the Pythagorean theorem but appeared not to have a geometrical image of it and did not make any connection with 'square' shapes in this or further discussion – this is not surprising of course if the theorem is simply represented as a numerical/algebraic formula without visualisation. But the point is that she does not consider this as an instance of a missing conception of 'square'.

## Comparison, Contrast and Limitations

Lorna and Charlene had very different mathematical backgrounds, levels of confidence and motivation to improve their subject knowledge. As a mature student, Lorna was highly motivated and aware of her "rusty" knowledge and particular areas of weakness. She had in fact underestimated her mathematical ability as measured on this test, which was above average for her cohort, whereas she had thought she had "mountains to climb". As a result of uncomfortable exposure of poor subject knowledge on her own school teaching practice, she had already targeted Shape and Space for study and was very pleased that her mathsmap indicated that she had achieved beyond her current expected ability level. It appeared she was also very motivated by her school mentor who had given her the opportunity to "start again". She was very independent and willing to put in a lot of extra time – she commented that the younger students wanted it all done for them. Lorna had targeted algebra from her mathsmap for personal study already and demonstrated in discussion that her confidence in articulating algebraic structure was growing. She seemed to be very positive about the sort of feedback the mathsmap gave her and considered her subject knowledge as a 'work in progress'.

Charlene was a high achieving science student who had recently completed AS-level. She was very confident about her mathematics ability and had quickly made sense of her mathsmap. She did not identify any areas of subject knowledge weakness, generally explaining most of her errors as simple processing errors due to her tendency to rush or to anticipate questions as more complex than they were. This seemed to be generally the case from discussion of her errors, though she exhibited some fundamental scale misconceptions related to linear graphing (in item 28, for example) with prototypical misreading of the scale. She did not appear to be alert to multi-step questions though she could identify them in discussion afterwards. Charlene did note that her mental mathematics skills needed further work, but predominantly diagnosed her errors as 'slips', and her narrative leaves little space for knowledge gaps or misconceptions. However she said she would prefer to have the actual test questions back to review to see whether she had just made a silly mistake or whether she did not actually understand something.

In both cases, the limitations of the mathsmap as a tool become apparent. Firstly, it was fortunate that Lorna was able to identify Shape and Space as an area of strength but it is not particularly well-designed to profile topic strength as it is a short, item-focussed diagnostic tool. Secondly, Charlene being a high-scorer receives less diagnostic feedback than Lorna. In a computer adaptive test format where items target the ability, Charlene would be presented with more challenging test items and would thus receive more diagnostic feedback from her mathsmap. Finally, for the same reason (that the items are generally distant from the ability), we might expect a particularly weak student to get less value out of the mapping tool as currently designed.

Subject knowledge is one component – but an important one – in building mathematical knowledge in teaching. We have shown here how two pre-service teachers made use of one subject knowledge audit tool to narrate their metacognition. We think that such opportunities for personal narration may provoke agency and provide a basis for further development of pedagogical knowledge.

## Conclusion and Discussion

In previous work we and others have shown how teacher errors can provide opportunities for pre-service teachers to examine the basis for their own understandings, as well as identifying areas for attention by teacher educators (for example, Rowland et al., 2001; Ryan & Williams, 2007b). We have offered here one method for encouraging teacher reflection by having pre-service teachers personally explore their responses, errors and misconceptions with a mathsmap. We are aware, however, that with such a focus, the deficit model can be predominant. The mathsmap is different from other feedback devices in drawing attention to non-normative responses of the two kinds. The unexpected correct and incorrect responses can be productively cast as 'trouble' to be explained, thus compelling stories to account for them (Bruner, 1996). Such accounts – it seems to us – provide opportunities to explore students' metacognitive knowledge, and even the sense of agency in the students' own learning.

Thus, Lorna narrated her unexpectedly correct responses with a story of her growth in competence and confidence in her capacity to learn. It is difficult not to interpret this as a very positive indicator. Charlene was a higher scorer and she narrated her unexpected errors with a story of 'slips' rather than considering a need to fill gaps in her subject knowledge.

We do not want to over-interpret these two limited cases, but rather point to the way 'accounting for the unexpected' in both cases impelled a story of themselves as learners or mathematicians. The resources they used – for example, whether they invoked 'misconceptions' or not – reflected their metacognitive knowledge of learning and hence tapped their pedagogical content knowledge. Interestingly, other work asking primary teachers to account for the unexpected errors of their children (as produced on the children's mathsmaps) have similarly provoked accounts from their teachers, which draw on explanations such as 'slips' or 'we've done a lot of that recently' (Petridou & Williams, 2007). This leads us to propose the mathsmap can be a tool for provoking students to 'story' their own learning and knowledge, and hence becomes a diagnostic of the cultural models available to them for narrating stories of 'learning' in general, which we argue is an important component of pedagogical knowledge and might be critical in the formation of professional identity.

Bruner (1996) suggests that cultural norms are constructed through canonical stories. One cultural norm identified above draws on the notion of learning as 'filling gaps' in knowledge – a norm that some have argued is dangerously reminiscent of the 'empty vessel' notion. Yet it 'works' for Lorna because she is able to see how her own efforts have 'filled the gap', and so the story reinforces her identity as an agentive, active self-improver. Perhaps we can identify other models in the data, or in others' stories of learning. If not, we suspect, these students will enter teaching with a very limited repertoire of models for learning and hence for their role in teaching.

Reconceptualising the stories in the literature about trainees' learning as 'canons' may make other narrative options available. For example, when students tell of the importance of multiple methods in their own learning of problem solving, might this be part of another canonical narrative of teaching, one which is more connectionist, and one which negates traditional teaching of procedures? If this connection can be firmly established, then we will understand why it is so important for teachers to experience such problem solving themselves as learners, and how these experiences can provide the resources for professional development of the connectionist teacher.

