1
A programme of general semantics

1.1 The programme in outline

Transparent Intensional Logic is a logical theory developed with a view to logical analysis of sizeable fragments of primarily natural language.

It is an unabashedly Platonist semantics that proceeds top-down from structured meanings to the entities that these meanings are modes of presentation of. It is a theory that, on the one hand, develops syntax and semantics in tandem while, on the other hand, keeping pragmatics and semantics strictly separate. It disowns *possibilia* and embraces a fixed domain of discourse. It rejects individual essentialism without quarter, yet subscribes wholeheartedly to intensional essentialism. It denies that the actual and present satisfiers of empirical conditions (possible-world intensions) are ever semantically and logically relevant, and instead replaces the widespread semantic actualism (that the actual of all the possible worlds plays a privileged semantic role) by a thoroughgoing anti-actualism. And most importantly, it unifies unrestricted referential transparency, unrestricted compositionality of sense, and all-out hyperintensional individuation of senses and attitudes in one theory.

The way we understand the enterprise of logical analysis of (natural) language, it is neither eliminative nor reductive, but selective. The analysis selects particular features of language, leaving all the remaining untouched and unscathed. We obviously acknowledge the pragmatic categories of (act of) assertion, language acquisition, communication, speaker's intention, etc. And we acknowledge no less the full range of pragmatic paraphernalia that keep natural language lubricated and running, including non-verbal winks and nods, hints and clues. But while they exist in their own right, they are immaterial to the project of, ideally, isolating all, and only, logically salient features of (natural) language. So we blot out what is in effect the vast bulk of natural language in order to zoom in on the remaining fragment and blow it large, as it were, with a view to studying it in more detail.

Yet the very name of our theory, 'Transparent Intensional Logic', is likely to strike one as being an oxymoron, like 'roaring silence'. How can there possibly be a *logic* that is *intensional* and at the same time *transparent*? Is not any intensional logic one which fails to heed various laws of extensional logic, such as referential transparency, substitution of identicals, and compositionality? Certainly, if 'intensional' is synonymous with 'non-extensional', then any logic is indeed intensional which fails to comply with one or more of the principles and rules of extensional logic. But 'intensional' may also mean—and this is the notion of intensionality germane to Transparent Intensional Logic—that the logic in question comes with an ontology of intensional entities and the means to logically manipulate these entities.

M. Duží et al., *Procedural Semantics for Hyperintensional Logic*, Logic, Epistemology, and the Unity of Science 17, DOI 10.1007/978-90-481-8812-3_1,

Transparent Intensional Logic flouts none of the principles of extensional logic and is, insofar, an extensional logic.

The underlying project is to operate with only one semantics for all kinds of logical-semantic context while adhering to the compositionality principle throughout. This universal semantics is obtained by developing a semantic theory for the hardest case (to wit, hyperintensional attitude contexts) and extending it to all the other cases (i.e., ‘generalising from the hardest case’). Less-hard cases demand less logical and semantic sophistication, and irrelevant subtleties are weeded out by installing cruder principles of individuation and substitution.

Referential transparency is the phenomenon that any term or expression—when used in a communicative act—expresses the same entity as its meaning and denotes the same entity as its denotation (or ‘semantic value’) regardless of the embedding context. This means rejecting so-called reference shift across the board. Instead the ‘shifts’ that reference-shift is intended to trigger are brought about by distinguishing between two different ways in which the meaning of a word may occur relative to a logical-semantic context, namely with either supposition *de dicto* or *de re*. The point is that the a priori relation between word and sense fixes a sense, which exhausts the function of the word. The so fixed sense may consequently be subjected to logical manipulation, for instance, by being made to occur with supposition *de dicto* or else *de re*.

Transparent Intensional Logic also contains the resources to distinguish in a principled manner between functions and their values. This is because the underlying logic is a (typed) lambda calculus (equipped with partial functions). Church’s logic of functions has been around for 70-odd years now, and is well-integrated into logical lore. Our main departure from how Church understood his calculus is that, in Transparent Intensional Logic, the terms for functional abstraction and functional application do not denote functions and functional values, respectively. Instead they denote multiple-step structures specifying how to form functions and functional values, respectively. We conceive of these structures as procedures whose products are either functions or functional values. Our theory’s own word for such structured procedures is *construction*.

Intuitively, constructions are procedures, of one or more steps, for inputting and outputting entities. Tichý often likens constructions to calculations.[1,2] Just as

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[1] The idea of linguistic sense as a calculation will be familiar not least from Moschovakis’ work on constructive semantics. See Section 1.5 for discussion.

[2] Muskens, in (2005, p. 474, n. 2), interprets constructions as ‘procedures that can be used to compute [Fregean] references]’ (ibid., p. 474), which is basically on the right track. We agree, with one proviso, with Muskens’ characterisation of a computational, or procedural, interpretation of Fregean sense: ‘If senses are a certain kind of algorithms, then two senses are identical if the corresponding algorithms are. While identity of algorithms itself is a non-trivial problem, this at least gives something to start with’ (Ibid.). The proviso is that constructions are allowed to be non-finitary. With this proviso in mind, we subscribe to the general ‘propositions-as-algorithms picture’ that Muskens sketches in (ibid., pp. 487ff). For an introduction to how reference-fixing along Fregean lines works in Martin-Löf’s type theory, see Primiero (2004) and (2008).

an arithmetic calculation takes numbers, processes them and yields other numbers, so constructions are, semantically speaking, calculations whose results may be, for instance, truth-values, truth-conditions, individuals, numbers, sets, properties, as well as other calculations. It is important not to confuse *procedures* (calculations) with the agent-, world-, and time-indexed *processes* of executing the procedures (i.e., individual cases of actual calculating) or with their *products* (results, output) or with the symbolic encoding of a computer *programme* in a programming language. *Construction* is the single most important notion of Transparent Intensional Logic, and its defining feature *par excellence*. It is anchored to an older notion of function as being more than a mere mapping from one set into another:

> In the 1920s, when [the lambda calculus and combinatory logic] began, logicians did not automatically think of functions as sets of ordered pairs, with domain and range given, as they are trained to do today. Throughout mathematical history, right through to modern computer science, there has run another concept of functions, less precise but strongly influential; that of a function as an operation-process (in some sense) which may be applied to certain objects to produce other objects. Such a process can be defined by giving a set of rules describing how it acts on arbitrary input-objects (The rules need not produce an output for every input.) (Hindley and Seldin, 1986, p. 44).

The constructions of Transparent Intensional Logic are intended precisely as such 'operation-processes' that receive an input and deliver an output (or in well-defined cases fail to deliver an output). The historical resources that Tichý acknowledges are first and foremost Frege's notions of sense (*Sinn*) and unsaturated function (*ungesättigte Funktion*), as opposed to modern-day functions, which are extensionally individuated mappings (akin to Frege's *Wertverläufe*), but also Russell's (not all-too crisp) notion of proposition. Tichý's objectualist take on 'operation-processes' may be seen in part as linguistic structures transposed into an objectual key; operations, procedures, structures are not fundamentally and inherently syntactic items, but fully-fledged, non-linguistic entities residing in a Platonic realm.

Still, the two most common misconceptions of constructions are that they are functions or formulae. True, functions (conceived of as mappings) are constructible by any of the different kinds of construction that their recursive definition enumerates (Definition 1.2); but constructions are distinct from what they construct (in particular, those constructions that construct nothing are still something). Especially, constructions are not what Tichý would begin to call 'determiners', which are just possible-world intensions.[3] Functions-in-extension (i.e., mappings) are set-theoretic entities; constructions not. Church-style functions-in-intension are much closer to constructions. But though functions-in-intension are construed, in mathematical logic and computer science, as rules, these are not clearly defined, whereas constructions are.[4] And, to be sure, constructions are encodable in artificial

[3] We have avoided the term 'determiner' in this book, because it is already in use in linguistics where it has a somewhat different meaning; e.g., articles are determiners.

[4] For functions-in-intension as rules or 'codes' for rules, see Mitchell (1990, p. 371) or Church (1941).

symbolic notation, but constructions are distinct from the formulae they are cloaked in. Formulae are linguistic entities; constructions not.

Apart from recasting the lambda-calculus procedurally, another departure from Church is that Transparent Intensional Logic includes not only a simple type theory but also a ramified type theory. The ramified type hierarchy serves to organise the constructions, together with functions with domain or range in constructions. Constructions offer a worked-out, positive answer to the open question of just how 'hyper' hyperintensions are. The leading principle is that any two procedurally isomorphic hyperintensions are identical. It turns out, though, that there are cases when two procedurally isomorphic constructions are just that—two constructions and not one. So a slightly coarser principle of individuation than the constructional one is called for to preserve the idea of hyperintensional individuation in terms of procedural individuation. Pavel Materna has introduced, in 1998, a rigorous notion of *concept* that identifies as one concept any two procedurally isomorphic constructions. This notion of concept has been incorporated into Transparent Intensional Logic, which therefore operates with four measures of individuation; extensional, intensional, conceptual, and constructional. Hyperintensional individuation is, in the final analysis, conceptual individuation. But since concepts are themselves constructions, we shall often speak in terms of constructions.

Having adumbrated the very basic ideas underlying Transparent Intensional Logic, here is how we locate Transparent Intensional Logic within the current landscape of formal semantics. Once the foundations of formal semantics seemed to have been firmly established. What remained to do was working out the subtleties of their applications to various problems concerning meaning and reference. David Kaplan puts it eloquently in this way:

> During the Golden Age of Pure Semantics we were developing a nice homogenous theory, with language, meanings, and entities of the world each properly segregated and related one to another in rather smooth and comfortable ways. This development probably came to its peak in Carnap's *Meaning and Necessity* (from 1947). Each *designator* has both an intension and an extension. Sentences have truth-values as extensions and propositions as intensions, predicates have classes as extensions and properties as intensions, terms have individuals as extensions and *individual concepts* as intensions, and so on. The intension of a compound is a function of the intensions of the parts and similarly the extension (except when intensional operators appear). There is great beauty and power in this theory (1990b, pp. 13–14).

However, Kaplan points out that already then there was trouble in paradise:

> [T]here remained some nagging doubts: proper names, demonstratives, and quantification into intensional contexts[5] (ibid., p. 14).

[5] Saarinen (1982, p. 131) offers the same list of trouble-makers, adding logical omniscience. As an aside, it is interesting to note that whereas epistemology has been preoccupied with skepticism (the spectre of knowing too little or nothing at all), epistemic logic has been preoccupied with omniscience (the spectre of knowing too much).

And Carnap himself observed in 1947 the problem of how to logically handle what Cresswell was later to dub 'hyperintensional' contexts:

> Hyperintensional contexts are simply contexts which do not respect logical equivalence (1975, p. 25).

Carnap asks whether a context might be neither extensional nor intensional, answering in the affirmative:

> Although [the sentences] 'D' and 'D″' have the same intension, namely, the L-true or necessary proposition, and hence the same extension, namely, the truth-value truth, their interchange transforms the [belief-reporting sentence 'John believes that D'] into the [belief-reporting sentence 'John believes that D″'], which does not have the same extension, let alone the same intension, as the first (1947, pp. 53–4).

So attitudes must be added to the list of nagging doubts, as soon as we are not content with holding, heroically but irrationally, that any two logically equivalent propositions (or whatever else plays the role of attitude *relata*) may always be validly substituted when figuring as complements of attitudes.

The over-all goal driving hyperintensional attitude logic is to avail ourselves of epistemic operators that are, in Dretske's wording, at most 'semi-penetrating' (see Dretske, 1977). For instance, it may be true that you know that if it is raining then the street gets wet and that you know that it is raining; but not that you, thereby, also know that the street gets wet. Or, since we favour relations over operators, the relation of knowing obtaining between knowers and hyperpropositions needs to have the effect of being at most 'semi-penetrating'. Much research in epistemic logic since Hintikka (1962) has centred on which restrictions to impose, and how to impose them, particularly with a view to solving the problem of logical omniscience. The solution we offer relates agents to constructions and equips each agent with one or more rules of inference that they are able to apply flawlessly to any appropriate set of premises. This way we are able to calculate the *inferable* knowledge of agents relative to their intelligence (*in casu*, their inferential capacities), and their individual inferable knowledge will be a proper subset of pieces of knowledge of all the constructions that are consequences of those already explicitly known by individual agents.

Nonetheless, despite the nagging doubts, formal semantics continued to blossom as a research discipline, really taking off in the late 1950s and early 1960s thanks to the advent of possible-world semantics. Kripke offered a semantics (several, in fact) for C.I. Lewis' naked modal syntax from the late 1910s. And Montague would soon afterwards develop an intensional logic based on Tarski-style semantics enriched with possible worlds by means of which to analyse large fragments of natural language. Kripke says that

> The main and the original motivation for the 'possible worlds analysis' — and the way it clarified modal logic — was that it enabled modal logic to be treated by the same set theoretic techniques of model theory that proved so successful when applied to extensional logic. It is also useful in making certain concepts clear (1980, p. 59, n. 22).

Kripke does not mention here which concepts he has in mind, but it seems safe to assume that they must be the notoriously elusive intensional entities like propositions, properties, relations-in-intension, individual concepts, magnitudes, etc. Possible-world semantics can tell us what an intension is and when any two intensions are identical. An intension is a function whose domain is made up of possible worlds, and *qua* functions intensions are individuated extensionally. If f, g range over intensions and w over possible worlds then if f, g return the same values for the same arguments then $f = g$. That is, co-intensionality is the principle of individuation of intensions:

$$\forall fg\ (\forall w\ (f_w = g_w) \supset f = g).$$

The upside of defining propositions and other intensions extensionally is that they become logically manageable and that we help ourselves to a clear notion, thanks to the fact that the logic of (total) functions is well understood.

Properly speaking, though, intensions are not functions, but pre-theoretic entities that are modelled intra-theoretically as functions. This slight correction is important to forestall an objection due to George Bealer. He launches in several places (variations of) what we would call ‘the argument from aroma’ (e.g., 1982, p. 90). The aroma of coffee is a property (an intension), but certainly not a mapping (mappings having no aroma); hence, properties are not mappings. We do not literally identify intensions with world-defined mappings—though for technical convenience we do identify the modelling and what is so modelled. The purpose of intensions is to capture empirical variability; such-and-such is the case, but might not have been the case, and vice versa. We construe intensions as functions from possible worlds to chronologies of entities, chronologies being functions from times to entities (including other intensions). Mathematics and logic, on the other hand, have no need for empirical variability; hence, they have no need for intensions.

Meanwhile, during the ascent of the model theory of possible-world semantics, in the non-model-theoretic quarters Heyting had long before formulated a constructivist semantics for mathematical language, Dummett would later extend, in a usually informal manner, constructivism to natural-language discourse, and Martin-Löf would put forward a detailed constructive type theory for mathematical language. Yet constructivism has so far not succeeded in framing a fully-fledged semantics for natural language, no least because it is far from obvious what the natural-language counterpart of a mathematical proof (-object) would be.[6]

Despite their initial success, the multifarious theories based on model theory eventually ground to a halt over the old problem of how to logically analyse attitudes. For the downside of intensions as mappings is that, though propositions and other intensions may have been logically murky prior to possible-world semantics,

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[6] See Ranta (1994) for an application of Martin-Löf’s type theory to natural-language discourse.

the corresponding notions were also somewhat richer.[7] The clarity of possible-world intensions comes at the price of impoverishing these notions. Put uncharitably, possible-world intensions are intensionality on the cheap. Though one of its champions, Kripke is alert to various shortcomings of possible-world semantics. Thus, after a remark on attitudes he vents an afterthought with far-reaching implications:

> How this relates to the question what 'propositions' are expressed by these [attitude-reporting] sentences [and] whether these 'propositions' are objects of knowledge and belief…are vexing questions. I have no 'official doctrine' concerning them, and in fact I am unsure that the apparatus of 'propositions' does not break down in this area. … Of course there may be more than one notion of 'proposition', depending on the demands we make of the notion (1980, p. 21; ibid., p. 21, n. 21).

The impasse over no least attitudes has lead to the re-discovery of so-called *structured* propositions. Kaplan may well have pioneered their revival in a 1970 talk that appeared as (1990b) when he urged that the analysis of 'John is tall' should include two components:

> [T]he property expressed by the predicate ['is tall'], and the individual John. That's right, John himself, trapped in a proposition (1990b, p. 13).

Along the same lines, Cresswell called for

> [An] analysis of propositions which assumes that they are structured entities…The most fully worked out account of structured meanings within a possible-worlds framework is that presented by David Lewis [in (1972)] (1975, p. 78).

Unfortunately, manoeuvring within a set-theoretic paradigm such as model theory, the only avenue open to Kaplan and Lewis was to identify structure with ordered *n*-tuples (or at least to model them as such). Tuples are a non-starter, for the simple reason that they are simple while structures are complex. Complexes have parts arranged in a particular way while sets only have elements.[8] The most a set can offer is a sequential ordering of its elements. So ⟨Is_Tall, John⟩, or ⟨John, Is_Tall⟩, is not a structured proposition. An additional objection is that either of these two two-tuples merely enumerates a property and an individual without specifying that the former is predicated of the latter. This is tantamount to the standard 'laundry list' objection that the items on the 'list' fail to hook up with one another so as to integrate into a whole, that is, it is left unexplained how sense atoms combine into one molecule. Yet another objection would be that ordered *n*-tuples most likely cannot do some of what propositions are intuitively expected to do. In particular, it is not clear in what sense a tuple can be said to be a truth-bearer

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[7] Three cases in point would be Fregean *Sinn* and Russellian *propositions*, and also Bolzanian *Sätze an sich*; see Materna (1998, 2004a).

[8] See also Simons (2007, §8): 'A complex whole is an object with more than one proper part, such that the parts are related together in the whole in a determinate way. This way of their being together in the whole is the *structure* of the whole.' Hence, 'a musket is not a *sum* of parts: it is a structured whole of parts put together in a certain way' (ibid., §7).

(i.e., something capable of being true/false) or an attitude relatum (i.e., something known/believed/hoped, etc., to be true/false).

Transparent Intensional Logic agrees with Cresswell, Kaplan, Richard and others that the meaning of a sentence must match, more or less, the structure of the sentence:

> [I]f the structure of propositions is as fine-grained as the structure of sentences, then it is hard to give to propositions any content but in terms of something analogous to sentence-like structured objects (Chierchia, 1989, p. 131).

For what other structure could arguably be a serious candidate? None that leaps readily to mind; especially not if, as in Transparent Intensional Logic, it is required that a logical analysis must treat of all, and only, those entities denoted in the analysandum. This constraint is called the *Parmenides Principle*, a forerunner of which would be Carnap's principle of subject-matter (1947, §24.2, §26.)

Apparently, mainstream analytic philosophy of language has bumped up against serious shortcomings in its foundations, with no obvious remedy in sight. True, when propositions are identified whenever materially equivalent or co-extensional, we have what we need for extensional logic, which validates the substitution of any two propositions having the same truth-value. And when propositions are identified whenever logically equivalent or co-intensional, we have what we need for intensional logic, which validates the substitution of any two propositions having, or being, the same truth-condition. But taking it to the third level of hyperintensions has seemed so far an insurmountable obstacle.

Little wonder, then, that much of what passes for analytic philosophy of language nowadays is shot through with semantic minimalism or even nihilism and an over-emphasis on pragmatic notions such as assertion, (act of) utterance, understanding, communication, language acquisition, etc. The glory days of Golden Age Semantics seem buried in the dim and distant past, with little hope of resurrection.

However, running alongside the mainstream of theories following in the slipstream of Kripke, Kaplan, Montague, etc., and the parallel mainstream of Dummett-style proof-theoretic semantics, we find a small group of lesser-known, worked-out theories of hyperintensional logic. These include, inter alia, George Bealer's, Edward Zalta's—and Pavel Tichý's. Tichý's is a theory that comes with a (very) 'big' semantics and a (very) 'small' pragmatics. The central concerns are only those a priori features of language that lend themselves to description and analysis in a purely logical manner. Thus, Tichý's theory is distinct both from those that 'pragmatize' their semantics and those that 'semanticize' their pragmatics. It observes a strict demarcation between semantics and pragmatics; so since even very sophisticated attitudes are to be analysed strictly semantically, it is obvious why a 'big' semantics is wanted. But whereas semantic and pragmatics are kept apart, semantics and syntax are developed in parallel. This turns the syntax of Transparent Intensional Logic into an *interpreted* one. We do not proceed as in model-theoretic semantics, in which first a lexicon and a set of rules of formation are introduced, followed by a syntax, and topped off with a semantics (interpretation). In particular, in TIL no expression may be introduced without typing the construction it expresses as its sense, which entails a typing of the entity that it denotes.

The puzzle-solving mettle of Transparent Intensional Logic comes at a high ontological price, due to its infinite hierarchy of higher-order entities; but it excels at parsimony in another respect. It contains but four essential constructions. They are called *Trivialization*, *Variable*, *Composition* (originally: *Application*), and *Closure* (originally: *Abstraction*).[9] These four key constructions can be divided into two groups of two. Composition and Closure are computation-like constructions; namely, the ('downward') application of a function to an argument, and the ('upward') formation of a function, respectively. The other two, Variable and Trivialization, provide the first two with input in each their way and independently of each other. Variables provide their values relative to a valuation function; Trivializations provide the entities they Trivialize by presenting them directly. The fact that constructions may themselves be Trivialized holds the key to how we obtain hyperintensional attitudes, by being able to distinguish between *using* and *mentioning* constructions.[10] These four constructions correspond to the syntax of a lambda calculus whose terms are variables, constants, applications and abstractions. Trivializations match constants, by picking out definite entities in just one step. The unusual ontological status of Variables should be underlined; they are objectual and not linguistic entities. The assignment of an entity to a Variable x does not relate this entity to a piece of language, unlike 'x', but completes an open construction that subsequently constructs a definite entity.

Constructions are arranged in a ramified, higher-order type theory that is based on a simple type theory of first-order objects. The simple type theory, when used for natural-language analysis, spans four ground types (individuals, truth-values, possible worlds, and reals doubling as times) and types of partial functions defined over them. The typing does not apply to linguistic entities, as in categorial grammar (cf. Montague, Leśniewski, Ajdukiewicz, Cresswell), but to abstract objects such as functions, truth-values, and higher-order entities, as in the constructivist type theory of Martin-Löf. Our bi-dimensional type theory fixes the objective relations among this multi-layered multitude of abstract entities. It thus enables the semanticist to control whether the input is type-theoretically internally coherent and whether the right type of output follows, so as to prevent categorial mismatches.

Transparent Intensional Logic eschews *possibilia* (possible worlds arguably the only exception). Instead the theory operates with a constant domain for all worlds

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9 It turns out, however, that we occasionally also need a fifth and a sixth construction, called *Execution* and *Double Execution*. Furthermore, the application of Transparent Intensional Logic to database theory has prompted two more constructions; one for constructing ordered n-tuples and another for constructing projections; see Duží (1992).

10 See Section 2.6.

and times. What varies are the values that (non-constant) intensions have in different worlds and at different times, and not the domains that different worlds and times have. The theory also rejects individual essentialism; no individual bears any purely non-constant property by any sort of necessity (including the enigmatic ‘metaphysical’ necessity). This is not to say, though, that we reject essentialism across the board; far from it. Taking a lead from a 1979 paper by Tichý, we have built up an essentialist theory, according to which relations of conceptual necessity obtain between various kinds of intension. The result is *intensional essentialism*, which says, roughly, that, necessarily, if *x* is a/the *F* then *x* is also a/the *G*, because being a/the *G* is in the essence of being a/the *F*. Intensional essentialism comes in handy, for instance, when spelling out the *de dicto*/*re* ambiguities besetting, e.g., ‘Necessarily, the King of Bhutan is a king’. Taken *de dicto*, it is true, for there is a necessary, a priori link between the intensions *the King of Bhutan* and *being a king*—you cannot have the former without also having the latter. Taken *de re*, it is false, for nothing of a logical or conceptual nature forces whatever individual is the King of Bhutan to be a king. It is neither true nor false, if there happens to be no King of Bhutan, for then there is nobody of whom it would be either true or false that *he* is a king.

Tichý began developing Transparent Intensional Logic simultaneously with Montague’s, around the mid-1960s, both attempting to get as much logical and semantic mileage as possible out of the possible-world paradigm. One tenet informing this project was that a natural language such as English is largely on an equal footing with the formal logical language in which it is analysed. This is a strong common point to share, and a major departure from the thoroughly suspicious attitude toward natural language that Russell, Frege, and Church, to mention but a few, championed. But Tichý and Montague parted company over some of the tenets that should inform the logical analysis of natural language. The most important difference is probably over whether natural language is permeated by shifts of reference (in the Fregean sense) and, if so, whether it should be replicated in the formal language in which the logical analysis is couched. Two other noteworthy differences between Tichý’s TIL and Montague’s IL are these. First, thanks to so-called explicit intensionalization and temporalization (see Section 2.4), TIL makes a fine-grained analysis of the *de re*/*de dicto* difference possible. For now, explicit intensionalization consists in explicit mention of variables ranging over worlds and times in the logical syntax proper. Moreover, each TIL analysis is fully compositional so that the ‘Church-Rosser diamond’ (the Koh-I-Noor of the lambda-calculi) holds, unlike IL.[11] Second, due to its hyperintensional procedural semantics, TIL offers a principle of individuation finer than logical equivalence, so that equivalent expressions may have different meanings. This feature enables us to analyse hyperintensional attitudes in an adequate manner (see Chapter 5).

[11] For further comparison of TIL and Montague’s IL, see Section 2.4.3.

It is vital to appreciate just how deep the issue of reference shift runs. Here is how we would rationally reconstruct how referential obliqueness came to be a theme pervading contemporary philosophy of language. What has become known as 'Frege's puzzle' can be summarised as follows. Historically, the puzzle turns essentially on judgements (*Urteile*). Frege's question is whether the *judgement* $\vdash Fa$ is identical to the *judgement* $\vdash Fb$ in case $a = b$. ('$\vdash$' is Frege's *Urteilsstrich*, judgement stroke, and not the symbol of validity.) For instance, is the judgement *the proposition (Gedanke) that the Morning Star is a heavenly body illuminated by the sun is true* identical to the judgement *the proposition that the Evening Star is a heavenly body illuminated by the sun is true*, in case the Morning Star is the same heavenly body as the Evening Star? Frege's answer is in the negative due to the manifest difference in epistemic value (*Erkenntniswert*) between the two judgements. Yet as an extensionalist logician (*Umfangslogiker*), Frege would have expected an answer in the affirmative. Hence his puzzlement. Frege's puzzle deals with the acquisition of knowledge by making judgements and the difference, puzzling at first, between knowing that the Morning Star is an *F* and knowing that the Evening Star is an *F*, even though the Morning Star is identical to the Evening Star. There are two things to know, not just one, and one may know the one without knowing the other.

However, the modern Anglo-Saxon reception of Frege has tended to neglect the differences between judgements and propositions in Frege, speaking of propositions only. Phrased in terms of propositions, the puzzle is why the proposition *that the F is the G* conveys non-trivial information, if true, while the proposition *that the F is the F* fails to. Or in terms of attitudes, an agent may believe the latter without believing the former and without being guilty of inconsistency or irrationality. In order to solve the puzzle, Frege attempts first to apply universal transparency to the puzzle, assuming that 'the *F*' and 'the *G*' refer to the same individual *a*. Call this 'Millian universal transparency'; 'Millian' because a singular term refers to an object, not a connotation, and because its reference is not mediated by a connotation.[12] Any account of the non-triviality of the former proposition is blocked, since it reduces to the triviality that *a* is self-identical. So, Millian universal transparency must be abandoned. Still two options apart from Millian universal transparency are open; Fregean systematic contextualism and Fregean universal transparency.

Frege, for extensionalist reasons, opted for contextualism. Tichý goes for universal transparency. The basic 'trick' behind the transparency of Transparent Intensional Logic is to universalise Frege's anomaly. Thus, universal transparency is obtained by means of universal obliqueness. If every context is oblique, or if every context is 'straight' (*gerade*), then it is pointless to uphold the distinction between oblique and straight context. Not that it would be a distinction without a difference, but the object under scrutiny—natural language—would fail to exemplify

[12] We are neglecting Mill's actual psychologistic semantic theory here.

the distinction. So Tichý takes Frege's semantics reserved for a marginal case and elevates it to the semantics for the universal case.

Interestingly, Transparent Intensional Logic agrees verbally with what Donald Davidson says about 'semantic innocence', that words maintain their meanings and denotations across shifts of context.[13] But Davidson's so-called 'paratactic' approach maintains that expressions invariably denote extensional entities, whereas our 'hypotactic' approach maintains that (empirical) expressions invariably denote intensional entities. Tichý was adamant from the outset that natural language does not display shifts of reference and that, even if it had, there should be no room for it in a logical formalism. The rationale for the latter claim is that, as already Frege himself emphasized, logical notation must be unambiguous. Logical notation must disambiguate language and not perpetuate ambiguities.[14] The rationale for the former claim is, briefly, that if the terms and expressions of natural language were to denote extensional entities (like individuals, truth-values, sets) then successful communication would require of speakers and hearers that they knew which possible world was the actual one.

The gist of the argument is this. Intensions are conditions satisfiable by possible worlds (and whatever other empirical indices we care to add, such as times). If empirical terms denoted the actual-world satisfiers of these conditions, then one must know which world is actual to know which entity is being so denoted. Successful communication would require not only understanding the meanings of terms and expressions, but also knowing their actual values. But then the empirically omniscient would have no epistemic need for communication, for they would already know everything there was to know; whereas the empirically non-omniscient would never know whom or what was being talked about.[15] We mortal language-users do possess much empirical knowledge (neglecting for now the challenges posed by radical scepticism), but even if mankind were to pool together all its current knowledge, what could be identified would at most be an equivalence class of possible worlds, of which the actual world would be a member. Hence it should not be a (mostly tacit) requirement that we be able to identify the actual world. At the same time, though, we do know that we live in the actual world and we do make our empirical assertions about the actual world. We agree with this portion of David Lewis' 'indexical' theory of actuality. But this is not to

[13] See Davidson (1968).

[14] As Muskens says, 'Why does [Montague's] IL show such exotic behaviour; why do Leibniz's Law, Universal Instantiation and Lambda Conversion not hold under the normal conditions? Because the logic was explicitly designed to reflect certain opacity phenomena in natural language' (1989, p. 10).

[15] In a recent comparison of Tichý and Zalta, Sierszulska says correctly that, '[K]nowing all the values of the [intensions] … would be the same as knowing all the facts … The proper analysis of a proposition cannot assume such [empirical, as opposed to logical] omniscience, and it stops at the point where all the possibilities are taken into account, but none is specified' (2006, p. 491).

say that we know of one particular possible world that it is actual, for this is exactly what we cannot know for want of empirical omniscience.

An additional point is that the widespread idea that empirical terms denote extensions fails to keep the factual relation between an intension and its world-relative satisfier apart from the semantic relation between an empirical expression and its denotation. Transparency is underpinned by an anti-actualist semantics founded upon a sharp demarcation between denotation and reference. The denotation relation holds a priori between a word and the entity (if any) identified by the meaning of the word (or *meanings* if the word is ambiguous, and *meaning* if unambiguous, at the level of logical analysis). Of course, it is a historically contingent fact that a configuration of letters of some alphabet and/or sounds constitutes a *word* of a language and expresses one meaning rather than another or none at all. Diachronically, such configurations may criss-cross in and out of a language and enter into different semantic relations at different points in time. Synchronically, however, the semantic relations characterising a certain language are fixed for any given point in time. When we use an expression in a communicative act we communicate its sense. The same configuration of letters or sounds might have had wildly different senses, since the relation between term/expression and sense is wholly arbitrary and not inherent. Only this fact is irrelevant to logic and semantics. It falls to linguistics and not logic or formal semantics to associate terms with senses. The starting-point of logical analysis of language presupposes both that the word/sense relations are in place and that the speakers of the language under scrutiny master these relations.[16]

As this book shows, this choice of starting-point dictates our analysis of, e.g., ‘Hesperus is Phosphorus’ and ‘Cicero is Tully’. If the terms are names of individuals, then the sentences merely express the self-identity of an individual bearing two names. What is to be known concerns not a worldly but a linguistic matter, then. But if the terms are instead names of individuals-in-intension—what Church, Carnap, Kaplan and others call ‘individual concepts’ and we call ‘individual offices or roles’—then what is to be known does concern a worldly matter; namely, that two differently named individuals-in-intension contingently coincide in the same individual (-in-extension), which or who bears neither name. As logical semanticists we adjudicate neither way. We enumerate the various possible semantic analyses of, e.g., ‘Hesperus is Phosphorus’ and ‘Cicero is Tully’, and chart their presuppositions and consequences.

The *denotation* of an empirical term is always an intension. The *reference* relation holds a posteriori between an empirical word and the value, if any, of its denotation at

[16] Tichý puts the point succinctly in a 1966 paper; ‘We assume, of course, a normal linguistic situation, in which communication proceeds between two people, both of whom understand the language. Logical semantics does not deal with other linguistic situations’ (2004, p. 55, n. 1). Likewise, C.A. Anderson says about Church’s Alternative (0): ‘Sense is what is known when the language is understood. In accordance with this, the intensional semantical rules should state essential facts about the semantics, the mastery of which constitutes (ideal) competence with the language. These may include the rules of synonymy [.]’ (1998, p. 163).

the actual world at the present moment. So, while an empirical word may lack a reference, it never lacks a denotation. For instance, the term ‘The King of France’ lacks a reference in the actual world at the present time. Yet the term has a denotation, namely the individual-in-intension (individual office) *the King of France*. The semantics of empirical words is such that no such word can be ‘empty’ in the sense of failing to pick out an entity; for it invariably picks out an intension. Whether a given intension lacks an extension at the actual world is a factual rather than semantic question. In the case of non-empirical words, the extra-semantic relation of reference drops out, since non-empirical terms and expressions do not pick out anything relative to empirical indices. They denote what—if anything—is constructed by their respective senses. The qualification ‘if anything’ is important, since some non-empirical words fail to denote. For instance, whereas there is a construction of the largest prime, there is no number for this construction to construct. Still the term ‘the largest prime’ is meaningful and has a meaning to contribute to a compound meaning, like the one of the sentence, ‘The largest prime is odd’. But the sentence fails to denote, because ‘the largest prime’ fails to.

So there is this one difference between empirical and non-empirical expressions. But let us stress the reason why both kinds of expression spring from the same source. *All* expressions, without exception, denote what is constructed by their senses. It is just that there are non-empirical cases where the sense fails to construct something for the relevant word to denote. The overarching semantic idea pertaining equally to mathematical and natural language is that sense is a calculation or procedure, while at the same time observing a thoroughgoing demarcation between these two compartments of language. Natural language descends from a calculation to an intension. Mathematical language descends either from a calculation to an extension or a lower-order calculation. The semantics of natural language demands an intensional intermediary between sense and (possible) extension due to the inherent anti-actualism informing Transparent Intensional Logic. The semantics of a natural-language term or expression terminates in the calculation of an intension. The sense is a manner of calculating the given intension so as to be able to arrive at its value at any world and time of evaluation. The semantics *does not* terminate in a calculation of the *actual and present value* of an intension, let alone in the value itself (if indeed any). There can be no final semantic, a priori step from intension to actual and present value on pain of reinstating empirical omniscience as a prerequisite for successful communication among non-omniscient language-users. The denotation is the same for all worlds and times, so words denoting intensions qualify as rigid designators. What varies is the reference; non-constant intensions do not return the same values at all worlds and times. But the reference relation is factual, a posteriori and extra-semantic; unlike the denotation relation, which is a priori and intra-semantic.

1.1.1 Semantic schemas

We are placing our procedural semantics within the general Fregean programme of explicating sense (*Sinn*) as the *mode of presentation* (*Art des Gegebenseins*) of the entity (*Bedeutung*) that a sense determines.[17] Muskens correctly points out that, 'The idea was provided with extensive philosophical justification in Tichý (1988)' and that '[Tichý's] notion of senses as *constructions* essentially captures the same idea' (2005, p. 474).

So our starting-point is Frege's well-known semantic diagram (FSD). This diagram is frequently accepted as one of the foundations of modern semantics. To explain why a true sentence of the form '$a = b$' can be informative, unlike a sentence of the form '$a = a$', Frege introduced an entity standing between an expression and the object denoted (*bezeichnet*) by the expression. He named this entity *Sinn* (sense) and explained the informative character of the true '$a = b$'-shaped sentences by saying that 'a' and 'b' denote one and the same object but differ in expressing (*ausdrücken*) distinct senses. Thus FSD can be visualized as in Fig. 1.1.

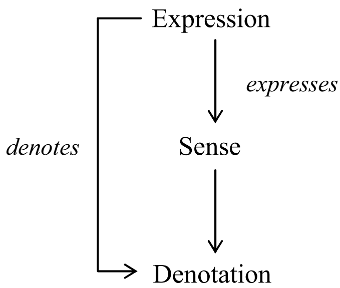

Fig. 1.1 Frege's semantic diagram (FSD)

So far, so good. The problem, though, is that Frege never defined *sense*. All he says is that it is a 'mode of presentation' (*Art des Gegebenseins*) of the denotation. The frequent interpretation of sense in contemporary semantics has it that sense is an *intension*. Thus, Kirkham says:

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[17] We know we are cutting corners here by paraphrasing 'Bedeutung' as 'entity'. We are doing so in order not to get bogged down in the ongoing discussion of how best to render 'Bedeutung'. The standard translation has been 'reference', but this does not do justice to Frege's idiosyncratic distinction between 'Sinn' and 'Bedeutung', which are more or less synonymous nouns in ordinary German, barring idiomatic usage; e.g., 'sinnlos' and 'bedeutungslos' are certainly not synonymous adjectives. The best verbatim translation would have been 'meaning', to be contrasted with 'sense'. But the idea of Frege being the *meaning* of 'Frege' sits very poorly indeed on the ears. Besides, 'Bedeutung' comes with a suggestion of pointing at an entity—'deuten auf'—that 'meaning' lacks. Fortunately, we can afford to be offhand about 'Bedeutung', since we are so strongly biased toward *Sinn*.

> Since the seminal work of Gottlob Frege (1892a) it has been a *commonplace* [italics ours] that the meaning of an expression has at least two components: the sense and the reference. The sense of an expression is often called the *connotation* or the *intension* of the expression, and the reference is often called the *denotation* or *extension* of the expression. The extension of an expression is the object or set of objects referred to, pointed to, or indicated by, the expression. ... The extension of 'the morning star' is a certain planet, Venus. The extension of a *predicate* is the set of all objects to which the predicate truly applies. The extension of 'red' is the set of all red things. The extension of 'vertebrate with a liver' is the set of all vertebrates with a liver (1992/1997, p. 4).

'Intension' can be interpreted in various ways. In the quotation above it is used as in Montague's theory, viz. as *the intension of* an expression. At the same time contemporary possible-world semantics takes intensions to be functions whose domain is made up of possible worlds. According to this view, an expression possesses an intension and an extension;[18] the former corresponding to Frege's 'Sinn', the latter to Frege's 'Bedeutung'.[19,20]

The intuition behind this interpretation is at first sight attractive. This can be shown by the classical Fregean example of 'The Morning Star' vs. 'The Evening Star'.[21] The senses of these expressions are distinct according to Frege. Now if we connect with either of these expressions an intension then the result is this: the sense of 'The Morning Star' is another possible-world intension than the sense of 'The Evening Star', but the value of both intensions in the actual world at the present moment is one and the same object—Venus, as it happens.

Of course, aspersions have been cast upon this view independently of the criticism that TIL had raised much earlier. For instance, van Lambalgen and Hamm say:

> In formal semantics for natural language it is not common practice to associate algorithms to expressions. …it is usually assumed that all one needs is the intension of an expression, defined as a function which maps a possible world into an extension of the expression in that possible world. It seems to us that this picture of meaning is too static, and by and large cognitively irrelevant (2004, p. 7).

As we argued above, the interpretation of *sense* as *intension* and *denotation* as *extension* in the case of empirical expressions (like 'The Morning Star', 'The Evening Star') is counterintuitive. Already Carnap (1947), knew that a logical analysis cannot provide the contingent values of intensions. If intensions are functions from possible worlds (and times, as in TIL) then we could logically determine the value of an intension in the actual world only if we knew which of the possible worlds is the actual one. On any rational explication of the notion of possible world, this knowledge cannot be a priori; therefore, determining the value of an

[18] See also Carnap (1947).

[19] Church (1956) has 'denotation'.

[20] Originally, Tichý also held to the view that Fregean sense may be explicated as a possible-world intension; cf. (1986a, p. 253, 2004, p. 651).

[21] See Section 3.3.

intension in the actual world must always be a matter of factual experience (rather than of logic).

The relation between an intension and its actual/present extension is beyond logical semantics. The spirit of TIL requires that the terms ‘denotation’ and ‘reference’ be semantically kept separate, at least in the case of empirical expressions. What is denoted are intensions, whereas the *value* of such an intension in the actual world (and at the present time) is called the *reference* (of the respective expression). Thus reference is not a matter of logical semantics, being ascertainable via experience only. The necessity of this decision is intuitively clear, as soon as we agree that logical analysis cannot contain any empirical elements. Consider an FSD where the expression is an empirical sentence. For Frege such a sentence denotes its truth-value. Take the sentence ‘Mars contains water’. If denoting is (as it should be) a logical relation then we could derive its actual truth-value. Then why send probes to Mars?

Accepting the view that empirical expressions denote possible-world intensions, the ‘The Morning/Evening Star’ problem might seem to be heading for a solution. The question, however, arises: do we need the notion of sense as a semantic category at all? Prevailing logical theories are *denotational* and *set-theoretic*:

> [T]he meanings, it should be stressed once more, are the semantic objects in the model, i.e., the individuals, properties, propositions, second-order properties and so on that we associate with the expressions. The logical expressions serve to represent these but are not to be confused with them (Gamut, 1991, p. 218).

We shall show that denotational and other set-theoretic approaches are too coarse-grained. Theories based on standard logic run together the meanings of terms and expressions that are classically equivalent, even if they are evidently not strictly synonymous. For an example, consider the two sentences

(1) ‘Bill walks’,
(2) ‘Bill walks and whales are mammals’.

Intuitively, (1) and (2) do not have the same meaning. Standardly, however, the meaning of (1) will be a certain set of possible worlds (the worlds in which Bill walks) and the meaning of (2) will be the intersection of this set with the set in which all whales are mammals. Since we presuppose full linguistic competence in language-users, sentences like ‘No bachelor is married’ and ‘Whales are mammals’ come out analytically true, i.e., true only in virtue of their meaning. Provided that we understand the meanings of the predicates ‘is a whale’ and ‘is a mammal’ as used in current English, when learning that whales are mammals we do not acquire factual information bearing on the state of the world. If you know that the individual before you is a whale, you need not examine the world in order to get to know that the individual is a mammal. Instead, an analytically true sentence is true in all possible worlds. Hence if the meaning of (2) is a certain set of possible worlds, then it is the same set as the set of worlds in which Bill walks.

