5

Basic geometric definitions for shells

In this chapter, the basic geometric definitions needed to develop the shell FEs are provided. The shell can be intuitively but imprecisely defined as a 3D body, which has one dimension much smaller than the other two. More precisely, the shell is a surface in a 3D space equipped with a thickness, which is much smaller than the size of the surface. This implies a specific geometrical description of shells.

5.1 Coordinates and position vector

Normal coordinates for shells. For the initial configuration of a shell, we use the *normal* coordinates, see Fig. 5.1, the characteristic feature of which is that one coordinate is normal to the reference surface. The coordinates involved are defined as follows:

1. The reference shell surface is parameterized by the coordinates ϑ^α $(\alpha = 1, 2)$. This surface is selected arbitrarily, but most often the middle surface is used for this purpose; this is not suitable, e.g., for composites with non-symmetric stacking sequence of layers. The *middle* surface is equidistant from the top and bottom surfaces bounding the shell. Various types of coordinates can be used as ϑ^α.
2. The direction normal to the reference surface is parameterized by the coordinate $\zeta \in [-h/2, +h/2]$, where h is the initial shell thickness. We can also use the natural coordinate $\xi^3 \in [-1, +1]$, which is more convenient in numerical integration over the thickness. The relation between these coordinates is $\zeta = (h/2)\, \xi^3$.

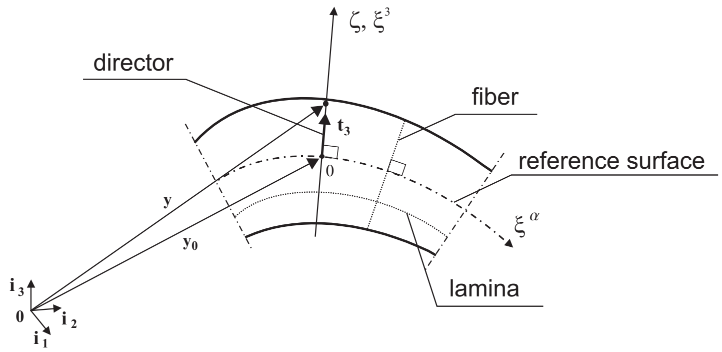

Fig. 5.1 Normal coordinates at a shell cross-section for initial configuration.

Selection of coordinates ϑ^α. Various types of coordinates can be used as ϑ^α.

1. In the FE method, the *natural* coordinates $\xi^\alpha \in [-1, +1]$ are used as ϑ^α. The corresponding tangent natural vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ are skew, i.e. neither unit nor perpendicular. The natural coordinates are arguments of the shape functions for finite elements, see Chap. 10.
2. In analytical derivations, the *orthonormal* coordinates S^α can be used as ϑ^α, see Chap. 6. They are associated with the orthogonal and unit vectors $\mathbf{t}_1$ and $\mathbf{t}_2$, in the plane tangent to the reference surface. Using them, we do not have to distinguish between co-variant and contra-variant components of vectors and tensors, and derivations are simplified.

Position vectors for shells. The position vector in the initial configuration is split as follows:

$$\mathbf{y}(\vartheta^\alpha, \zeta) = \mathbf{y}_0(\vartheta^\alpha) + \zeta\ \mathbf{t}_3(\xi^\alpha), \qquad \alpha = 1, 2, \tag{5.1}$$

where $\mathbf{y}_0$ is the position of the reference surface and $\mathbf{t}_3$ is the vector normal to this surface, called the *director*, see Fig. 5.2. Besides, $\mathbf{y}(\vartheta^\alpha, \zeta = \text{const.})$ defines the *lamina* while $\mathbf{y}(\vartheta^\alpha = \text{const.}, \zeta)$ defines the *fiber* of a shell.

We also assume that the normal coordinates are *convected*, which means that a position of a selected point is identified by the same pair $(\vartheta^\alpha, \zeta)$ in the initial configuration and in each deformed configuration. The position vector in the deformed configuration is split as follows:

$$\mathbf{x}(\vartheta^\alpha, \zeta) = \mathbf{x}_0(\vartheta^\alpha) + \mathbf{d}(\vartheta^\alpha, \zeta), \tag{5.2}$$

where $\mathbf{x}_0$ is the current position of the reference surface and $\mathbf{d}$ is the out-of-plane vector defined by kinematical assumptions. Note that $\mathbf{d}$ is also called the deformed or current director. For the Reissner hypothesis, $\mathbf{d}$ is not normal to the current reference surface, see Sect. 6, but it is normal for the Kirchhoff hypothesis, see Sect. 6.3.4.

