Strange Non-chaotic Attractors in Noisy FitzHugh-Nagumo Neuron Model

Guang-Jun Zhang, Jue Wang, Jian-Xue Xu, Hong Yao, and Xiang-Bo Wang

Abstract Strange Non-chaotic attractors in noisy FHN neuron model periodically driven are researched in this paper. Here we show, based on a nonlinear dynamical analysis and numerical evidence, that under the perturbation of weak noise strange non-chaotic attractor can be induced in FHN neuron model. And the mechanism of strange non-chaotic attractor is related to transitions among chaotic attractor, periodic-3 attractor and chaotic saddle in two sides of crisis point of system respectively.

Keywords Strange non-chaotic attractor · Chaotic saddle · Periodic-3 attractor · Chaotic attractor

1 Introduction

It is well known that neurons work in a noisy environment, and it is therefore of great interest to study how information is encoded and transmitted when neurons work in such a noisy environment [1]. The subject of strange non-chaotic attractors (SNAs) has attracted continuous interest in the nonlinear and statistical physics community [2, 3]. Here "strange" refers to the nontrivial, complicated geometry of the attractor, and "nonchaotic" indicates that the maximum Lyapunov exponent of the attractor is non-positive and there is thus no sensitive dependence on initial conditions [2, 3]. In principle, strange non-chaotic attractors occur in all dissipative dynamical systems that exhibit the period-doubling route to chaos, where the attractors formed at the accumulation points of period-doubling cascades are fractal sets with zero Lyapunov exponent [3]. In some neuron model which leads to

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chaos by the route of periodic-doubling bifurcation, such as autonomous, periodically driven and quasiperiodically driven dynamical system, as typical dissipative dynamical systems, there should exist robust strange non-chaotic attractors. But this has not attracted interest of researchers. The results of neurophysiology experiment show that non-periodic firing neurons are more sensitive to stimulis than periodic firing neurons [4]. For some chaotic spike neuron model, the strange non-chaotic attractor is of particular importance in the research of encoding and transmission of neural information in noisy environment. There probably are some relations between some phenomena in neuron spike, such as chaotic synchronization, and strange non-chaotic attractors. So in the paper, strange non-chaotic attractors in periodically driven FHN neuron model under the perturbation of weak noise are researched.

2 The Bifurcation Characteristic of the Periodically Driven FHN Neuron Model

We consider the periodically driven FitzHugh-Nagumo neuron model in the following form Ref. [5]:

$$\begin{cases} \varepsilon \frac{dv}{dt} = v(v-a)(1-v) - w \\ \frac{dw}{dt} = v - dw - b + r\sin\beta t \end{cases} \tag{1}$$

The variable v is the fast voltagelike variable and w is the slow recovery variable. Throughout the paper we fix the values of the constants to $\varepsilon = 0.02,\ d = 0.78,\ a = 0.5,\ r = 0.27,\ \beta = 14.0$. And parameter b is as controlling parameter. A firing or spike is considered to occur if the variable v has a positive-going crossing of the firing threshold vth, chosen as vth $= 0.5$. According to Ref. [6], by modified generalized cell mapping, the nonlinear dynamics global characteristic of FHN neuron model is obtained. The bifurcation figure of system is shown in Fig. 1. As shown in Fig. 1, it can be seen that with the change of b system leads to chaos by the route of periodic-doubling bifurcation. When b is some certain value crisis in FHN neuron model occurs. And when $b = 0.2404$, boundary crisis occurs. A chaotic saddle in FHN neuron model appears after boundary crisis occurs. In a certain parameter region before boundary crisis there coexist a chaotic attractor and a period-3 attractor. But after boundary crisis chaotic attractor disappears, the system displays a period-3 spiking, and a chaotic saddle remains.

