ARCHIMEDES' QUADRATURES

Jean Christianidis, Apostolos Demis
Department of Philosophy & History of Science
University of Athens
University Campus, 157 71 Athens, Greece
e-mail: ichrist@phs.uoa.gr, demkap@otenet.gr

ABSTRACT In the paper we discuss the three methods that Archimedes employs to deal with the problem of the quadrature of a parabolic segment. We characterize the three approaches as heuristic, mechanical and geometric respectively. We investigate Archimedes' own attitude towards the three methods, and we conclude with a critical presentation of the prevalent views concerning the matter, which have been expressed in the past by historians of mathematics.

1. INTRODUCTION

Euclid (*fl. ca*. 300 B.C.) and Archimedes (*ca*. 287–212 B.C.) are the two most prominent figures in the history of mathematics of Antiquity. Both of them are part of that group of Greek mathematicians who, through their work, deeply influenced the development of early modern mathematics. The fact that they lived geographically and chronologically close to each other could lead one to infer that their scientific work, too, is completely analogous in breadth and content and that Archimedes also worked, at least up to a point, within the framework of the research program that Euclid had initiated a few decades earlier. Such a conclusion, however, is far from the truth. More specifically, as far as mathematics is concerned, the study of the work of the two men reveals that Euclid and Archimedes belong to two mathematical traditions which, although not entirely irrelevant to each other, are nevertheless distinct from each other, while both of them can be traced back to the times of the classical Greek mathematics.

The first was the "tradition of *stoicheiôsis* of mathematics." It was a tradition which focused on studying the logical structure of mathematical reasoning, ensuring the rigor and simplicity of mathematical proof and generally organizing and systematizing the structure of the mathematical edifice. Euclid was the main representative of this tradition and his *Elements* was the crowning achievement of the researches in the field.

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Alongside the tradition of *stoicheiôsis*, a second one was developed among the Greek mathematicians, more or less in the same period, which would be called the "metric tradition". Without being wholly unrelated to the former one, the metric tradition focused mainly on measuring geometric shapes, that is on discovering techniques to develop formulas, as we would say today, in order to measure the area or the volume for two- and three-dimensional shapes, as well as on developing arguments to prove these formulas. The metric tradition was established by Eudoxus (390–337 B.C. or 408–355 B.C.) although its origins can be traced even further back in time, in the second half of the 5th century, in the work of Democritus. Nevertheless, the tradition came to full fruition in the 3rd century, with the work of Archimedes. (Knorr 1993, 151–152)

Archimedes is the foremost representative of the metric tradition among the Greek mathematicians. An important part of his work is about quadrature and cubature, and indeed most of his treatises are devoted to such issues. These treatises could be characterized, by analogy with the tradition to which they belong, as "metric," and they are the following: *"Measurement of a Circle*, *Quadrature of the Parabola*, *On the Sphere and the Cylinder*, *On Conoids and Sheroids*, *On Spirals*, and *The Method of Mechanical Theorems*. Relevant to the above mentioned works, and more specifically to *The Method of Mechanical Theorems* and the *Quadrature of the Parabola*, is also the treatise *On the Equilibrium of Planes"*.

Among the works of Archimedes which survive in the Greek language, only *Sand-Reckoner, Floating Bodies* and *The Cattle Problem* are not related to the issue of quadrature and cubature, while the subject matter of his treatise *Stomachion* still remains uncertain, despite recent progress. It is worth noting, however, that even some of these works deal, in a way, with measurement issues. Thus, *Sand Reckoner* talks about the number of specks of sand which could fill the universe, *The Cattle Problem* is about determining the number of four sets of bulls and cows which satisfy certain conditions, while *Stomachion* is about the combinatory problem of the number of solutions to the problem of making a square out of the rearrangement of the 14 puzzle pieces into which the square had originally been divided. (Netz, Acerbi, Wilson 2005)

The statement that the mathematical work of Archimedes belongs to the "metric" tradition of Greek mathematics and that Archimedes is the leading representative of the tradition is by no means a new conclusion. It is common ground in the history of Greek mathematics nowadays, that Archimedes made use of the infinitesimal methods developed by Eudoxus, he refined and expanded them further and applied them skillfully to a great number of quadrature and cubature cases of geometric shapes.

