Chapter 8
Gyrotriangle Exgyrocircles

Abstract Hyperbolic triangle excircles are called, in gyrolanguage, gyrotriangle exgyrocircles. These are determined in this chapter in terms of their gyrocenters and gyroradii. Their gyrocenters, in turn, are determined in terms of their gyrobarycentric coordinate representations with respect to the reference gyrotriangle. Moreover, relationships between the exgyroradii of a gyrotriangle exgyrocircles, and the gyrotriangle ingyroradius and circumgyroradius are obtained.

8.1 Introduction

Each gyrotriangle has an ingyrocircle, defined in Definition 7.11, p. 171, and up to three exgyrocircles the definition of which follows.

Definition 8.1 An exgyrocircle of a gyrotriangle is a gyrocircle lying outside the gyrotriangle, tangent to one of its sides and tangent to the extensions of the other two, see Figs. 8.1–8.2. The gyrocenter and the gyroradius of an exgyrocircle of a gyrotriangle are called the gyrotriangle exgyrocenter and exgyroradius.

Two additional gyrotriangle gyrocenters that are associated with ingyrocircles and exgyrocircles are the Nagel gyropoint and the Gergonne gyropoint, which are studied in this chapter.

8.2 Gyrotriangle Exgyrocircles and Ingyrocircles

In this section, we obtain gyrobarycentric coordinates for gyrotriangle exgyrocenters. Strikingly, as a byproduct we obtain gyrobarycentric coordinates for gyrotriangle ingyrocenters as well.

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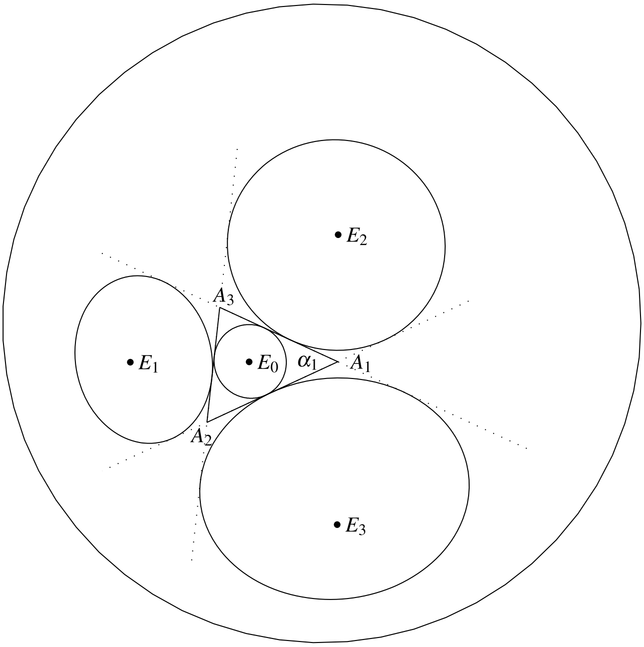


$E_1: \quad e_1 := -\cos\frac{\alpha_1}{2} + \sin\frac{\alpha_2}{2} + \sin\frac{\alpha_3}{2} = 0.1320$

$E_2: \quad e_2 := -\cos\frac{\alpha_2}{2} + \sin\frac{\alpha_1}{2} + \sin\frac{\alpha_3}{2} = 0.1176$

$E_3: \quad e_3 := -\cos\frac{\alpha_3}{2} + \sin\frac{\alpha_1}{2} + \sin\frac{\alpha_2}{2} = 0.0626$

Fig. 8.1 The gyrotriangle exgyrocircles, and exgyrocenters E_k, $k = 1, 2, 3$, in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$. Exgyrocircle with exgyrocenter E_k exists if and only if $e_k > 0$, $k = 1, 2, 3$. When e_k tends to 0, E_k approaches the boundary of the ball $\mathbb{R}^n_s$, as shown here for $n = 2$

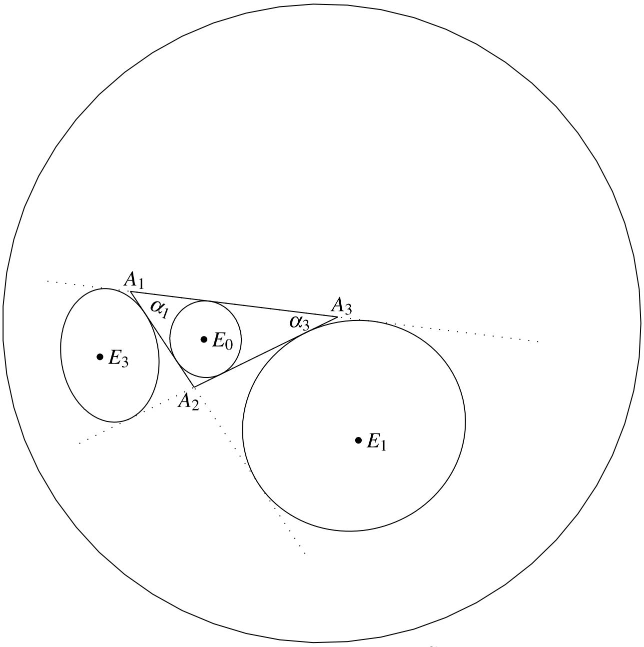
Let E be a generic exgyrocenter or ingyrocenter of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, shown in Figs. 8.1–8.3. Furthermore, let

$$E = \frac{m_1\gamma_{A_1}A_1 + m_2\gamma_{A_2}A_2 + m_3\gamma_{A_3}A_3}{m_1\gamma_{A_1} + m_2\gamma_{A_2} + m_3\gamma_{A_3}} \tag{8.1}$$

be the gyrobarycentric coordinate representation of E with respect to the set $S = \{A_1, A_2, A_3\}$ of the gyrotriangle vertices, where the gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ of E in (8.2) are to be determined in (8.13a), (8.13b), (8.13c), (8.13d), p. 224.

By Identity (4.29c), p. 91, with $X = \ominus A_k$ $k = 1, 2, 3$, we have, respectively,

$$\gamma_{\ominus A_1\oplus E} = \frac{m_1\gamma_{\ominus A_1\oplus A_1} + m_2\gamma_{\ominus A_1\oplus A_2} + m_3\gamma_{\ominus A_1\oplus A_3}}{m_0} = \frac{m_1 + m_2\gamma_{12} + m_3\gamma_{13}}{m_0},$$



$$E_1: \quad e_1 := -\cos\frac{\alpha_1}{2} + \sin\frac{\alpha_2}{2} + \sin\frac{\alpha_3}{2} = \quad 0.1084$$

$$E_2: \quad e_2 := -\cos\frac{\alpha_2}{2} + \sin\frac{\alpha_1}{2} + \sin\frac{\alpha_3}{2} = -0.0091$$

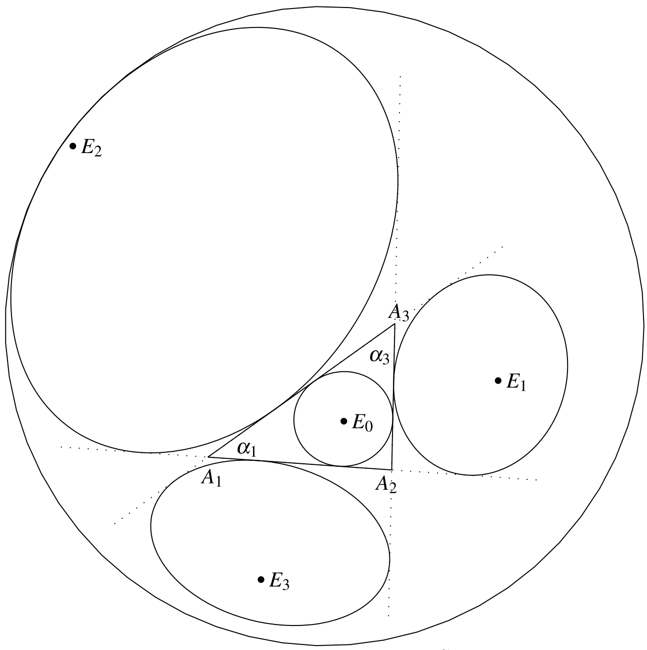
$$E_3: \quad e_3 := -\cos\frac{\alpha_3}{2} + \sin\frac{\alpha_1}{2} + \sin\frac{\alpha_2}{2} = \quad 0.1583$$