Therefore, (1) and (2) are predicted to be synonymous, which obviously they are not.[22]

This inaccuracy might seem not to be that important, though. After all, the theory gives a correct prediction of the relation of *entailment* here. The two sentences entail each other, and this fact correctly follows from Montague-like set-theoretical theories. So why not embrace co-entailment, although more coarse-grained than strict synonymy, as a good approximation to meaning and synonymy in natural language? Here is why not. Natural language is rich enough to express the differences in the meanings of co-entailing sentences. Attitudes are typical examples. One can easily believe that Bill walks without believing that Bill walks and that whales are mammals. Though ‘All whales are mammals’ denotes a constant intension, the sentence is far from being meaningless.[23] Only a more fine-structured notion of meaning than co-entailment will capture the meaning of ‘All whales are mammals’. But, of course, if an empirical expression denotes an intension then what would its sense be? And, furthermore, what would the sense of a *mathematical* expression be?[24]

Consider, e.g., the expression

$$‘(2 \times 2) - 3’$$

It will probably be agreed that this expression denotes the number 1. But why is that? What is its sense? This problem is eloquently formulated by Tichý:

> If the term ‘(2 × 2) – 3’ is not diagrammatic of anything, in other words, if the numbers and functions mentioned in this term do not themselves combine into any whole, then the term is the only thing which holds them together. The numbers and functions hang from it like Christmas decorations from a branch. The term, the linguistic expression, thus becomes more than a way of *referring* to an independently specifiable subject matter: it becomes *constitutive* of it. An arithmetical finding must, on this approach, be construed as a finding *about* a linguistic expression. ... But since an expression is always part of a particular notational system, our theorist must construe the arithmetician as being concerned specifically with a definite notation (1988, p. 7).

Now if we wish to retain Frege’s idea that between an expression and its denotation there is some abstract entity (*Sinn*) serving as intermediary then such an entity cannot be a possible-world intension. In the empirical case an intension is what the expression *denotes*; in the non-empirical case, either no intensions are needed or they are always going to be constant functions. Possible-world intensions serve the purpose of modelling empirical variability, and are out of place in mathematics. Yet there are theories that attempt to account for, e.g., inconsistent beliefs and absurd

[22] See Section 2.1.2 for another aspect of this problem, and Section 2.2.1 for the definitions of *synonymy* and *equivalence*.

[23] This is not to say that it would have *empirical* information to offer; see Section 5.4.

[24] Note that the first place in Frege (1892a) where he introduces the notion of sense is not the famous one involving ‘The Morning Star’ and ‘The Evening Star’, but one involving the medians of a triangle. Here we chose a still simpler example.

objects (like round squares) by introducing a parallel logical space of logically impossible worlds (cf. Priest, 1992). But just as little as the number five belongs to the domain of possible worlds and just as little as mathematical sentences are evaluated at possible worlds, so round squares should not be assigned to the domain of any impossible world. The very idiom of worlds, whether possible or impossible, is out of place, as soon as non-empirical objects like numbers and figures are involved. We will show that terms like 'round square' and 'the greatest prime' are not meaningless expressions and that we can handle them without the category of impossible worlds.[25] So, which kind of entity can play the role of sense and possibly be captured by logical analysis?

The example of a simple arithmetical expression shows that the sense should be an extra-linguistic entity, whose existence would explain the connection between an expression and the object denoted. As we have already pointed above, we have such an entity at hand. It is *the* key notion of TIL, the one of *construction*. Our neo-Fregean semantic schema is the adjusted version of FSD as visualized by Fig. 1.2.

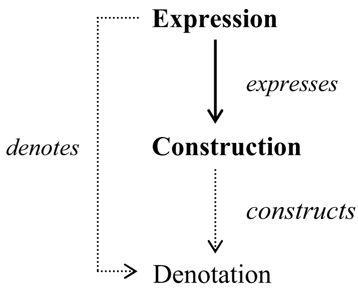

Fig. 1.2 TIL semantic schema

The most important relation in this schema is between an expression and what is expressed by it: its meaning, i.e., a construction. Once we exactly define *construction*, we can logically examine it; we can investigate what (if anything) the construction constructs, what is entailed by it, etc. Thus constructions are semantically primary, denotations secondary. Once a construction is explicitly given, the entity (if any) it constructs is already implicitly given, but will have to be teased out by means of logical analysis. As a limiting case, the logical analysis may reveal that the construction fails to construct anything; we will say that it is *improper*.

It might be tempting to say that the references of empirical terms and expressions were tertiary. But they are not. The preceding discussion of denotation versus reference served to make the point that the relation of denotation is intra-semantic and the relation of reference extra-semantic. Given a denotation, logical

[25] They express empty concepts, the former identifying an empty class of geometrical figures, the latter identifying no number at all. See Section 2.2.

analysis cannot tease out its reference. So there is no room for reference in our semantic schema.

As for *terminology*, Tichý himself did not use Fregean expressions; he did not refer (at least in his mature works, in particular in his 1988) to constructions as 'meanings'. So he did not use the term 'concept' in our sense.[26] Further, Tichý's final semantic schema, in (1988, p. 224), reduces all semantic relations to *denotation*; what is denoted is, without exception, a construction. True, as mentioned above, from the semantic point of view a construction is primary and the product of the construction secondary. Thus the above semantic schema of Fig. 1.2 is *impure*. Our *pure semantic schema* (Fig. 1.3) comes down to this (see also Section 3.2.1):

$$\begin{array}{l} \text{Expression } E \\ \quad \Big\downarrow \textit{ expresses} \\ \text{Construction (the sense of } E\text{)} \end{array}$$

Fig. 1.3 TIL pure semantic schema

We have one, methodological, reason for not going along with Tichý's final schema. TIL is a procedural semantics and as such opposed to denotational semantics. So Tichý's final schema represents a hybrid between the procedural and the denotational approaches, by having terms directly denote procedures without a procedure being a stepping-stone between term and entity. Moreover, according to well-entrenched terminology, 'denotation' is reserved for a relation between terms and set-theoretic entities, yet procedures are none such. Hence our preference for a three-tiered impure semantic schema to make the relation between what is expressed and what is denoted explicit, and a pure semantic schema to go with our procedural semantics. So we say that expressions express their meanings and denote (or fail to denote) entities identified (constructed) by the respective construction. The impure semantic schema must help us achieve the goal of this book, which is to assign constructions to expressions as their meanings and the products of the constructions as their denotations. This is also to say that being impure does not detract from a semantic schema's standing.

The viability of the thesis that empirical terms and expressions denote intensions presupposes that we possess of a means to obtain an extension from an intension. For surely we do not want to end up claiming that the sentence, 'The King of Bhutan is a benign ruler' ascribes the property of being a benign ruler to the intension *The King of Bhutan*. Two standard options are in circulation in the literature; a special extensionalization operation/operator or functional application. We use functional application, so we have no need for an operation/operator earmarked

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[26] The general idea that concepts are procedures was, however, advanced by Tichý already in 1968 and 1969. We will deal with concepts (i.e. closed constructions in normal form) as *procedural meanings* in Section 2.2.

specially for extensionalization. Nor do we need a special operation/operator of predication, as functional application fits the bill. Hence, the logical analysis of 'The King of Bhutan is a benign ruler' will contain multiple instances of functional application; one from *The King of Bhutan* to an individual and another from *being a benign ruler* to a set. An additional instance of functional application takes the set to a truth-value, according as the individual is a benign ruler or not. If no individual is forthcoming, nor is a truth-value.[27]

The anti-actualism permeating Transparent Intensional Logic is what motivates *explicit intensionalization* and *temporalization*. The syntactic form of explicit intensionalization and temporalization consists in lambda abstraction over variables ranging over possible worlds and instants of time:

$$\lambda w \lambda t\, [\ldots w \ldots t \ldots].$$

Any formula matching this schema is to be read as follows: In any possible world (λw), at any time (λt), evaluate $[\ldots w \ldots t \ldots]$.

A Closure such as the above may be completed in this or that manner. Whichever way, though, the Closure will be a construction of a denotation (intension), which, if defined at the particular world and time of evaluation, will yield a reference. In other words, our semantics is *top-down*, from structured senses to empirical conditions. From this point there is an extra-semantic transition from empirical conditions to satisfiers (if any). As is seen, explicit intensionalization and temporalization operates with a set of worlds, whereas semantic actualism operates with one particular world. Still, the assertion that the sun is shining is obviously not to the effect that the sun is shining in some possible world or other. Rather the assertion is targeted at the actual world. And that is just the point—the link from possible-world propositions to the actual world is not mediated semantically, but pragmatically. It is by asserting a proposition (by assertorically uttering a sentence denoting it) that a speaker anchors the proposition to the actual world. Communication about matters empirical proceeds on the understanding that assertions are assertions about the actual world and the present time. Propositions (or any other types of intension) are not in and by themselves anchored to the actual world or the present time. Consider again the example of the King of Bhutan being a benign ruler. In case a truth-value is forthcoming, it is abstracted over to obtain a function from worlds and times to truth-values. Such a function is a proposition, and the assertion that the King of Bhutan is a benign ruler is to the effect that the proposition thus asserted is true in a set of possible worlds that includes the actual world at the present time.

Having introduced explicit intensionalization and temporalization, here is, briefly, how Trivialization helps us to a notion of hyperintensional attitudes. If an agent is related to $\lambda w \lambda t\, [\ldots w \ldots t \ldots]$, then the agent is related to what this Closure constructs, i.e. an intension, typically a proposition (in the case of 'propositional attitudes') or

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[27] See Section 2.4.2.

else an individual role or a property (in the case of 'notional attitudes'). We say that the Closure occurs *used*, because it is used to yield an entity different from itself, namely the entity it constructs. But the whole Closure may itself be constructed, in this manner:

$$^0[\lambda w \lambda t\, [\ldots w \ldots t \ldots]].$$

We say that the entire Closure $[\lambda w \lambda t\, [\ldots w \ldots t \ldots]]$ occurs *mentioned*, because it itself is the object of discourse. Recalling the semantic schema of Fig. 1.2, the Closure is now in the position of denotation, whereas the Trivialization $^0[\lambda w \lambda t\, [\ldots w \ldots t \ldots]]$ is in the position of a construction that constructs the Closure. What the agent is related to is no longer what the Closure constructs, but the Closure itself (i.e., a procedure and not its product). Whereas empirical attitudes come in two variants, intensional and hyperintensional, mathematical attitudes are invariably hyperintensional. For instance, the attitude of *calculating* relates an individual to a Composition (rather than the outcome of the Composition). So the relevant construction must again be Trivialized: $^0[\ldots]$.

In general, since Closures and Compositions are hyperintensionally individuated, substitution of attitude *relata* will be much more restrictive than is the case with attitude logics based on set-theoretic modal logic.

The rejection of reference shift by no means implies that Tichý was blind to various both subtle and entrenched distinctions in logic. Only he accommodates them differently. Tichý claims that empirical terms and expressions exhaust their role by expressing a sense and denoting the intension that the sense yields. This holds for all contexts, such that empirical terms and expressions denote intensions and not extensions, whatever sort of semantic context they are embedded in. Once an intension has been picked out by a word, the word has fulfilled its task, and the so denoted intension can be logically manipulated. The intension may be either extensionalized or not. If extensionalized, it yields its value, if any, at the given world and time of evaluation. If un-extensionalized, it yields itself. The distinction between extensionalized and un-extensionalized intensions concerns two different ways of using (as opposed to mentioning) constructions as constituents of larger constructions. Constituent constructions occur with supposition *de dicto* or *de re*. Briefly, if *de dicto*, the so constructed intension is not extensionalized. If *de re*, it is. If the constructions do not construct intensions, then the *de dicto*/*de re* distinction is the distinction between the either intensional* or extensional* supposition that a constituent construction can occur with. Intensional* and extensional* are not the same as intensional and extensional, as the latter pair is used in possible-world semantics. The former pair applies to all constructions; the latter exclusively to constructions of intensions. When a constituent construction occurs with extensional* supposition, then the so constructed function is applied to an argument in order to obtain the corresponding value, if any. This way a property becomes attributable to a functional value. When occurring with intensional* supposition, then the so constructed function is not applied. This way a property becomes attributable to the function itself.[28]

All in all, the particular use that Transparent Intensional Logic makes of the distinction between *de dicto* and *de re* substitutes reference shift. It is of vital importance to the project of Transparent Intensional Logic that a very sophisticated and detailed conception of supposition *de dicto*/*re* be in place. Elaborating this conception has been the focus of intense research the last some years, and in this book we present the most elaborate conception to this day.[29]

In a wider context, the typed universe of Transparent Intensional Logic, with its ever-ascending hierarchy of constructions, can be seen in part as a counter-reaction to the frugal ontologies propagated by Quine and a host of others, not least under the banner of nominalism. Quine combines his pragmatism-flavoured nominalism with an extensionalist conception of semantics, according to which only extensional entities are ever denoted. Quine's final verdict on denotation is unfavourable to modalities and attitudes, not to extensionalism; Tichý draws the opposite conclusion.

One of many ways of summing up this clash is as the clash between *bottom-up* and *top-down* approaches to semantic analysis. The parallel clash over ontology is then the clash between an approach that starts out with concrete particulars and stays as close as possible to *terra firma* and an approach that starts out with abstract modes of presentation and only introduces concrete particulars in their capacity as whatever is presented in a particular manner. To express the difference metaphorically, if the former approach to semantics and ontology is *terrestrial*, the latter approach is *celestial*. So, tongue-in-cheek, whereas Isaac Newton founded a modern celestial mechanics, Tichý founded a modern *celestial semantics*.

1.2 The top-down vs. bottom-up approach to logical semantics

1.2.1 The bottom-up approach

In its broadest sense, logic is the science of correct reasoning and the art of argumentation.

Today's logic is *formal* logic. This is to say that logic investigates the *validity* of arguments irrespective of what the premise(s) and the conclusion of a given argument *mean*. It is quite another issue whether the premise(s) and the conclusion form a *sound* argument; i.e., whether the premises are *true*. The notion of truth presupposes the notion of meaning. And in order to reason we have to understand particular sentences. Since we understand a sentence by knowing its *meaning*, we

[28] Moreover, intensional* supposition is *dominant* with respect to the extensional* one. For details, see Section 2.6.

[29] This marks an advance over Tichý's stance as expounded in 1986a and 1988 (§41).

need to know what the premises of an argument mean. We agree with Frege that drawing inferences must be from sound arguments, since the point of inferring is to obtain new knowledge (the conclusion) from old knowledge (the premises). Thus analysis of language (i.e., discovering the meanings of particular expressions) is a necessary precondition for reasoning.

Historically, many logical systems developed from the simplest cases to increasingly more complicated ones. Beginning in ancient times with the logic of Aristotle and the Stoics, currently characterised as fragments of first-order predicate logic and propositional logic, respectively, many specialised logical systems have since emerged. These include, inter alia, modal logic, epistemic logic, doxastic logic, deontic logic, fuzzy logic, paraconsistent logic, many-valued logic, provability logic, temporal logic, and intuitionistic logic. How is that possible, though? Isn't there just one logic? Yes and no. In the broadest sense, there is just one logic. In a much more narrow sense, there are many logical theories of this or that. Beginning with atomic sentences, propositional logic specialises in how to compose atomic sentences into compound ones. Predicate logic investigates the structure of atomic sentences with quantifiers. If you add modalities you enter the sphere of modal logic. If you add other operators like epistemic or doxastic ones, still other logics emerge. Thus it is natural to start with the simple cases first. Let us consider some examples.

(1) 'Some prime numbers are even.'
(2) 'Some odd numbers are even.'
(3) 'Some clever students are lazy.'

If analyzed in first-order predicate logic, one formula analyses all three sentences:

$$\exists x\, (P(x) \wedge Q(x)).$$

As it stands, the formula is neither true nor false. It is only a syntactically well-formed formula, which cannot be evaluated unless and until meanings have been assigned to P and Q and a functional range to x, it is just a pattern for applying particular symbolic inference rules. Thus we can infer, e.g., the formulae '$\exists x\, P(x)$' and '$\exists x\, Q(x)$'.

In order to decide whether the formula is true or false, we have to interpret it first. On some interpretations it is true, on others it is false. Interpreting P, Q over the universe of numbers as the set of prime numbers and even numbers, respectively, it come out true. Interpreting the same symbols as representing odd numbers and even numbers, it comes out false. And interpreting the symbols P, Q, e.g., as a set of clever students and lazy students, respectively, over some universe of individuals, it is either true or else false according as these sets share a non-empty intersection.

This sort of analysis is worrisome. First, why do all the above sentences receive one and the same analysis? Sentence (1) is analytically and provably (hence, necessarily) true, whereas sentence (2) is analytically and provably (hence, necessarily) false. Sentence (3) is only contingently true, and so requires empirical inquiry to establish its actual truth-value. The formula is true on some interpretations and false on others. Second, in what way does such a translation of a perfectly well-understandable natural-language sentence into a symbolic formula make its *meaning* clear?

Consider further the sentences

(4) 'No bachelor is married.'
(5) 'No bachelor is rich.'

The identical formula analyzing both sentences would be

$$\forall x\ (P(x) \supset \neg Q(x)),$$

or equivalently,

$$\neg\exists x\ (P(x) \wedge Q(x)).$$

While (4) is analytically true, (5) is contingently true or false. Since neither formula is logically valid, one may again wonder how it is possible that two so semantically different sentences lend themselves to one and the same logical analysis (whether the analysis be $\forall x\ (P(x) \supset \neg Q(x))$ or $\neg\exists x\ (P(x) \wedge Q(x))$).

The standard answer is that it is not the point of first-order predicate logic to deal with *empirical sentences* like (3) and (5). This logic was designed for the purpose of mathematical reasoning. First-order predicate logic was designed to prove theorems, not to spell out what theorems mean, so as long as (1) and (2) have the same consequences, there is no need to assign different formulae to them.

But first-order predicate logic is standardly used to analyse empirical sentences. This practice creates a mismatch between the analytic tool and what is to be analysed. The analyses above are too coarse-grained, as well as being ambiguous. These difficulties would be neglectable if we could always infer the correct consequences from the premises. Unfortunately, we cannot. An up-dated puzzle of old shows why:

Necessarily, 8 is greater than 5
The number of planets equals 8

Necessarily, the number of planets is greater than 5.

We just used Leibniz's law of substitution of identicals to infer from true premises a false conclusion. Paradox! Modal logic sorts out the fallacy, though:

$$\frac{\begin{array}{c}\Box\, G(8, 5) \\ n(p) = 8\end{array}}{\Box\, G(n(p), 5).}$$

The conclusion is not derivable, just as we desired. ‘$G(8, 5)$’ occurs within the scope of a modal operator, and we must not substitute co-extensional terms into contexts governed by a modal operator. But we are left in the dark as to *why* not. A rule is required that suspends the applicability of Leibniz’s Law in precisely circumscribed cases. Without such a rule available to us, blocking an argument such as this remains ad hoc. As with solutions ad hoc in general, while they may succeed in alerting us to the fact *that* there is a problem, they fail to show *how* to solve the problem. Little logical insight is to be garnered from a mere ban on substituting into modal contexts.

Another problem concerning this solution is what the *meaning* of the modal operator $\Box$ is. Obviously, it is not a property of the truth-value **T**, though ‘$(8 > 5)$’ denotes **T**. One may grant that the ‘language’ of modal logic is a handy shorthand and still suspect that it hardly provides a transparent analysis. Furthermore, the following fallacies cannot be blocked by modal logic:

John McCain wanted to become the President of the USA
Barack Obama is the President of the USA

John McCain wanted to become Barack Obama.

Oedipus sought the murderer of his father
Oedipus is the murderer of his farther

Oedipus sought Oedipus.

We have to switch to a system of some intensional logic in order to render the fact that ‘to become’ and ‘to seek’ establish intensional contexts that are not to be substituted into. If B is an attitudinal operator, the shared analysis is

$$\frac{\begin{array}{c}B(a, f(b)) \\ a = f(b)\end{array}}{B(a, a).}$$

Again, the undesirable substitution is said to be blocked, because the substitution of ‘a’ for ‘$f(b)$’ in a context preceded by B is banned. But *why* and *how*? What is the meaning of the operator B? Obviously, B does not stand for a relation between two individuals; an individual cannot become another individual, unless it would somehow bizarrely alter its identity. Yet ‘$f(b)$’ does denote an individual.

In general, a ban on substitution will cure the symptom, but not the disease. Addressing the underlying problem requires formulating a non-circular, independently motivated rule to regulate substitution in intensional contexts.

Another fallacy is this famous example calling for deontic logic:

The letter ought to be delivered
If the letter is delivered, then it is delivered or burnt

The letter ought to be delivered or burnt.

O a deontic operator, the argument goes into

$$\frac{\begin{array}{c} O(d(a)) \\ d(a) \supset (d(a) \vee b(a)) \end{array}}{O(d(a) \vee b(a)).}$$

O blocks the undesirable application of *modus ponendo ponens*—somehow. However, consider this variant:

The letter ought to be written and delivered
If the letter is written and delivered, then it is delivered

The letter ought to be delivered.

$$\frac{\begin{array}{c} O(w(a) \wedge d(a)) \\ w(a) \wedge d(a) \supset d(a) \end{array}}{O(d(a)).}$$

Why it is that this time around O does not block the application of *modus ponens*? What is the meaning of O? What does the operator operate on? Certainly not on a truth-value; the property of being ordered has to be ascribed to a *proposition*, not to a truth-value. Thus, though the standard version of deontic logic is an extensional first-order logic, it should actually be an intensional logic.

However, none of the standard logics deal with the problem of *existence*, since existence is simply assumed. Consider Russell's classical example:

The King of France does not exist.

As the King of France does not exist, it is not true that the King of France is bald. And since it is not true that the King of France is bald, the King of France is not bald. Since the King of France is not bald, it follows that there is somebody who is the King of France and who is not bald. Finally, from this it follows that the King of France exists.

What went wrong? First-order logic can provide no diagnosis of the fallacy involved. The formula corresponding to both the sentence 'It is not true that the King of France is bald' and the sentence 'The King of France is not bald' is '$\neg B(k(a))$'. The only standard answer would be that '$k(a)$' is not a well-formed term, because it is non-denoting. But what is, in fact, needed to block Russell's argument from going through is a logic of *partial* functions. Only this involves a departure from a logic that tolerates only total functions.

This example mixes existence and modality:

$$\frac{\text{Necessarily, the King of France is a king}}{\text{The King of France is necessarily a king.}}$$

The premise is (necessarily) true if read *de dicto*. The conclusion is (necessarily) false or else undefined if read *de re*. So the argument is invalid. But the notation of modal logic analyses both the premise and the conclusion as '$\Box P(k(a))$', which does not render the difference between necessity *de dicto* and necessity *de re*. So the invalidity of the argument is obfuscated by the notation.

This is not to say that modal logic cannot distinguish, in general, between necessity *de dicto* and *de re*; of course, it can. For instance, it easily manages to distinguish between necessitating a consequence and necessitating a consequent, as in

$$\frac{\Box\ \forall x\ (x \text{ is the King of France } \supset x \text{ is a king})}{\forall x\ (x \text{ is the King of France} \supset \Box\ (x \text{ is a king})).}$$

The argument comes out invalid, because it trades a premise sporting necessity *de dicto* for a conclusion sporting necessity *de re*. So that is good. What is not good is that this argument is an analysis of another pair of sentences than {'Necessarily, the King of France is a king', 'The King of France is necessarily a king'}, namely {'Necessarily, for all *x*, *if x* is the King of France *then x* is a king', 'For all *x*, *if x* is the King of France *then*, necessarily, *x* is a king'}. These two pairs are nowhere near to being equivalent, not least because the second pair incorporates implication and universal quantification, and the first one does not. The second argument simply does not qualify as a logical analysis of the first pair of sentences and is insofar irrelevant.

A*ttitudes* are another notorious troublemaker. They force us to switch to some epistemic, doxastic, etc., logic. Here is a standard example.

Charles believes that if it is raining then the street is wet
(If it is raining then the street is wet) iff
(if the street is not wet then it is not raining)

Charles believes that if the street is not wet then it is not raining.

A case can be made for the validity of this argument, as well as for its invalidity. If Charles' attitude concerns an empirical state-of-affairs then his attitude is not sensitive to whether its complement (what is believed) is a proposition or its contraposition. If, on the other hand, his attitude concerns a particular way of conceptualising or presenting an empirical state-of-affairs, then there are strong reasons for blocking the argument. One thing is to believe one conceptualisation or presentation of a state-of-affairs, quite another thing is to believe another such conceptualisation. *Ex hypothesi*, Charles agrees to the first conceptualisation, but he may dissent from, or have no opinion about, the one occurring as complement in the conclusion.

However, consider another example:

Charles knows that Thelma is happy

Charles knows that (Thelma is happy and whales are mammals).

It may be the case that the first sentence is true whereas the second is false. Yet the standard possible-world semantics of epistemic logic yields the result that the second sentence must be true as well, 'Thelma is happy' and 'Thelma is happy and whales are mammals' being analytically equivalent. This is due to the fact that the proposition that whales are mammals is the necessary proposition *TRUE*, which takes the truth-value **T** for all possible worlds and times. Provided (as we are supposing) we understand the meaning of 'is a whale' and 'is a mammal' as these predicates are used in current English, if an individual is known to be a whale, we need not (empirically) examine the state of the world in order to get to know that the individual is a mammal.

In the standard notation of epistemic logic, the premise and the conclusion above become

$$\frac{K_a\, H(b)}{K_a\, [H(b) \wedge \forall x\, (W(x) \supset M(x))].}$$

But in the epistemic systems based on Kripkean possible-world semantics, this variant of *epistemic closure* holds:

If $(M, w) \models K_a\, \varphi$ and $(\varphi \models \psi)$, then $(M, w) \models K_a\, \psi$.

If a knows an empirical proposition, then a also knows everything logically implied by it. And a immediately knows all analytical truths as well, because they follow from the empty set of assumptions; or semantically put, they are true in every possible world.

Hence, when knowing that Thelma is happy, Charles is bound to know that Thelma is happy and that whales are mammals. And he is bound to know all mathematical truths as well, because they are analytically true, hence either true throughout all logically possible worlds or true independently of worlds altogether.

Here is an example demonstrating the difference between beliefs *de dicto* and *de re*:

'Charles believes that the King of France is a king.'

'Charles believes of the King of France that he is a king.'

Whereas the first sentence may be true, the second sentence cannot be true, as long as there is no King of France. The standard advice is to turn to doxastic logic:

$$B_b P[k(a)] \qquad (\textit{de dicto})$$
$$\lambda x\, B_b P[x]\, k(a) \qquad (\textit{de re}).$$

Again, worrisome questions arise. β-reduction converts the two analyses into one and the same formula. Why aren't we allowed to execute *the* basic computational rule of the λ-calculi in this case? The standard answer would be, 'Because the term '$k(a)$' is non-denoting'. But how can we know that the term is non-denoting and, thus, not well-formed? On another interpretation the same term will be a perfectly well-formed term. It does not seem right that the vicissitudes of the empirical world should make a difference as to whether a term is well-formed.

Or for a variant analysis:[30]

$$B_b P[k(a)] \qquad (\textit{de dicto})$$
$$(\exists x)\, (x = k(a) \wedge B_b P[k(a)] \qquad (\textit{de re}).$$

Where does the existential quantifier come from in the *de re* case? There is no trace of it in the original sentence. How can the logical forms of two similar sentences differ so radically? Hintikka and Sandu propose in 1996 a remedy by means of Independence Friendly first-order logic:

> Independence Friendly (IF) first-order logic deals with a frequent and important feature of natural language semantics. Without the notion of *independence*, we cannot fully understand the logic of such concepts as belief, knowledge, questions and answers, or the *de dicto* vs. *de re* contrast (1996, p. 173).

[30] See Hintikka and Sandu (1989).

They solve the *de dicto* case as above, and propose the *de re* solution with the independence indicator '/':

$$B_b\, P[k(a) \,/\, B_b].$$

This is certainly a more plausible analysis, closer as it is to the syntactic form of the original sentence. Furthermore, the independence indicator indicates the essence of the matter; there are two *independent* questions: 'Who is the King of France (if *k(a)* is interpreted as the King of France)?' and 'What does Charles think of that person?'. Of course, Charles needs to have a relation of 'epistemic intimacy' (cf. Chisholm, 1976) to a certain individual, but he need not make the connection between this person and the office of King of France (though the ascriber must). Still, the semantics of '$/B_b$' is not pellucid, which tells against it as a tool suitable for logical analysis. We will show that informational independence can be precisely captured by means of TIL's explicit intensionalization and temporalization without invoking any new non-standard operators.[31]

We consider it a non-negotiable datum to be respected by any viable attitude logic that attitudes *de dicto* and *de re* do not turn out to be equivalent. But it won't suffice for a given theory of attitude logic to simply point out the non-equivalence and ban conversion, again because a ban must be backed up by a logical insight into *why* conversion will fail to preserve equivalence. The following example serves to motivate the non-equivalence between attitudes *de dicto* and *de re*:

'Charles believes that the President of Zimbabwe is an absolute despot.'
'Charles believes of the President of Zimbabwe that he is an absolute despot.'

These two sentences do not denote the same proposition, for their truth-conditions differ. Charles might have read in a reliable newspaper, and so have come to believe, that the President of Zimbabwe is an absolute despot, thus making the first sentence true. However, Charles may have no idea as to who the President of Zimbabwe is, nor whether this particular individual is a despot. In such a situation the second sentence is not true. Or, another scenario is imaginable: Charles is acquainted with someone who happens to be the President of Zimbabwe, and Charles believes that his acquaintance is a despot, without having the slightest idea that this person is the President of Zimbabwe. In such a situation the second sentence is true and the first false.

Regrettably, the standard notation of doxastic logic deployed above does not reveal the difference in meaning between these two sentences. If '$k(a)$' is a denoting term, then the two formulae come out equivalent. The only way out of this predicament seems to be to heed the advice not to use the β-rule here, because the variable x occurs within the scope of the doxastic operator 'B'. The fact *that x* occurs within the scope of B is unquestionably the source of the trouble. But *why*

[31] See Section 5.1.2.

does x's occurrence within the scope of B invalidate β-transformation? This is the question that the logical semanticist must answer.

Qualms about substitution within attitude contexts motivate the need to ascend from intensional logic to *hyperintensional* logic. Here is an example in which it is indisputable that hyperintensional attitude complements are called for.

Charles calculates 2+5
2+5 = 7

Charles calculates 7.

It is no option to relate Charles to possible-world intensions. Their granularity is far too crude for them to figure as complements in mathematical attitudes. Thus, Charles would be related to a constant function from possible worlds and instants of time to a number. This grossly misrepresents what the activity of calculating is all about, which is to apply arithmetic operations to numbers. Finer granularity that would block the undesirable derivation would relate Charles to the expression '2+5'. Yet Charles cannot be related to a piece of mathematical *notation.* The argument does not say what syntactic transformation Charles performs in order to calculate the sum of 2 and 5. In the case at hand Charles calculates 2+5 by applying the addition function to the pair of numbers (2, 5). Besides, the conclusion is either false or nonsensical, depending on what sense can imaginably be made of calculating an individual number. Yet also this argument has the airs of a valid argument.

All the arguments above are *puzzles*. If there is a definition of *puzzle*, it is that a puzzle is an argument that takes premises individually considered true to conclusions that are indisputably false or else nonsensical. Hence, a puzzle threatens to trade (seeming) truths for either falsehoods or nonsense. In general, puzzles flow from two different sources. Either the logical form of one or more premises is ill-understood, or an otherwise valid rule of inference is applied outside its domain. (Of course, a puzzle may well flow from both sources.) The solution to a puzzle consists, thus, in blaming either the analysis of one or more of the premises or the rule of inference (or both). If one blames the rule of inference, one thereby claims to have discovered that, in the cases at hand, Leibniz's Law is valid only in some contexts. If one blames the analysis of the premises, one thereby claims to have discovered that Leibniz's Law does not apply, because the argument in question fails to have the appropriate logical form for it to apply. Our strategy throughout is to find fault, not with Leibniz's Law, but with how one or more premises of a given argument are logically analysed. The logical forms of the premises of the arguments above (as well as those of many others considered in this book) will turn out to be somewhat more complicated than predicted by first-order logic. This is in itself hardly a revolutionary claim; but what is innovative about our approach is that it offers an exact calibration of the degree of complexity of particular premises and conclusions.

If we start with first-order predicate logic (FOL), then what we have is a system that is broadly known, well-researched and profoundly elaborated. There are sound and complete calculi for this logic, such that all the logically valid formulas of FOL are provable. Though the system is not decidable, it is partially decidable: if a formula is logically true then there are algorithms that would answer Yes in a finite number of steps when inputting such a formula. The language of FOL has become the language of mathematics. Attractive mathematical theories have been couched in this language, and their properties are well-known.

But, there is only so much one can use FOL to. The shortcomings of FOL can be briefly summarised as follows. First, it is an *extensional* system. Though this is in itself no shortcoming, this fact does not make it possible to distinguish between analytical and empirical expressions. The difference is that the reference of the latter is dependent on modal and/or temporal parameters. Thus there is a need for an intensional system in the vein of possible-world semantics.

Second, FOL is a *first-order* system. This fact does not make it possible to systematically distinguish between ascribing a property to a function as a whole (like in ‘Sinus is a periodic function’) and ascribing a property to a particular functional value (as in ‘$\sin(\pi) = 1$’). Another example: ‘Charles is incorruptible’ versus ‘Being incorruptible is an honourable property’. We need a higher-order system.

Third, FOL is a system working with *total* functions only. However, in order to work with empty concepts and functions not returning values at some arguments, as well as the problems of empirical (non)existence, and value gaps in mathematics, what is needed is a logic of partial functions.

Fourth, FOL is a system whose universe is always *one-sorted*, while allowing one sort to be replaced by another. However, one needs to be able to distinguish distinct types of entities that the system talks about. There is certainly a categorial difference between an individual role such as The King of Bhutan and any of the extensions of this intension, which are individuals. Similarly, there is certainly a categorial difference between a numerical function and any of its arguments or values, which are numbers. Thus, one is better off switching to many-sorted logics. And if, moreover, one needs to distinguish between modal and temporal parameters, as in ‘The President of the USA might not have been a president’ and ‘The President of the USA is often a Republican’, one needs to switch to modal logics, temporal logics, etc.

Thus we need increasingly expressive logical systems—only to realize sooner or later that there is always something missing. Today, as a result, we have ended up with a sprawling tree whose branches are particular logics. Certainly, no single logic can render all the features of natural language. Furthermore, these individual logics are well elaborated from the formal point of view. Starting with an alphabet, grammatical rules determine a set of well-formed formulae. Having thus defined the syntax of a formal language, we choose a subset of the set of well-formed formulae as axioms, and specify the rules of inference by choosing a finite set of sequences of formulae. Finally, the so defined theory is investigated for its interesting mathematical/logical properties. We ask whether a theory is consistent and, thus, has a model, whether it is complete, whether the underlying calculus is complete, etc. As a result, instead of natural language we find ourselves studying the formal language itself.

This is unquestionably an interesting and legitimate task of logic and mathematics. Indeed, some of the greatest achievements of twentieth-century logic and mathematics are meta-mathematical, including meta-logical, insights into the properties of particular sets of well-formed formulae (*wff*'s). Yet you may ask: How does such a translation of a natural-language sentence into a shorthand formula contribute to the *analysis* of the sentence? In what way does it cultivate our *reasoning*? The answer would be, 'By following the formal axioms and rules of a given theory you obtain the logical consequences of its axioms'. But then one has to correctly *interpret* the theory in order to use it to solve a particular problem. Moreover, *which* particular theory should an agent apply in this or that case, and *how* should the resulting formulae be interpreted?

Still, if this panoply of logics is indispensable for something beginning to look like a full theory of natural language, and if the individual logics are technically precise, do we not have, as working logicians, all we need to go about our business of logically analysing fragments of natural language? Yes and no. We do have some logic or other available for almost all particular kinds of context involving particular problematic expressions. But what we do not have is an overarching, unitary logic.

Imagine one is building up a multi-agent system of autonomous, intelligent agents who are to communicate by exchanging messages, and who make decisions based on the content of these messages. Each message may concern a particular problem; thus the agents would have to keep switching between logical systems. They would have to combine modal logics, epistemic logics, temporal logics, provability logics, and so on and so forth. But inter-translatability forms a stumbling-block, since the same connectives may not preserve meaning when switching between logics. Agents may end up speaking at cross purposes.

Thus, in our opinion, in a multi-agent world of the Semantic Web, information and communication technologies (ICT), artificial intelligence (AI), and other such facilities, there is a pressing need for a *universal framework* informed by one philosophical logic making all the semantically salient features of natural language explicit. Consequently, such a universal logical framework would and should make a fine-grained logical analysis of relevant premises possible to create a platform for an ideal inference machine that neither over-infers (yielding consequences not entailed by the premises) nor under-infers (failing to yield consequences entailed by the premises).

The ambition of TIL is to provide such a universal framework. The purpose of this book is to display the framework in all its might. The TIL 'language of constructions' is not a formal language of non-interpreted terms. It is formal, if by 'formal' we mean rigorously defined and employing a special notation. But the individual terms and the entire language are themselves not the subject of our study. Rather the terms of the ‘language of constructions’ unambiguously encode logical constructions, and these extra-linguistic procedures are the ultimate subject matter of our study.

1.2.2 The top-down approach

We mentioned in Section 1.1 that TIL generalises from the hardest case and obtains the less-hard cases by lifting various restrictions that apply only higher up. This way of proceeding is opposite to how semantic theories tend to be built up. As we illustrated in Section 1.2.1, the standard approach consists in beginning with atomic sentences, proceeding to molecular sentences formed by means of truth-functional connectives or by quantifiers, and from there to sentences containing modal operators and, finally, attitudinal operators.

Thus, to use a simple case for illustration, once a vocabulary and rules of formation have been laid down, a semantics gets off the ground by analysing an atomic sentence as follows:

(1) ‘Charles is happy’
Fa

And further upwards:

(2) ‘Charles is happy, and Thelma is grumpy’
$Fa \wedge Gb$

(3) ‘Somebody is happy’
$\exists x\ (Fx)$

(4) ‘Possibly, Charles is happy’
$\Diamond\ (Fa)$

(5) ‘Thelma believes that Charles is happy’
$\mathrm{B}b\ (Fa)$.

In non-hyperintensional (including non-procedural) theories of formal semantics, attitudinal operators are swallowed by the modal ones, typically with ‘$\Box$’ standing for knowledge and ‘$\Diamond$’ for belief (as in the so-called modal logic of knowledge and belief). But when they are not, we have three levels of granularity: the coarse level of truth-values, the fine-grained level of truth-conditions (propositions, truth-values-in-intension), and the hyper-fine-grained level of hyperpropositions (propositional constructions).

TIL operates with these three levels of granularity (in fact, adding a fourth level of granularity, slightly coarser than that pertaining to constructions, in terms of *concepts*; see Section 2.2). We start out by analysing sentences from the uppermost end, furnishing them with a hyperintensional semantics, and working our way downwards, furnishing even the lowest-end sentences (as well as non-sentential expressions) with a hyperintensional semantics. That is, the sense of an atomic sentence such as ‘Charles is happy’ is a hyperproposition, i.e., a propositional *construction*, due to the trickle-down effect of our top-down approach. Likewise, the sense of ‘1+2=4’ is a *construction* of a truth-value.

Our motive for working top-down is pivoted on anti-contextualism: any given term or expression expresses the same construction as its sense in whatever sort of context the term or expression is embedded within. As for denotation, in the case of non-denoting expressions (mathematical expressions expressing improper constructions) it holds that such an expression does not denote anything in any context. Further, some terms, like indexicals, express only what we call ‘pragmatically incomplete meanings’[32] and, therefore, only denote relative to a valuation, being insofar sensitive to which context they are embedded in. All remaining terms do denote, though, and have context-insensitive denotations.

Furthermore, the sentence ‘Charles is happy’ is an intensional context, in the sense that its logical analysis must involve reference to empirical parameters, in this case both possible worlds and instants of time. One reason is because Charles is only *contingently* happy; i.e., he is only happy at some worlds and only sometimes. The other reason is because the *analysans* must be capable of figuring as an argument for functions whose domain is made up of propositions rather than truth-values. Construing ‘Fa’ as a name of a truth-value works only in the case of extensional contexts like (1) and (2). It won’t work in modal contexts like (4), since truth-values are not the sort of thing that can be possible. Nor will it work in a *hyperintensional* context of knowing or believing, since truth-values are not the sort of thing that can be known or believed. The sentence ‘Charles is happy’ is a hyperintensional context, as soon as Thelma’s art of believing relates her to a hyperproposition.

A logical syntax cannot tolerate ambiguous terms. The historical culprit for the notation found in the *analysantes* of (3), (4) and (5) must, in our view, be the conception of modalities due to the original syntax of ‘$\Box$’, ‘$\Diamond$’, which treats ‘$\Box$’, ‘$\Diamond$’ as being syntactically on a par with ‘$\neg$’; both ‘$\neg p$’ and ‘$\Box p$’ are well-formed formulae. This makes for handy notation, but it remains implicit that the argument of $\neg$ is a truth-value of p and the argument of $\Box$, p itself, i.e., the entire function. If ‘K’ (denoting an epistemic operator) is introduced as a notational variant of ‘$\Box$’ we get formulae like ‘Kp’, and we are allowed to generate strings like, ‘$\neg p \wedge K\neg p$’, where the extension/intension ambiguity of the notation is manifest. Moreover, if K is a hyperintensional operator, and $\Box$ an intensional operator, then we are in for three-way ambiguity as in, ‘$(\Box p \rightarrow p) \wedge Kp$’.

Tichý also bemoans the inherent ambiguity of the syntax of modal logic:

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[32] See Section 3.4. Though *incomplete* is, strictly speaking, a privative modifier, such that an incomplete meaning would not be a meaning, by ‘pragmatically incomplete meaning’ we intend, stipulatively, a meaning that is an open construction with free variables.

> [T]he modal logician keeps us in the dark…about [the logical type of □]. His axioms are framed in terms of p's and q's – as in '$\Box(p\supset q)\supset(\Box p\supset\Box q)$' – but it is entirely unclear what these variable-letters are meant to range over. The fact that they combine with truth-functional connectives like '$\supset$' might suggest that they range over the truth-values. This, however, is hardly compatible with their combinableness with '□' (Tichý, 1988, p. 279).

And he notes elsewhere that

> [S]tandard first-order logic is only capable of dealing with propositional constructions *de re*: negation, conjunction, alternation and the like. Propositional construction[s] *de dicto*, especially modal, probabilistic, epistemic, deontic, subjunctive, and causal constructions, are far beyond the reach of first[-]order logic. All attempts to force such constructions on to the Procrustean bed of first-order idiom are, in my view, doomed to failure (Tichý, 1978a, p. 10; 2004, p. 258).