Various formalisms in shell description. Typically, the displacement and rotation vectors are represented in the reference ortho-normal basis $\{\mathbf{i}_k\}$, to enable linking of finite elements of various spatial orientation. Different formalisms are obtained as a result of the following two choices:

1. Various bases can be used to represent the position vectors $\mathbf{y}$ and $\mathbf{x}$.
   a) The local Cartesian basis $\{\mathbf{t}_k^c\}$ at the element center. Then, first, the displacement and rotation components must be transformed from the reference basis to this local basis and, later, the tangent stiffness matrix and the residual vector generated in this local basis must be transformed back to the reference basis $\{\mathbf{i}_k\}$.
   b) The reference Cartesian basis $\{\mathbf{i}_k\}$. Then, to apply various shell assumptions (and techniques related to the FE method), we must transform strain components to the local Cartesian basis $\{\mathbf{t}_k\}$.
2. Various coordinates can be used to parameterize the position vectors $\mathbf{y}$ and $\mathbf{x}$ and, as a consequence, as intermediate variables for differentiation in the deformation gradient:
   a) For natural coordinates $\{\xi^\alpha, \zeta\}$, the current position vector $\mathbf{x} = \mathbf{x}(\xi^\alpha(\mathbf{y}), \zeta(\mathbf{y}))$, and the deformation gradient is as follows:

   $$\mathbf{F} \doteq \frac{\partial \mathbf{x}}{\partial \mathbf{y}} = \frac{\partial \mathbf{x}}{\partial \xi^\alpha} \otimes \frac{\partial \xi^\alpha}{\partial \mathbf{y}} + \frac{\partial \mathbf{x}}{\partial \zeta} \otimes \frac{\partial \zeta}{\partial \mathbf{y}}, \tag{5.3}$$

   this form is used, e.g., in Sect. 10.4.
   b) For orthonormal coordinates $\{S^\alpha, \zeta\}$, the current position vector $\mathbf{x} = \mathbf{x}(S^\alpha(\mathbf{y}), \zeta(\mathbf{y}))$, and the deformation gradient is as follows:

   $$\mathbf{F} \doteq \frac{\partial \mathbf{x}}{\partial \mathbf{y}} = \frac{\partial \mathbf{x}}{\partial S^\alpha} \otimes \frac{\partial S^\alpha}{\partial \mathbf{y}} + \frac{\partial \mathbf{x}}{\partial \zeta} \otimes \frac{\partial \zeta}{\partial \mathbf{y}}, \tag{5.4}$$

   this form is used, e.g., in Chap. 6.

   Besides, the natural coordinate $\xi^3 \in [-1, +1]$ can be used instead of $\zeta \in [-h/2, +h/2]$.

5.2 Basic geometric definitions

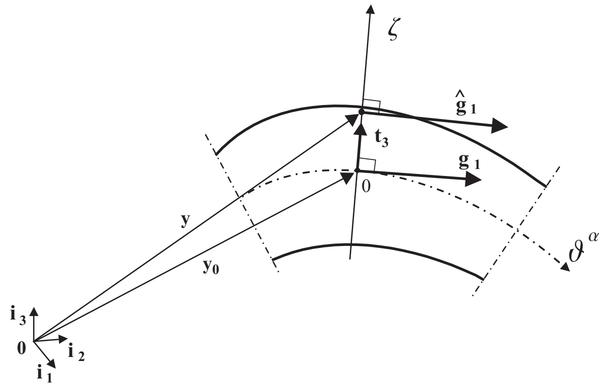

Fig. 5.2 Local bases $\{\hat{\mathbf{g}}_\alpha, \mathbf{t}_3\}$ and $\{\mathbf{g}_\alpha, \mathbf{t}_3\}$ for initial configuration.

Tangent basis varying over thickness. For the initial (non-deformed) configuration, the position vector $\mathbf{y} = \mathbf{y}(\vartheta^\alpha, \zeta)$ is given by eq. (5.1). The vectors tangent to the reference surface at arbitrary lamina ζ are obtained by differentiation of eq. (5.1),

$$\hat{\mathbf{g}}_\alpha(\zeta) \doteq \frac{\partial \mathbf{y}(\zeta)}{\partial \vartheta^\alpha} = \mathbf{g}_\alpha + \zeta\, \mathbf{t}_{3,\alpha}, \qquad \alpha = 1, 2, \tag{5.5}$$

where $\mathbf{g}_\alpha \doteq \partial \mathbf{y}_0 / \partial \vartheta^\alpha$. These vectors are neither unit nor mutually orthogonal, i.e.

