ON ARCHIMEDEAN ROOTS IN TORRICELLI'S MECHANICS

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ABSTRACT In recent papers we analyzed the historical development of the foundations of the centres of gravity theory during the Renaissance. Using these works as a starting point, we shall briefly present a progression of knowledge with cultural and mathematical Archimedean roots in Torricelli's mechanics.

1. INTRODUCTION

Archimedes (287–212 B.C.) was a deeply influential author for Renaissance mathematicians according to the two main traditions. The humanistic tradition, adhering strictly to philological aspects, followed by Willem van Moerbeke (1215–1286), Regiomontanus (1436–1476) and Federigo Commandinus (1509–1575). The pure mathematical tradition followed by Francesco Maurolico (1694–1575), Luca Valerio (1552–1618), Galileo Galilei (1564–1642), Evangelista Torricelli (1608–1647).

The investigation into Archimedes's influence on Torricelli has a particular relevance because of its depth. Also it allows us to understand in which sense Archimedes' influence was still relevant for most scholars of the seventeenth century (Napolitani 1988). Besides there being a general influence on the geometrization of physics, Torricelli was particularly influenced by Archimedes with regard to mathematics of indivisibles. Indeed, it is Torricelli's attitude to confront geometric matter both with the methods of the ancients, in particular the exhaustion method, and with the indivisibles, so attempting to compare the two, as is clearly seen in his letters with Cavalieri (Torricelli 1919–1944; see mainly vol. 3). Torricelli, in particular, solved twenty one different ways the squaring a parabola (Heath 2002; *Quadrature of the parabola*, Propositio 17 and 24, p. 246;

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p. 251), a problem already studied by Archimedes: eleven times with exhaustion, ten with indivisibles. The *reductio ad absurdum* proof is always present.

Based on previous works (Pisano 2008) we can claim that the Archimedean approach to geometry is different from the Euclidean one. The object is different, because Archimedes mainly deals with metric aspects, which was quite new, also the aim is different, being more oriented towards solving practical problems. In addition, mainly the theory organization is different, because Archimedes does not develop the whole theory axiomatically, but sometimes uses an approach for problems, characterized by *reductio ad absurdum*. Moreover, the epistemological status of the principles is different. Archimedean principles are not always as self evident as those of the Euclidean tradition and may have an empirical nature. Some of the Archimedean *principles* have a clear methodological aim, and though they may express the daily feeling of the common man, they have a less cogent evidence then the principles of Euclidean geometry.

Knowledge of Archimedes' contribution is also fundamental to an historical study of Torricelli's mechanics. Archimedes was the first scientist to set *rational criteria* for determining centres of gravity of bodies and his work contains physical concepts formalised on mathematical basis. In *Book I* of the *On Plane Equilibrium* (Heath 2002) Archimedes, besides studying the rule governing the law of the lever also finds the centres of gravity of various geometrical plane figures (Heath 2002, Clagett 1964–1984; Heiberg 1881). By means of his *Suppositio* (principles) Archimedes (Heath 2002, pp. 189–202) is able to prove *Propositio* (theorems) (Heath 2002, pp. 189–202) useful in finding the centres of gravity of composed bodies. In particular, the sum of all the components may require the adoption of the method of exhaustion.

Archimedes's typical method of arguing in mechanics was by the use of the reduction *ad absurdum*, and Torricelli in his study on the centres of gravity resumes the same approach.

With regard to Torricelli's works, we studied mainly his mechanical theory (Capecchi and Pisano 2004; Idem 2007; Pisano 2009) in the *Opera geometrica* (Torricelli 1644), Table 1 and Fig. 1. We focused in detail on his discourses upon centres of gravity (Pisano 2007) where he enunciated his famous principle:

"It is impossible for the centre of gravity of two joined bodies in a state of equilibrium to sink due to any possible movement of the bodies".

The *Opera geometrica* is organized into four parts. Particularly, parts 1, 2, 3, are composed of *books* and part 4 is composed of an *Appendix*. Table 1 shows the index of the text:

Table 1. An index of *Opera geometrica* (Torricelli's manuscripts are now preserved at the Biblioteca Nazionale of Florence. Galilean Collection, n° 131–154).