It is worth dwelling on the topic of typing for a minute. Our perhaps pedantic-seeming harping on notational tidiness is grounded in a contentual issue of wide-ranging importance; namely, what we just said, that a logical syntax cannot tolerate ambiguous terms.

On our diagnosis, a bottom-up approach to modalities and attitudes is bound, it seems, to acquiesce in ambiguous notation and context-sensitive reference shift. This amounts, in effect, to operating with several semantic theories, one for each sort of semantic context. A top-down approach holds out the prospect of one semantic theory for all sorts of semantic context. The methodology consists in starting out on the top floor with a hyperintension and then either staying there or, if the semantic analysis requires it, taking the lift down and getting off either at the floor of intensions or at the floor of extensions.[33] Since we start out at the top, we start out with *constructions*, which we define next.

1.3 Foundations of TIL

1.3.1 Functional approach

The fundamental notions in terms of which a system is built up cannot be defined *in* the system itself, but must be understood prior to the theory and are introduced into the theory as primitives. So, for example, predicate logics are built up in terms of sets and relations. By contrast, the fundamental notion for TIL is the one

[33] We are making a simplification here to get the top-down picture clear. As a matter of fact, there are several floors of hyperintensions, intensions and extensions to get off at. In particular, while you always start out at the top, at *a* level of hyperintensions, there are going to be floors of hyperintensions above the floor you are on. Furthermore, the floor you get off at may itself be one of hyperintensions (though a floor one level down from where you started out). On the other hand, the vast bulk of empirical cases that we analyse in this book conform to the picture of starting out with a hyperintension, descending to the intension it presents and then descending from intension to extension. 'Charles is happy' would be a case in point.

of *function*.[34] This seemingly banal fact *is* important. Functions—unlike relations or sets—are *procedure-friendly* in the following sense:

(i) for any n-ary function *qua* mapping $M_1 \times \ldots \times M_n \rightarrow N$ there is an abstract procedure (often called *abstraction*) that produces at every n-tuple of elements of $M_1,\ldots,M_n$, respectively, at most one member of N;
(ii) the reverse procedure *applies* the mapping $M_1 \times \ldots \times M_n \rightarrow N$ to a particular n-tuple of elements of $M_1,\ldots,M_n$, respectively, and produces either nothing (if the mapping is undefined at that tuple) or the value of the mapping at that tuple.

Moreover, contemporary mathematics and logic define functions as *mappings*; i.e., as a special kind of set. The principle of extensionality is what guarantees this set-theoretical character of functions. Where f, g are functions the Principle says:

$$\forall x_1 \ldots x_n\ (f(x_1,\ldots,x_n) = g(x_1,\ldots,x_n)) \supset f = g.$$

On the other hand, as it is documented, e.g., in Tichý,

> Originally functions were understood as particular ways or methods of proceeding from numbers to numbers, i.e., as incomplete numerical constructions (1988, p. 3).

So

> [I]n order to properly grasp the modern notion of function one must keep it strictly apart from the notion of schematic calculation. ... one must always remember that the method is extraneous to the function itself (ibid).

Indeed, any function *qua* mapping can be constructed in infinitely many ways. Not distinguishing functions from methods is a source of many wrong turns in semantics, as will be shown when applying TIL to puzzle-solving.

Another reason for preferring functions to relations is *partiality*. A partial function f may return no value at some n-tuples. The corresponding relation R_f is the set of (n+1)-tuples, i.e., the subset of the respective Cartesian product. But among the (n+1)-tuples that are elements of the *complement* relation, one is not able to distinguish those which do not belong to the relation R_f (due to the fact that the respective entity is not a value of f at the argument) from those at which the function is undefined.

A simple example. Let f be a function that maps $M = \{a, b, c, d\}$ onto $N = \{\alpha, \beta, \gamma\}$ as follows: $a \rightarrow \beta$, $b \rightarrow \gamma$, $d \rightarrow \alpha$; at argument c function f is undefined. The respective relation R_f contains three of the twelve possible couples: $\{\langle a, \beta\rangle, \langle b, \gamma\rangle, \langle d, \alpha\rangle\}$. Now, although we know that, e.g., $\neg R_f(a, \gamma)$ and $\neg R_f(c, \alpha)$, the difference

[34] Also Montague (1974a), together with other semanticists, has opted for the functional approach and adopted a typed λ-calculus for his logical analysis of natural language.

between *f* being defined at *a* and undefined at *c* is lost. We cannot deduce whether the value of *f* exists at *c* or not.[35]

Finally, the functional approach is connected with the idea that any logical analysis of natural language should obey *compositionality*, which comes down to explaining the semantic behaviour of compounds in terms of the semantic behaviour of their components.[36] Obviously, our concern with partiality is part of a wider concern with compositionality. A term that has no reference (as opposed to denotation) affects the semantic behaviour of the compound it is part of. The challenge for a theory like ours which wishes to heed both the partiality constraint and the compositionality constraint becomes how to avoid that the semantic analysis of a compound comes to a standstill if one or more constituents contribute nothing at the level of denotation or reference. The way we tackle the challenge is, not surprisingly, by having non-referring terms contribute something at another level. *All* terms contribute a sense to the compounds they are constituents of; but some terms contribute only a sense.

The reasons just outlined explain why we are using a Frege-Church-style function/argument logic. The philosophical as well as logical advantage of a logic based on functions is that it can model interlocking logical structures in terms of functional dependencies. Functional dependencies are modelled by how the value of one function becomes the argument of another function, or how a function applicable to some particular argument is handed down by another function. A logic of functions is erected on the idea that one operation typically presupposes that another operation has already been executed so as to provide something to work with. As mentioned *ad* (i) and (ii) above, the functional operations are two in number—*application* and *abstraction*—of which the former 'descends' from a function to a value, while the latter 'ascends' to a function from other entities (perhaps including other functions). It is a key characteristic of the logic we are advocating that the outcome of the execution of an operation may itself be an operation. Otherwise the machinery would grind to a halt far too soon.

Our functional approach affects also how we think of language. We adhere to the Fregean tenet that every sentence contains at least one functor. For instance, we construe predicates as functors.[37] Predicates denote functions whose argument(s) must be picked out by some other expression(s) of the sentence. For instance, in 'Charles is happy', 'is happy' is the functor and 'Charles' the argument expression.

So why not settle for functions as meanings? For several reasons, each of which is conclusive. First, functions are too crudely individuated to qualify as hyperintensions. Functions are extensionally individuated, so possible-world intensions are

[35] We will deal with partiality in detail in Sections 2.6 and 2.7, where the need for partial functions is demonstrated together with a specification of inference rules for working with them.

[36] See Section 2.1.2 and also Tichý (1988, p. 287).

[37] —which is to say that we adhere to 'the Fregean doctrine that predicates name functions', as Bealer says (1982, p. 89).

individuated up to co-intensionality. Second, the operations of abstraction and application are exterior to functions and cannot be captured in terms of functions. Functions are not themselves procedures; functions can, and do, instead figure as input and output of procedures. Third, functions are set-theoretic entities and so cannot have parts. So it is not obvious how the account of compositionality, including partiality, is supposed to proceed. Fourth, functions cannot figure as modes of presentation. For sure, one can attempt to strain the notion of function and make it play the role of mode of presentation. But who wants a poor man's modes of presentation? Functions are sets, so it takes some charity to accept that the Cartesian product $A \times B$ would qualify as a presentation of, say, the mapping of a particular argument $a \in A$ onto a particular value $b \in B$. Any such correspondence between a and b records merely the fact *that* a is mapped onto b, but not *how*. Countless many procedures for mapping a onto b can be reconstructed; but none in particular. Yet a key reason for introducing modes of presentation is that there may be two or more clearly circumscribed modes of presentation of the same thing.

To anticipate a possible misunderstanding, note that in the semantics of mathematics, the terms 'function-in-intension' and 'function-in-extension' are sometimes used. For instance, Church (1941) broaches the question under which circumstances two functions are to be considered the same. He says:

> The most immediate and, from some points of view, the best way to settle this question is to specify that two functions f and g are the same if they have the same range of arguments and, for every element a that belongs to this range, (fa) is the same as (ga). When this is done we shall say that we are dealing with *functions in extension.*
>
> It is possible, however, to allow two functions to be different on the ground that the *rule* of correspondence is *different in meaning* in the two cases although always yielding the same result when applied to any particular argument. When this is done we shall say that we are dealing with *functions in intension*. The notion of difference in meaning between two rules of correspondence is a vague one, but, in terms of some system of notation, it can be made exact in various ways. We shall not attempt to decide what the true notion of difference in meaning is but shall speak of functions in intension in any case where a more severe criterion of identity is adopted than for functions in extension. There is thus not one notion of function in intension, but many notions; involving various degrees of intensionality (1941, pp. 2–3; emphasis ours).

Function-in-extension corresponds to the modern notion of function as a mapping, and function-in-intension could arguably correspond to our notion of *construction of a function.* However, since the notion of function-in-intension is a vague one, and obviously dependent on the formal system in which the meaning of the correspondence rule is captured, we will *not* use the term 'function-in-intension'. There is no reason for us to trade the crisp notion of construction (of a function) for the vague one of function-in-intension. But vague though it may be, its vagueness is in part owed to the 'various degrees of intensionality' that Church wants his overarching notion of function-in-intension to encompass.

Two degrees are minimally required to get the programme of a general semantics for natural-language discourse off the ground. The first is the degree made available by possible-world semantics, which individuates its intensions up to logical equivalence (cf. Church's functions-in-extension). The second is the hyperintensional degree. Only there is no such thing as *the* hyperintensional degree. Both Church and Cresswell define hyperintensionality negatively as any individuation finer than logical equivalence. The question, then, is how austere or how lax a degree of individuation we as semanticists need to impose when analysing a piece of language. We neither want the possible need for very fine-grained hyperintensionality to outstrip our logical resources to meet the need, nor do we want to arbitrarily impose just one degree of hyperintensionality.

So what we do is take the TIL constructions and have them serve as the most fine-grained hyperintensions available to us. If one imagines a hierarchy of hyperintensions, with the most fine-grained ones at the top, then one moves down the hierarchy by forming equivalence classes of more fine-grained hyperintensions and obliterating the differences among their individual members. This is how we arrive at our rigorous notion of *concept*. *Concepts* have a particular degree of hyperintensionality, and this degree seems, by and large, to be what we are looking for. What we are looking for are higher-order objects that satisfy the following criterion of hyperintensional individuation: any two hyperintensions are identical exactly when they are procedurally indistinguishable. The idea of procedure that guides us is, in general, that a procedure prescribes what to do to what entity or entities in what order to obtain what sort of entity. It seems natural to us to hold, then, that two expressions are synonymous just in case their respective meanings prescribe one and the same procedure. We find it hard to imagine what might be the semantic or logical import of a principle of hyperintensional individuation finer than procedural individuation.[38] Procedural individuation is pretty fine-grained, anyway. Yet the research project of laying down just how hyper hyperintensionality is must respect the fact, *vide* Church, that hyperintensionality is an open-ended cluster concept. What TIL contributes to this project is an intuitive principle of individuation (a procedural one) and higher-order entities that satisfy the principle (constructions, especially concepts), together with the possibility of a hierarchy of hyperintensions with constructions at the top.

The verdict is that functions are no good, if we want to assign hyperintensionally individuated, structured procedures to terms and expressions as their meanings. Constructions, in contrast, do the trick.

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[38] In particular, we are not going to draw distinctions that reflect notational differences that are not backed up by abstract procedural differences. So *Mates' puzzle* is not a puzzle for us; see Section 5.1.

1.3.2 Constructions and types

Constructions are procedures, or instructions, specifying how to arrive at less-structured entities. Qua procedures, constructions are structured, unlike set-theoretical objects, which are devoid of structure. Qua abstract, extra-linguistic entities, constructions are reachable only via a verbal definition. The 'language of constructions' is a modified hyperintensional version of the typed λ-calculus, where Montague-like λ-terms denote, not the functions constructed, but the constructions themselves. The modification is extensive. Church's λ-terms form part of his *simple* type theory, whereas our λ-terms belong to a *ramified* type theory. Constructions *qua* procedures operate on input objects (of any type, even constructions of any order) and yield as output (or, in well-defined cases, fail to yield) objects of any type. This way constructions construct *partial* functions.

When claiming that constructions are algorithmically structured, we mean the following. A construction C consists of one or more particular steps, or *constituents*, that are to be individually executed in order to execute C. The entities a construction operates on are not constituents of the construction. Similarly as the constituents of a computer program are its subprograms, so the constituents of a construction are again constructions. Thus on the lowest level of non-constructions, the objects that constructions work on have to be supplied by other (albeit trivial) constructions. The constructions hosting these trivial constructions may occur not only as constituents to be executed, but also as entities that still other constructions operate on. Therefore, one should not conflate *using* constructions as constituents of Composed constructions (where a Composed construction is what results from applying the operation of composition/application to a construction) and *mentioning* constructions that enter as input entities into Composed constructions. So we must distinguish strictly between *using* and *mentioning* constructions. We will deal with the use/mention distinction in Section 2.6; for now just briefly this. The constituents of a construction C, which are to be individually executed in order to execute C, are *used in C*. On the other hand, the entities (constructions or non-constructional objects) a construction C operates on are *mentioned in C*. Mentioning is, in principle, achieved by *using* atomic constructions. A construction C is atomic if it does not contain any other construction as a used subconstruction (a 'constituent of C') than C. There are two atomic constructions that supply entities (of any type) on which complex constructions operate: *Variables* and *Trivializations*.

Variables are constructions that construct an object dependently on *valuation*: they *v-construct*, where v is the parameter of valuation. With the important difference that we construe variables as extra-linguistic objects and not as expressions, our theory of variables is otherwise identical to Tarski's. Thus, in TIL variables construct objects of the respective types dependently on valuation in the following way. For each type α there are countably infinitely many variables x_1, x_2, …. The members of α (unless α is a singleton) can be organised in infinitely many infinite sequences. Let the sequences be given (as one is allowed to assume in a realist semantics). The valuation v takes a sequence $\langle s_1, s_2, \dots \rangle$ and assigns s_1 to the variable x_1, s_2 to the variable x_2; and so on.[39]

When X is an object of any type (including a construction), the Trivialization of X, denoted '${}^0X$', constructs X without the mediation of any other constructions. ${}^0X$ is the unique atomic construction of X that does not depend on valuation: it is a primitive, non-perspectival mode of presentation of X.

The other constructions are *compound*, as they consist of other constituents apart from themselves. These are *Composition*, *Closure*, *Execution* and *Double Execution.* Composition is the procedure of applying a function f to an argument A to obtain the value (if any) of f at A. Closure is the procedure of constructing a function by abstracting over variables; i.e., the procedure of abstracting, or extracting, a function from a context, as when abstracting $\lambda x(\varphi\, x)$ from $\varphi\,(a)$. Finally, higher-order constructions can be used twice over as constituents of Composed constructions. This is achieved by the construction called *Double Execution*, which we are going to need later. (Tichý adds also a simple construction called *Execution*, see Definition 1.2.)

TIL constructions, as well as the entities they construct, all receive a type. Thus TIL has a liberal ontology, accommodating both intensions of whatever degree n whose values are intensional entities of degree n–1, as well as constructions of whatever order $m > 1$ that construct entities of order m–1. Intensions may come in different orders, due to type rising, and in different degrees. An intension is higher-order if its range is made up of higher-order entities. For instance, a relation-in-intension relating individuals to constructions, as in the case of hyperintensional attitudes, is higher-order. An intension is first-order, but of a higher degree than zero, if its range is made up of first-order intensions; i.e., any such intensions as do not include constructions. For instance, *the tallest mountain* is of degree one, because its (world- and time-relative) values are themselves extensional entities (individuals), while *the most characteristic property of a war criminal* is of degree two, because its values are themselves intensional entities of degree one. Extensional entities also come in different orders. For instance, the set of all n-order constructions with some particular property is an extensional n-order entity.[40]

The definitions proceed inductively. First, we define simple types of order 1; second, constructions operating on types; finally, the whole ontology of entities as organised into a ramified hierarchy of types.

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[39] Tichý (1988) devotes an entire chapter to variables, explaining their objectual role as constructions; for details see (1988, pp. 47–62).

[40] The *degree* of a first-*order* entity corresponds roughly to an order in predicate logics. For instance, in order to ascribe properties to individual properties in predicate logic, we need to work within second-order logic. However, in TIL, properties of individuals are 1st-order objects of degree 1. Properties of properties of individuals are 1st-order objects of degree 2; and so on.

Definition 1.1 (*types of order 1*) Let B be a *base*, where a base is a collection of pair-wise disjoint, non-empty sets. Then:

(i) Every member of B is an elementary *type of order 1 over B*.
(ii) Let α, β_1, ..., β_m ($m > 0$) be types of order 1 over B. Then the collection $(\alpha\beta_1...\beta_m)$ of all m-ary partial mappings from $\beta_1 \times ... \times \beta_m$ into α is a functional *type of* order *1 over B*.
(iii) Nothing is a *type of order 1 over B* unless it so follows from (i) and (ii). □

Remark. For the purposes of natural-language analysis we choose the so-called *objectual base* described and motivated in the following Section 1.4. The objectual base B consists of the following atomic types:

o	the set of truth-values $\{\mathbf{T}, \mathbf{F}\}$;
ι	the set of individuals (the universe of discourse);
τ	the set of real numbers;
ω	the set of logically possible worlds (the logical space).

TIL is an open-ended system. The above objectual base $\{o, \iota, \tau, \omega\}$ was chosen, because it is apt for natural-language analysis, but in the case of mathematics a (partially) distinct base would be appropriate; for instance, the base consisting of natural numbers, of type ν, and truth-values. The derived functional types would then be defined over $\{\nu, o\}$.

Remark. An object O belonging to a type α is an α-object, denoted 'O/α'.

Remark. α-sets of elements of type α are modelled by their characteristic functions. Thus they are $(o\alpha)$-objects. For instance, a set of individuals is an object of type $(o\iota)$, a set of real numbers is an object of type $(o\tau)$, a set of couples of real numbers (i.e., a binary relation over reals) is an object of type $(o\tau\tau)$.

Example 1.1 *Types of extensional mathematical objects (non-constructions)*

- *Prime* is the set of prime numbers. It is an object of type $(o\nu)$.
- The factor set of sets of numbers that have the same remainder when dividing by 5 is an object of type $(o(o\nu))$.
- Binary functions defined on reals, like +, –, ×, :, are objects of type $(\tau\tau\tau)$.
- Binary relations-in-extensions on reals, like >, <, having the same remainder when dividing by 5 with an integer quotient, are objects of type $(o\tau\tau)$.

Definition 1.2 (*construction*)

(i) The *variable* x is a *construction* that constructs an object O of the respective type dependently on a valuation v; it v-constructs O.

(ii) Where X is an object whatsoever (an extension, an intension or a *construction*), ${}^0X$ is the *construction Trivialization.* It constructs X without any change.

(iii) The *Composition* $[X\ Y_1 \ldots Y_m]$ is the following *construction*. If X *v-constructs* a function f of a type $(\alpha\beta_1 \ldots \beta_m)$, and $Y_1, \ldots, Y_m$ *v-construct* entities $B_1, \ldots, B_m$ of types $\beta_1, \ldots, \beta_m$, respectively, then the *Composition* $[X\ Y_1 \ldots Y_m]$ *v-constructs* the value (an entity, if any, of type α) of f on the tuple-argument $\langle B_1, \ldots, B_m \rangle$. Otherwise the *Composition* $[X\ Y_1 \ldots Y_m]$ does not *v-construct* anything and so is *v-improper.*

(iv) The *Closure* $[\lambda x_1 \ldots x_m\ Y]$ is the following *construction*. Let $x_1, x_2, \ldots, x_m$ be pairwise distinct variables v-constructing entities of types $\beta_1, \ldots, \beta_m$ and Y a construction v-constructing an α-entity. Then $[\lambda x_1 \ldots x_m\ Y]$ is the *construction* λ-*Closure* (or *Closure*). It *v-constructs* the following function $f/(\alpha\beta_1 \ldots \beta_m)$. Let $v(B_1/x_1, \ldots, B_m/x_m)$ be a valuation identical with v at least up to assigning objects $B_1/\beta_1, \ldots, B_m/\beta_m$ to variables $x_1, \ldots, x_m$. If Y is $v(B_1/x_1, \ldots, B_m/x_m)$-improper (see iii), then f is undefined on $\langle B_1, \ldots, B_m \rangle$. Otherwise the value of f on $\langle B_1, \ldots, B_m \rangle$ is the α-entity $v(B_1/x_1, \ldots, B_m/x_m)$-constructed by Y.

(v) The *Execution* ${}^1X$ is the *construction* that either v-constructs the entity v-constructed by X or, if X v-constructs nothing, is v-improper.

(vi) The *Double Execution* ${}^2X$ is the following *construction*. Let X be any entity; the *Double Execution* ${}^2X$ is *v-improper* (yielding nothing relative to v) if X is not itself a construction, or if X does not v-construct a construction, or if X v-constructs a v-improper construction. Otherwise, let X v-construct a construction X' and X' v-construct an entity Y. Then ${}^2X$ v-constructs Y.

(vii) Nothing is a *construction*, unless it so follows from (i) through (vi). □

Remark. That a variable x constructs an entity dependently on valuation v will be referred to as 'v-constructing'. That a variable x v-constructs entities of a type α will be referred to as 'ranging over α', denoted by '$x \rightarrow_v \alpha$'.

Remark. ${}^1X$ is the procedure of executing X. Thus if X is a construction then the execution of ${}^1X$ consists in executing X. However, if X is not a construction then ${}^1X$ is the abortive construction whose input is X and whose output is nothing. A non-construction cannot be executed. Thus if X is a v-improper construction or a non-construction, ${}^1X$ is v-improper. Similarly, ${}^2X$ is the instruction to execute X and go on and execute the result. Thus ${}^2X$ is a v-improper construction if X is a v-improper construction or if the object v- constructed by X is a v-improper construction or a non-construction.

Remark. In principle, also *Triple Execution* could be defined, as could any other multiple *Execution*. But, pragmatically speaking, we as practising TIL-ians have had no need so far for Executions beyond Double Execution. And, methodologically speaking, we observe the constraint that the different kinds of construction should not be multiplied beyond what we know to be necessary. But, should Triple (or whatever) Execution turn out to be indispensable, it will be defined and added to the open-ended recursive definition of *construction*.

Remark. We use the terms 'mapping' and 'function' synonymously. By 'partial mapping' we mean a mapping that associates every argument (of the respective type) with at most one value (of the respective type); a total function is then a limiting case of the former; namely, a mapping that associates every argument with just one value. By 'properly partial mapping' we mean a partial mapping that is not total.

Remark. The names of constructions are written with upper-case first letters, to distinguish them from regular English words. The exception is 'variable', since it is already a well-established technical term in logical and mathematical literature.

Remark. Outer brackets of Closure will be omitted whenever no confusion can arise. We will say that a construction C *constructs* an entity E if C v-constructs E for all valuations v. Similarly, we will say that a construction C is *improper* if C is v-improper for all valuations v.

Definition 1.3 (*subconstruction*) Let C be a construction. Then

(i) C is a *subconstruction of* C.
(ii) If C is ${}^0X$, ${}^1X$ or ${}^2X$ and X is a construction then X is a *subconstruction of* C.
(iii) If C is $[X X_1 \ldots X_n]$ then $X, X_1, \ldots, X_n$ are *subconstructions of* C.
(iv) If C is $[\lambda x_1 \ldots x_n\ Y]$ then Y is a *subconstruction of* C.
(v) If A is a *subconstruction of* B and B is a *subconstruction of* C then A is a *subconstruction of* C.
(vi) A construction is a *subconstruction of* C only if it so follows from (i) to (v). □

Above we warned against the confusion that might arise from not distinguishing two ways in which a subconstruction D of a construction C may occur. The two ways were *using* D as a constituent of C and *mentioning* D by means of another constituent of C. Constructions are *used* in extensional or intensional contexts, and *mentioned* in hyperintensional (i.e., *constructional* or *conceptual*) contexts. These three kinds of context and the difference between using and mentioning constructions will be rigorously defined in Section 2.6, once more notions have been defined. Now we only briefly characterise the use/mention distinction.

> Let D be a subconstruction of a construction C. Then an occurrence of D *is mentioned in* C if the execution of C does not include the execution of D. Otherwise the occurrence of D *is used in* C *as a constituent.*

The simplest way to mention a construction C is by using the Trivialization of C. Thus in the Trivialization ${}^0[{}^0+\ {}^02\ x]$ the Composition $[{}^0+\ {}^02\ x]$ is not used; it is mentioned by using its Trivialization ${}^0[{}^0+\ {}^02\ x]$, which constructs $[{}^0+\ {}^02\ x]$ independently of valuation. The variable x is not free for substitution in $^0[^0{+}\ ^02\ x]$, as it is bound by the outer Trivialization. Thus we define:

Definition 1.4 (*free variable, bound variable, open/closed construction*) Let C be a construction with at least one occurrence of a variable ξ.

(i) Let C be ξ. Then the *occurrence of ξ in C* is *free*.
(ii) Let C be ^0X. Then every *occurrence of ξ in C* is ^{0}*bound* ('Trivialization-bound').
(iii) Let C be $[\lambda x_1...x_n\ Y]$. Any *occurrence of ξ* in Y that is one of x_i, $1 \le i \le n$, is *λ-bound* in C unless it is ^{0}*bound in Y*. Any *occurrence of ξ in Y* that is neither ^{0}*bound* nor *λ-bound* in Y is *free* in C.
(iv) Let C be $[X\ X_1...X_n]$. Any *occurrence of ξ* that is *free*, ^{0}*bound*, *λ-bound* in one of $X, X_1,...,X_n$ is, respectively, *free*, ^{0}*bound*, *λ-bound in C*.
(v) Let C be ^1X. Then any *occurrence of ξ* that is *free*, ^{0}*bound*, *λ-bound* in X is, respectively, *free*, ^{0}*bound*, *λ-bound in C*.
(vi) Let C be ^2X. Then any *occurrence of ξ* that is *free*, *λ-bound* in a constituent of C is, respectively, *free*, *λ-bound in C*. If an occurrence of ξ is ^{0}*bound* in a constituent ^0D of C and this occurrence of D is a constituent of X' v-constructed by X, then if the occurrence of ξ is free, λ-bound in D it is *free*, *λ-bound in C*. Otherwise, any other occurrence of ξ in C is ^{0}*bound* in C.
(vii) An *occurrence* of ξ is *free*, *λ-bound*, ^{0}*bound* in C only due to (i)–(vi).

A construction with at least one occurrence of a free variable is an *open construction*. A construction without any free variables is a *closed construction*. □

TIL has two kinds of binding, either by Trivialization or by lambda. In both cases variables behave in harmony with the general principle that a bound variable is not free for substitution. The distinction between ^{0}-binding and λ-binding can be best illuminated as follows. Consider the following Closures (variables x, y v-constructing elements of type τ):

(a) $\lambda x\ [^0{\le}\ x\ ^00]$

(b) $\lambda y\ [^0{\le}\ y\ ^00]$.

The Closures (a), (b) are equivalent in that they construct the same class of numbers. The variables x, y are λ-bound in (a), (b). By contrast, consider

(c) $^0[\lambda x\ [^0{\le}\ x\ ^00]]$

(d) $^0[\lambda y\ [^0{\le}\ y\ ^00]]$.

The Trivializations (c), (d) are not equivalent, since they construct distinct (albeit equivalent) constructions. The variables x, y are ^{0}bound in (c), (d) according to the points (ii) and (iii) in Definition 1.4. Note, however, that x has only *one* occurrence in (a), as well as in (c), the former occurrence being λ-bound, the latter ^{0}bound. Similarly, there is one occurrence of y in (b) and in (d). 'λx', 'λy' are not improper symbols; they denote instructions to abstract over occurrences of x, y, respectively. And even if there is no occurrence of x, as in $\lambda x\ [^0\!+\ ^01\ ^01]$, the instruction specified by 'λx' is a one-step instruction. For instance, the Closure

$$\lambda x\ [^0\!+\ ^01\ ^01]$$

does not construct the number 2, but the constant function, of type $(\tau\tau)$, that associates the value 2 with all arguments.

Concerning point (vi) of Definition 1.4, consider $^2(^0x)$, which is the Double Execution of the Trivialization of x. If $x \rightarrow_v \tau$ and x v-constructs the number 1, then $^2(^0x)$ v-constructs what is v-constructed by the result of executing ^0x, i.e., by x. Thus $^2(^0x)$ v-constructs the number 1 and it is equivalent to x. In general, $^2(^0x)$ v-constructs what x v-constructs. Hence x is *free* in $^2(^0x)$. However, the Double Execution $^2(^0(^0x))$ constructs what ^0x constructs, namely the variable x; the variable x is thus ^{0}bound in $^2(^0(^0x))$.

Definition 1.5 (*congruency and equivalence of constructions*) Let C, $D/*_n \rightarrow \alpha$ be constructions, and $\approx_v/(\text{o}*_n*_n)$, $\approx/(\text{o}*_n*_n)$ binary relations between constructions of order n. Using infix notation $^0C \approx_v {}^0D$, $^0C \approx {}^0D$, we define:

C, D are *v-congruent*, $^0C \approx_v {}^0D$, iff either C and D v-construct the same α-entity, or both C and D are v-improper;

C, D are *equivalent*, $^0C \approx {}^0D$, iff C, D are v-congruent for all valuations v. □

Corollaries.

If C, D are *identical*, $^0C =_* {}^0D$, then C, D are equivalent, $^0C \approx {}^0D$, but not vice versa.

If C, D are equivalent, $^0C \approx {}^0D$, then C, D are v-congruent, $^0C \approx_v {}^0D$, but not vice versa.

Remark. Recall that C, D are *identical*, $^0C =_* {}^0D$, if C and D are exactly the same procedure. Thus, for instance, though $^0[\lambda x\ [^0\!+\ x\ ^01]] \approx {}^0[\lambda y\ [^0\!+\ y\ ^01]]$, the two constructions are not identical. They construct one and the same function *Successor*/$(\tau\tau)$, i.e.,

$$\lambda x\ [^0\!+\ x\ ^01] =_{(\tau\tau)} \lambda y\ [^0\!+\ y\ ^01],$$

but in two different ways, because $x, y \rightarrow \tau$ are two different procedures. Different variables are not even equivalent and may be only v-congruent. On the other hand, if 'is sky-blue' and 'is azure' denote one and the same property of individuals, then not only ^{0}*Sky-blue* $=_{(((o\iota)\tau)\omega)}$ ^{0}*Azure*, but also ^{00}*Sky-blue* $=_*$ ^{00}*Azure*, i.e., ^{0}*Sky-blue* is identical to ^{0}*Azure.*

Types: $=_{(\tau\tau)}/(o(\tau\tau)(\tau\tau))$; $=_{(((o\iota)\tau)\omega)}/(o(o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$; $=_*/(o*_1*_1)$; *Sky-blue*, *Azure*$/(o\iota)_{\tau\omega}$.

Example 1.2 *Equivalent and v-congruent constructions.*

(a) Let $v(5/x,1/y)$ be a valuation identical to v at least up to assigning the number 5 to the variable x and the number 1 to the variable y. Then the constructions $[{}^0\!+ x\ {}^01] \to_v \tau$, $[\lambda x\ [{}^0\!+ x\ y]\ {}^05] \to_v \tau$, $[{}^0Succ\ x] \to_v \tau$, are $v(5/x,1/y)$-congruent, because they $v(5/x,1/y)$-construct the number 6.
Types: $+/(\tau\tau\tau)$; $x, y/*_1\to_v\tau$; $Succ/(\tau\tau)$, the successor function.

(b) The constructions $[{}^0Divide\ {}^05\ x] \to_v \tau$, $[{}^0Square\_root\ [{}^0\!-\ [x\ {}^05]]] \to_v \tau$ are $v(0/x)$-congruent, because they are $v(0/x)$-improper.
Types: $Divide/(\tau\tau\tau)$, the division function; $x/*_1\to\tau$; $Square\_root/(\tau\tau)$, the positive square root function.

(c) The constructions $[{}^0\!+ {}^05\ {}^01] \to \tau$, $[\lambda x\ [{}^0\!+ x\ {}^01]\ {}^05] \to \tau$, $[{}^0Succ\ {}^05] \to \tau$, are equivalent. They construct the number 6;

(d) The constructions $[{}^0Divide\ x\ {}^00] \to_v \tau$, $[{}^0Square\_root\ [{}^0\!-\ [{}^00\ {}^05]]] \to \tau$ are equivalent, because they are v-improper for every valuation v.

In TIL—as also in Montague Grammar—*quantifiers* denote functions of type $(o(o\alpha))$, α an arbitrary type. Quantifiers are not ‘improper symbols’, ‘syncategorematic signs’, and suchlike. Note that TIL quantifiers do not bind variables. ‘$\forall x$’, ‘$\exists y$’ are shorthand for ‘$\forall\lambda x$’, ‘$\exists\lambda y$’, so the binding is done exclusively by λ.

The phenomenon of λ-binding arises due to λ-abstraction, i.e., Closure. The semantics of a formula of the form ‘$\forall x\ A$’ is in TIL deciphered as $[{}^0\forall\ \lambda xA]$, x v-constructing (‘ranging over’) objects of type α and A (v-)constructing a truth-value.

Quantifiers are thus defined as follows.

Definition 1.6 (*quantifiers* $\forall$ *and* $\exists$*, singulariser Sing*) The *quantifiers* $\forall^\alpha$, $\exists^\alpha$ are type-theoretically polymorphic total functions of type(s) $(o(o\alpha))$ defined as follows:

The *universal quantifier* $\forall^\alpha$ is a function that associates a class C of α-elements with **T** if C contains all elements of the type α, otherwise with **F**. The *existential quantifier* $\exists^\alpha$ is a function that associates a class C of α-elements with **T** if C is a non-empty class, otherwise with **F**.

The *singulariser* $Sing^\alpha$ is a partial, type-theoretically polymorphic function of type(s) $(\alpha(o\alpha))$ that associates a class C with the only α-element of C if C is a singleton, otherwise the function $Sing^\alpha$ is undefined. □

If $A \rightarrow$ ο and $x \rightarrow_v \alpha$, we will often use the abbreviated notation

$$\text{`}\forall x\, A\text{'},\ \text{`}\exists x\, A\text{'} \text{ and } \text{`}\iota x\, A\text{'}$$

instead of

$$\text{`}[{}^0\forall^\alpha\ \lambda x\, A]\text{'},\ \text{`}[{}^0\exists^\alpha\ \lambda x\, A]\text{'},\ \text{`}[{}^0Sing^\alpha\ \lambda x\, A]\text{'},$$

respectively, when no confusion can arise.

Remark. Classes of elements of type α are modelled by their characteristic functions, of type (οα). Hence there are *several empty classes*, of types ($ο\alpha_1$), ($ο\alpha_2$), etc., and not just one empty class *simpliciter*. Moreover, due to partiality there may be different kinds of emptiness; the respective characteristic function can be either *false* at a given argument or *undefined*. We can even obtain a degenerate class by using a function undefined at all arguments. An example would be the class of numbers that are equal to the result of dividing the number two by zero, constructed by $\lambda x\ [{}^0{=}\ x\ [{}^0\mathbf{:}\ {}^02\ {}^00]]$.

***Example 1.3** Mathematical constructions*

(a) The function +, defined on the natural numbers (of type ν), is not a construction. It is a mapping of type (ν νν), i.e., a set of triples, the first two members of which are natural numbers, while the third member is their sum. The simplest construction of this mapping is ${}^0+$ (See Definition 1.2, (ii)).

(b) The function + can be constructed by infinitely many equivalent, yet distinct constructions; for instance, the following Closures are equivalent by constructing the same function +:
$\lambda xy\ [{}^0+\ x\ y]$, $\lambda yx\ [{}^0+\ x\ y]$, $\lambda xz\ [{}^0+\ x\ z]$, $\lambda xy\ [{}^0+\ [{}^0-\ [{}^0+\ x\ y]\ y]\ y]$ (See Definition 1.2 (iii) and (iv)).

(c) The Composition $[{}^0+\ {}^02\ {}^05]$ constructs the number 7, i.e., the value of the function + (constructed by ${}^0+$) at the argument $\langle 2, 5\rangle$ constructed by ^{02} and ^{05} (See Definition 1.2 (iii)).
Note that the numbers 2, 5 are not constituents of this Composition, nor is the function +. Instead, the Trivialisations ${}^0+$, ^{02}, ^{05} are the constituents of the Composition $[{}^0+\ {}^02\ {}^05]$.

(d) The Composition $[{}^0+\ x\ {}^01]$ *v*-constructs the successor of any number *x*.
Note that the number 1 is not a constituent of this Composition. Instead, the Trivialisation ^{01} is a constituent; the other two constituents are ${}^0+$, *x*.

(e) The Closure $\lambda x\ [{}^0+\ x\ {}^01]$ constructs the successor function (See Definition 1.2 (iv)). The successor function can be constructed by infinitely many constructions, the simplest one of which is the Trivialisation of the function: ${}^0Succ$. Thus $\lambda x\ [{}^0+\ x\ {}^01]$ and ${}^0Succ$ are equivalent by constructing the same function. Yet the Trivialization ${}^0Succ$ is not a finitary, executable procedure. It is a one-step procedure producing an *infinite mapping* as its product. On the other hand, the Closure $\lambda x\ [{}^0+\ x\ {}^01]$ is an easily executable procedure. The instruction to execute this procedure can be decomposed into the following steps:

Take any number x and the number 1*; apply the function* + *to the couple of numbers obtained at the previous step; abstract from the value of x.*

(f) The Composition of this closure with $^0 5$, i.e., $[\lambda x\ [^0{+}\ x\ ^0 1]\ ^0 5]$, constructs the number 6 (See Definition 1.2 (iii)).

(g) The Composition $[^0{:}\ x\ ^0 0]$ does not *v*-construct anything for any valuation of *x*; it is *v-improper* for any valuation *v* (See Definition 1.2 (iii)). We will say 'improper', for short.

(h) The closure $\lambda x\ [^0{:}\ x\ ^0 0]$ is not improper, as it constructs something, even though it is only a degenerate function, *viz.* one undefined at all its arguments (See Definition 1.2 (iv)).

(i) If *x* is a variable *v*-constructing real numbers of type τ, then the Compositions $[^0\exists\lambda x\ [^0{>}\ x\ ^0 5]]$, $[^0\forall\lambda x\ [^0{>}\ x\ ^0 5]]$ construct the truth-value **T** and **F**, respectively, because the class of real numbers greater than 5 constructed by the Closure $\lambda x\ [^0{>}\ x\ ^0 5]$ is not empty, but is not the whole type τ.

(j) If $Sing^{\tau}/(\tau(o\tau))$ is a singularizer, then the following construction (the meaning of 'the greatest prime') is *v*-improper for all valuations *v*, i.e., improper: $[^0 Sing^{\tau}\ \lambda x\ [^0{\wedge}\ [^0 Prime\ x]\ [^0\forall\lambda y\ [^0{\supset}\ [^0 Prime\ y]\ [^0{\geq}\ x\ y]]]]]]$, or for short, $\iota x\ [[^0 Prime\ x] \wedge \forall y\ [[^0 Prime\ y] \supset [^0{\geq}\ x\ y]]]$.

So much for examples for now. As mentioned above, constructions can not only be *used* to construct objects of a lower-order type, they can also be *mentioned* by other constructions. Constructions can in this manner themselves serve as input/output objects, on which higher-order constructions operate. However, within the simple hierarchy of types, as defined in Definition 1.1, there is no type to be assigned to constructions themselves. For instance, the Composition $[^0{+}\ ^0 2\ ^0 5]$ constructs the number 7, an entity of type τ (or ν, depending on the choice of objectual base). But when Charles calculates 2+5, he is related to the Composition $[^0{+}\ ^0 2\ ^0 5]$ and not to its product 7. What is then the type of the activity of *calculating*? It is a relation (-in-intension) of an individual to the respective construction itself. And this constructional type has to be of a higher order than the type of its product.

Typical examples of hyperintensional contexts are attitude reports involving mathematical knowledge and belief. For instance, in

> 'Charles believes that arithmetic is recursively axiomatizable
> and that Gödel proved it'

the meanings of 'that arithmetic is recursively axiomatizable' and 'that Gödel proved it' are only mentioned, because Charles does not believe the truth-value **F**. Instead, he believes that the meaning of the embedded clause yields **T.** In other words, Charles is related to a *construction* of **F**.

At its most fundamental level, the formal ontology of TIL is bi-dimensional. One dimension is made up of constructions, while the other dimension encompasses non-constructions. The ontology of entities of TIL organised in a *ramified* *hierarchy of types* enables us to logically handle *structured meanings* as higher-order, hyperintensional, abstract entities, thus avoiding inconsistency problems stemming from the need to mention these entities within the theory itself. Any higher-order entity can be safely, not only *used*, but also *mentioned* within the theory.

On the ground level of the type-hierarchy, there are entities unstructured from the algorithmic point of view belonging to a *type of order* 1. Given an *objectual base* of atomic types, molecular complexity is increased by the induction rule for forming partial functions. Where α, $\beta_1,\ldots,\beta_n$ are types of order 1, the set of partial mappings from $\beta_1 \times\ldots\times \beta_n$ to α, denoted '$(\alpha\ \beta_1\ldots\beta_n)$', is a type of order 1 as well. (See Definition 1.1.)

Constructions that construct entities of order 1 are constructions of order 1. They belong to a type of order 2, denoted '$*_1$'. The type $*_1$ serves as a base for the induction rule: any collection of partial functions, of type $(\alpha\ \beta_1\ldots\beta_n)$, involving $*_1$ in their domain or range is a *type of order* 2. Constructions belonging to a type $*_2$, which construct entities of order 1 or 2, and partial functions involving such constructions, belong to a *type of order* 3; and so on *ad infinitum*.

Definition 1.7 (*ramified hierarchy of types*) Let B be a base. Then:

$\mathbf{T_1}$ (*types of order* 1) defined by Definition 1.1.

$\mathbf{C}_n$ (*constructions of order n*)

(i) Let x be a variable ranging over a type of order n. Then x is a *construction of order n over B*.
(ii) Let X be a member of a type of order n. Then ^{0}X, ^{1}X, ^{2}X are *constructions of order n over B*.
(iii) Let $X, X_1, \ldots, X_m$ ($m > 0$) be *constructions of order n over B*. Then $[X\,X_1 \ldots X_m]$ is a *construction of order n over B*.
(iv) Let $x_1, \ldots, x_m, X$ ($m > 0$) be *constructions of order n over B*. Then $[\lambda x_1 \ldots x_m\, X]$ is a *construction of order n over B*.
(v) Nothing is a *construction of order n over B* unless it so follows from $\mathbf{C}_n$ (i) to (iv).