Of course, such stories need to be evaluated and sifted. The cultural model of 'magical influences' which is common in many cultures – including our own – whereby errors are 'just slips' may not be a helpful model with which to narrate pedagogy. But a critical approach to identity formation would ask that even these intuitive stories need to be written and examined by educational criteria.

The research issue which might arise is how teachers' identities, dispositions towards, and knowledge of, learning and teaching may benefit from such reflections on their own experiences of learning. We argue that the task is to study narratives of learner identities, and even *professional identities*, and narratives of learners-becoming-teachers, and to understand the critical events that mediate this long term professional identity work.

Chapter 10 by Williams (this volume) argues that Shulman's 'propositional knowledge' arises from scientific reflection on practice and is largely mediated by academic, formal conceptual language, but that 'case knowledge' is embedded in practical teaching situations and is typically mediated by conceptual language from the classroom and staffroom. By extension from this we can argue that 'case knowledge' for teaching can be generated from learning experiences too, and that this is the most obvious and appropriate source of this for pre-service teachers. It is pertinent and compelling to note that 'case study' knowledge is traditionally told through narrative accounts, sometimes even biographies, and is as close to a 'story telling' genre of research reportage as one finds in social and educational research.

We conclude with some research questions that should help frame the next steps in this line of research:

- What are the canonical stories of learning and teaching we want our students/teachers to be able to tell?
- What experiences do our pre-service teachers need to reflect on to generate this cultural knowledge?
- There may be many ways of narrating learning and teaching. How might professional identities result from teachers positioning themselves in relation to these canons? By what criteria might we best evaluate the canonical stories of learning and teaching?

These questions reformulate old questions in the conceptual framework of cultural narrative – old wine in new bottles. But it might be helpful to think of teacher education in this way. Stories, narratives, parables, folk tales and the like have been the chief means by which cultural knowledge has been shared for many thousands of years and arguably continues to be so; how better can we explain that the ancient common sense views of pedagogy continue to thrive in schools despite the volumes of book-knowledge 'available' to educators?

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# Chapter 16
# Building Mathematical Knowledge in Teaching by Means of Theorised Tools

**José Carrillo**

## Introduction

The literature on mathematical knowledge in/for teaching demonstrates a concern in the mathematics education community to deepen understanding of what knowledge a teacher has, or should have, for the teaching of mathematics, and how this knowledge is acquired or constructed. Recent examples include the work of Ball and her colleagues (e.g., Ball, Thames, & Phelps, 2008), contributions to the Research Forum on teacher knowledge[1] at the 2009 Conference of the International Group for the Psychology of Mathematics Education, and various other papers presented there (e.g., Charalambous, 2009; Gilbert & Gilbert, 2009; Klymchuk & Thomas, 2009; Turner, 2009). Other significant works include volume 1 of *The International Handbook of Mathematics Teacher Education* (Sullivan & Wood, 2008[2]); and the chapters by Llinares and Krainer (2006), and Ponte and Chapman (2006), in the *Handbook of Research on the Psychology of Mathematics Education* (Gutiérrez & Boero, 2006). But in order to make progress from the present state of affairs, this community concern must be translated into efforts to provide the necessary tools to further the understanding it aims to achieve.[3]

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[1]Research forum1: Teacher knowledge and teaching: Considering a complex relationship through three different perspectives. Several authors in Tzekaki, Klardrimidou, & Sakonidis (2009).

[2]The introduction by Sullivan is titled "Knowledge for Teaching Mathematics: An Introduction". Section 1 is headed "Mathematics Discipline Knowledge for Teaching".

[3]The application of these tools is typically realised through activities or tasks promoting learning on the part of the teacher. These tasks are not the object of analysis in this section. A repertoire of tasks oriented towards teacher training can be found in Clarke, Grevholm, & Millman (2008), including Section A, "Tasks as a Tool for Exploring the Cyclical Nature of Learning and Developing Reflection in the Teaching of Mathematics", which considers the importance of reflection in mathematics teaching (to which I shall return), Section B, "Tasks as a Tool for Developing Mathematical Knowledge for Teaching", and Section C, "Tasks as a Tool for Developing Knowledge through and for Practice", tackling the importance of starting from the

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The chapters comprising this section offer a variety of such tools, with their corresponding theoretical frameworks and research methods, while showing a high degree of homogeneity with regard to the objective of building mathematical knowledge. In drawing out the similarities and differences, we can differentiate the tools or the studies presented in the foregoing chapters in terms of their *scope*, and in particular the contexts and the limitations involved. For example, a study might be focused on the primary system, but the theoretical framework, or the research tools employed in the study, could be equally applicable to teachers of other age groups. The participants in the study include both trainee and practising teachers, and range across all levels from kindergarten to secondary. Likewise the tools employed are either *generic* or *specific* to the study of mathematical knowledge in teaching (or to the professional knowledge of a mathematics teacher). The teacher's learning environment or the context of the study can be *individual* or *collective*, and the latter may, or may not, feature the role of *knowledgeable other*. To this extent, it is important to analyse what role is given to teacher *reflection* in these chapters. We might also think closely about the *potential of a tool to promote professional knowledge and development*, in addition to confirming its *analytical virtues for research*. Finally, it is important to make evident the *theoretical framework involved,* and the extent to which the *approach and tools adopted are complementary*.