Fig. 8.2 Exgyrocircle with exgyrocenter E_k exists if and only if $e_k > 0$, $k = 1, 2, 3$. When e_k tends to 0, E_k approaches the boundary of the ball $\mathbb{R}^n_s$, as shown here for $n = 2$. Indeed, $e_2 < 0$ and, accordingly, exgyrocircle with exgyrocenter E_2 does not exist

$$\begin{aligned}
\gamma_{\ominus A_2 \oplus E} &= \frac{m_1\gamma_{\ominus A_2\oplus A_1} + m_2\gamma_{\ominus A_2\oplus A_2} + m_3\gamma_{\ominus A_2\oplus A_3}}{m_0} = \frac{m_1\gamma_{12} + m_2 + m_3\gamma_{23}}{m_0}, \\
\gamma_{\ominus A_3 \oplus E} &= \frac{m_1\gamma_{\ominus A_3\oplus A_1} + m_2\gamma_{\ominus A_3\oplus A_2} + m_3\gamma_{\ominus A_3\oplus A_3}}{m_0} = \frac{m_1\gamma_{13} + m_2\gamma_{23} + m_3}{m_0},
\end{aligned} \tag{8.2}$$

where we use the standard gyrotriangle index notation, shown in Fig. 6.1, p. 128, and in (6.1), p. 127, and where m_0 is the constant of the gyrobarycentric coordinate representation of E in (8.1) which, according to (4.29d), p. 91, is given by the equation

$$m_0^2 = m_1^2 + m_2^2 + m_3^2 + 2m_1m_2\gamma_{12} + 2m_1m_3\gamma_{13} + 2m_2m_3\gamma_{23}. \tag{8.3}$$



$$E_1: \quad e_1 := -\cos\tfrac{\alpha_1}{2} + \sin\tfrac{\alpha_2}{2} + \sin\tfrac{\alpha_3}{2} = 0.1502$$

$$E_2: \quad e_2 := -\cos\tfrac{\alpha_2}{2} + \sin\tfrac{\alpha_1}{2} + \sin\tfrac{\alpha_3}{2} = 0.0003$$

$$E_3: \quad e_3 := -\cos\tfrac{\alpha_3}{2} + \sin\tfrac{\alpha_1}{2} + \sin\tfrac{\alpha_2}{2} = 0.0694$$

Fig. 8.3 Exgyrocircle with exgyrocenter E_k exists if and only if $e_k > 0$, $k = 1, 2, 3$. When e_k tends to 0, E_k approaches the boundary of the ball $\mathbb{R}_s^n$, as shown here for $n = 2$. Indeed, $e_2 \approx 0$ and, accordingly, exgyrocenter E_2 lies close to the boundary of the disk

1. The gyrodistance of E (note that E represents each of E_k, $k = 0, 1, 2, 3$, in Fig. 8.1) from the gyroline that passes through points A_1 and A_2, Fig. 8.1, is the gyroaltitude r_{12} of gyrotriangle A_1A_2E drawn from base A_1A_2. Hence, by Theorem 7.17, p. 184, r_{12} is given by the equation

$$\gamma_{r_{12}}^2 = \frac{2\gamma_{12}\gamma_{\ominus A_1 \oplus E}\gamma_{\ominus A_2 \oplus E} - \gamma_{\ominus A_1 \oplus E}^2 - \gamma_{\ominus A_2 \oplus E}^2}{\gamma_{12}^2 - 1}. \tag{8.4a}$$

2. The gyrodistance of E from the gyroline that passes through points A_1 and A_3, Fig. 8.1, is the gyroaltitude r_{13} of gyrotriangle A_1A_3E drawn from base A_1A_3. Hence, by Theorem 7.17, p. 184, r_{13} is given by the equation

$$\gamma_{r_{13}}^2 = \frac{2\gamma_{13}\gamma_{\ominus A_1 \oplus E}\gamma_{\ominus A_3 \oplus E} - \gamma_{\ominus A_1 \oplus E}^2 - \gamma_{\ominus A_3 \oplus E}^2}{\gamma_{13}^2 - 1}. \tag{8.4b}$$

3. The gyrodistance of E from the gyroline that passes through points A_2 and A_3, Fig. 8.1, is the gyroaltitude r_{23} of gyrotriangle A_2A_3E drawn from base A_2A_3. Hence, by Theorem 7.17, p. 184, r_{23} is given by the equation

$$\gamma_{r_{23}}^2 = \frac{2\gamma_{23}\gamma_{\ominus A_2\oplus E}\gamma_{\ominus A_3\oplus E} - \gamma_{\ominus A_2\oplus E}^2 - \gamma_{\ominus A_3\oplus E}^2}{\gamma_{23}^2 - 1}. \tag{8.4c}$$

The gyrodistances from E to each of the three gyrolines A_1A_2, A_1A_3 and A_2A_3 are equal, implying

$$\begin{aligned} \gamma_{r_{12}}^2 &= \gamma_{r_{13}}^2, \\ \gamma_{r_{12}}^2 &= \gamma_{r_{23}}^2. \end{aligned} \tag{8.5}$$

Substituting successively (8.4a), (8.4b), (8.4c) and (8.2) into (8.5), along with the convenient normalization condition

$$m_1^2 + m_2^2 + m_3^2 = 1, \tag{8.6}$$

we obtain from (8.5)–(8.6) the following system of three equations for the three unknowns m_1^2, m_2^2 and m_3^2:

$$\begin{aligned} m_1^2\big(\gamma_{12}^2 - 1\big) - m_3^2\big(\gamma_{23}^2 - 1\big) &= 0, \\ m_2^2\big(\gamma_{12}^2 - 1\big) - m_3^2\big(\gamma_{13}^2 - 1\big) &= 0, \\ m_1^2 + m_2^2 + m_3^2 &= 1. \end{aligned} \tag{8.7}$$

The system (8.7) can be written as a matrix equation,

$$\begin{pmatrix} 1 & 1 & 1 \\ \gamma_{12}^2 - 1 & 0 & -(\gamma_{23}^2 - 1) \\ 0 & \gamma_{12}^2 - 1 & -(\gamma_{13}^2 - 1) \end{pmatrix} \begin{pmatrix} m_1^2 \\ m_2^2 \\ m_3^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{8.8}$$

so that

$$\begin{aligned} \begin{pmatrix} m_1^2 \\ m_2^2 \\ m_3^2 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \\ \gamma_{12}^2 - 1 & 0 & -(\gamma_{23}^2 - 1) \\ 0 & \gamma_{12}^2 - 1 & -(\gamma_{13}^2 - 1) \end{pmatrix}^{-1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{1}{D} \begin{pmatrix} (\gamma_{12}^2 - 1)(\gamma_{23}^2 - 1) & (\gamma_{12}^2 - 1)(\gamma_{13}^2 - 1) & -(\gamma_{23}^2 - 1) \\ (\gamma_{12}^2 - 1)(\gamma_{33}^2 - 1) & -(\gamma_{13}^2 - 1) & (\gamma_{12}^2 - 1)(\gamma_{23}^2 - 1) \\ (\gamma_{12}^2 - 1)^2 & -(\gamma_{12}^2 - 1) & (\gamma_{12}^2 - 1) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{8.9}$$

where D is the determinant of the coefficient matrix in (8.8), given by

$$D = \{(\gamma_{12}^2 - 1) + (\gamma_{13}^2 - 1) + (\gamma_{23}^2 - 1)\}(\gamma_{12}^2 - 1) > 0. \tag{8.10}$$

Hence, by (8.9),

$$\begin{aligned} m_1^2 &= \frac{1}{D}(\gamma_{12}^2 - 1)(\gamma_{23}^2 - 1), \\ m_2^2 &= \frac{1}{D}(\gamma_{12}^2 - 1)(\gamma_{13}^2 - 1), \\ m_3^2 &= \frac{1}{D}(\gamma_{12}^2 - 1)^2, \end{aligned} \tag{8.11}$$

where the gyrobarycentric coordinates are normalized by (8.6).

