7

Shell-type constitutive equations

In this chapter we assume that the constitutive equation for a 3D body is given and we derive various forms of constitutive equation for shells. The elastic materials, compressible and incompressible, are considered.

The shell-type constitutive equations are discussed as follows: for the 3D shells in Sect. 7.1, while for the shells for which the normal strain is recovered in Sect. 7.2. The correction factors for the transverse shear are derived in Sect. 7.3.

7.1 Constitutive equations for 3D shells

Introduction. The 3D shells, by definition, have a non-zero normal strain and two formulations belong to this class:

1. the so-called "solid-shells", which have nodes on the top and bottom surfaces bounding the shell and use the translational degrees of freedom but not rotations. The "solid-shell" element, which is a counterpart of the four-node shell element, has eight nodes and three dofs per node, see [100, 199, 241]. The normal strain κ_{33} must be enhanced and properly approximated, see [31].
2. the shells based on Reissner kinematics with two additional normal stretch parameters, see Sect. 6.6.1. The normal stretch parameters enhance the shell kinematics but require additional equilibrium equations. They can be treated as elemental variables and eliminated at the element's level.

The 3D shells are not within the scope of this book but the constitutive equations for them have relatively simple forms and, hence, they are instructive. The constitutive equations for 3D shells can assume the following forms:

1. *The 3D constitutive equations*, written for 3D stresses and strains, can be used without any modification, see Sect. 7.1.1. They are particularly useful for non-linear complicated constitutive laws, e.g. for plasticity.
2. *The shell-type constitutive equations*, which are written for shell stress and couple resultants. They can be formulated either in the incremental form, see Sect. 7.1.2, or in the general form, see Sect. 7.1.3. They are particularly useful for linear constitutive laws, especially when they can be integrated over the thickness either analytically as for the linear SVK material or numerically as for layered composites.

7.1.1 Incremental 3D constitutive equations

Assume that the strain $\mathbf{E}$ is a polynomial of the normal coordinate $\zeta \in [+h/2, -h/2]$, i.e. $\mathbf{E}(\zeta)$, and all components of $\mathbf{E}$ are non-zero. Let $\mathbf{S}$ designate the stress which is work-conjugate to $\mathbf{E}$. The VW of the stress $\mathbf{S}$ is

$$\delta \mathcal{W} = \int_V \delta \mathbf{E} \cdot \mathbf{S} \, \mathrm{d}V. \tag{7.1}$$

To define the tangent matrix, we calculate the directional derivative of $\delta \mathcal{W}$ which yields

$$\Delta \delta \mathcal{W} = \int_V [\delta \mathbf{E} \cdot (\mathbb{C} \, \Delta \mathbf{E}) + \mathbf{S} \cdot \Delta \delta \mathbf{E}] \, \mathrm{d}V, \tag{7.2}$$

where $\mathbb{C} \doteq \partial \mathbf{S} / \partial \mathbf{E}$ denotes the 3D constitutive operator. Note that

1. To calculate the integral over the volume V we have to integrate over the thickness h and over the reference surface A, see Sect. 10.5.
2. The stress $\mathbf{S}$ is updated by using the incremental constitutive equation

   $$\Delta \mathbf{S} = \mathbb{C} \, \Delta \mathbf{E} \tag{7.3}$$

   at Gauss points. This form is general and applies to arbitrary non-linear materials.

In this approach, the modifications related to shells are minimal:

1. For shells based on the Reissner kinematics, the strain $\mathbf{E}(\zeta) = \boldsymbol{\varepsilon} + \zeta \boldsymbol{\kappa}$ is used,

2. The shell stress and couple resultants can be obtained in the post-processing phase but are not required to obtain the solution. The shell stress and couple resultants are defined as the following integrals:

$$\mathbf{N}_i \doteq \int_h \zeta^i \, \mathbf{S}(\zeta) \; \mu \, \mathrm{d}\zeta, \qquad i = 0, ..., L, \tag{7.4}$$

where ζ^i denotes the i-th power of ζ, $\mu \doteq \det \mathbf{Z}$, and $\mathbf{Z}$ is the shifter tensor, see eq. (5.10). The same Gauss points over the thickness are used as in the integration of $\Delta\delta\mathcal{W}$.

7.1.2 Incremental constitutive equations for shell resultants

Below, we derive the shell resultants and the constitutive (stiffness) matrices assuming that strain $\mathbf{E}$ is represented as the polynomial of the normal coordinate ζ,

$$\mathbf{E}(z) = \mathbf{E}_0 + \zeta \, \mathbf{E}_1 + ... + \zeta^L \, \mathbf{E}_L, \tag{7.5}$$

where the number of terms L is arbitrary. This form encompasses the first-order as well as the second-order kinematics of a shell as special cases. Separating the integration over the thickness from the integration over the reference surface and using eq. (7.5), the VW of stress $\mathbf{S}$ of eq. (7.1) becomes

$$\delta\mathcal{W} = \int_A [\delta\mathbf{E}_0, \delta\mathbf{E}_1, ..., \delta\mathbf{E}_L] \begin{bmatrix} \mathbf{N}_0 \\ \mathbf{N}_1 \\ \vdots \\ \mathbf{N}_L \end{bmatrix} \; \mathrm{d}A, \tag{7.6}$$

where A is the area of the reference surface. The shell stress resultants for the stress $\mathbf{S}$ are defined as

$$\mathbf{N}_i \doteq \int_h \zeta^i \, \mathbf{S}(\zeta) \; \mu \, \mathrm{d}\zeta, \qquad i = 0, ..., L. \tag{7.7}$$

The shell form of $\Delta\delta\mathcal{W}(\zeta)$ is defined as $\Delta\delta\Sigma \doteq \int_h \Delta\delta\mathcal{W}(\zeta) \; \mu \, \mathrm{d}\zeta$, and, upon integration of eq. (7.2) over the thickness and by using eq. (7.5), we obtain

$$\Delta\delta\Sigma = \int_A [\delta\mathbf{E}_0, \delta\mathbf{E}_1, ..., \delta\mathbf{E}_L] \begin{bmatrix} \mathbb{C}_0 & \mathbb{C}_1 & ... & \mathbb{C}_L \\ \mathbb{C}_1 & \mathbb{C}_2 & ... & \mathbb{C}_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{C}_L & \mathbb{C}_{L+1} & ... & \mathbb{C}_{L+L} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{E}_0 \\ \Delta\mathbf{E}_1 \\ \vdots \\ \Delta\mathbf{E}_L \end{bmatrix} + [\mathbf{N}_0, \mathbf{N}_1, ..., \mathbf{N}_i] \begin{bmatrix} \Delta\delta\mathbf{E}_0 \\ \Delta\delta\mathbf{E}_1 \\ \vdots \\ \Delta\delta\mathbf{E}_i \end{bmatrix} \mathrm{d}A, \quad (7.8)$$

where the shell constitutive operators are defined as

$$\mathbb{C}_k \doteq \int_h \zeta^k \, \mathbb{C}(\zeta) \, \mu \, \mathrm{d}z, \qquad k = 0, ..., 2L, \quad (7.9)$$

where k indicates the power of the thickness coordinate ζ.

The stress and couple resultants are updated by the incremental constitutive equations

$$\begin{bmatrix} \Delta\mathbf{N}_0 \\ \Delta\mathbf{N}_1 \\ \vdots \\ \Delta\mathbf{N}_L \end{bmatrix} = \begin{bmatrix} \mathbb{C}_0 & \mathbb{C}_1 & ... & \mathbb{C}_L \\ \mathbb{C}_1 & \mathbb{C}_2 & ... & \mathbb{C}_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{C}_L & \mathbb{C}_{L+1} & ... & \mathbb{C}_{L+L} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{E}_0 \\ \Delta\mathbf{E}_1 \\ \vdots \\ \Delta\mathbf{E}_L \end{bmatrix}. \quad (7.10)$$

This form is effective if we calculate the shell constitutive operators $\mathbb{C}_k$ only once, as for linear materials.

For strain linear over shell thickness. For shells based on the Reissner kinematics, we use a linear representation of strain

$$\mathbf{E}(\zeta) = \mathbf{E}_0 + \zeta\, \mathbf{E}_1 = \boldsymbol{\varepsilon} + \zeta\boldsymbol{\kappa} \quad (7.11)$$

and the shell stress and couple resultants are

$$\mathbf{N}_0 \doteq \int_h \mathbf{S}(\zeta) \, \mu \, \mathrm{d}\zeta = \mathbf{N}, \qquad \mathbf{N}_1 \doteq \int_h \zeta \, \mathbf{S}(\zeta) \, \mu \, \mathrm{d}\zeta = \mathbf{M}. \quad (7.12)$$

The shell form of the VW of stress, eq. (7.8), becomes

$$\Delta\delta\Sigma = \int_A \left\{ [\delta\boldsymbol{\varepsilon}, \delta\boldsymbol{\kappa}] \begin{bmatrix} \mathbb{C}_0 & \mathbb{C}_1 \\ \mathbb{C}_1 & \mathbb{C}_2 \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{\varepsilon} \\ \Delta\boldsymbol{\kappa} \end{bmatrix} + [\mathbf{N}, \mathbf{M}] \begin{bmatrix} \Delta\delta\boldsymbol{\varepsilon} \\ \Delta\delta\boldsymbol{\kappa} \end{bmatrix} \right\} \mathrm{d}A, \quad (7.13)$$

where the shell constitutive operators are

$$\mathbb{C}_0 \doteq \int_h \mathbb{C}(\zeta)\, \mu\, d\zeta, \quad \mathbb{C}_1 \doteq \int_h \zeta\, \mathbb{C}(\zeta)\, \mu\, d\zeta, \quad \mathbb{C}_2 \doteq \int_h \zeta^2\, \mathbb{C}(\zeta)\, \mu\, d\zeta. \tag{7.14}$$

In general, the integrals in the definitions of $\mathbf{N}$, $\mathbf{M}$, and $\mathbb{C}_k$ are evaluated numerically, although for simple materials analytical integration is also possible.