## Theorised Tools from Teachers' Knowledge: KQ, SMK & PCK, MKT

The theoretical foundations underlying the chapters by Turner and Rowland, Tirosh, Tsamir and Levenson, and Corcoran and Pepperell derive from the theories concerning the *Knowledge Quartet* (Rowland, Huckstep, & Thwaites, 2005), *Subject-Matter Knowledge* and *Pedagogical Content Knowledge* (Shulman, 1986, 1987), and *Mathematical Knowledge for Teaching* (Ball et al., 2008). All of these draw on Shulman's seminal work (also cited in the Chapter 15 by Ryan and Williams), incorporating adaptations appropriate to the domain of mathematics teaching (see Section 1).

Analysis of teaching often focuses on management issues, at both primary and secondary levels, in the contexts of both trainee and practising teachers. The Knowledge Quartet offers a tool for overcoming such a limited analysis, and foregrounds mathematical content knowledge within the study of mathematics teaching.

Turner and Rowland make the case for the need to increase teachers' mathematical content knowledge as limitations in this area bear close relationship to poor student achievement. To a certain extent, researchers into teachers' professional knowledge, and in particular the authors of the other chapters, assume the

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practice of teaching in order to construct teachers' knowledge, including a chapter relating tasks in groups of practising teachers with tasks in initial teacher training (Carrillo & Climent, 2008).

need to develop teachers' knowledge as a means of improving student learning. What Turner and Rowland bring to the debate, is the notion that this correlation holds not in respect of the teachers' formal mathematical qualifications, but in respect of the classroom practice. As a result, one of the distinctive features of the Knowledge Quartet is the consideration of teachers' mathematical content knowledge *in teaching*.

It is knowledge in action, which has greatest impact on student learning outcomes, and hence, it is the knowledge that can be observed in the actual practice of teaching, which forms the starting point for reflection, and the end point in the improvement process. This approach to the development of elementary mathematics teaching is firmly classroom-based, and shuns purely theoretical findings deriving from scrutiny of the research literature and handing down prescriptions as to the ideal characteristics and knowledge the teacher should possess/acquire. The focus on knowledge *in teaching* is of such significance that Turner and Rowland adopt the central argument of Mason and Spence (1999) in which "*knowing-about* mathematics and mathematics teaching is only realised as *knowing-to* in the act of teaching".

Pursuing this perspective, Rowland et al. (2005) take a grounded approach to the data with the aim of generating a theory capable of capturing the dimensions of teachers' mathematical content knowledge, not in terms of what might be desirable, but with the intention of developing a framework for reflection on teaching and teacher knowledge, so as to develop both.

Note that the development of the KQ and Ball's scheme for mathematical content knowledge, in the context of a teacher training programme, can be conceptualised in terms of the notions of 'theoretical loop' and 'practical loop' (Skott, 2005). Skott warns us against any kind of idea of linearity between theory and practice in which the teacher is considered a mere implementer of theoretical constructs (Skott, 2008). In training contexts such as those described here, practice is both the source of problems and the space for implementing possible solutions and approaches. A clear distinction is made between the researcher's interests and those of the teacher, in terms of, on the one hand, the theoretical loop (with the emphasis on theory drawing on practice and thence returning to theory, undertaken by the researcher), and on the other, the practical loop (with the emphasis on practice illuminated by theory, undertaken by the teacher).

Similarly, Ball and her colleagues' work on mathematical knowledge for teaching also starts from analysis of teaching. It is through analysis of the practice of teaching that these researchers gain insight into the nature and dimensions of mathematical knowledge for teaching, adapting Shulman's framework for teachers' knowledge concerning subject-matter knowledge and pedagogical content knowledge to the area of mathematics teaching.

In their chapter, Tirosh et al. adopt the domains proposed by Ball and her colleagues for mathematical content knowledge for teaching: common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS) and knowledge of content and teaching (KCT). In their study, these authors use the most developed part of Ball's well-known framework on the domains of mathematical knowledge for teaching, in which CCK, SCK and horizon knowledge (HK) are situated within SMK, and KCT, KCS and knowledge of content and curriculum (KCC) within PCK. Ball and colleagues themselves state that the scheme is still a work in progress and that HK and KCC have yet to be developed with greater precision and width (Ball et al., 2008).

Both theoretical frameworks, those of mathematical knowledge for teaching (MKT, Ball and colleagues) and the Knowledge Quartet (KQ, Rowland and colleagues) represent developments of Shulman's frame. Could they perhaps share something else in common which might help us explore the complementariness of the approaches in the chapters by Turner and Rowland, and by Corcoran and Pepperell on the one hand, and Tirosh, Tsamir and Levenson on the other? And, what of the differences?

As Turner and Rowland illustrate, the KQ (Foundation, Transformation, Connection, Contingency) is a useful tool for reflecting on the ways in which content knowledge is mobilised in the classroom, and indeed, as mentioned above, it is the classroom which becomes the most typical context of its application. This is not to say that lesson observation is the only means that the KQ recognises for bringing mathematical content knowledge into focus. Other appropriate sources include video recordings and written reports of lessons, group or individual interviews, and seminar sessions involving teachers, mentors and researchers; the essential point is that analysis begins with the classroom. It is by basing itself on the practice of teaching that reflection, configured along the dimensions of the KQ, can contribute to enhancing mathematical content knowledge.

For their part, Tirosh et al. present an analysis of the MKT displayed by teachers participating in a professional development programme, drawing their data from examples supplied by the teachers at various points during the course. Although in this respect there is no analysis of classroom practice itself, they do make reference to how to present activities to children. Furthermore, they make a comparison between the results of pupils whose teachers participated in the course and those who did not.

Despite the differences in the backgrounds to the two analyses, it is clear that the study by Tirosh et al. could be complemented by a consideration of practice in the same way that Turner and Rowland's work might be enriched by the incorporation of concrete examples and how these might translate into activities for the pupils.