However, while the normalization (8.6) was temporarily convenient, it is irrelevant since gyrobarycentric coordinates are homogeneous by definition, Definition 4.2, p. 86. Hence, for the sake of simplicity, we drop the common factor $(\gamma_{12}^2 - 1)/D$ in (8.11), obtaining the following equations for gyrobarycentric coordinates m_k, $k = 1, 2, 3$, of E in (8.1) in which we employ Identity (1.9), p. 5, and introduce the common factor s^2 for convenience:

$$\begin{aligned} m_1^2 &= s^2(\gamma_{23}^2 - 1) = \gamma_{23}^2 a_{23}^2, \\ m_2^2 &= s^2(\gamma_{13}^2 - 1) = \gamma_{13}^2 a_{13}^2, \\ m_3^2 &= s^2(\gamma_{12}^2 - 1) = \gamma_{12}^2 a_{12}^2. \end{aligned} \tag{8.12}$$

Finally, in order to determine gyrobarycentric coordinates for the exgyrocenter E in (8.1) it remains to determine the signs of the gyrobarycentric coordinates m_k, $k = 1, 2, 3$, in (8.12). Being homogeneous, the two selections of a positive sign for each m_k and a negative sign for each m_k are indistinguishable. Similarly, the two selections of one positive and two negative signs for the m_k's and two positive and one negative signs for the m_k's are indistinguishable.

Hence, there are precisely four distinct sign selections for the m_k's, which turn out to correspond to the three exgyrocenters and the ingyrocenter of gyrotriangle $A_1A_2A_3$. These are:

$$E_1: \quad (m_1 : m_2 : m_3) = (-\gamma_{23}a_{23} : \gamma_{13}a_{13} : \gamma_{12}a_{12}), \tag{8.13a}$$

$$E_2: \quad (m_1 : m_2 : m_3) = (\gamma_{23}a_{23} : -\gamma_{13}a_{13} : \gamma_{12}a_{12}), \tag{8.13b}$$

$$E_3: \quad (m_1 : m_2 : m_3) = (\gamma_{23}a_{23} : \gamma_{13}a_{13} : -\gamma_{12}a_{12}), \tag{8.13c}$$

$$E_0: \quad (m_1 : m_2 : m_3) = (\gamma_{23}a_{23} : \gamma_{13}a_{13} : \gamma_{12}a_{12}). \tag{8.13d}$$

1. Substituting (8.13a) into (8.1), the point E of (8.1) becomes E_1, $E = E_1$, where

$$E_1 = \frac{-\gamma_{23}a_{23}\gamma_{A_1}A_1 + \gamma_{13}a_{13}\gamma_{A_2}A_2 + \gamma_{12}a_{12}\gamma_{A_3}A_3}{-\gamma_{23}a_{23}\gamma_{A_1} + \gamma_{13}a_{13}\gamma_{A_2} + \gamma_{12}a_{12}\gamma_{A_3}}. \tag{8.14a}$$

   The point E_1 is the exgyrocenter of gyrotriangle $A_1A_2A_3$ opposite to vertex A_1, as shown in Fig. 8.1.
2. Substituting (8.13b) into (8.1), the point E of (8.1) becomes E_2, $E = E_2$, where

$$E_2 = \frac{\gamma_{23}a_{23}\gamma_{A_1}A_1 - \gamma_{13}a_{13}\gamma_{A_2}A_2 + \gamma_{12}a_{12}\gamma_{A_3}A_3}{\gamma_{23}a_{23}\gamma_{A_1} - \gamma_{13}a_{13}\gamma_{A_2} + \gamma_{12}a_{12}\gamma_{A_3}}. \tag{8.14b}$$

   The point E_2 is the exgyrocenter of gyrotriangle $A_1A_2A_3$ opposite to vertex A_2, as shown in Fig. 8.1.
3. Substituting (8.13c) into (8.1), the point E of (8.1) becomes E_3, $E = E_3$, where

$$E_3 = \frac{\gamma_{23}a_{23}\gamma_{A_1}A_1 + \gamma_{13}a_{13}\gamma_{A_2}A_2 - \gamma_{12}a_{12}\gamma_{A_3}A_3}{\gamma_{23}a_{23}\gamma_{A_1} + \gamma_{13}a_{13}\gamma_{A_2} - \gamma_{12}a_{12}\gamma_{A_3}}. \tag{8.14c}$$

   The point E_3 is the exgyrocenter of gyrotriangle $A_1A_2A_3$ opposite to vertex A_3, as shown into Fig. 8.1.
4. Substituting (8.13d) into (8.1), the point E of (8.1) becomes E_0, $E = E_0$, where

$$E_0 = \frac{\gamma_{23}a_{23}\gamma_{A_1}A_1 + \gamma_{13}a_{13}\gamma_{A_2}A_2 + \gamma_{12}a_{12}\gamma_{A_3}A_3}{\gamma_{23}a_{23}\gamma_{A_1} + \gamma_{13}a_{13}\gamma_{A_2} + \gamma_{12}a_{12}\gamma_{A_3}}. \tag{8.14d}$$

   The point E_0 is the ingyrocenter of gyrotriangle $A_1A_2A_3$, shown in Fig. 8.1.

The gyrobarycentric coordinates of E_0 with respect to the set $S = \{A_1, \dots, A_N\}$ in (8.14d) are all positive, so that m_0^2 in (8.3) is positive. Hence, by Corollary 4.9, p. 93, the point E_0 lies on the interior of gyrotriangle $A_1A_2A_3$. Accordingly, E_0 is the ingyrocenter of gyrotriangle $A_1A_2A_3$.

The gyrobarycentric coordinate representation (8.14d) of the ingyrocenter of gyrotriangle $A_1A_2A_3$ is obtained in (7.103), p. 178, by a different method. Here, in contrast, it is obtained as a byproduct of the exgyrocenters determination.

8.3 Existence of Gyrotriangle Exgyrocircles

According to Corollary 4.9, p. 93, a point with a gyrobarycentric coordinate representation exists if and only if its constant m_0 is real, that is, if and only if $m_0^2 > 0$. Let us, therefore, calculate the constant m_0^2 of the gyrobarycentric coordinate representation of each of the gyrotriangle exgyrocenters and its ingyrocenter.

Following (4.27), p. 90, of Definition 4.5, the constant m_0 of the gyrobarycentric coordinate representation of the exgyrocenter E in (8.1), p. 220, is given by the equation

$$m_0^2 = m_1^2 + m_2^2 + m_3^2 + 2m_1m_2\gamma_{12} + 2m_1m_3\gamma_{13} + 2m_2m_3\gamma_{23}. \tag{8.15}$$

1. Substituting the gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ from (8.13a) into (8.15), we obtain the constant m_0^2 of exgyrocenter E_1 in (8.14a),

$$m_0^2 = (\gamma_{12}^2 - 1) + (\gamma_{13}^2 - 1) + (\gamma_{23}^2 - 1) + 2\gamma_{12}\gamma_{13}\gamma_{23}(a_{12}a_{13} - a_{12}a_{13} - a_{12}a_{13}). \tag{8.16a}$$

2. Substituting the gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ from (8.13b) into (8.15), we obtain the constant m_0^2 of exgyrocenter E_2 in (8.14b),

$$m_0^2 = (\gamma_{12}^2 - 1) + (\gamma_{13}^2 - 1) + (\gamma_{23}^2 - 1) + 2\gamma_{12}\gamma_{13}\gamma_{23}(-a_{12}a_{13} + a_{12}a_{13} - a_{12}a_{13}). \tag{8.16b}$$

3. Substituting the gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ from (8.13c) into (8.15), we obtain the constant m_0^2 of exgyrocenter E_3 in (8.14c),

$$m_0^2 = (\gamma_{12}^2 - 1) + (\gamma_{13}^2 - 1) + (\gamma_{23}^2 - 1) + 2\gamma_{12}\gamma_{13}\gamma_{23}(-a_{12}a_{13} - a_{12}a_{13} + a_{12}a_{13}). \tag{8.16c}$$

4. Substituting the gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ from (8.13d) into (8.15), we obtain the constant m_0^2 of ingyrocenter E_0 in (8.14d),

$$m_0^2 = (\gamma_{12}^2 - 1) + (\gamma_{13}^2 - 1) + (\gamma_{23}^2 - 1) + 2\gamma_{12}\gamma_{13}\gamma_{23}(a_{12}a_{13} + a_{12}a_{13} + a_{12}a_{13}). \tag{8.16d}$$

Hence,

1. The exgyrocircle with exgyrocenter E_1 of gyrotriangle $A_1A_2A_3$, opposite to vertex A_1, Fig. 8.1, exists if and only if

$$(\gamma_{12}^2 - 1) + (\gamma_{13}^2 - 1) + (\gamma_{23}^2 - 1) > 2\gamma_{12}\gamma_{13}\gamma_{23}(-a_{12}a_{13} + a_{12}a_{13} + a_{12}a_{13}), \tag{8.17a}$$

as we see from (8.16a).