3 The Strange Non-chaotic Attractor in FHN Neuron

In this paper, the Gaussian distributed white noise $\xi(t)$ is used for perturbing the system. The mean and autocorrelation function are as follows respectively:

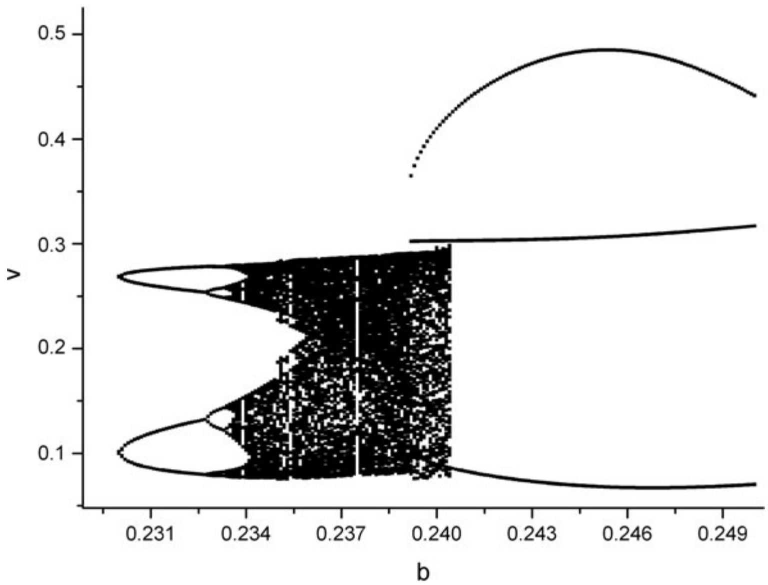

Fig. 1 The bifurcation characteristic of FHN neuron model

$$\begin{cases} <\xi(t)>=0 \\ <\xi(t)\xi(s)>=2D\delta(t-s) \end{cases} \tag{2}$$

where D is the noise intensity.

After the noise is added to Eq. (2), the equation becomes:

$$\begin{cases} \varepsilon\frac{dv}{dt}=v(v-a)(1-v)-w \\ \frac{dw}{dt}=v-dw-b+r\sin\beta t+D\cdot\xi(t) \end{cases} \tag{3}$$

Ref. [2] researched the strange non-chaotic attractor in logistic map and kicked duffing oscillator when the system bifurcation parameter is in the right of boundary crisis. In this case, a periodic attractor and a non-attracting chaotic invariant set (chaotic saddle) coexist, and the asymptotic attractor of system is periodic at the absence of noise. Under the perturbation of appropriate noise, the transition between periodic attractor and chaotic saddle will occur. Then strange non-chaotic attractor is induced. The attractor appears to be geomertrically strange, but its nontrivial maximum Lyapunov exponent is negative. There are fluctuations of the finite-time Lyapunov exponent into the positive side.

According to the Section 2, a boundary crisis occurs in FHN neuron model when b is 0.2404. When the b is in the right of boundary crisis point, a periodic-3 attractor and a non-attracting chaotic invariant set (chaotic saddle) coexist, and system displays periodic-3 spiking at the absence of noise. At the presence of appropriate noise, the same transition between periodic attractor and chaotic saddle will occur.

Then strange non-chaotic attractor should be induced to occur. The mechanism of strange non-chaotic attractor in this case is the same as the case in Ref. [2].

When the b is in the left of boundary crisis, the asymptotic attractor of system is a chaotic attractor. Then, the Lyapunov exponent of system response is positive. According to Ref. [7], at the presence of weak noise, the system motion may be in transition between attractors in two sides of continuous bifurcation point. Appropriate noise can induce the system motion to move towards attractors in the right of crisis point. Trajectory from a random initial condition typically moves toward the chaotic attractor, stays near the attractor for a finite time, and noise induced it to transit to the periodic attractor or chaotic saddle. If the system motion transit to periodic attractor, it will stay finite time in periodic attractor until noise enough to induce it to transit to chaotic saddle or chaotic attractor in the left of crisis point. If the system motion transit to chaotic saddle, there is thus transient chaos. Trajectory typically moves toward the chaotic saddle along its stable manifold, stays near the saddle for a finite time, and leaves the saddle along its unstable manifold before finally approaching the periodic attractor. After finite time, the appropriate noise also moves the system motion to the chaotic attractor in the left of boundary crisis point. Then strange non-chaotic attractor should be also induced to occur. The mechanism of strange non-chaotic attractor is related to the random transitions between chaotic attractor in the left of boundary crisis point and the periodic attractor in the right of crisis point. Because in the right of crisis point there coexist chaotic saddle and periodic attractor, the system motion also transits probably between periodic attractor and chaotic saddle. Because chaotic spike neurons are more sensitive to stimulis than periodic firing neurons [4], strange non-chaotic attractor in this case is of much more significance to reveal some phenomena in life science.