$\mathbf{T}_{n+1}$ (*types of order n* + 1)
Let $*_n$ be the collection of all constructions of order n over B.
(i) $*_n$ and every type of order n are *types of order* $n + 1$.
(ii) If $0 < m$ and α, $\beta_1,\ldots,\beta_m$ are *types of order* $n + 1$ *over B*, then $(\alpha\ \beta_1 \ldots \beta_m)$ (see T_1 (ii)) is a *type of order* $n + 1$ *over B*.
(iii) Nothing is a *type of order* $n + 1$ *over B* unless it so follows from $\mathbf{T}_{n+1}$ (i) and (ii). □

Example 1.4 *Entities of higher-order types*

(a) The constructions
$^0+$, $[^0+\ x\ ^01]$, $\lambda x\ [^0+\ x\ ^01]$, $[\lambda x\ [^0+\ x\ ^01]\ ^05]$, $[^0\!:\ x\ ^00]$, $\lambda x\ [^0\!:\ x\ ^00]$, all mentioned in Example 1.3, construct objects of types of order 1. They are constructions of order 1 (see Definition 1.7, C_n), and belong, thus, to the type $*_1$ (see Definition 1.7, T_{n+1}); i.e., to the type of order 2 (See Definition 1.7, T_{n+1} (i)).

(b) Let *Improper* be the set of constructions of order 1 that are v-improper for all valuations v; then *Improper* is an object belonging to $(o*_1)$, the type of order 2 (See Definition 1.7, T_{n+1} (ii)).

(c) The Composition $[^0\mathit{Improper}\ ^0[^0\!:\ x\ ^00]]$ is a member of $*_2$, the type of order 3. It constructs the truth-value **T**. The constituent $^0[^0\!:\ x\ ^00]$ of this Composition is a member of $*_2$; it is an atomic *proper* construction that constructs $[^0\!:\ x\ ^00]$, a member of $*_1$. It is atomic, because the construction $[^0\!:\ x\ ^00]$ is not used here as a constituent but only mentioned as an input object.

(d) Let *Arithmetic* be a set of unary arithmetic functions defined on natural numbers, making *Arithmetic* an entity of type $(o(\nu\nu))$, and let $x \rightarrow_v \nu$. Then the Composition $[^0\mathit{Arithmetic}\ [\lambda x\ [^0+\ x\ ^01]]]$ belonging to $*_1$, the type of order 2, constructs **T** (an entity of type o, the type of order 1), because the Closure $[\lambda x\ [^0+\ x\ ^01]]$ constructs the unary function *Successor*, and this function is arithmetic. It belongs to the set *Arithmetic*.

(e) The Composition $[^0\mathit{Arithmetic}\ ^2c]$ v-constructs the truth-value **T** if c v–constructs, for instance, the Closure $[\lambda x\ [^0+\ x\ ^01]]$. The Double Execution ^2c then v-constructs what is v-constructed by this Closure; namely, the arithmetic successor function. The Composition $[^0\mathit{Arithmetic}\ ^2c]$ is an object belonging to $*_3$, the type of order 4; the variable c v-constructing the Closure of type $*_1$ is an entity of type $*_2$, the type of order 3. Since Double Execution increases the order of a construction (see Definition 1.7., C_n (ii) and T_{n+1} (i)), ^2c belongs to $*_3$, the type of order 4. Therefore, the Composition $[^0\mathit{Arithmetic}\ ^2c]$ belongs to $*_3$, the type of order 4. This exemplifies the phenomenon of *type raising*.

Note that every construction C belongs to $*_n$, so that C is an entity of a type of order $n > 1$, and (v–) constructs an entity belonging to a type α of a lower order. We will use the notation '$C/*_n \rightarrow_v \alpha$'. For instance, '$x/*_1 \rightarrow_v \tau$' reads 'The variable x belongs to the type $*_1$ and v-constructs reals'. For the variable c of the above example we write '$c/*_2 \rightarrow_v *_1$'.

Typing not only enables us to avoid vicious-circle problems, it also makes it possible to avoid another kind of 'improperness'. If X is not a construction of order n ($n \geq 1$), then ^1X does not construct anything and so is improper; if X is not a construction of order n ($n \geq 2$), then ^2X is improper; finally, if $X, X_1, \ldots, X_n$ are not constructions of types according to Definition 1.2 (iii), then $[X\ X_1 \ldots X_n]$ does not construct anything and so is improper. If a construction C is type-theoretically improper, then it does not v-construct an entity of any type α due to *wrong typing*.

The notion of construction is both the most important and most misunderstood of all TIL notions. This is little wonder, considering the fact that the model-theoretic paradigm of doing semantics continues to be overwhelmingly dominant and set theory continues to be the background theory of most analytic ontology. Constructivist logicians and computer scientists, in contrast, tend to find it easier to tap into TIL. Again, this is little wonder, since constructivists have their own notion of construction and computer scientists are trained in reasoning in procedures. Perhaps a Platonic dialogue (*sans comparaison*!) is as good a means as any to lay to rest the most common misconceptions of the Platonist notion of construction. Imagine the following dialogue taking place between a TILian and a non-TILian during a coffee break at a conference:

Question: Are constructions formulae of some type logic?
Answer: No!
Q: Are they equivalence classes of such formulae?
A: No!
Q: Are they denotations of closed formulae?
A: No!
Q: So what are they?
A: They are what Definition 1.2 says they are.
Q: Sure, I understand the formalities of your definition, but saying what the particular constructions construct you're not saying what they *are*!
A: So an informal, pre-theoretical characterisation is what you're after? Well, the fundamental idea is that of *abstract procedure*.
Q: Procedures are set in time, so how can they be abstract, as constructions are supposed to be?
A: The execution of a procedure (or algorithm, if you like) is a time-consuming process, all right, whereas the procedure itself is beyond time and space.
Q: So what about your symbolic language, the 'language of constructions'—why do you not simply say that its expressions are constructions?
A: These expressions serve only to represent, or encode, constructions; as expressions they cannot construct anything. What is important about expressions is only what they mean and not their syntactic shapes.
Q: But constructions outside time and space can construct something? How can abstract objects *do* anything?
A: They don't *do* anything, for sure. But *agents* can *execute* them. We do this sort of thing every day when executing algorithms or following instructions. When agents execute constructions, they follow an intellectual path that is already laid out. Agents, or any of their artefacts, do not construct constructions. This is why TIL is a realist and not an idealist theory.
Q: But you could do it like Montague did—translating expressions of natural language into the language of intensional logic, and then interpreting the result in the standard manner. What you achieve by using your constructions you would

get using a meta-language. So it seems like your superstructure of higher-order objects is not needed at all.[41]

A: Okay, this calls for a longer reply. Montague's and other intensional logics interpret the expressions of their language in terms of functions. However, from our perspective these mappings are only the products of the respective procedures. In terms of conceptual priority, there is an instance preceding functions. Montague does not make it possible to *mention* the procedures as objects *sui generis* or to make a shift to hyperintensions. Yet we do need a hyperintensional semantics. Procedures—our constructions—can be not only executed in order to obtain a product but also talked about in their own right, by using other higher-order constructions. It is not by chance that mathematicians did not always use the term 'function' in its contemporary sense, as standing for mappings, which are mere set-theoretic objects. Functions were previously thought of as *calculation procedures*. Also, the original interpretation of the terms of lambda calculus was procedural. For instance, Barendregt says,

> [I]n this interpretation the notion of a function is taken to be intensional, i.e., as an algorithm (1997, p. 184).

We would say, '... is taken to be *hyperintensional*, i.e., as an algorithm', because the term 'intensional' is currently reserved for mappings from possible worlds (if not among proof-theoretic semanticists, then at least among model-theoretic semanticists). Besides, our approach to semantic analysis is simpler and more direct. We do not pair expressions from, say, English off with symbols stemming from an artificial symbolism, interpret this symbolism and then couch our analysis in terms of what these symbols mean. Rather we pair English words and phrases off with their meanings straightaway, using our 'language of constructions' to encode these meanings. TIL does not need a meta-language, since we have a ramified type hierarchy instead.[42]

Q: You don't have a meta-language? That's somewhat unusual in modern logic.

A: It is. Yet we do have a parallel notion of *using* and *mentioning*, only what is used and mentioned are constructions and not words (though, of course, we're also able to quote words, by means of quotation marks). But let me quote Tichý on why TIL does not need a meta-language. Look:

> The whole linguistic outlook of modern logic and metamathematics, the preoccupation with symbols and strings of symbols as objects of study, results from the parsimonious decision to dispense with all entities other than first-order ones […]. The mathematician averts his eyes from constructions, which constitute his real subject matter, and looks at pieces of notation instead. This approach may satisfy his craving for ontological economy, but let it not be thought that it *simplifies* matters. If a range of entities is studied obliquely by means of proxies, rather than directly, the cognitive situation is complicated by the gratuitous intrusion of the proxy relation.

[41] See Montague (1974a).

[42] For a more detailed comparison of Tichý's TIL with Montague's IL, see Section 2.4.3.

> There is no *intrinsic* relation between a formula and the construction it represents. Hence if anything said about the formula is to have a bearing on things mathematical, the relation of the formula as a whole, or of its constituents, to mathematical objects must be explicitly stipulated. In order to put a stipulation into words, one has to *name* entities of both kinds: the mathematical objects and the linguistic expressions corresponding to them. Hence the need for a metalanguage, distinct and separate from the original notation in question. But the metalinguistic expressions themselves signify constructions. One thus faces a choice: one can either acquiesce in these higher-order constructions, or one can ignore them too and look instead at the meta-meta-expressions corresponding to them. If the first option is chosen the question arises why the same treatment cannot be applied at the bottom level, thus avoiding the original linguistic detour as well. And if the second option is taken one is obviously caught in an infinite regress of ever higher metalanguages (1988, p. 71).

Q: But that direct route to meanings comes with a completely *objectual* vision of logic, right?

A: Right. To get your head around TIL, don't think in terms of language-meets-language; think in terms of language-meets-reality. This reality is the Platonic realm of realist logic and semantics. In fact, what we're studying, at the end of the day, is not language, whether natural or artificial, but the simple and complex objects populating this realm. Language is a gateway, even if it's of independent interest. TIL *is* a philosophy of language, it's just that we think one can't, ultimately, study language by means of language.

Q: Okay, so that's why you replace other people's upper-level languages, or metalanguages, by a sphere of upper-level abstract objects?

A: Exactly. That's what TIL is pretty much all about. Leşniewski and Tarski were good Polish nominalists, so they wouldn't dream of admitting higher-level objects. Instead they erected higher-level languages. We're Platonists, on the other hand, so we agree with Frege that a third realm must be acknowledged. Only we're actually telling you what's in that realm.

Q: Constructions?

A: Constructions!

1.4 Possible-world intensions vs. extensions

1.4.1 Epistemic framework

TIL operates with a single *procedural semantics*, as explained above. TIL constructions are, without exception, assigned to expressions as their structured meanings. But within this one semantics TIL observes a strict demarcation between two kinds of subsidiary semantics: one for logical and mathematical languages and another for empirical languages, whether colloquial or scientific. The demarcation hinges not on formal vs. natural, but on empirical vs. non-empirical. The defining difference is that empirical languages incorporate an element of *contingency* that the non-empirical ones lack. Empirical languages must be able to denote *empirical conditions* that may or may not be satisfied. Non-empirical languages have no need for an additional category of expressions for empirical conditions. Roughly, the semantics for non-empirical languages is simpler, because the intensional level has been lopped off; yet also more complicated, because constructions constructing constructions (for instance, variables of type $*_{n+1}$ constructing constructions of type $*_n$), rather than intensions, are often needed.

For instance, the predicate 'is a student' does not denote each individual that is a student, nor a class of students. Rather, it denotes a *property* of individuals, the 'populations' of which are particular sets of individuals depending on particular states-of-affairs. To master 'is a student' is not to know a particular set of individuals; rather, it is to know how, for any state-of-affairs, to determine whether a given individual satisfies the condition of being a student. We model these empirical conditions as possible-world *intensions* which are functions with domain in possible worlds and values in chronologies of elements of a given type α. Thus we distinguish between the *hyperintension* (i.e., a construction of an intension *I*) assigned to an empirical expression *E* as its meaning, and the possible-world *intension I* denoted by *E*. However, as soon as we introduce what we shall call an *epistemic framework* for a given empirical language, the procedural semantics of the language operates in the same way as in the case of mathematical language. This is so because the epistemic framework assigned to a language confines what can possibly be talked about within that language.

In order to specify the objectual base of TIL over which an infinite ramified hierarchy of types is defined (see Definition 1.7), we must explicate the category of possible worlds. To this end we first need to explain the informal, pre-theoretical epistemic framework of a given empirical language.[43] First of all, the main methodological principle of TIL-based logical analysis of natural language has been formulated in Tichý as follows:

> To explicate a system of intuitive, pre-theoretical, notions is to assign to them, as surrogates, members of the functional hierarchy over a definite objectual base. Relations between the intuitive notions are then represented by the mathematically rigorous relationships between the functional surrogates (1988, pp. 194–95).

Everybody has a pre-theoretical understanding concerning reality, according to which there are things doing things and doing things to other things. A first approximation of a theoretical explication of this intuition would amount to saying that reality consists of individuals exemplifying properties and occurring in relations. By even just beginning to offer an explication along these lines, the formal semanticist has embarked upon the enterprise of providing a logical surrogate of reality. This surrogate is not supposed to render reference to reality superfluous; instead it must run in parallel to reality. The surrogate is the framework within which a semantic theory is stated. The things, in the widest possible sense, which

[43] For further background, see Tichý (1988, pp. 177–200).

are represented by a surrogate in the framework are the things that can possibly be talked about in some given language *L*. The overall project of TIL is (nothing less than) the explication of the framework underlying natural language, so *L* is not a particular national language, but any natural language.

Any successful linguistic communication between language-users makes use of a shared framework.[44] Tichý says,

> Communication between speakers and their audiences can only succeed on the basis of a shared logical space (1988, p. 201).
> To tell someone that Ali is sick I must somehow draw his attention to the [propositional] construction $\lambda w \lambda t\ [{}^0 Sick_{wt}\ {}^0 Ali]$. Communication is exchange of linguistic constructions over [an objectual base] (1986a, p. 264, 2004, p. 662).

To account for the expressive power of a given language shared by a community of language-users, Tichý introduces the concept of *epistemic framework* and the concepts of intensional and objectual bases affiliated with it. The goings-on of extra-theoretical reality make up the pre-theoretic *intensional base*, and the intensions defined over an *objectual base* attempt to capture them intra-theoretically. They do so by means of assignments to the functions defined over the objectual base. Tichý calls the totality of these assignments an 'explication' of the intensional base by means of the objectual base. An epistemic framework is then an intensional base garnished with an explication.

For instance, the string 'Ali is sick' presupposes, in order to have the sense it has in English, that it belongs to a language interpreted over an epistemic framework that comes with individuals, properties and a vehicle of predication.

Why is it important to point out that successful communication presupposes a shared epistemic framework common to all the parties to a discourse? Because the framework reconstructs the range of expressions a speaker or hearer can possibly make sense of. An expression which falls outside the purview of the framework is without sense (i.e., strictly speaking, not an expression at all, but a string of letters or sounds).[45,46]

The pre-theoretically understood elements of the *objectual base B* may in principle be pretty much whatever. But for the purposes of natural-language analysis, it has turned out that the elements must include, at least, *truth-values*, *individuals*, *times*, and *possible worlds*. Formally, $B = \{o, \iota, \tau, \omega\}$, each element of which is a non-empty set and disjoint from any of the three other sets. These four kinds of

[44] This holds no less for communication between solitary language-users and themselves in the form of inner soliloquies, as ought to be uncontroversial as far as philosophical theses go. We also tend to think that unverbalized thinking is impossible without the use of (a non-private) language; but we are not broaching this issue here.

[45] Cmorej calls such a string a 'semi-expression' in his 2005 discussing the thesis that semantics is a priori.

[46] In a wider philosophical context, the notion of epistemic framework might be of use to *hermeneutics*; e.g., with respect to Gadamer-like melting-together (*Verschmelzung*) of two or more different epistemic frameworks. We have not attempted to take the notion into this direction, though.

object are all non-functions, and cannot be defined (though characterised) within TIL. They are, in a word, logically primitive relative to *B*. However, the functions arising from *B* by combining elements drawn from it can be defined within TIL; this is required if we wish to display functional dependencies in accordance with our functional approach.

Explication of pre-theoretical intuitions consists, by and large, in offering an analysis of how α-objects are functionally dependent on β-objects. In hyperintensional contexts the analysis becomes more involved, since it must be spelt out how the relevant function(s) is (are) constructed. However, not everything can be either a function or a construction. Some objects serve as functional arguments or values without themselves being functions;[47] they are the ‘rock-bottom’ objects of the given epistemic framework. The elements of the members of *B* serve as arguments for intensions, and cannot be analysed within TIL without incurring circularity. It is particularly important that a state-of-affairs which is said to obtain at some world *W* not be conceived as a function from worlds to (chronologies of) extensions, but as entities being atomic relative to the given epistemic framework. The objectual base *B*, for its part, can be thought of as being among the fundamental ontological assumptions—or ‘ontological commitments’, if you like—of TIL.

A most important part of the explication is the interpretation of possible worlds. It goes as follows:

> By an *intension/time* I shall understand an ordered couple consisting of a member of the intensional base and a moment of time. A *determination system* is then an assignment which assigns to (some) intension/times unique objects over $\{\iota, o, \tau, \omega\}$ in such a way that if the type corresponding to the intension is $\xi_{\tau\omega}$[48] then the object assigned to the intension/time is ξ. Briefly, a determination system specifies one combinatorial possibility as to what objects are determined…by what intensions at what times. Now to interpret the basic category ω is to assign to each of its members a unique determination system (Tichý, 1988, p. 199).

The notion of epistemic framework is indispensable within TIL—as well as within any other theory of philosophical logic directed toward natural-language analysis—as it regulates the relationship between artificial and natural language. If the intensional base was skipped and the starting-point was the objectual base instead, TIL would be exactly what Tichý takes other intensional systems to task for being; namely, nothing but a logical game. In the form of a rhetorical question, what would be the purpose of defining an infinity of functions of type(s) $\alpha_{\tau\omega}$ if they were not somehow anchored to (fragments of) reality external to the system? In brief, intensions are not to be made sense of by means of another language that natural-language terms are translated into, but by being paired off with the pre-theoretical grasp of reality we all have to the effect that things do things to things.

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47 To be sure, in mathematics we can *model* them as zero-arity functions. But this hardly *makes* them functions.

48 See below; it is the type of a function $(\omega \rightarrow (\tau \rightarrow \xi))$ for a type ξ.

Still, any explication will have to cut corners in order to match, at least to some tolerable degree, extra-systematic reality and so cannot be expected to cut it cleanly by the joints. Fine-tuning an explication will come down to making the type-theoretic analysis more sophisticated or, more drastically, adding new types to *B*. In fact, the latter has already happened more than once. Whereas the original type-theories included only individuals and truth-values (or only individuals, if truth-values were just individuals), every possible-world semantics with a type-theory will have to add worlds as a type (if only in the half-hearted manner of Montague). Later on, when Tichý realised the need for numbers and times as an independent type, type τ was added. Kaplan (1975) is another possible-world semanticist with the same simple type theory as TIL, since also his intensions are defined over times as well.

In a nutshell, the enterprise of logical analysis of natural language has as its ultimate (and extremely ambitious) goal the exhaustive explication of the intensional base underlying natural language; i.e., its epistemic framework *in totō*. As Frege famously said in a not all-too dissimilar connection, *dahin gelangen wir nie*. So the goal is of a regulative nature. In what follows the *epistemic framework* that TIL assigns to natural language is described.[49]

Universe of discourse (type ι). The members of the universe are *individuals.* The individuals are *bare individuals*. This means that all the properties possessed by an individual necessarily are, roughly, trivial. In Section 1.4.2 we will explain in which sense some properties are trivial. For now, trivial properties are either *constant* functions (i.e., properties that have a constant extension—a set of individuals—as value in all possible worlds and times) or *partially constant* functions (whose extension varies for some possible worlds/times) with a *constant subset* of their possible extensions. All purely non-constant properties (without a non-empty constant subset of all possible extensions) are had by an individual only contingently. A bare individual is, then, what remains if one abstracts from all its non-trivial properties. From a logico-semantic point of view, a bare individual is simply a peg on which to hang properties. Another important feature of the universe is that it is one in number; there are no other universes/domains in other possible worlds, so there are no *possibilia* (‘possible individuals’).

Truth-values (type ο). There are just two truth-values, **T** and **F**. So TIL is a bivalent logic and insofar classical. TIL comes with truth-value gaps, however, and is insofar not classical. Any abstract objects can serve as surrogates, but we have to interpret them, so we say that **T** is the truth-value True and **F** the truth-value False.

Times or *real numbers* (type τ). The easy interpretation is described in Tichý (1988, p. 199); choosing the origin 0 of the time scale and a specific duration of time between 0 and the time represented by 1, we get the result that every real number will represent a unique instant of time, and vice versa. In TIL time forms a *continuum*. Alternatively, times could have been paired off with natural numbers,

[49] As of early 2010.

making times *discrete* instead. And, in fact, discrete times will often suffice for the purposes of analysis of natural language. But in order to avoid that the times we are analysing should outstrip our capacity to model them (to avoid running out of time(s), as it were), we are playing safe and modelling times as continuous straightaway.

Possible worlds (type ω). Consider an intensional base (relative to a given language). Every member of the intensional base conjugated with a time singles out some object, and every *possible world* is interpreted as specifying 'one combinatorial possibility as to what objects are [singled out]… by what intensions (i.e., members of the intensional base) at what times' (Tichý, 1988, p. 199).

This construal of possible worlds is distinct from many other conceptions, not least D. Lewis', according to which all possible worlds are concrete and actual *sub specie aeternitatis* (see his 1986). Nor are our possible worlds sets of formulae or Carnap-style state descriptions. Our construal is *Tractarian* in that it takes possible worlds as collections of *states-of-affairs* rather than of objects. Possible worlds, as we understand them, are the maximal consistent sets of chronologies of possible states-of-affairs.[50]

1.4.2 Intensions and extensions

The previous section provided the philosophy of intensions. In this section their logic follows.

Definition 1.8 ((α-)*intension*, (α-)*extension*) (*α-*)*intensions* are members of a type (αω): functions from possible worlds to the arbitrary type α; (*α-*)*extensions* are objects of the type α, where α is not equal to (βω) for any β; i.e., extensions are α-objects that are not functions from possible worlds. □

Remark. Intensions are frequently functions of the type ((ατ)ω), i.e., functions from possible worlds to *chronologies* of type α (in symbols: $\alpha_{\tau\omega}$), where an α-chronology is a function of type (ατ).

Remark. It is a noteworthy upshot of our general top-down approach that extensional entities are defined negatively and in terms of intensional entities; namely, as those objects that are not intensions. In case of an ordinary language extensional entities are of logical and semantic interest only insofar as they figure as values (or in the values) of intensions.

[50] Also Hintikka seems to accept this conception, but his possible worlds are *epistemic*, dependent on particular language-users (See, e.g., Hintikka and Hintikka, 1989).

We will use variables w, w_1, w_2,… as v-constructing elements of type ω (possible worlds), and t, t_1, t_2, … as v-constructing elements of type τ (times). If C v-constructs an α-intension, the frequently used Composition of the form $[[C\ w]\ t]$, v-constructing the intensional descent, or extensionalization, of an α-intension, will be abbreviated as 'C_{wt}'.

Intensions may come in different *orders*, due to type raising, and in different *degrees*.

An intension is a *higher-order* entity if its range is made up of higher-order entities. For instance, a relation-in-intension relating individuals to constructions, as in the case of hyperintensional attitudes, is higher-order. E.g., *Believe**, *Know** are entities of type $(o\iota *_n)_{\tau\omega}$, i.e., entities belonging to a type of order n+1, n≥1.

An intension is first-order, but of a *higher degree* than zero, if its range is made up of first-order intensions; i.e., any such intensions as do not include constructions. For instance, *the tallest_mountain*/$\iota_{\tau\omega}$ is of degree 1, because its (world- and time-relative) values are themselves extensional entities (individuals), while *the most characteristic property of a war criminal* is an entity of type $((o\iota)_{\tau\omega})_{\tau\omega}$, i.e. an intension of order 1 and degree 2, because its values are themselves intensional entities of degree 1 (properties of individuals).

Extensional entities also come in different orders. For instance, the *set* of all n-order constructions with some particular property is an extensional n-order entity of type $(o*_n)$.

Some important kinds of intension are:

Proposition/$o_{\tau\omega}$. They are denoted by empirical (declarative) sentences. Propositions are truth-values-in-intension.

Property of members of a type α, or simply α-*property*/$(o\alpha)_{\tau\omega}$.[51] General terms (some nouns intransitive verbs, adjectives) usually denote properties, mostly of individuals. Properties are sets-in-intension.

Relation-in-intension/$(o\beta_1\ldots\beta_m)_{\tau\omega}$. For example, transitive empirical verbs and attitudinal verbs denote such relations. If omitting $_{\tau\omega}$, we get the type $(o\beta_1\ldots\beta_m)$ of *relation-in-extension* (to be found mainly in mathematics and logic).

α-*role*/α-*office*/$\alpha_{\tau\omega}$, $\alpha \neq o$, $\alpha \neq (o\beta)$, $\alpha \neq (o\beta_1\ldots\beta_m)$, frequently $\iota_{\tau\omega}$; often denoted by the concatenation of a superlative and a noun ('the highest mountain'). An individual role corresponds to what Church (1956) calls an 'individual concept'. This word could cause misunderstandings, since *concept* in TIL is no intension, so we shan't use it.[52] Individual offices are individuals-in-intension.

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[51] Remember that collections, sets, classes of α-objects are members of type $(o\alpha)$; TIL handles classes (subsets of a type) as characteristic functions. Similarly, relations (-in-extension) are of type(s) $(o\beta_1\ldots\beta_m)$.

[52] For the theory of concepts, see Section 2.2.

Example 1.5 *Types of intensional objects*

- ‘Being happy’, or ‘is happy’, denotes a property of individuals/$(o\iota)_{\tau\omega}$. Given a possible world and a time, we are given the class of individuals that are happy at that world/time pair.
- ‘The President of the Czech Republic’ denotes an individual office, a.k.a. individual role/$\iota_{\tau\omega}$. Given a possible world and a time, we are given the individual, if any, who occupies the office, or plays the role, of President of the Czech Republic at that world/time pair. At some world/time pairs, there is no such individual (the function being properly partial).
- ‘The King of France is happy’ denotes a proposition/$o_{\tau\omega}$. If $\langle w, t\rangle$ is such a pair of worlds and times where the role of King of France is occupied by an individual X and X is happy at $\langle w, t\rangle$ then the proposition is true at $\langle w, t\rangle$. If X is not happy at $\langle w, t\rangle$ the proposition is false at $\langle w, t\rangle$. If the office of King of France is not occupied at $\langle w, t\rangle$ (as in the actual world now), the proposition lacks a truth-value.[53]
- ‘Calculating’ denotes an attitude of an individual to a construction, i.e., a relation-in-intension that is a higher-order intension of type $(o\iota *_n)_{\tau\omega}$.
- ‘Knowing* (explicitly)’ denotes an attitude of an individual to a construction, i.e., a relation-in-intension of a higher-order type $(o\iota *_n)_{\tau\omega}$.
- ‘Knowing (implicitly)’ denotes an attitude of an individual to a proposition, i.e., a relation-in-intension that is a higher-degree intension of the first-order type $(o\iota o_{\tau\omega})_{\tau\omega}$.[54]

For an example of the distinction between mathematical and ordinary language, consider the sentence

‘The number of the planets is 8.’

This sentence does not denote a truth-value, but a proposition/$o_{\tau\omega}$, and its meaning is a construction of the denoted proposition, namely a *hyperproposition*:

$$\lambda w \lambda t\ [{}^0\!= [{}^0\!Number\_of\ {}^0\!Planet_{wt}]\ {}^0 8].$$

Types: *Number_of*/$(\tau(o\iota))$: the cardinality function that returns the number of elements of an $(o\iota)$-set; *Planet*/$(o\iota)_{\tau\omega}$; =/$(o\tau\tau)$; 8/τ.

The denoted proposition is an empirical truth-condition that is satisfied only by those worlds and times at which the number of planets is 8.[55] Provided these are post-Plutonic times then (for all that is commonly known) there are exactly eight planets in the Solar system. If so, then it is a contingent truth. If not, then it is a

[53] See Section 3.1 dealing with definite descriptions.

[54] See Section 5.1 dealing with propositional attitudes.

[55] We are presupposing—naïvely, as it happens—the existence of a definition of the property of planethood that will decide unequivocally for any celestial body in our solar system whether it is a planet.

contingent falsehood. The example demonstrates that 'The number of the planets' cannot be a name of 8, nor that 'The number of the planets is 8' can be a name of the truth-value **T** (or **F**, for that matter). For then the semantic naming relation would fluctuate in accordance with either astronomical facts or our presumed knowledge of such facts.

What does denote a truth-value is the sentence

'The number of elements in {Mercury, Earth, …, Neptune} is 8.'

It denotes the truth-value **T**/ο, and its meaning is the Composition

$$[{}^{0}{=}\ [{}^{0}Number\_of\ {}^{0}S]\ {}^{0}8].$$

Type: S = {Mercury, …, Neptune}/(οι).

Whatever, if any, the planets of a solar system may be, it is a mathematical truth that the set {Mercury, Earth, …, Neptune} has 8 elements. Making an inventory of the planets of a solar system does not consist in counting the number of elements in sets of planets. It consists in applying the empirical condition of being a planet to the celestial bodies of the solar system in question.

1.4.2.1 Classification of empirical properties

In Chapter 4 we will explain in detail how two intensions may be conceptually related in such a way that having one necessitates having the other as well. When there is such necessitation, we say that one intension is *essential* of the other. It is intensions, and not extensions such as individuals, that are the bearers of essential properties. Instead our individuals are 'bare' in the sense that no non-trivial intension is necessarily true of them.

However, it remains at this point in time an open issue whether it is possible that a ι-object may lack all non-trivial properties at some $\langle w, t\rangle$. If this is possible, then such an individual will be 'bare' in a more dramatic sense than just not possessing any non-trivial properties necessarily (which is already considered dramatic enough in several quarters).[56]

Consider three ways of analysing 'the man without properties' (example courtesy of Robert Musil).

First analysis:

$$\lambda w \lambda t\ [{}^{0}Sing\ \lambda x\ [[{}^{0}Man_{wt}\ x] \wedge [\forall p\ \neg[p_{wt}\ x]]]].$$

Second analysis:

$$\lambda w \lambda t\ [{}^{0}Sing\ \lambda x\ [\forall p\ \neg[p_{wt}\ x]]].$$

[56] Now we are using 'trivial' and 'non-trivial' intuitively. By 'trivial' we do not mean epistemically trivial. Once we explain what is meant by 'trivial', we will use rigorous terms instead.

Third analysis:

$$\lambda w \lambda t\ [{}^0 Sing\ \lambda x\ [\forall p\ [[p_{wt}\ x] \supset [{}^0 Triv\ p]]]].$$

Types: *Sing*/$(\iota(o\iota))$: the singulariser function that associates a singleton *S* with its only member and is otherwise undefined;[57] $\forall/(o(o(o\iota)_{\tau\omega}))$: the general quantifier over ι-properties; *Man*/$(o\iota)_{\tau\omega}$; $p \rightarrow (o\iota)_{\tau\omega}$; $x \rightarrow \iota$; *Triv*/$(o(o\iota)_{\tau\omega})$: the class of trivial ι-properties.

The first analysis is a construction of a ι-office occupiable by any individual who has the property of being a man and at the same time no properties at $\langle w, t\rangle$. Since *Man* is in the domain of *p*, the conjuncts cannot both be true.

The second analysis is a construction of a ι-office occupiable by any individual who has no properties at $\langle w, t\rangle$. Since every individual has the property of being self-identical, this office is necessarily vacant. Hence both constructions construct the 'impossible' ι-office, which is necessarily vacant. (Similarly, the property of being an *x* such that *x* has no *p* is paradoxical, since it is in the range of *p*.)

The third analysis is a construction of an ι-office occupiable by an individual that does not have any non-trivial properties. The question is whether this office is ever occupied. The answer will depend on how restrictive or how liberal a notion of *non-trivial* ι-property is used; i.e., what the class *Triv* is taken to be. It certainly contains all *constant* properties, i.e., the properties that have a constant set of individuals as a value at all $\langle w, t\rangle$. One of them is self-identity, which every individual necessarily possesses. However, should we take on board Cambridge-like properties in the vein of *being an x such that x is the same height as Kim Jong Il*? Whatever height Comrade Kim may have at this or that $\langle w, t\rangle$, it is necessary that he have exactly the same height as Kim Jong Il. The trick is to index a property to a specific individual *a*, such that, necessarily, *a* must have that property, without using a trivial, constant property such as being self-identical. *Being the same height as Kim Jong Il* is a contingent property, for it is not a constant function. Not all individuals have the same height as Kim Jong Il at all worlds and times, so the sets that are its extensions at various $\langle w, t\rangle$-pairs will not always have the same members. But, due to the indexing, one individual can always be relied upon to be in whatever set is the extension at whatever $\langle w, t\rangle$; to wit, Kim himself. So the intension *being the same height as Kim Jong Il* is insofar *partially constant*. The property has an *essential core*: namely, the set {Kim Jong Il}.[58] Similarly, the contingent, i.e. non-constant, property *being the same age as a or b* has the essential core $\{a, b\}$. All individuals but *a*, *b* have this property contingently; only *a*, *b* have it necessarily. If the intension is non-trivial, its non-triviality is 'partial' or 'impure'; and if trivial, then its triviality is also impure. We will call such a property 'partially constant'.

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[57] See Definition 1.6.

[58] The term 'essential core' was coined by Pavel Cmorej (1996). See also Cmorej (1988, 2006).

Intensions that have constant values in all worlds and times are certainly trivial. However, as explained above, some non-constant, contingent properties can also be necessarily applicable or inapplicable to some individuals (though not to just any individual), and are in some sense also trivial. Thus the characterisation of the class *Triv* has to be extended. The general direction in which to look for an answer is indicated by Tichý's distinction between *primary* and *parasitic* properties.

> A change in a thing clearly consists in the acquisition or loss of a property. But if any property is as good as any other, we get the odd result that a thing cannot change without every other thing changing as well. Suppose object *X* becomes red and consider another object, *Y*. *Y* will be spatially related to *X* in a definite way; suppose it is 50 miles due south from *X*. Then as *X* acquires redness, *Y* acquires the property of being 50 miles due south from a red object. This change in *Y*, however, is obviously a phoney change, because the property of being 50 miles due south from a red object is a phoney, parasitic property. It is a property which will not figure in the specification of a possible world. To specify a possible world, one has to specify, *inter alia,* where each object is and what colour it is. Once all this has been fixed, there is no need to specify which objects have the property of being 50 miles due south from red objects; for all this has been implicitly specified already. While the extension of redness is part of what makes a world the world it is, the extension of the property of being 50 miles due to south from a red object is not. It is a parasitic property, a mere logical shadow cast by genuine—or, as we will say, *primary*—properties like being red and being at a certain place. For a thing to change, it must acquire or lose not any arbitrary property, but a primary one. We have seen that the possible worlds of a logical space are generated as distributions of the attributes in the intensional base through things. It is thus natural to identify primary properties, relations, etc. with those which correspond to the members of the intensional base
> (Tichý, 1980b, p. 271, 2004, p. 419).

As explained in Section 1.4.1, every language is based on a definite universe of discourse (i.e., a collection of *individuals*) and an *intensional base*, which is the collection of primary intensions[59] that the given language has predicates for. The objectual base ($\omicron$, ι, τ, ω), together with a definite interpretation of $\omicron$, τ, ω, forms an epistemic framework. Possible worlds are then possible chronologies of distributions of members of the intensional base over individuals.

Hence primary properties are certainly *contingent*, *non-constant* and thus non-trivial. No individual has a primary property of the intensional base necessarily, i.e., in all $\langle w, t\rangle$. So there is no non-empty constant subset of the possible extensions of a primary property.

Some of the derivative properties parasitic upon the *primary properties* are also contingent, like the above property *being* 50 *km due south from a red object.* It is a contingent fact that an object *X* possesses at some time the property *being red.* This fact implies infinitely many facts where *derivative* properties play a role; for example, an object *Y* that happens to be 50 km due south from *X* gets the derivative property *being* 50 *km due south from a red object.* And *Y* does not have this

[59] The distinction between 'primary' vs. 'secondary' intensions is not to be confused with some other distinctions like, e.g. Evans' 'deep' vs. 'superficial' intensions or what also goes under the name 'primary and secondary intensions' in two-dimensional semantics. See Evans (1977).

property of *logical* necessity. However, *Y* necessarily has the derivative property of *not* being 50 km due south from *itself*.[60]

Note that the 'derivateveness' of a property does not concern a construction of the property. Any property can be constructed in infinitely many ways. Rather, it concerns necessary dependencies between the respective facts and thus properties as well. For instance, the fact that an individual *a* is this or that age is logically contingent. But there is a necessary correlation between *a* being 50 and *a* not being younger than 30. It is impossible that *a* be 50 and at the same time younger than 30. As we will explain in Chapter 4, there are so-called *requisite* relations between intensions. On the other hand, there are no such dependencies between primary properties of the intensional base; the respective basic facts are independent, parallel to the *Tractarian* conception of *Tatsachen*.[61]

As explained above, non-constant, contingent properties with an essential core are *partly constant*. They are *essential of some* individuals, namely of those belonging to the relevant essential core. All other individuals contingently have, or do not have, these properties. Hence, if *P* is a partly constant property, then there are at least two world/time pairs $\langle w, t\rangle$, $\langle w', t'\rangle$, such that P_{wt} is not the same set as $P_{w't'}$. There is, however, a constant subset of the varying extensions of *P*, namely the essential core of *P*.[62]

Our hypothesis is that *partly constant properties* with an essential core are parasitic on *reflexive* relations-in-intension, where a reflexive relation-in-intension is an entity $R/(o\iota\iota)_{\tau\omega}$ such that, necessarily, its value in $\langle w, t\rangle$ is a reflexive relation-in-extension:

$$\forall w \forall t\, [\forall x\, [{}^0R_{wt}\, x\, x]].$$

The relations of being the same height as some individuals, of being of the same age, of not being 20 years older than, etc., can serve as examples. Of course, since *being the same age as* is necessarily reflexive, an individual *a* cannot be a different age than *a*, unless *a* would, bizarrely, lose its identity.

On the other hand, *purely constant properties* are functions having the same set of individuals as value in all worlds *w* at all times *t*.[63] Thus if *P* is a purely constant property, the set P_{wt} is the same in all $\langle w, t\rangle$, and it is the essential core of *P*. Every individual belonging to P_{wt} has *P* at all $\langle w, t\rangle$, and every individual not belonging to P_{wt} lacks *P* at every $\langle w, t\rangle$. The essential core of a purely constant

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[60] We do not consider here subatomic particles of quantum physics, of course. After all, Heisenberg's uncertainty principle has a negligible effect on objects of macroscopic scale.

[61] The claim that there are no dependencies between primary properties of the intensional base requires qualification, however. Consider *being red* and *being blue*. Neither is parasitic upon the other, but at the same time they are dependent, by being defined in terms of their respective positions in a spectrum.

[62] Cmorej (2006) calls these properties *partly essential*.

[63] Cmorej (2006) calls these properties *essential.*

property P is either equal to the whole universe or is a proper subset of the universe. An example of the former would be the property of being self-identical, constructed by $\lambda w \lambda t\ \lambda x\ [x = x]$; examples of the latter would be the properties of being identical to a particular individual a, $\lambda w \lambda t\ \lambda x\ [x = a]$, being identical to an individual a or b, $\lambda w \lambda t\ \lambda x\ [[x = a] \vee [x = b]]$, being identical to neither a nor b, $\lambda w \lambda t\ \lambda x\ [\neg[x = a] \wedge \neg[x = b]]$; etc.

To sum up, a property P belongs to the class *Triv* iff P has a non-empty essential core *EC*. Individuals belonging to *EC* have P necessarily. So the property P is essential of the elements of *EC*. Properties with a non-empty essential core are either purely constant or partly constant. The former are constant intensions and the latter contingent.

Now we can classify individual properties according to different criteria into the following categories.

Partiality criterion:

- *Purely partial properties.* A property P is *purely partial* iff there is a world w and a time t at which P has no extension: $[{}^0P_{wt}]$ is v-improper.[64]
- *Partial properties.* A property P is *partial* iff there is a world w and a time t at which the characteristic function v-constructed by ${}^0P_{wt}$ is purely partial; equivalently, there is an individual a such that $[{}^0P_{wt}\ {}^0a]$ is v-improper.

For instance, the property of having stopped smoking is partial. If $StopSmoking/(o\iota)_{\tau\omega}$ is this property, then $[{}^0StopSmoking_{wt}\ x]$ v-constructs **T** if individual x used to smoke and stopped smoking, **F** if x used to smoke and did not stop smoking. Finally, $[{}^0StopSmoking_{wt}\ x]$ is v-improper if x never smoked.

Now let P be a property that is not purely partial. Then we can further apply the

Criterion of contingency or non-contingency:

- A property P is *constant* (or *non-contingent*) iff P has the same extension in all worlds and times, where the extension is defined as follows:

$$[\iota c\ \forall w \forall t\ [c = {}^0P_{wt}]], \text{ where } c/*_1 \rightarrow_v (o\iota).$$

If a property P is constant, then its extension is its essential core.

The property of being *self-identical*, constructed by $\lambda w \lambda t\ \lambda x\ [x = x]$, $x \rightarrow \iota$, is an example of a constant property; the essential core of this property is the set of all individuals. An example of a constant property with an empty essential core is the property of not being identical with itself, $\lambda w \lambda t\ \lambda x\ [x \neq x]$.

The property of being identical to a or b, constructed by $\lambda w \lambda t\ \lambda x\ [[x = a] \vee [x = b]]$, is another example of a constant property. Its essential core is the set $\{a, b\}$. The other individuals necessarily lack this property.

[64] We add this category just for completeness. Purely partial properties are bizarre properties like the one defined as follows: $\lambda w \lambda t\ \iota c\ [[c = \varnothing] \wedge \neg\ [c = \varnothing]]$, where $c/*_1 \rightarrow_v (o\iota)$.

- A property P is *non-constant* (or *contingent*) iff there are at least two distinct extensions of P. In other words, there are world/time pairs $\langle w_1, t_1\rangle$, $\langle w_2, t_2\rangle$ such that ${}^0P_{w1t1} \neq {}^0P_{w2t2}$.

If P is a non-constant (contingent) property, then we can further distinguish between a partially constant and a purely contingent property:

– A non-constant property P is *partially constant* iff there is a non-empty essential core of P. The essential core of a non-constant property P is defined as follows:

$$\iota c\ [\exists x\ [cx] \wedge [c = \lambda x\ [\forall w \forall t\ [{}^0P_{wt}\, x]]]], \text{ where } c/{*_1} \rightarrow (o\iota).$$

Obviously, the essential core of a non-constant property P is the smallest non-empty subset of all the possible extensions of P.