2. The exgyrocircle with exgyrocenter E_2 of gyrotriangle $A_1A_2A_3$, opposite to vertex A_2, Fig. 8.1, exists if and only if

$$(\gamma_{12}^2 - 1) + (\gamma_{13}^2 - 1) + (\gamma_{23}^2 - 1) > 2\gamma_{12}\gamma_{13}\gamma_{23}(a_{12}a_{13} - a_{12}a_{13} + a_{12}a_{13}), \tag{8.17b}$$

as we see from (8.16b).

3. The exgyrocircle with exgyrocenter E_3 of gyrotriangle $A_1A_2A_3$, opposite to vertex A_3, Fig. 8.1, exists if and only if

$$(\gamma_{12}^2 - 1) + (\gamma_{13}^2 - 1) + (\gamma_{23}^2 - 1) > 2\gamma_{12}\gamma_{13}\gamma_{23}(a_{12}a_{13} + a_{12}a_{13} - a_{12}a_{13}), \tag{8.17c}$$

as we see from (8.16c).

4. The ingyrocircle with ingyrocenter E_0 of gyrotriangle $A_1A_2A_3$, Fig. 8.1, exists if and only if

$$\left(\gamma_{12}^2-1\right)+\left(\gamma_{13}^2-1\right)+\left(\gamma_{23}^2-1\right)>2\gamma_{12}\gamma_{13}\gamma_{23}(-a_{12}a_{13}-a_{12}a_{13}-a_{12}a_{13}), \tag{8.17d}$$

as we see from (8.16d). But, Inequality (8.17d) is valid for any gyrotriangle $A_1A_2A_3$. Hence, the ingyrocenter E_0 exists for any gyrotriangle $A_1A_2A_3$, as expected.

8.4 Exgyroradius and Ingyroradius

Following (8.4a) and (1.9), p. 5, we have

$$r_{12}^2=s^2\frac{\gamma_{r_{12}}^2-1}{\gamma_{r_{12}}^2}. \tag{8.18}$$

1. Substituting, successively, (8.4a), (8.2), and (8.13a) into (8.18), r_{12} becomes r_1, $r_{12}=r_1$, where r_1 is given by the equation

$$r_1^2=\frac{s^2}{2}\frac{1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2}{D_1}, \tag{8.19a}$$

where

$$\begin{aligned}D_1=\gamma_{12}\gamma_{13}\gamma_{23}-1-\gamma_{12}\sqrt{\gamma_{13}^2-1}\sqrt{\gamma_{23}^2-1}-\sqrt{\gamma_{12}^2-1}\gamma_{13}\sqrt{\gamma_{23}^2-1}\\+\sqrt{\gamma_{12}^2-1}\sqrt{\gamma_{13}^2-1}\gamma_{23}.\end{aligned}$$

2. Substituting, successively, (8.4a), (8.2), and (8.13b) into (8.18), r_{12} becomes r_2, $r_{12}=r_2$, where r_2 is given by the equation

$$r_2^2=\frac{s^2}{2}\frac{1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2}{D_2}, \tag{8.19b}$$

where

$$\begin{aligned}D_2=\gamma_{12}\gamma_{13}\gamma_{23}-1-\gamma_{12}\sqrt{\gamma_{13}^2-1}\sqrt{\gamma_{23}^2-1}+\sqrt{\gamma_{12}^2-1}\gamma_{13}\sqrt{\gamma_{23}^2-1}\\-\sqrt{\gamma_{12}^2-1}\sqrt{\gamma_{13}^2-1}\gamma_{23}.\end{aligned}$$

3. Substituting, successively, (8.4a), (8.2), and (8.13c) into (8.18), r_{12} becomes r_3, $r_{12}=r_3$, where r_3 is given by the equation

$$r_3^2=\frac{s^2}{2}\frac{1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2}{D_3}, \tag{8.19c}$$

where

$$D_3 = \gamma_{12}\gamma_{13}\gamma_{23} - 1 + \gamma_{12}\sqrt{\gamma_{13}^2 - 1}\sqrt{\gamma_{23}^2 - 1} - \sqrt{\gamma_{12}^2 - 1}\gamma_{13}\sqrt{\gamma_{23}^2 - 1}$$
$$- \sqrt{\gamma_{12}^2 - 1}\sqrt{\gamma_{13}^2 - 1}\gamma_{23}.$$

4. Substituting, successively, (8.4a), (8.2), and (8.13d) into (8.18), r_{12} becomes r_0, $r_{12} = r_0$, where r_0 is given by the equation

$$r_0^2 = \frac{s^2}{2}\frac{1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2}{D_0}, \tag{8.19d}$$

where

$$D_0 = \gamma_{12}\gamma_{13}\gamma_{23} - 1 + \gamma_{12}\sqrt{\gamma_{13}^2 - 1}\sqrt{\gamma_{23}^2 - 1} + \sqrt{\gamma_{12}^2 - 1}\gamma_{13}\sqrt{\gamma_{23}^2 - 1}$$
$$+ \sqrt{\gamma_{12}^2 - 1}\sqrt{\gamma_{13}^2 - 1}\gamma_{23}.$$

Formalizing the results of Sects. 8.2–8.4, we have the following theorem:

Theorem 8.2 (The Gyrotriangle Ingyrocircle and Exgyrocircles Theorem) *Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, let E_0 and r_0 be the gyrotriangle ingyrocenter and ingyroradius, and let E_k and r_k be the gyrotriangle exgyrocenter and exgyroradius of the exgyrocircle opposite to vertex A_k, $k = 1, 2, 3$.*

Then, in the standard gyrotriangle index notation, shown in Fig. 6.1, *p.* 128, *and in* (6.1), *p.* 127,

0. *Ingyrocenter E_0 of gyrotriangle $A_1A_2A_3$ always exists. Equivalently, ingyrocenter E_0 of gyrotriangle $A_1A_2A_3$ exists if and only if*

$$\left(\gamma_{12}^2 - 1\right) + \left(\gamma_{13}^2 - 1\right) + \left(\gamma_{23}^2 - 1\right) > 2\gamma_{12}\gamma_{13}\gamma_{23}(-a_{12}a_{13} - a_{12}a_{13} - a_{12}a_{13}).$$

1. *Exgyrocenter E_1 of gyrotriangle $A_1A_2A_3$ exists if and only if*

$$\left(\gamma_{12}^2 - 1\right) + \left(\gamma_{13}^2 - 1\right) + \left(\gamma_{23}^2 - 1\right) > 2\gamma_{12}\gamma_{13}\gamma_{23}(-a_{12}a_{13} + a_{12}a_{13} + a_{12}a_{13}).$$

2. *Exgyrocenter E_2 of gyrotriangle $A_1A_2A_3$ exists if and only if*

$$\left(\gamma_{12}^2 - 1\right) + \left(\gamma_{13}^2 - 1\right) + \left(\gamma_{23}^2 - 1\right) > 2\gamma_{12}\gamma_{13}\gamma_{23}(a_{12}a_{13} - a_{12}a_{13} + a_{12}a_{13}).$$

3. *Exgyrocenter E_3 of gyrotriangle $A_1A_2A_3$ exists if and only if*

$$\left(\gamma_{12}^2 - 1\right) + \left(\gamma_{13}^2 - 1\right) + \left(\gamma_{23}^2 - 1\right) > 2\gamma_{12}\gamma_{13}\gamma_{23}(a_{12}a_{13} + a_{12}a_{13} - a_{12}a_{13}).$$