De sphaera et solidis sphaeralibus, Liber primus, 3–46; Liber secondus, 47–94.
De motu gravium naturaliter descendentium et proiectorum, Liber Primus, 97–153; Liber secundus, 154–243.
De dimensione parabolae Solidique Hyperbolici, 1–84.
Appendix: *De Dimensione Cycloidis*, 85–92.
De Solido acuto Hyperbolico, 93–135.
De Dimensione Cochlea, 136–150.

OPERA
GEOMETRICA
EVANGELISTÆ
TORRICELLII

De Solidis Sphæralibus
De Motu.
De Dimensione Parabolæ
De solido Hyperbolico
Cum Appendicibus de Cycloide, & Cochlea.

Fig. 1. The front page of Torricelli's Opera geometrica with the index of content.

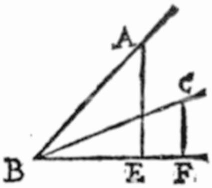
Torricelli in his theory on the centre of gravity, following Archimedes' approach, uses

a) *Reductio ad absurdum* as a particular instrument for mathematical proof.
b) Geometrical representation of physical bodies: weightless beams and reference in geometrical form to the law of the lever.
c) Empirical evidence to establish principles.

We focused mostly upon the exposition of studies contained in *Liber primis. De motu gravium naturaliter descendentium,* where Torricelli's principle is exposed, Fig. 2 and 3. In Galileo's theory on dynamics, Torricelli present problems which, according to him, remain unsolved. His main concern is to prove a Galileo's supposition, which states: velocity degrees for a body are directly proportional to the inclination of the plane over which it moves:

Liber Primus

demonstratione confirmabimus: protinus ad ostendendum id quod Galileo principium siue petitio est, accedemus.



Præmittimus.

Duo grauia simul coniuncta ex se moueri non posse, nisi centrum commune grauitatis ipsorum descendat.

Quando enim duo grauia ita inter se coniuncta fuerint, vt ad motum vnius motus etiam alterius consequatur, erunt duo illa grauia tamquam graue vnum ex duobus compositum, siue id libra fiat, siue troclea, siue qualibet alia Mechanica ratione, graue autem huiusmodi non mouebitur vnquam, nisi centrum grauitatis ipsius descendat. Quando verò ita constitutum fuerit vt nullo modo commune ipsius centrum grauitatis descendere possit, graue penitus in sua positione quiescet: alias enim frustra moueretur; horizontali, scilicet latione, qua nequaquam deorsum tendit.

PROPOSITIO I.

SI in planis inæqualiter inclinatis, eandem tamen eleuationem habentibus, duo grauia constituantur, quæ inter se eandem homologè rationem habeant quam habent longitudines planorum, grauia æquale momentum habebunt.

Fig. 2. Torricelli's principle. Opera Geometrica. De motu gravium naturaliter descendentium et proiectorum, p. 99.

The speeds acquired by one and the same body moving down planes of different inclinations are equal when the heights of these planes are equal (Galilei 1890–1909, Vol., VIII, p. 205)

Torricelli seems to suggest that this supposition may be *proved* beginning with a "theorem" according to which "the momentum of equal bodies on planes unequally inclined are to each other as the perpendicular lines of equal parts of the same planes" (Torricelli 1644, *De motu gravium naturaliter descendentium et proiectorum*, p. 99). Moreover, Torricelli also assumes that this theorem has not yet been demonstrated (Note, in the first edition of the Galileo's *Discorsi* in 1638, there is no proof of the "theorem". It was added only in 1656 to the *Opere di Galileo Galilei linceo*, (Galilei 1656). However Torricelli knew it, as is clear in some letters from Torricelli to Galileo regarding the "theorem"; Torricelli 1919–1944, Vol. III, p. 48, pp. 51, 55, 58, 61).

2. ARCHIMEDEAN THINKING

Torricelli frequently declares and explains his Archimedean background.

> Inter omnia opera Mathematics disciplinas pertinentia, iure optimo Principem sibi locum vindicare videntur Archimedis; quae quidem ipso subtilitatis miraculo terrent animos (Torricelli 1644, *Proemium*, p. 7).