If P is a contingent property with a non-empty essential core, then P is *partially contingent*; or equivalently, *partially constant*. We have decided in favour of the latter characterization in order to stress that P is constant with respect to *some* individual(s) and contingent with respect to others.

For example, the property of being of the same height as a or b is constant with respect to a and b. Its essential core is the set $\{a, b\}$. The other individuals contingently have this property or contingently lack it. It seems that all partially constant properties are based on a reflexive relation. But we are not going to assume, let alone attempt to prove that this is so, we treat it only as a hypothesis.

– A property P is *purely contingent* (or *purely non-constant*) iff P is neither constant nor partially constant. In other words, there is no non-empty essential core of P.

As examples of purely contingent properties, think of *being happy, weighing 88 kg.*

Our *individual anti-essentialism* thus qualifies as a 'modest' one:[65]

If an individual a has a property P necessarily (i.e., at all w, t), then P has a non-empty essential core *Ess* and the individual a is an element of *Ess* (i.e., P is a constant or partly constant function). Formally,

$$\forall p\ [[\exists x\ \forall w \forall t\ [p_{wt}\, x]] \supset [[{}^0\mathit{Constant}\ p] \vee [{}^0\mathit{Partially\_constant}\ p]]]$$

[65] The idea of *modest anti-essentialism* owes much to Pavel Cmorej.

where $x \rightarrow \iota$; $p \rightarrow (o\iota)_{\tau\omega}$; *Constant, Partially_Constant*/$(o(o\iota)_{\tau\omega})$ are the classes of constant or partially constant properties, respectively.

Figure 1.4 illustrates particular kinds of properties (*Ess* is here the essential core of *P*).

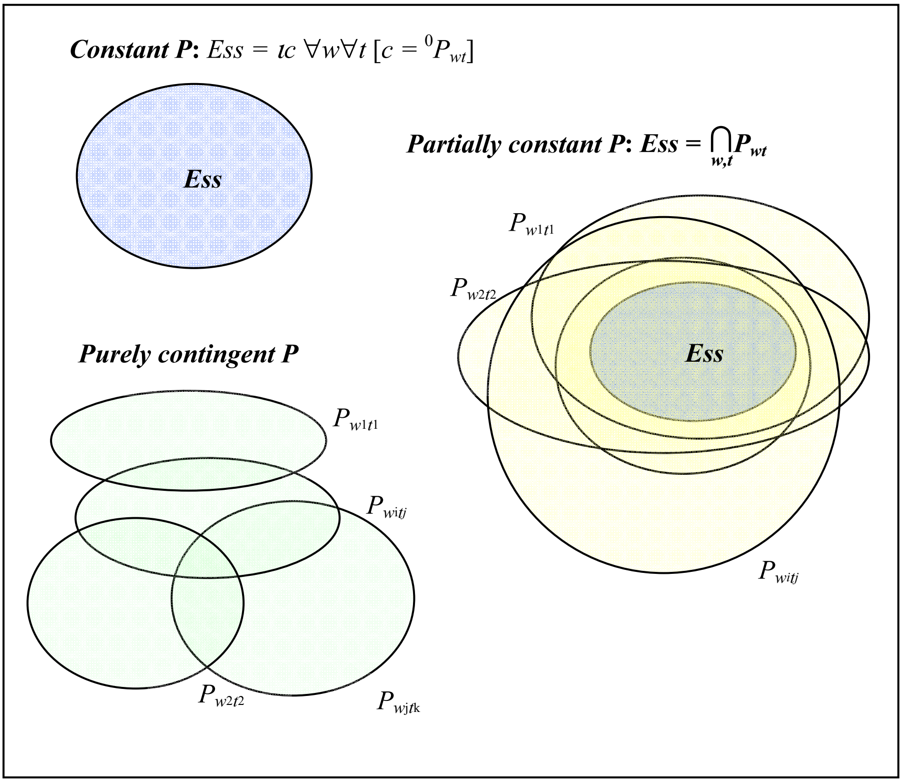

Fig. 1.4 Schema of constant, partially constant/contingent, and purely contingent properties

Now we are in a position to answer the question raised at the outset of this section of whether it is possible that an individual may lack all purely contingent (non-constant) properties at some $\langle w, t\rangle$. The answer is No. To show why, we use an example of a more outlandish property than *being the same height as King Jong Il*, namely, the property *being self-identical and the time is T* (for instance, noon on April 1, 2010).[66] One of its constructions is (T/$(o\tau)$ being some fixed interval of times)

$$\lambda w \lambda t\, \lambda x\, [[x = x] \wedge [{}^0T\, t]].$$

[66] This example is due to Pavel Cmorej.

(The construction $[{}^0T\ t]$ suffices, because it is immaterial how the proposition that the time is 12 o'clock on April 1, 2010 is constructed.) An individual satisfies this property if it is self-identical and the time is T when it is tested for self-identity. The time is not always T, so the property is not constant. But each x is self-identical. Hence, each individual has such properties, and there are no strictly bare individuals. However, as explained above, such a phoney property is derivative and not a member of the intensional base.

Apart from dividing properties into constant and non-constant, partly constant and purely contingent, there is another criterion, according to which properties divide into *empirical* and *analytical*. An empirical property is a property P such that for *no* individual a is it decidable a priori whether P applies to a. It must always be established a posteriori. On the other hand, an analytic property P' is decidable a priori for all individuals. Obviously, purely constant properties are analytic, and purely contingent properties are empirical. Partly constant/contingent properties should be decidable analytically a priori with respect to the individuals belonging to the essential core. Of course, we do not need experience in order to decide whether an individual a is the same age as a or b.[67]

A note on *self-predication*. Muskens cites 'Having fun is fun' as an example of self-predication (2005, p. 485). We do not think it qualifies as one, though. The first occurrence of 'fun' is as a noun and the second as an adjective (like 'funny'). Better examples of apparent self-predication would be, 'Being nice is nice' and 'It is fine to be fine'. A type-theoretic analysis shows that the two respective occurrences of 'nice' and 'fine' denote entities of different types. One occurrence denotes entities of type $(o(o\iota)_{\tau\omega})_{\tau\omega}$, which are empirical properties of ι-properties. The other occurrence denotes ι-properties/$(o\iota)_{\tau\omega}$. If $F/(o\iota)_{\tau\omega}$ and $F^*/(o(o\iota)_{\tau\omega})_{\tau\omega}$, the analysis is $\lambda w \lambda t\ [{}^0F^*_{wt}\ {}^0F]$.

Self-predication is never an option in TIL, unlike what type-free logics like Bealer's allow for.

[67] However, as Cmorej points out in 1988, it is an open question whether there are properties that are partly constant in a less obvious way, for which the respective essential core would be decidable only a posteriori. The thoughts on how to categorize properties arose from a discussion with Cmorej in 2005.

1.4.2.2 The part-whole relation

In Section 1.4.2.1 above, we broached the thesis of *modest individual anti-essentialism*:

> If an individual I has a property P necessarily (i.e., at all worlds and times), then P has a non-empty essential core Ess and I is an element of Ess (i.e., P is a constant or partly constant function).

There is, however, a frequently voiced objection to individual anti-essentialism. If, for instance, Tom's only car is disassembled into its elementary physical parts, then Tom's car no longer exists; hence, the property of being a car is essential of the individual referred to by 'Tom's only car'. Our response to the objection is this. First, what is *denoted* (as opposed to *referred to*) by 'Tom's only car' is not an individual, but an individual office, which is an intension having occasionally different individuals, and occasionally none, as values in different possible worlds at different times. Whenever Tom does buy a car, it is not logically necessary that Tom buy some one particular car rather than any other. Second, the individual *referred to* as 'Tom's only car' does not cease to exist even after having been taken apart into its most elementary parts. It has simply lost some properties, among which the property of being a car, the property of being composed of its current parts, etc, while acquiring some other properties. Suppose somebody by chance happened to reassemble the parts so that the individual would regain the property of being a car. Then Tom would have no right to claim that this individual was *his* car, in case it was allowed that the individual had ceased to exist. Yet Tom should be entitled to claim the reassembled car as his.[68] Therefore, when disassembled, Tom's individual did not cease to exist; it had simply (unfortunately) obtained the property of completely disintegrating into its elementary physical parts. So much for modest individual anti-essentialism.

The second thesis we are going to argue for is this. A material entity that is a mereological sum of a number of parts, such as a particular car, is—from a logical point of view—a simple, hence unstructured individual. Only its design, or *construction*, is a complex entity, namely a structured procedure. This is to say that a car is not a structured whole that organizes its parts in a particular manner. Tichý says:

> [A] *car* is a simple entity. But is this not a *reductio ad absurdum*? Are cars not complex, as anyone who has tried to fix one will readily testify?
>
> No, they are not. If a car were a complex then it would be legitimate to ask: Exactly how complex is it? Now how many parts does a car consist of? One plausible answer which may suggest itself is that it has three parts: an engine, a chassis, and a body. But an equally plausible answer can be given in terms of a much longer list: several spark plugs, several pistons, a starter, a carburettor, four tyres, two axles, six windows, etc. Despite

[68] As Tichý argues in 1987, where he uses the example of a watch being 'repaired' by a watch-maker in such a way as to become a key.

> being longer the latter list does not overlap with the former: neither the engine, nor the chassis nor the body appears on it. How can that be? How can an engine, for example, both be and not be a part of one and the very same car?
>
> There is no mystery, however. It is a commonplace that a car can be *decomposed* in several alternative ways. … Put in other words, a car can be *constructed* in a very simple way as a mereological sum of three things, or in a more elaborate way as a mereological sum of a much larger set of things (1995, pp. 179–80).

It is a contingent fact that this or that individual consists of other individuals and thereby creates a mereological sum. Importantly, *being a part of* is a relation between *individuals*, not between intensions. There can be no inheritance or implicative relation between the respective properties ascribed to a whole and its individual parts. It is vital not to confuse the *requisite* relation, which obtains between intensions, with the *part-whole* relation, which obtains between individuals. The former relation obtains of necessity (e.g., necessarily, any individual that is an elephant is a mammal), while the latter relation obtains contingently.[69] Logically speaking, any two individuals can enter into the part-whole relation. One possible combination has Saturn a part of Socrates (or vice versa). There will be restrictions on possible combinations, but these restrictions are anchored to nomic necessity (provided a given possible world at which a combination of individuals is attempted has laws of nature at all).[70] One impossible combination would have the largest mountain on Saturn be a part of π (or vice versa). Why impossible? Because of wrong typing: the arguments of the part-whole relation must be individuals (i.e., entities of type ι), but the largest mountain on Saturn is an individual office while π is a real number.

Still, *which parts are essential* for an individual in order to have a *property P*? The property of having an engine is essential for the property of being a car, because something designed without an engine does not qualify as a car, but at most as a toy car, which is not a car. The answer to the question which parts are essential in order to have a property *P* is, in the car/engine example, that the *property* of having an engine is a *requisite* of the *property* of being a car. What is necessary is that a car, *any* car, should have an engine. It is even necessary that it should have a particular kind of engine, where *being a kind of engine* is a property of a property of individuals. What is not necessary is that any car should have some one particular engine belonging to a particular kind of engine: *mutatis mutandi*, any two members of a particular kind of engine will be mutually replaceable.[71] Thus the relation *Part_of* is of type $(o\iota\iota)_{\tau\omega}$.

The sort of unrestricted mereological combinations that we are adumbrating and advocating gives rise to a more fundamental problem that Cmorej takes on in

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[69] The full logic of requisites is set out in Chapter 4.

[70] See Duží (2007) for a discussion of *wharrots*. A wharrot is an individual consisting of a carrot and a whale. Unless further restrictions are laid down, wharrots exist as soon as whales and carrots do. (We are indebted to Maarten Franssen for the example of wharrots.)

[71] This problem is connected with the analysis of property modification, including *being a malfunctioning P*, dealt with in Section 4.4.

1988.[72] The problem is this. If a composition of a physical individual is contingent and allows parts to be replaced or lost, then *which unique part of such an individual is essential for the individual's identity?* Cmorej argues that the assumption of variable composition of a mereological sum leads to absurd consequences. Let us briefly summarise his arguments.

Cmorej presents two puzzling thought experiments. The first puzzle can be called, 'Did, or did not, an individual have the property *P*?'; the second, 'Where is the individual?'

Here is the first puzzle. Imagine an individual *X* that has the property *P*. The property *P* is stipulated to be *penetrating*, which means that, necessarily, if *X* has *P* then all its parts have *P*.

Formally, *P* is *penetrating* iff

$$\forall w \forall t\ \forall x\ [[{}^0P_{wt}x] \supset \forall y\ [[{}^0Part\_of_{wt}\, y\, x] \supset [{}^0P_{wt}y]]].$$

Types: $P/(\omicron\iota)_{\tau\omega}$; $Part\_of/(\omicron\iota\iota)_{\tau\omega}$; $x, y \rightarrow \iota$.

For instance, the property of weighing less than 50 kg is penetrating. An individual cannot weigh less than 50 kg if some of its parts weigh more than 50 kg.

Let *X* have a penetrating property *P* at time t_1. During the time interval $\langle t_1, t_2\rangle$, $t_1 < t_2$, *X* loses all its *proper* parts, as well as the property *P*, so that at t_2 *X* does not have *P* anymore, and *X* also does not contain any *proper* parts that *used* to have *P*.[73] Now the question is whether at t_2 we can truly ascribe to *X* the property of having had *P*. Cmorej uses a past-tense operator ***Pt*** that is applied to the proposition that *X* has *P*, forming the proposition that *X* had *P* in the past. Thus the operator denotes a property *Pt* of propositions, *Pt* of type $(\omicron\omicron_{\tau\omega})_{\tau\omega}$, which is defined as follows: Let $p \rightarrow \omicron_{\tau\omega}$ be a variable *v*-constructing a proposition. Then

$${}^0Pt = \lambda w \lambda t\ \lambda p\ \exists t'\ [[t' < t] \wedge p_{wt'}].$$

Intuitively, the answer should be in the affirmative. It is true at t_2 that *X* used to have *P*, because *what is done cannot be undone* (as Macbeth learnt the hard way). But how are we to evaluate the truth-conditions of the proposition constructed by $\lambda w \lambda t\ [{}^0Pt_{wt}\ \lambda w \lambda t\ [{}^0P_{wt}X]]$ at t_2? When evaluating the proposition constructed by $\lambda w \lambda t\ [{}^0P_{wt}X]$, we must consider all the parts of *X*, because *P* is penetrating. Cmorej argues that, similarly, when evaluating the truth-conditions of $\lambda w \lambda t\ [{}^0Pt_{wt}\ \lambda w \lambda t\ [{}^0P_{wt}X]]$ at t_2, we must take into account the parts that *X* consists of *at time* t_2. But, there is no trace of *P* in *X* at t_2; no proper part of *X* used to have *P*. This is peculiar, indeed. Could *X* have been, for instance, inside a room, or in a magnetic field, or submerged into a liquid, if there is not even a tiny proper part of *X* to which the respective property could have been ascribed? Hardly. Thus Cmorej comes to the conclusion that $\lambda w \lambda t\ [{}^0Pt_{wt}\ \lambda w \lambda t\ [{}^0P_{wt}X]]$ is, at t_2, both true (according to the principle that what is

[72] See also Geach (1972, pp. 215–16) for the related problem of 'the cat on the mat'.

[73] A proper part of *X* is an individual *Y* such that *Y* is a part of *X* and $Y \neq X$.

done cannot be undone) and false, because none of its parts used to have the property P. Contradiction!

First, however, we disagree with Cmorej's argument on grounds of analogy. He argues that when evaluating whether 'The world champion of 100 m race used to be a smoker' we examine the *current* world champion, not any of the previous ones. Of course, we have to examine the individual that *currently* and *actually* plays the role of world champion of 100 m sprint race—but we should examine his/her *history*. Though the current champion may have stopped smoking, we should ask whether he/she *previously smoked.* Similarly, when asking whether X used to have P we have to examine the *history* of X, which includes the proper parts that X used to consist of. We have to ask which parts X consisted of *in the past*, and whether any of these parts previously used to have P in the interval $\langle t_1, t_2\rangle$.

Thus we must use the *Past* function, which we will define in Section 2.5.2. Simplifying a bit, the result of applying *Past* to the proposition constructed by $\lambda w\lambda t\ [{}^0P_{wt}X]$ and to the interval $\langle t_1, t_2\rangle$ referring to the past is this:

$$\lambda w\lambda t\ \exists t'\ [[t' < t] \wedge [t_1 \leq t' \leq t_2] \wedge [{}^0P_{wt'}X]].$$

Evaluating the truth-conditions in a world w at a time t comes down to empirically searching for the truth-value v-constructed by $\exists t'\ [[t' < t] \wedge [t_1 \leq t' \leq t_2] \wedge [{}^0P_{wt'}X]]$. In other words, we have to examine the history of X in the interval $\langle t_1, t_2\rangle$ *preceding* time t.

But, secondly, there is another, more alarming question. If no current proper part of X can help us examine the history of X, how are we to examine its history at all? We need to abstract from all the current proper parts of X, as well as all *their* properties, and consider only the properties that the *bare* individual X used to have. What, then, determines the *numerical identity* of the *bare* individual X?

This problem ties in with the second puzzle. The second puzzle is this. Imagine that a person a owns a golden fountain pen (i.e., a pen, all of whose parts are golden) and a person b owns a pen that looks exactly like a's, except that it is not made of gold but of fool's gold (i.e., all its parts being made of fool's gold). Moreover, b's pen and all its parts function in exactly the same way as a's pen and its parts and, so, are functionally equivalent. At time t_1 a's pen is located at the place L_a and b's pen at the place L_b. During the time interval $\langle t_1, t_2\rangle$ b gradually replaces, part by part, the *proper* parts of a's pen by the *proper* parts of b's pen, so that at t_2 all the *proper* parts of a's pen are located at L_b and all the *proper* parts of b's pen are located at L_a. As a result, a's pen and b's pen look and function in the same way at t_2 as they did at t_1, except that a's pen is made of fool's gold and b's pen is made of gold.

The conclusion of the thought experiment has an air of plausibility. Yet we are not convinced that a's pen is made of fool's gold and b's pen is made of gold. To see why, imagine that the interval $\langle t_1, t_2\rangle$ is very short and that *all* the parts have been interchanged *at once*. Wouldn't most people be inclined to say that b simply *stole* a's pen and replaced it by his junk pen? We would, at least. Furthermore, even if the swap was performed part by part, how could *all* the proper parts of a's pen be transferred from L_a to L_b without the whole individual being *ipso facto* transferred?

Hence the questions arise: Where is a's pen and where is b's pen at t_2? Which of the pens is golden at t_2? There are two mutually incompatible answers:

(i) a's pen is located at L_a and is made of fool's gold, whereas b's pen is located at L_b and is made of gold; b did not steal a's pen, b only drastically lowered the value of a's pen.
(ii) a's pen is located at L_b and is made of gold, whereas b's pen is located at L_a and is made of fool's gold; b stole a's pen, and replaced it by his pyrites pen.

Now let someone unaware of the swap examine the two pens at t_2. In *both cases* the result of the examination would be as follows. The pen located at L_a is made of fool's gold, because *all* its parts are made of fool's gold, whereas the pen located at L_b is golden, because *all* its parts are made of gold. Since the examiner is unaware of the swap, he naturally assumes that the golden pen at L_b is a's pen. Consequently, the variant *ad* (i) will seem impossible to the examiner.

Cmorej thus arrives at the conclusion that the assumption of unrestricted variation of an individual's composition is unacceptable. In other words, given an individual X, the property of being a part of X must be essential of X. Hence, for any individual X it must hold that the property constructed by

$$\lambda w \lambda t\, \lambda y\, [{}^0 Part\_of_{wt}\, y\, X]$$

is an *essential* property of X, i.e., a constant function. But at the same time this property is, intuitively, *empirical*, for we cannot know a priori which parts X consists of.

What are we to make of Cmorej's conclusion that some properties of X are both essential and empirical? We wish to reject it. Here is why. A consequence of Cmorej's conclusion is that X would consist of the very same parts in each world w at each time t. This would mean that the material composition of X must be constant, such that each time X loses some part and obtains a new one, a *new* individual X' comes into being. As a result, the universe of discourse would have to vary accordingly. Moreover, we could not a priori distinguish between individuals X, X', X'', X''', etc. For instance, your cells are continuously being renewed, yet your numerical identity should certainly not hinge on one particular pool of cells. You would not be the same individual in the morning as the one who went to bed the night before. This is certainly untenable as a criterion of numerical individuation of individuals.

As the above thought experiments show, if we embrace variable composition of a mereological sum, then we face the problem of the identity of individuals. To dramatize the problem, imagine that somebody is gradually stealing proper parts of your car (rather than stealing the whole car in one go). If the thief steals one molecule he has not stolen your car. If he steals the steering wheel, he has not stolen your car. If he steals all four wheels, he has not stolen your car. But if the thief steals *all* proper parts of your car, wouldn't you say that he had stolen your car? Of course, you would, and so would your insurance company (hopefully). The car thief has committed diachronic theft, as it were, the same way an embezzler may gradually drain an account. If one goes along with our view, the question *which* part is essential of your car's identity turns out to be ill-posed.

This example suggests that the only way out is to say that *no proper* physical part is essential of your car (or of any other concrete individual). But this is to say that an individual may lose all its proper physical parts without losing its identity, making the identity of an individual a purely *abstract object*. A bare individual is an abstract object of a transcendental nature, and Cmorej's proposed proof that bare individuals do not exist is correct, because existence is a property of *intensions*, namely the property of being instantiated or being occupied. As we showed in Section 1.4.2.1, we cannot specify the property of *not* having any properties. We can only *abstract away* the properties an individual has. We must presuppose pre-theoretically that there is a fixed domain of individuals whose identity is given to us a priori, regardless of whether we are able to determine *which* particular individual we are examining on some occasion. Within our theory, individuals are *logically primitive* relative to a base *B* (see Section 1.4.1).

1.4.2.3 The top-down approach to semantics revisited

In Section 1.2 we critically examined the standard bottom-up way of analysing terms and expressions. We adduced the following five examples and explained why their standard analyses are too coarse-grained:

(1) 'Charles is happy'
Fa

And further upwards:

(2) 'Charles is happy, and Thelma is grumpy'
$Fa \wedge Gb$

(3) 'Somebody is happy'
$\exists x\ (Fx)$

(4) 'Possibly, Charles is happy'
$\Diamond\ (Fa)$

(5) 'Thelma believes that Charles is happy'
$\mathrm{B}b\ (Fa)$.

Now we have the tools to analyze these sentences in a fine-grained way. As we explained above, we aim at assigning propositional *constructions* to the analysed sentences. We are going to illustrate the method of analysis by analysing first the sentences (1) and (2). Our method consists in three steps.

First, we assign types to the objects mentioned by the sentences: *Charles*, *Thelma*/ι; *Happy*, *Grumpy*/$(o\iota)_{\tau\omega}$; $\wedge/(ooo)$.

Second, by Composing constructions of these objects (here, Trivializations) we aim at constructing the propositions denoted by (1) and (2), respectively:

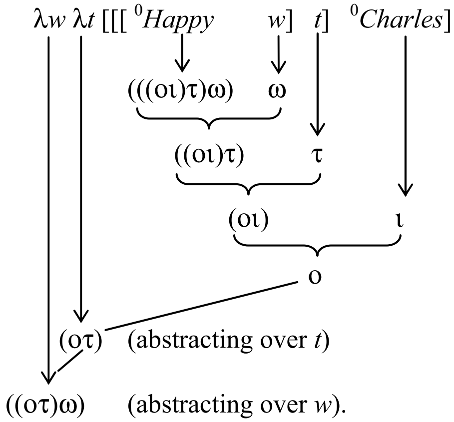
(1′) $\lambda w \lambda t\ [{}^0Happy_{wt}\ {}^0Charles]$.

(2′) $\lambda w \lambda t\ [{}^0\wedge\ [\lambda w \lambda t\ [{}^0Happy_{wt}\ {}^0Charles]]_{wt}\ [\lambda w \lambda t\ [{}^0Grumpy_{wt}\ {}^0Thelma]]_{wt}]$.

Note that our uniform semantics works smoothly top-down and back up again, involving all three kinds of context, to wit, hyperintensional, intensional and extensional. The Closures $[\lambda w \lambda t\ [{}^0Happy_{wt}\ {}^0Charles]]$ and $[\lambda w \lambda t\ [{}^0Grumpy_{wt}\ {}^0Thelma]]$ construct the propositions that Charles is happy and that Thelma is grumpy, respectively. However, propositions are not arguments of the right type for truth-value functions. They are intensional objects and have to be extensionalized first in order to yield an extension. That is, the proposition that Charles is happy has to be subjected to intensional descent: $\lambda w \lambda t\ [{}^0Happy_{wt}\ {}^0Charles]_{\boldsymbol{wt}}$.

The Composition $[\lambda w \lambda t\ [{}^0Happy_{wt}\ {}^0Charles]]_{wt}$ is a construction *v*-constructing a truth-value; i.e., the type of the value of the proposition constructed by $\lambda w \lambda t\ [{}^0Happy_{wt}\ {}^0Charles]$ at $\langle w, t\rangle$. Similarly, $\lambda w \lambda t\ [{}^0Grumpy_{wt}\ {}^0Thelma]$ constructs the proposition that Thelma is grumpy, and its Composition with $\langle w, t\rangle$, as in $[\lambda w \lambda t\ [{}^0Grumpy_{wt}\ {}^0Thelma]]_{wt}$, *v*-constructs the value (of type o) of this proposition at $\langle w, t\rangle$. So a conjunction receives two truth-values as input, yielding a third as output. Finally, we need to abstract from the values of *w*, *t* in order to construct the proposition that Charles is happy and Thelma is grumpy.

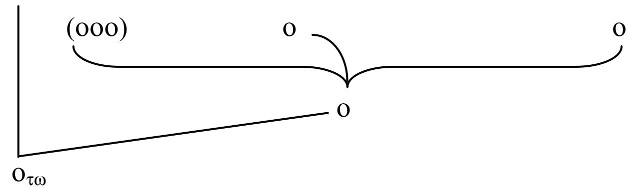
Third, via type-theoretical checking we verify that the individual constructions have been combined in a type-theoretically coherent way:



The Composition $[{}^0Happy_{wt}\ {}^0Charles]$ v-constructs **T**, according as the individual constructed by ${}^0Charles$ (i.e., Charles) belongs to the extension of the property *Happy* (v-constructed by ${}^0Happy_{wt}$) at a given $\langle w, t\rangle$. Abstraction over the values of w, t constructs a proposition/$o_{\tau\omega}$. In other words, the sense of 'Charles is happy' is a procedure the evaluation of which in any world w (λw) at any time t (λt) consists in checking whether Charles has the property of being happy at that $\langle w, t\rangle$-pair.

The type-theoretical checking of $[\lambda w\lambda t\ [{}^0Grumpy_{wt}\ {}^0Thelma]]$ proceeds in the same way. Finally, we check the whole (2′).

$$\lambda w\lambda t\ [{}^0\wedge\ \ [\lambda w\lambda t\ [{}^0Happy_{wt}\ {}^0Charles]]_{wt}\ \ [\lambda w\lambda t\ [{}^0Grumpy_{wt}\ {}^0Thelma]]_{wt}]$$



We have just verified that what this Closure constructs is a proposition, which is the right type of object to be denoted by a sentence.

Now we are going to analyse (3), (4) and (5) along the same lines. Quantifiers were defined in Definition 1.6. Thus the analysis of sentence (3) is this:

$$(3')\qquad \lambda w\lambda t\ [{}^0\exists^\iota\ \lambda x\ [\lambda w\lambda t\ [{}^0Happy_{wt}\ x]]_{wt}].$$

The Closure $[\lambda x\ [\lambda w\lambda t\ [{}^0Happy_{wt}\ x]]_{wt}]$ v-constructs the set of individuals instantiating the property *Happy* at $\langle w, t\rangle$. $\exists^\iota$ is here a function of type $(o(o\iota))$ inputting the set just constructed and outputting a truth-value, according as the set is empty or not. Finally, by abstracting over the values of w, t we construct the proposition that somebody is happy.

The analysis of sentence (4) depends on the type of possibility ascribed to the proposition that Charles is happy. If possibility is understood as *logical possibility* then $\Diamond^L$ is a function of type $(oo_{\tau\omega})$: the class of logically possible propositions. In such a case we have:

$$(4')\qquad \lambda w\lambda t\ [{}^0\Diamond^L\ [\lambda w\lambda t\ [{}^0Happy_{wt}\ {}^0Charles]]].$$

This construction constructs the trivial proposition *TRUE*. It is certainly logically possible that Charles be happy; in the possible-world idiom, there is a world w and a time t at which Charles has the property of being happy. Thus logical possibility can be defined by the following construction:

$$\lambda p\ [{}^0\exists^\omega\ \lambda w\ [{}^0\exists^\tau\ \lambda t\ p_{wt}]]$$

or for short,

$$\lambda p\ [\exists w \exists t\ p_{wt}].$$

Types: $p \rightarrow_v o_{\tau\omega}$; $\exists^{\omega}/(o(o\omega))$; $\exists^{\tau}/(o(o\tau))$.

This definition yields (by performing equivalent β-reductions):

$$(4'') \qquad \lambda w \lambda t\, [\exists w' \exists t'\, [^0Happy_{w't'}\, ^0Charles]].$$

Obviously, (4″) constructs *TRUE*. A more natural analysis can be obtained by construing *empirical possibility* $\Diamond^{em}$ as a *property of propositions*, an $(oo_{\tau\omega})_{\tau\omega}$-object. This yields

$$(4''') \qquad \lambda w \lambda t\, [^0\Diamond^{em}{}_{wt}\, [\lambda w \lambda t\, [^0Happy_{wt}\, ^0Charles]]].$$

This Closure constructs the contingent proposition that Charles' being happy is possible at the given $\langle w, t\rangle$ of evaluation.

In Section 1.2.2 we claimed that logical syntax cannot tolerate ambiguous terms. We explained that the handy notation of modal logics found in the *analysantes* of (3), (4) and (5) treats '□', '◇' as being syntactically on a par with truth-functional connectives like '¬', both '$\neg p$' and '$\Box p$' being well-formed formulae. Also we are allowed to generate strings like '$(\Box p \rightarrow p) \wedge Kp$', '$K$' standing for *knowing*. However, since what is necessary is not a truth-value but a proposition, and what is known is not a truth-value but a hyperproposition (in the case of explicit knowledge, see Section 5.1.2), we face here three-way ambiguity mixing together an extensional, an intensional and a hyperintensional context.

Now our context-*in*variant semantics begins to pay off. We need not analyse 'Charles is happy' any differently, nor are we forced to hold that 'Fa', hitherto denoting a truth-*value*, now denotes a truth-*condition* (proposition) instead.

Similarly, when analysing (5), the meaning of 'Charles is happy' is the same as above, namely the Closure $\lambda w \lambda t\, [^0Happy_{wt}\, ^0Charles]$, and we get

$$(5') \qquad \lambda w \lambda t\, [^0Believe_{wt}\, ^0Thelma\ \lambda w \lambda t\, [^0Happy_{wt}\, ^0Charles]].$$

This is the analysis of attitudes germane to classical possible-world semantics, according to which the object of an attitude is a proposition. Thus, *Believe* is a function of type $(o\iota o_{\tau\omega})_{\tau\omega}$. Again it is now paying off that 'Charles is happy' was paired off with a proposition straightaway, despite the fact that in (2) we need two truth-values as functional arguments.

If we analyse 'to believe' in (5) as a case of *explicit belief*, then *Believe** is a function of type $(o\iota *_n)_{\tau\omega}$. An agent, Thelma in our case, is now related to a hyperproposition. Again, it is paying off that 'Charles is happy' was analysed as expressing a hyperproposition, viz. the above Closure $\lambda w \lambda t\, [^0Happy_{wt}\, ^0Charles]$. Thelma is related to this very Closure, which can be constructed most directly by its Trivialization. We obtain

$$(5'') \qquad \lambda w \lambda t\, [^0Believe^*_{wt}\, ^0Thelma\ ^{\mathbf{0}}[\lambda w \lambda t\, [^0Happy_{wt}\, ^0Charles]]].$$

Sometimes it is said that the value of an intension in a possible world and at a time is an extension. As a general claim this is not true, however, because, as was pointed out above, there are intensions of a higher *degree* and of a higher *order.* Examples of the latter would be hyperintensional attitudes like *Believe**, *Know**, *Calculate*, all of type $(o\iota *_n)_{\tau\omega}$. As an example of a higher-degree intension, consider, for instance, the expression 'Einstein's favourite proposition'. This definite description obviously does not *refer* rigidly: in some equivalence classes of worlds/times Einstein will favour one proposition, in another equivalence class he will favour another proposition, and in yet another equivalence class he will favour none at all. So the type of the *denotation* of 'Einstein's most favourite proposition' is $(o_{\tau\omega})_{\tau\omega}$: a 2nd-*degree proposition*, the type of whose values is $o_{\tau\omega}$.

Type-theoretical analysis, which is the first part of our logical analysis of natural language (see Section 2.1), consists in associating types with meaningful expressions.[74] As competent users of our native language we know which expressions are empirical and we should be able to find the adequate type. (Montague's associating categories and then types with particular classes of expressions corresponds to this stage of logical analysis of natural language.) Sometimes the situation is not immediately clear, though. For instance, compare 'colour' and 'colour of'. The empirical character of the latter is obvious. What may be less obvious is what sort of intension it denotes. Now, it denotes an intension whose type is $((o\iota)_{\tau\omega}\,\iota)_{\tau\omega}$: in any world/time the outcome of applying this function to an individual is at most one colour (black, red, blue, etc.; i.e., a property of individuals). But from the fact that particular colours are properties, and so intensions, it does not follow that 'colour' denotes an intension. Actually, whereas asserting that an object is blue involves uttering an empirical sentence that denotes a contingent proposition, asserting that blue is a colour involves uttering an analytically true sentence denoting the proposition *TRUE*. This is because 'colour' denotes a *set* (rather than a property) of properties (colours). Black, red, blue, etc., are colours at all worlds and times, or rather independently of worlds and times. What varies are their extensions at various world/time pairs. Thus the type of the entity *Colour* is $(o(o\iota)_{\tau\omega})$: the word 'colour' denotes an extension. Whether a property belongs to this *set* of properties is true or false independently of empirical facts.

Another example of a 2nd-*degree intension* would be *the highest US executive office*. This role is occupied by individual offices, currently by the office of US

[74] As this point about typing also shows, TIL requires that the objects that are to be logically manipulated be typed and defined before any (possible) axiomatization. Of course, proposing some axioms involves running a risk, for it could be objected that the chosen axioms do not truly describe the nature of the objects. But this risk is only what characterises scientific work when carried out in a realist manner, according to which axioms do not prescribe what the objects of a domain are, but instead try to describe some properties that are ontologically and conceptually prior to the axioms. Analogously, 'Poincaré, like Kronecker, thought one does not have to define the whole numbers or construct their properties on an axiomatic foundation. Our intuition precedes such a structure' (Kline, 1980, p. 233).

President. So the type of the intension denoted by 'The highest US executive office' is $(\iota_{\tau\omega})_{\tau\omega}$.

If the type of the values of an intension is, say, $(o_{\tau\omega})_{\tau\omega}$, then the type of that intension has got to be $((o_{\tau\omega})_{\tau\omega})_{\tau\omega}$ to form a 3rd-degree intension. The rule for forming higher-degree intensions is straightforward: whenever a world index w is added as an argument to an intension of degree n, the degree of the resulting intension is n+1. Adding only a temporal index t won't suffice, since intensions are, strictly speaking, defined as functions from possible worlds. But adding t next to adding w may be called for to capture the temporal variability of the value distribution of a particular higher-degree intension.

To illustrate, the analysis of 'Einstein's most favourite proposition' is as follows. Einstein may have favoured many propositions, so the type of *Favour_prop_of (somebody)* needs to be $((oo_{\tau\omega})\iota)_{\tau\omega}$: A function that associates, dependently on $\langle w, t\rangle$, an individual with the set of propositions the individual favours at $\langle w, t\rangle$. The Composition $[{}^0\mathit{Favour\_prop\_of}_{wt}\ {}^0\mathit{Einstein}]$ *v*-constructs the set of propositions that Einstein favours at $\langle w, t\rangle$. Which of these propositions is the most favourite one depends again on the circumstances at $\langle w, t\rangle$. Thus the type of *The_Most* turns out to be $(o_{\tau\omega}(oo_{\tau\omega}))_{\tau\omega}$. A function of this type associates, dependently on $\langle w, t\rangle$, a set of propositions with a proposition, to wit, the most favoured one of them all. Thus the Composition

$$[{}^0\mathit{The\_Most}_{wt}\ [{}^0\mathit{Favour\_prop\_of}_{wt}\ {}^0\mathit{Einstein}]]$$

v-constructs the proposition that is Einstein's most favourite one at a given $\langle w, t\rangle$. Finally, by abstracting over the values of w, t, we construct the propositional role of Einstein's most favourite proposition:

$$\lambda w\lambda t\ [{}^0\mathit{The\_Most}_{wt}\ [{}^0\mathit{Favour\_prop\_of}_{wt}\ {}^0\mathit{Einstein}]].$$

To illustrate the distinction between 'colour' and 'colour of', we analyse the sentences

(6) 'The colour of Charles' most favourite shirt is green'

and

(7) 'Charles' most favourite colour is green'.

To trim the notation, let π be the type of an individual property, i.e. $(o\iota)_{\tau\omega}$. Then the types of entities that receive mention in (6) and (7) are:

$\mathit{Charles}/\iota$; $\mathit{Favour\_of}^1/((o\iota)\iota)_{\tau\omega}$; $\mathit{Favour\_of}^2/((o\pi)\iota)_{\tau\omega}$; $\mathit{Colour\_of}/(\pi\iota)_{\tau\omega}$; $\mathit{Colour}/(o\pi)$; $\mathit{Shirt\_of}/((o\iota)\iota)_{\tau\omega}$; Green/π; $\mathit{Most}^1/(\iota(o\iota))_{\tau\omega}$; $\mathit{Most}^2/(\pi(o\pi))_{\tau\omega}$.

The definite description 'Charles' most favourite shirt' denotes the individual office $ChFS/\iota_{\tau\omega}$ occupiable by the shirt, if any, that Charles happens to favour the most at some world/time of evaluation. Thus a coarse-grained analysis of (6) is

(6′) $$\lambda w \lambda t\, [[{}^0Colour\_of_{wt}\; {}^0ChFS_{wt}] = {}^0Green].$$

On the other hand, the definite description 'Charles' most favourite colour' denotes the property office $ChFC/\pi_{\tau\omega}$ occupiable by the property, if any, that happens to be Charles' most favourite. A coarse-grained analysis of (7) is

(7′) $$\lambda w \lambda t\, [{}^0ChFC_{wt} = {}^0Green].$$

Now, in order to refine the above analyses, we define the entities *ChFS* and *ChFC* in terms of the simpler entities the sentences talk about, i.e., *Shirt_of*, $Favour\_of^1$, $Favour\_of^2$. The individual office *ChFS* is defined as follows ($x \rightarrow \iota$):

$$\lambda w \lambda t\, [{}^0Most^1{}_{wt}\; \lambda x\, [[[{}^0Shirt\_of_{wt}\; {}^0Charles]\; x] \wedge [[{}^0Favour\_\, of^1{}_{wt}\; {}^0Charles]\; x]]].$$

The Closure $\lambda x\, [[[{}^0Shirt\_of_{wt}\; {}^0Charles]\; x] \wedge [[{}^0Favour\_\, of^1{}_{wt}\; {}^0Charles]\; x]]$ *v*-constructs the set of individuals that are Charles' favourite shirts at $\langle w, t\rangle$; $Most^1{}_{wt}$ selects from this set the individual, if any, that is the most favourite one at $\langle w, t\rangle$.

The individual office *ChFC* is defined as follows ($p \rightarrow \pi$):

$$\lambda w \lambda t\, [{}^0Most^2{}_{wt}\; \lambda p\, [[{}^0Colour\; p] \wedge [[{}^0Favour\_of^2{}_{wt}\; {}^0Charles]\; p]]].$$

The Closure $\lambda p\, [[{}^0Colour\;\; p] \wedge [[{}^0Favour\_of^2{}_{wt}\; {}^0Charles]\; p]]$ *v*-constructs the set of properties which belong to the set of colours (the first conjunct) and are Charles' favourite properties (the second conjunct). The application of $Most^2$ to this set yields the property selected from this set, if any, namely the one that is the most favourite one at $\langle w, t\rangle$.

By substituting these definitions into (6′) and (7′), we get these fine-grained analyses of (6) and (7):

(6″) $$\lambda w \lambda t\, [[{}^0Colour\_of_{wt}\; [{}^0Most^1{}_{wt}\; \lambda x\, [[[{}^0Shirt\_of_{wt}\; {}^0Charles]\; x] \wedge [[{}^0Favour\_of^1{}_{wt}\; {}^0Charles]\; x]]]] = {}^0Green].$$

(7″) $$\lambda w \lambda t\, [[{}^0Most^2{}_{wt}\; \lambda p\, [[{}^0Colour\; p] \wedge [[{}^0Favour\_of^2{}_{wt}\; {}^0Charles]\; p]]] = {}^0Green].$$

1.4.3 Logical objects

In this section we specify those important extensions that are classified as *logical objects* in TIL. We are aware of the problem of determining which objects are logical and which are extra-logical. For our purposes, we consider as logical objects only the extensions defined in this Section 1.4.3, i.e., truth-functions, quantifiers, singularizers, identities, and the functions *Sub* and *Tr*.

(a) *Truth-functions.* Unary (negation, $\neg$), type (oo); binary ($\wedge$, $\vee$, $\supset$, etc.), type (ooo). TIL is a classical logic in that it works with just two truth-values, **T**, **F**. This does not mean that every sentence of natural language must be true or false, though. Since some truth-bearers are neither true nor false, TIL has adopted *partial* functions, which associate with each argument *at most* one value. Thus the sentence

'The King of France is bald'

denotes a properly partial proposition (of type $o_{\tau\omega}$) that lacks a truth-value in, among others, the actual world at the present time. And the sentence

'The greatest prime is even or not even'

does not denote a truth-value, because 'the greatest prime' expresses an improper construction (see Example 1.3 (j)).

Remark. The third and further values in so-called many-valued logics cannot be construed as *truth*-values. They can be interpreted in various other ways (uncertainty and fuzziness being the most famous cases). The way TIL handles partiality bears similarities to Bochvar's three-valued logic (see Bochvar, 1939), where the 'third value' associated with one variable is the reason why it must be associated with the entire complex formula. Thus, if phrased in TIL jargon, if p v-constructs **T** and a construction Q constructs the third value, then the disjunction ($p \vee Q$) gets **T** in Łukasiewicz, the 'third value' in Bochvar, and no value in TIL. The matrices of Bochvar's three-valued logic will coincide with the matrices of a theory like TIL, which operates with three options: **T**, **F**, neither ('gap').