The ingyrocenter E_0, *and each exgyrocenter* E_k, *when exists*, $k = 1, 2, 3$, *are given by*

$$
\begin{aligned}
E_0 &= \frac{\gamma_{23}a_{23}\gamma_{A_1}A_1 + \gamma_{13}a_{13}\gamma_{A_2}A_2 + \gamma_{12}a_{12}\gamma_{A_3}A_3}{\gamma_{23}a_{23}\gamma_{A_1} + \gamma_{13}a_{13}\gamma_{A_2} + \gamma_{12}a_{12}\gamma_{A_3}},\\
E_1 &= \frac{-\gamma_{23}a_{23}\gamma_{A_1}A_1 + \gamma_{13}a_{13}\gamma_{A_2}A_2 + \gamma_{12}a_{12}\gamma_{A_3}A_3}{-\gamma_{23}a_{23}\gamma_{A_1} + \gamma_{13}a_{13}\gamma_{A_2} + \gamma_{12}a_{12}\gamma_{A_3}},\\
E_2 &= \frac{\gamma_{23}a_{23}\gamma_{A_1}A_1 - \gamma_{13}a_{13}\gamma_{A_2}A_2 + \gamma_{12}a_{12}\gamma_{A_3}A_3}{\gamma_{23}a_{23}\gamma_{A_1} - \gamma_{13}a_{13}\gamma_{A_2} + \gamma_{12}a_{12}\gamma_{A_3}},\\
E_3 &= \frac{\gamma_{23}a_{23}\gamma_{A_1}A_1 + \gamma_{13}a_{13}\gamma_{A_2}A_2 - \gamma_{12}a_{12}\gamma_{A_3}A_3}{\gamma_{23}a_{23}\gamma_{A_1} + \gamma_{13}a_{13}\gamma_{A_2} - \gamma_{12}a_{12}\gamma_{A_3}}.
\end{aligned}
\tag{8.20}
$$

The ingyroradius r_0, *and each exgyroradius* r_k *that exists*, $k = 1, 2, 3$, *are given by*

$$
r_k^2 = \frac{s^2}{2}\frac{1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2}{\gamma_{12}\gamma_{13}\gamma_{23} - 1 + G_k}, \tag{8.21}
$$

$k = 0, 1, 2, 3$, *where*

$$
\begin{aligned}
G_0 &= \gamma_{12}\sqrt{\gamma_{13}^2 - 1}\sqrt{\gamma_{23}^2 - 1} + \sqrt{\gamma_{12}^2 - 1}\gamma_{13}\sqrt{\gamma_{23}^2 - 1} + \sqrt{\gamma_{12}^2 - 1}\sqrt{\gamma_{13}^2 - 1}\gamma_{23},\\
G_1 &= -\gamma_{12}\sqrt{\gamma_{13}^2 - 1}\sqrt{\gamma_{23}^2 - 1} - \sqrt{\gamma_{12}^2 - 1}\gamma_{13}\sqrt{\gamma_{23}^2 - 1} + \sqrt{\gamma_{12}^2 - 1}\sqrt{\gamma_{13}^2 - 1}\gamma_{23},\\
G_2 &= -\gamma_{12}\sqrt{\gamma_{13}^2 - 1}\sqrt{\gamma_{23}^2 - 1} + \sqrt{\gamma_{12}^2 - 1}\gamma_{13}\sqrt{\gamma_{23}^2 - 1} - \sqrt{\gamma_{12}^2 - 1}\sqrt{\gamma_{13}^2 - 1}\gamma_{23},\\
G_3 &= \quad \gamma_{12}\sqrt{\gamma_{13}^2 - 1}\sqrt{\gamma_{23}^2 - 1} - \sqrt{\gamma_{12}^2 - 1}\gamma_{13}\sqrt{\gamma_{23}^2 - 1} - \sqrt{\gamma_{12}^2 - 1}\sqrt{\gamma_{13}^2 - 1}\gamma_{23}.
\end{aligned}
\tag{8.22}
$$

Clearly,

$$
\sum_{k=0}^{3} G_k = 0, \tag{8.23}
$$

as we see from (8.22) of Theorem 8.2.

Hence, by (8.21), and by (7.148) and (7.152), p. 188, and by (7.35), p. 163,

$$
\begin{aligned}
&\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\\
&\quad=\frac{8}{s^2}\frac{\gamma_{12}\gamma_{13}\gamma_{23}-1}{1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2}\\
&\quad=2\frac{1+\cos\alpha_1\cos\alpha_2\cos\alpha_3}{s^2F}\\
&\quad=2\frac{1+\cos\alpha_1\cos\alpha_2\cos\alpha_3}{s^2\cos\frac{\alpha_1+\alpha_2+\alpha_3}{2}\cos\frac{\alpha_1-\alpha_2-\alpha_3}{2}\cos\frac{-\alpha_1+\alpha_2-\alpha_3}{2}\cos\frac{-\alpha_1-\alpha_2+\alpha_3}{2}}\\
&\quad=2\frac{1+\cos\alpha_1\cos\alpha_2\cos\alpha_3}{R^2\cos^2\frac{\alpha_1-\alpha_2-\alpha_3}{2}\cos^2\frac{-\alpha_1+\alpha_2-\alpha_3}{2}\cos^2\frac{-\alpha_1-\alpha_2+\alpha_3}{2}}
\end{aligned}
\tag{8.24}
$$

where F is given by (7.144), p. 187.

8.5 In-Exgyroradii Relations

By substituting gyrotriangle gyrotrigonometric identities from Sect. 7.12, p. 187, into (8.21)–(8.22) and employing the result of Theorem 7.4, p. 163, we obtain gyrotrigonometric relations for the gyrotriangle ingyroradius r_0, exgyroradii r_k, $k = 1, 2, 3$, and circumradius R, which are presented in the following theorem:

Theorem 8.3 (The In-Exgyroradii Theorem) *Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ with ingyroradius r_0, exgyroradii r_k, $k = 1, 2, 3$, and circumgyroradius R. Then in the gyrotriangle index notation in Fig.* 6.1, *p.* 128,

$$
\begin{aligned}
r_0&=\frac{1}{2}R\frac{\cos\frac{\alpha_1-\alpha_2-\alpha_3}{2}}{\cos\frac{\alpha_1}{2}}\frac{\cos\frac{-\alpha_1+\alpha_2-\alpha_3}{2}}{\cos\frac{\alpha_2}{2}}\frac{\cos\frac{-\alpha_1-\alpha_2+\alpha_3}{2}}{\cos\frac{\alpha_3}{2}},\\
r_1&=\frac{1}{2}R\frac{\cos\frac{\alpha_1-\alpha_2-\alpha_3}{2}}{\cos\frac{\alpha_1}{2}}\frac{\cos\frac{-\alpha_1+\alpha_2-\alpha_3}{2}}{\sin\frac{\alpha_2}{2}}\frac{\cos\frac{-\alpha_1-\alpha_2+\alpha_3}{2}}{\sin\frac{\alpha_3}{2}},\\
r_2&=\frac{1}{2}R\frac{\cos\frac{\alpha_1-\alpha_2-\alpha_3}{2}}{\sin\frac{\alpha_1}{2}}\frac{\cos\frac{-\alpha_1+\alpha_2-\alpha_3}{2}}{\cos\frac{\alpha_2}{2}}\frac{\cos\frac{-\alpha_1-\alpha_2+\alpha_3}{2}}{\sin\frac{\alpha_3}{2}},\\
r_3&=\frac{1}{2}R\frac{\cos\frac{\alpha_1-\alpha_2-\alpha_3}{2}}{\sin\frac{\alpha_1}{2}}\frac{\cos\frac{-\alpha_1+\alpha_2-\alpha_3}{2}}{\sin\frac{\alpha_2}{2}}\frac{\cos\frac{-\alpha_1-\alpha_2+\alpha_3}{2}}{\cos\frac{\alpha_3}{2}},
\end{aligned}
\tag{8.25}
$$

and

$$\frac{1}{r_0} - \left\{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right\} = \frac{2}{R} \frac{\cos\frac{\alpha_1+\alpha_2+\alpha_3}{2}}{\cos\frac{\alpha_1-\alpha_2-\alpha_3}{2}\cos\frac{-\alpha_1+\alpha_2-\alpha_3}{2}\cos\frac{-\alpha_1-\alpha_2+\alpha_3}{2}} \tag{8.26}$$

and

$$\begin{aligned} & r_1 + r_2 + r_3 - (4R + r_0) \\ &\quad = R\frac{\cos\frac{\alpha_1+\alpha_2+\alpha_3}{2}}{\sin\alpha_1\sin\alpha_2\sin\alpha_3}\left\{3\sin\frac{\alpha_1+\alpha_2+\alpha_3}{2} + \sin\frac{3\alpha_1-\alpha_2-\alpha_3}{2}\right. \\ &\qquad \left. + \sin\frac{-\alpha_1+3\alpha_2-\alpha_3}{2} + \sin\frac{-\alpha_1-\alpha_2+3\alpha_3}{2}\right\}. \end{aligned} \tag{8.27}$$

Proof The results of the theorem follow straightforwardly by substitutions of gyrotriangle gyrotrigonometric identities from Sect. 7.12, p. 187, into (8.21)–(8.22) and employing the result of Theorem 7.4, p. 163.