The stress and couple resultants can be updated using the incremental constitutive equations

$$\begin{bmatrix} \Delta\mathbf{N} \\ \Delta\mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbb{C}_0 & \mathbb{C}_1 \\ \mathbb{C}_1 & \mathbb{C}_2 \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{\varepsilon} \\ \Delta\boldsymbol{\kappa} \end{bmatrix}. \tag{7.15}$$

If cross-sectional properties are symmetric w.r.t. $\zeta = 0$, then $\mathbb{C}_1 = \mathbf{0}$ and the constitutive equations are uncoupled, which means that $\Delta\mathbf{N}$ depends only on $\Delta\boldsymbol{\varepsilon}$, and $\Delta\mathbf{M}$ only on $\Delta\boldsymbol{\kappa}$, i.e.

$$\begin{bmatrix} \Delta\mathbf{N} \\ \Delta\mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbb{C}_0 & \mathbf{0} \\ \mathbf{0} & \mathbb{C}_2 \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{\varepsilon} \\ \Delta\boldsymbol{\kappa} \end{bmatrix}. \tag{7.16}$$

Finally, we recall that this formulation requires non-zero normal components of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$, so it is suitable only for 3D shells.

7.1.3 General form of constitutive equations for shell resultants

In this section, the constitutive equations are derived in a general (non-incremental) form and the shell stress and couple resultants are used. The shell strain energy is obtained by the analytical integration over the thickness and two types of material are considered: (A) the linear SVK material and (B) the incompressible material.

A. Linear SVK material

The first-order isotropic elastic St.Venant–Kirchhoff (SVK) material is linear and is applicable only to small strain problems. The standard form of the strain energy function for the SVK material is

$$\mathcal{W}(\mathbf{E}) \doteq \tfrac{1}{2}\lambda\,(\mathrm{tr}\mathbf{E})^2 + \mathrm{G}\,\mathrm{tr}\mathbf{E}^2, \tag{7.17}$$

where $\mathbf{E}$ is a symmetric strain, and λ, G are Lamé constants. The energy is defined per unit volume of the initial (non-deformed) configuration. The constitutive equations are $\mathbf{S} \doteq d\mathcal{W}(\mathbf{E})/d\mathbf{E}$ and, for the SVK material, we obtain the Hooke's law

$$\mathbf{S} = \lambda\,\mathrm{tr}(\mathbf{E})\mathbf{I} + 2\mathrm{G}\,\mathbf{E}, \tag{7.18}$$

where the identities $\mathrm{d}(\mathrm{tr}\mathbf{E})/\mathrm{d}\mathbf{E} = \mathbf{I}$ and $\mathrm{d}(\mathrm{tr}\mathbf{E})^2/\mathrm{d}\mathbf{E} = 2\mathbf{E}$ were used.

The form of the strain energy of eq. (7.17) is only valid for a symmetric $\mathbf{E}$ because, for a non-symmetric $\mathbf{E}$, it yields $S_{12} = 2G\,E_{21}$. For non-symmetric $\mathbf{E}$, we should replace $\mathrm{tr}\mathbf{E}^2$ by $\mathrm{tr}(\mathbf{E}\mathbf{E}^{\mathrm{T}})$.

Strain energy for shell. Assume that the strain is expressed as a linear polynomial of the thickness coordinate ζ, i.e. $\mathbf{E} = \boldsymbol{\varepsilon} + \zeta\boldsymbol{\kappa}$, where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ are symmetric shell strains. For this derivation, we assume that all components of these strain are non-zero. Then

$$\mathrm{tr}\mathbf{E} = \mathrm{tr}\boldsymbol{\varepsilon} + \zeta\mathrm{tr}\boldsymbol{\kappa}, \qquad (\mathrm{tr}\mathbf{E})^2 = (\mathrm{tr}\boldsymbol{\varepsilon})^2 + 2\zeta(\mathrm{tr}\boldsymbol{\varepsilon})(\mathrm{tr}\boldsymbol{\kappa}) + \zeta^2(\mathrm{tr}\boldsymbol{\kappa})^2, \tag{7.19}$$

$$\mathbf{E}^2 = \boldsymbol{\varepsilon}^2 + \zeta(\boldsymbol{\varepsilon}\boldsymbol{\kappa} + \boldsymbol{\kappa}\boldsymbol{\varepsilon}) + \zeta^2\boldsymbol{\kappa}^2, \qquad \mathrm{tr}\mathbf{E}^2 = \mathrm{tr}\boldsymbol{\varepsilon}^2 + 2\zeta\mathrm{tr}(\boldsymbol{\varepsilon}\boldsymbol{\kappa}) + \zeta^2\mathrm{tr}\boldsymbol{\kappa}^2. \tag{7.20}$$

Substituting these expressions into the strain energy (7.17) and integrating over the thickness, we obtain

$$\Sigma \doteq \int_{-\frac{h}{2}}^{+\frac{h}{2}} \mathcal{W}(\mathbf{E}(\zeta))\;\mu\,\mathrm{d}\zeta = h\;\mathcal{W}(\boldsymbol{\varepsilon}) + \frac{h^3}{12}\;\mathcal{W}(\boldsymbol{\kappa}), \tag{7.21}$$

which is the shell strain energy per unit area of the reference surface in the initial configuration. Note that the couplings $(\mathrm{tr}\boldsymbol{\varepsilon})(\mathrm{tr}\boldsymbol{\kappa})$ and $\mathrm{tr}(\boldsymbol{\varepsilon}\boldsymbol{\kappa})$ dropped out because the integral of terms depending linearly on ζ is zero.

Shell constitutive equations. A kinematically admissible variation of the shell strain energy is

$$\delta\Sigma = \frac{\partial\Sigma}{\partial\boldsymbol{\varepsilon}} \cdot \delta\boldsymbol{\varepsilon} + \frac{\partial\Sigma}{\partial\boldsymbol{\kappa}} \cdot \delta\boldsymbol{\kappa}. \tag{7.22}$$

The stress and couple resultants are defined as

$$\mathbf{N} \doteq \frac{\partial\Sigma}{\partial\boldsymbol{\varepsilon}} = h\,\frac{d\mathcal{W}(\boldsymbol{\varepsilon})}{d\boldsymbol{\varepsilon}}, \qquad \mathbf{M} \doteq \frac{\partial\Sigma}{\partial\boldsymbol{\kappa}} = \frac{h^3}{12}\,\frac{d\mathcal{W}(\boldsymbol{\kappa})}{d\boldsymbol{\kappa}} \tag{7.23}$$

and then the variation of the shell strain energy can be concisely written as

$$\delta\Sigma = \mathbf{N} \cdot \delta\boldsymbol{\varepsilon} + \mathbf{M} \cdot \delta\boldsymbol{\kappa}. \tag{7.24}$$

Because, the derivatives $d\mathcal{W}(\boldsymbol{\varepsilon})/d\boldsymbol{\varepsilon}$ and $d\mathcal{W}(\boldsymbol{\kappa})/d\boldsymbol{\kappa}$ have analogous forms as $d\mathcal{W}(\mathbf{E})/d\mathbf{E}$, hence, the constitutive equations for the stress resultants and the couple resultants are

$$\mathbf{N} = h\ [\lambda\,(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I} + 2\mathrm{G}\,\boldsymbol{\varepsilon}]\,, \qquad \mathbf{M} = \frac{h^3}{12}\ [\lambda\,(\mathrm{tr}\boldsymbol{\kappa})\mathbf{I} + 2\mathrm{G}\,\boldsymbol{\kappa}]\,. \tag{7.25}$$

Note that the distribution of the shear stresses over the thickness is parabolic and the constitutive equations for the components $N_{\alpha 3}$ and $M_{\alpha 3}$ have to be corrected, see Sect. 7.3.

B. Incompressible Mooney–Rivlin material

Consider a class of second-order hyper-elastic materials which undergo an isochoric (or volume-preserving) deformation. Because the relation between the initial volume dV and the current volume dv is $dv = \det\mathbf{F}\ dV$, the incompressibility of the material is defined by the condition

$$\det\mathbf{F} = 1, \tag{7.26}$$

where $\mathbf{F}$ is the deformation gradient. This definition implies that the third invariant of the right Cauchy–Green tensor $\mathbf{C}$ is equal to one, $I_3(\mathbf{C}) \doteq \det\mathbf{C} = (\det\mathbf{F})^2 = 1$. Thus, the strain energy of incompressible materials depends only on the two first principal invariants of $\mathbf{C}$,

$$I_1(\mathbf{C}) \doteq \mathrm{tr}\mathbf{C}, \qquad \mathrm{I}_2(\mathbf{C}) \doteq \tfrac{1}{2}\left[(\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}\mathbf{C}^2\right]. \tag{7.27}$$

Below, we define two classical incompressible materials:

1. The so-called neo-Hookean material is defined by the following strain energy function:
$$\tilde{\mathcal{W}}(I_1(\mathbf{C})) \doteq c_1\,[I_1(\mathbf{C}) - 3]\,, \tag{7.28}$$
where c_1 is a material constant. This energy function depends only on the first invariant of $\mathbf{C}$.
2. The Mooney–Rivlin is the material is defined by the following strain energy function,
$$\tilde{\mathcal{W}}(I_\alpha(\mathbf{C})) \doteq c_1\,[I_1(\mathbf{C}) - 3] + c_2\,[I_2(\mathbf{C}) - 3]\,, \qquad \alpha = 1,2, \tag{7.29}$$
where c_1, c_2 are material constants. This energy function depends on the two first invariants of $\mathbf{C}$.

For more details on incompressible materials, see [81, 159, 93, 116].

For membranes, the strain energy for the incompressible material can also be expressed in terms of principal stretches. Then the incompressibility condition is applied to the Ogden's form of the strain energy of eq. (7.93).