To verify the theoretical analysis above, the Eq. (3) is integrated using the fourth-order Runge-Kutta method with time step $\Delta t = 0.01$ second. And the Poincare maps of several cases are obtained respectively by section $t = n^{*}2\pi/\beta(n = 1, 2, 3, \ldots)$. The Lyapunov exponent λ_1 of system in several cases are calculated respectively. The results of calculation are shown in Figs. 2, 3, 4, and 5. From the

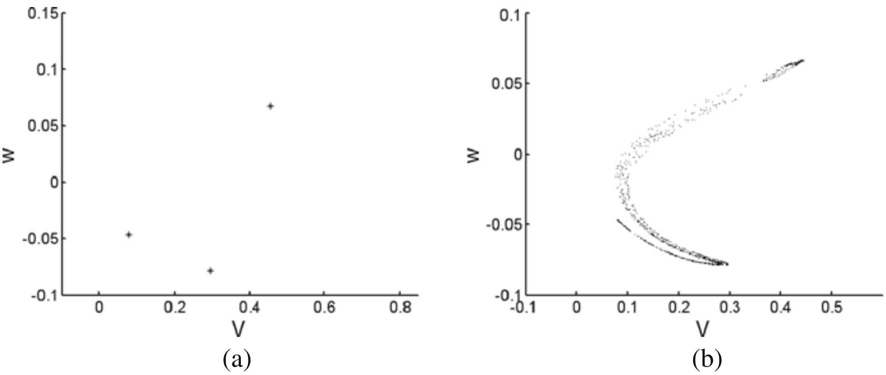

Fig. 2 The attractors of FHN neuron model (Eq. 3) in the *right* of crisis point when $b = 0.241$, $\varepsilon = 0.02$, $d = 0.78$, $a = 0.5$, $r = 0.27$, $\beta = 14.0$. (**a**) $D = 0.0$, (**b**) $D = 0.005$

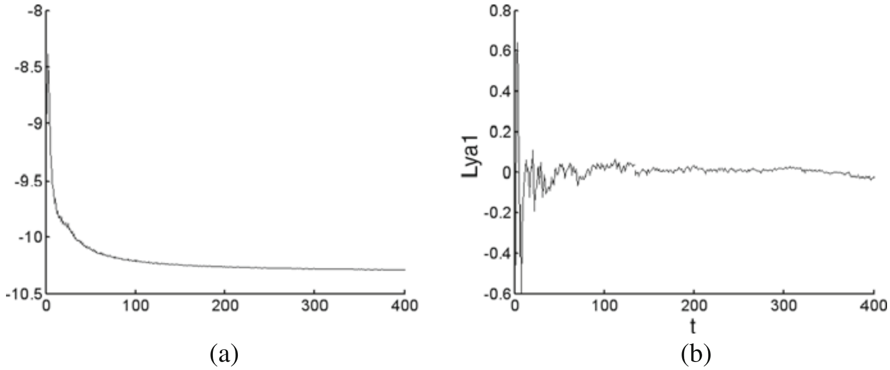

Fig. 3 The evolution of the finite-time Lyapunov exponent in two cases of noise in the *right* of crisis point when $b = 0.241,\ \varepsilon = 0.02,\ d = 0.78,\ a = 0.5,\ r = 0.27,\ \beta = 14.0$. (**a**) D=0.00, (**b**) $D = 0.005$