Ryan and Williams report on their study into teachers' reflection about their results in a diagnostic assessment of mathematics SMK. Their interest lies in the promotion of pedagogical content knowledge and, in particular, how to mediate the subject for one's pupils. They suggest that the first step to becoming a reflective teacher is to understand the processes involved in one's own comprehension of the subject. In this way, the researchers relate the act of making knowledge comprehensible to the pupils to that of knowing oneself as a mathematician. To achieve such metacognitive knowledge demands the development of a reflective capacity and this, in turn, should extend to the classroom (reflective teaching).

The first dimension of the KQ, Foundation, comprises the teacher's theoretical knowledge and beliefs. The authors consider that the other three dimensions stem from this, as it is the configuration of their accumulated propositional knowledge which guides how they set about the task of teaching. One part of this background is mathematical knowledge, which Ryan and Williams explore. They concur with Turner and Rowland regarding the influence of this dimension on the development of pedagogical content knowledge, although their study into this domain of the KQ is not grounded in practice. Nevertheless, teaching remains their ultimate objective as their work is guided by the following questions: "What sort of subject knowledge informs more effective teaching . . . how does a novice teacher take control of their identity as a mathematics learner themselves and use this productively in their teaching?"

Foundation concerns the knowledge and conceptions that are typically acquired during initial (and continuing)[4] training. The other three dimensions, however, focus on knowledge-in-action at both the planning stage and in terms of classroom performance.

There is a parallelism between the way the dimensions Transformation, Connection and Contingency interface with that of Foundation, and the focus of the study by Tirosh et al., in which the teacher's mathematical knowledge is analysed and promoted in relation to the knowledge that their students are required to learn and the corresponding approach to its acquisition.

While Foundation is related to Shulman's (1987) comprehension, the second dimension, Transformation, refers to the teacher's ability to reconfigure their knowledge in such a way as to give pupils access to it (Ball, 1988; Shulman, 1987). The third dimension, Connection, refers to the knowledge which enables connections to be made between procedures and concepts, anticipate complexities, decide learning sequences and recognise the cognitive demands of concepts and procedures. The final dimension, Contingency, refers to the decisions and responses by the teacher to unexpected events and answers on the part of the pupils.

The Knowledge Quartet was originally developed as a framework for reflection on mathematics content knowledge, as evidenced by actual classroom practice. Its scope was later broadened to develop teachers' SMK and PCK (e.g. Turner, 2008). It was through reflection on their own teaching, guided by the KQ, that participating teachers in this latter study illustrated the utility of the KQ and the development of its conceptions of mathematics teaching and content knowledge, especially PCK, but also SMK.

Corcoran and Pepperell also make use of the KQ in their work and, following Tall (2008), do not aim to provide models of ideal lessons to follow, but rather to involve teachers in a process of deepening understanding of how mathematics is learned. This might include teaching standard mathematical methods, such as techniques for calculation, and also non-routine problems, but the focus is always on the pupils, their anticipated responses and potential obstacles, two aspects at the heart of KCT and KCS.

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[4] Note that the KQ was originally developed in the context of intial primary teacher training, although it has subsequently been applied to in-service contexts (Turner, 2008).

This relationship between the themes under analysis in the chapter by Corcoran and Pepperell, and the dimensions of MKT is clear and to be expected in any study into learning to teach mathematics. Furthermore, the authors provide a new perspective on implementing the KQ. They note how participants felt somewhat constrained by a perceived linearity of the four dimensions, and so resequenced the components in a way that was more consistent with their experience of Lesson Study. Hence, taking the pupils' learning as their starting point, they first gave prominence to Contingency. This naturally led on to the Connection dimension as they reflected on particular cases, and ultimately to the Transformation and Foundation dimensions. The point is not to advocate one particular sequence over another, but to illustrate the legitimacy of reordering of the KQ dimensions in order to focus on and improve MKT.

Turner and Rowland highlight how the use of the KQ can challenge teachers' conceptions and content knowledge. For them, the usefulness of the KQ lies precisely in the centrality it confers on mathematical content knowledge, beyond issues of classroom management. The authors concede that the participating teachers would most likely have developed their practice in one form or another without their intervention. For them, the value of the KQ lay in the focus it directed on mathematical content knowledge, bringing to teachers' attention questions concerning this knowledge which they were then able to address. And these aspects of mathematical content knowledge can be the object of reflection based on the dimensions of MKT: CCK, SCK, KCT, KCS.

In summary, the chapters considered so far have presented, in respect of the theoretical framework of teacher knowledge, the Knowledge Quartet and the adaptation of SMK and PCK in terms of MKT, with its dimensions of CCK, SCK, KCT and KCS, along with the 'delinearisation' of the KQ dimensions suggested by Corcoran and Pepperell.

## The CICD – A Theorised Tool from Mathematics Knowledge

Tirosh et al. propose combining theories of teacher knowledge with theories of mathematics knowledge so as to promote the construction of teachers' mathematical knowledge for teaching. As mentioned above, the authors employ the domains proposed by Ball and her colleagues for mathematical content knowledge for teaching. In addition, they also incorporate Tall and Vinner's (1981) notions of concept image and concept definition (CICD), paying special attention to the relationship between formal and intuitive knowledge. Formal knowledge has been mediated by an authority, whereas intuitive knowledge is immediate and usually experiential, and may not always be consistent with the logical reasoning required for comprehending mathematical concepts.

