ARCHIMEDES IN SECONDARY SCHOOLS: A TEACHING PROPOSAL FOR THE MATH CURRICULUM

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ABSTRACT The aim is to propose, at various levels in secondary schools, Archimedes' idea for calculating π using the computer as programming tool. In this way, it will be possible to remember the work of one of the greatest geniuses in history and, at the same time, carry out an interdisciplinary project, particularly relevant to the current debate on the Math curriculum.

1. INTRODUCTION AND PRELIMINARY

In the current teaching practice in Italian secondary education, as is apparent from available textbooks, there is little evidence in the math curriculum, bar a few exceptions, of reference to the genius of Archimedes (Syracuse 287–212BC). This is a serious gap not merely on account of the fact that he was one of the greatest mathematicians of all time, but also, and above all, because such omission deprives students of an opportunity to investigate vital areas of debate in an interdisciplinary context. One argument linked to the name of Archimedes, of great interest to mathematicians ever since, is the calculation of π [Beckmann, 1971], in other words the constant ratio between the length of the circumference and its diameter. Archimedes' contribution [Frajese, 1974] to this constitutes a milestone that lends itself for its simplicity to explanation and experimentation at various levels within secondary schools. For its infinite mathematical depth, Archimedes' idea has a didactic relevance that goes beyond its practical utility; indeed, it can be proposed as a problem and approached from various angles with diverse instruments, also including the formulation of an algorithm to be tested in a programming environment tailored to the capabilities of the students. Thus a link can be formed not only between the different branches of Mathematics such as algebra, geometry, trigonometry and calculus, but also with information science, in particular with

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the computer as programming tool. In his book "*Regarding the Measurement of the Circle*" [Frajese, 1974] Archimedes establishes that the ratio between the surface of a circle and the square of its radius is equal to the ratio between its circumference and diameter; then, considering the polygons inscribed and circumscribed with 6, 12, 24, 48, and 96 sides he calculates the successive approximations of π that lead him to the following values:

$$3+\frac{10}{71}<\pi<3+\frac{1}{7} \qquad \text{o} \qquad \frac{223}{71}<\pi<\frac{22}{7}$$

In other terms, he obtains: $3.1408<\pi<3.1429$. This is an extraordinary achievement because at the time there were no algebraic notations available; moreover, Archimedes did not use our positional system to elaborate his calculations nor any calculating instrument. The geometrical method used by Archimedes involves pure abstract calculations (not measurements!). He considers a circle of radius 1 circumscribed and inscribed with polygons of 3×2^n sides. Let us indicate a_n the semi-perimeter of the circumscribed polygon and b_n that of the inscribed polygon. Geometrically, it can be demonstrated [Delahaye, 1997] that for n=1 (hexagon)

$$a_1 = 2\sqrt{3} \qquad b_1 = 3$$

and for every n:

$$\frac{1}{a_n}+\frac{1}{b_n}=\frac{2}{a_{n+1}} \qquad b_n \times a_{n+1} = \left(b_{n+1}\right)^2$$

Thus:

$$\begin{cases} a_{n+1}=\dfrac{2a_n b_n}{a_n+b_n} \\ b_{n+1}=\sqrt{b_n a_{n+1}} \end{cases} \qquad n=1,2,...... \tag{1.1}$$

By utilizing this recurring formula, it is possible to approximate π with the desired precision, provided that we are capable of calculating the square roots (that which, at that time, was realizable only by 'trial and error'). Archimedes' calculations stop at the values a_5 and b_5.

On the basis of these considerations, the rest of the work is organized as follows: in the second section an exemplification is proposed for lower secondary school (age of student 10–13) and in sections 4 and 5 for higher secondary school - for the first two year and second three year period, respectively. Finally, the article is concluded with a number of didactic reflections.