Of the multitude of quadrature and cubature cases included in the works of Archimedes, two of them are especially interesting. The first such case is the quadrature of a parabolic segment. What is interesting about this quadrature is the fact that Archimedes deals with it three times, in two different treatises: in the *Quadrature of the Parabola* (hereafter, the *Quadrature*) and in *The Method of Mechanical Theorems* (hereafter, *The Method*). Indeed, the former is solely devoted to the subject of the quadrature of a parabolic segment, which Archimedes investigates using two different methods (a mechanical one and a geometric one), while the latter features the quadrature of the parabolic segment as an example of the application of the heuristic method, which Archimedes had devised in order to determine the surface area and the volume of geometric shapes, independently of the formal, rigorous proof of the conclusions he drew. The second such case appears in *The Method*. More specifically, in Propositions 12–15 of this treatise, Archimedes deals with the same problem of the cubature of a cylindrical segment (a "hoof") three times and in three different ways. So, in Propositions 12–13 he investigates the problem using a mechanical method, in Proposition 14 he employs the use of "indivisibles" (which, as we shall see, is an essential part of his heuristic method) and in Proposition 15 by a geometric method. (Saito 2006, 36 n° 3).

Obviously, the existence of multiple ways of handling these two problems in the works of Archimedes, even within the same work, poses a number of historiographical questions about the content, the role and the weight that the Syracusan mathematician placed on the various methods of quadrature that he employed. Later on in this paper, we shall draw on the example of the three ways which Archimedes employs to deal with the quadrature of a parabolic segment, as a means to investigate such questions raised by the historiographical research. Before doing so, however, it would be useful to mention a few facts concerning the two treatises in which Archimedes studies the quatradure of the parabolic segment.

2. THE *QUADRATURE* AND *THE METHOD*: SOME FEATURES OF THE TWO TREATISES

The *Quadrature* and *The Method* differ from each other both in style and use. The *Quadrature* has the style of a formal publication. It was written by Archimedes in order, as we would say, to have it published as a book addressing a reading public. *The Method's* form, on the other hand, gives the impression of a text extracted from Archimedes' personal records. It is attached to a letter to Eratostenes and it addresses, at most, a little circle of mathematicians associated with Eratosthenes. This work has a much stronger personal touch. It's more like an elaborate version of the notes a geometer keeps during his research project. Repeating a phrase by Dijksterhuis, we may say that, through this work, Archimedes allows us to have a look inside his mathematical study room. (Dijksterhuis 1987, 315) So, the two treatises have significant differences between them in form and intention.

The second point worth mentioning is that the writing of the *Quadrature* chronologically precedes *The Method*. This derives directly from a passage in Archimedes' letter to Eratosthenes, where he writes: "I now wish to describe the method in writing, partly, because I have already spoken about it before, that I may not impress some people as having uttered idle talk, partly because I am convinced that it will prove very useful for mathematics." (Dijksterhuis' translation) By saying that he has spoken in the past about it, Archimedes actually refers to a phrase in the preface of the *Quadrature* where he mentions that he first discovered the theorem about the square area of a parabolic segment by means of mechanics and then proved it by means of geometry. By juxtaposing the two extracts, it is concluded that Archimedes wrote the *Quadrature* before writing *The Method*. The same conclusion arises from the closing phrase of the first proposition of *The Method*, where, after stating that he found, by means of mechanics, the area of a segment of a parabola to be $^4/_3$ of the triangle which has the same base as the segment and equal height, Archimedes adds: "This has not therefore been proved by the above, but a certain impression has been created that the conclusion is true. Since we thus see that the conclusion has not been proved, but we suppose it is true, we shall mention the previously published geometrical proof, which we ourselves have found for it, in its appointed place." (Dijksterhuis' translation) In this extract Archimedes once more refers to the *Quadrature*, which leads to the conclusion that the writing of this work chronologically precedes the writing of *The Method*.