Formulation for right stretching tensor U. The incompressibility condition can also be formulated in terms of the third invariant of the right stretching tensor $\mathbf{U}$,

$$I_3(\mathbf{U}) \doteq \det \mathbf{U} = 1. \tag{7.30}$$

This form is obtained from the polar decomposition $\mathbf{F} = \mathbf{QU}$, for which

$$\det \mathbf{F} = \det(\mathbf{RU}) = (\det \mathbf{R})(\det \mathbf{U}) = \det \mathbf{U} = 1, \tag{7.31}$$

as $\det \mathbf{R} = 1$ for $\mathbf{R} \in \mathrm{SO}(3)$. Hence, for the incompressible material, the strain energy depends on the two first principal invariants of $\mathbf{U}$, i.e. $\tilde{\mathcal{W}} = \tilde{\mathcal{W}}(I_1(\mathbf{U}), I_2(\mathbf{U}))$, where the principal invariants of $\mathbf{U}$ are

$$I_1(\mathbf{U}) \doteq \mathrm{tr}\mathbf{U}, \qquad \mathrm{I}_2(\mathbf{U}) \doteq \tfrac{1}{2}\left[(\mathrm{tr}\mathbf{U})^2 - \mathrm{tr}\mathbf{U}^2\right]. \tag{7.32}$$

The constitutive equation for the symmetric Biot stress tensor is

$$\mathbf{T}_s^B \doteq \frac{\partial \mathcal{W}(\mathbf{U})}{\partial \mathbf{U}} + p\,\mathbf{I} = \frac{\partial \tilde{\mathcal{W}}(I_1(\mathbf{U}), I_2(\mathbf{U}))}{\partial \mathbf{U}} + p\,\mathbf{I}. \tag{7.33}$$

Using the chain rule, we obtain

$$\frac{\partial \tilde{\mathcal{W}}}{\partial \mathbf{U}} = \frac{\partial \tilde{\mathcal{W}}}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{U}} + \frac{\partial \tilde{\mathcal{W}}}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{U}}, \tag{7.34}$$

where $\partial I_1/\partial \mathbf{U} = \mathbf{I}$ and $\partial I_2/\partial \mathbf{U} = I_1\mathbf{I} - \mathbf{U}$. Thus, the constitutive equation can be rewritten as a linear polynomial of $\mathbf{U}$, i.e. $\mathbf{T}_s^B = \beta_0 \mathbf{I} + \beta_1 \mathbf{U}$, where β_0 and β_1 are scalar coefficients depending on the invariants.

The invariants of $\mathbf{C}$ in eq. (7.29) can be written as functions of the invariants of $\mathbf{U}$,

$$I_1(\mathbf{C}) = I_1^2(\mathbf{U}) - 2I_2(\mathbf{U}), \qquad I_2(\mathbf{C}) = I_2^2(\mathbf{U}) - 2I_1(\mathbf{U})\, I_3(\mathbf{U}), \tag{7.35}$$

where, for $I_3(\mathbf{U}) = 1$, the second one is reduced to $I_2(\mathbf{C}) = I_2^2(\mathbf{U}) - 2I_1(\mathbf{U})$. Thus, the Mooney–Rivlin strain energy of eq. (7.29) is, in terms of the invariants of $\mathbf{U}$, as follows:

$$\tilde{\mathcal{W}}(I_\alpha(\mathbf{U})) = c_1\left[I_1^2(\mathbf{U}) - 2I_2(\mathbf{U}) - 3\right] + c_2\left[I_2^2(\mathbf{U}) - 2I_1(\mathbf{U}) - 3\right]. \quad (7.36)$$

Assume that the right stretching tensor is a linear polynomial of the thickness coordinate ζ of the shell, i.e. $\mathbf{U} = \mathbf{e} + \zeta\mathbf{k}$, where $\mathbf{e}$ and $\mathbf{k}$ are the symmetric shell strains. Then the invariants of $\mathbf{U}$ can be expressed as

$$I_1(\mathbf{U}) = I_1(\mathbf{e}) + \zeta I_1(\mathbf{k}), \qquad I_2(\mathbf{U}) = I_2(\mathbf{e}) + \zeta A + \zeta^2 I_2(\mathbf{k}), \quad (7.37)$$

$$I_3(\mathbf{U}) = \frac{1}{6}\left[I_3(\mathbf{e}) + \zeta B(\mathbf{e},\mathbf{k}) + \zeta^2 B(\mathbf{k},\mathbf{e}) + \zeta^3 I_3(\mathbf{k})\right], \quad (7.38)$$

where the auxiliary scalars are

$$A \doteq I_1(\mathbf{e})\, I_1(\mathbf{k}) - \mathrm{tr}(\mathbf{ek}), \quad (7.39)$$

$$B(\mathbf{a},\mathbf{b}) \doteq 6\left[I_2(\mathbf{a})\, I_1(\mathbf{b}) + \mathrm{tr}(\mathbf{a}^2\mathbf{b}) - \mathrm{I}_1(\mathbf{a})\,\mathrm{tr}(\mathbf{ab})\right], \quad (7.40)$$

for the second rank tensors $\mathbf{a}$ and $\mathbf{b}$. Note the presence of the coupling terms in A and $B(\mathbf{a},\mathbf{b})$, which render that the second and third invariant of $\mathbf{U}$ are not expressible in terms of the invariants of $\mathbf{e}$ and $\mathbf{k}$. For the squares of invariants of $\mathbf{U}$, which are also present in eq. (122), we have

$$I_1^2(\mathbf{U}) = I_1^2(\mathbf{e}) + 2\zeta I_1(\mathbf{e})\, I_1(\mathbf{k}) + \zeta^2 I_1^2(\mathbf{k}), \quad (7.41)$$

$$I_2^2(\mathbf{U}) = I_2^2(\mathbf{e}) + \zeta^2 A^2 + \zeta^4 I_2^2(\mathbf{k}) + 2\zeta I_2(\mathbf{e})\, A + 2\zeta^2 I_2(\mathbf{e})\, I_2(\mathbf{k}) + 2\zeta^3 I_2(\mathbf{k})\, A, \quad (7.42)$$

where

$$A^2 = I_1^2(\mathbf{e})\, I_1^2(\mathbf{k}) - 2I_1(\mathbf{e})\, I_1(\mathbf{k})\,\mathrm{tr}(\mathbf{ek}) + [\mathrm{tr}(\mathbf{ek})]^2. \quad (7.43)$$

Note that for the assumed approximations of $\mathbf{U}$, the strain energy is the second order polynomial of ζ for the neo-Hookean material and the fourth order polynomial for the Mooney–Rivlin material.

Strain energy for shell. Let us define the shell strain energy density per unit area of the middle surface in the initial configuration, as the integral of the strain energy over the thickness, i.e.

$$\tilde{\Sigma}(I_\alpha(\mathbf{U})) \doteq \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tilde{\mathcal{W}}(I_\alpha(\mathbf{U}))\; \mu\, \mathrm{d}\zeta. \quad (7.44)$$

The integration over the thickness renders that the terms of $\tilde{\mathcal{W}}(I_\alpha(\mathbf{U}))$ multiplied by even powers of ζ are equal to zero and the shell energy splits as follows:

$$\tilde{\Sigma} = c_1\ \tilde{\Sigma}_1\ +\ c_2\ \tilde{\Sigma}_2, \tag{7.45}$$

where

$$\tilde{\Sigma}_1 = h\left[I_1^2(\mathbf{e}) - 2I_2(\mathbf{e}) - 3\right] + \frac{h^3}{12}\left[I_1^2(\mathbf{k}) - 2I_2(\mathbf{k})\right], \tag{7.46}$$

$$\tilde{\Sigma}_2 = h\left[I_2^2(\mathbf{e}) - 2I_1(\mathbf{e}) - 3\right] + \frac{h^3}{12}\left[A^2 + I_2(\mathbf{e})I_2(\mathbf{k})\right] + \frac{h^5}{80}I_2^2(\mathbf{k}). \tag{7.47}$$

In the neo-Hookean component $\tilde{\Sigma}_1$, the terms depending on $\mathbf{e}$ and $\mathbf{k}$ are separated, leading to uncoupled constitutive equations. On the other hand, the component $\tilde{\Sigma}_2$ contains coupling terms such as $I_1(\mathbf{e})I_1(\mathbf{k})$, $I_2(\mathbf{e})I_2(\mathbf{k})$, $\mathrm{tr}(\mathbf{ek})$, and some products and powers of them.

Shell constitutive equations. For symmetric $\mathbf{e}$, $\delta\mathbf{e}$, $\mathbf{k}$, and $\delta\mathbf{k}$, the variation of the shell strain energy may be written as

$$\delta\tilde{\Sigma}(\mathbf{e},\mathbf{k}) = \mathbf{N}_s^B\cdot\delta\mathbf{e} + \mathbf{M}_s^B\cdot\delta\mathbf{k}, \tag{7.48}$$

where the stress and couple resultants are defined as

$$\mathbf{N}_s^B \doteq \frac{\mathrm{d}\tilde{\Sigma}}{\mathrm{d}\mathbf{e}}, \qquad \mathbf{M}_s^B \doteq \frac{\mathrm{d}\tilde{\Sigma}}{\mathrm{d}\mathbf{k}}. \tag{7.49}$$

To facilitate further differentiation, we calculate the following derivatives:

$$\frac{\partial\mathrm{tr}(\mathbf{ek})}{\partial\mathbf{e}} = \mathbf{k}, \qquad \frac{\partial\left[\mathrm{tr}(\mathbf{ek})\right]^2}{\partial\mathbf{e}} = 2\mathrm{tr}(\mathbf{ek})\ \mathbf{k}, \qquad \frac{\partial\mathrm{A}^2}{\partial\mathbf{e}} = 2\mathrm{A}\ \mathbf{D}(\mathbf{k}), \tag{7.50}$$

$$\frac{\partial\mathrm{tr}(\mathbf{ek})}{\partial\mathbf{k}} = \mathbf{e}, \qquad \frac{\partial\left[\mathrm{tr}(\mathbf{ek})\right]^2}{\partial\mathbf{k}} = 2\mathrm{tr}(\mathbf{ek})\ \mathbf{e}, \qquad \frac{\partial\mathrm{A}^2}{\partial\mathbf{k}} = 2\mathrm{A}\ \mathbf{D}(\mathbf{e}), \tag{7.51}$$

where the auxiliary tensor is defined as

$$\mathbf{D}(\mathbf{A}) \doteq I_2(\mathbf{A})_{,\mathbf{A}} = I_1(\mathbf{A})\,\mathbf{I} - \mathbf{A}. \tag{7.52}$$