Archimedes, in the *Quadratura parabolae*, first obtains results using the mechanical approach and then reconsiders the discourse with the classical methods of geometry to confirm in a rigorous way the correctness of his results (Heath 2002). Similarly, Torricelli, with the compelling idea of duplicating the procedure, devotes many pages to proving certain theorems on the "parabolic segment", by following, the geometry used in pre-history ancients (Torricelli (1644), *Quadratura parabolae pluris modis per duplicem positionem more antiquorum absoluta*, pp. 17–54)[1] and then proving the validity of the thesis also with the "indivisibilium" (Heath 2002, *Quadratura parabolae,* pp. 253–252; pp. 55–84; Torricelli 1644, *De solido acuto hyperbolico problema alterum*, pp. 93–135). In this respect, it is interesting to note that he underlines the "concordantia" (Torricelli 1644, *De solido acuto hyperbolico problema alterum*, p. 103) of methods of varying rigour.

> Hactenus de dimensione parabolae more antiquorum dictum sit; Reliquum est eandem parabolae mensuram nova quedam, sed mirabili ratione aggrediamur; ope scilicet Geometriae Indivisibilium, et hoc diversis modis: Suppositis enim praecipui Theorematib. antiquorum tam Euclidis, quam Archimedis, licet de rebus inter se diversissimis sint, mirum est ex unoquoque eorum quadraturam parabolae facili negotio elici posse; et vive versa. Quasi ea sit commune quoddam vinculum veritatis. […] Contra vero: supposita parabolae quadratura, praedicta omnia Theoremata facile demonstrari possunt. Quod autem haec indivisibilium Geometria novum penitus inventum sit equidem non ausim affirmare. Crediderim potius veteres Geometras hoc metodo usos in inventione Theorematum difficillimorum quamquam in demonstrationibus aliam viam magis probaverint, sive ad occultandum artis arcanum, sive ne ulla invidis detractoribus proferretur occasio contradicendi (Torricelli 1644, *Quadratura parabolae per novam indivisibilium Geometriam pluribus modis absoluta*, p. 55, *op. cit.*).

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[1] In the original manuscripts of *Opera geometrica* there are some glosses to Eulid's *Elements*, to Apollonius' *Conic sections*, to Archimedes, Galileo, Cavalieri's works, et al., autograph by Torricelli.

From the previous passage there appears not only the desire to give the reader results and methods, but also to say that the indivisibles technique was not completely unknown to the ancient Greek scholars. Besides, Torricelli seems to hold onto the idea that the method of demonstration of the ancients, such as the Archimedes' method, was intentionally kept secret. He states that the ancient geometers worked according to a method "in invenzione" suitable "ad occultandum artis arcanum" (Torricelli 1644, *Quadratura parabolae per novam indivisibilium Geometriam pluribus modis absoluta*, p. 55).

However the Archimedean influence in Torricelli goes further. The well known books *De sphaera et solidis sphaeralibus* (Torricelli 1644, *Liber primus*, 3–46) present an enlargement of the Archimedean proofs of books I–II of *On the sphere and cylinder* (Heath 2002, pp. 1–90).

> […] In quibus Archimedis Doctrina de sphaera & cylindro denuo componitur, latius promovetur, et omni specie Solidorum, quae vel circa, vel intra, Sphaeram, ex conversione polygonorum regularium gigni possint, universalius Propagatur (Torricelli 1644, *De sphaera et solidis sphaeralibus*, p. 2).

In other parts Torricelli faces problems not yet solved by Archimedes, or by the other mathematicians of antiquity. With the same style as Archimedes, he does not try to arrive at the first principles of the theory and does not limit himself to a single way of demonstrating a theory.

> Veritatem praecedentis Theorematis satis per se claram, et per exempla ad initium libelli proposita confirmatam satis superque puto. Tamen ut in hac parte satisfaciam lectori etiam Indivisibilium parum amico, iterabo hanc ipsam demonstrationis in calce operis, per solitam veterum Geometrarum viam demonstrandi, longiorem quidem, sed non ideo mihi certiorem (Torricelli 1644, *De solido hyperbolico acuto problema secundum,* p. 116).