They claim that this combined theory is useful for teachers and teacher educators, enabling one to focus on mathematical knowledge and its implementation in teaching. On the one hand, the framework allows teacher educators to give specific attention to areas of knowledge, and on the other, it helps to focus teachers' attention on the particular knowledge they are attempting to build in their classrooms. The unique feature of this study, then, is this – the implementation of a theory of mathematics knowledge which guides reflection on teachers' mathematical knowledge.

In the context of an in-service training programme, the authors relate the construction of kindergarten teachers' SCK regarding concept images and concept definitions of triangles. The instructor used the combined framework as a means of probing the teachers' current knowledge and directing reflection upon it.

The chapter also considers the differences between SCK and KCT and likewise between KCT and KCS. For example, a formal concept definition is one thing, corresponding to SCK, but its adaptation to the age and level of the pupils is quite another, and in this case corresponds to KCT. The authors show how their framework helped in alerting teachers to the types of knowledge they needed to be able deploy in the classroom. The relevant mathematical knowledge was necessary, but not sufficient; the pedagogical knowledge of how to make this comprehensible to their pupils was also required. Once this difference was made explicit, teachers could first clarify their own understanding of concept definitions, and then consider the task of successfully conveying them.

Participating teachers were thus provided with a theoretical tool for focusing on their mathematical knowledge in relation to the specific topic through which their MKT[5] was being developed. "A crucial trait of a valuable framework of teacher knowledge is the extent to which it identifies that knowledge needed for student learning and understanding" (Graeber & Tirosh, 2008, p. 124). In addition to reflecting on what the teacher must know, it is also important to reflect on what the pupils must achieve. The double viewpoint of this study, mathematics knowledge and teacher's knowledge, enabled the authors to pinpoint what types of knowledge are required by teachers to teach mathematics.

The chapter by Ryan and Williams also analyses the mathematics knowledge of the participating teachers. Unlike the study by Tirosh et al., the authors do not approach mathematics knowledge in relation to what the pupils are required to learn, or the design of teaching activities. In Ball's terms, we are not talking here of SCK, but CCK.

It was suggested above that the combined scheme is useful for both instructor and teacher educators. What is the role of these latter? What is the role of the 'knowledgeable other'?

[5]Other theoretical tools can be employed according to the focus of the analysis of mathematical knowledge, such as: van Hiele levels for geometrical concept formation (Gutiérrez & Jaime, 1998; Van Hiele, 1986), the APOS theory (Dubinsky, 1994), and Sfard's stages in the process of acquisition of mathematical notions (Sfard, 1991). Although not taking a combinatory approach to theory, as in the chapter by Tirosh et al., see the essay by Carrillo, Climent, Contreras, and Muñoz (2007) and Muñoz-Catalán, Carrillo, and Climent (2009) on applying Sfard's work to the professional development of mathematics teachers.

## The Role of the Researcher/Instructor/Teacher Educator in Building Mathematical Knowledge in Teaching

For Turner and Rowland, a key role of the teacher educator/researcher is to provide beginning teachers with the necessary theoretical tools (the KQ) to be able to reflect on their teaching. The starting point for such reflection in their study was the viewing of video recordings of lessons. The researcher guided the joint reflection in group meetings, conducted individual interviews, and promoted the participants' reflective written accounts. In addition, drawing on data from observations, interviews and reports, the researcher attempted to focus on the mathematical knowledge in teaching displayed by the participants and to consider the theoretical implications in order to consolidate the KQ. For their part, the participating teachers were interested in improving their mathematical knowledge in teaching, again underlining the difference between the theoretical loop and the practical loop (Skott, 2005) noted above. On the other hand, in Tirosh et al., the instructors explicitly introduced kindergarten teachers to CICD theory, requiring them to apply CICD theory to examples from their classes and to distinguish between the mathematical knowledge required of them as teachers and the pedagogical knowledge required to convey it to their pupils in the kindergarten. In contrast, Turner and Rowland do not introduce the MKT frameworks with the intention that teachers apply them for themselves, but rather as a theoretical tool for the instructors to evaluate the teachers' knowledge.

Nevertheless, their chapter does give a central role to reflection:

> The Knowledge Quartet . . . provides a means of reflecting on teaching and teacher knowledge, with a view to developing both
>
> The study was based on a model of teacher professional development through reflection both *in* and *on* teaching action (Schön, 1983)
>
> The Knowledge Quartet was used to focus the teachers' reflections on the mathematics content knowledge realised in their teaching.

This selection of excerpts illustrates the usefulness of the KQ in focusing teachers' reflection on mathematical content knowledge and its potential for promoting professional development, with the researcher acting as guide in this reflective process.

In the chapter by Tirosh et al., the word *reflection* appears just once (as a disjunct in the expression 'Upon reflection'). Nevertheless, the tasks and instructions given to the teachers illustrate the aim of the instructors not so much to evaluate the teachers' knowledge, as to explore it through reflective situations:

> Look at the definitions (now written on the board) and try to think which are correct and which are incorrect . . . if there are definitions which are unacceptable, explain why

In the context of the research presented by Tirosh et al., the instructors take a more 'direct' or 'closed' role than the researcher in Turner and Rowland's study: "Although the instructor's approach may be considered quite direct, it became the norm with these kindergarten teachers that the instructor gave the closing argument of each discussion."

Reflection is also a central notion in the chapter by Corcoran and Pepperell: "The Lesson Study approach is built on the collective development of teaching effectiveness through collaborative work and reflection on practice . . . bringing together subject and pedagogy in reflecting on and refining practice."

Corcoran and Pepperell employ the KQ theoretical framework and present an experience of learning mathematics based on Lesson Study. The chapter concerns the use of Japanese Lesson Study (JLS) to improve mathematical knowledge for teaching in pre-service teacher education. It starts from the idea that JLS allows teachers to focus their attention on knowledge *for*, and *in*, mathematics teaching. However, for this to happen the participating teachers are required to develop a critical approach to their own practice and to take responsibility for their mathematics learning (Fernandez, Cannon, & Chokshi, 2003).