2. EXEMPLIFICATION FOR LOWER SENCODARY SCHOOLS

After the pupils have understood (for instance through basic experiments with simple materials) that the ratio between the length of the circumference and its diameter is constant, that is, indicated with c the length of the circumference and with r its radius, the ratio does not vary with the

$$\frac{c}{2r} \tag{2.1}$$

varying circumference. There remains the problem of numeric calculus of such constant. The said constant value has been traditionally indicated with the Greek letter π (pi), so we have

$$\frac{c}{2r} = \pi \tag{2.2}$$

Archimedes' idea, as we have mentioned, consists of inscribing and circumscribing regular polygons to the circumference and of approximating the length of the circumference with the perimeters of these polygons. Bearing in mind the actual level of learning, as a first step we propose to circumscribe a square to a circumference. This can be easily done using the programming environment MatCos [Costabile and Serpe, 2003 and 2009; The MatCos software can be in demand to the first author], Fig. 1. Indeed, the following programming reaches the goal:

Code MC1: square circumscribed to a circumference

```
axe=straightlineNum; A=Point_su(axe); B=Point_su(axe);
r=straightline(A,B); cancel(axe); PenStyle(5);
m=distance(A,B);p=Perpendicular(r,A);
p1=Perpendicular(r,B);
PenColour(128,0,0); PenThickness(2); PenStyle(1);
l=segment(A,B); l1=segment(A,m,p); l2=segment(B,m,p1);
C=l2.extreme(2); D=l1.exstreme(2); l3=segment(D,C);
PenStyle (5); PenThickness(1); cancel(P,P1,r);
s=straightline(A,C); s1=straightline(B,D);
Q=intersection(s,s1);
PenStyle(1); PenColour(0,0,255); PenThickness(2);
g=circ(Q,M/2); f=diameter(g); cancel(s,s1);
```

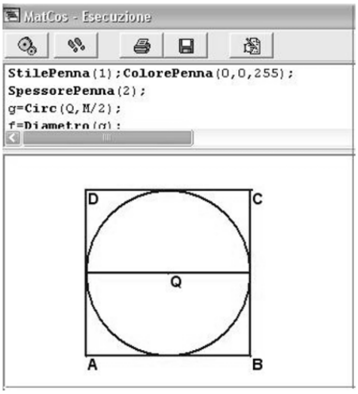

Fig. 1. Output of a square circumscribed to a circumference.

It can be easily ascertained, both with geometrical considerations and with practical verifications using the software, that the side of the square is equal to the diameter. Moreover, observing that the length of the circumference is less than the perimeter of the square we can write the following inequality:

$$c < 8r \quad \text{and thus} \qquad \frac{c}{2r} < \frac{8r}{2r} = 4$$

that is in (2.2) we have the first result:

$$\pi < 4 \tag{2.3}$$

This result can be improved considering the regular hexagon inscribed and circumscribed to the circumference, Fig. 2. This can be done with the following program:

Code MC2: hexagon inscribed and circumscribed to a circumference

```
P=point; r=readnumber("radius"); c=circ(P,r);
A=list; B=list; D=list; A(1)=point_on(c);
   for(i from 2 to 6) execute;
   A(i)=rotation(A(i-1),P,60,anticlockwise);
   end;
PenColour (128,0,64); polygon(A); segment(A(1),P);
segment(A(2),P); PenStyle(5);
r=straightline(a(1),a(2));
s=perpendicular(r,p); h=intersection(r,s);
B(1)=intersection(c,s); t1=tangent(C,B(1));
   for(i from 2 to 6) execute;
   b(i)=rotation(b(i-1),P,60,anticlockwise);
```

```
    t2=tangent(c,b(i)); t3=tangent(c,b(i-1));
    d(i-1)=intersection(t2,t3);
    cancel(t2,t3);
    end;
n=distance(d(1),d(2));
print("the side of the circumscribed hexagon is" , "
",n);
t4=tangent(c,b(6)); d(6)=intersection(t1,t4);
cancel(r,s,t1,t4); PenColour(128,0,64); polygon(d);
```