The following Table 1.1 is a TIL matrix of truth-functions and their Composition with truth-values and truth-value gaps. By the sign '$\perp$' we do not mark a third value, but a truth-value gap. P, Q are constructions v-constructing truth-values, and P, Q may be v-improper. The sign '*' marks rather peculiar rows, to be explained below.

According to Definition 1.2 (iii), the Composition $[X\ X_1 \ldots X_m]$ is v-improper whenever one or more of the constructions X, X_1, …, X_m are v-improper. This is in accordance with the compositionality constraint: once a construction X_i does not supply an object on which the construction X is to operate, the whole Composition fails to v-construct anything, making it v-improper. In this way partiality is being propagated upwards. This holds also for the Compositions of the constructions of truth-functions. Thus, e.g., $[^0\vee P\ Q]$ is *v*-improper if *P* or *Q* is *v*-improper.

Table 1.1 TIL matrix of truth-values

P	Q	$[^0\wedge P\ Q]$	$[^0\vee P\ Q]$	$[^0\supset P\ Q]$	$[^0\equiv P\ Q]$	
1	1	1	1	1	1	
1	0	0	1	0	0	
0	1	0	1	1	0	
0	0	0	0	1	1	
1	⊥	⊥	⊥	⊥	⊥	*
0	⊥	⊥	⊥	⊥	⊥	*
⊥	1	⊥	⊥	⊥	⊥	*
⊥	0	⊥	⊥	⊥	⊥	*
⊥	⊥	⊥	⊥	⊥	⊥	

Rows marked by ‘*’ might seem peculiar. Aren’t we used to a disjunction being true iff at least one disjunct is true? Aren’t we used to an implication being true iff the antecedent is false or the consequent true? Imagine a situation in which Charles does not smoke. Ostensibly, in such a situation we may truly claim the following:

‘Charles stopped smoking or he never smoked.’

Alas, analysing the sentence in this careless way yields a construction of a proposition that goes *undefined* at such $\langle w, t\rangle$ pairs at which Charles never smoked:

$$\lambda w\lambda t\ [^0\vee\ [\lambda w\lambda t\ [^0StopSmoking_{\mathrm{wt}}\ {}^0Ch]_{wt}\ [\lambda w\lambda t\ [^0NeverSmoked_{wt}\ {}^0Ch]_{wt}].$$

Types: *StopSmoking*, *NeverSmoked*/$(o\iota)_{\tau\omega}$; *Ch(arles)*/ι.

The problem is created by the proposition constructed by $[\lambda w\lambda t\ [^0StopSmoking_{wt}\ {}^0Ch]$ that does not have a truth-value at those $\langle w, t\rangle$ at which Charles never smoked. Hence the Composition $[\lambda w\lambda t\ [^0StopSmoking_{wt}\ {}^0Ch]_{wt}$ is *v*-improper for any such $\langle w, t\rangle$ pairs. This is due to the fact that the proposition that Charles stopped smoking comes with the *presupposition* that he used to smoke.[75] Thus the first argument of the function $\vee$ is missing and so the application fails. For this reason the Closure constructs a proposition that has truth-value gaps at those $\langle w, t\rangle$ pairs at which Charles never smoked.

A remedy is within reach, fortunately. In those cases where an extensionalized proposition enters as argument of a truth-function, we should use the totalizing

[75] *Presupposition* will be defined in Definition 1.14

propositional property $True/(o o_{\tau\omega})_{\tau\omega}$, which returns **T** for those $\langle w, t\rangle$ pairs at which the argument proposition is true, and **F** in all the remaining cases. The resulting analysis is:

$$\lambda w \lambda t\ [^0\!\vee\ [^0True_{wt}\ \lambda w \lambda t\ [^0StopSmoking_{wt}\ ^0Ch]]\ [\lambda w \lambda t\ [^0NeverSmoke_{wt}\ ^0Ch]_{wt}].$$

Gloss: 'It is true that Charles stopped smoking, or he never smoked'.

We will discuss the problem of partial functions and truth-value gaps in more details in Sections 2.6 and 2.7.[76]

(b) *Quantifiers.* The standard universal ($\forall^\alpha$) and existential ($\exists^\alpha$) quantifiers were defined in Definition 1.6. They are not 'improper symbols' for TIL; rather, they are type-theoretically polymorphous total functions of a type $(o(o\alpha))$ for the given type α, so they are classes of classes.[77]

- The universal quantifier $\forall^\alpha$ is the class of those classes that are not proper subclasses of α, so $\forall$ is a singleton.
- The existential quantifier $\exists^\alpha$ is the class of all non-empty subclasses of the class α.

Some sentences cannot be literally analysed using these standard quantifiers, unless we reformulate them. For instance,

(8) 'Some students are clever'

and

(9) 'All students are lazy'

can be analysed in the standard way as follows:

$$\lambda w \lambda t\ [^0\exists \lambda x\ [^0\!\wedge\ [^0Student_{wt} x]\ [^0Clever_{wt} x]]]$$

and

$$\lambda w \lambda t\ [^0\forall \lambda x\ [^0\!\supset\ [^0Student_{wt} x]\ [^0Lazy_{wt} x]]].$$

Types: $\forall, \exists/(o(o\iota))$; $\wedge, \supset/(ooo)$; *Student*, *Clever*, $Lazy/(o\iota)_{\tau\omega}$; $x/*_1 \to \iota$.

However, the above sentences (8) and (9) do not mention conjunction and implication. Thus these analyses are not in accordance with the principle of subject matter, which says, roughly, that each subconstruction of a given meaning of an

[76] See also Duží (2003a).

[77] By 'type-theoretically polymorphous functions' we mean a set of functions that are defined and thus behave in the same way, independently of their type. For instance, any member of the set of functions *Cardinality* associates a finite class with the number of its elements. Hence this definition is polymorphous; there are actually infinitely many cardinality functions, one for each type: $Card_1/(\tau(o\iota))$—the number of a set of individuals, $Card_2/(\tau(o\tau))$—the number of a set of numbers, etc., which we indicate by using a type variable α in the type of $Cardinality/(\tau(o\alpha))$.

expression E has to be assigned to a meaningful subexpression of E as its meaning. In other words, each subconstruction of the meaning assigned to E must construct an object denoted by a subexpression of E.[78] Therefore, these constructions are the meanings of different, albeit equivalent, sentences, namely 'There are individuals who are students and who are lazy' and 'It holds for all individuals x that if x is a student then x is lazy'.

In order to analyse sentences like (8) and (9) literally, in accordance with the principle of subject matter, we must use another type of quantifier, for example *All*, *Some*, and *No*, which are known as *restricted quantifiers.* These are type-theoretically polymorphous functions of type $((o(o\alpha))(o\alpha))$, defined as follows:

- All^{α} is the function which associates a class A of α-objects with the class of all those classes that contain A as a subset.
- $Some^{\alpha}$ is the function which associates a class A of α-objects with the class of all those classes that have a non-empty intersection with A.
- No^{α} is the function which associates a class A of α-objects with the class of all those classes that have an empty intersection with A.

All^{ι} and $Some^{\iota}$ of type $((o(o\iota))(o\iota))$ enable us to analyse (8) and (9) as expressing the Closures

(8′) $\lambda w\lambda t\ [[{}^{0}Some^{\iota}\ {}^{0}Student_{wt}]\ {}^{0}Clever_{wt}]$

and

(9′) $\lambda w\lambda t\ [[{}^{0}All^{\iota}\ {}^{0}Student_{wt}]\ {}^{0}Lazy_{wt}]$.

The Composition $[{}^{0}All^{\iota}\ {}^{0}Student_{wt}]$ v-constructs the set M of those sets of individuals which contain the population of students at a given $\langle w, t\rangle$ as a subset. The Composition $[[{}^{0}All^{\iota}\ {}^{0}Student_{wt}]\ {}^{0}Lazy_{wt}]$ v-constructs **T** for those $\langle w, t\rangle$ at which the set of individuals who are lazy at $\langle w, t\rangle$ belongs to M. In other words, it v-constructs **T** for a given $\langle w, t\rangle$ if the population of students is a subset of the population of lazy individuals at that $\langle w, t\rangle$. Abstraction over the values of w, t constructs the proposition that all students are lazy. It takes **T** at those $\langle w, t\rangle$ pairs at which all students are lazy.

For a mathematical example, consider the sentence

'It holds for all numbers that if the number is a prime then it is odd'.

The construction expressed by this sentence constructs **F**:

$$[{}^{0}\forall\ \lambda x\ [{}^{0}\supset [{}^{0}Prime\ x][{}^{0}Odd\ x]]].$$

The class constructed by

[78] This principle, and its relevance to semantic analysis, is discussed in Section 2.1.

$$\lambda x\,[{}^0\!\supset [{}^0\!Prime\ x][{}^0\!Odd\ x]]$$

is not the class of all real numbers, of course, because the Composition

$$[{}^0\!\supset [{}^0\!Prime\ x][{}^0\!Odd\ x]]$$

$v(2/x)$-constructs the truth-value **F**.

Types: $\forall/(o(o\tau))$; $\supset/(ooo)$; *Prime, Odd*$/(o\tau)$.

Similarly, the construction expressed by ‘No prime number is even’ constructs **F**:

$$[[{}^0\!No\ {}^0\!Prime]\ {}^0\!Even].$$

The type of *No* is here $((o(o\tau))(o\tau))$.

The class of even numbers does not belong to the class of all those classes that have an empty intersection with the class of prime numbers. On the other hand, the construction expressed by ‘All primes greater than 2 are odd’ constructs **T**:

$$[[{}^0\!All\ \lambda x\,[{}^0\!\wedge [{}^0\!Prime\ x]\ [{}^0\!> x\ 2]]]\ {}^0\!Odd].$$

The type of *All* is here $((o(o\tau))(o\tau))$.

The set of numbers constructed by $\lambda x\,[{}^0\!\wedge [{}^0\!Prime\ x]\ [{}^0\!> x\ 2]]$ is a subset of the set of odd numbers. Note that, for instance, the last sentence (and its corresponding meaning) is equivalent to, ‘It holds for all numbers that if the number is a prime greater than 2, then it is odd’, the analysis of which is:

$$[{}^0\forall\ \lambda x\,[{}^0\!\supset [{}^0\!\wedge [{}^0\!Prime\ x]\ [{}^0\!> x\ 2]]\ [{}^0\!Odd\ x]]].$$

The class of numbers constructed by $\lambda x\,[{}^0\!\supset [{}^0\!\wedge [{}^0\!Prime\ x]\ [{}^0\!> x\ 2]]\ [{}^0\!Odd\ x]]$ is the whole type τ.

(c) *Singulariser*. The function *Sing* was defined in Definition 1.6. If a construction C v-constructs a singleton whose only member is a then $[{}^0\!Sing\ C]$ v-constructs a. Otherwise (i.e., if C v-constructs an empty class or a class containing more than one element) $[{}^0\!Sing\ C]$ is v-improper (See Definition 1.2 (iii)).

Remark. Often the abbreviated notation ‘$\iota x\ A$’ will be preferred to ‘$[{}^0\!Sing\ [\lambda x\ A]]$’.

Examples: The analysis of ‘The only even prime number’ is the Composition $(x \to \tau)$

$$[\iota x\,[{}^0\!\wedge [{}^0\!Even\ x]\ [{}^0\!Prime\ x]]].$$

It constructs the number 2, because the class of numbers constructed by

$$\lambda x\, [^0\wedge\, [^0Even\ x]\, [^0Prime\ x]]$$

is the singleton {2}.

The analyses of ‘The only man to ever run 100 m in less than 9 s’, ‘The only man to ever run 100 m in less than 10 s’ are the respective constructions of ι-offices:

$$\lambda w \lambda t\ \iota x\, [^0\wedge\, [^0Man_{wt}\ x][^0{<}\, [^0Run\_in_{wt}\ x\ ^0100]\ ^09]],$$
$$\lambda w \lambda t\ \iota x\, [^0\wedge\, [^0Man_{wt}\ x][^0{<}\, [^0Run\_in_{wt}\ x\ ^0100]\ ^010]].$$[79]

Types: $x \to \iota$; $Man/(o\iota)_{\tau\omega}$; $</(o\tau\tau)$; $Run\_in/(\tau\ \iota\tau)_{\tau\omega}$: an empirical function that assigns to an individual and a number (the distance in metres) the number (of seconds) that it takes the given individual to run the respective distance.

Both offices are currently vacant, because the ι-class *v*-constructed by

$$\lambda x\, [^0\wedge\, [^0Man_{wt}\ x][^0{<}\, [^0Run_{wt}\ x\ ^0100]\ ^09]]$$

is empty in the actual world now, and the class *v*-constructed by

$$\lambda x\, [^0\wedge\, [^0Man_{wt}\ x][^0{<}\, [^0Run_{wt}\ x\ ^0100]\ ^010]]$$

is a multi-element class. Its elements are, in 2009, Jim Hines, Ronnie Ray Smith, Charles Greene, Steve Williams, Eddie Hart, Reynaud Robinson, Silvio Leonard, Carl Lewis, Maurice Greene, Asafa Powell, Usain Bolt, and others.

(d) *Identity*. The type-theoretically polymorphic function = of type $(o\alpha\alpha)$, occasionally with an index pointing to the type α, is *identity*. We have, e.g.,

$$[^0{=}_\tau\, [^0{+}\ ^07\ ^05]\ ^012],$$
$$[^0\neg\, [^0{=}_{*1}\ ^0[^0{+}\ ^07\ ^05]\ ^{00}12],$$
$$[^0{=}_{(o\tau)}\ \lambda x\, [^0{\geq}\ x\ ^00]\ \lambda x\, [^0\neg[^0{<}\ x\ ^00]]]$$

(all constructing **T**).

(e) *Sub* and *Tr* functions. In Definition 1.4 we specified two ways of binding variables in TIL, λ-binding and ^{0}binding. In both cases, a bound variable is not free for substitution, which brings technical trouble with it. To appreciate what sort of trouble, here are two examples of reckless deriving.

[79] We are disregarding here the problem of physical units.

Types: $B^*/(o\iota{*_1})_{\tau\omega}$; $B/(o\iota o_{\tau\omega})_{\tau\omega}$; $F/(o\iota)_{\tau\omega}$; a/ι; $x/{*_1} \rightarrow_v \iota$; $C/{*_1} \rightarrow_v \iota$.

$$(\mathrm{A}_1) \quad \frac{\lambda w \lambda t\, [{}^0B^*_{wt}\ {}^0a\ {}^0[\lambda w \lambda t\, [{}^0F_{wt}\ C]]]}{\lambda w \lambda t\, [{}^0\exists \lambda x\, [{}^0B^*_{wt}\ {}^0a\ {}^0[\lambda w \lambda t\, [{}^0F_{wt}\ x]]]].}$$

$$(\mathrm{A}_2) \quad \frac{\lambda w \lambda t\, [{}^0B_{wt}\ {}^0a\ [\lambda w \lambda t\, [{}^0F_{wt}\ C]]]}{\lambda w \lambda t\, [{}^0\exists \lambda x\, [{}^0B_{wt}\ {}^0a\ [\lambda w \lambda t\, [{}^0F_{wt}\ x]]]].}$$

Why are the conclusions no good? The occurrence of x in

$${}^0[\lambda w \lambda t\, [{}^0F_{wt}\ x]]$$

of the conclusion of (A_1) is ^{0}bound, so the variable x is *mentioned* and not used, hence not available for manipulation. It is, as it were, shielded from $\exists$ by the first Trivialization in ${}^0[\lambda w \lambda t\, [{}^0F_{wt}\ x]]$. In other words, x and C occur in (A_1) in a *hyperintensional context*.[80] A linguistic parallel would be to attempt to quantify into a quotational context, where the quotation marks would have an analogous shielding effect.

The argument (A2) is also invalid, for similar though slightly different reasons. Although the occurrence of x in

$$[\lambda w \lambda t\, [{}^0F_{wt}\ x]]$$

of the (A_2)-conclusion is free, the conclusion is not entailed by the premise. There are $\langle w, t\rangle$-pairs at which the proposition constructed by the premise is true, while the proposition constructed by the conclusion is false. The construction C v-constructing individuals occurs in the *intensional* context of $[\lambda w \lambda t\, [{}^0F_{wt}\ C]]$; thus C may be v-improper while the Closure $[\lambda w \lambda t\, [{}^0F_{wt}\ C]]$ is always proper (see Definition 1.2 (iv)) and the Composition $[{}^0B_{wt}\ {}^0a\ [\lambda w \lambda t\, [{}^0F_{wt}\ C]]]$ may v-construct **T** even if there is no C.

A parallel would be to attempt to quantify into an intensional or a hyperintensional context. For instance, from the truth of

Charles believes that Santa Claus is generous

[80] *Intensional* and *hyperintensional context* were characterized in Section 1.3, and will be formally defined in Section 2.6 together with valid rules for inferring existence. Here just briefly: a hyperintensional context is one in which constructions are mentioned, whereas an intensional context is one in which constituents are used with intensional (or *de dicto*) supposition.

we cannot validly infer that Santa Claus exists.[81] What we can infer is that there is an individual *office* (a.k.a. individual *role*) such that Charles believes that its occupant is generous.

However, sometimes we do need to quantify and/or substitute into a hyperintensional or intensional context; for instance, when analysing *de re* attitudes or sentences with anaphoric reference.[82] The solution is to substitute for the variable x the Trivialization of the entity v-constructed by the respective construction C instead of substituting the construction C itself. To this end, we need the functions Sub^n and Tr^α, which make variables amenable to manipulation by, first, untying them from the context they occur in and, second, substituting Trivialization of an appropriate entity for them.

Let X, Y, Z be constructions of order n, Y a variable. Then the function $Sub_n/(*_n\ *_n\ *_n\ *_n)$ is a mapping which, when applied to $\langle X, Y, Z\rangle$, returns the construction that is the result of correctly substituting X for Y in Z. Correct substitution will be defined in Definition 2.22. For now it suffices to say that a substitution is correct if no free variable occurring in X becomes bound in the resulting construction. Thus, for instance, the Composition

$$[{}^0Sub_1\ {}^{00}2\ {}^0x\ {}^0[{}^0{+}\ x\ {}^01]]$$

constructs the result of substituting ^{02} for x into $[{}^0{+}\ x\ {}^01]$, so the result is the Composition $[{}^0{+}\ {}^02\ {}^01]$. Therefore, the Composition $[{}^0Sub_1\ {}^{00}2\ {}^0x\ {}^0[{}^0{+}\ x\ {}^01]]$ is equivalent to ${}^0[{}^0{+}\ {}^02\ {}^01]$, both constructing as they do the Composition $[{}^0{+}\ {}^02\ {}^01]$:

$$[{}^0Sub_1\ {}^{00}2\ {}^0x\ {}^0[{}^0{+}\ x\ {}^01]] =_{*1} {}^0[{}^0{+}\ {}^02\ {}^01]].$$

Next, let α be a type of order n, a an object of type α. Then $Tr_\alpha/(*_n\ \alpha)$ is a function which, when applied to a, returns the Trivialization of a.[83]

Note that there is an essential difference between using Trivialization and applying the Tr_α function. For instance, whereas ^{03} constructs the number 3, the Composition $[{}^0Tr_\tau\ {}^03]$ constructs the construction ^{03}. Whereas the Trivialization ${}^0x$ binds the variable x and constructs just x, the variable x is free in the Composition $[{}^0Tr_\tau\ x]$, which v-constructs the Trivialization of the number that v assigns to x. For instance, $[{}^0Tr_\tau\ x]$ $v(2/x)$-constructs the construction ^{02}.

To illustrate the application of the Sub function, consider the schematic Composition

$$[{}^0Sub_1\ [{}^0Tr_\iota\ {}^0A_{wt}]\ {}^0y\ {}^0[\ldots y\ldots]].$$

Types: $A/\iota_{\tau\omega}$; $y \rightarrow_v \iota$; a/ι.

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[81] For the propositional attitudes of knowing and believing, see Sections 5.1 and 5.3.

[82] For attitudes and anaphoric sentences, see Chapter 5 and Section 3.5, respectively.

[83] See Tichý (1988, pp. 74–5).

This Composition either v-constructs the construction $[\ldots{}^0a\ldots]$, in case ${}^0A_{wt}$ v-constructs a, or is v-improper, in case ${}^0A_{wt}$ is v-improper.

We will often omit the lower-index when using the polymorphic functions Sub_n and Tr_α, writing simply '*Sub*' and '*Tr*', when the typing is obvious.

We will deal with quantifying into intensional and hyperintensional contexts in Section 5.3. To get a first feel for how TIL approaches quantifying in, consider again the above example

$$\frac{\text{Charles believes that Santa Claus is generous}}{\text{There is an office such that Charles believes that its occupant is generous.}}$$

We will analyse Charles' attitude as one of explicit belief, which is a relation-in-intension of an individual to a hyperproposition (a propositional construction). First, type-theoretical analysis:

$Charles/\iota$; $Believe/(o\iota *_1)_{\tau\omega}$; $Santa\_Claus/\iota_{\tau\omega}$: an individual office; $Generous/(o\iota)_{\tau\omega}$; $\exists/(o(o\iota_{\tau\omega}))$.

The premise says that Charles explicitly believes that Santa Claus is generous. To construct the proposition, we have to ascribe the property of being generous to the occupant of the office of Santa Claus. To this end we use the Trivialization of the office and its intensional descent, ${}^0Santa\_Claus_{wt}$, which v-constructs the individual (if any) that plays the role of Santa Claus at the $\langle w, t\rangle$-pair of evaluation. The proposition is now constructed by the Closure $\lambda w\lambda t\ [{}^0Generous_{wt}\ {}^0Santa\_Claus_{wt}]$. Since Charles bears the relation of explicit belief to this *construction*, we must *mention* it by means of Trivialization. The analysis of the premise is

$$\text{(P)} \qquad \lambda w\lambda t\ [{}^0Believe_{wt}\ {}^0Charles\ {}^0[\lambda w\lambda t\ [{}^0Generous_{wt}\ {}^0Santa\_Claus_{wt}]]].$$

Now, we cannot frivolously derive that Santa Claus exists, of course, for the office of Santa Claus is not occupied. But we can derive that there is such an office. Here is how. Let variable $r/*_1$ v-construct individual offices, of type $\iota_{\tau\omega}$. Then for any $\langle w, t\rangle$ such that the Composition

$$[{}^0Believe_{wt}\ {}^0Charles\ {}^0[\lambda w\lambda t\ [{}^0Generous_{wt}\ {}^0Santa\_Claus_{wt}]]]$$

v-constructs $\mathbf{T}$, the Composition

$$[{}^0\exists\lambda r\ [{}^0Believe_{wt}\ {}^0Charles\ [{}^0Sub\ [{}^0Tr\ r]\ {}^0r\ {}^0[\lambda w\lambda t\ [{}^0Generous_{wt}\ r_{wt}]]]]]$$

v-constructs **T** as well. To show this, let $v(Santa\_Claus/r)$ be a valuation identical to v up to assigning the office *Santa_Claus* to the variable r. Then $[^0Tr\ r]$ $v(Santa\_Claus/r)$-constructs $^0Santa\_Claus$, and

$$[^0Sub\ [^0Tr\ r]\ ^0r\ ^0[\lambda w\lambda t\ [^0Generous_{wt}\ r_{wt}]]]$$

$v(Santa\_Claus/r)$-constructs the Closure

$$[\lambda w\lambda t\ [^0Generous_{wt}\ ^0Santa\_Claus_{wt}]].$$

So the Composition

$$[^0{=}_{*1}\ [^0Sub\ [^0Tr\ r]\ ^0r\ ^0[\lambda w\lambda t\ [^0Generous_{wt}\ r_{wt}]]]$$
$$^0[\lambda w\lambda t\ [^0Generous_{wt}\ ^0Santa\_Claus_{wt}]]]$$

$v(Santa\_Claus/r)$-constructs **T**. Hence the class of individual offices v-constructed by the Closure

$$\lambda r\ [^0Believe_{\mathrm{wt}}\ ^0Charles\ [^0Sub\ [^0Tr\ r]\ ^0r\ ^0[\lambda w\lambda t\ [^0Generous_{wt}\ r_{wt}]]]]$$

is not empty. The analysis of the conclusion entailed by the premise (P) is then:

(C) $\lambda w\lambda t\ [^0\exists\lambda r\ [^0Believe_{wt}\ ^0Charles\ [^0Sub\ [^0Tr\ r]\ ^0r$
$^0[\lambda w\lambda t\ [^0Generous_{wt}\ r_{wt}]]]]].$

For a mathematical example, consider the sentence

(10)
'There is a number x such that dividing any number y by x is improper'.

If objects of higher-order types were not admitted, we would have no means to analyse this true sentence. The procedure of dividing y by x is improper for some number x, because it does not yield a product for some x, namely 0.

Let $Div/(\tau\tau\tau)$ be the function of dividing and $Improper/(o*_1)$ the class of constructions of order 1 that are v-improper for any valuation v. Finally, let the variables x, y range over the type τ. Then to express that dividing y by x is improper amounts to expressing the Composition

$$[^0Improper\ ^0[^0Div\ y\ x]].$$

Now, we cannot recklessly quantify over x and y, because x, y are ^{0}bound here. There is a way out, however. We use *Sub* and *Tr* to pre-process, as it were, the Composition

$$[{}^0 Improper\ {}^0[{}^0 Div\ y\ x]]$$

to make it construct **T**. First, by means of *Tr*, we untie *x* and *y*, and then substitute the resulting Trivialization of the numbers *v*-constructed by *x* and *y* into the Composition $[{}^0 Div\ y\ x]$. Here is how:

$$[{}^0 Sub\ [{}^0 Tr\ x]\ {}^0 x\ [{}^0 Sub\ [{}^0 Tr\ y]\ {}^0 y\ {}^0[{}^0 Div\ y\ x]]].$$

Note that in this Composition *x* and *y* are free for manipulation; the result is a *construction*, *in casu* the procedure of applying the division function to the numbers *v*-constructed by *x* and *y*. Now we want to express that this construction is improper for some number *v*-constructed by *x* and for all numbers *v*-constructed by *y*. The resulting analysis is thus

(10′) $[{}^0\exists\lambda x\ [{}^0\forall\lambda y\ [{}^0 Improper\ [{}^0 Sub\ [{}^0 Tr\ x]\ {}^0 x\ [{}^0 Sub\ [{}^0 Tr\ y]\ {}^0 y\ {}^0[{}^0 Div\ y\ x]]]]]].$

To see that this Composition constructs **T**, it suffices to realise that the Composition

$$[{}^0 Sub\ [{}^0 Tr\ x]\ {}^0 x\ [{}^0 Sub\ [{}^0 Tr\ y]\ {}^0 y\ {}^0[{}^0 Div\ y\ x]]]$$

behaves as follows. It $v(0/x)$-constructs the construction $v(0/x)$-constructed by $[{}^0 Sub\ [{}^0 Tr\ y]\ {}^0 y\ {}^0[{}^0 Div\ y\ x]]$, i.e. by $[{}^0 Sub\ [{}^0 Tr\ y]\ {}^0 y\ {}^0[{}^0 Div\ y\ {}^0 0]]$. The latter $v'(0/x)$-constructs a $v'(0/x)$-improper construction for any valuation $v'(0/x)$ identical to $v(0/x)$ up to assigning any number to *y*. For instance, for $v(0/x,1/y)$ we obtain the construction $[{}^0 Div\ {}^0 1\ {}^0 0]$. For $v(0/x, 2/y)$ we obtain the construction $[{}^0 Div\ {}^0 2\ {}^0 0]$; and so on. Thus, the class $v(0/x)$-constructed by

$$\lambda y\ [{}^0 Improper\ [{}^0 Sub\ [{}^0 Tr\ x]\ {}^0 x\ [{}^0 Sub\ [{}^0 Tr\ y]\ {}^0 y\ {}^0[{}^0 Div\ y\ x]]]]]$$

is the whole type τ, and the Composition

$$[{}^0\forall\lambda y\ [{}^0 Improper\ [{}^0 Sub\ [{}^0 Tr\ x]\ {}^0 x\ [{}^0 Sub\ [{}^0 Tr\ y]\ {}^0 y\ {}^0[{}^0 Div\ y\ x]]]]]]$$

$v(0/x)$-constructs **T**. Therefore, the class of numbers constructed by

$$\lambda x\ [{}^0\forall\lambda y\ [{}^0 Improper\ [{}^0 Sub\ [{}^0 Tr\ x]\ {}^0 x\ [{}^0 Sub\ [{}^0 Tr\ y]\ {}^0 y\ {}^0[{}^0 Div\ y\ x]]]]]]$$

is non-empty (because its element is the number 0), and the Composition (10′) constructs the truth-value **T**.

1.5 Constructions as structured meanings

1.5.1 Structured meanings

The contemporary mainstream method of logically analyzing expressions of a natural language consists in building up an artificial language and defining some rules of translation that make it possible to find for every expression of the given language its translated counterpart in the artificial language. The latter is unambiguous (unlike the former) and is interpreted in a model in the usual way.[84]

Tichý calls this method *formalization*. Formalization itself, if thought of as a means to make ideas precise, is indispensable. The method deployed by TIL to make ideas precise is a method of *direct* analysis. The notion of construction enables us to justify this direct transition from expressions to their meanings.

In a wider perspective, an important difference between Tichý and Montague is preceded by a famous difference Schopenhauer saw between himself and Kant. Schopenhauer said that,

> [Kant] is comparable to a person who measures the height of a tower from its shadow; but I am like one who applies the measuring rod directly to the tower itself.[85] (1819, p. 555.)

Montague, like other model-theoretic ('Tarskian') semanticists, translates natural-language phrases into shapes belonging to a pure syntax which are subsequently valuated. Tichý translates natural-language phrases into a likewise artificial symbolism. But TIL's symbolism is importantly different from IL's. TIL's 'language of constructions' is an *interpreted* formalism, so syntax and semantics work in tandem. The syntax of the λ-terms of TIL is provided by the existing λ-calculus, while the formalism is inherently interpreted, because its λ-terms are introduced as terms denoting constructions. The TIL analysis of a natural-language expression does not tell us which expression belonging to some other language it is synonymous with. Instead it tells us which its sense is. Montague's approach to analysis is indirect, Tichý's direct. The TIL λ-terms are in themselves of no interest and serve only as gateways or stepping-stones to non-linguistic entities, namely senses (constructions).[86] The only way to talk about senses is to avail oneself of terms denoting them. But the only task that the symbolic 'language of constructions' has to fulfil is to denote (atomic and compound) constructions. Metaphorically,

[84] See, e.g., Gamut (1991) or Montague (1974d).

[85] The original German text can be found in the *Anhang: Kritik der Kantischen Philosophie* in the 2nd Book of (1819), and goes,'[Kant] ist demjenigen zu vergleichen, der die Höhe des Thurmes aus dessen Schatten mißt, ich aber dem, welcher den Maaßstab unmittelbar anlegt.'

[86] Cf. Russell, who famously talked about thinking about logical objects for 2 s every 6 months, the rest of the time thinking about notation (1953, p. 185).

the symbols are transparent in the sense that we look through them to look at the constructions they denote.[87]

By way of illustration, the TIL analysis of

'1 + 2 = 3'

is the Composition

$$[{}^0\!= [{}^0\!+ {}^0 1\ {}^0 2]\ {}^0 3].$$

The term '$[{}^0\!= [{}^0\!+ {}^0 1\ {}^0 2]\ {}^0 3]$' denotes the sense of '1 + 2 = 3', i.e. the procedure of applying the identity function to two arguments, the first being the result of applying the plus function to 1 and 2, the second argument being the number 3. In general terms, a logical analysis of a given language consists in establishing such pairs of expressions and constructions. The *code* function underlying a given natural language, at a given phase of its historical development, will have been cracked, once all the expressions of the language have been paired off with constructions. That meanings are conceptually prior to their encoding in a language is summed up thus:

> The notion of a *code* [our emphasis] presupposes that prior to, and independently of, the code itself there is a range of items to be encoded in it. Hence…meanings cannot be conceived of as products of the language itself. They must be seen as logical rather than linguistic structures, amenable to investigation quite apart from their verbal embodiments in any particular language. To investigate logical constructions in this way is the task of logic. The linguist's brief is to investigate how logical constructions are encoded in various vernaculars (Tichý, 1994b, pp. 804–05).

Coupling all the expressions of an actual natural language with constructions would be no mean achievement for field linguistics. Logical analysis does not aspire to crack the code for *all* the expressions of a language, but it must aspire to be able to crack the code of *any* expression. When the idealization is made that language-users are perfectly competent, the idealization amounts to the language-users mastering every ⟨expression, construction⟩ pair of a given language.

TIL, and so its adjacent conception of logical analysis of natural language, is strictly opposed to any theories that maintain that meanings are produced by language.

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[87] Tichý suggested construing his λ-formalism as an iconography or pictorial script (see especially 1988, p. 224). This construal is buttressed by a strict enforcement of the principle of subject matter, which in turn might suggest something like a homomorphism between the set of λ-terms and the set of constructions of a given order (though an isomorphism is excluded, since there are more constructions of a given order than there are λ-terms). However, we have not attempted to develop this sketchy idea of iconography into a theory of λ-terms as something like *logical pictures* of constructions, mainly because the project of logical analysis of language does not need it and because any such theory would have to be embedded within the vast discussion on perfect languages, the expressive power of pictures, etc. For a discussion of the notion of pictorial script (without reference to TIL), see Jespersen and Reintges (2008).

Language, instead, is a code, and a code is a mapping of linguistic entities into non-linguistic entities. The latter are not inherently meanings, but become meanings in virtue of the code. That is, entities existing conceptually prior to pieces of language are made to serve in the office of linguistic meaning.[88] The idea of code is squarely incompatible with all theories that share with Quine the view that taking language as a code for certain objective operations means being a naïve advocate of 'the myth of the museum'.[89]

Constructions, then, are the primary subject-matter of our logical study. Their encoding in particular languages is of secondary importance. How constructions are encoded is fixed by sets of linguistic conventions, and field linguistics studies a posteriori the conventions germane to different languages at particular stages in their historical development. But the properties of and relations between particular constructions are a priori. For instance, the argument

$$\frac{\begin{array}{c}\text{There is no } x \text{ such that } x \text{ is a prime greater than 2 and } x \text{ is even}\\ \text{5 is a prime greater than 2}\end{array}}{\text{5 is not even}}$$

is logically valid independently of in which natural language these constructions are encoded. If we choose Czech instead of English, the encoding of the above valid argument would trivially be different:

$$\frac{\begin{array}{c}\text{Neexistuje } x \text{ takové, že } x \text{ je prvočíslo větší než 2 a } x \text{ je sudé}\\ \text{5 je prvočíslo větší než 2}\end{array}}{\text{5 není sudé číslo.}}$$

Yet the underlying constructions are *identical*. For instance, ${}^{00}\mathit{Prime} =_{*1} {}^{00}\mathit{Prvočíslo}$, ${}^{00}\mathit{Even} =_{*1} {}^{00}\mathit{Sudé}$. What is Trivialized is not a symbol, but an object (here, the set of prime numbers and the set of even numbers). In fact, an identical construction is what two synonymous expressions (whether of the same language or different languages) owe their synonymy to.[90]

The very *term* 'construction' is not entirely felicitous, connected as it is with many potentially misleading connotations, chief among which are the ones of mental procedure and constructivist proof (–object).[91] However, TIL constructions and those of intuitionism/constructivism share some noteworthy common ground.

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[88] —where 'office' is used as in normal English and not as in TIL.

[89] See Materna (2004b).

[90] More precisely, synonymous expressions express a common concept; see Section 2.2.

[91] In choosing the term 'construction', Tichý was inspired by geometry 'where we speak of various constructions of, say, the center of a circle, using rule and compass' (1986b, p. 514, 2004, p. 601).

For instance, verbally, at least, we agree with the intuitionist Fletcher when he says,

> If one had to define constructions in general, one would surely say that a type of construction is specified by some *atoms* and some *combination rules* of the form "Given constructions $x_1, \ldots, x_k$ one may form the construction $C(x_1,\ldots,x_k)$, subject to certain conditions on $x_1, \ldots, x_k$" (1998, p. 51).

TIL constructions are in themselves abstract, objective procedures. When made to serve as meanings, they are procedures detailing how to arrive at denoted entities.

What in part characterizes semantic realism is exactly that 'thoughts [*in casu* constructions. *Our insertion*] are independent of their expression in any language' (Tichý, 1988, p. vii). Yet, although TIL is semantic realism with a vengeance, TIL fails to qualify as such according to Dummett's entrenched definition of semantic realism. According to Dummett, realism construes sentential meanings as truth-conditions, while Dummett's own proof-theoretic anti-realism is cast in terms of assertability conditions. To qualify as realism in Dummett's sense, since empirical truth-conditions are possible-world propositions, TIL would have to construe propositions as the senses of empirical sentences; but we have argued at length why we are not pursuing this tack. One tenet, though, that TIL shares with realism as Dummett understands it is that truth-conditions obtain or fail independently of human cognitive means to establish which way they go. It is evident, however, that Dummett's conception of realism is too narrow to capture TIL, or indeed any other realist theory based on a procedural rather than truth-conditional semantics.[92]

Thus one of the advocates of procedural semantics, W.A. Woods, sums up

> two extreme interpretations of procedural semantics – a black-box approach in which the internal structure of a meaning function is inaccessible (only the input-output relations are available), and a low-level detail approach in which every detail of the operation of the meaning function procedure is considered a 'part of the meaning'. The former gives rise to a sense of equivalence between meaning functions that is too weak ..., in that it counts as equivalent meaning functions whose input-output relations are the same (in all possible situations) regardless of the means by which those extensions are determined [thus identifying, e.g., tautologies]. The low-level detail interpretation is at the opposite extreme of this spectrum. Its sense of equivalence is so strong that it counts two meaning functions as different if they differ in any detail of their operation regardless of the extent to which they effectively do the same thing. The notion of abstract procedure that is required for the characterization of meaning functions appears to lie somewhere between these extremes – providing a degree of internal structure that is considered significant, while leaving certain low-level details unspecified (or specified with suitable don'-care conditions) (1981, p. 329).

When assigning a *construction* to an expression as its meaning, we specify *procedural know-how*, which must not be confused with the respective *performatory know-how*. Distinguishing performatory know-how from procedural know-how,

[92] For the notion of procedural semantics, see Johnson-Laird (1977) and Woods (1981). For a defence of denotational semantics against procedural semantics, see Fodor (1975).

Rescher says of the latter that a knower *x* 'knows how *A* is done in the sense that *x* can spell out instructions for doing *A*' (2005, p. 6). Thus,

> *x* knows that people swim by moving their arms and legs in a certain cycle of rhythmic motions. But, of course, *x* can know how *A* is done without being able to do *A*—that is, without *x* having the performatory skills that enable *x* to do *A*. (For instance, *x* may know that a certain result is produced when a text is translated from one language to another without actually knowing how to make such a translation.) (ibid., p. 7).

If linguistic meaning is procedural, then to know what a given expression of a given language means is to possess procedural know-how. Linguistic competence is to know what particular procedure is encoded by an expression and how to execute the procedure. It is not required of the linguistically competent either that they should execute the procedure or even have the performatory know-how to do so.

For instance, to know what '1 + 2' means is to understand the instruction to add 1 and 2. It does not include either actually adding 1 and 2 (whether by following a procedure or by luck) or possessing the skill to do so. Similarly, we do understand the formulation of the Goldbach Conjecture (i.e., we do know the meaning of 'All positive even integers ≥ 4 can be expressed as the sum of two primes') without being able to execute the instruction in order to obtain the respective truth-value. In other words, we know the following construction without knowing what this construction constructs:[93]

$$\forall x \left[[[{}^0Even\ x] \wedge [{}^0{>}\ x\ {}^02]] \supset \exists yz\ \mathbf{[}[{}^0Prime\ y] \wedge [{}^0Prime\ z] \wedge [x = [{}^0{+}\ y\ z]]\mathbf{]}\right].$$

Types: ν (the type of natural numbers); $\forall/(o(o\nu))$; *Even*/$(o\nu)$; *Prime*/$(o\nu)$; x, y, $z/{*_1} \to \nu$.

Constructions are structured from the algorithmic point of view. We will now illustrate the way in which they are so structured.

Let us again consider a simple arithmetical expression, say,

'7 + 5'.

Bearing in mind that language is a code, we see that the above expression can be construed as encoding the meanings of particular simple subexpressions, but—and this is most important—also *the way these particular meanings combine* to form the meaning of the whole expression. In other words, the meaning **M** of the whole expression

M('7 + 5')

is not the same as the set of meanings of particular subexpressions of *E*, here

[93] The respective hypothesis expresses an ineffective procedure.

$$\{\mathbf{M}(\text{‘7’}), \mathbf{M}(\text{‘5’}), \mathbf{M}(\text{‘+’})\}.$$

(Remember Tichý’s metaphor of ‘Christmas decorations hanging from the branch’.) In general, constructions are abstract procedures that integrate particular subprocedures (‘steps’) into one whole. A mere set of meanings could not integrate individual meanings into the meaning of a molecule. Constructions consist of parts that are themselves constructions. So since constructions are procedures, one could equally well say that procedures consist of parts that are themselves procedures. The meaning of ‘7 + 5’ is the procedure $[^0\!+\ ^0 7\ ^0 5]$ decomposable into constituents as follows:

(1) $^0 7$: identify the number 7
(2) $^0 5$: identify the number 5
(3) $^0\!+$: identify the function +
(4) $[^0\!+\ ^0 7\ ^0 5]$: apply the product of step (3) to the products obtained at steps (1) and (2), respectively, in order to obtain the value of the function at this pair of arguments.

At least since Frege’s days there have been logicians who strove to avail themselves of fine-grained and structured meanings.[94] Analytic philosophy of language has pretty much since its inception been characterised in part by this quest. For instance, Russell’s structured propositions were not unlike our constructions. Unlike sets, they consisted of *parts*, but some of these parts were (due to Russell’s theory of acquaintance) concrete particulars. This leads to consequences that do not tally with our intuitive use of the term ‘proposition’; for instance, that propositions must be mind-friendly. Thus, we would definitely side any day with Frege against Russell over whether Mont Blanc can be in any sensible way part of anything deserving the name ‘proposition’. Moreover, language-users understand many sentences without being acquainted with the concrete particulars that the sentences talk about by means of abstract objects. The parts of a procedure have to be other procedures and cannot be the objects themselves, though the procedure may lead up to a non-procedure as its final output. A procedure (including any procedure figuring as a constituent subprocedure) is a presentation of an object rather than a presented object. But when knowing a procedure we need not know its output before actually executing it. We need to be acquainted with the procedure first before being able to execute it so as to arrive at the result. And some procedures may even fail to provide an output. A procedure is a different object than its product (if any), which is why exhaustive knowledge of the procedure does not include knowledge of its product. One thing is to know what to do (to know the procedure), quite another thing is to actually execute the procedure, and yet another thing is to know and understand what sort of object, if any, is the output.