$$\hat{\mathbf{g}}_\alpha(\zeta) \cdot \hat{\mathbf{g}}_\alpha(\zeta) = 1 + 2\zeta \mathbf{t}_{3,\alpha} \cdot \mathbf{g}_\alpha + \zeta^2 \mathbf{t}_{3,\alpha} \cdot \mathbf{t}_{3,\alpha} \neq 1 \quad \text{(no sum. over } \alpha),$$

$$\hat{\mathbf{g}}_1(\zeta) \cdot \hat{\mathbf{g}}_2(\zeta) = \zeta(\mathbf{t}_{3,1} \cdot \mathbf{g}_2 + \mathbf{t}_{3,2} \cdot \mathbf{g}_1) + \zeta^2 \mathbf{t}_{3,1} \cdot \mathbf{t}_{3,2} \neq 0,$$

but still $\hat{\mathbf{g}}_\alpha$ is normal to $\mathbf{t}_3$ because

$$\hat{\mathbf{g}}_\alpha \cdot \mathbf{t}_3 = \mathbf{g}_\alpha \cdot \mathbf{t}_3 + \zeta \mathbf{t}_{3,\alpha} \cdot \mathbf{t}_3 = 0,$$

where $\mathbf{g}_\alpha \cdot \mathbf{t}_3 = 0$ by definition, and $\mathbf{t}_{3,\alpha} \cdot \mathbf{t}_3 = 0$, as a result of differentiation of $\mathbf{t}_3 \cdot \mathbf{t}_3 = 1$ w.r.t. ϑ^α. Hence, $\hat{\mathbf{g}}_\alpha$ is parallel to $\mathbf{g}_\alpha$, and tangent to the reference surface.

Co-basis to tangent basis varying over thickness. The co-basis $\{\hat{\mathbf{g}}^\alpha, \mathbf{t}_3\}$ is also designated as the basis dual (or reciprocal) to $\{\hat{\mathbf{g}}_\alpha, \mathbf{t}_3\}$. The vectors $\hat{\mathbf{g}}^\alpha$ are defined as

$$\hat{\mathbf{g}}^\alpha \cdot \hat{\mathbf{g}}_\beta = \delta^\alpha_\beta, \qquad \hat{\mathbf{g}}^\alpha \cdot \mathbf{t}_3 = 0. \tag{5.6}$$

This definition provides three equations for $\hat{\mathbf{g}}^1$ and three for $\hat{\mathbf{g}}^2$, from which they can be directly determined. Alternatively, we can construct the co-basis as follows.

The conditions $\hat{\mathbf{g}}^1 \cdot \hat{\mathbf{g}}_2 = 0$ and $\hat{\mathbf{g}}^1 \cdot \mathbf{t}_3 = 0$ imply that $\hat{\mathbf{g}}^1$ is normal to $\hat{\mathbf{g}}_2$ and $\mathbf{t}_3$. Similarly, $\hat{\mathbf{g}}^2$ is normal to $\hat{\mathbf{g}}_1$ and $\mathbf{t}_3$. Hence, we can construct

$$\bar{\mathbf{g}}^1 = \bar{\mathbf{g}}_2 \times \mathbf{t}_3, \qquad \bar{\mathbf{g}}^2 = \mathbf{t}_3 \times \bar{\mathbf{g}}_1, \tag{5.7}$$

where $\bar{\mathbf{g}}_\alpha \doteq \hat{\mathbf{g}}_\alpha / \|\hat{\mathbf{g}}_\alpha\|$ are auxiliary unit vectors. The so-defined $\bar{\mathbf{g}}^\alpha$ have a proper direction, but their length is incorrect, i.e. $\bar{\mathbf{g}}^1 \cdot \hat{\mathbf{g}}_1 \neq 1$ and $\bar{\mathbf{g}}^2 \cdot \hat{\mathbf{g}}_2 \neq 1$. Hence, we define, $\hat{\mathbf{g}}^1 \doteq A\,\bar{\mathbf{g}}^1$ and $\hat{\mathbf{g}}^2 \doteq B\,\bar{\mathbf{g}}^2$, and from the conditions $\hat{\mathbf{g}}^1 \cdot \hat{\mathbf{g}}_1 = 1$ and $\hat{\mathbf{g}}^2 \cdot \hat{\mathbf{g}}_2 = 1$, we obtain $A = 1/(\bar{\mathbf{g}}^1 \cdot \hat{\mathbf{g}}_1)$ and $B = 1/(\bar{\mathbf{g}}^2 \cdot \hat{\mathbf{g}}_2)$. Finally, the vectors of the co-basis are as follows:

$$\hat{\mathbf{g}}^1 = \frac{\hat{\mathbf{g}}_2 \times \mathbf{t}_3}{(\hat{\mathbf{g}}_2 \times \mathbf{t}_3) \cdot \hat{\mathbf{g}}_1}, \qquad \hat{\mathbf{g}}^2 = \frac{\mathbf{t}_3 \times \hat{\mathbf{g}}_1}{(\mathbf{t}_3 \times \hat{\mathbf{g}}_1) \cdot \hat{\mathbf{g}}_2}, \tag{5.8}$$

and they belong to the plane spanned by $\hat{\mathbf{g}}_\alpha$.

