1

Introduction

1.1 Subject of the book

An imprecise but intuitively clear definition of the *shell structure* is that it is a 3D body, with one dimension much smaller that the other two. In other words, it is a surface in a 3D space equipped with a thickness which is much smaller than the size of the surface. There are numerous examples of shells among engineering structures, such as bodies of cars, hulls of ships, and fuselages of airplanes. Also some civil engineering structures, such as tanks and roofs can often be considered and analyzed as shells.

However, we must see the difference between shell structures and the shells defined in mechanics.

In mechanics, the *shell* designates the 2D governing equations obtained from equations for a 3D body when using some kinematical assumptions. The most often used assumption is either the Kirchhoff hypothesis or the Reissner hypothesis, and each of them implies a different structure of governing equations. The first hypothesis is preferred in theoretical works, while the second one is used in computational mechanics, and also in this book. When the shell equations are treated by the finite element method, which is the most popular and convenient method for engineering analysis, we then obtain *shell elements*. Mostly due to the presence of the rotational degrees of freedom, the shell elements involve specific problems not encountered in displacement-type elements.

This book is concerned with finite elements for Reissner shells and its subject is defined by the following *keywords*:

1. Computational mechanics of non-linear shells,
2. Shell equations: Reissner kinematics, finite rotations, finite strains,

3. Shell finite elements: four-node, enhanced or mixed or mixed/enhanced,
4. Drilling rotation: drill rotation constraint or Allman's shape functions,
5. Normal strain: recovered or parameterized,
6. Constitutive equations: incremental, plane stress or 3D.

The basic information on linear shell elements may be found in some textbooks on finite elements, but this book contains several advanced topics related to non-linear shells such as, e.g., the parametrization of finite rotations, the methods of inclusion of the drilling rotation, various methods of treating the normal strain, and the mixed/enhanced finite elements. Some of these topics have been the subject of our research for years and all the described methods have been implemented in our own elements and tested.

A wide range of applications of shell elements implies that they should be versatile, i.e. account for finite rotations and strains, admit the incorporation of various constitutive laws, and enable convenient linking with other elements. This is a serious challenge which requires various aspects of the element's formulation to be well advanced.

We must be aware that the use of the Reissner hypothesis has not only positive but also negative consequences. The positive ones are that the shell elements admit large thickness/size ratios, adequately represent bending modes, and can be based on C^o approximations. The negative consequences are related to the normal strain and the drilling rotation.

1. *Normal strain.* The standard Reissner (or Kirchhoff) hypothesis yields the normal strain component equal to zero which provides an unrealistic constraint on bending deformation so that the 3D constitutive laws cannot be directly used. For this reason, the normal strain must be either recovered from a suitable auxiliary condition or the shell kinematics must be enhanced by additional stretch parameters.
2. *Drilling rotation.* The shell strains obtained from the Green strain by using the Reissner hypothesis do not depend on the drilling rotation which is the rotation about the vector normal to the shell surface. The drilling rotation is needed to use three parameters for increments of rotations and conveniently link the shell elements with 3D beam and shell elements. For this reason, we derive shell equations from the 3D mixed equations which incorporate the drilling rotation as an independent variable.

The current state of the shell equations and shell elements has been achieved gradually through the efforts of many researchers which yielded thousands of works on the subject. This book benefitted from many of them, although not all of them have been cited here.

1.2 Notation

General rules of notation

1. Small bold letters - vectors, e.g. $\mathbf{v}$.
2. Capital bold letters - second-rank tensors, e.g. $\mathbf{A}$.
3. Open-face letters - fourth-rank tensors, e.g. $\mathbb{C}$.
4. Arrays of components of vectors and tensors are denoted by the same letters as vectors and tensors. Sometimes, sans serif fonts are used, e.g. v, A.
5. Superscript asterisk (*) - forward-rotated objects, e.g. $\mathbf{A}^*$,
6. Subscript asterisk ($_*$) - backward-rotated objects, e.g. $\mathbf{A}_*$,
7. "$\cdot$", "$\times$", "$\otimes$" - scalar product, cross product, tensorial product,
8. Symmetric part - $\operatorname{sym}\mathbf{T} \doteq \frac{1}{2}(\mathbf{T}+\mathbf{T}^T)$. Besides, $\mathbf{T}_s \doteq \operatorname{sym}\mathbf{T}$,
9. Skew-symmetric part - $\operatorname{skew}\mathbf{T} \doteq \frac{1}{2}(\mathbf{T}-\mathbf{T}^T)$. Besides, $\mathbf{T}_a \doteq \operatorname{skew}\mathbf{T}$,
10. Components of tensors - $A_{ij} = \mathbf{A}\cdot(\mathbf{t}_i \otimes \mathbf{t}_j) = (\mathbf{A}\mathbf{t}_j)\cdot\mathbf{t}_i$,
11. Gradient of a scalar A and gradient of a vector $\{v_i\}$ w.r.t. coordinates S^k,