[94] For example, see Sundholm (1994) on Frege’s epistemological motivations for a fine-grained individuation of *Gedanken*.

As pointed out in Section 1.1, Carnap (1947) rightly recognised that his intensions cannot handle all cases of synonymy and attempted to define the concept of intensional isomorphism. Church (1954) launched a counterexample involving two intensionally isomorphic sentences, one of which can be easily believed and the other not. A criticism of Carnap's attempt can be also found in Tichý (1988, pp. 8–9), where it is pointed out that the notion of intensional isomorphism is too dependent on a particular choice of notation. The structured character of meaning was later urged by David Lewis (1972), where non-structured intensions are generated by finite, ordered trees. This idea of 'tree-like' meanings obviously influenced George Bealer's idea of 'intensions of the second kind' in his (1982).[95]

The idea of structured meaning was propagated also by M.J. Cresswell (1975) and (1985), in which meaning is defined as an ordered *n*-tuple. Cresswell would construe the meaning of the above expression as a triple, viz.,

$$\langle \mathbf{M}(\text{'+'}), \mathbf{M}(\text{'7'}), \mathbf{M}(\text{'5'})\rangle.$$

That this is far from being a satisfactory solution is shown in Tichý (1994a) and Jespersen (2003). In brief, these tuples are again set-theoretic entities structured at most from a mereological point of view, by having elements or parts (though one balks at calling elements 'parts', since sets, including tuples, are not complexes). Besides, tuples are of the wrong making to serve as truth-bearers and objects of attitudes, since a tuple cannot be true or be known, hoped, etc., to be true. The above tuple is 'flat' from the procedural or algorithmic point of view. The *way* of combining particular parts together is missing here. For instance, the instruction to *apply* the function *plus* to a particular argument could have been one such way. It is to no avail to add the operation of application to a tuple to somehow create propositional unity, since the operation would merely be an element alongside other elements.[96] Moreover, the procedure specifying a function remains the same when other arguments are supplied as input for the function to be applied to. It is uncontroversial that tuples are set-theoretic objects; and all sets, unlike procedures, are algorithmically simple, have no 'input/output gaps', and are flat mappings.

[95] Van Heijenoort attempts to interpret Fregean *Sinn* in terms of trees. He suggests (1977, pp. 99–100) that the Fregean *Sinn* of a formula *T* is to be identified with a tree *T'*, whose semantic structure will be isomorphic to the syntactic structure of *T*. The suggestion is *prima facie* appealing, not least because the diagrammatic structure of trees is in the vicinity of the syntactic structure of Frege's *Begriffsschrift* notation. However, as Van Heijenoort himself points out, 'a tree is a mapping… Thus, in Fregean terms, a tree would be the object that is the *Werthverlauf* of a certain function. This conclusion may seem quite odd.' Indeed it does. But, worse, if Fregean *Sinn* is to be sliced in terms of cognitive significance rather than merely logical equivalence, then a mapping won't do as *analysans* due to the crude individuation of mappings.

[96] See Cocchiarella (2003, p. 51) for a recent statement of this objection. For a philosophical and historical discussion of propositional unity, see Gaskin (2008).

We agree with Moschovakis' idea of meaning as algorithm (see Moschovakis (1994, 2006), van Lambalgen and Hamm (2004)). In Moschovakis (2006) the meaning of a term *A* is 'an (abstract, idealized, not necessarily implementable) algorithm which computes the denotation of *A*.' (2006, p. 27; see also 1994).[97] The later version (2006) works with a formal language that extends the typed λ-calculus and so can accommodate, *per* Montague, reasonably large fragments of natural language. Moschovakis outlines his conception thus:

> The starting point … [is] the insight that a correct understanding of programming languages should explain the relation between a program and the algorithm it expresses, so that the basic interpretation scheme for a programming language is of the form
>
> (50) program $P \rightarrow$ algorithm(P) $\rightarrow$ den(P).
>
> It is not hard to work out the mathematical theory of a suitably abstract notion of algorithm which makes this work; and once this is done, then it is hard to miss the similarity of (50) with the basic Fregean scheme for the interpretation of a natural language,
>
> (51) term $A \rightarrow$ meaning(A) $\rightarrow$ den(A).
>
> This suggested at least a formal analogy between algorithms and meanings which seemed worth investigating, and proved after some work to be more than formal: when we view natural language with a programmer's eye, it seems almost obvious that we can represent the meaning of a term *A* by the algorithm which is expressed by *A* and which computes its denotation (ibid., p. 42).

In modern jargon, TIL belongs to the paradigm of *structured meaning*. However, Tichý does not reduce structure to set-theoretic sequences, as do Kaplan and Cresswell. Nor does Tichý fail to explain how the sense of a molecular term is determined by the senses of its atoms and their syntactic arrangement, as Moschovakis objects to 'structural' approaches in (2006, p. 27).

The notion of TIL construction is bound to elude the followers of holistic theories (Quine, the later Wittgenstein, etc.). In fact, the idea of construction is an anti-holistic idea, supposing as it does that the meaning of an expression can be in principle *composed* from the meanings of its subexpressions.

TIL is opposed to various nominalist trends in contemporary philosophy, not least their misuse of Occam's razor. Tichý' succinctly sums up the lie of the land:

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[97] Moschovakis' notion of algorithm borders on being too permissive, since algorithms are normally understood to be effective. (See Cleland (2002) for discussion.) Tichý separates algorithms sharply from constructions: 'The notion of construction is…correlative not with the notion of algorithm itself but with what is known as a particular algorithmic *computation*, the sequence of steps prescribed by the algorithm when it is applied to a particular input. But not every construction is an algorithmic computation. An algorithmic computation is a sequence of *effective* steps, steps which consist in subjecting a manageable object…to a feasible operation. A construction, on the other hand, may involve steps which are not of this sort' (1986b, p. 526 2004, p. 613).

> [T]he vision informing 20th century philosophy has been aptly described as one of a desert landscape. Philosophers behave as if in expectation of an ontological tax collector to whom they will owe the less the fewer entities they declare. The metaphysical purge is perpetrated under a banner emblazoned with Occam's Razor. But Occam never counselled ontological genocide at all cost. He only cautioned against multiplying entities *beyond necessity.* His Razor is thus in full harmony with the complementary principle, known as Menger's Comb, which cautions against trying to do with less what requires more. The two methodological precepts are just two sides of the same coin (1995, p. 175, 2004, p. 875).

Thus one should bear in mind that there is a complementary warning in the shape of *Menger's comb*. Another pointed criticism of the abuse of Occam's razor is this:

> To satisfy the constraints of ontological parsimony, one should add as few objects as possible in a nonarbitrary way. But with abstract objects, the only way to add as few objects as possible in a nonarbitrary way is to add them all! … Platonized naturalism acknowledges that a maximal ontology of abstracta is the simplest because a plenum is not an arbitrary selection from some larger class (Linsky and Zalta, 1995, p. 552).

Morale: logical analysis of natural language must take the form of a procedural semantics in order to succeed. So, in keeping with Menger's comb, nothing less than a 'maximal ontology of abstracta' is going to be plentiful enough to contain procedures as fully-fledged entities.

1.5.1.1 Analytic vs. logical

There has been a long philosophical dispute concerning the definition of analytic truth and the relation between analytic and synthetic truths. The distinction goes as far back as Leibniz, at least. For now it is sufficient to adopt the explication that an analytically true sentence is true solely in virtue of its meaning. Since we presuppose full linguistic competence in language-users, sentences like 'No bachelor is married', 'Whales are mammals', and also mathematical sentences like 'The problem of logical validity is not decidable in first-order predicate logic' come out analytically true. Provided that we understand the meanings of the predicates 'is a whale' and 'is a mammal' as used in current English, when learning that whales are mammals we do not acquire information bearing on the state of the world. If you know that the individual before you is a whale, you need not examine the world in order to get to know that the individual is a mammal.

Our procedural semantics enables us to easily define the difference between analytically and logically true sentence, as well as the difference between analytically and logically valid argument. Recall that *TRUE* is the proposition that takes value **T** in all worlds at all times.

Definition 1.9 (*analytically true sentence*) A mathematical *sentence* is *analytically true* iff it expresses a construction constructing the truth-value **T**. A *sentence* involving empirical expressions is *analytically true* iff it expresses a construction constructing the proposition *TRUE.*

Yet the literal analysis of the sentence ‘No bachelor is married’ does not reveal the fact that it is analytically true.

The types are: *Bachelor*, *Married*/$(o\iota)_{\tau\omega}$; *No*/$((o(o\iota))(o\iota))$: the quantifier that assigns to a given set *M* the set of those sets of individuals which have an empty intersection with *M.*

Thus the synthesis is:

(*) $\lambda w\lambda t\ [[{}^0No\ {}^0Bachelor_{wt}]\ {}^0Married_{wt}]$.

Type-checking:

$$\begin{array}{ccc} \lambda w\lambda t & [\underbrace{\underbrace{[\underset{((o(o\iota))(o\iota))}{{}^0No}\quad \underset{(o\iota)}{{}^0Bachelor_{wt}}]}_{(o(o\iota))}\quad \underset{(o\iota)}{{}^0Married_{wt}}}_{o}] \\ \downarrow & \\ o_{\tau\omega} & \longleftarrow o \end{array}$$

This Closure constructs a proposition, as it should, but it is not obvious that the so constructed proposition is identical to *TRUE*.[98]

On the other hand, the sentence ‘It is not true that there is an individual *x* such that *x* is not married and *x* is a man and *x* is married’ is also analytically true; but not only that: it is also *logically* true, as its analysis shows:

(**) $\lambda w\lambda t\ [\forall w\forall t\ [\neg\exists x\ [\neg[{}^0Married_{wt}\,x] \wedge [{}^0Man_{wt}\,x] \wedge [{}^0Married_{wt}\,x]]]]$.

Since the Composition $[\neg\exists x\ [\neg[{}^0Married_{wt}\ x] \wedge [{}^0Man_{wt}\ x] \wedge [{}^0Married_{wt}\ x]]]$ obviously and provably *v*-constructs **T** for any valuation *v*, the generalisation

$$[\forall w\forall t\ [\neg\exists x\ [\neg[{}^0Married_{wt}\,x] \wedge [{}^0Man_{wt}\,x] \wedge [{}^0Married_{wt}\,x]]]]$$

constructs **T**. Therefore, the proposition constructed by the above Closure is the proposition *TRUE.*

But, which of the two equivalent constructions (*), (**) should be assigned to ‘No bachelor is married’ as its meaning? Provided the predicate ‘is a bachelor’ is a

[98] This three-step analysis anticipates Section 2.1.1.

semantically simple expression, the *literal meaning* of this sentence is (*).[99] Thus we define:

Definition 1.10 (*literal meaning of an expression*) Let E be an expression whose semantically simple subexpressions are $S_1, \ldots, S_n$, and let $S_1, \ldots, S_n$ denote the objects $X_1, \ldots, X_m$. Let C_E be a construction that is assigned to E as its meaning such that there is no closed subconstruction of C_E constructing an object that is not denoted by a subexpression of E. Then C_E is the *literal meaning of* E iff ${}^0X_1, \ldots, {}^0X_m$ are all closed subconstructions of C_E constructing the objects $X_1, \ldots, X_m$, respectively.

Definition 1.10 imposes the constraint that the objects that receive mention by *simple* meaningful subexpressions should be constructed by their Trivialisations. If the expression E is semantically simple, then the Trivialisation of the denoted object is assigned to E as its literal meaning. On the other hand, if E is semantically complex, then the Trivialisations of objects denoted by simple subexpressions of E are combined into a complex construction assigned to E as its literal meaning in the manner complying with the set-theoretical conditions imposed by E.[100]

In order to define the notion of logical truth, we must first define the notion of *literal logical form*:

Definition 1.11 (*literal logical form of an expression*) Let C_E be the literal logical analysis of E, whose subconstructions construct (by Trivialisation) the extra-logical objects $X_1, \ldots, X_n$, X_i/α_i. Let $V_1 \rightarrow \alpha_1, \ldots, V_n \rightarrow \alpha_n$ be variables not occurring in C_E. Then the *literal logical form* (*LLF*) of E is the construction LC_E that differs from C_E only in replacing all occurrences of ${}^0X_i$ by V_i.

It is important to note that according to Definition 1.11 only Trivialisations of *extra-logical* objects are replaced by type-theoretically appropriate variables in order to obtain the literal logical form of the relevant expression. Construction of logical objects like truth-functions and quantifiers are left unchanged.[101] Thus the literal logical form of a sentence corresponds to a formula of a formal language. The formulae of a formal language are associated with their models by means of an interpretation of special non-logical symbols. A formula is then logically true if it is true on every interpretation.

As we explained at the outset of this section, we do not translate sentences of a natural language into a formal language with a view to interpreting this language. Instead, by means of 'the language of constructions' we directly examine constructions expressed by natural-language sentences. Yet there is a similarity with

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[99] The other option amounts to conceiving 'is a bachelor' as a semantically complex expression. See also Section 2.2.1.

[100] See Section 2.1 for the method of semantic analysis.

[101] See Section 1.4.3 for the list of logical objects.

the formal approach. If a sentence is logically true, it is true in virtue of its logical form, regardless of any particular extra-logical objects receiving mention in the sentence.[102] For instance, the sentence ‘No number is even and not even’ is logically true, unlike the sentence ‘No number is even and odd’, which is only analytically true. The literal logical form assigned to the former is

$$\neg\exists x\ [[E\ x] \wedge \neg[E\ x]],$$

whereas the literal logical form assigned to the latter is

$$\neg\exists x\ [[E\ x] \wedge [O\ x]].$$

Types: $x \to \tau$; $E, O \to (o\tau)$.

The construction $\neg\exists x\ [[E\ x] \wedge \neg[E\ x]]$ v-constructs **T** for *all* valuations of the variable E, whereas the construction $\neg\exists x\ [[E\ x] \wedge [O\ x]]$ v-constructs **F** for some valuations of variables E and O. These are those valuations for which E and O v-construct sets with a non-empty intersection.[103]

Thus Definition 1.11 enables us to easily define *logically true sentence*.

Definition 1.12 (*logically true sentence*) A mathematical *sentence S* is *logically true* iff the *LLF* of *S* v-constructs the truth-value **T** for every valuation v. A *sentence S* involving empirical expressions is *logically true* iff the *LLF* of *S* v-constructs the proposition *TRUE* for every valuation v.

Obviously, any logically true sentence is analytically true. It is a well-known fact that the converse does not hold, as indeed the ‘bachelor’ example showed. The same holds also for mathematical sentences, as showed by the above mathematical example. For another mathematical example, the sentence T_1

T_1 ‘If $2 < 5$ and $5 < 11$ then $2 < 11$’

is analytically, but not logically, true. The LLF of T_1 is ($L \to (o\tau\tau)$, $k, m, n \to \tau$):

[102] This problem was tackled as early as in 1837 by Bolzano, who introduced a modern method of variation of (objective) representations (‘Vorstellungen an sich’) and defined *generally valid* sentences *with respect to representations* $r_1,\dots,r_m$ such that the sentence remains true if these representations are changed or varied (See 1837, §§147–48).

[103] Similarly, the formula ‘$\neg\exists x\ [E(x) \wedge \neg E\ (x)]$’ of first-order predicate logic is true on every interpretation assigning a subset of the universe to the symbol ‘*E*’, whereas there are interpretations of ‘*E*’ and ‘*O*’ on which the formula ‘$\neg\exists x\ [E(x) \wedge O(x)]$’ is false, viz. those interpretations that assign non-disjoint sets to the symbols ‘*E*’ and ‘*O*’.

T_1' $[[[L\ k\ m] \wedge [L\ m\ n]] \supset [L\ k\ n]]$.[104]

There is a valuation v such that the antecedent v-constructs **T** and the consequent **F**. (For instance, the valuation v that assigns the relation $\neq$ to the variable L, and the numbers 2, 5, 2 to variables k, m, n, respectively.) For the same reason, even the sentence T_2 is not logically true:

T_2 'If 2 < 5 and 5 < 11 and if < is transitive then 2 < 11'.

Though T_2 specifies a more detailed procedure than T_1, it leaves it open what is the definition of the transitive relation. *LLF* of T_2 is (the variable $T \rightarrow (o\ (o\tau\tau))$ v-constructing a class of binary relations)

T_2' $[[[L\ k\ m] \wedge [L\ m\ n] \wedge [T\ L]] \supset [L\ k\ n]]$,

which is not the form of a logically true sentence. Only when we explicitly define the class of transitive binary relations by

$$\lambda r\ \forall x \forall y \forall z\ \big[[r\ x\ y] \supset [[r\ y\ z] \supset [r\ x\ z]]\big]$$

is the logically true sentence T_3 obtained:

T_3 'If 2 < 5 and 5 < 11 and if $\forall x \forall y \forall z\ (x < y \supset (y < z \supset x < z))$ then 2 < 11'.

Additional types: $r \rightarrow (o\tau\tau)$; $x, y, z \rightarrow \tau$.
The LLF of T_3 is the form of a logically true sentence:

T_3' $[[[L\ k\ m] \wedge [L\ m\ n] \wedge \forall x \forall y \forall z\ \big[[L\ x\ y] \supset [[L\ y\ z] \supset [L\ x\ z]]\big]] \supset [L\ k\ n]]$.

These definitions make it possible to easily define the difference between analytically and logically valid arguments. For instance, the following argument is analytically, but not logically, valid:

No bachelor has ever been married

Whales are mammals.

Since both the premise and the conclusion are analytically true sentences, the argument is analytically valid; there is no possible world w and time t at which the premise would be true and the conclusion false. Similarly, the following mathematical argument is analytically, but not logically, valid:

[104] For the sake of simplicity we are now omitting the symbol of Trivialization of logical objects and using the standard notation of quantifiers and infix notation for the truth-functions.

$$\frac{\begin{array}{c}\text{No prime number greater than 2 is even;}\\ \text{9 is not a prime number greater than 2}\end{array}}{\text{9 is not even.}}$$

Since every true mathematical sentence is true *only* in virtue of its meaning, there is no world/time pair at which the premises were true and the conclusion false. Any argument with premises $S_1,\ldots,S_n$ and conclusion S corresponds to a conditional sentence of the form 'If S_1 and … and S_n then S'. If the argument is analytically valid, then there is no possible world w and time t such that the premises would be true and the conclusion false. Hence, the conditional sentence is analytically true. And vice versa, if the conditional sentence is analytically true, the corresponding argument is analytically valid. Thus we define:

Definition 1.13 (*analytically/logically valid argument*) Let $S_1, \ldots, S_n$ be premises and S the conclusion of an argument A, and let S_A be the respective implicative statement of the form 'If S_1 and … and S_n then S'. Then

(i) *A is analytically valid* iff S_A is analytically true.
(ii) *A is logically valid* iff S_A is logically true.

For instance, the following argument is not only analytically, but also logically valid:

$$\frac{\begin{array}{c}\text{There is no } x \text{ such that } x \text{ is a prime number greater than 2 and } x \text{ is even;}\\ \text{5 is a prime number greater than 2}\end{array}}{\text{5 is not even.}}$$

The literal analysis of the premises and the conclusion is as follows:

$$\frac{\begin{array}{c}[{}^0\neg\ [{}^0\exists\lambda x\ [{}^0\wedge\ [{}^0\wedge\ [{}^0Prime\ x]\ [{}^0{>}\ x\ {}^02]]\ [{}^0Even\ x]]]]\\ [[{}^0\wedge\ [{}^0Prime\ {}^05]\ [{}^0{>}\ {}^05\ {}^02]]\end{array}}{[{}^0\neg\ [{}^0Even\ {}^05]].}$$

Types: $\exists/(o(o\tau))$; *Prime*, *Even*$/(o\tau)$, $>/(o\tau\tau)$; 5, $2/\tau$; $x/*_1\to\tau$.

And the corresponding literal logical form is:

$$\frac{\begin{array}{c}[{}^0\neg\ [{}^0\exists\lambda x\ [{}^0\wedge\ [{}^0\wedge\ [P\ x]\ [R\ x\ a]]\ [E\ x]]]];\\ [[{}^0\wedge\ [P\ b]\ [R\ b\ a]]\end{array}}{[{}^0\neg\ [E\ b]].}$$

Now it is easy to prove that the corresponding implicative sentence is logically true (to make this fact easier to see, we are again using standard infix notation without Trivialisation for logical connectives):

$$[\neg\exists x\,[[P\,x] \wedge [R\,x\,a] \wedge [E\,x]] \wedge [[P\,b] \wedge [R\,b\,a]]] \supset \neg[E\,b] =$$
$$\Big[\forall x\Big[[[P\,x] \wedge [R\,x\,a]] \supset \neg[E\,x]\Big] \wedge \Big[[P\,b] \wedge [R\,b\,a]\Big]\Big] \supset \neg[E\,b]$$

Variables: P, $E/*_1\rightarrow(o\tau)$; $R/*_1\rightarrow(o\tau\tau)$; a, $b/*_1\rightarrow\tau$.

As we have argued in Section 1.2, an argument is valid or invalid in virtue of the meanings of its premises and conclusion. Therefore, the type of the *entailment relation* obtaining between the set of premises and the conclusion of an argument is $(o(o*_n)*_n)$. It is a relation-in-extension between a set of constructions (the meanings of the premises) and a construction (the meaning of the conclusion).[105] Thus the entailment relation can be defined as follows:

Let $S_1, \ldots, S_n$ be the premises and S the conclusion of an argument involving the empirical expressions $S_1, ..., S_n$, S thus expressing the propositional constructions $\boldsymbol{C_1}, \ldots, \boldsymbol{C_n}, \boldsymbol{C} \rightarrow o_{\tau\omega}$. Then $S_1,\ldots, S_n$ *entail* S if $\{C_1, \ldots, C_n\} \models C$. As a corollary of definition 1.13, this is so iff

$$\forall w \forall t\, [[[{}^0True_{wt}\, \boldsymbol{C_1}] \wedge \ldots \wedge [{}^0True_{wt}\, \boldsymbol{C_n}]] \supset [{}^0True_{wt}\, \boldsymbol{C}]].$$

$True/(oo_{\tau\omega})_{\tau\omega}$ is the propositional property of being true at $\langle w, t\rangle$.

Let $S_1, \ldots, S_n$ be the premises and S the conclusion of a mathematical argument, $S_1, ..., S_n$, S thus expressing the truth-value constructions $\boldsymbol{C_1}, \ldots, \boldsymbol{C_n}, \boldsymbol{C} \rightarrow o$. Then $S_1, \ldots, S_n$ *entail* S if the set of constructions $\boldsymbol{C_1}, \ldots, \boldsymbol{C_n}$ entails the construction $\boldsymbol{C}$. As a corollary of definition 1.13, this is so iff

$$[[[{}^0True^*\, {}^0\boldsymbol{C_1}] \wedge \ldots \wedge [{}^0True^*\, {}^0\boldsymbol{C_n}]] \supset [{}^0True^*\, {}^0\boldsymbol{C}]].$$

$True^*/(o*_n)$ is the function that, when applied to a truth-value construction C, returns the value **T** if C v-constructs **T**, otherwise **F**.

Remarks.

(a) *Empirical case.*
Since the propositions denoted by the premises and the conclusion of a valid argument may lack a truth-value in some world w at a time t, we have to use the propositional property *True*.

[105] See Tichý (1988, p. 235).

(b) *Mathematical case.*
Since the premises or conclusion of a mathematical argument may express *v*-improper constructions, we need to use the function *True**.[106]
If partiality were not involved, then the Composition $[^0True^* \, ^0\boldsymbol{C}]$ would be equivalent to $[^{20}\boldsymbol{C}]$ or simply to $\boldsymbol{C}$.

1.5.2 Supposition de dicto and de re vs. reference shift

The term 'transparent' in 'transparent intensional logic' is to be interpreted in an *anti-contextualistic* manner. The point is that various alternative approaches lead to a seemingly necessary limitation of the compositionality principle. 'Oblique contexts' are standardly cited as a motive for restraining the principle. Intentional contexts are typical instances of 'oblique contexts'. Example: Since it was Sir Walter Scott who wrote the novels *Waverley* and *Ivanhoe*, Frege would have held that the definite descriptions

'The author of *Waverley*'

and

'The author of *Ivanhoe*',

denoted Sir Walter Scott. Evidently, the sentence

'Charles believes that the author of *Waverley* is a poet'

can be true whereas the sentence

'Charles believes that the author of *Ivanhoe* is a poet'

can be false at the same time. Frege wanted to observe compositionality, which would be obviously violated if 'The author of *Waverley*' denoted the same individual as 'The author of *Ivanhoe*'; the truth-value of both sentences would necessarily be the same. Wishing to save compositionality, Frege made the semantics of an expression depend on the linguistic context in which it is embedded. In atomic and molecular contexts 'The author of *Waverley*' and 'The author of *Ivanhoe*' both denote Sir Walter Scott, but in 'oblique contexts' like the one above both descriptions denote what in atomic and molecular contexts (e.g., 'The author of *Waverley* is happy and the Sun is shining') is their sense. Compositionality is

[106] Note also that due to the ramified hierarchy of types, no inconsistency problems arise when introducing truth predicates like *True* and *True**. In our higher-order typed approach there is no need to use disquotation like True('walks(Bill)') $\Leftrightarrow$ walks(Bill) and a hierarchy of meta-languages with their established grounded truths. The sentence 'Bill walks' is true in world *w* at time *t* if the proposition constructed by $\lambda w \lambda t \, [^0Walk_{wt} \, ^0Bill]$ takes value $\mathbf{T}$ in *w* at *t*.

saved (the expressions possessing distinct senses); the price exacted is *contextualism*.

The price is very high indeed. No expression can denote an object, unless a particular kind of context is provided. Yet such a solution is far from being natural. There are cases of real ambiguity, witness homonymous expressions. Which of the denotations is relevant in such cases (e.g., 'is a bank') can be detected by a particular context (cf. 'A bank was robbed' vs. 'A woman walks along the banks of the Dnepr'), but would anybody say that 'The author of *Waverley*' were another such case of homonymy? Hardly; unless, of course, their intuitions had been warped by Fregean contextualism. Furthermore, expressions can be embedded within other expressions to various degrees; consider the sentence

'Charles knows that Tom believes that the author of *Waverley* is a poet.'

The expression 'The author of *Waverley*' should now denote the 'normal' sense of the 'normal sense' of itself. Adding still further layers of embedding sets off an infinite hierarchy of senses, which is to say that 'The author of *Waverley*' has the potential of being infinitely ambiguous. This seems plain wrong, and is first and foremost an awkward artefact of Frege-Churchian semantics.

One well-known form of contextualism consists in distinguishing two kinds of context. In one kind ('referential context') a definite description refers to the object that satisfies the uniqueness condition, in the other context a definite description denotes something else. The problem with the distinction between two kinds of semantic context is that their definition is *circular*. Someone who propounds it wants to say that the descriptive term refers to the object that occupies the respective individual office in the respective kind of context. But this kind of context is defined just *via* the way the term is supposed to function in such a context:

Q: When is a context extensional?
A: A context is extensional if it validates the rules of (i) substitution of co-referential singular terms and (ii) existential generalisation.
Q: And when are (i), (ii) valid?
A: These rules are valid if all the contexts they are applied to are extensional.

Hence, the notions of extensional context and the validity of (i), (ii) are interdefined, the respective *definiendum* and *definiens* presupposing one another. This argument, which Tichý merely drops in passing,[107] is a potent one. In general the obvious move is to either define the semantic notion of extensional context (partly) in terms of the logical notion of the validity of one or more rules or else define the logical notion (partly) in terms of the semantic one. But to do either, it is required that the respective *definiens* be already determinate.

[107] See Tichý (1986a, p. 256, 2004, p. 654).

In this book we proceed in the following manner:[108] We first define the occurrence of a meaning-endowed constituent with extensional and intensional *supposition*, respectively. Thus we speak of extensional contexts in which constructions occur with extensional supposition, and of intensional contexts in which constructions occur with intensional supposition. Then we go on to prove that the rules (i) and (ii) are valid in extensional contexts.

Besides, even if reference shift is embraced, it is insufficient to let 'the *F*' denote a *Sinn* in an oblique context. If *a* believes that *the F* is a *G* then 'the *F*' denotes a *Sinn*—but *a* does not believe that some *Sinn* is a *G*. For instance, if Charles believes that the author of *Ivanhoe* is a Dutchman then Charles does not believe that the *Sinn* of 'The author of *Ivanhoe*' is a Dutchman. The advocates of reference shift need to explain how, in an oblique context, the *Sinn* of a term is to descend to an entity capable of being a Dutchman. In other words, what is needed is an account of extensionalization, or intensional descent.

The way out of the circle consists in (disambiguated) expressions denoting objects *independently of context*. In our example we say that 'The author of *Waverley*' *never* denotes the individual Sir Walter Scott; it *always* denotes the individual office that an individual must occupy to be the author of *Waverley*.

In TIL we construe this office as an ι-intension of type $\iota_{\tau\omega}$; a function from possible worlds and times to the *universe* (the set of individuals) . In a so-called 'direct' context (*oratio recta*) like

'The author of *Waverley* is a poet'

we predicate the respective property of whomever individual (if any) occupies this office in the given world/time of evaluation. Thus the truth-value of the proposition denoted by the sentence at the given $\langle w, t\rangle$ depends only on the particular individual who occupies the office at that $\langle w, t\rangle$; it is irrelevant who occupies it at worlds/times other than $\langle w, t\rangle$. In an 'oblique' context (*oratio obliqua*) we do not use the office in this manner, we just mention it, and the truth-value of the proposition is dependent on the occupancy of the office in all worlds at all times. The former case is known as using the definite description 'The author of *Waverley*' with *de re* supposition, the latter as using it with *de dicto* supposition. *Its meaning and denotation are, however, the same in both cases*.

Thus the meaning of 'The author of Waverley' is a construction of an individual office:

$$\lambda w \lambda t\ [{}^0 Author\_of_{wt}\, {}^0 Waverley] \rightarrow \iota_{\tau\omega}.$$

[108] See Section 2.6.

Types: $Author\_of/(\iota\iota)_{\tau\omega}$; $Waverley/\iota$.[109]

The meaning of 'The author of *Waverley* is a poet' is the propositional construction

$$\lambda w \lambda t\ [{}^0Poet_{wt}\ \lambda w \lambda t\ [{}^0Author\_of_{wt}\ {}^0Waverley]_{wt}] \rightarrow o_{\tau\omega}.$$

Additional type: $Poet/(o\iota)_{\tau\omega}$.

The meaning of 'Tom believes that the author of Waverley is a poet' is a construction of another proposition:

$$\lambda w \lambda t\ [{}^0Believe_{wt}\ {}^0Tom\ [\lambda w \lambda t\ [{}^0Poet_{wt}\ \lambda w \lambda t\ [{}^0Author\_of_{wt}\ {}^0Waverley]_{wt}]]] \rightarrow o_{\tau\omega},$$

(if the sentence is construed as expressing an implicit belief), or alternatively

$$\lambda w \lambda t\ [{}^0Believe*_{wt}\ {}^0Tom\ {}^{\mathbf{0}}[\lambda w \lambda t\ [{}^0Poet_{wt}\ \lambda w \lambda t\ [{}^0Author\_of_{wt}\ {}^0Waverley]_{wt}]]] \rightarrow o_{\tau\omega},$$

(if the sentence is construed as expressing an explicit belief).[110]

Additional types: $Believe/(o\iota o_{\tau\omega})_{\tau\omega}$: a relation(-in-intension) of an individual to a proposition; $Believe*/(o\iota *_n)_{\tau\omega}$: a relation(-in-intension) of an individual to a hyperproposition, i.e. a propositional construction; Tom/ι.

Finally, the meaning of 'Charles knows that Tom believes that the author of *Waverley* is a poet' is again a construction of a proposition. Implicit knowledge first:

$$\lambda w \lambda t\ [{}^0Know_{wt}\ {}^0Charles\ [\lambda w \lambda t\ [{}^0Believe_{wt}\ {}^0Tom$$
$$[\lambda w \lambda t\ [{}^0Poet_{wt}\ \lambda w \lambda t\ [{}^0Author\_of_{wt}\ {}^0Waverley]_{wt}]]]]].$$

Explicit knowledge:

$$\lambda w \lambda t\ [{}^0Know*_{wt}\ {}^0Charles\ {}^{\mathbf{0}}[\lambda w \lambda t\ [{}^0Believe_{wt}\ {}^0Tom$$
$$[\lambda w \lambda t\ [{}^0Poet_{wt}\ \lambda w \lambda t\ [{}^0Author\_of_{wt}\ {}^0Waverley]_{wt}]]]]].$$

Additional types: $Know/(o\iota o_{\tau\omega})_{\tau\omega}$: a relation(-in-intension) of an individual to a proposition; $Know*/(o\iota *_n)_{\tau\omega}$: a relation(-in-intension) of an individual to a propositional construction; $Charles/\iota$.

Our top-down approach furnishing all the expressions with a hyperintensional semantics—i.e., assigning *constructions* (of intensions) to (empirical) expressions as their meanings in all kinds of context—makes it possible to adhere to the

[109] To assign the type ι to a novel is a crass philosophical simplification, of course; here it is logically innocuous, since we are not going to draw inferences.

[110] 'Propositional' attitudes divide into relations (-in-intension) to propositions/$o_{\tau\omega}$ and propositional constructions/$*_n \rightarrow o_{\tau\omega}$. The former are often called *implicit* attitudes, the latter *explicit* attitudes. We will deal with propositional attitudes in detail in Section 5.1.

compositionality principle. In a word, compositionality is saved without resorting to contextualism.

In TIL, there is no such contextual thing as the *intension/extension of an expression*. Instead every expression either denotes an extension or an intension, independently of contextual embedding. What is dependent on context is the *supposition*, which comes in a *de dicto* and a *de re* variant. In general, *empirical expressions* denote non-constant intensions. We will rigorously define the *de dicto*/*de re* distinction in Section 2.7. Now we explicate the difference only informally.

Compare the following sentences:

(S_1) 'The President of the Czech Republic is an economist.'

(S_2) 'The President of the Czech Republic is eligible.'

First, neither sentence talks about Václav Klaus, though the office of President of the Czech Republic is currently (2010) occupied by Klaus. The individual named 'Václav Klaus' does not receive mention here. Instead, both sentences talk about the individual *office* denoted by 'The President of the Czech Republic'. The definite description 'The President of the Czech Republic' never *denotes* the individual (if any) that occupies the office; it only contingently *refers to* a particular individual. We language-users understand the expression in exactly the same way regardless of the embedding context. Moreover, we understand it even if we do not know which individual occupies the office in the actual world at time *t,* and we do understand it even with respect to such a state of affairs $\langle w, t\rangle$ at which *no* individual is occupying the office. Hence the definite description 'The President of the Czech Republic' denotes the office $PresCR/\iota_{\tau\omega}$ itself, and its meaning is a construction of that office:

$$\lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR] \rightarrow \iota_{\tau\omega}.$$

Types: $Pres\_of/(\iota\iota)_{\tau\omega}$; CR/ι.

Yet there is a substantial difference between how the meaning of 'The President of the Czech Republic' occurs in (S_1) and (S_2). The property of being an economist cannot be ascribed to an office but only to an individual. On the other hand, the property of being eligible can only be ascribed to the *office* itself. That the President is eligible means that the presidency acquires a holder by election. It would appear as though (S_1) were about, inter alia, the individual occupying the office *PresCR*, anyway. But 'The President of the Czech Republic' is used here as a pointer to an individual, so the office must be extensionalized via application to the values of *w, t* to provide an individual:

$$[\lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]]_{\boldsymbol{wt}} \rightarrow_v \iota$$

This Composition *v*-constructs relative to a world/time parameter the individual (if any) occupying the office at the given $\langle w, t\rangle$. (Remember that *denotation* is a semantic relation a priori between expressions and entities, and *reference* an extra-semantic, factual relation between expressions and world-time relative entities.)

Thus the analysis of (S_1) comes down to this construction:

(S_1') $\lambda w\lambda t\ [{}^0Economist_{wt}\ [\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]]_{wt}]$

Additional type: *Economist*/$(o\iota)_{\tau\omega}$.

Individuals can be economists, but they cannot be eligible; individual offices can. Though a particular individual, say Klaus, can be elected *for a* presidential *office*, the individual itself is not eligible. (If individuals were eligible, it would mean that one could become a particular individual by election: a fascinating thought, perhaps.) Instead, the office is currently eligible by the Czech Parliament; but the office could be hereditary, or eligible by referendum. *Eligible* is of type $(o\iota_{\tau\omega})_{\tau\omega}$, and the analysis of ($S_2$) is this:

(S_2') $\lambda w\lambda t\ [{}^0Eligible_{wt}\ [\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]]]$.

We say that the meaning of 'The President of the Czech Republic' is used with supposition *de re* in (S_1') and supposition *de dicto* in (S_2'). However, the meaning of 'The President of the Czech Republic', namely the Closure $\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]$, remains the same. Again, the shift concerns neither the meaning nor the denotation, but only the *supposition* with which the (same) meaning is used.

The proposition constructed by (S_1') takes the value **T** at those $\langle w, t\rangle$ at which the individual that occupies *PresCR* belongs to the class of individuals that instantiate the property of being an economist, and **F** if the individual does not belong to the class. It might seem that in such a state-of-affairs where there is no President of the Czech Republic the proposition should be false. (This would be the Russellian tack.) However, if it was so, the proposition that the President of the Czech Republic is *not* an economist would have to be true, which would in turn entail that there were indeed a President of the Czech republic.[111] In other words, that the President of the CR is an economist not only *entails* but also *presupposes* that the President of the CR exists. Remember that our logic is one of *partial* functions. Once a constituent—$\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}$ in our case—of a Composed construction is *v*-improper, the whole Composition is *v*-improper, and the function (here, a proposition) constructed by the respective Closure is undefined at its argument (See Definition 1.2). Therefore in those states of affairs where *PresCR* is vacant, the proposition has *no truth-value.*

On the other hand, the proposition denoted by (S_2) may be false even in the states-of-affairs lacking a President of the Czech Republic. Its truth-value does not depend on the occupancy of *PresCR* in those states-of-affairs. In particular, we

[111] See Strawson (1950).

cannot substitute a construction of the current occupant of the office. For if we could do this, we could deduce, absurdly, that Klaus is eligible.

In Section 1.1 we argued that empirical expressions rigidly *denote* intensions. Later we added that empirical expressions non-rigidly *refer* to particular values of the intensions denoted by them. However, there are expressions that never refer to an extension. For instance, when we claim that the President of the USA is eligible, we should, properly speaking, say that the *office* of President of the USA is eligible. Eligibility is a property of the office (of type $(o\iota_{\tau\omega})_{\tau\omega}$). The expression 'The office of the President of the USA' (or 'The American presidency' for short) never refers to an individual. It rigidly denotes the office itself and can be used only with *de dicto* supposition.[112] Similarly, the predicate 'is happy' denotes a property of individuals ($Happiness/(o\iota)_{\tau\omega}$) and when used (in the *de re* way) in order to be predicated of an individual it refers at $\langle w, t\rangle$ to a particular class of individuals. However, 'happiness' rigidly denotes the property *Happiness* but cannot be predicated of individuals. It can be used only in the *de dicto* way, like in the sentence 'Happiness is Charles' ultimate goal in life'. In general, the intensional semantics of TIL enables us to say that some empirical expressions like 'happiness', 'the American presidency', 'the proposition that G.W. Bush is the President of the USA', etc., which rigidly denote intensions, are *names* given to those entities by a linguistic convention. They are rigid designators *de jure* and they never non-rigidly refer to particular extensions.[113]

The *de dicto/de re* distinction can be summarized as follows:

De dicto supposition:
A construction $C_E \to \alpha_{\tau\omega}$ (and derivatively the subexpression E whose meaning C_E is) occurring in the analysis C_S of a sentence S is used with *de dicto* supposition in C_S iff the truth-value of the proposition *v*-constructed by C_S in a world w at a time t does not depend only on the particular value of the α-intension I_E *v*-constructed by C_E at *this particular* $\langle w, t\rangle$. Rather, it depends on the whole I_E. In other words, the intension I_E is a *dictum* and is *not used* to point to a value.

De re supposition:
There is *de re* supposition when the reference of E (namely, the α-value, the *res*, *v*-constructed by C_{Ewt}) of the denoted α-intension I_E comes into play. The truth-value of the proposition denoted by S in a world w at a time t depends on the value of the α-intension I_E denoted by E at *this particular* $\langle w, t\rangle$, while the values of I_E at other $\langle w', t'\rangle$ are irrelevant.

This preliminary characterization could serve almost as a definition, though not quite. According to it, the sentence S alone would be in *de re* supposition in itself, which is not so. The sentence talks about (denotes) the whole *dictum*—a

[112] More precisely, its meaning occurs always intensionally, see Section 2.6.2, in particular Definition 2.20.

[113] See Zouhar (2009), where he deals with the Kripkean distinction between rigid designators *de jure* and *de facto*.

proposition—and never its reference (*res*)—its truth-value in the actual world-time. The sentences (S_1), (S_2), and the constructions (S_1'), (S_2'), respectively, occur with *de dicto* supposition in themselves.

Note that in a compound sentence particular clauses may occur with *de re* as well as with *de dicto* supposition. Consider the following example:

(S_3) 'If the President of the Czech Republic is a playwright then the President of the Czech Republic is Václav Havel.'

An analysis of the antecedent and consequent sentences yields the following propositional constructions, respectively:

(Ca) $\lambda w \lambda t\ [{}^0 Playwright_{wt}\ \lambda w \lambda t\ [{}^0 Pres\_of_{wt}\ {}^0 CR]_{wt}]$

(Cb) $\lambda w \lambda t\ [\lambda w \lambda t\ [{}^0 Pres\_of_{wt}\ {}^0 CR]_{wt} = {}^0 Havel]$.

Additional types: $Playwright/(o\iota)_{\tau\omega}$; $Havel/\iota$.

However, the propositional connective '$\supset$' (implication) denotes a truth-function of type (ooo); it must be applied to truth-values and cannot be applied to propositions. Thus the propositions constructed by (Ca), (Cb) have to undergo intensional descent, and the truth-value (in w at t) of the proposition denoted by (S_3) does depend on the truth-values of these propositions at the same particular $\langle w, t \rangle$:

(S_3') $\lambda w \lambda t\ [{}^0 \supset [\lambda w \lambda t\ [{}^0 Playwright_{wt}\ \lambda w \lambda t\ [{}^0 Pres\_of_{wt}\ {}^0 CR]_{wt}]]_{wt}$
$[\lambda w \lambda t\ [\lambda w \lambda t\ [{}^0 Pres\_of_{wt}\ {}^0 CR]_{wt} = {}^0 Havel]]_{wt}]$.

Both sentences and their meanings (Ca), (Cb) occur with supposition *de re* in (S_3), (S_3'), respectively. Again, at those $\langle w, t \rangle$ at which *PresCR* is vacant, the sentence (S_3) does not have a truth-value. The fact is even more evident if we consider the β-reduced construction ($S_{3\beta}$) equivalent to (S_3'):

($S_{3\beta}$) $\lambda w \lambda t\ [[{}^0 Playwright_{wt}\ [{}^0 Pres\_of_{wt}\ {}^0 CR]] \supset [[{}^0 Pres\_of_{wt}\ {}^0 CR] = {}^0 Havel]]$.