From $\hat{\mathbf{g}}_\beta(\zeta) \doteq \partial \mathbf{y} / \partial \vartheta^\beta$ of eq. (5.5) and $\hat{\mathbf{g}}^\alpha \cdot \hat{\mathbf{g}}_\beta = \delta^\alpha_\beta$, we can deduce the following definition of a vector of the co-basis:

$$\hat{\mathbf{g}}^\alpha(\zeta) \doteq \frac{\partial \vartheta^\alpha}{\partial \mathbf{y}(\zeta)}. \tag{5.9}$$

Shifter (translation) tensor Z. The tangent vectors of eq. (5.5) can be alternatively expressed as

$$\hat{\mathbf{g}}_\alpha(\zeta) = \mathbf{g}_\alpha + \zeta \mathbf{t}_{3,\alpha} = (\mathbf{G}_0 - \zeta \mathbf{B})\,\mathbf{g}_\alpha = \mathbf{Z}(\zeta)\ \mathbf{g}_\alpha, \tag{5.10}$$

where $\mathbf{G}_0 \doteq \mathbf{g}_\alpha \otimes \mathbf{g}^\alpha$ is the metric tensor and $\mathbf{B} \doteq -\mathbf{t}_{3,\alpha} \otimes \mathbf{g}^\alpha$ is the curvature tensor, both for the reference surface and symmetric. Hence, the shifter tensor, $\mathbf{Z}(\zeta) \doteq \mathbf{G}_0 - \zeta \mathbf{B}$, maps the vectors $\mathbf{g}_\alpha$ at the reference surface onto the vectors $\hat{\mathbf{g}}_\alpha$ at an arbitrary lamina ζ, accounting for the curvature of the reference surface. For a flat geometry, i.e. when the curvature $\mathbf{B} = \mathbf{0}$, we have $\mathbf{Z}(\zeta) = \mathbf{G}_0$ i.e. the dependence on ζ vanishes.

The shifter tensor for the co-basis vectors $\hat{\mathbf{g}}^\alpha$ can be found by making use of the condition $\hat{\mathbf{g}}_\alpha(\zeta) \cdot \hat{\mathbf{g}}^\alpha(\zeta) = 1$ (no summation over α). Using the shifter tensor $\mathbf{Z}$ for $\hat{\mathbf{g}}_\alpha$ and an auxiliary (unknown) tensor $\mathbf{A}$ for $\hat{\mathbf{g}}^\alpha$, we have to satisfy the condition $(\mathbf{Z}\mathbf{g}_\alpha) \cdot (\mathbf{A}\mathbf{g}^\alpha) = 1$ or transforming further, $(\mathbf{A}^T \mathbf{Z}\, \mathbf{g}_\alpha) \cdot \mathbf{g}^\alpha = 1$. As $\mathbf{g}_\alpha \cdot \mathbf{g}^\alpha = 1$, hence $\mathbf{A}^T \mathbf{Z}\, \mathbf{g}_\alpha = \mathbf{g}_\alpha$ must hold. Therefore, a symmetric $\mathbf{A} \doteq \mathbf{Z}^{-1}$ is a shifter for the co-basis vectors, i.e.

$$\hat{\mathbf{g}}^\alpha(\zeta) = \mathbf{Z}^{-1}(\zeta)\; \mathbf{g}^\alpha. \tag{5.11}$$

The inverse $\mathbf{Z}^{-1}(\zeta)$ can be easily found in terms of components of $\mathbf{G}_0$ and $\mathbf{B}$,

$$(\mathbf{Z})_{ij} = \begin{bmatrix} G_{11} - \zeta B_{11} & G_{12} - \zeta B_{12} \\ \text{sym.} & G_{22} - \zeta B_{22} \end{bmatrix},$$

$$(\mathbf{Z})^{-1}_{ij} = \mu^{-1} \begin{bmatrix} G_{22} - \zeta B_{22} & -G_{12} + \zeta B_{12} \\ \text{sym.} & G_{11} - \zeta B_{11} \end{bmatrix}, \tag{5.12}$$

where $\mu \doteq \det \mathbf{Z} = \det \mathbf{G}_0 - \zeta(G_{11}B_{22} + G_{22}B_{11} - 2G_{12}B_{12}) + \zeta^2 \det \mathbf{B}$. The inverse of the shifter can be rewritten as

$$\begin{aligned} \mathbf{Z}^{-1}(\zeta) &= \mu^{-1} \left[(\det \mathbf{G}_0)\mathbf{G}_0^{-1} - \zeta(\det \mathbf{B})\, \mathbf{B}^{-1}\right] \\ &= \mu^{-1} \left[\mathrm{tr}(\mathbf{G}_0 - \zeta\mathbf{B})\mathbf{I} - (\mathbf{G}_0 - \zeta\mathbf{B})\right], \end{aligned} \tag{5.13}$$

where the last form does not use the inverse of $\mathbf{G}_0$ and $\mathbf{B}$. It is obtained from the Cayley–Hamilton formula, which, e.g., for $\mathbf{B}$ is as follows:

$$\mathbf{B}^2 - I_1\, \mathbf{B} + I_2\, \mathbf{I} = \mathbf{0}, \tag{5.14}$$

where $I_1 = \mathrm{tr}\mathbf{B} = 2H$ and $I_2 = \frac{1}{2}(\mathrm{tr}\mathbf{B} - \mathrm{tr}\mathbf{B}^2) = \det \mathbf{B} = \mathrm{K}$. Besides, $H \doteq \frac{1}{2}\mathrm{tr}\mathbf{B}$ is the mean curvature and $K \doteq \det \mathbf{B}$ is the Gaussian curvature. Multiplying eq. (5.14) by $\mathbf{B}^{-1}$, we obtain $I_2\, \mathbf{B}^{-1} = I_1\mathbf{I} - \mathbf{B}$, which provides the last form of eq. (5.13). In a similar way, we modify the term for $\mathbf{G}_0$.

For a flat geometry, i.e. when the curvature $\mathbf{B} = \mathbf{0}$, we obtain $\mu = \det \mathbf{G}_0$, and $\mathbf{Z}^{-1}(\zeta) = \mathbf{G}_0^{-1}$.

Deformation gradient and identity tensor. Assume the initial position vector of the shell as in eq. (5.1). For the current position vector $\mathbf{x} = \mathbf{x}(\vartheta^\alpha(\mathbf{y}), \zeta(\mathbf{y}))$, the deformation gradient can be written as

$$\mathbf{F} \doteq \frac{\partial \mathbf{x}}{\partial \mathbf{y}} = \frac{\partial \mathbf{x}}{\partial \vartheta^\alpha} \otimes \frac{\partial \vartheta^\alpha}{\partial \mathbf{y}} + \frac{\partial \mathbf{x}}{\partial \zeta} \otimes \frac{\partial \zeta}{\partial \mathbf{y}} = \mathbf{x}_{,\alpha} \otimes \hat{\mathbf{g}}^\alpha + \mathbf{x}_{,\zeta} \otimes \mathbf{t}^3, \tag{5.15}$$

where $\partial\vartheta^\alpha/\partial\mathbf{y}(\zeta) = \hat{\mathbf{g}}^\alpha$ by eq. (5.9) and $\partial\zeta/\partial\mathbf{y} = \mathbf{t}^3 = \mathbf{t}_3$ by eq. (5.1). Note that

$$\mathbf{F}\,\hat{\mathbf{g}}_\alpha = \mathbf{x}_{,\alpha}, \qquad \mathbf{F}\,\mathbf{t}_3 = \mathbf{x}_{,\zeta}. \tag{5.16}$$

The identity tensor is defined as the second-rank tensor obtained from the deformation gradient for the current position vector $\mathbf{x}$ assumed as equal to the initial position vector $\mathbf{y}$, i.e.

$$\mathbf{I} \doteq \mathbf{F}|_{\mathbf{x}=\mathbf{y}} = \hat{\mathbf{g}}_\alpha \otimes \hat{\mathbf{g}}^\alpha + \mathbf{t}_3 \otimes \mathbf{t}^3. \tag{5.17}$$

This definition guarantees that the approximations of $\mathbf{I}$ and $\mathbf{F}$ over ζ are consistent, and that the approximated $\mathbf{F} = \mathbf{I}$ for a rigid body motion. A similar reasoning can be applied to the rotation tensor $\mathbf{Q} \in \mathrm{SO}(3)$, see the application in eqs. (6.13) and (6.15).

To express eq. (5.11) in the basis on the reference surface, we use $\hat{\mathbf{g}}^\alpha(\zeta) = \mathbf{Z}^{-1}(\zeta)\,\mathbf{g}^\alpha$ and then the simplicity of the above forms of $\mathbf{F}$ and $\mathbf{I}$ disappears.

Restriction on curvature of a shell. Let us estimate the contribution of the term related to the shell curvature to the norm of the tangent vector. Using eq. (5.5), we obtain

$$\|\,\hat{\mathbf{g}}_\alpha\| = \|\mathbf{g}_\alpha + \zeta\mathbf{t}_{3,\alpha}\| \leq \|\mathbf{g}_\alpha\| + \|\zeta\mathbf{t}_{3,\alpha}\|, \tag{5.18}$$

where $\|\mathbf{g}_\alpha\| = (\mathbf{g}_\alpha \cdot \mathbf{g}_\alpha)^{\frac{1}{2}}$, $\|\mathbf{t}_{3,\alpha}\| = (\mathbf{t}_{3,\alpha} \cdot \mathbf{t}_{3,\alpha})^{\frac{1}{2}}$. We may safely omit the second term, related to curvature, when

$$\frac{h}{2}\,\|\mathbf{t}_{3,\alpha}\| \ll \|\mathbf{g}_\alpha\|. \tag{5.19}$$

For a cylindrical surface, this restriction becomes

$$\frac{h}{2R} \ll 1, \tag{5.20}$$

see the example of Sect. 5.3 and eq. (5.36). If eq. (5.19) holds, then the ζ-dependent part of the shifter $\mathbf{Z}(\zeta)$ can be omitted, i.e. we use $\zeta\mathbf{B} \approx \mathbf{0}$, which implies

$$\mathbf{Z}(\zeta) \approx \mathbf{G}_0, \qquad \mathbf{Z}^{-1}(\zeta) \approx \mathbf{G}_0^{-1}, \qquad \mu \doteq \det\mathbf{Z} = \det\mathbf{G}_0. \tag{5.21}$$

Further simplifications are obtained for the orthonormal coordinates S^α, see the next paragraph.