Of course, the chronological order of the writing of the two treatises does not coincide with the order in which Archimedes conceived what is included in them. On the contrary, the discovery of a theorem always precedes its formal, rigorous proof. The problem of the classification of Archimedes' works according to the chronological writing order, from the one hand, and according to the order they occupied in Archimedes' research agenda, on the other hand, is, indeed, a problem hard to solve, which still preoccupies the historians of Greek mathematics. (Knorr 1978; Vitrac 1992)

3. THE QUADRATURES OF THE PARABOLIC SEGMENT: A HISTORIOGRAPHICAL DISCUSSION

In this paper we shall not enter into the detailed technical presentation of the three methods Archimedes employs to deal with the problem of the quadrature of a parabolic segment. Such a presentation can be found in any book of the history of Greek mathematics, and, in particular, in (Dijksterhuis 1987), which is still considered the best review of Archimedes' complete works. We merely note that Archimedes investigates the problem in proposition 1 of *The Method*, by using an approach which can be characterized – anticipating in a way the discussion which will now follow– heuristic; in propositions 14 and 16 of the *Quadrature of the Parabola*, by using an approach which can be characterized as mechanical; and in proposition 24 of the same treatise, by using a geometric approach. We shall now examine Archimedes' own attitude towards the three methods, in other words, how he perceives the role and the importance of each one, so as to conclude the article with a critical presentation of the prevalent views concerning the matter, which have been expressed in the past by historians of mathematics. (Knorr 1982; 1993; 1996; Dijksterhuis 1987; Netz, Saito, Tchernetska 2001; Saito 2006)

Archimedes uses two pairs of terms to characterize, and distinguish from each other, the three methods which he employs in order to investigate the problem of the quadrature. These pairs could be rendered as: "heuristic" – "demonstrative", and "mechanical" – "geometric". It is true that not all of these words occur *en personne* in Archimedes' texts. The word "heuristic," for example, does not occur anywhere. Instead, other equivalent forms appear in expressions such as "*dia mechanikôn heurethen*," "*tou nun ekdidomenou theorêmatos tên heuresin*," "*fanentôn mêchanikôs*," "*fanen dia tôn mêchanikôn*." Similarly, instead of the word "mechanical," the phrase "*dia tôn mêchanikôn*," or the adverbial form "*mechanikôs*," occur. In the same way, variants of the other two words, "demonstrative" and "geometric," appear in the texts. For brevity's sake, from now on we shall render all the variants using the word pairs that we mentioned above.

Let us now see how Archimedes characterizes each method. The quadrature expounded in *The Method* is characterized as "heuristic" and "mechanical." Its heuristic character lies, he notes, in the fact that it gives one the ability to know in advance, by means of mechanics, some mathematical properties, a knowledge which is useful in finding proof for the relevant theorem. For "it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge." (Dijksterhuis' translation). Nevertheless, Archimedes is keen to point out that the quadrature achieved by this method does not constitute proof of the conclusion. This is deduced from the clarification at the end of the proposition 1 of *The Method*, where he mentions: “This has not been proved by the above, but a certain impression has been created that the conclusion is true.” (Dijksterhuis’ translation). There is no doubt, therefore, that Archimedes considers the first method of quadrature, that is the quadrature expounded in his treatise *The Method*, as mechanical, heuristic, but not demonstrative.

Similarly clear is the way Archimedes treats the last of the two quadratures exposed in *Quadrature*. He characterizes it as geometric and demonstrative. This derives right from the following extract taken from the preface: “I have therefore written out the proofs (he is referring to the quadrature of the parabolic segment), and now send them, first as they were investigated by means of mechanics, and also as they may be proved by means of geometry.” (translation by Ivor Thomas) This extract is similar to another one taken from the same preface: “I set myself the task of communicating to you … a certain geometrical theorem, which had not been investigated before, but has now been investigated by me, and which I first discovered by means of mechanics and later proved by means of geometry.” (translation by Ivor Thomas). In both extracts Archimedes is referring to the geometric proof exposed in the second part of the *Quadrature of the Parabola*, in which the main theorem is proposition 24. The fact that, in the latter he uses the expression “exhibited (*epideichthen*) by means of geometry,” instead of the expression “proved (*apodeiknytai*) by means of geometry” used in the former, does not, in any way, change the conclusion that Archimedes considers the method he employs to deal with the problem of the quadrature of a parabolic segment in the second part of his treatise, as both geometric and demonstrative.