Equation (8.26) follows from (8.25) straightforwardly owing to the elegant trigonometric/gyrotrigonometric identity

$$\begin{aligned} &\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2} - \cos\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2} \\ &\quad - \sin\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2} - \sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2} \\ &= \cos\frac{\alpha_1+\alpha_2+\alpha_3}{2}. \end{aligned} \tag{8.28}$$

Similarly, (8.27) follows from (8.25) straightforwardly; see Problem 8.6, p. 266. □

In the Euclidean limit, $s \to \infty$, gyrolengths and gyroangles tend to corresponding lengths and angles. Accordingly, gyroangle gyrotriangle sum tends to angle triangle sum, π, so that the gyrocosine function of a gyroangle,

$$\cos\frac{\alpha_1+\alpha_2+\alpha_3}{2}, \tag{8.29a}$$

tends to the corresponding cosine function of an angle,

$$\cos\frac{\pi}{2} = 0. \tag{8.29b}$$

Hence, the elegant relations (8.26)–(8.27) for gyrotriangles reduce in that limit to the familiar relations

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_0} \quad \text{(Euclidean Geometry)} \tag{8.30}$$

and

$$r_1 + r_2 + r_3 = 4R + r_0 \quad \text{(Euclidean Geometry)} \tag{8.31}$$

for triangles in Euclidean geometry.

Equation (8.30) is found, for instance, in [9, p. 13], and (8.31) is found, for instance, in [8, p. 13].

The incircle and the three excircles, each touching all the three sides of their reference triangle, are called the four *tritangent circles* of the triangle.

8.6 In-Exradii Relations

Interestingly, the elegant relations (8.25) remain invariant in form under the Euclidean limit $s \to \infty$, so that they are valid in Euclidean geometry as well. However, for application in Euclidean geometry the relations (8.25) can be simplified owing to the fact that triangle angle sum in π.

Indeed, under the condition

$$\alpha_1 + \alpha + \alpha_3 = \pi, \tag{8.32a}$$

we have the following trigonometric identities for triangle angles:

$$\begin{aligned} \cos\frac{\alpha_1 - \alpha - \alpha_3}{2} &= \sin\alpha_1 = 2\sin\frac{\alpha_1}{2}\cos\frac{\alpha_1}{2}, \\ \cos\frac{-\alpha_1 + \alpha_2 - \alpha_3}{2} &= \sin\alpha_2 = 2\sin\frac{\alpha_2}{2}\cos\frac{\alpha_2}{2}, \\ \cos\frac{-\alpha_1 - \alpha_2 + \alpha_3}{2} &= \sin\alpha_3 = 2\sin\frac{\alpha_3}{2}\cos\frac{\alpha_3}{2}. \end{aligned} \tag{8.32b}$$

Substituting the simplifications offered by (8.32a), (8.32b) into (8.25) we obtain the following corollary of Theorem 8.3:

Corollary 8.4 *Let $A_1A_2A_3$ be a triangle with angles α_k, exradii r_k, $k = 1, 2, 3$, inradius r_0 and circumradius R in a Euclidean space $\mathbb{R}^n$. Then,*

$$\begin{aligned} r_0 &= 4R\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2}, \\ r_1 &= 4R\sin\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2}, \\ r_2 &= 4R\cos\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2}, \\ r_3 &= 4R\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2}, \end{aligned} \tag{8.33}$$

$$r_1 + r_2 + r_3 = 4R + r_0, \tag{8.34}$$

and

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_0}. \tag{8.35}$$

Proof Substituting the simplifications offered by (8.32a), (8.32b) into (8.25) we obtain (8.33).

Equation (8.34) follows from (8.33) straightforwardly by means of the elegant trigonometric/gyrotrigonometric identity

$$\begin{aligned}
&\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2} - \sin\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2} \\
&\quad - \cos\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2} - \cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2} \\
&= \sin\frac{\alpha_1+\alpha_2+\alpha_3}{2},
\end{aligned} \tag{8.36}$$

and (8.35) is established in (8.30). □

8.7 In-Exgyrocenter Gyrotrigonometric Gyrobarycentric Representations

It is useful to express the gyrobarycentric coordinate representations of the ingyrocenter and the exgyrocenters of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ gyrotrigonometrically. Being homogeneous, and following the law of gyrosines (6.44), p. 140, the gyrobarycentric coordinates in (8.13a), (8.13b), (8.13c), (8.13d), p. 224, are, respectively, equivalent to

$$\begin{aligned}
E_0&: \quad (m_1 : m_2 : m_3) = \left(\frac{\gamma_{23}a_{23}}{\gamma_{12}a_{12}} : \frac{\gamma_{13}a_{13}}{\gamma_{12}a_{12}} : 1\right) = \left(\frac{\sin\alpha_1}{\sin\alpha_3} : \frac{\sin\alpha_2}{\sin\alpha_3} : 1\right), \\
E_1&: \quad (m_1 : m_2 : m_3) = \left(-\frac{\gamma_{23}a_{23}}{\gamma_{12}a_{12}} : \frac{\gamma_{13}a_{13}}{\gamma_{12}a_{12}} : 1\right) = \left(-\frac{\sin\alpha_1}{\sin\alpha_3} : \frac{\sin\alpha_2}{\sin\alpha_3} : 1\right), \\
E_2&: \quad (m_1 : m_2 : m_3) = \left(\frac{\gamma_{23}a_{23}}{\gamma_{12}a_{12}} : -\frac{\gamma_{13}a_{13}}{\gamma_{12}a_{12}} : 1\right) = \left(\frac{\sin\alpha_1}{\sin\alpha_3} : -\frac{\sin\alpha_2}{\sin\alpha_3} : 1\right), \\
E_3&: \quad (m_1 : m_2 : m_3) = \left(\frac{\gamma_{23}a_{23}}{\gamma_{12}a_{12}} : \frac{\gamma_{13}a_{13}}{\gamma_{12}a_{12}} : -1\right) = \left(\frac{\sin\alpha_1}{\sin\alpha_3} : \frac{\sin\alpha_2}{\sin\alpha_3} : -1\right).
\end{aligned}$$

These, in turn, give rise to the following gyrotrigonometric gyrobarycentric coordinates for the exgyrocenters E_k, $k = 1, 2, 3$, and the ingyrocenter E_0 of gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$:

$$E_0: \quad (m_1 : m_2 : m_3) = (\sin\alpha_1 : \sin\alpha_2 : \sin\alpha_3), \tag{8.37a}$$

$$E_1: \quad (m_1 : m_2 : m_3) = (-\sin\alpha_1 : \sin\alpha_2 : \sin\alpha_3), \tag{8.37b}$$

$$E_2: \quad (m_1 : m_2 : m_3) = (\sin\alpha_1 : -\sin\alpha_2 : \sin\alpha_3), \tag{8.37c}$$

$$E_3: \quad (m_1 : m_2 : m_3) = (\sin\alpha_1 : \sin\alpha_2 : -\sin\alpha_3). \tag{8.37d}$$

We have thus obtained the following theorem:

Theorem 8.5 (In-Exgyrocenters) *Let $S = \{A_1, A_2, A_3\}$ be a pointwise independent set of three points in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$. The ingyrocenter E_0 and the exgyrocenters E_k, $k = 1, 2, 3$, of gyrotriangle $A_1A_2A_3$ have the following gyrotrigonometric gyrobarycentric coordinate representations*:

$$E_0 = \frac{\sin\alpha_1\gamma_{A_1}A_1 + \sin\alpha_2\gamma_{A_2}A_2 + \sin\alpha_3\gamma_{A_3}A_3}{\sin\alpha_1\gamma_{A_1} + \sin\alpha_2\gamma_{A_2} + \sin\alpha_3\gamma_{A_3}} \tag{8.38a}$$

$$E_1 = \frac{-\sin\alpha_1\gamma_{A_1}A_1 + \sin\alpha_2\gamma_{A_2}A_2 + \sin\alpha_3\gamma_{A_3}A_3}{-\sin\alpha_1\gamma_{A_1} + \sin\alpha_2\gamma_{A_2} + \sin\alpha_3\gamma_{A_3}}, \tag{8.38b}$$

$$E_2 = \frac{\sin\alpha_1\gamma_{A_1}A_1 - \sin\alpha_2\gamma_{A_2}A_2 + \sin\alpha_3\gamma_{A_3}A_3}{\sin\alpha_1\gamma_{A_1} - \sin\alpha_2\gamma_{A_2} + \sin\alpha_3\gamma_{A_3}}, \tag{8.38c}$$

$$E_3 = \frac{\sin\alpha_1\gamma_{A_1}A_1 + \sin\alpha_2\gamma_{A_2}A_2 - \sin\alpha_3\gamma_{A_3}A_3}{\sin\alpha_1\gamma_{A_1} + \sin\alpha_2\gamma_{A_2} - \sin\alpha_3\gamma_{A_3}}. \tag{8.38d}$$