We note that the exposition of the mechanical argumentation present in Archimedes's *Method* was not known at Torricelli's time because Johan Heiberg only discovered it in 1906 (Heath 1912). Therefore, in Archimedes's writing there were lines of reasoning which, because a lack of justification, were labelled as mysterious by most scholars. Thus in such instances it was necessary to assure the reader of the validity of the thesis and also to convince him about the strictness of Archimedes' approaches, particularly exhaustion reasoning and *reductio ad absurdum*, by proving his results with some other technique.

> The appearance of approximation [in Archimedes's proofs] is surely a substantial innovation in the mathematical demonstrations and the difference between

Elements and Archimedes' work is a sign of a mentality more opened towards applications, and perhaps that the classical epoch of geometry was closed (Marchini 2005, pp. 189–190).

3. ON PROOFS

It is well known from the *Method* (Heiberg 1913) that Archimedes studied a given problem whose solution he anticipated by means of crucial propositions which were then proved by the *reductio ad absurdum* or by exhaustion. Indeed Archimedes's himself did not attribute the same amount of certainty to his *Method* proofs, as he attributes to classical mathematical proofs. His reasoning on *Quadratura parabolae* (Heath 2002, Proposition 24, p. 251) is exemplary. Addressing Eratosthenes (276–196 B.C.), Archimedes wrote at the beginning of his *Method*:

> […] I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge (Heath 1913, p. 13).

One of the characteristics of Torricelli's proofs was the syntactic return to the demonstration approach followed by the ancient Greeks, with the explicit description of the technique of reasoning actually used. Besides the well known *ad absurdum* there were also the *permutando* and *the ex aequo*. In *De proportionibus liber* he defines them explicitly:

> Propositio IX. Si quatuor magnitudines proportionales fuerint, et permutando proportionales erunt. Sint quatuor rectae lineae proportionales *AB, BC, CD, DE*. Nempe ut *AB* prima ad *BC* secundam, ita sit *AD* tertia ad *DE* quartam. Dico primam *AB* ad tertiam *AD* ita esse ut secunda *BC* ad quartam *DE*. Qui modus arguendi dicitur permutando (Torricelli 1919–1944, *De Proportionibus liber*, p. 313)
>
> Propositio X. Si fuerint quotcumque, et aliae ipsis aequales numero, quae binae in eadem ratione sumantur, et ex aequo in eadem ratione erunt. Sint quotcumque magnitudines A, B, C, H, et aliae ipsis aequales numero D, E, F, I, quae in eadem ratione sint, si binae sumantur, nempe ut A ad B ita sit D ad E, et iterum ut B ad

C, ita sit E ad F, et hoc modo procedatur semper. Dico ex equo ita esse primam A ad ultimam H, uti est prima D ad ultimam I. Qui modus arguendi dicitur ex aequo (Torricelli 1919–1944, *De Proportionibus liber*, p. 314).

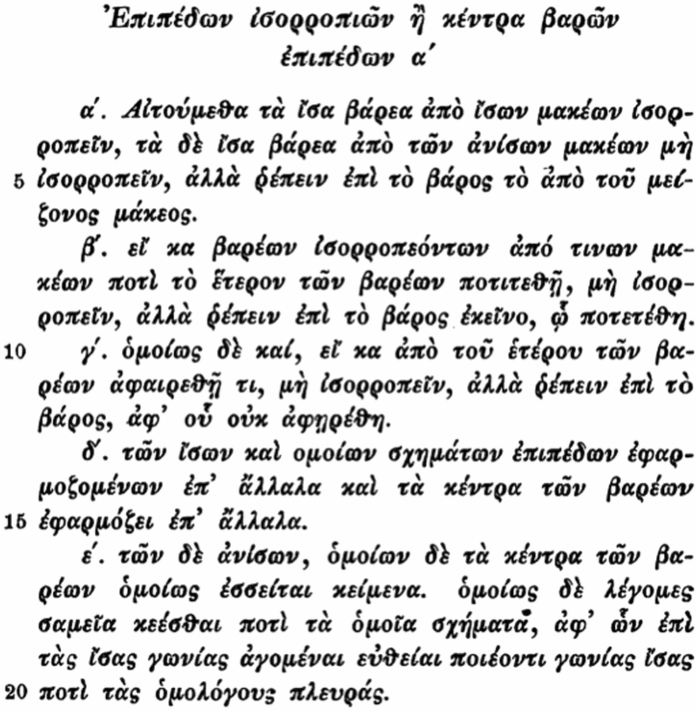
Torricelli seems to neglect the algebra of his time and adheres to the language of proportions. He dedicated a book to this language, *De Proportionibus liber* (Torricelli 1919–1944, pp. 295–327), where he only deals with the theory of proportions to be used in geometry. In such a way he avoids the use of the *plus* or *minus*, in place of which he utilizes the *composing* (Torricelli 1919–1944, p. 316) and *dividing* (Idem, p. 313). Such an approach allows him to work always with the ratio of segments. By following the ancients to sum up segments he imagines them as aligned and then translated and connected, making use of terms like "simul", "et" or "cum" (Torricelli 1919–1944, Prop. XV, p. 318). In what follows we present a table which summarizes the most interesting part of *Proportionibus liber* where Torricelli proves again theorems by referring to reasoning in the Archimedean manner, Table 2.