Now, as far as the method of quadrature expounded in the first part of the *Quadrature of the Parabola* is concerned, things are not so clear as in the previous two cases. Of course, there is no doubt about its mechanical nature. The problem lies in whether Archimedes believes that this reasoning constitutes a convincing and acceptable proof. As we shall see later on, this is exactly the point, which has led to disagreement among modern historians of mathematics. In our opinion, the extract quoted above (“I have therefore written out the proofs, and now send them, first as they were investigated by means of mechanics, and also as they may be proved by means of geometry”), in which Archimedes uses the plural form “proofs” to characterize both of the quadratures exposed in *Quadrature*, constitutes a piece of evidence which should not be overlooked.

However, the matters concerning this last mentioned method of quadrature are more complicated than what we presented earlier on, because the question is raised as to how valid and mathematically rigorous can a proof be which uses notions and reasonings taken from mechanics. The argument developed by Archimedes in this quadrature, is, in a way, a hybrid endeavor, which draws both on geometry and mechanics. It is true that the argument which is developed in the quadrature of *The Method* has similar characteristics. Archimedes, however, does not consider this quadrature as demonstrative; he considers it as heuristic, so, in this sense, no question of its demonstrative validity is raised. The quadrature, on the other hand, exhibited in the first part of *Quadrature*, does not have the characteristics of a heuristic procedure. Its development requires the conclusion to be known in advance. From the other hand, its mechanical character is indisputable and it consists in: a) considering the geometric magnitudes as physical (namely, as having weight), b) the use of the weighing balance, c) the application of the law of the lever, d) the use of properties concerning the centers of gravity. Is, however, the investigation of geometric properties, using arguments taken from mechanics, an acceptable method? To the pure mathematician, imbued with the Euclidean ideal of the rigor of proof, it would be unacceptable. Vitrac writes about this: "Pour un puriste ceci n'est pas recevable comme démonstration géométrique car il y a un problème par rapport aux principes de la démonstration utilisée." (Vitrac 1992, 75) Vitrac also refers to sections I, 6-7 from Aristotle's *Analytica posteriora*, adding that, "Une figure en mécanique a poids et grandeur; en géométrie elle a seulement une grandeur." (Vitrac 1992, 75 n° 56) Indeed, according to Aristotle, principles taken from different scientific disciplines should not be used within the same proof; one should not enter a field of study using means and techniques belonging to another field. However, Archimedes himself, as it has been mentioned, calls, even if once, the quadrature demonstrative. Based on the above observations, it is clear that there are some open historiographical questions pertaining to the role and the character of the quadrature exhibited in the first part of the *Quadrature of the Parabola*. Let us now see how the historians of mathematics approached this subject.

The starting point of our investigation will be the quadrature set forth in *The Method*. As it has been mentioned, Archimedes always characterizes this method as heuristic, he adds that it does not constitute proof, and refers for its proof to his treatise *Quadrature of the Parabola*, which had been published earlier. The question which naturally arises is why the method employed in the quadrature of *The Method* is not viewed as sufficient in order for the conclusion to be considered valid and rigorously proved. Is there some lack of mathematical rigor in the method, and, if so, where exactly is this lack traced?

A decoding of the method reveals that there are two different types of arguments used in it:

1. Firstly, arguments taken from mechanics are used. The geometric magnitudes are considered as having weight, they are suspended from the beam of a hypothetical weighing balance, the law of the lever is applied to deduce relationships between the geometric magnitudes, and properties related to the centers of gravity are used.
2. Secondly, a plane figure is considered as made up of “all” the parallel segments of straight lines drawn along a given direction and whose endpoints lie on the perimeter of the figure. So, the figure may be decomposed into such parallel chords of a given length, and be reconstructed again by them. The “sum” of the segments of straight lines gives the area of the plane figure. We shall call these segments of straight lines “indivisibles.” This notion can also be extended to solid figures, which can be decomposed into parallel cross-sections.