Remark. The above restriction on the curvature of the reference surface is not used in the shell FEs derived in this work. It is used only in some analytical derivations, e.g. in Chap. 6.

Note, however, that there are FEs in use where this restriction is applied for efficiency. The curved shell structures can be analyzed by such elements provided that the discretization error is minimized by using a sufficiently large number of elements and by a suitable choice of their shapes and positions.

Simplifications for orthonormal coordinates S^α. The orthonormal coordinates S^α are often used as ϑ^α in analytical derivations, see e.g. Chap. 6. These coordinates are associated with the tangent orthonormal vectors $\mathbf{t}_\alpha$, which are used instead of $\mathbf{g}_\alpha$.

For the reference surface, $\zeta = 0$, we denote $\mathbf{g}_\alpha = \mathbf{t}_\alpha$, where $\mathbf{t}_\alpha$ are unit and orthogonal by the definition of coordinates S^α. Defining $\mathbf{t}^\alpha \doteq \hat{\mathbf{g}}^\alpha(\zeta)|_{\zeta=0}$, we obtain from eq. (5.8)

$$\mathbf{t}^1 = \frac{\mathbf{t}_2 \times \mathbf{t}_3}{(\mathbf{t}_2 \times \mathbf{t}_3) \cdot \mathbf{t}_1} = \mathbf{t}_1, \qquad \mathbf{t}^2 = \frac{\mathbf{t}_3 \times \mathbf{t}_1}{(\mathbf{t}_3 \times \mathbf{t}_1) \cdot \mathbf{t}_2} = \mathbf{t}_2, \tag{5.22}$$

i.e. the basis and the co-basis on the reference surface are identical. Hence, we do not distinguish between co-variant and contra-variant components of vectors and tensors, and derivations are simplified.

For the orthonormal coordinates, the metric tensor $\mathbf{G}_0 = \mathbf{I}$, and $\det \mathbf{G}_0 = 1$. The shifter tensor and its inverse of eq. (5.12) become simpler,

$$(\mathbf{Z})_{ij} = \begin{bmatrix} 1 - \zeta B_{11} & -\zeta B_{12} \\ \text{sym.} & 1 - \zeta B_{22} \end{bmatrix}, \quad (\mathbf{Z})^{-1}_{ij} = \mu^{-1} \begin{bmatrix} 1 - \zeta B_{22} & \zeta B_{12} \\ \text{sym.} & 1 - \zeta B_{11} \end{bmatrix} \tag{5.23}$$

or

$$\mathbf{Z}^{-1}(\zeta) = \mu^{-1} \left(\mathbf{I} - \zeta K \, \mathbf{B}^{-1} \right) = \mu^{-1} \left[\mathbf{I} - \zeta (2H\mathbf{I} - \mathbf{B}) \right], \tag{5.24}$$

where $\mu \doteq \det \mathbf{Z} = 1 - \zeta(2H) + \zeta^2 K$. For the restriction on curvature of eq. (5.19), we obtain $\mu \approx \det \mathbf{G}_0 = 1$, $\mu^{-1} = 1$, and $\mathbf{Z}^{-1}(\zeta) \approx \mathbf{G}_0^{-1} = \mathbf{G}_0 = \mathbf{I}$. As a consequence, some expressions are significantly simplified.

Remark. Geometry of the four-node finite element is approximated by the bilinear shape functions, so it is either flat (planar) or a hyperbolic paraboloid (h-p) surface. For a planar element, $H = 0$, and $K = 0$, i.e. it consists of only parabolic points. For the h-p element, H is a complicated function and $K < 0$, i.e. it consists of hyperbolic points only.

5.3 Example: Geometrical description of cylinder

Consider a cylindrical shell shown in Fig. 5.3. Its middle surface can be parameterized in a standard manner by cylindrical coordinates: the radius R, the angle θ (measured in the $\{\mathbf{i}_1, \mathbf{i}_3\}$-plane, and starting from $\mathbf{i}_1$) and the generator coordinate, t. The reference Cartesian basis is denoted by $\{\mathbf{i}_k\}$.