Table 2. Some Torricelli's Archimedean proofs in *Quadratura parabolae.*

Lemma II,V,VI, X–XI,XII–XIII, XVII – Propositio IV	Ad absurdum proofs
Lemma XIV	Ex aequo et dividendo et permutando
Lemma XVI, XVIII	Ex aequo
Lemma XIX	Ex aequo et Ad absurdum
Propositio III[2]	Componendo
Propositio V	Ad absurdum et Componendo
Propositio IX	Ex aequo et Ad absurdum

We notice that proofs by means of indivisibles are not *reductio ad absurdum*. This is so because these proofs are algebraic. Instead, in nearly all other proofs Torricelli uses the technique typical of proportions, *dividendo, permutando* and *ex aequo*.

[2] In proposition III Torricelli, referring to Luca Valerio, proves a Lemma differently from him: "Libet hic demonstrare Lemma Lucae Valerij, nostro tamen modo, diversisque penitus Mechanicae principijs. Ipse enim utitur propositione illa, qua ante demonstraverat centrum gravitatis hemisphereij. Nos autem simili ratione ac in praecedentibus [I–II], demonstrabimus et Lemma, et ipsam Valerij conclusionem" (Torricelli1644, *Quadratura parabolae pluris modis per duplicem positionem more antiquorum absoluta*, p. 33; Valerio 1604, book II, p. 12).



Ἐπιπέδων ἰσορροπιῶν ἢ κέντρα βαρῶν
ἐπιπέδων α΄

α΄. Αἰτούμεθα τὰ ἴσα βάρεα ἀπὸ ἴσων μακέων ἰσορροπεῖν, τὰ δὲ ἴσα βάρεα ἀπὸ τῶν ἀνίσων μακέων μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος τὸ ἀπὸ τοῦ μείζονος μάκεος.

β΄. εἴ κα βαρέων ἰσορροπεόντων ἀπό τινων μακέων ποτὶ τὸ ἕτερον τῶν βαρέων ποτιτεθῇ, μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος ἐκεῖνο, ᾧ ποτετέθη.

γ΄. ὁμοίως δὲ καί, εἴ κα ἀπὸ τοῦ ἑτέρου τῶν βαρέων ἀφαιρεθῇ τι, μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος, ἀφ' οὗ οὐκ ἀφῃρέθη.

δ΄. τῶν ἴσων καὶ ομοίων σχημάτων ἐπιπέδων ἐφαρμοζομένων ἐπ' ἄλλαλα καὶ τὰ κέντρα τῶν βαρέων ἐφαρμόζει ἐπ' ἄλλαλα.

ε΄. τῶν δὲ ἀνίσων, ὁμοίων δὲ τὰ κέντρα τῶν βαρέων ὁμοίως ἐσσεῖται κείμενα. ὁμοίως δὲ λέγομες σαμεῖα κεέσθαι ποτὶ τὰ ὁμοῖα σχήματα, ἀφ' ὧν ἐπὶ τὰς ἴσας γωνίας ἀγομέναι εὐθείαι ποιέοντι γωνίας ἴσας ποτὶ τὰς ὁμολόγους πλευράς.

Fig. 3. Archimedes' first suppositio: On plane equilibrium, Heiberg 1881, p. 142.