The derivatives of the shell strain energy are

$$\frac{\partial\tilde{\Sigma}_1}{\partial\mathbf{e}} = 2h\mathbf{e}, \qquad \frac{1}{2}\frac{\partial\tilde{\Sigma}_2}{\partial\mathbf{e}} = h\left[I_2(\mathbf{e})\,\mathbf{D}(\mathbf{e}) - \mathbf{I}\right] + \frac{h^3}{12}\boldsymbol{\pi}(\mathbf{k},\mathbf{e}), \tag{7.53}$$

$$\frac{\partial \tilde{\Sigma}_1}{\partial \mathbf{k}} = 2\frac{h^3}{12}\mathbf{k}, \qquad \frac{1}{2}\frac{\partial \tilde{\Sigma}_2}{\partial \mathbf{k}} = \frac{h^3}{12}\boldsymbol{\pi}(\mathbf{e},\mathbf{k}) + \frac{h^5}{80}I_2(\mathbf{k})\,\mathbf{D}(\mathbf{k}), \tag{7.54}$$

where the term which couples the contribution of $\mathbf{e}$ and $\mathbf{k}$ is defined as follows:

$$\boldsymbol{\pi}(\mathbf{a},\mathbf{b}) \doteq A\ \mathbf{D}(\mathbf{b}) \ + \ \tfrac{1}{2}I_2(\mathbf{b})\ \mathbf{D}(\mathbf{a}). \tag{7.55}$$

Using the above equations, the following coupled constitutive equations for the shell are obtained:

$$\mathbf{N}_s^B = c_1\,[2h\mathbf{e}] + c_2 2\left[h\,(I_2(\mathbf{e})\,\mathbf{D}(\mathbf{e}) - \mathbf{I}) + \frac{h^3}{12}\boldsymbol{\pi}(\mathbf{k},\mathbf{e})\right], \tag{7.56}$$

$$\mathbf{M}_s^B = c_1\left[2\frac{h^3}{12}\mathbf{k}\right] + c_2 2\left[\frac{h^3}{12}\boldsymbol{\pi}(\mathbf{e},\mathbf{k}) + \frac{h^5}{80}I_2(\mathbf{k})\,\mathbf{D}(\mathbf{k})\right]. \tag{7.57}$$

Note that these constitutive equations for the hyper-elastic incompressible material have been obtained without any simplifications of the strain energy and have quite a complicated form. In numerical implementations, the incremental forms are much more convenient.

7.2 Reduced shell constitutive equations

The reduced shell constitutive equations are obtained by using the normal strain recovered from an auxiliary condition. This recovery is performed because the standard Reissner hypothesis yields the normal strains ε_{33} and κ_{33} equal to zero, as we can see in

1. Eqs. (6.40)–(6.41) for the *non-symmetric relaxed* right stretch strain,
2. Eqs. (6.48)–(6.49) for the *symmetric relaxed* right stretch strain, and
3. Eqs. (6.54)–(6.55) for the Green strain.

The zero values of the normal strains are non-physical and inaccurate, which can be easily shown for membranes, see eq. (7.73) and Fig. 7.1.

For the Kirchhoff shells, the components of strain are usually evaluated as follows:

$$\varepsilon_{\alpha\beta} \sim h\kappa_{(\alpha\beta)} = \mathrm{O}(\eta), \quad \varepsilon_{3\beta} \sim h\kappa_{3\beta} = \mathrm{O}(\eta\theta), \quad \varepsilon_{33} = \mathrm{O}(\nu\eta). \tag{7.58}$$

where η is the maximum eigenvalue of the Green in-plane strains and ν is the Poisson's ratio. The small parameter θ is defined in [171], p. 111, eq. (6.3.4). We note that a special methodology, proposed in [119, 120] and later successfully developed in [129, 173, 174], must be used to construct consistent approximations to the strain energy. Obviously, ε_{33} is not negligible compared to the other strain components.

To improve accuracy, we can calculate the normal strains from auxiliary conditions, such as (a) the zero normal stress (ZNS) condition, see Sect. 7.2.1, or (b) the incompressibility condition, see Sect. 7.2.2. This approach using the recovery is classical and most often used despite several difficulties involved such as:

1. the auxiliary conditions are not always fully physically justified, e.g. the ZNS condition in case of the multi-layer shells,
2. the reduced constitutive laws are often difficult to derive for some constitutive equations and are very complicated. For the ZNS condition, this problem can be alleviated by using the incremental form of the constitutive equations, see Sect. 7.2.1.
3. the reduced constitutive equations can be more difficult to solve than the original 3D equations, e.g. for the J_2 plasticity, where the 2D yield surface is not spherical and the radial return algorithm cannot be applied.

The recovery of the normal strain renders that the constitutive equations are more accurate but also more complicated.

7.2.1 Reduced constitutive equations for ZNS condition

For thin membranes, we can use the plane stress conditions

$$S_{31}(z) = 0, \qquad S_{32}(z) = 0, \qquad S_{33}(z) = 0, \tag{7.59}$$

where S_{31} and S_{32} are transverse shear stresses and S_{33} is the normal stress, all in the local Cartesian basis $\{\mathbf{t}_k\}$. However, for the Reissner shells, only the condition for the normal stress is acceptable,

$$S_{33}(z) = 0, \tag{7.60}$$

while the transverse shear strains must remain unconstrained. This ZNS condition was used for the Kirchhoff shells in the classical works [153, 171, 172].

A. Incremental formulation in stresses

In the 3D formulation of Sect. 7.1.1, the incremental constitutive equation (7.3) is written for stresses as $\Delta\mathbf{S} = \mathbb{C}\,\Delta\mathbf{E}$ and can be rewritten as

$$\begin{bmatrix} \Delta \mathbf{S}_v \\ \Delta S_{33} \end{bmatrix} = \begin{bmatrix} \mathbb{C}_{vv} & \mathbb{C}_{v3} \\ \mathbb{C}_{3v} & C_{33} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{E}_v \\ \Delta E_{33} \end{bmatrix}, \tag{7.61}$$

where $(\cdot)_v$ denotes a vector of tangent components arranged in the order $\{11, 22, 12\}$. Besides, $\dim \mathbb{C}_{vv} = 3 \times 3$, $\dim \mathbb{C}_{v3} = 3 \times 1$, and $\dim \mathbb{C}_{3v} = 1 \times 3$. This equation involves only the tangent components and the normal components 33, while the transverse shear components were omitted for simplicity.

From the condition of a zero increment of the normal stress, i.e. $\Delta S_{33} = 0$, and the last (scalar) equation of eq. (7.61), we can calculate the normal strain increment

$$\Delta E_{33} = -\frac{1}{C_{33}} \mathbb{C}_{3v}\, \Delta \mathbf{E}_v, \tag{7.62}$$

for which the first (matrix) equation of eq. (7.61) becomes

$$\Delta \mathbf{S}_v = \mathbb{C}_{vv} \Delta \mathbf{E}_v + \mathbb{C}_{v3} \Delta E_{33} = \mathbb{C}^*\ \Delta \mathbf{E}_v, \tag{7.63}$$

where the constitutive matrix is defined as

$$\mathbb{C}^* \doteq \mathbb{C}_{vv} - \frac{1}{C_{33}} \mathbb{C}_{v3} \mathbb{C}_{3v}. \tag{7.64}$$

The above-reduced incremental constitutive equation (7.63) and the reduced constitutive matrix $\mathbb{C}^*$ are for tangent components and both account for the increment of normal strain, i.e. the change of thickness. The normal strain is updated as

$$E_{33}^i = E_{33}^{i-1} + \Delta E_{33}, \tag{7.65}$$

where ΔE_{33} is given by eq. (7.62), and E_{33}^{i-1} is the value for the previous iteration.

Remark. Note that this incremental procedure is quite general and can be applied to any non-linear hyper-elastic materials, e.g. to the compressible neo-Hookean materials, which are generalizations of the incompressible neo-Hookean material of eq. (7.28). For instance, in [212], the strain energy function has the form

$$\mathcal{W} \doteq \underbrace{\frac{\lambda}{2}(\ln J)^2 - G \ln J}_{\text{compressible part}} + \frac{G}{2}(\mathrm{tr}\mathbf{C} - 3), \tag{7.66}$$

where $J \doteq \det \mathbf{F}$. For $\mathbf{F} = \mathbf{I}$, we obtain $\mathcal{W} = 0$ and the constitutive operator is reduced to the one for the SVK material. A simpler form of the compressible part is obtained when the first term is replaced by the first term of its series expansion at $J = 1$, i.e. $(\ln J)^2 = (J-1)^2 + \mathrm{O}(J-1)^3$; other forms of this part are listed in [116], p. 160.

Remark. A very simple scheme of treating the normal strain was used for shells in [101]. The nonlinear governing equations are solved iteratively and the normal strain is evaluated for the last available solution, so its value lags one iteration behind. This certainly somehow impairs the convergence rate, but it is acceptable as long as the iterations converge.

B. Incremental formulation in stress resultants and couple resultants

For the constitutive equations of Sect. 7.1.2, we use the recovery procedure for the shell stress and couple resultants, which is analogous to that for stresses. Note that the condition $S_{33}(z) = 0$ implies, by eq. (7.12), the zero values of shell resultants, i.e. $N_{33} = 0$ and $M_{33} = 0$. We use these conditions in the incremental form

$$\Delta N_{33} = 0, \qquad \Delta M_{33} = 0. \tag{7.67}$$

We assume that the shell constitutive equations are decoupled, as in eq. (7.16), and consider them separately.

For the first of eq. (7.16), $\Delta \mathbf{N} = \mathbb{C}_0 \, \Delta \boldsymbol{\varepsilon}$, we use the condition $\Delta N_{33} = 0$, and the results are analogous to these for stresses, if we replace

$$\mathbf{E} \rightarrow \boldsymbol{\varepsilon}, \quad \Delta S_{33} \rightarrow \Delta N_{33}, \quad \Delta E_{33} \rightarrow \Delta \varepsilon_{33}, \qquad \mathbb{C} \rightarrow \mathbb{C}_0, \quad \mathbb{C}^* \rightarrow \mathbb{C}_0^*. \tag{7.68}$$

For the second of eq. (7.16), i.e. $\Delta \mathbf{M} = \mathbb{C}_2 \, \Delta \boldsymbol{\kappa}$, we use the condition $\Delta M_{33} = 0$, and the results are analogous to these for stresses, if we replace

$$\mathbf{E} \rightarrow \boldsymbol{\kappa}, \quad \Delta S_{33} \rightarrow \Delta M_{33}, \quad \Delta E_{33} \rightarrow \Delta \kappa_{33}, \qquad \mathbb{C} \rightarrow \mathbb{C}_2, \quad \mathbb{C}^* \rightarrow \mathbb{C}_2^*. \tag{7.69}$$

For the recovered $\Delta\varepsilon_{33}$ and $\Delta\kappa_{33}$, the normal strain of a shell is linearly approximated in z,

$$\Delta E_{33}(z) = \Delta\varepsilon_{33} + z \, \Delta\kappa_{33}. \tag{7.70}$$

The example of the 2D beam indicates that the recovery of both normal strains is beneficial, although each one for a different deformation, see Table 7.1.