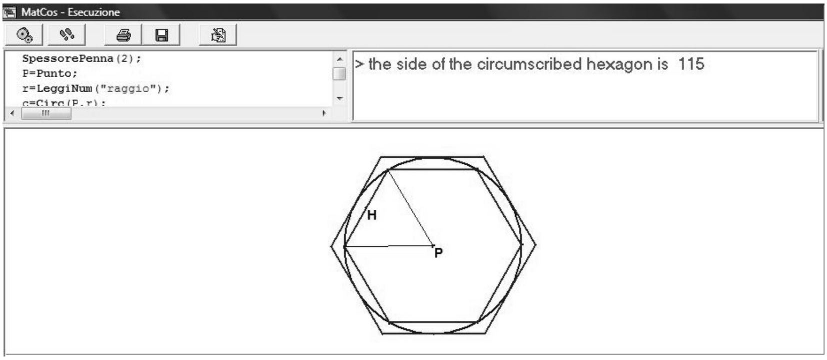

Fig. 2. Output of the hexagon inscribed and circumscribed to a circumference whit radius r=100.

By executing this program many times it becomes apparent that the length of the side of the hexagon inscribed is the same as the radius, thus the perimeter is *6r*, from which the inequality:

$$p < c \;\Rightarrow\; \frac{p}{2r} < \frac{c}{2r} = \pi \;\Rightarrow\; 3 < \pi \tag{2.4}$$

As far as the side of the circumscribed hexagon is concerned, we can obtain the measurement directly with the software. For example, for a circumference with radius *100* we find *l=115* and so the perimeter equals:

$$p = 6 \times 115 \tag{2.5}$$

Keeping in mind that in this case the inequality is $c < p$, one easily finds

$$\pi = \frac{6 \times 115}{2 \times 100} \approx 3,4$$

Thus, the regular hexagon inscribed and circumscribed produces the inequality

$$3 < \pi < 3,4 \tag{2.6}$$

The theoretical justification that the radius of the circumference equals the side of the regular hexagon inscribed can be obtained also for lower secondary schools. The measurement of the side of the circumscribed hexagon is more difficult. A better result can be obtained inscribing and circumscribing a regular dodecagon. The code of the program follows the previous idea, though it should be noted that in this case the angle of rotation should be 30°, Fig.3.

Thus we have the following code:

Code MC3: dodecagon inscribed and circumscribed to a circumference

```
P=point; r=readnumber("radius"); c=circ(P,r);
A=list; B=list; D=list; A(1)=pointatrandom_on(c);
for(i from 2 to 12) execute;
  A(i)=rotation(A(i-1),P,30,anticlockwise);
end;
m=distance(a(2),a(3)); print(m*6/r);
PenColour(128,0,64);polygon(A); PenColour (0,0,255);
segment(A(1),P); segment(A(2),P);
PenStyle(5); r1=straightline(a(1),a(2));
s=perpendicular(r1,p); h=intersection(r1,s);
B(1)=intersection(c,s); t1=tangent(C,B(1));
for(i from 2 to 12) esecute;
  b(i)=wheel(b(i-1),P,30,anticlockwise);
  t2=tangent(c,b(i)); t3=tangent(c,b(i-1));
  d(i-1)=intersection(t2,t3);
  cancel(t2,t3);
end;
m1=distance(d(2),d(3)); print(m1*6/r);
t4=tangent(c,b(12)); d(12)=intersection(t1,t4);
cancel(r,s,t1,t4); PenColour(128,0,64); PenStyle(1);
polygon(d);
```

By analogy with what has been said so far, we find the inequality

$$3,1 < \pi < 3,2 \tag{2.7}$$

which is narrower than (2.6).

Didactic activity in the lower secondary school can thus be concluded stating that π is an irrational number, that is, it has unlimited non periodic decimal representation, whose approximation to the cents is *3,14*.