4. CONCLUSION

We focused on conceptual aspects of Archimedes' and Torricelli's studies of the centre of gravity theory based on previous investigations on Archimedes' *On the Equilibrium of Plane* and Torricelli's *Opera geometrica*. In the present work we have outlined some of the fundamental concepts common to the two scholars: the logical organization and the paradigmatic discontinuity with respect to the Euclidean technique. Indeed Archimedes' theory (mechanical and geometrical) does not appear to follow a unique pattern. It maintains two kinds of organization, one problematic the other axiomatic deductive.

In conclusion, to compare the science of Archimedes and Torricelli from an epistemological point of view, we resume in Table 3 the crucial aspects of their theory organization.

Table 3. Archimedes' and Torricelli's foundations of theory.

	Archimedes	Torricelli
Organization of the theory	– Problematic (mechanics) – Axiomatic (geometry)	– Problematic (mechanics) – Axiomatic (geometry)
Body systems	– Without explicating the type of connection	– Aggregate – Tied up way or untied
Foundational concept	– Centre of gravity	– Centre of gravity of Archimedes
Type of infinite	– Potential Infinitum – Toward Actual Infinitum	– Potential Infinitum – Actual Infinitum (indivisibles)
Central problem of the theory	– Criteria to determinate the centre of gravity for single and composed geometrical bodie	– Galileo's ballistic theory by means of Archimedean equilibrium theory
Techniques of arguing	– *Reductio ad Absurdum*	– *Reductio ad Absurdum*
Techniques of calculus	– Method of exhaustion	– Archimedes's method of exhaustion – Indivisible method

The breaking of the Euclidean paradigm, in the Khun sense (Khun 1962), by Archimedes could offer, with the limitation implicit in the concept of paradigm, a first possible lecture key. We pass from a normal science composed of axioms and self–evidence to a new science where proof also means also to find a field of applicability of a new theory, the centrobarica.

REFERENCES

Capecchi D., Pisano R., Il principio di Torricelli prima di Torricelli in: L'eredità di Fermi, Majorana e altri temi, Proceedings XXIV SISFA, Napoli, 2004, Bibliopolis, Napoli, 2007, pp. 107–112.

Capecchi D., Pisano R., La teoria dei baricentri di Torricelli come fondamento della statica, Physis, Vol. XLIV, 1, 2007, pp. 1–29.

Clagett M., Archimedes in the Middle Ages, Madison–Philadelphia, Clarendon University Press, Oxford, 1964–1984.

Galilei G., Opere di Galileo Galilei, by A. Favaro, Barbera, Firenze, 1890–1909.

Galilei G., Opere di Galileo Galilei linceo, Dozza, Bologna, 1656.

Heath T.L., The Method of Archimedes Recently Discovered by Heiberg. A Supplement to the Works of Archimedes, (1897), Cambridge University Press, Cambridge, 1912.

Heath T.L., The works of Archimedes, Dover Publications INC, Mineola, N.Y., 2002.

Heiberg J.L., Archimedes Opera omnia, BG Teubneri, Studgardiae, 1881.

Kuhn T.S., The Structure of scientific revolutions, University of Chicago Press, Chicago, 1962.

Napolitani P.D., La tradizione archimedea in: Luca Pacioli e la matematica del Rinascimento, Fondazione Piero della Francesca–Petruzzi, San Sepolcro, 1998, pp. 77–102.

Pisano R., Il ruolo della scienza archimedea nei lavori di meccanica di Galilei e di Torricelli, in: Da Archimede a Majorana: La fisica nel suo divenire, XXVI SISFA, Roma 2006, Guaraldi, Rimini, 2009, pp. 65–74.

Pisano R., Brief history of centre of gravity theory. Epistemological notes in: Proceedings of ESHS Congress, Polish Academy of Science, 2007, pp. 934–941.

Pisano R., The role played by mechanical science in the architects and engineers in the Renaissance (in Italian), Ph.D. Thesis, University of Roma "La Sapienza", 2008, 2 vols.

Torricelli E., Opera geometrica, Masse & de Landis, Florentiae, 1644.

Torricelli E., Opere di Evangelista Torricelli, by Loria G., Vassura G., vols. I–II–II–IV, Montanari, Faenza, 1919–1944.

Valerio L., De centris gravitatis solidorum libritres. Typis B. Bonfadini, Romae, 1604.