C. Constitutive equations for SVK material using ZNS condition

For the linear SVK material, the procedure of the previous section is reduced to the classical procedure using the general form of the ZNS condition. In terms of components, the SVK strain-energy function of eq. (7.17) has the following form:

$$\begin{aligned}\mathcal{W}(\mathbf{E}) \doteq {} & \frac{\lambda}{2}\,(E_{11}+E_{22}+E_{33})^2 \\ & + G\left(E_{11}^2+2E_{12}^2+E_{22}^2+E_{33}^2+2E_{13}^2+2E_{23}^2\right)\end{aligned} \tag{7.71}$$

and the constitutive equations are

$$\begin{gathered} S_{11}=\lambda\,(E_{11}+E_{22}+E_{33})+2G\,E_{11}, \qquad S_{22}=\lambda\,(E_{11}+E_{22}+E_{33})+2G\,E_{22}, \\ S_{33}=\lambda\,(E_{11}+E_{22}+E_{33})+2G\,E_{33}, \\ S_{12}=4G\,E_{12}, \qquad S_{13}=4G\,E_{13}, \qquad S_{23}=4G\,E_{23}. \end{gathered} \tag{7.72}$$

For simplicity, in the sequel we neglect the transverse shear strain components. From the ZNS condition $S_{33}=0$ and for S_{33} of eq. (7.72), we can calculate the normal strain

$$E_{33}=-c_0\,(E_{11}+E_{22}), \qquad c_0 \doteq \frac{\lambda}{\lambda+2G}=\frac{\nu}{1-\nu}, \tag{7.73}$$

where c_0 is plotted for $\nu\in[0,\frac{1}{2}]$ in Fig. 7.1. We see that $0\le c_0\le 1$, and always is greater than ν, which is shown as a straight line in this figure. Note that ν is used in the estimation $\varepsilon_{33}=\mathrm{O}(\nu\eta)$ of eq. (7.58).

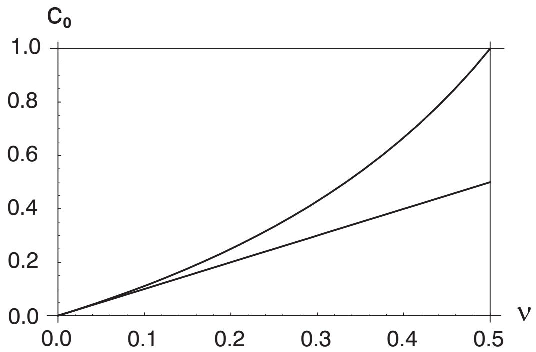

Fig. 7.1 Coefficient c_0 for $\nu\in[0,\frac{1}{2}]$.

Using eq. (7.73), the strain energy (7.71) becomes

$$\mathcal{W}^*(\mathbf{E}_v) \doteq \frac{1}{2}c_1(E_{11}^2 + E_{22}^2) + c_2\, E_{11}E_{22} + \frac{1}{2}c_3\, E_{12}^2, \tag{7.74}$$

where

$$c_1 \doteq \frac{4G(\lambda+G)}{\lambda+2G} = \frac{E}{1-\nu^2}, \quad c_2 \doteq \frac{2G\lambda}{\lambda+2G} = \frac{E\nu}{1-\nu^2}, \quad c_3 \doteq 4G = \frac{2E}{1+\nu}. \tag{7.75}$$

This form of strain energy depends only on the tangent components $\{11, 22, 12\}$.

The reduced constitutive equations $\mathbf{S}_v(\boldsymbol{\varepsilon}_v) \doteq \partial\mathcal{W}^*(\mathbf{E}_v)/\partial\mathbf{E}_v$ and the reduced constitutive matrix $\mathbb{C}_{vv} \doteq \partial\mathbf{S}_v(\mathbf{E}_v)/\partial\mathbf{E}_v$ are as follows:

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{12} \end{bmatrix} = \begin{bmatrix} c_1\, E_{11} + c_2\, E_{22} \\ c_2\, E_{11} + c_1\, E_{22} \\ c_3\, E_{12} \end{bmatrix}, \qquad \mathbb{C}_{vv} = \begin{bmatrix} c_1 & c_2 & 0 \\ c_2 & c_1 & 0 \\ 0 & 0 & c_3 \end{bmatrix}. \tag{7.76}$$

The inverse of the constitutive matrix is

$$\mathbb{C}_{vv}^{-1} = \begin{bmatrix} d_1 & d_2 & 0 \\ d_2 & d_1 & 0 \\ 0 & 0 & d_3 \end{bmatrix}, \qquad d_1 = \frac{\lambda+G}{G(3\lambda+2G)},$$
$$d_2 = \frac{-\lambda}{2G(3\lambda+2G)}, \qquad d_3 = \frac{1}{4G}. \tag{7.77}$$

The eigenvalues of the constitutive matrix are

$$\begin{aligned} \operatorname{eigv} \mathbb{C}_{vv} &= \left\{ \frac{2G}{\lambda+2G}(3\lambda+2G),\ 4G,\ 2G \right\} \\ &= \left\{ \frac{E}{1-\nu},\ \frac{2E}{1+\nu},\ \frac{E}{1+\nu} \right\}. \end{aligned} \tag{7.78}$$

For $\nu \in [0, \frac{1}{2}]$, the smallest eigenvalue of $\mathbb{C}_{vv}$ is $E/(1+\nu) = 2G$.

Finally, we note that the strain energy of eq. (7.74) can be expressed using the constitutive matrix $\mathbb{C}_{vv}$ as follows:

$$\mathcal{W}^*(\mathbf{E}_v) \doteq \frac{1}{2}\,\mathbf{E}_v \cdot (\mathbb{C}_{vv}\mathbf{E}_v). \tag{7.79}$$

For in-plane strain linear over shell thickness. For shells, we use the in-plane strains which are linear in the normal coordinate, i.e. $\mathbf{E}_v(\zeta) = \boldsymbol{\varepsilon}_v + \zeta \boldsymbol{\kappa}_v$. Using the strains of this form in eq. (7.73), we obtain the normal strain of the shell as a linear polynomial of ζ,

$$E_{33}(\zeta) = \varepsilon_{33} + \zeta \kappa_{33}, \tag{7.80}$$

where $\varepsilon_{33} = c_0\,(\varepsilon_{11} + \varepsilon_{22})$ and $\kappa_{33} = c_0\,(\kappa_{11} + \kappa_{22})$. Hence, we obtain a linear approximation of the normal strain when the in-plane strains are linear in ζ.

Using $\mathbf{E}_v(\zeta) = \boldsymbol{\varepsilon}_v + \zeta \boldsymbol{\kappa}_v$, and integrating the strain energy of eq. (7.74) over the thickness, we obtain the shell strain energy as a sum of the membrane energy and the bending energy

$$\Sigma \doteq \int_{-\frac{h}{2}}^{+\frac{h}{2}} \mathcal{W}^*(\mathbf{E}_v(\zeta))\ \mu\,\mathrm{d}\zeta = h\ \mathcal{W}^*(\boldsymbol{\varepsilon}_v)\ +\ \frac{h^3}{12}\ \mathcal{W}^*(\boldsymbol{\kappa}_v). \tag{7.81}$$

We can rewrite eq. (7.76) as $\mathbf{S}_v = \mathbb{C}_{vv}\,\mathbf{E}_v$ and use it in the definition of eq. (7.12) to obtain the constitutive equations for the shell stress and couple resultants

$$\mathbf{N}_v \doteq \int_{-\frac{h}{2}}^{+\frac{h}{2}} \mathbf{S}_v(\zeta)\ \mu\,\mathrm{d}\zeta = h\,\mathbb{C}_{vv}\boldsymbol{\varepsilon}_v, \quad \mathbf{M}_v \doteq \int_{-\frac{h}{2}}^{+\frac{h}{2}} \zeta\,\mathbf{S}_v(\zeta)\ \mu\,\mathrm{d}\zeta = \frac{h^3}{12}\mathbb{C}_{vv}\boldsymbol{\kappa}_v. \tag{7.82}$$

D. Effects of normal strain recovery for 2D beam

Consider a straight 2D beam in the $\{\mathbf{t}_1, \mathbf{t}_3\}$-plane, where $\{\mathbf{t}_j\}$ $(j = 1, 3)$ is the local ortho-normal basis associated with the initial configuration. For the standard Reissner hypothesis, $\mathbf{x}(\zeta) = \mathbf{x}_0 + \zeta \mathbf{Q}_0 \mathbf{t}_3$, we can split the *non-symmetric relaxed* right stretch strain, $\tilde{\mathbf{H}}_n \doteq \mathbf{Q}_0^T \mathbf{F} - \mathbf{I}$, as follows: $\tilde{\mathbf{H}}_n(\zeta) = \boldsymbol{\varepsilon} + \zeta \boldsymbol{\kappa}$, where the components in $\{\mathbf{t}_i\}$ are

$$\varepsilon_{11} = \mathbf{x}_{0,1} \cdot \mathbf{a}_1 - 1, \qquad 2\,\varepsilon_{13} = \mathbf{x}_{0,1} \cdot \mathbf{a}_3, \qquad \varepsilon_{33} = 0, \tag{7.83}$$

$$\kappa_{11} = \omega_{,1}, \qquad \kappa_{13} = 0, \qquad \kappa_{33} = 0,$$

where $\mathbf{a}_i \doteq \mathbf{Q}_0\,\mathbf{t}_i$. The 3D formulation is reduced to a 2D formulation by setting the 21 and 23 components of stress and strain to zero and recovering ε_{22} from the condition $\sigma_{22} = 0$. Then, for the SVK material, we obtain the following beam-type constitutive equations:

$$N_{11} = Ch\,(\varepsilon_{11} + \nu\varepsilon_{33})\,, \quad N_{33} = Ch\,(\nu\varepsilon_{11} + \varepsilon_{33})\,,$$
$$N_{13} = k\,Ch\frac{1-\nu}{2}(2\varepsilon_{13}), \tag{7.84}$$
$$M_{11} = C\frac{h^3}{12}\,(\kappa_{11} + \nu\kappa_{33})\,, \qquad M_{33} = C\frac{h^3}{12}\,(\nu\kappa_{11} + \kappa_{33})\,,$$

where $C = E/(1-\nu^2)$, E is the Young's modulus, ν is the Poisson's ratio, and k is the shear correction factor. Note that only the components $\{11, 33, 13\}$ are involved.