Taking into consideration these two different types of arguments involved in Archimedes’ reasoning, let us now examine where, according to the historians of mathematics, the lack of mathematical rigor of this method of quadrature is located. The most frequently expressed view in the bibliography is the one formulated by E.J. Dijksterhuis. According to Dijksterhuis, the lack of mathematical rigor is due to the employment of the “indivisibles” and not, by any means, to the mechanical aspects of the method. (Dijksterhuis 1987, 319, 336) On the contrary, Dijksterhuis says that Archimedes assigned demonstrative validity to the mechanical aspects of the method, and this is inferred, first of all, by the fact that Archimedes himself had established mechanics (statics) as a demonstrative science in *On the Equilibrium of the Planes*, and secondly by the fact that, in the first part of *Quadrature*, he proves the conclusion about the parabolic segment applying mechanical considerations, but not indivisibles. In the following years, Dijksterhuis’ point of view was adopted by other scholars and today it is considered as dominant.

Apart from the scholars who embraced Dijskterhuis’ position, there were others who disagreed, claiming that the lack of rigor of the method is due not only to the use of indivisibles, but to its mechanical nature, as well. As a consequence, those scholars maintained that, because of its mechanical attributes, the quadrature in the first part of *Quadrature* does not constitute a valid and rigorous proof either, and that neither Archimedes considers it as such. An earlier historian who expressed such a view was Oskar Becker, from whom we quote the following extract: “Archimède ne tient pas cette méthode pour rigoureuse, tout d’abord à cause de considérations infinitésimales qui remontaient partiellement à Démocrite (B 155) …, mais aussi à cause de l'emploi de la Statique. Ainsi dans la *Quadrature de la parabole* il remplace dans ses considérations par exhaustion (le segment de parabole est décomposé non en un nombre infini de segments mais en un nombre d'éléments qui, fini à l'origine, est progressivement porté à l'infini). Cela conduit encore à une démonstration purement géométrique au cours de laquelle il utilise en même temps que certaines intégrales définies des séries infinies convergentes." (Becher, Hofmann 1956, 81–82) Becker repeated his view a year later, in a critique that he wrote on the English edition of Dijksterhuis' book. (Becker 1957)

The historian of mathematics, however, who most forcefully expressed his objections to the position of Dijksterhuis, was W.R. Knorr. (Knorr 1982; 1996) According to Knorr, the main weakness of Archimedes' method, as far as its mathematical rigor is concerned, is exactly its mechanical nature and not the indivisibles. As he notes, "in Archimedes's account the indivisibles are merely a secondary aspect; for the essence of his method lies in its appeal to mechanical principles." (Knorr 1982, 73) The arguments that Knorr appeals to in order to justify his view are the following: 1) Archimedes always refers to the heuristic method using the expression "*dia tôn mechanikôn*;" he never uses anything which would imply the indivisibles. 2) In the extracts 218.11–12 and 220.17–20 from *Quadrature* quoted above, Archimedes juxtaposes the "demonstrative" to the "mechanical." In this setting, when he mentions in *The Method* that some of the theorems he originally found by means of mechanics, he later proved by means of geometry, because "the investigation using this procedure does not constitute proof," he can only refer, Knorr claims, to the mechanical attributes of the method. 3) Finally, the inclusion of the geometric proof in the second part of the *Quadrature of the Parabola*, is due to Archimedes' wish to forestall possible objections raised in the name of pure mathematics about the legitimacy of the use of mechanical elements (such as weight, the weighing balance, equilibrium) in proofs which concern exclusively geometric properties of geometric figures, as in the case of the quadrature which takes place in the first part of the *Quadrature of the Parabola*.