At those worlds and times where the Presidency is vacant, the construction $[{}^0 Pres\_of_{wt}\ {}^0 CR]$ fails to construct an occupant of *PresCR*. Due to the definition of Composition, both Composed subconstructions of ($S_{3\beta}$), namely $[{}^0 Playwright_{wt}\ [{}^0 Pres\_of_{wt}\ {}^0 CR]]$ and $[[{}^0 Pres\_of_{wt}\ {}^0 CR] = {}^0 Havel]$, are also v-improper. Thus the construction of the implication function $\supset$ does not receive an argument to work on, and it also fails to v-construct a truth-value. The proposition constructed by (S_3') is undefined for those worlds and times at which the Presidency goes vacant. This is so because (S_3) comes with an *existential presupposition*: for (S_3) to take a truth-value at a given $\langle w, t \rangle$, the President of the Czech Republic has to exist at that $\langle w, t \rangle$. Again, (S_3) not only entails but also presupposes the existence of the President of the Czech Republic.

Remark. This kind of a β-reduction has been called in Duží (2003a, b, 2004) *β_i-reduction* ('i' meaning 'innocuous'). It consists simply in substituting variables for variables (of the same type), in our case *w*, *t* for *w*, *t.* Since a variable can never be *v*-improper, such a reduction is always an equivalent transformation. In this sense it is 'innocuous'. However, in a logic of partial functions like TIL it must be taken into account that a simple 'syntactic version' of the β-reduction rule is generally *not* valid. We will deal with the problem in Section 2.7.

1.5.2.1 Two principles *de re*

Existential presupposition is a special case of presupposition. For instance, the sentence 'Charles stopped smoking' not only entails that Charles previously smoked, but also presupposes it. One cannot stop doing something that one has not previously done. Strawson's test makes this clear. Being asked whether you stopped smoking, you are not entitled to give a Yes/No answer unless you previously smoked.

To define the notion of presupposition, we make use of the three propositional properties *True*, *False*, and *Undef*, all of type $(o o_{\tau\omega})_{\tau\omega}$. They are defined as follows.[114] Let *P* be a propositional construction ($P/{*}_n \rightarrow o_{\tau\omega}$). Then

$[{}^0True_{wt}\,P]$ *v*-constructs the truth-value **T** iff P_{wt} *v*-constructs **T**, otherwise **F**.

$[{}^0False_{wt}\,P]$ *v*-constructs the truth-value **T** iff $[\neg P_{wt}]$ *v*-constructs **T**, otherwise **F**.

$[{}^0Undef_{wt}\,P]$ *v*-constructs the truth-value **T** iff $[[\neg[{}^0True_{wt}\,P]] \wedge [[\neg{}^0False_{wt}\,P]]]$ *v*-constructs **T**, otherwise **F**.

Hence $[{}^0Undef_{wt}\,P] = [[\neg{}^0True_{wt}\,P] \wedge [\neg{}^0False_{wt}\,P]]$.

Note that, e.g., $[\neg[{}^0True_{wt}\,P]]$ is not equivalent to $[{}^0False_{wt}\,P]$, though our logic is bivalent. We do not work with a third truth-value. If $[{}^0Undef_{wt}\,P]$ *v*-constructs **T**, then P_{wt} is *v*-improper, and the proposition P constructed by *P* does not have *any* truth-value at $\langle w, t\rangle$.[115]

Now we define:

Definition 1.14 (*presupposition*) Let *P*, $Q \rightarrow o_{\tau\omega}$ be constructions constructing propositions P, Q. Then Q is a *presupposition* of P iff the truth of Q at $\langle w, t\rangle$ is a necessary condition for P having a truth-value at $\langle w, t\rangle$:

$$\forall w \forall t\, [[{}^0True_{wt}\,P] \vee [{}^0False_{wt}\,P]] \supset [{}^0True_{wt}\,Q]].$$

[114] Now we use this convention: '*P*' for a construction of a proposition, 'P' for the proposition *v*-constructed by *P.*

[115] Cf. Table 1.1: truth-value matrix, Section 1.4.3.

Corollary. Q is a *presupposition* of P iff *Q* is entailed both by *P* and *non-P*. If Q is not true at $\langle w, t\rangle$, then P is undefined at $\langle w, t\rangle$:

$$\forall w \forall t\ [\neg[{}^0True_{wt}\ Q] \supset [{}^0Undef_{wt}\ P]].$$

One should not confuse the notion of presupposition with the notion of *commitment*, for the latter is weaker than the former. In order to exactly determine the difference, we recall the definition of the entailment relation. Let *P, Q* be propositional constructions as above. Then the *P entails Q* ($P \models Q$) iff

$$\forall w \forall t\ [[{}^0True_{wt}\ P] \supset [{}^0True_{wt}\ Q]].$$

We will often use the notation '$(P \models Q)$' instead of '$[{}^0\!\models {}^0P\ {}^0Q]$'. Note that *P*, *Q* must be Trivialized, since these very constructions, rather than the propositions they construct, are the arguments of $\models$.

Schematically, the difference between presupposition and commitment is this. Let *non-P* be a propositional construction of the form $\lambda w \lambda t\ [\neg P_{wt}]$. Then

(i) Q is a *presupposition* of P iff $(P \models Q)$ and $(non\text{-}P \models Q)$
(ii) Q is a *commitment* of P iff $(P \models Q)$ and neither $(non\text{-}P \models Q)$ nor $(non\text{-}P \models non\text{-}Q)$

An example of commitment would be, for instance:

'Ground zero was visited by the Pope in April of 2008.'

The sentence is multiply ambiguous. The ambiguity concerns the supposition with which the definite descriptions 'ground zero' and 'the Pope' occur, where 'ground zero' goes short for 'the ground zero in New York City'.[116] On one reading both occur with *de re* supposition. In such a case the sentence presupposes that both ground zero and the Pope exist *now.* Yet there are other readings. Among them is the reading on which 'ground zero' occurs *de re* and 'the Pope' occurs *de dicto* with respect to the temporal parameter.[117] In such a case the sentence presupposes the existence of ground zero, but not of the Pope now. It only entails that the Pope *existed* in April 2008. Hence, if it were true that

'Ground zero was *not* visited by the Pope in April 2008',

one could *not* deduce that the Pope exists now or existed in April, 2008. That ground zero was not visited by the Pope in April of 2008 might have been either because the office of Pope was vacant at the time or that the Pope did exist but its

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[116] It is interesting to note that '[the] ground zero [of New York City]' has now been elevated to the status of proper name, which requires capitalizing both words, as in 'Ground Zero'. Many sites are ground zero, but only one is Ground Zero, relative to the status that current American English has bestowed upon 'Ground Zero'. In journalese 'Ground Zero' refers to one particular ground zero. So if the Pope visits the NYC ground zero then the New York Times et al. are likely to write 'The Pope to visit Ground Zero'.

[117] We will deal with temporal *de dicto* vs. *de re* cases in Section 2.5.2.3.

occupant was not among the visitors of ground zero. This goes to show why commitment is weaker than presupposition.

We are now able to formulate *the first principle de re*:

> ***Principle of existential presupposition.*** If a construction C of an α-office $C/\alpha_{\tau\omega}$ occurs with *de re* supposition in the propositional construction P, then the proposition constructed by P has the *presupposition* that C exist (that the α-office C be occupied): $\lambda w\lambda t\, [{}^0Exist_{wt}\, C]$, $Exist/(o\iota_{\tau\omega})_{\tau\omega}$.

The office of President of the Czech Republic is certainly a *properly* partial function: there are worlds/times at which the President of the Czech Republic does not exist; for instance, in the actual world and at all times before 1993. However, if $[{}^0 =$ is true, then so is the proposition that the President of the Czech Republic exists. In Section 2.3 we show that existence can be analysed as a property of intensions, in this case of individual offices, $Exist/(o\iota_{\tau\omega})_{\tau\omega}$. Hence the following argument is valid:

$$\frac{\lambda w\lambda t\, [{}^0Economist_{wt}\, \lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0CR]_{wt}]}{\lambda w\lambda t\, [{}^0Exist_{wt}\, \lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0CR]].}$$

Similarly, the President of the Czech Republic *not* being an economist entails the existence of the President of the Czech Republic.

Since the property of existence (in the sense of occupancy of an office) can be defined by means of the existential quantifier ($x \to \iota$; $r \to \iota_{\tau\omega}$; $=_\iota/(o\iota\iota)$),

$$\lambda w\lambda t\, \lambda r\, [{}^0\exists\, \lambda x\, [{}^0=_\iota\, x\, r_{wt}]],$$

the conclusion can be equivalently expressed by the construction

$$\lambda w\lambda t\, [{}^0\exists\, \lambda x\, [{}^0=_\iota\, x\, \lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0CR]_{wt}]].$$

Valid logical forms of the arguments are thus easily obtained by existential generalisation:

$$\frac{\lambda w\lambda t\, [P_{wt}\, r_{wt}]}{\lambda w\lambda t\, [{}^0\exists\, \lambda x\, [{}^0=_\iota\, x\, r_{wt}]]} \qquad \frac{\lambda w\lambda t\, \neg[P_{wt}\, r_{wt}]}{\lambda w\lambda t\, [{}^0\exists\, \lambda x\, [{}^0=_\iota\, x\, r_{wt}]].}$$

Additional type: ($P \to (o\iota)_{\tau\omega}$):[118]

Of course, if the proposition constructed by the premise takes value **T** at $\langle w, t\rangle$ then the individual occupying at $\langle w, t\rangle$ the office constructed by $\lambda w\lambda t\, [{}^0Pres\_of_{wt}\, {}^0CR]$

[118] See Section 1.5.1 for details on the notion of logical form.

belongs to the class v-constructed by $^0Economist_{wt}$. Hence the office of President must be occupied at $\langle w, t\rangle$, and the conclusion is true at $\langle w, t\rangle$. In other words, the argument *is truth-preserving* from premises to conclusion.

However, due to partiality, a valid argument may fail to be *falsity-preserving* from conclusion to premises.[119] If at $\langle w, t\rangle$ the conclusion is false, then it does not mean that at least one of the premises is false at $\langle w, t\rangle$. For, if the office is not occupied at a particular world W and a particular time T, then the construction $\lambda w\lambda t$ $[^0Pres\_of_{wt}\ ^0CR]_{wt}$ is *v-improper* for the valuation assigning W to w and T to t. Therefore, the Composition in which the construction of the office occurs *de re,* namely

$$[^0Economist_{wt}\ \lambda w\lambda t\ [^0Pres\_of_{wt}\ ^0CR]_{wt}]$$

is also *v(W/w, T/t)*-improper, and the proposition P constructed by the premise has *no truth-value* at this $\langle W, T\rangle$. It is neither true nor false, because in the absence of a President at $\langle W, T\rangle$ there is no fact of the matter as to whether the President is an economist at $\langle W, T\rangle$. The proposition P is a properly partial function, because it has truth-value gaps. In order that P have a truth-value, the President of the Czech Republic has to exist; P comes with an *existential presupposition*.

Remember that we do not introduce a third truth-value in order to handle partiality. Thus we do not follow Muskens' theory of *partial possible worlds* or Barwise and Perry's *situation semantics*, nor do we introduce partiality whenever it might seem to be technically convenient.[120] TIL is a Platonist semantics, ideally aiming at cutting reality at its joints, as the saying goes. Propositions simply *are* true, false or neither, independently of our 'allowing' them to be so, and they are never both true and false. (There is no room for paraconsistent truth-value gluts in TIL.)

For example, Muskens (1995, pp. 42–50) introduces four combinations of truth-values: **T** = 'true and not false', **F** = 'false and not true', **N** = 'neither true nor false' and **B** = 'both true and false', in order to handle synonymy in terms of co-entailment. In Muskens' partial logic, the sentences

(1) 'John walks'

and

(2) 'John walks and Bill talks or does not talk'

are not equivalent, though 'Bill talks or does not talk' is a classical tautology and as such denotes the necessary proposition true in all possible worlds. According to Muskens, the reason is because in a situation where Mary sees John but not Bill, the sentence 'Mary sees John walk' can be true or false, unlike the sentence 'Mary

[119] A valid argument need not be truth-preserving from conclusion back up to its premises, either; namely, if the argument is unsound.

[120] See Muskens (1995), Barwise and Perry (1983).

sees John walk and Bill talk or not talk' which is undefined in a situation where Mary does not see Bill. Thus (2) does not follow from (1).[121]

We disagree on this point. If Mary does not see Bill at all, then, of course, she cannot see him talk or doing anything else, which does *not mean* (*contra* Muskens) that '[T]he sentence "Bill talks" will be undefined, that is, neither true nor false, in the part of the world that is seen by her'. Nor does it mean that as a consequence the sentence 'Bill talks or doesn't talk' and 'John walks and Bill talks or doesn't talk' are both undefined in that situation as well. Sentences (1) and (2) *are* equivalent (as they denote the same proposition), and the sentence 'Bill talks or does not talk' is a tautology, independently of whether Mary knows it.[122] Note that Muskens uses classical entailment to argue that (2) does not follow from (1). But (2) does follow from (1), independently of Mary's cognitive abilities and *independently of situations*. And (1) and (2) are true or false, dependently on states-of-affairs, but independently of Mary's seeing that they are. There is no reason to introduce partiality here.

According to Muskens, co-entailment in a partial theory will be a better approximation to synonymy than classical co-entailment is. In our opinion, Muskens is in effect modelling our cognitive abilities, and his theory can be treated as a cognitive theory. The new 'truth-values' he introduces, namely **N** and **B**, are actually not (objective) truth-values of propositions, but, say, subjective degrees of *knowledge* in a particular situation. We can even introduce infinitely many such 'truth-values', for instance, an interval between 0 and 1, to map 'degrees of preciseness of measurement', or 'degrees of our conviction in the truth', or any other (subjective) degrees, and build up fuzzy logics, etc. We can even introduce new (objectively correct) *inference rules* within our logic that would better map the relation of logical consequence. Still, the relation of co-entailment, or co-denotation, will always be just an *approximation* to synonymy, and a counter-example could always be found. Notoriously well-known ones are attitudinal sentences (see Chapter 5). No intensional semantics can properly handle synonymy, because its finest individuation is equivalence. We need a hyperintensional semantics to properly handle synonymy[123] and to construe meaning as an algorithmically structured *procedure.*

Now we are going to explain the second principle *de re*, namely the principle of substitution of co-*referential* expressions. First, what does it mean that the truth-value at $\langle w, t\rangle$ of a proposition *depends* on the value of another intension? Consider again the sentence

(S_1) 'The President of the Czech Republic is an economist'

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[121] See Muskens (1995, pp. 1–3).

[122] The semantics of proper names is simplified here, allowing 'Bill' to be simply a label of an individual. See, however, Section 3.2. Moreover, on the TIL conception, there are no non-existing individuals: we work with a constant domain of individuals.

[123] For the definition of synonymy, see Section 2.2, Definition 2.10.

and its analysis:

(S_1') $\lambda w \lambda t\, [{}^0Economist_{wt}\ \lambda w \lambda t\, [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]$.

'The President of the Czech Republic' occurs *de re* in (S_1), as does the occurrence of the construction $\lambda w \lambda t\, [{}^0Pres\_of_{wt}\ {}^0CR]$ in (S_1'). If the President is Václav Klaus, then (S_1) and this additional premise entail that Václav Klaus is an economist; hence the following argument is valid:

$$\frac{\begin{array}{c}\lambda w \lambda t\, [{}^0Economist_{wt}\ \lambda w \lambda t\, [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}] \\ \lambda w \lambda t\, [{}^0{=}\ \lambda w \lambda t\, [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}\ {}^0Klaus]\end{array}}{\lambda w \lambda t\, [{}^0Economist_{wt}\ {}^0Klaus].}$$

Similarly, if the President is the husband of Livie Klausová then (S_1) and this additional premise entail that the husband of Livie Klausová is an economist ($Husband\_of/(\iota\iota)_{\tau\omega}$):

$$\frac{\begin{array}{c}\lambda w \lambda t\, [{}^0Economist_{wt}\ \lambda w \lambda t\, [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}] \\ \lambda w \lambda t\, [{}^0{=}\ \lambda w \lambda t\, [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}\ \lambda w \lambda t\, [{}^0Husband\_of_{wt}\ {}^0Livie]_{wt}]\end{array}}{\lambda w \lambda t\, [{}^0Economist_{wt}\ \lambda w \lambda t\, [{}^0Husband\text{-}of_{wt}\ {}^0Livie]_{wt}].}$$

This is no surprise, of course, because Leibniz's law of substitution law is uncontroversially valid in these cases, and the following is the schema of a valid argument:

$$\frac{\begin{array}{c}\lambda w \lambda t\, [\dots\ C\ \dots] \\ \lambda w \lambda t\, [{}^0{=}\ \ C\ D]\end{array}}{\lambda w \lambda t\, [\dots.\ D\dots].}$$

The principle of substitution of co-*referential* expressions is an instance of Leibniz's Law.

Tichý formulates the principle as follows.

> Let 'X', 'Y' denote individual offices. Let '…Y…' be a sentence arising from sentence '…X…' by putting the term 'Y' for some *de re* occurrences of 'X' in '…X…'. Then the argument
>
> $$\frac{\begin{array}{l}X \text{ at } \langle W, T\rangle \text{ is } Y \text{ at } \langle W, T\rangle \\ \dots X \text{ at } \langle W, T\rangle\dots\end{array}}{\dots Y \text{ at } \langle W, T\rangle\dots}$$
>
> is valid. (1978a, p. 9, 2004, p. 257).

The rationale behind the substitution is that what is predicated of the occupant of X at $\langle w, t\rangle$ is what is predicated of the occupant of Y at $\langle w, t\rangle$ on condition of co-occupation of X and Y at $\langle w, t\rangle$. That is, even though '…X at $\langle w, t\rangle$…' and '…Y at

$\langle w, t\rangle$…' may have different truth-conditions, their truth-values coincide at every $\langle w, t\rangle$ at which 'X at $\langle w, t\rangle$ is Y at $\langle w, t\rangle$' expresses a truth.

Hence *the second principle de re* is the following:

Principle of substitution of co-referential expressions. If an expression E occurs in a sentence S with *de re* supposition, then the substitution (*salva veritate*) of a co-referential expression E' for the occurrence of E in S is valid.

The corresponding rule of substitution *de re* is then:

Rule of substitution of v-congruent constructions. Let $C \to \alpha_{\tau\omega}$, $D \to \alpha_{\tau\omega}$ and let C_{wt}, D_{wt} be v-congruent constructions (i.e., $C_{wt} = D_{wt}$) and let $S(D/C)$ be a construction that arises from S by substituting D for one or more *de re* occurrences of C in S. Then S_{wt} and $S(D/C)_{wt}$ are v-congruent as well (i.e., $S_{wt} = S(D/C)_{wt}$).

For another example, the denoted office can be a second-degree office (an office of an individual office), like, for instance *the highest executive office of the USA*. The following argument is valid:

The highest executive office of the USA is the President, not the King
The highest executive office of the USA is the most respectable office in the USA

The most respectable office of the USA is the President, not the King.

Type-theoretical analysis: $HEO/(\iota_{\tau\omega})_{\tau\omega}$: the highest executive office of the USA; $MRO/(\iota_{\tau\omega})_{\tau\omega}$: the most respectable office of the USA; $PresUSA$, $KingUSA/\iota_{\tau\omega}$; $=_{\iota\tau\omega}/(o\iota_{\tau\omega}\iota_{\tau\omega})$.

Synthesis:

$$\frac{\lambda w\lambda t\,[[{}^0{=}_{\iota\tau\omega}\,{}^0HEO_{wt}\,{}^0PresUSA] \wedge [\neg[{}^0{=}_{\iota\tau\omega}\,{}^0HEO_{wt}\,{}^0KingUSA]]] \qquad \lambda w\lambda t\,[{}^0{=}_{\iota\tau\omega}\,{}^0HEO_{wt}\,{}^0MRO_{wt}]}{\lambda w\lambda t\,[[{}^0{=}_{\iota\tau\omega}\,{}^0MRO_{wt}\,{}^0PresUSA] \wedge [\neg[{}^0{=}_{\iota\tau\omega}\,{}^0MRO_{wt}\,{}^0KingUSA]]].}$$

Since ${}^0HEO$ and ${}^0MRO$ occur with *de re* supposition in the premises (unlike the constituents ${}^0PresUSA$, ${}^0KingUSA$), the substitution *salva veritate* is valid.

A classical puzzle from around 1970 due to Barbara Partee can also be resolved by sorting out the interplay between *de dicto* and *de re* supposition.[124] Partee's puzzle is this:

[124] For discussion, see Yagisawa (2001), Moschovakis (2006, p. 43), and Partee (2005, p. 43).

The temperature is 90°F
The temperature is rising

90°F is rising.

The argument seems at first blush to invite a smooth substitution of '90°F' for 'the temperature' in the context '…is rising…' by Leibniz's Law. Yet the conclusion is indisputably either false or nonsensical. Partee did intend, however, to come up with a flawed argument to make a particular point within a particular discussion at the time to do with so-called intensional positions for singular terms to occur in, such that these positions would be distinct from (overtly) modal contexts. And her argument obviously is flawed. The challenge that her argument presents is to construct a logical analysis that will block the inference. Here is how we go about this.

As always, we begin with a type-theoretical analysis of the objects mentioned by the premises: *Temperature*/$\tau_{\tau\omega}$: a magnitude;[125] *Rising*/$(o\tau_{\tau\omega})_{\tau\omega}$: a property of a magnitude; $=_\tau/(o\tau\tau)$; $90/\tau$.

(P_1) $\lambda w\lambda t\ [{}^0{=_\tau}\ {}^0Temperature_{wt}\ {}^090]$

(P_2) $\lambda w\lambda t\ [{}^0Rising_{wt}\ {}^0Temperature]$

The diagnosis of the invalidity of the argument is now straightforward. The Trivialization ^{0}*Temperature* occurs *de re* in (P_1), but *de dicto* in (P_2). In other words, the object of predication in (P_2) is the entire function *Temperature* rather than its particular value. So the substitution of the construction ^{090} for ^{0}*Temperature* into (P2) would be invalid.

1.5.2.2 Interplay between *de dicto* and *de re*

Consider now another sentence:

(S_4) 'If the President of the Czech Republic is a playwright then Charles believes that the President of the Czech Republic is Václav Havel.'

An adequate analysis of the consequent has to respect the fact that Charles can believe that the President is Václav Havel even if the President is instead Václav Klaus, or even if the President does not exist. Charles may simply not be up on

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[125] It is understood that the temperature is not just any temperature (of something), but a particular temperature, and most likely the temperature at the location of whoever says the temperature is rising.

Czech public affairs. Thus the meaning of the clause expressed by the consequent is ($Believe/(o\iota o_{\tau\omega})_{\tau\omega}$):[126]

(S_{4emb})
$\lambda w\lambda t\ [{}^0Believe_{wt}\ {}^0Charles\ \lambda w_1\lambda t_1\ [\lambda w_2\lambda t_2\ [{}^0Pres\_of_{w2t2}\ {}^0CR]_{w1t1} = {}^0Havel]]$.

The Closure $\lambda w_1\lambda t_1\ [\lambda w_2\lambda t_2\ [{}^0Pres\_of_{w2t2}\ {}^0CR]_{w1t1} = {}^0Havel]$ occurs *de dicto* in (S_{4emb}). Also the Closure $\lambda w_2\lambda t_2\ [{}^0Pres\_of_{w2t2}\ {}^0CR]$ occurs *de dicto* in (S_{4emb}), even though it is Composed with w_1, t_1, which triggers intensional descent of the office *PresCR*. The truth-value of the proposition constructed by (S_{4emb}) at a particular $\langle w, t\rangle$ may well depend on *PresCR* being occupied at worlds other than w or at times other than t.

The sentence (S_4) expresses the construction:

(S_4') $\lambda w\lambda t\ [{}^0\supset [\lambda w\lambda t\ [{}^0Playwright_{wt}\ \lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]]_{wt}$
$[\lambda w\lambda t\ [{}^0Believe_{wt}\ {}^0Charles\ [\lambda w\lambda t\ [\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt} = {}^0Havel]]]]_{wt}]$.

The construction $\lambda w\lambda t\ [{}^0Playwright_{wt}\ \lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt}]$ is used with *de re* supposition in (S_4'), and so is the first occurrence of $\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]$. The construction $\lambda w\lambda t\ [\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt} = {}^0Havel]$ is used with *de dicto* supposition in (S_4'), and so is the second occurrence of $\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]$.

This goes to show that the *de dicto* context is *dominant* over the *de re* context. In the Closure $\lambda w\lambda t\ [\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt} = {}^0Havel]$ the construction of the presidency, viz. $\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]$, occurs with *de re* supposition, such that the individual value of the office at a given $\langle w, t\rangle$-pair of evaluation is the object of predication, whereby the values of the office at $\langle w', t'\rangle$-pairs other than the $\langle w, t\rangle$-pair of evaluation become irrelevant. By contrast, the occurrence of the Closure $\lambda w_2\lambda t_2\ [{}^0Pres\_of_{w2t2}\ {}^0CR]$ in (S_{4emb}), as well as the second occurrence of the Closure $\lambda w\lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]$ in (S_4'), is intensional, i.e. with *de dicto* supposition. This is so, because in (S_{4emb}) the whole proposition that the President of the Czech Republic is Havel is the object of predication. Thus it is not so that the individual values of the presidency at $\langle w', t'\rangle$-pairs other than the $\langle w, t\rangle$-pair of evaluation are irrelevant.

Tichý sums it up thus:

> In general, a *de re* constituent of D is a *de re* constituent of any application in which D appears as a *de re* constituent; a *de re* constituent of D is a *de dicto* constituent of any application in which D appears as a *de dicto* constituent. A *de dicto* constituent is a *de dicto* constituent of any application in which D appears as a (*de re* or *de dicto*) constituent. Briefly, *de dicto* is the dominant one of the two suppositions (1988, p. 217).

Examples of sentences with 'the *F*' occurring with *de re* supposition:

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[126] We conceive of *believing* as a relation-in-intension between an individual and a proposition here, making believing an implicit attitude. See, however, Chapter 5. In order to mark the scope of particular λ-bindings of variables *w* and *t* we use numerical subscripts here.

- simple sentences: 'The *F* is a *G'*.
- modalities: 'The *F* is necessarily a *G'*.
- attitudes: 'The *F* is believed by Charles to be a *G*'.

Modalities will be resumed in Chapter 4 and attitudes in Chapter 5.

Simple sentences of the form 'The *F* is a *G'* as dealt with above are, however, ambiguous between *de re* and *de dicto* readings. Consider, for instance, the sentence

'Kurt Gödel's most favourite argument is analytically valid.'

On its *de re* reading the sentence has the existential presupposition that there be exactly one argument that is Gödel's favourite. If Gödel favoured more arguments to the same degree or if he had no one favourite argument, the sentence would have no truth-value. The reading *de dicto* mentions a necessary condition to be satisfied by an argument in order to qualify as Gödel's favourite argument. The *de dicto* reading can be loosely paraphrased as

'Being analytically valid is indispensable for an argument to be
Gödel's most favourite one.'

The truth-condition of this sentence does not require that Gödel have a favourite argument.

Types of the objects mentioned by the sentence:
Argument/$*_n$: a hyperproposition (a construction of a proposition);[127]
Gödel's favourite argument/$*_{n\tau\omega}$: a constructional office (an office occupiable by constructions of order *n*);
Favour_arg_of/$((o*_n)\iota)_{\tau\omega}$: an empirical function assigning a set of arguments to an individual;
Most/$(*_n\ (o*_n))_{\tau\omega}$: an empirical function associating a set of arguments with an argument, the most favourite one;
Analytical/$(o*_n)$: the class of analytically valid arguments;
Indispensable/$(o(o*_n)*_{n\tau\omega})_{\tau\omega}$: a relation (-in-intension) between a class of arguments and a constructional office.

Now the Closure $\lambda w\lambda t\ [{}^0Most_{wt}\ [{}^0Favour\_arg\_of_{wt}\ {}^0\text{Gödel}]] \rightarrow *_{n\tau\omega}$ constructs the constructional office, and we have:

(a) *de re* reading:

$$\lambda w\lambda t\ [{}^0Analytical\ \lambda w\lambda t\ [{}^0Most_{wt}\ [{}^0Favour\_of_{wt}\ {}^0G\ddot{o}del]]_{wt}]$$

(b) *de dicto* reading (rephrased):

$$\lambda w\lambda t\ [{}^0Indispensable_{wt}\ {}^0Analytical\ \lambda w\lambda t\ [{}^0Most_{wt}\ [{}^0Favour\_of_{wt}\ {}^0G\ddot{o}del]]].$$

[127] For details on *arguments*, see Sections 1.5.1 and 5.4.

Let $Occ^*/(o*_{n\tau\omega})_{\tau\omega}$ be the property of a constructional office of being occupied. The relation of being indispensable can be defined as follows:

$$[{}^0Indispensable_{wt}\ C\ H] = [[{}^0Occ^*_{wt}\ H] \supset [{}^0True_{wt}\ \lambda w\lambda t\ [C\ H_{wt}]]].$$

Types: $C \rightarrow (o*_n)$, $H \rightarrow *_{n\tau\omega}$.

Finally, using this refinement, the *de dicto* reading of the sentence expresses the construction:

(c) *de dicto* reading:

$$\lambda w\lambda t\ [[{}^0Occ^*_{wt}\ \lambda w\lambda t\ [{}^0Most_{wt}\ [{}^0Favour\_of_{wt}\ {}^0G\ddot{o}del]]] \supset$$
$$[{}^0True_{wt}\ \lambda w\lambda t\ [{}^0Analytical\ \lambda w\lambda t\ [{}^0Most_{wt}\ [{}^0Favour\_of_{wt}\ {}^0G\ddot{o}del]]_{wt}]]].$$

Another example of the ambivalence of simple sentences of the form ‘The *F* is a *G’* is the sentence

‘The King of France is a king.’

On its *de re* reading it expresses the construction ($King/(o\iota)_{\tau\omega}$; $King\_of/(\iota\iota)_{\tau\omega}$; $France/\iota$)

$$\lambda w\lambda t\ [{}^0King_{wt}\ \ \lambda w\lambda t\ [{}^0King\_of_{wt}\ {}^0France]_{wt}],$$

β-reducible to

$$\lambda w\lambda t\ [{}^0King_{wt}\ [{}^0King\_of_{wt}\ {}^0France]],$$

both of which construct a proposition that has no truth-value in the actual world now (as well as in any of the world/time at which the King of France does not exist). The *de re* reading of the sentence comes with the existential presupposition that the King of France exist. In those worlds/times at which the King of France exists, the proposition is true. Hence, on its *de re* reading the sentence does not express an analytically true proposition, though one that *almost* is. It does not denote the proposition *TRUE*, but a properly partial proposition that is true at some $\langle w, t\rangle$, and undefined at all the rest (hence nowhere and never false).

On its *de dicto* reading the sentence rather expresses a necessary relation between the property of being a king and the office of King of France. Necessarily, whenever somebody or other occupies the office of King of France, that individual is a king. (Or in plain English, if you are the king of something, then you are a king.) We call such a relation between intensions a *requisite*. Here the property of being a king is a requisite of the office of King of France, such that every occupant must have the relevant property. Thus the analysis of the *de dicto* reading of the above sentence is

$$[{}^0Requisite\ {}^0King\ \lambda w\lambda t\ [{}^0King\_of_{wt}\ {}^0France]].$$

Additional type: $Requisite/(o(o\iota)_{\tau\omega}\iota_{\tau\omega})$.

Each office may have indefinitely many such requisites. For instance, the office of President of the USA has the properties of being born in the United States, being above 35 years of age, etc., as its requisites. The set of all the requisites of an office is called its *essence,* and the office is fully characterised by its essence.[128]

A broader problem arises when we consider the *context* in which a particular construction occurs. We tackled the problem above, when we analysed the sentence (S_4) and concluded that the *de dicto* context is the dominant one of the two suppositions.

Now we are going to show that there are *three* contexts: hyperintensional (constructional), intensional (*de dicto*) and extensional (*de re*). Of these three the hyperintensional context is dominant over both the intensional and the extensional context, and the intensional context is dominant over the extensional context.

Consider again the sentence

(S_4) 'If the President of the Czech Republic is a playwright then Charles believes that the President of the Czech Republic is Václav Havel.'

Above we analysed Charles's belief as a relation-in-intension of an individual to a proposition. However, an alternative belief relation is an option. When belief is explicit belief, the believer enters into a relation-in-intension to a hyperproposition. Where $Believe^*/(o\iota *_n)_{\tau\omega}$, we have:[129]

(S_{4emb*}) $\lambda w \lambda t\ [{}^0Believe^*_{wt}\ {}^0Charles\ {}^0[\lambda w \lambda t\ [\lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt} = {}^0Havel]]]$.

Now it no longer holds that the Closure $\lambda w \lambda t\ [\lambda w \lambda t\ [{}^0Pres\_of_{wt}\ {}^0CR]_{wt} = {}^0Havel]$ is used with *de dicto* supposition in (S_{4emb*}), because it is *not used as a constituent* of (S_{4emb*}). It is *mentioned* here. Moreover, its constituents are *mentioned* in (S_{4emb*}) as well.

For this reason we must distinguish between *using* a construction as a constituent of another construction and *mentioning* a construction. If a construction is used as a constituent, it can be used in two different ways: intensionally or extensionally. The three kinds of context are as follows:[130]

- *Hyperintensional context*: the sort of context in which a construction is not used to *v*-construct an object. Instead, the *construction* itself is an argument of another function; the construction is just *mentioned.*

Example: 'Charles calculates 2+5' expresses as its meaning the Closure

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[128] For more on requisites and essence, see Chapter 4.

[129] See Chapter 5 for details on propositional attitudes.

[130] Here we only briefly characterize the three contexts. Precise definitions will be provided in Section 2.6. Note that the notions 'intensional' and 'extensional' are used here in a broader sense than in possible-world semantics. To distinguish these notions from possible-world intension and extension, we will often add the asterisk '*' when talking about (hyper-) intensional/extensional occurrence of a construction.

$$\lambda w \lambda t\ [{}^0\mathit{Calculate}_{wt}\ {}^0\mathit{Charles}\ {}^0[{}^0\!+\ {}^02\ {}^05]].$$

The Composition $[{}^0\!+\ {}^02\ {}^05]/*_1$ is not used to construct the number 7 here. Instead, it is an argument of the function $\mathit{Calculate}/(o\iota *_1)_{\tau\omega}$. Thus $[{}^0\!+\ {}^02\ {}^05]$ occurs in the *hyperintensional* context of $\lambda w \lambda t\ [{}^0\mathit{Calculate}_{wt}\ {}^0\mathit{Charles}\ {}^0[{}^0\!+\ {}^02\ {}^05]]$.

- *Intensional context*: the sort of context in which a construction is used to *v*-construct a function and *not a particular value* of the function. Moreover, the construction does not occur within another hyperintensional context.

Example: 'Sinus is a periodical function' expresses the Composition

$$[{}^0\mathit{Periodical}\ {}^0\mathit{Sinus}],$$

where $\mathit{Periodical}/(o(\tau\tau))$ is the class of periodical functions of type $(\tau\tau)$; $\mathit{Sinus}/(\tau\tau)$.

${}^0\mathit{Sinus}$ occurs in the *intensional* context of the Composition $[{}^0\mathit{Periodical}\ {}^0\mathit{Sinus}]$. It is not Composed with a τ-argument in order to construct a value of the sinus function. Instead the function is just mentioned, as it must be if a property is to be predicated of it.

On the other hand, 'Charles knows that sinus is periodical' expresses the construction $\lambda w \lambda t\ [{}^0\mathit{Know}^*_{wt}\ {}^0\mathit{Charles}\ {}^0[{}^0\mathit{Periodical}\ {}^0\mathit{Sinus}]]$, $\mathit{Know}^*/(o\iota *_1)_{\tau\omega}$. Here the Composition $[{}^0\mathit{Periodical}\ {}^0\mathit{Sinus}]$ occurs hyperintensionally; therefore also all its subconstructions, including ${}^0\mathit{Sinus}$, occur in a hyperintensional context.

In the empirical case, intensional constructions usually occur in intensional contexts. Consider 'Charles wants to become the President of the USA'. Charles is related here to the presidential *office*; he wants to occupy it. Thus the analysis comes down to this:

$$\lambda w \lambda t\ [{}^0\mathit{Want\_to\_become}_{wt}\ {}^0\mathit{Charles}\ \lambda w \lambda t\ [{}^0\mathit{President\_of}_{wt}\ {}^0\mathit{USA}]].$$

Types. $\mathit{Want\_to\_become}/(o\iota\iota_{\tau\omega})_{\tau\omega}$; $\mathit{President\_of}/(\iota\iota)_{\tau\omega}$; $\mathit{Charles}, \mathit{USA}/\iota$;

The whole Closure occurs intensionally; it is not used to *v*-construct the truth-value of the so constructed proposition. Moreover, the construction of the presidency, namely $\lambda w \lambda t\ [{}^0\mathit{President\_of}_{wt}\ {}^0\mathit{USA}]$, occurs intensionally (i.e., with *de dicto* supposition) in the intensional context of the whole Closure.

- *Extensional context*: the sort of context in which a construction of a function is used to construct a particular value of the function at a given argument, and the construction does not occur within another intensional or hyperintensional context.

Example: 'sin(π) = 0' expresses the Composition $[[{}^0\mathit{Sinus}\ {}^0\pi] = {}^00]$, where ${}^0\mathit{Sinus}$ occurs extensionally; the Composition is used to construct the value of the sinus function at the argument π.

As mentioned above, constructions of intensions usually occur intensionally; if occurring extensionally, then they usually *v*-construct a particular value of an intension. For instance, $[\lambda w \lambda t\ [^0President\_of_{wt}\ ^0USA]]_{wt}$ v-constructs an individual; the Closure $\lambda w \lambda t\ [^0President\_of_{wt}\ ^0USA]$ occurs extensionally, since the so constructed office is extensionalized.

However, in the Closure

$$\lambda w \lambda t\ [^0Republican_{wt}\ [\lambda w \lambda t\ [^0President\_of_{wt}\ ^0USA]]_{wt}]$$

(which is the meaning of 'The President of the USA is a Republican') the construction of the presidency occurs extensionally (i.e., with *de re* supposition), but in the intensional context of the whole Closure.

The topics of *de dicto*/*de re* supposition and hyperintensional, intensional and extensional contexts are resumed in Section 2.6.

1.5.3 Important entities and notational conventions: summary

Below follows a summary of the main features of our semantic schemas which we introduced in Section 1.1, as well as the main notational conventions. In this chapter we defined, among others, *construction*, *ramified hierarchy of types*, important extensions like *quantifiers* and the notion of *literal meaning* of an expression. We also illustrated how constructions are assigned to semantically self-contained expressions, whereby an expression invariably expresses a construction as its meaning. Whenever an expression does have a *denotation*, the denotation can be any entity of the ontology of TIL:

- an α-*intension* (an object of type α_ω, typically $\alpha_{\tau\omega}$) when the expression is *empirical*;
- an α-*extension*, i.e., an α-object, where $\alpha \neq (\beta\omega)$ for any β;
- a *construction* of type $*_n$, when the expression is mathematical or logical.

Empirical expressions invariably denote α-intensions. The sense of the sentence 'Charles is a bachelor' is a procedure for evaluating, in any possible world at any time, the truth-conditions of this sentence. The sense is the Closure $\lambda w \lambda t$ $[^0Bachelor_{wt}\ ^0Charles]$. The denotation of this sentence is the proposition $P/o_{\tau\omega}$ constructed by this construction. P is true in a subset of logical space; namely, at those worlds and times at which Charles has the property of being a bachelor. If the sentence is true *simpliciter*, then the pair made up of the actual world and the current time is a member of this subset. The reference of this sentence (its truth-value) is beyond the purview of the a priori discipline of logical semantics. (See Sections 1.1 and 2.4.1 for the details of the argument from omniscience in favour of anti-actualism.)

Mathematical expressions denote α-extensions. But even in this case the respective extension is only of secondary semantic interest. What is of primary semantic interest is the respective construction. This is especially clear in the case of expressions lacking denotation, like ‘the greatest prime’. Mathematicians had to first understand the expression, i.e., to know the respective instruction detailing *how* to seek the product; only then were they able to prove that there is no product of the procedure expressed by the expression:

$$\iota x\ [{}^0\wedge\ [{}^0 Prime\ x]\ \forall y\ [{}^0\supset\ [{}^0 Prime\ y]\ [{}^0\geq x\ y]]].$$

We now recapitulate the most important entities and notational conventions occurring throughout the book.

- An arbitrary object X of the arbitrary type α is an *α-object*, denoted ‘X/α’.
- The notation for the type $((\alpha\tau)\omega)$ of α-intensions is abbreviated ‘$\alpha_{\tau\omega}$’.
- The constant proposition that takes value **T** in all possible worlds at all times will be referred to as ‘*TRUE*’.
- The propositional properties of being true, false, undefined are the functions $True/(o o_{\tau\omega})_{\tau\omega}$, $False/(o o_{\tau\omega})_{\tau\omega}$, $Undef/(o o_{\tau\omega})_{\tau\omega}$, respectively.
- Every construction C belongs to $*_n$: C is an entity of a type of order $n > 1$, and (v-) constructs an entity (if any) belonging to a type α of a lower order. That a construction C v-constructs an α-object will be denoted ‘$C/*_n \to_v \alpha$’, or sometimes ‘$C \to_v \alpha$’. For instance, ‘$x/*_1 \to_v \tau$’ reads, ‘The variable x belongs to the type $*_1$ and constructs reals relative to a valuation.’
- If a construction C v-constructs an α-object a independently of valuation, we simply say that C constructs a and write ‘$C \to \alpha$’.
- We often write ‘$\forall x\ A$’, ‘$\exists x\ A$’, ‘$\iota x\ A$’, instead of ‘$[{}^0\forall^\alpha\ \lambda x\ A]$’, ‘$[{}^0\exists^\alpha\ \lambda x\ A]$’, ‘$[{}^0 Sing^\alpha\ \lambda x\ A]$’, respectively, when it is not urgent to highlight typing and lambda-binding.
- We also often use infix notation without Trivialization when using constructions of the truth-functions $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\equiv$ (equivalence) and negation ($\neg$), and when using a construction of an identity relation.
- Variables w, w_1, w_2, … v-construct elements of type ω (possible worlds), and t, t_1, t_2, … v-construct elements of type τ (times ordered in a continuum).
- If C v-constructs an α-intension, the frequently used Composition of the form $[[C\ w]\ t]$, v-constructing the *intensional descent* (a.k.a. *extensionalization*) of an α-intension, is abbreviated ‘C_{wt}’.