$$\left[\frac{\partial A}{\partial S^k}\right] = \begin{bmatrix} \frac{\partial A}{\partial S^1} \\ \frac{\partial A}{\partial S^2} \\ \frac{\partial A}{\partial S^3} \end{bmatrix}, \qquad \left[\frac{\partial v_i}{\partial S^k}\right] = \begin{bmatrix} \frac{\partial v_1}{\partial S^1} & \frac{\partial v_2}{\partial S^1} & \frac{\partial v_3}{\partial S^1} \\ \frac{\partial v_1}{\partial S^2} & \frac{\partial v_2}{\partial S^2} & \frac{\partial v_3}{\partial S^2} \\ \frac{\partial v_1}{\partial S^3} & \frac{\partial v_2}{\partial S^3} & \frac{\partial v_3}{\partial S^3} \end{bmatrix}, \qquad i,k = 1,2,3. \tag{1.1}$$

List of symbols

1. h - initial shell thickness,
2. $\zeta \in [-h/2, +h/2]$ - thickness coordinate,
3. $\mathbf{y}$ - position vector in the non-deformed (initial) configuration,
4. $\mathbf{x}$ - position vector in the deformed (current) configuration,
5. $\boldsymbol{\chi}$: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{y})$ - deformation function,
6. $\mathbf{F} \doteq \partial\mathbf{x}/\partial\mathbf{y}$ - deformation gradient,
7. $\mathbf{Q}, \mathbf{R} \in \mathrm{SO}(3)$ - orthogonal (rotation) tensors,
8. $\mathbf{Q}_0 \in \mathrm{SO}(3)$ - rotation constant over the shell thickness,
9. $\mathbf{C} \doteq \mathbf{F}^T\mathbf{F}$ - Cauchy–Green tensor,
10. $\mathfrak{C} \doteq \operatorname{skew}(\mathbf{Q}^T\mathbf{F})$ - Rotation Constraint (RC),

11. $\mathbf{E} \doteq \frac{1}{2}(\mathbf{C} - \mathbf{I})$ - Green strain,
12. $\mathbf{n}_\alpha^B, \mathbf{m}_\alpha^B$ - shell stress and couple resultant vectors,
13. $\boldsymbol{\varepsilon}_\alpha, \boldsymbol{\kappa}_\alpha$ - strain vectors of zeroth and first order,
14. E, ν, G - Young's modulus, Poisson's ratio, shear modulus,
15. λ, μ - Lamé coefficients.

Reference bases

1. $\{\mathbf{i}_k\}$ - global reference ortho-normal basis, $k = 1, 2, 3$,
2. $\{\mathbf{g}_\alpha\}$ - local natural basis at the middle surface for the initial configuration, $\alpha = 1, 2$,
3. $\{\mathbf{t}_k\}$ - local ortho-normal basis at the middle surface for the initial configuration,
4. $\{\mathbf{a}_k\}$ - local ortho-normal forward-rotated basis at the middle surface for the deformed configuration.

Abbreviations

1. AD - Automatic Differentiation
2. AMB - Angular Momentum Balance
3. BC - Boundary Conditions
4. BVP - Boundary Value Problem
5. CL - Constitutive Law
6. DK - Discrete Kirchhoff (elements)
7. dof - degree of freedom
8. FE - Finite Element
9. FD - Finite Difference
10. HR - Hellinger–Reissner (functional)
11. HW - Hu–Washizu (functional)
12. LMB - linear momentum balance
13. PE - Potential Energy
14. PS - Pian–Sumihara (element)
15. RBF - Residual Bending Flexibility (correction)
16. RC - Rotation Constraint
17. RI - Reduced Integration
18. VW - Virtual Work
19. ZNS - Zero Normal Stress (condition)
20. 1D, 2D, 3D - one-dimensional, two-dimensional, three-dimensional
21. 1-F, 2-F, 3-F, 4-F - one-field, two-field, three-field, four-field
22. 2nd PK - second Piola–Kirchhoff (stress)