Our comments on the above arguments are the following: Archimedes uses the expression "*dia tôn mechanikôn*" both when he describes the heuristic method, and when he refers to the mechanical quadrature of the *Quadrature of the Parabola*. In the fist case, however, he is careful to always add to this expression a participle such as "*heurethen*" or "*fanen*" (found), something he does not do in the second case. The juxtaposition, therefore, is not between the "demonstrative" and the "mechanical" as such, as Knorr claims, but between the "demonstrative" and the "found *dia tôn mechanikôn*", in other words between the "demonstrative" and the "heuristic." On the contrary, Archimedes by no means juxtaposes the "demonstrative" with the mechanical quadrature in *Quadrature of the Parabola*, since, as we have seen, he characterizes the latter as proof. Besides, the last argument does not state anything as to how Archimedes himself evaluated the mechanical quadrature of the *Quadrature of the Parabola*. The possibility that some mathematicians might raise objections over the legitimate use of mechanical elements in geometric proofs, does not mean that Archimedes himself shared their views. On the contrary, it is plausible to assume that Archimedes considered as legitimate and convincing the mechanical proof of the quadrature of the parabolic segment, which he himself invented, and that the inclusion of the geometric proof in the second part of the *Quadrature of the Parabola*, aimed at making the mechanical proof more easily acceptable by a public which might have had some disbelief and objections about the latter, but had no reservations whatsoever about the validity of the former.

The main issue in the preface to *Quadrature of the Parabola* is not to juxtapose the mechanical with the geometric treatment of quadrature problems, as Knorr claims. The main issue is to address the question of what kind of propositions should be taken as lemmas (axioms) in order for the proofs to be considered as valid. Archimedes notes that, in the past, there had been geometers who tried to solve (and to justify the solution, to prove) problems such as the quadrature of the circle, the quadrature of a segment of a circle, or the quadrature of an area bounded by an ellipse section and the chord at its ends, using lemmas which were not easily acceptable. The use of such lemmas had the result that the solutions proposed by those geometers were not recognized, by most of their colleagues, as having validity. However, nobody, Archimedes notes, had tried to square the parabolic segment, in the past. This problem was solved for the first time by himself, "and for the proof this lemma is assumed: given [two] unequal areas, the excess by which the greater exceeds the less can, by being added to itself, be made to exceed any given finite area." (translation by Ivor Thomas)

Archimedes explains that this lemma –often referred to as the "continuity axiom" in the bibliography–, had also been employed by earlier geometers, because, by its use or the use of similar lemmas, they showed several theorems that are included in Book XII of Euclid's *Elements*. By saying that Archimedes refers, from the one hand, to Eudoxus, and, from the other, to Proposition X, 1 of Euclid's *Elements*. So, after acknowledging the theorems which Eudoxus had proved, in the past, using a similar version to his continuity axiom, Archimedes adds the following critical phrase: "In the event, each of the aforesaid theorems has been accepted, no less than those proved without this lemma; and it will satisfy me if the theorems now published by me obtain the same degree of acceptance." (translation by Ivor Thomas).

The first conclusion drawn from this phrase is that the broader subject matter which occupies Archimedes in the preface to the *Quadrature* is proofs and their validity. Secondly, the question which preoccupies him is not the validity of the mechanical, as opposed to the purely geometric proofs, but the validity of the proofs which make use of the continuity axiom (independently of whether they are mechanical or purely geometric) as opposed to those which do not make use of this axiom. Thirdly, Archimedes states that he himself believes that the proofs which make use of the continuity axiom (in any version) are no less valid than the common geometric proofs which are carried out without the use of the aforementioned axiom.

In the context of the above discussion, Archimedes presents in the main body of his treatise two methods of treating the quadrature of a parabolic segment, a mechanical one and a purely geometric one, which he calls "proofs" and which, both of them, use the continuity axiom. Taking into consideration all of the above, we reach the conclusion that Archimedes included the mechanical treatment in the *Quadrature of the Parabola* as an entirely legitimate proof of the theorem of the quadrature of a parabolic segment, for which he only claims it to be considered as valid as the purely geometric proof, or proofs, of Eudoxus, which are included in the twelfth book of Euclid's *Elements*. Finally, as far as the quadrature of *The Method* is concerned, we are in accord with Dijksterhuis' view, namely, that its lack of rigor is due to the use of the indivisibles, and not to its mechanical aspects, and that this is the reason why Archimedes considers this method as heuristic and not demonstrative.

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