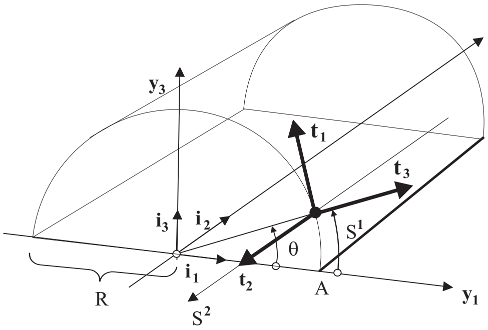

Fig. 5.3 Local basis $\{\mathbf{t}_k\}$ for a cylinder.

A position vector of an arbitrary point on the surface is given by $\mathbf{y} = y_k \mathbf{i}_k$, where $y_1 = R\cos\theta$, $y_2 = -t$, $y_3 = R\sin\theta$. The length of a circumferential arc on the cylinder is

$$S^1 = \int_0^\theta \sqrt{y_{1,\theta}^2 + y_{3,\theta}^2}\, \mathrm{d}\theta = \theta R. \tag{5.25}$$

Next, we introduce the arc-length surface coordinates: one along a circumference, $S^1 = \theta R$, and the other along a generator, $S^2 = t$. Then, the components of the position vector are

$$y_1 = R\cos\frac{S^1}{R}, \qquad y_2 = -S^2, \qquad y_3 = R\sin\frac{S^1}{R}, \tag{5.26}$$

and their non-zero derivatives are $\partial y_1/\partial S^1 = -\sin(S^1/R)$, $\partial y_2/\partial S^2 = -1$, $\partial y_3/\partial S^1 = \cos(S^1/R)$. Hence, the tangent vectors of the local basis associated with the arc-length coordinates are

$$\mathbf{t}_1 = \frac{\partial \mathbf{y}}{\partial S^1} = -\sin\frac{S^1}{R}\mathbf{i}_1 + \cos\frac{S^1}{R}\mathbf{i}_3, \qquad \mathbf{t}_2 = \frac{\partial \mathbf{y}}{\partial S^2} = -\mathbf{i}_2, \tag{5.27}$$

i.e. $\mathbf{t}_1$ and $\mathbf{t}_2$ are unit and orthogonal. Components of the metric tensor, $\mathbf{G}_0 \doteq \mathbf{t}_\alpha \otimes \mathbf{t}_\alpha$, are

$$G_{11} = \mathbf{t}_1 \cdot \mathbf{t}_1 = 1, \qquad G_{12} = \mathbf{t}_1 \cdot \mathbf{t}_2 = 0, \qquad G_{22} = \mathbf{t}_2 \cdot \mathbf{t}_2 = 1. \tag{5.28}$$

For the arc-length coordinates, a unit length of tangent vectors is a general property, see [230] p. 6, while their orthogonality is implied here by a specific choice of S^1 and S^2. The unit vector normal to the surface can be obtained as

$$\mathbf{t}_3 = \mathbf{t}_1 \times \mathbf{t}_2 = \cos\frac{S^1}{R}\mathbf{i}_1 + \sin\frac{S^1}{R}\mathbf{i}_3, \tag{5.29}$$

and its derivatives are

$$\mathbf{t}_{3,1} = \frac{1}{R}\left(-\sin\frac{S^1}{R}\mathbf{i}_1 + \cos\frac{S^1}{R}\mathbf{i}_3\right) = \frac{1}{R}\mathbf{t}_1, \qquad \mathbf{t}_{3,2} = \mathbf{0}. \tag{5.30}$$

Hence, the curvature tensor is $\mathbf{B} \doteq -\mathbf{t}_{3,\alpha} \otimes \mathbf{t}_\alpha = -\frac{1}{R}\mathbf{t}_1 \otimes \mathbf{t}_1$, at its components in the basis $\{\mathbf{t}_\alpha\}$ are

$$B_{11} = -\mathbf{t}_{3,1} \cdot \mathbf{t}_1 = -\frac{1}{R}, \qquad B_{12} = -\mathbf{t}_{3,1} \cdot \mathbf{t}_2 = 0, \qquad B_{22} = -\mathbf{t}_{3,2} \cdot \mathbf{t}_2 = 0. \tag{5.31}$$

Then, the mean curvature $H \doteq \frac{1}{2}\mathrm{tr}\mathbf{B} = -1/(2\mathrm{R})$ and the Gaussian curvature $K \doteq \det\mathbf{B} = 0$.