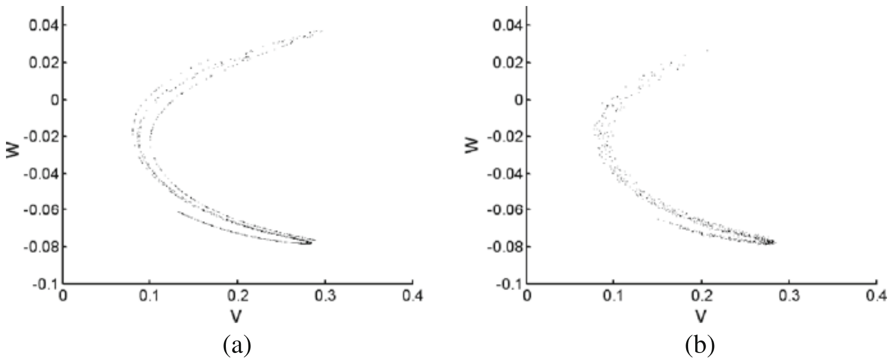

Fig. 4 The attractors of FHN neuron model (Eq. 3) in the *left* of crisis point when $b = 0.240,\ \varepsilon = 0.02,\ d = 0.78,\ a = 0.5,\ r = 0.27,\ \beta = 14.0$. (**a**) $D = 0.0$, (**b**) $D = 0.03$

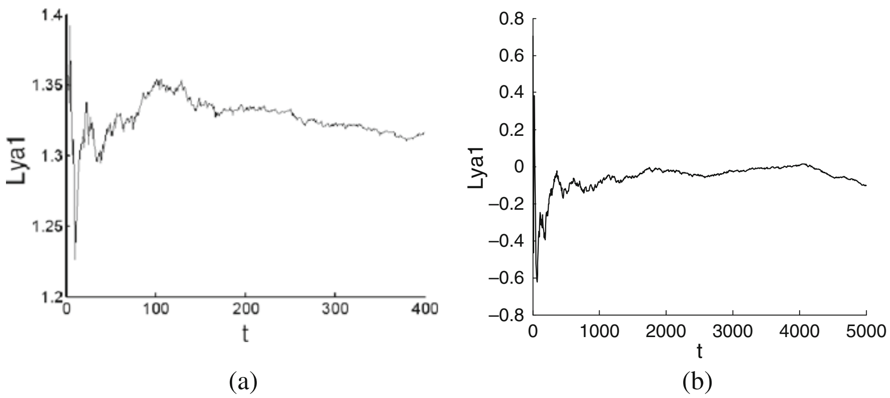

Fig. 5 The evolution of the finite-time Lyapunov exponent in two cases of noise in the *left* of crisis point when $b = 0.240,\ \varepsilon = 0.02,\ d = 0.78,\ a = 0.5,\ r = 0.27,\ \beta = 14.0$. (**a**) $D = 0.00$, (**b**) $D = 0.03$

Figs. 2 and 3, it can be seen that when b is in the right of crisis point the asymptotic attractor is a period-3 attractor without noise, the Lyapunov exponent λ_1 is negative. And at the presence of weak noise the asymptotic attractor is a strange attractor, but it is not chaotic because its Lyapunov exponent is non positive at the most time. From the Fig. 4 and 5, it can be seen that when b is in the left of crisis point the asymptotic attractor is a chaotic attractor without noise, the Lyapunov exponent λ_1 is positive. And at the presence of weak noise the asymptotic attractor is a strange attractor, but it also is not chaotic because its Lyapunov exponent is non positive at the most time.

4 Conclusions

In summary, we have shown that strange non-chaotic attractor can occur in periodically driven FHN neuron. With the change of b system leads to chaos by the route of periodic-doubling bifurcation. When b is some certain value crisis occurs. And when $b = 0.2404$, boundary crisis occurs. A chaotic saddle in FHN neuron model appears after boundary crisis occurs. In a certain parameter region before boundary crisis there coexist a chaotic attractor and a period-3 attractor. But after boundary crisis chaotic attractor disappears, the system displays a period-3 spiking, and a chaotic saddle remains. Not only can strange non-chaotic attractor appear at the appropriate noise when there coexist periodic attractor and chaotic saddle in nonlinear dynamical system just like Ref. [3], but also when the asymptotic attractor of system is a chaotic attractor in the neighborhood of left of crisis point.

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