The Lesson Study cycle goes beyond critical consideration of the pupils' responses during an observation. It is characterised (Chapter 13 by Corcoran & Pepperell, this volume) by two processes: "*kyozai kenkyu*, – a process in which teachers collaboratively investigate all aspects of the content to be taught and instructional materials available – and *jyugyo kentuikai* – the post-lesson review session (Takahashi, Watanabe, Yoshida, & Wang-Iverson, 2005)". In preparing these lessons, prominence was given to the specific characteristics of the school where they were to be imparted. The participating teachers set aside time to clarify the mathematics involved in planning the lessons, relying on their own knowledge, published resources and research documents, and drawing on the expertise of 'knowledgeable others' such as university teachers.

The role of 'knowledgeable other' is of great importance in the research of Corcoran and Pepperell and bears similarities with the researcher, instructor or teacher educator in the other chapters. As the name suggests, a 'knowledgeable other' is a person in possession of a greater degree of expertise than the teachers participating in the research project, who is able to stimulate discussion and reflection on mathematics knowledge and the participants' classroom practice. In short, he or she represents a vital resource in the process, although they do not necessarily participate in each stage of the Lesson Study cycle.

The study stems from the conjecture made by Lewis, Perry, and Murata (2006), by which "Lesson study strengthens three pathways to instructional improvement: teachers' knowledge, teachers' commitment and community, and learning resources" (p. 5). Corcoran and Pepperell place their research within in a theory of social practice, employing the notion of *legitimate peripheral participation* (Lave & Wenger, 1991), and considering the group formed by the trainee teacher participants and one of the researchers a *community of practice* (Wenger, 1998). As such, the collective building of knowledge, and the role of the knowledgeable other as catalyst within the group are especially important. Chapter 13 by Corcoran and Pepperell (this volume) write: "Each member of the elective group was involved in planning, teaching, analysing and revising mathematics lessons that would promote children's mathematical reasoning . . . Fernandez's work (2005) underlines the need for a knowledgeable other to act as catalyst and to properly challenge accustomed ways of working ". The chief benefit of participation in a Lesson Study community was seen to be that in the process of learning about mathematics teaching, members also developed their mathematical knowledge for teaching at the same time. The knowledgeable other helped participants to adopt a “researcher perspective”, akin to the role of the Knowledge Quartet as a theoretical tool for promoting teachers’ reflection on (non-management) aspects of their practice.

In Ryan and Williams, a central role is taken by the teacher educator, who gathers information on the mathematical knowledge of the pre-service teachers participating in the study via a diagnostic assessment instrument along with the trainees’ reflections upon feedback about their responses to the test items. The role of teacher educator, as they see it, is to supply the tools and create the opportunities for pre-service teachers to reflect on their mistakes and set about bridging the perceived gaps.

The other three chapters report on studies in which participating teachers are offered tools for analysing, reflecting on and improving their practice, in consonance with Sullivan’s (2009) perception of the need for “offering prospective and practising teachers experiences that can enrich their subsequent teaching” (p. 231). Among the key elements identified by Sullivan as most significant when using innovative teaching approaches with pupils are mathematical knowledge and knowledge of what it means to teach mathematics, consistent with the central role accorded to mathematical knowledge (including beliefs) in this collection. The experiences Sullivan refers to can derive from the teacher’s own class, that of a colleague, or indeed from a simulation. One noteworthy context for providing teachers with experiences is that of Lesson Study: “lesson study improves instruction by developing teachers’ knowledge (of content, pedagogy, and student thinking)” (Lewis, Perry, & Hurd, 2009, p. 302).[6]

## The Role of the Mathsmaps in Building Mathematical Knowledge and PCK

Responding to unforeseen circumstances is a common concern for many teachers and researchers: for example, Ribeiro, Carillo, and Monteiro (2010) focus on professional knowledge at moments of improvisation, when the teacher is forced to react to unexpected responses or suggestions from his or her pupils. Among the chapters in this section, Turner and Rowland establish the Contingency dimension to cater for this in the KQ, while Corcoran and Pepperell highlight, in their discussion of Lesson Study, the opportunities to discuss the unexpected ways children can go about activities.

The perspective offered by Ryan and Williams in their chapter is different. Here, the unexpected refers to the correct answers given by participating student teachers to questions they were expected to get wrong, as much as it does to incorrect answers

[6]Concern for student learning and for promoting the teacher’s learning from this is also a feature of Learning Study, a combination of Lesson Study and design experiment: “In a learning study teachers get the opportunity to observe colleagues *teach the same thing*. This is one of the features of a learning study that makes it appropriate to mathematics teacher education” (Runesson, 2008, p. 170).

they were expected to get right. The authors present the use of an assessment feedback tool (mathsmap) on the part of two pre-service primary school teachers in England as a means to reflect on their subject knowledge in mathematics (Ryan & Williams, 2007a, 2007b).

Ryan and Williams place considerable emphasis on metacognitive reflection, and use the unexpected results in the tests to encourage trainees to consider their own identity as mathematics learners. They recognise the difficulties many trainees experience in marrying subject matter knowledge to pedagogical content knowledge, and consider that reflecting on their own systems of learning, and in particular the way they overcome difficulties, is a useful aid to achieving this difficult balance.