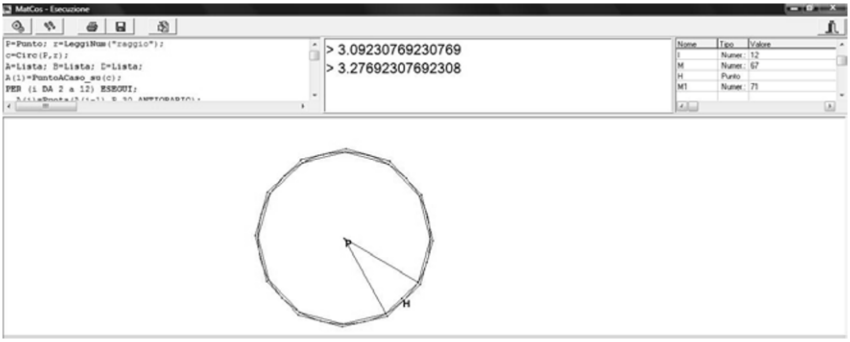

Fig. 3. Output of a dodecagon inscribed and circumscribed to a circumference with radius r=130.

3. EXEMPLIFICATION FOR HIGHER SECONDARY SCHOOL: FIRST TWO YEAR PERIOD

In the first two year period of higher secondary school the topic of calculus of π should be dealt with, and can be thoroughly investigated using Archimedes' idea with the aid of the computer as a programming tool. Indeed, especially in the 'Liceo Scientifico' (Scientific Lyceum), you can first of all demonstrate with the Method of Exhaustion that the ratio

$$\frac{c}{2r} \quad \text{is constant} \tag{3.1}$$

Thus, inscribing and circumscribing to the circumference regular polygons with an increasing number of sides, indicating with p_n e P_n the perimeters of the polygons inscribed and circumscribed respectively, with n sides, we have the inequality

$$\frac{p_n}{2} < \pi < \frac{P_n}{2} \tag{3.2}$$

where, to simplify, the radius of the circumference has been supposed equal to 1.

The inequality (3.2) enables the calculation, at least in theory, of increasingly accurate approximations of π. In order to calculate p_n e P_n you obviously need to know the measure of the side of the regular polygon with n sides inscribed and circumscribed. In broad terms, it is not easy to approach a similar problem in the first two year period of secondary school. Nevertheless, we can still reach our aim with a useful simplification which takes into consideration polygons of 3×2^n $n=1,2,\ldots$sides, as originally suggested by Archimedes.

In fact, with simple applications of Pythagoras and Euclid's theorems one can demonstrate recurring formulae which are equivalent to (1.1) and easy to program. In more precise terms, with the use of Pythagoras' theorem alone we have:

Theorem 1 [see e.g. Costabile, 1992] - If l_n *(n=3,4,5,...)* indicates the side of the regular polygon of *n* sides inscribed in the circumference with radius 1 (for convenience), then the side l_{2n} of the regular polygon inscribed of *2n* sides is given by:

$$l_{2n} = \sqrt{2-\sqrt{4-l_n^2}} \tag{3.3}$$

This recurring formula, keeping in mind that the side of the hexagon inscribed in the circumference is equal to the radius, enables us to calculate the side of the polygon of *12, 24, 48, 96,....* sides. For example we have:

$$l_6 = 1$$

$$l_{12} = \sqrt{2-\sqrt{3}}$$

$$l_{24} = \sqrt{2-\sqrt{2-\sqrt{3}}}$$

....

As regards the side of the regular polygon of *n* circumscribed to the circumference, using Euclid and Pythagoras' theorems we can demonstrate the following:

Theorem 2 [see e.g. Costabile, 1992] - Let L_n be the side of the regular polygon of *n* sides circumscribed to the circumference with radius 1 and l_n the side of the regular polygon with the same number of sides inscribed, then we have the following relation

$$L_n = \frac{2l_n}{\sqrt{4-l_n^2}} \tag{3.4}$$

Combining (3.3) and (3.4) we derive the inequality

$$\frac{nl_n}{2} < \pi < \frac{nl_n}{\sqrt{4-l_n^2}} \tag{3.5}$$

which enables us to calculate easily approximations -rounding up and rounding down- of π, starting from *n=6* and doubling the number of sides at each step. The formulae (3.3) and (3.4) give the same results as formula (1.1), that is with the Archimedes method. We can obtain an alternative demonstration of (1.1) with trigonometric identities [see e.g. Weisstein, 2010 – From MathWorld – A Wolfram Web]. On the chain of inequalities (3.5) we can construct an algorithm which calculates π with a previously established precision; we can assume as estimate of the error:

$$\frac{1}{2}(P_n - p_n)$$

that is
$$nl_n\left[\frac{2-\sqrt{4-l_n^2}}{2\sqrt{4-l_n^2}}\right] \tag{3.6}$$

The codification in MatCos of the algorithm in question can be the following:

Code MC4: Archimedes' algorithm

```
n=6; L=1; eps=readnumber("desidered precision");
d=10;
Execute Until (d>eps);
   Pinf=n*L/2;  Psup=n*L/SquareRoot(4-L^2);
   d=(Psup-Pinf);
   n=n*2;
   L= SquareRoot(2-SquareRoot(4-L^2));
   end;
Print("for  n=  ",n,  "  the  rounding  down  value  is
",pinf);
Print("per n= ",n, " the rounding up value is ",psup);
Print("Pi is comprised between ", Pinf, " and ", Psup);
```

Assuming $\varepsilon = 10^{-3}$ we obtain the following values in output (Fig. 4):

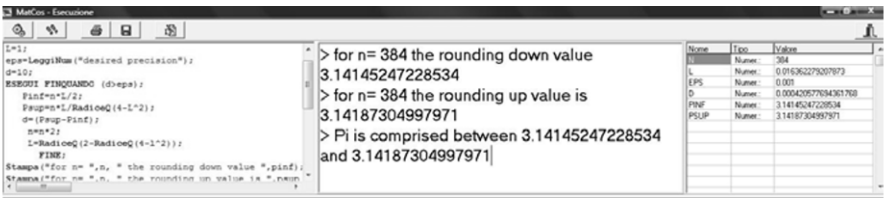

Fig. 4. Output of π with (2.6) and precision $\varepsilon=10^{-3}$.

The precision can be increased up to $\varepsilon = 10^{-13}$, thus the following values in output can be obtained (Fig. 5):

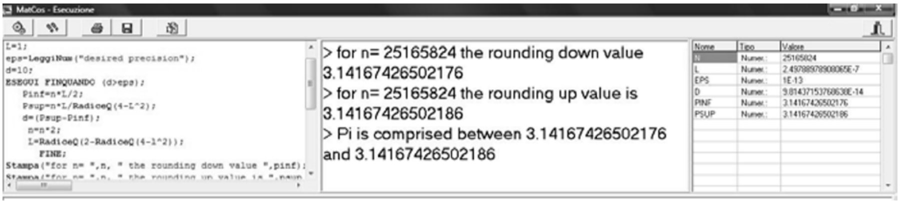

Fig. 5. Output of π with (2.6) and precision ε=10^{-13}.

For $\varepsilon = 10^{-14}$ the result we obtain is wrong, Fig. 6. This is not due to errors in the algorithm or in the MatCos code, but for cause of the error of rounding and his propagation. This gives us the chance to introduce and investigate the relevant topic of the error of rounding which is due to the nature of numbers and the use of Finite Arithmetic. In fact, substituting the formula (3.3) with the other equivalent

$$l_{2n} = \frac{l_n}{\sqrt{2+\sqrt{4-l_n^2}}} \tag{3.7}$$

the previous program (Code MC4) produces positive results also for $\varepsilon = 10^{-14}$, that is it gives 15 exact digits.