The normal strains ε_{33} and κ_{33} are equal to zero in eq. (7.83), but we can recover them as follows. The condition $N_{33} = 0$ yields $\varepsilon_{33} = -\nu\,\varepsilon_{11}$. Similarly, κ_{33} can be recovered using the condition $m_{33} = 0$, which yields $\kappa_{33} = -\nu\,\kappa_{11}$. Due to the recovery, the normal strain is linearly approximated over ζ,

$$\tilde{H}_{n33}(\zeta) = \varepsilon_{33} + \zeta\kappa_{33} = -\nu(\varepsilon_{11} + \zeta\kappa_{11}). \tag{7.85}$$

Using the recovered normal strains, the constitutive relations of eq. (7.84) become

$$N_{11} = Ch\left(1-\nu^2\right)\varepsilon_{11} = Eh\,\varepsilon_{11}, \quad N_{11} = C\frac{h^3}{12}\left(1-\nu^2\right)\kappa_{11} = E\frac{h^3}{12}\kappa_{11}. \tag{7.86}$$

Comparing these forms with N_{11} of eq. (7.84) for $\varepsilon_{33} = 0$ and M_{11} of eq. (7.84) for $\kappa_{33} = 0$, we see that, in both cases, the strain recovery renders that the stiffness is reduced by the factor $\left(1-\nu^2\right)$.

Numerical test. The slender cantilever test is described in Sect. 15.3.1, Fig. 15.13. The cantilever is modeled by 100 two-node beam elements and loaded by either the stretching force $P_x = 1$ or the bending moment $M_z = 1$, or by the transverse force $P_y = 1$.

The linear solutions are presented in Table 7.1, where the tip's displacement and rotation are reported. We see that for $P_x = 1$, the recovery of ε_{33} is beneficial, while the recovery of κ_{33} has no effect. On the other hand, for $M_z = 1$ and $P_y = 1$, the situation is opposite and only the recovery of κ_{33} is beneficial. Without the recovery of κ_{33}, the solutions are too stiff and the error is 9%, as $\nu = 0.3$.

Table 7.1 Slender cantilever. Effect of recovery of normal strains for different loads.

Recovered strains	$P_x = 1$	$M_z = 1$		$P_y = 1$	
	$u_x \times 10^4$	$u_y \times 10^2$	$\omega \times 10^3$	u_y	$\omega \times 10^2$
none	0.91	5.46	1.0920	3.6402	5.46
ε_{33}	1.00	5.46	1.0920	3.6402	5.46
κ_{33}	0.91	6.00	1.2000	4.0002	6.00
Ref.	1.00	6.00	1.2000	4.0000	6.00

7.2.2 Reduced constitutive equations for incompressibility condition

The incompressibility condition, eq. (7.26) or (7.30), can be exploited in two ways:

1. It can be appended to the potential energy, i.e. $\Pi'(\boldsymbol{\chi}, p) \doteq \Pi(\boldsymbol{\chi}) + \int_V p\ (\det \mathrm{Grad}\boldsymbol{\chi} - 1)\ \mathrm{d}V$, where the pressure p serves as the Lagrange multiplier. Note that unless p is included as a variable, the calculated stress is determined up to the pressure, see [239], pp. 70–72. This method is generally applicable, see [218] and the literature cited therein.
2. It can be treated as an auxiliary equation to recover the normal strain for shells made of an incompressible material. This application is of interest in this section and two formulations are presented below:
   a) For membranes, the description is given in terms of principal stretches and we assume that all strains are constant over the thickness, see Sect. 7.2.2A,
   b) For arbitrary shells, we assume that all components, except the normal one, are linear polynomials of ζ. Hence, the recovered U_{33} is a rational function of ζ, and the question arises of how many terms in the expansion should be retained, see Sect. 7.2.2B.

A. Membranes. Description in principal stretches

The principal directions of the right stretching tensor $\mathbf{U}$ are defined as follows:

$$\mathbf{Q} \in \mathrm{SO}(3): \qquad \mathbf{Q}\mathbf{U}\mathbf{Q}^{\mathrm{T}} = \hat{\mathbf{U}}, \tag{7.87}$$

where $\hat{\mathbf{U}} = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\}$ and $\lambda_i\ (i = 1, 2, 3)$ are the principal stretches. For isotropic materials, the Biot stress $\hat{\mathbf{T}}_s^B$ is coaxial with $\hat{\mathbf{U}}$ and, hence, also $\mathbf{Q}\mathbf{T}_s^B\mathbf{Q}^T = \hat{\mathbf{T}}_s^B$ holds, where $\hat{\mathbf{T}}_s^B = \mathrm{diag}\{t_1, t_2, t_3\}$ and t_i are the principal values of the Biot stress. Note that $\mathbf{Q}$ can vary during deformation.

For membranes, the stretches λ_i are constant over the thickness. Besides, the transverse shear stresses and strains are equal to zero, so one principal direction is normal to the membrane; we designate it as λ_3. As a consequence a one-parameter rotation $\mathbf{Q}$ describes the orientation of the tangent principal axes.

To find the principal directions in the tangent plane, we note that the Cauchy–Green tensor $\mathbf{C} \doteq \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ has the same principal directions as $\mathbf{U}$, but has a simpler form. Hence, instead of eq. (7.87), we use the equation $\hat{\mathbf{C}} = \mathbf{Q}\mathbf{C}\mathbf{Q}^T$, where

$$\hat{\mathbf{C}} \doteq \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{12} & \hat{C}_{22} \end{bmatrix}, \qquad \mathbf{Q} \doteq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad \mathbf{C} \doteq \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix} \tag{7.88}$$

and θ is the angle defining the first principal direction. From the condition $\hat{C}_{12} = 0$, we find $\theta(C_{\alpha\beta})$ and, next, $\hat{C}_{11}(C_{\alpha\beta})$ and $\hat{C}_{22}(C_{\alpha\beta})$, where $\alpha, \beta = 1, 2$. Besides, we have the relations to stretches

$$\lambda_1^2 = \hat{C}_{11}(C_{\alpha\beta}), \qquad \lambda_2^2 = \hat{C}_{22}(C_{\alpha\beta}), \tag{7.89}$$

which are used to calculate the derivatives needed in constitutive equations, see eq. (7.100).

For incompressible materials, we can use the incompressibility condition of eq. (7.30) written in terms of the principal stretches, $\det \hat{\mathbf{U}} = \lambda_1\lambda_2\lambda_3 = 1$, to calculate the normal stretch

$$\lambda_3 = (\lambda_1\lambda_2)^{-1}, \tag{7.90}$$

and, next, to obtain the reduced strain energy.

Ogden's strain energy. The Ogden form of the strain energy is an isotropic function of principal stretches

$$\mathcal{W}(\lambda_i) = \sum_r \frac{\mu_r}{\alpha_r} \left[\lambda_1^{\alpha_r} + \lambda_2^{\alpha_r} + \lambda_3^{\alpha_r} - 3\right], \qquad i = 1, 2, 3, \tag{7.91}$$

where μ_r and α_r are the material constants, see [158, 159]. The number of terms r is selected to characterize a particular material, e.g. for rubber $r = 3$ is used. The principal values of the Biot stress are obtained as

$$t_i \doteq \frac{\partial \mathcal{W}(\lambda_j)}{\partial \lambda_i}, \qquad i, j = 1, 2, 3. \tag{7.92}$$

For incompressible materials, we can use λ_3 of eq. (7.90), and then the reduced strain energy depends only on two stretches

$$\mathcal{W}^*(\lambda_\alpha) = \sum_r \frac{\mu_r}{\alpha_r} \left[\lambda_1^{\alpha_r} + \lambda_2^{\alpha_r} + (\lambda_1\lambda_2)^{-\alpha_r} - 3\right], \qquad \alpha = 1, 2 \tag{7.93}$$

and the principal values of the Biot stress are

$$t_\alpha^* \doteq \frac{\partial \mathcal{W}(\lambda_\beta)}{\partial \lambda_\alpha}, \qquad \alpha, \beta = 1, 2. \tag{7.94}$$

Strain energy depending on invariants of $\mathbf{U}$. Assume that the strain energy is a function of the principal invariants of $\mathbf{U}$ which, in turn, are expressed by stretches λ_i,

$$\tilde{\mathcal{W}}(I_i(\mathbf{U})) = \hat{\mathcal{W}}(I_i(\hat{\mathbf{U}})), \qquad i = 1, 2, 3, \tag{7.95}$$

where the principal invariants of $\hat{\mathbf{U}}$ are as follows:

$$I_1(\hat{\mathbf{U}}) = \lambda_1 + \lambda_2 + \lambda_3, \quad I_2(\hat{\mathbf{U}}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad I_3(\hat{\mathbf{U}}) = \lambda_1\lambda_2\lambda_3. \tag{7.96}$$

The constitutive equation for the principal values of the Biot stress is calculated as

$$t_i \doteq \frac{\partial \hat{\mathcal{W}}(I_j(\hat{\mathbf{U}}))}{\partial \lambda_i} = \frac{\partial \hat{\mathcal{W}}}{\partial I_j} \frac{\partial I_j}{\partial \lambda_i}, \qquad i, j = 1, 2, 3. \tag{7.97}$$

Using λ_3 of eq. (7.90) in the two first invariants, we obtain

$$I_1^*(\hat{\mathbf{U}}) = \lambda_1 + \lambda_2 + (\lambda_1\lambda_2)^{-1}, \qquad I_2^*(\hat{\mathbf{U}}) = \lambda_1\lambda_2 + (\lambda_1)^{-1} + (\lambda_2)^{-1}. \tag{7.98}$$

The strain energy becomes a function of λ_1 and λ_2, i.e. $\tilde{\mathcal{W}}(I_i(\mathbf{U})) = \hat{\mathcal{W}}(I_\alpha^*(\hat{\mathbf{U}}))$, $\alpha = 1, 2$, and the constitutive equation is calculated as

$$t_\alpha^* \doteq \frac{\partial \hat{\mathcal{W}}(I_\beta^*(\hat{\mathbf{U}}))}{\partial \lambda_\alpha} = \frac{\partial \hat{\mathcal{W}}}{\partial I_\beta^*} \frac{\partial I_\beta^*}{\partial \lambda_\alpha}, \qquad \alpha, \beta = 1, 2. \tag{7.99}$$

Remark. The above formulas are simple, but the computational procedure for rubber-like membranes is not trivial because the relation between the stretches and strain components is complicated. For instance, the constitutive equation for the Ogden energy is

$$S^{\alpha\beta} = \frac{\partial \mathcal{W}(\lambda_\gamma)}{\partial E_{\alpha\beta}} = \frac{\partial \mathcal{W}(\lambda_\gamma)}{\partial \lambda_1} \frac{\partial \lambda_1}{\partial E_{\alpha\beta}} + \frac{\partial \mathcal{W}(\lambda_\gamma)}{\partial \lambda_2} \frac{\partial \lambda_2}{\partial E_{\alpha\beta}}, \qquad \alpha, \beta, \gamma = 1, 2, \tag{7.100}$$

where the derivatives $\partial\lambda_1/\partial E_{\alpha\beta}$ and $\partial\lambda_2/\partial E_{\alpha\beta}$ are computed by using eq. (7.89) and are complex. The computational procedure for the formulation in terms the second Piola–Kirchhoff stress and Green strain is given in [86].