The authors present the case of a student teacher who can be seen as representative of a process of improving one's practice. Starting from her own experience on teaching practice, this trainee realised her lack of knowledge in certain areas, and decided to take the appropriate steps to remedy the situation, which in turn bolstered her confidence and improved her teaching: teaching practice  awareness of lack of knowledge  improvement in subject knowledge  self-confidence  improved teaching practice. The authors locate one's identity as a mathematics learner at the heart of teacher training. It is not just a matter of being in possession of subject knowledge, but of being aware of one's own characteristics as a mathematics learner. For this purpose, the authors provide the pre-service teachers with a tool (mathsmap) and ask them to give an account of why they felt they made mistakes or responded correctly when they were not expected to. This metacognitive awareness of one's own knowledge and learning is fundamental to pedagogical knowledge and quite possibly instrumental in the formation of one's professional identity. Ryan and Williams, and Corcoran and Pepperell both talk about identity and narratives, sharing a recognition of the need to promote the building of a professional identity. Corcoran and Pepperell summarise identity as the learning of individuals within a joint enterprise. In such a community of practice, work is characterised by the narratives shared by the individuals. Hence, for these authors, learning has a clear social component: ("the responsibility for development of mathematical knowledge for teaching is not an individual but a collective one, which participation in the practice of lesson study appears to meet"), while Ryan and Williams present the use of narrative accounts as a kind of report which the teacher undertakes in an individual learning process.

The process is neither linear nor simple; references to knowledge distinct from subject knowledge are lacking, but as stated above, Ryan and Williams consider the process a first step towards developing other kinds of knowledge and a more reflective teaching.

## Final Remarks

In the introduction I presented various dimensions which it is worth reflecting on and which provide material for future research.

First, it is useful to reflect on the *scope* of the tools and the research presented in these chapters. The KQ was presented in the context of pre-service primary mathematics teaching, the MKT was developed basically in the context of in-service primary mathematics teaching, the CICD was applied to the context of in-service kindergarten teachers, the JLS was applied to the context of pre-service primary mathematics teaching, and mathsmaps were implemented in the context of pre-service primary mathematics teaching. But we must distinguish between the contexts and limitations expounded in these chapters and the potential of these tools. The KQ has been applied to the context of in-service education (Turner, 2008, see note to p. 4), and could likewise be applied to mathematics teachers at secondary level, albeit with significant adaptations to the foundation dimension (Rowland, 2010).

In the same way, the theoretical framework of SMK, PCK and MKT represents a useful tool for secondary teachers, and within the context of pre-service education, both for researchers wishing to study teachers' knowledge, and teachers themselves aiming to develop their reflective capacity. This theoretical framework offers dimensions which orient the trainee teacher with respect to the aspects and characteristics which should configure his or her professional knowledge and which should therefore be taken into account in the process of becoming a teacher. The secondary mathematics teacher, with their greater store of mathematical knowledge, yet needs to develop this knowledge in relation to their role as a teacher (SCK).

Although applied to basic concepts by Tirosh et al., the CICD was originally developed for advanced mathematical thinking, and, as a tool for facilitating reflection on learning mathematics notions, is likewise applicable to pre-service education.

The Lesson Study cycle was conceived with practising teachers in mind. Nevertheless, Corcoran and Pepperell have shown its utility in building Mathematical Knowledge in Teaching in a pre-service context, at the same time as noting certain limitations concerning the lesson given by the student teachers. Essentially, these lessons were 'one-offs', provided for the purpose of the research, rather than snapshots of a prolonged relationship with the same class. As such, the student teachers enjoyed limited potential to learn about their pupils. Nevertheless, the researchers were able to emphasise the overall positive aspects of experiencing the act of teaching and of observing colleagues across a range of ages and different school settings. The intrinsic nature of the JLS means that it could be applied to any level of education.

With respect to mathsmaps, because this is essentially a tool for assessing the mathematical performance of an individual, it is context independent, whether pre- or in-service education and whatever the level. Nevertheless, it is as well to note the reluctance of many practising teachers, especially at secondary level, to undergo a questionnaire specifically designed to question their mathematics knowledge.

Regarding the *specificity* and *potential* of the tools, the KQ and MKT are specific to the mathematics teacher's professional knowledge, and provide a set of dimensions for its study and development. That is, they have much to offer both researchers and teachers. The JLS, for its part, is a system of teacher training, based on a principle of collective effort, and promoting reflection on the pupils' learning. One advantage in its favour is that, through participation as a 'knowledgeable other', the researcher can take an active part in the training process at the same time as studying it. The CICD is not a tool for Mathematical Knowledge in Teaching, but for mathematics knowledge, as noted above. Tirosh et al. highlight its potential for use by both the researcher and teacher. The mathsmap and the concomitant reflection generated by it, have also been seen to offer both researchers and teachers advantages in the study of mathematics knowledge.

I have also commented on *individual* versus *collective* learning on the part of the teacher, and on the role of *reflection* and the *knowledgeable other*. Although, as mentioned above, the studies are grounded in different learning contexts, collective actions and reflection on one's own practice and knowledge are given prominence in the chapters making up this section. Various roles are available for the researcher/instructor, from that focusing particularly on evaluating the student teachers' learning to promoting reflection amongst student teachers and teachers, including that of *knowledgeable other* encouraging discussion and reflection by teachers on their mathematics knowledge and the practice of teaching.

Finally, regarding the extent to which these theoretical frameworks are complementary, if our aim is to focus on and promote the mathematics knowledge of the teacher(s), then we can deploy the KQ to promote their reflection upon CCK, SCK, KCT and KCS (dimensions of MKT). To do so, we can start with our own experiences in the field of education, or those of other teachers, or indeed scenarios involving the pupils' learning, which might imply a non-linear treatment of the KQ dimensions. The analysis of these experiences, and even their design could be undertaken using the JLS model, although it would be necessary to shift the focus of analysis from the JLS pupils' learning to the teacher's learning. Consideration of the pupils' learning could be effected via the CICD. This could be complemented by evaluation of the mathematics knowledge of the teacher(s) using mathsmap.