Following (8.15), p. 226, and the gyrotrigonometric gyrobarycentric coordinates m_k, $k = 1, 2, 3$, in (8.37a), (8.37b), (8.37c), (8.37d), the constant m_0^2 of each of the gyrotrigonometric gyrobarycentric coordinate representations of E_k, $k = 1, 2, 3, 0$, in (8.38a), (8.38b), (8.38c), (8.38d) is, respectively,

$$\begin{aligned}
E_0: \quad m_0^2 &= \sin^2\alpha_1 + \sin^2\alpha_2 + \sin^2\alpha_3 \\
&\quad + 2(\cos\alpha_1\cos\alpha_2 + \cos\alpha_1\cos\alpha_3 + \cos\alpha_2\cos\alpha_3) \\
&\quad + 2(\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3) \\
&= 4\left\{\cos^2\frac{\alpha_1}{2} - \left(\cos\frac{\alpha_2}{2} - \cos\frac{\alpha_3}{2}\right)^2\right\} \\
&\quad \times \left\{-\cos^2\frac{\alpha_1}{2} + \left(\cos\frac{\alpha_2}{2} + \cos\frac{\alpha_3}{2}\right)^2\right\},
\end{aligned} \tag{8.39a}$$

$$\begin{aligned}
E_1: \quad m_0^2 &= \sin^2\alpha_1 + \sin^2\alpha_2 + \sin^2\alpha_3 \\
&\quad + 2(-\cos\alpha_1\cos\alpha_2 - \cos\alpha_1\cos\alpha_3 + \cos\alpha_2\cos\alpha_3) \\
&\quad + 2(\cos\alpha_1 - \cos\alpha_2 - \cos\alpha_3)
\end{aligned}$$

$$= 4\left\{\cos^2\frac{\alpha_1}{2} - \left(\sin\frac{\alpha_2}{2} - \sin\frac{\alpha_3}{2}\right)^2\right\}$$
$$\times\left\{-\cos^2\frac{\alpha_1}{2} + \left(\sin\frac{\alpha_2}{2} + \sin\frac{\alpha_3}{2}\right)^2\right\}, \qquad (8.39b)$$

$$E_2:\quad m_0^2 = \sin^2\alpha_1 + \sin^2\alpha_2 + \sin^2\alpha_3$$
$$x \qquad + 2(-\cos\alpha_1\cos\alpha_2 + \cos\alpha_1\cos\alpha_3 - \cos\alpha_2\cos\alpha_3)$$
$$+ 2(-\cos\alpha_1 + \cos\alpha_2 - \cos\alpha_3)$$
$$= 4\left\{\cos^2\frac{\alpha_2}{2} - \left(\sin\frac{\alpha_1}{2} - \sin\frac{\alpha_3}{2}\right)^2\right\}$$
$$\times\left\{-\cos^2\frac{\alpha_2}{2} + \left(\sin\frac{\alpha_1}{2} + \sin\frac{\alpha_3}{2}\right)^2\right\}, \qquad (8.39c)$$

$$E_3:\quad m_0^2 = \sin^2\alpha_1 + \sin^2\alpha_2 + \sin^2\alpha_3$$
$$+ 2(\cos\alpha_1\cos\alpha_2 - \cos\alpha_1\cos\alpha_3 - \cos\alpha_2\cos\alpha_3)$$
$$+ 2(-\cos\alpha_1 - \cos\alpha_2 + \cos\alpha_3)$$
$$= 4\left\{\cos^2\frac{\alpha_3}{2} - \left(\sin\frac{\alpha_1}{2} - \sin\frac{\alpha_2}{2}\right)^2\right\}$$
$$\times\left\{-\cos^2\frac{\alpha_3}{2} + \left(\sin\frac{\alpha_1}{2} + \sin\frac{\alpha_2}{2}\right)^2\right\}. \qquad (8.39d)$$

The ingyrocenter E_0 of any gyrotriangle $A_1A_2A_3$ exists. Hence, its constant m_0^2 must be positive. Hence, by the extreme right-hand side of the fourth equation in (8.39a), (8.39b), (8.39c), (8.39d), we have the inequality

$$\left|\cos\frac{\alpha_2}{2} - \cos\frac{\alpha_3}{2}\right| < \cos\frac{\alpha_1}{2} < \cos\frac{\alpha_2}{2} + \cos\frac{\alpha_3}{2} \qquad (8.40)$$

for the gyroangles α_k, $k = 1, 2, 3$, of any gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space.

It can be shown that in addition to (8.40), gyroangles α_k, $k = 1, 2, 3$, of any gyrotriangle $A_1A_2A_3$ satisfy the inequality

$$\left|\sin\frac{\alpha_2}{2} - \sin\frac{\alpha_3}{2}\right| < \cos\frac{\alpha_1}{2}. \qquad (8.41)$$

Inequality (8.41), along with the identities in (8.39a), (8.39b), (8.39c), (8.39d) for the constant m_0^2 of each of the gyrotriangle incenter and excenters E_k, $k = 0, 1, 2, 3$, imply the following existence conditions for E_k:

0. Ingyrocenter E_0 of gyrotriangle $A_1A_2A_3$ always exists. Equivalently, ingyrocenter E_0 of gyrotriangle $A_1A_2A_3$ exists if and only if

$$\cos\frac{\alpha_1}{2} < \cos\frac{\alpha_2}{2} + \cos\frac{\alpha_3}{2}. \tag{8.42}$$

1. Exgyrocenter E_1 of gyrotriangle $A_1A_2A_3$ exists if and only if

$$\cos\frac{\alpha_1}{2} < \sin\frac{\alpha_2}{2} + \sin\frac{\alpha_3}{2}. \tag{8.43}$$

2. Exgyrocenter E_2 of gyrotriangle $A_1A_2A_3$ exists if and only if

$$\cos\frac{\alpha_2}{2} < \sin\frac{\alpha_1}{2} + \sin\frac{\alpha_3}{2}. \tag{8.44}$$

3. Exgyrocenter E_3 of gyrotriangle $A_1A_2A_3$ exists if and only if

$$\cos\frac{\alpha_3}{2} < \sin\frac{\alpha_1}{2} + \sin\frac{\alpha_2}{2}. \tag{8.45}$$

8.8 In-Excenter Trigonometric Barycentric Representations

The gyrotrigonometric gyrobarycentric coordinates (8.37a), (8.37b), (8.37c) (8.37d) remain invariant in form under the Euclidean limit $s \to \infty$, resulting in the following corollary of Theorem 8.5:

Corollary 8.6 (In-Excenters) *Let $S = \{A_1, A_2, A_3\}$ be a pointwise independent set of three points in a Euclidean vector space $\mathbb{R}^n$. The incenter E_0 and the excenters E_k, $k = 1, 2, 3$, of triangle $A_1A_2A_3$ have the following trigonometric barycentric coordinate representations*:

$$E_0 = \frac{\sin\alpha_1 A_1 + \sin\alpha_2 A_2 + \sin\alpha_3 A_3}{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3}, \tag{8.46a}$$

$$E_1 = \frac{-\sin\alpha_1 A_1 + \sin\alpha_2 A_2 + \sin\alpha_3 A_3}{-\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3}, \tag{8.46b}$$

$$E_2 = \frac{\sin\alpha_1 A_1 - \sin\alpha_2 A_2 + \sin\alpha_3 A_3}{\sin\alpha_1 - \sin\alpha_2 + \sin\alpha_3}, \tag{8.46c}$$

$$E_3 = \frac{\sin\alpha_1 A_1 + \sin\alpha_2 A_2 - \sin\alpha_3 A_3}{\sin\alpha_1 + \sin\alpha_2 - \sin\alpha_3}. \tag{8.46d}$$