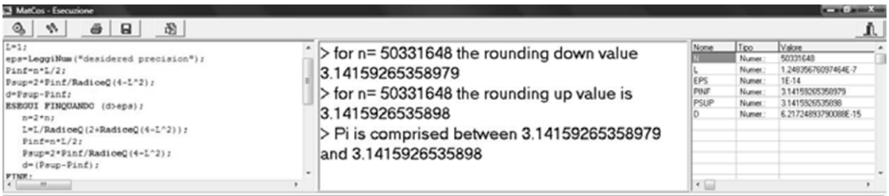

Fig. 6. Output of π with (2.7) and precision ε=10^{-14}.

4. EXEMPLIFICATION FOR HIGHER SECONDARY SCHOOL: THREE YEAR PERIOD

During the three year period it is advisable to go back to the calculus of π after the fundamental notions of trigonometry have been acquired, as well as the Taylor series, if a more in depth study of error is required [see e.g. Costabile, 2003]. Applying the Law of Sines and well-known trigonometric identities, the side of the regular polygon inscribed in the circumference with radius 1 is:

$$l_n = 2\sin\frac{\pi}{n}$$

and thus:
$$\frac{p_n}{2} = \frac{nl_n}{2} = n\sin\frac{\pi}{n} \tag{4.1}$$

Taking into account Taylor – McLaurin's development of the function $\sin x$ we have:

$$\frac{p_n}{2} = n\left[\frac{\pi}{n} - \frac{1}{3!}\left(\frac{\pi}{n}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{n}\right)^5 + \dots\dots\dots\right] \tag{4.2}$$

from which
$$\pi - \frac{p_n}{2} = O\left(n^{-2}\right) \tag{4.3}$$

which provides the asymptotic development of the error. Finally, keeping in mind that for $x > 0$ $\sin x > x - \frac{x^3}{6}$, from (4.1) we can obtain the increase of the error:

$$\pi - \frac{p_n}{2} < \frac{\pi^3}{6n^2} < \frac{\left(\frac{22}{7}\right)^3}{6n^2} \approx \frac{5.17}{n^2} \tag{4.4}$$

Combining (4.1) and (4.4) we can obtain an algorithm with a-priori estimate of the error, which is easy to implement in MatCos since the command $\sin(\alpha)$ is available with α allocated in degrees, Fig. 7. A possible code is the following:

Code MC5: Archimedes' algorithm with a-priori estimate of the error

```
n=readnumber("number of sides of initial polygon");
eps=readnumber("required error");
d=10; p=n*sin(180/n);
   Execute Unitile (d>eps);
   n=n*2;
   p=n*sin(180/n); d=5.17/(n^2);
   end;
print(" the approximated value is ", p);
print(" number of sides is ", n);
print(" effective error is ", pi-p);
print(" estimated error is ", d);
```

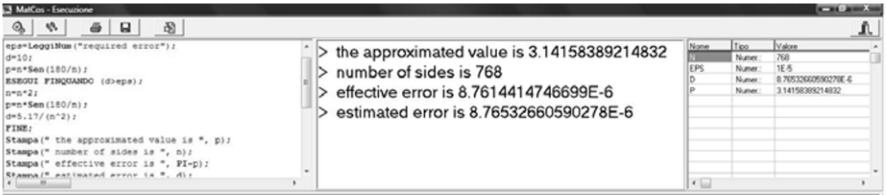

Fig. 7. Output with n=24 assigned and precision $\varepsilon = 10^{-5}$.

5. CONCLUSION

The outlined process appears to reach the aims set out in the introduction. The suggested didactic activity is varied, with ideas ranging across different branches of Mathematics. Furthermore, the use of the computer within a programming environment on the one hand explores an interdisciplinary approach between Mathematics and Information Technology, while on the other it offers a modern approach which connects the past to the present and is in line with recent scientific-technological innovations. Also, the high educational value of programming, albeit at a simple level, in terms of the development of logical-rational skills, creativity, deduction and intuition are well-known. Finally, the themes proposed here represent the opportunity for an historical overview of mathematical thought, and point out how Archimedes can still be a master in the field of Mathematics, and how his genius still inspires modern science.

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