Incompressibility condition for small strains. For small strains, we can obtain an alternative expression for the incompressibility condition. Consider the following linear Taylor expansions at $\lambda_1 = \lambda_2 = \lambda_3 = 1$,

$$\det \mathbf{U} \doteq \lambda_1\lambda_2\lambda_3 = 1 + \mathrm{d}\lambda_1 + \mathrm{d}\lambda_2 + \mathrm{d}\lambda_3 + \mathrm{O}(\mathrm{d}\lambda_1^2, \mathrm{d}\lambda_2^2, \mathrm{d}\lambda_3^2), \tag{7.101}$$

$$\mathrm{tr}(\mathbf{U} - \mathbf{I}) \doteq \lambda_1 + \lambda_2 + \lambda_3 - 3 = \mathrm{d}\lambda_1 + \mathrm{d}\lambda_2 + \mathrm{d}\lambda_3 + \mathrm{O}(\mathrm{d}\lambda_1^2, \mathrm{d}\lambda_2^2, \mathrm{d}\lambda_3^2). \tag{7.102}$$

Hence,

$$\det \mathbf{U} - 1 = \mathrm{tr}(\mathbf{U} - \mathbf{I}) = \mathrm{tr}\mathbf{H}, \tag{7.103}$$

with the second-order accuracy. Thus, for small strains, the incompressibility condition $\det \mathbf{U} = 1$ can be replaced by the condition $\mathrm{tr}\mathbf{H} = 0$.

B. Arbitrary shells

Consider the incompressibility condition of eq. (7.30), which is expressed in terms of the right stretching tensor. Let us write this condition as

$$\det \mathbf{U} = U_{31}D_{31} - U_{32}D_{32} + U_{33}D_{33} = 1, \tag{7.104}$$

where the minors are

$$D_{31} \doteq U_{12}U_{23} - U_{13}U_{22}, \quad D_{32} \doteq U_{11}U_{23} - U_{13}U_{12}, \quad D_{33} \doteq U_{11}U_{22} - U_{12}^2. \tag{7.105}$$

The normal component U_{33} appears in this equation only once and can be calculated as

$$U_{33} = \frac{1}{D_{33}}(1 - U_{31}D_{31} + U_{32}D_{32}), \tag{7.106}$$

where the r.h.s. depends on all components of $\mathbf{U}$ except the normal one.

We denote the $\mathbf{U}$ without the 33 component by $\mathbf{U}^*$, and assume that it is a linear polynomial of ζ, i.e. $\mathbf{U}^* = \mathbf{e}^* + \zeta\mathbf{k}^*$. Then U_{33} is a rational function of ζ with a polynomial of the third order in the nominator, and a polynomial of the second order in the denominator. Hence, unless the denominators are equal to zero, $U_{33}(\zeta)$ is infinitely times differentiable and we can perform the Taylor series expansion of it around the middle surface, retaining as many terms as necessary,

$$U_{33}(\zeta) = (U_{33})_0 + (U_{33,\zeta})_0\, \zeta + \tfrac{1}{2}\, (U_{33,\zeta\zeta})_0\, \zeta^2 + \mathrm{O}(\zeta^3), \tag{7.107}$$

where we can denote $(U_{33})_0 \doteq e_{33}$ and $(U_{33,\zeta})_0 \doteq k_{33}$ to keep the notation consistent. The so-recovered e_{33} and k_{33} can be used in $\mathbf{U}$.

A rigorous analysis of the question of how many terms of the expansion should be retained is difficult, see [202], because we must define, in advance, the class of deformation and geometry which is analyzed. Some insight provides the example given below in which we determine accuracy of the linear expansion of the normal strain.

Example. Inversion of a spherical cap. The example of an inversion of a spherical cap is solved analytically in [232]. The current position vector is assumed as $\mathbf{x}(\zeta) = \mathbf{x}_0 + \lambda(\zeta)\, \bar{\mathbf{n}}$, where $\bar{\mathbf{n}}$ is a unit vector normal to the deformed middle surface and $\lambda(\zeta)$ is the extension function. The deformed configuration also has a spherical shape, so $\lambda(\zeta)$ is obtained analytically, see eq. (6.4) therein. The obtained normal strain has the following form:

$$\varepsilon_{33}(\hat{\xi}) \doteq \frac{\partial \lambda}{\partial \zeta} = \frac{(1 + \frac{h}{R}\hat{\xi})^2}{\left[1 + \eta^3 - (1 + \frac{h}{R}\hat{\xi})^3\right]^{2/3}}, \tag{7.108}$$

where $\hat{\xi} = -(\zeta + h/2)/h$, $\hat{\xi} \in [-1, 0]$ and η is the in-plane stretch of the reference surface. We see that ε_{33} is a complicated function of $\hat{\xi}$, but a constant approximation of it is sufficient for the following limit cases:

1. Thin and/or flat shells. For $h/R \rightarrow 0$, we have $\varepsilon_{33}(\hat{\xi}) \rightarrow 1/\eta^3$.
2. Large stretches. For $\eta \rightarrow \infty$, we have $\varepsilon_{33}(\hat{\xi}) \rightarrow 0$, i.e. the normal strain vanishes. In reality, for the inflated structures made of rubber-like materials, $\eta \leq 10$.

The relative error of a linear expansion of $\varepsilon_{33}(\hat{\xi})$ at the middle surface $(\hat{\xi} = -0.5)$ is given in Table 7.2. We see that, when the 1% error at the external surface is acceptable, the linear approximation of ε_{33} can be used for a range of values of η and h/R.

Table 7.2 Relative error [in %] for linear expansion of normal strain ε_{33}.

η	h/R				
	0.2	0.1	0.05	0.01	0.001
0.5	14.95	4.98	1.53	0.070	0.00080
1.0	8.57	2.42	0.67	0.030	0.00030
1.5	3.54	0.83	0.20	0.008	0.00008
2.0	2.37	0.51	0.12	0.005	0.00005
5.0	1.61	0.32	0.07	0.003	0.00003
10.0	1.57	0.31	0.07	0.003	0.00003

7.3 Shear correction factor

The value of the shear correction factor k can be determined in several ways, see [259, 260]. Below, for the assumption that the distribution of the in-plane stresses is linear across thickness, we find that the transverse shear is parabolic and determine the value of the shear correction factor.

3D equilibrium equations and traction boundary conditions. The 3D equilibrium equations in a local Cartesian basis $\{\mathbf{t}_i\}$ at the reference surface of a shell are as follows:

$$\sigma_{\alpha\beta,\alpha} + \sigma_{3\beta,3} = 0, \qquad \sigma_{\alpha 3,\alpha} + \sigma_{33,3} = 0, \tag{7.109}$$

where σ_{ij} $(i, j = 1, 2, 3)$ is the stress (symmetric). We assume that the body force $b_i = 0$. The indices $\alpha, \beta = 1, 2$ correspond to the tangent (in-plane) directions and the index 3 to the normal direction of the basis $\{\mathbf{t}_k\}$, see Fig. 7.2.

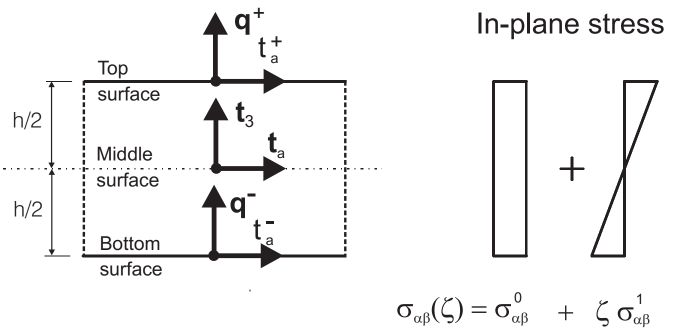

Fig. 7.2 The external loads and distribution of in-plane stress.

The transverse shear stress $\sigma_{3\beta}$ and the transverse normal stress σ_{33} have to satisfy the traction boundary conditions at surfaces bounding the shell,

$$\sigma_{3\beta}|_{\zeta=+\frac{h}{2}} = \tau_\beta^+, \qquad \sigma_{3\beta}|_{\zeta=-\frac{h}{2}} = \tau_\beta^-, \tag{7.110}$$

$$\sigma_{33}|_{\zeta=+\frac{h}{2}} = q^+, \qquad \sigma_{33}|_{\zeta=-\frac{h}{2}} = q^-, \tag{7.111}$$

where τ_β^+ and τ_β^- are the tangent components while q^+ and q^- are the normal components of the external load on the top and bottom surfaces, respectively.

Distribution of transverse shear stress. We assume that the in-plane stresses are linear over the thickness, i.e. $\sigma_{\alpha\beta}(\zeta) = \sigma^0_{\alpha\beta} + \zeta\sigma^1_{\alpha\beta}$, where $\zeta \in [-h/2, +h/2]$. Using the equilibrium equations (7.109), we determine the distribution of the transverse shear stress $\sigma_{3\beta}$ over the thickness.