Let us construct a shell-like body by equipping the cylindrical surface with the thickness h. Then the position vector is $\mathbf{y}(\zeta) = \mathbf{y}_0 + \zeta\,\mathbf{t}_3$, where $\zeta \in [-h/2, +h/2]$. For an arbitrary ζ, the basis vectors defined by eq. (5.5) are

$$\hat{\mathbf{t}}_1(\zeta) = \left(1 + \frac{\zeta}{R}\right)\mathbf{t}_1, \qquad \hat{\mathbf{t}}_2(\zeta) = \mathbf{t}_2, \tag{5.32}$$

where the mid-surface tangent vectors of eq. (5.27) and the derivatives of eq. (5.30) are used. We see that the basis vector $\hat{\mathbf{t}}_1(\zeta)$ has a direction of $\mathbf{t}_1$, but its length varies with ζ, see Fig. 5.4. This has a obvious consequence that, e.g. for a displacement vector $\mathbf{u}$ constant over ζ, the component $u_1(\zeta) \doteq \mathbf{u} \cdot \hat{\mathbf{t}}_1(\zeta)$ varies with ζ. This also implies a nontrivial form of the shifter tensor of eq. (5.10), which becomes

$$\mathbf{Z}(\zeta) = \left(1 + \frac{\zeta}{R}\right)\mathbf{t}_1 \otimes \mathbf{t}_1 + \mathbf{t}_2 \otimes \mathbf{t}_2. \tag{5.33}$$

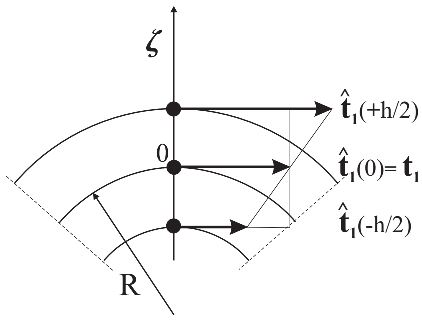

Fig. 5.4 Tangent vector $\hat{\mathbf{t}}_1$ at characteristic values of coordinate ζ.

We can easily check that $\hat{\mathbf{t}}_1(\zeta) = \mathbf{Z}(\zeta)\,\mathbf{t}_1$ and $\hat{\mathbf{t}}_2(\zeta) = \mathbf{Z}(\zeta)\,\mathbf{t}_2$, indeed. Besides, $\mu \doteq \det \mathbf{Z} = 1 + (\zeta/R)$.

Now, we can examine the basis vector $\hat{\mathbf{t}}_\alpha(\zeta) = \mathbf{t}_\alpha + \zeta \mathbf{t}_{3,\alpha}$, and estimate a contribution of the second term resulting from the shell curvature. Thus,

$$\|\,\hat{\mathbf{t}}_\alpha\| = \|\mathbf{t}_\alpha + \zeta \mathbf{t}_{3,\alpha}\| \leq \|\mathbf{t}_\alpha\| + \|\zeta \mathbf{t}_{3,\alpha}\|, \tag{5.34}$$

where $\|\mathbf{t}_1\| = (\mathbf{t}_1 \cdot \mathbf{t}_1)^{\frac{1}{2}} = 1$, $\|\mathbf{t}_2\| = (\mathbf{t}_2 \cdot \mathbf{t}_2)^{\frac{1}{2}} = 1$, and $\|\mathbf{t}_{3,1}\| = (\mathbf{t}_{3,1} \cdot \mathbf{t}_{3,1})^{\frac{1}{2}} = 1/R$, $\|\mathbf{t}_{3,2}\| = (\mathbf{t}_{3,2} \cdot \mathbf{t}_{3,2})^{\frac{1}{2}} = 0$. For $\zeta = \pm\frac{h}{2}$, we obtain

$$\|\hat{\mathbf{t}}_1\| \leq 1 + \frac{h}{2R}, \qquad \|\hat{\mathbf{t}}_2\| = 1. \tag{5.35}$$

The second term of $\|\hat{\mathbf{t}}_1\|$ is negligible when

$$\frac{h}{2R} \ll 1, \tag{5.36}$$

which illustrates the restriction of eq. (5.19).

The vector $\hat{\mathbf{t}}_2(\zeta)$ given by eq. (5.32) is a unit vector and, hence, we can easily obtain the co-basis, i.e.

$$\hat{\mathbf{t}}^1(\zeta) = \left(1 + \frac{\zeta}{R}\right)^{-1} \mathbf{t}_1, \qquad \hat{\mathbf{t}}^2(\zeta) = \mathbf{t}_2, \tag{5.37}$$

and check that $\hat{\mathbf{t}}^1 \cdot \hat{\mathbf{t}}_1 = 1$, $\hat{\mathbf{t}}^2 \cdot \hat{\mathbf{t}}_2 = 1$, $\hat{\mathbf{t}}^1 \cdot \hat{\mathbf{t}}_2 = 0$, and $\hat{\mathbf{t}}^2 \cdot \hat{\mathbf{t}}_1 = 0$, indeed. The inverse of the shifter is

$$\mathbf{Z}^{-1}(\zeta) = \left(1 + \frac{\zeta}{R}\right)^{-1} \mathbf{t}_1 \otimes \mathbf{t}_1 + \mathbf{t}_2 \otimes \mathbf{t}_2, \tag{5.38}$$

and, using eq. (5.33), we can check that indeed $\mathbf{Z}^{-1}(\zeta)\,\mathbf{Z}(\zeta) = \mathbf{I}$.