It is not my intention in the above paragraph on the complementariness of perspectives, nor indeed in any preceding sections, to suggest that it would be desirable to undertake research using all the theoretical tools presented in this section. The idea is simply to present a multi-faceted view of the building of mathematical knowledge in teaching. Full consideration of the compatibility of the different approaches would require a far more extensive discussion.

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# Chapter 17
# Conclusion

**Kenneth Ruthven and Tim Rowland**

An important goal of this book has been to highlight how our understanding of mathematical knowledge in teaching has progressed over recent years, and with it the means for supporting the development of such knowledge.

The first section of the book has outlined important advances in understanding how mathematical knowledge is expressed in teaching, and how it is functionally adapted to the teaching role. There has been particular progress in differentiating facets of subject knowledge that support teachers in planning and enacting mathematics teaching, and a start has been made to mapping out these various facets in greater detail.

Likewise, progress has been made in understanding the complex reciprocal relationship between teaching and learning, particularly in classroom approaches that are organised around the collective (re)construction of mathematical knowledge. This has brought out previously hidden aspects of the mathematical knowledge on which such teaching approaches depend, notably the epistemic and interactional competences that teachers employ in animating and supporting active negotiation of knowledge within more authentic modes of communal mathematical enquiry.

Such research, then, has shown that there are important variations in the range and depth of mathematical knowledge required by different types of teaching approach. For example, forms of practice in which classroom interaction focuses on students' mathematical thinking depend on teachers being proficient in supporting, eliciting, analysing and responding to such thinking. Moreover, the intelligent use of such forms of practice calls for teachers to understand (and identify with) the models of mathematical activity and thought, teaching and learning, that lie behind them.

While the personal knowledge of individual teachers remains a central focus of work in this field, a broadening of perspective has recognised significant ways in which mathematical knowledge is situated within teaching practices and distributed across pedagogical resources and professional networks. This perspective has drawn

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attention to how the mediation of classroom activity and teacher action by curricular materials and pedagogical forms affects the mathematical knowledge that comes into play. Likewise, it has shown how development of knowledgeable teaching and teacher knowledge can be scaffolded by educative use of such cultural tools and reflective membership of practitioner communities.

If teaching practices are, to some significant degree, contextually embedded and culturally shaped, then this is likely to extend to the mathematical knowledge expressed within them. Recognising that current thinking about mathematical knowledge in teaching has developed primarily in certain quarters of the English-speaking world, the second section of the book has sought to take a broader view, examining how particular facets of mathematical knowledge are expressed within teaching practices in specific systems and institutions.

Comparison has revealed noteworthy differences in practice and knowledge, shaped by the contexts and cultures not just of national education systems but of particular types of school (and teacher education) institution. Equally, attention has been drawn to the way in which ideas (and policies) relating to mathematical knowledge in teaching are formulated within the overweening discourse of a particular society and historical period, through analysing how the contemporary discourse of professional audit and individual accountability has shaped operative conceptions of mathematical knowledge in teaching and the operational tools through which such knowledge is made visible.

The third section of the book has shown a range of ways in which the evolving system of ideas about mathematical knowledge in teaching that was examined in the first part can be brought to bear on the development of such knowledge within teacher education and professional development. Each of the chosen examples presented a theorised tool system in which some conceptual framework for analysing mathematical knowledge was coupled with some pedagogical organisation of joint activity involving prospective or serving teachers and a ‘knowledgeable other’.

Thus, in one of the examples presented, the Knowledge Quartet provides the central conceptual framework for analysing mathematical knowledge in teaching within a form of lesson study adapted to the conditions of initial teacher education. In another example, this same framework guides a process of personal reflection on teaching, initially scaffolded by feedback from a supervisor. In both these examples, the grounded framework of the Knowledge Quartet provides a detailed and explicit model of the facets of mathematical knowledge in teaching to be attended to by the participants. Another of the examples crosses two simple taxonomies to provide a similarly detailed and explicit framework for analysing mathematical knowledge for teaching, introduced and applied within a taught course for serving teachers. In the final example, the ‘mathsmap’ is employed to generate personalised discrepancies between a teacher’s actual performance and an idealised model of mathematical capability, providing a stimulus for self-diagnosis and reflection by prospective teachers, organised around their construction of personal narratives on learning mathematics.

The growing attention to the educative use of theorised tools in developing mathematical knowledge for and in teaching appears to be a promising development, creating the potential to marry practical action and innovation more closely to the development and refinement of theory. Equally, exploring variation in the coupling of analytic frameworks and pedagogical structures within teacher education shows promise in providing a more systematic basis for synthesising studies with a view to developing more robust and transposable understanding of how particular aspects of mathematical knowledge in teaching can be developed effectively.

This book has sought to contribute to the development of a more programmatic approach to research on mathematical knowledge in teaching. In each section, a cognate range of conceptual frameworks and developmental tools have been presented, and then triangulated in the syntheses which look back over each section. We intend that the critical reflection and speculative integration offered there will contribute to developing a more systematic and reflexive research programme, capable of building a stronger knowledge base for designing teacher education and professional development provision. Equally, we have already noted how research has only started to map out the detail of mathematical knowledge in teaching. There is scope for a more comprehensive research programme to extend scrutiny beyond the particular phases, systems and topics that have received most attention to date: to examine mathematical knowledge in secondary and tertiary teaching as much as primary, beyond a small group of anglophone cultures, and in relation to areas and aspects of mathematics other than arithmetic.

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