Integrating eq. $(7.109)_1$ w.r.t. ζ (or 3), we have

$$\sigma_{3\beta}(\zeta) = C - \zeta\sigma^0_{\alpha\beta,\alpha} - \frac{\zeta^2}{2}\sigma^1_{\alpha\beta,\alpha}. \tag{7.112}$$

By the boundary condition at the bottom boundary, $\sigma_{3\beta}|_{\zeta=-\frac{h}{2}} = \tau_\beta^-$, we obtain

$$\sigma_{3\beta}(\zeta) = \tau_\beta^- - \left(\frac{h}{2} + \zeta\right)\sigma^0_{\alpha\beta,\alpha} + \left(\frac{h^2}{8} - \frac{\zeta^2}{2}\right)\sigma^1_{\alpha\beta,\alpha}. \tag{7.113}$$

There is no another constant to account for the condition at the top boundary $\zeta = +h/2$, but it is satisfied, as shown below. The integral of eq. $(7.109)_1$ over the thickness yields the relation

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{\alpha\beta,\alpha}\,\mathrm{d}\zeta + (\tau_\beta^+ - \tau_\beta^-) = h\sigma^0_{\alpha\beta,\alpha} + (\tau_\beta^+ - \tau_\beta^-) = 0. \tag{7.114}$$

On the other hand, for the top boundary, eq. (7.113) yields

$$\sigma_{3\beta}\,(+h/2) = \tau_\beta^- - h\sigma^0_{\alpha\beta,\alpha} \tag{7.115}$$

and, by eq. (7.114), the r.h.s. of this equation is equal to τ_β^+. Hence, $\sigma_{3\beta}(\zeta)$ of eq. (7.113) satisfies both the boundary conditions.

We can rewrite eq. (7.113) in several equivalent forms. By using $\sigma^0_{\alpha\beta,\alpha}$ calculated from eq. (7.114), we rewrite eq. (7.113) as

$$\sigma_{3\beta}(\zeta) = \frac{1}{2}\left(1 - \frac{2\zeta}{h}\right)\tau_\beta^- + \frac{1}{2}\left(1 + \frac{2\zeta}{h}\right)\tau_\beta^+ + \left(\frac{h^2}{8} - \frac{\zeta^2}{2}\right)\sigma^1_{\alpha\beta,\alpha} \tag{7.116}$$

or, using the natural coordinate $\bar{\zeta} \doteq 2\zeta/h \in [-1, +1]$,

$$\sigma_{3\beta}(\bar{\zeta}) = S_1(\bar{\zeta})\, \tau_\beta^- + S_2(\bar{\zeta})\, \tau_\beta^+ + \frac{h^2}{8} S_3(\bar{\zeta})\, \sigma^1_{\alpha\beta,\alpha}, \tag{7.117}$$

where the component functions are

$$S_1(\bar{\zeta}) \doteq \frac{1}{2}\left(1 - \bar{\zeta}\right), \quad S_2(\bar{\zeta}) \doteq \frac{1}{2}\left(1 + \bar{\zeta}\right), \quad S_3(\bar{\zeta}) \doteq 1 - \bar{\zeta}^2, \tag{7.118}$$

see Fig. 7.3. For the zero boundary conditions, $\tau_\beta^+ = \tau_\beta^- = 0$, we obtain a very simple formula

$$\sigma_{3\beta}(\bar{\zeta}) = \frac{h^2}{8}(1 - \bar{\zeta}^2)\, \sigma^1_{\alpha\beta,\alpha}. \tag{7.119}$$

Concluding, for the in-plane stress linearly distributed over the thickness, the distribution of the transverse shear stress is parabolic in $\bar{\zeta}$.

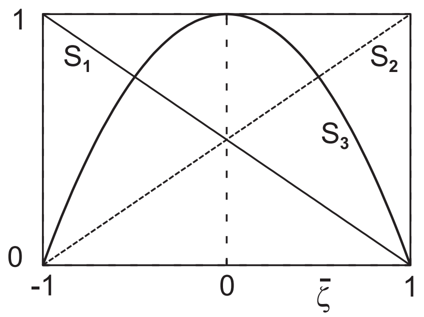

Fig. 7.3 Component functions for transverse shear stress.

Transverse shear stress in terms of shell resultants. We can express $\sigma_{3\beta}$ of eq. (7.117) in terms of the stress and couple resultants.

For the membrane stress $\sigma_{\alpha\beta}(\zeta) = \sigma^0_{\alpha\beta} + \zeta\sigma^1_{\alpha\beta}$, the in-plane stress and couple resultants are

$$N_{\alpha\beta} \doteq \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{\alpha\beta}(\zeta)\, \mathrm{d}\zeta = h\, \sigma^0_{\alpha\beta}, \qquad M_{\alpha\beta} \doteq \int_{-\frac{h}{2}}^{+\frac{h}{2}} \zeta\sigma_{\alpha\beta}(\zeta)\, \mathrm{d}\zeta = \frac{h^3}{12}\, \sigma^1_{\alpha\beta}. \tag{7.120}$$

We calculate $\sigma^1_{\alpha\beta} = (12/h^3)\, M_{\alpha\beta}$ from the last formula and use it in the transverse shear stress of eq. (7.117),

$$\sigma_{3\beta}(\bar{\zeta}) = S_1(\bar{\zeta})\,\tau_\beta^- + S_2(\bar{\zeta})\,\tau_\beta^+ + \frac{3}{2h}S_3(\bar{\zeta})\,M_{\alpha\beta,\alpha}. \tag{7.121}$$

For this form of $\sigma_{3\beta}$, the transverse shear stress and couple resultants are

$$N_{3\beta} \doteq \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{3\beta}(\zeta)\ \mathrm{d}\zeta = \frac{h}{2}(\tau_\beta^+ + \tau_\beta^-) + M_{\alpha\beta,\alpha}, \tag{7.122}$$

$$M_{3\beta} \doteq \int_{-\frac{h}{2}}^{+\frac{h}{2}} \zeta\,\sigma_{3\beta}(\zeta)\ \mathrm{d}\zeta = \frac{h^2}{12}(\tau_\beta^+ - \tau_\beta^-). \tag{7.123}$$

For the zero tangent loads, $\hat{\tau}_\beta^- = \hat{\tau}_\beta^+ = 0$, these resultants are reduced to

$$N_{3\beta} = M_{\alpha\beta,\alpha}, \qquad M_{3\beta} = 0, \tag{7.124}$$

where the first equation is a well-known formula linking the bending moment and the transverse shear resultant. By using it in the transverse shear stress of eq. (7.121), we obtain

$$\sigma_{3\beta}(\bar{\zeta}) = \frac{3}{2h}S_3(\bar{\zeta})\,M_{\alpha\beta,\alpha} = \frac{3}{2h}S_3(\bar{\zeta})\,N_{3\beta}, \tag{7.125}$$

which depends on the transverse shear resultant. The last form is identical to eq. $(20.5)_2$ of [153], p. 573.

Remark. Note that for the zero tangent loads, we have $M_{3\beta} = 0$ in eq. (7.124) and, hence, by the inverse constitutive equation, the first-order shell strain $\kappa_{3\beta} = 0$. Then we can omit the term with $\kappa_{3\beta}$ in the shell strain energy.

Shear correction factor. We can use the parabolic transverse shear stress to derive the shear correction factor. Note that, for the Reissner kinematics, the transverse shear strain is linear in ζ and cannot match the parabolic shear stress of eq. (7.125).

For the SVK material, the complementary energy density is

$$\begin{aligned}\mathcal{W}_c \doteq\ & \frac{1+\nu}{2E}(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{21}^2 + 2\sigma_{31}^2 + 2\sigma_{32}^2)\\ & - \frac{\nu}{2E}(\sigma_{11} + \sigma_{22} + \sigma_{33})^2,\end{aligned} \tag{7.126}$$

where E is Young's modulus and ν is the Poisson's ratio. For simplicity, we separate the term for the transverse shear stress,

$$\mathcal{W}_c^{3\beta} \doteq \frac{1+\nu}{E}\,\sigma_{3\beta}^2, \qquad \beta = 1,2. \tag{7.127}$$

For the transverse shear stress of eq. (7.125), the shell (integral) counterpart of $\mathcal{W}_c^{3\beta}$ becomes

$$\Sigma_c^{3\beta} \doteq \int_{-\frac{h}{2}}^{+\frac{h}{2}} \mathcal{W}_c^{3\beta}(\zeta)\ \mathrm{d}\zeta = \frac{1+\nu}{E}\,\frac{6}{5h} N_{3\beta}^2. \tag{7.128}$$

Then the inverse constitutive equations for the transverse shear strain is

$$\varepsilon_{3\beta} \doteq \frac{\partial \Sigma_c^{3\beta}}{\partial N_{3\beta}} = \frac{6}{5h}\,\frac{2(1+\nu)}{E}\ N_{3\beta}, \tag{7.129}$$

from which we can obtain the constitutive equation for the transverse shear stress resultant

$$N_{3\beta} = \frac{5}{6}\,\frac{E}{2(1+\nu)} h\, \varepsilon_{3\beta} = k\, Gh\ \varepsilon_{3\beta}, \tag{7.130}$$

where $G \doteq E/[2(1+\nu)]$ is the shear modulus and $k = 5/6$ is the shear correction factor. This factor accounts for the parabolic distribution of $\sigma_{3\beta}$ corresponding to the linear distribution of $\sigma_{\alpha\beta}$ over the thickness, and was obtained in [190]. Equation (7.130) corresponds to eq. $(20.12)_2$ of [153], p. 574.

Finally, we note that the shear correction factor can also be derived for the shearing moment $M_{3\beta}$ but it is rarely used, as usually the strain energy of $\kappa_{3\beta}$ is omitted in shell elements, as the second order quantity.

Summarizing, three results were obtained for shells in this section:

1. the formula for distributions of $\sigma_{3\beta}$ over the shell thickness, eq. (7.116) or (7.117),
2. the motivation for omitting the first-order shell strain $\kappa_{3\beta}$ in the strain energy, see eq. (7.124), and the remark which follows,
3. the shear correction factor for constitutive equation for $N_{3\beta}$, eq. (7.130).