8.9 Exgyrocircle Points of Tangency, Part I

The exgyrocircle points of tangency where the A_k-excircle of the gyrotriangle meets the opposite side of A_k, Fig. 8.4, are associated with the gyrotriangle gyrocenter called Nagel gyropoint, N_a, shown in Fig. 8.4, and studied in Sect. 8.12

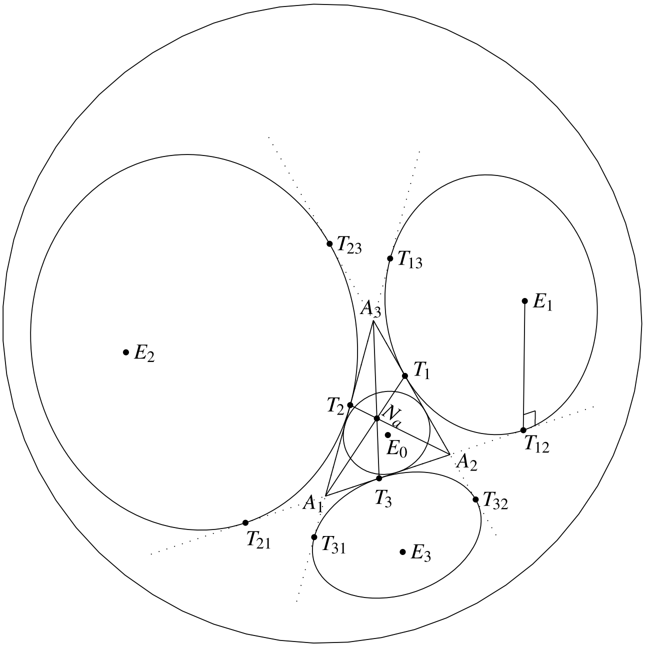


Fig. 8.4 Nagel Gyropoint, N_a, of a gyrotriangle in an Einstein gyrovector space. The point E_k, $k = 1, 2, 3$, is the A_k-exgyrocenter of gyrotriangle $A_1A_2A_3$ opposite to A_k, and the point T_k is the point in which the A_k-excircle of the gyrotriangle meets the gyrotriangle side opposite to A_k. The Nagel gyropoint of gyrotriangle $A_1A_2A_3$ is the point of concurrency of the three gyrolines A_kT_k, determined in (8.69), p. 245. The points of tangency T_1, etc., are given by Theorem 8.7, p. 239, and the points of tangency T_{12}, etc., are given by Theorem 8.12, p. 247

Let us consider the point of tangency T_3 where the A_3 excircle of a gyrotriangle $A_1A_2A_3$ meets the gyrotriangle side opposite to A_3, shown in Fig. 8.4. It is the perpendicular foot of the gyrotriangle exgyrocenter E_3 on the gyroline A_1A_2. Accordingly, T_3 is the gyroaltitude foot of gyrotriangle $A_1A_2E_3$, drawn from E_3, as shown in Fig. 8.4.

Hence, by Theorem 7.16, p. 183, the gyroaltitude foot T_3 possesses the gyrobarycentric coordinate representation

$$T_3 = \frac{m_1\gamma_{A_1}A_1 + m_2\gamma_{A_2}A_2}{m_1\gamma_{A_1} + m_2\gamma_{A_2}} \tag{8.47a}$$

with respect to the set $\{A_1, A_2\}$, with gyrobarycentric coordinates

$$\begin{aligned} m_1 &= \gamma_{\ominus A_1\oplus A_2}\gamma_{\ominus A_2\oplus E_3} - \gamma_{\ominus A_1\oplus E_3} = \gamma_{12}\gamma_{\ominus A_2\oplus E_3} - \gamma_{\ominus A_1\oplus E_3}, \\ m_2 &= \gamma_{\ominus A_1\oplus A_2}\gamma_{\ominus A_1\oplus E_3} - \gamma_{\ominus A_2\oplus E_3} = \gamma_{12}\gamma_{\ominus A_1\oplus E_3} - \gamma_{\ominus A_2\oplus E_3}. \end{aligned} \tag{8.47b}$$

The gyrobarycentric coordinates m_1 and m_2 in (8.47b) involve the gamma factors $\gamma_{\ominus A_1 \oplus E_3}$ and $\gamma_{\ominus A_2 \oplus E_3}$, which we calculate below.

Being the A_3-exgyrocenter of gyrotriangle $A_1A_2A_3$, E_3 is given by, (8.14c), p. 225,

$$E_3 = \frac{\gamma_{23}a_{23}\gamma_{A_1}A_1 + \gamma_{13}a_{13}\gamma_{A_2}A_2 - \gamma_{12}a_{12}\gamma_{A_3}A_3}{\gamma_{23}a_{23}\gamma_{A_1} + \gamma_{13}a_{13}\gamma_{A_2} - \gamma_{12}a_{12}\gamma_{A_3}}. \tag{8.48}$$

Hence, by Theorem 4.6, p. 90,

$$\gamma_{X \oplus E_3} = \frac{\gamma_{23}a_{23}\gamma_{X\oplus A_1} + \gamma_{13}a_{13}\gamma_{X\oplus A_2} - \gamma_{12}a_{12}\gamma_{X\oplus A_3}}{m_0} \tag{8.49}$$

for all $X \in \mathbb{R}^n_s$, where $m_0 > 0$ is the constant of the gyrobarycentric coordinate representation of E_3 in (8.48). This constant need not be specified as we will see below in the transition from (8.51) to (8.52).

Following (8.49) with $X = \ominus A_k$, $k = 1, 2, 3$, we have, respectively,

$$\begin{aligned}
\gamma_{\ominus A_1 \oplus E_3} &= \frac{\gamma_{23}a_{23} + \gamma_{13}a_{13}\gamma_{12} - \gamma_{12}a_{12}\gamma_{13}}{m_0}, \\
\gamma_{\ominus A_2 \oplus E_3} &= \frac{\gamma_{23}a_{23}\gamma_{12} + \gamma_{13}a_{13} - \gamma_{12}a_{12}\gamma_{23}}{m_0}, \\
\gamma_{\ominus A_3 \oplus E_3} &= \frac{\gamma_{23}a_{23}\gamma_{13} + \gamma_{13}a_{13}\gamma_{23} - \gamma_{12}a_{12}}{m_0}.
\end{aligned} \tag{8.50}$$

Substituting from (8.50) into (8.47b), we have

$$\begin{aligned}
m_1 &= \frac{(\gamma_{12}^2 - 1)\gamma_{23}a_{23} - (\gamma_{12}\gamma_{23} - \gamma_{13})\gamma_{12}a_{12}}{m_0}, \\
m_2 &= \frac{(\gamma_{12}^2 - 1)\gamma_{13}a_{13} - (\gamma_{12}\gamma_{13} - \gamma_{23})\gamma_{12}a_{12}}{m_0}.
\end{aligned} \tag{8.51}$$

Being homogeneous, a common nonzero factor of gyrobarycentric coordinates is irrelevant, so that convenient gyrobarycentric coordinates m_1 and m_2 of the point T_3 in (8.47a) are obtained from (8.51):

$$\begin{aligned}
m_1 &= \big(\gamma_{12}^2 - 1\big)\gamma_{23}a_{23} - (\gamma_{12}\gamma_{23} - \gamma_{13})\gamma_{12}a_{12}, \\
m_2 &= \big(\gamma_{12}^2 - 1\big)\gamma_{13}a_{13} - (\gamma_{12}\gamma_{13} - \gamma_{23})\gamma_{12}a_{12}.
\end{aligned} \tag{8.52}$$

Substituting from (7.143)–(7.147) into (8.52), along with the abbreviation $F = F(\alpha_1, \alpha_2, \alpha_3)$, we have

$$
\begin{aligned}
m_1 &= \frac{8F^{3/2}}{\sin^2\alpha_1 \sin^2\alpha_2 \sin\alpha_3} \frac{1-\cos\alpha_2}{\sin\alpha_2} = \frac{8F^{3/2}}{\sin^2\alpha_1 \sin^2\alpha_2 \sin\alpha_3} \tan\frac{\alpha_2}{2}, \\
m_2 &= \frac{8F^{3/2}}{\sin^2\alpha_1 \sin^2\alpha_2 \sin\alpha_3} \frac{1-\cos\alpha_1}{\sin\alpha_1} = \frac{8F^{3/2}}{\sin^2\alpha_1 \sin^2\alpha_2 \sin\alpha_3} \tan\frac{\alpha_1}{2}.
\end{aligned}
\tag{8.53}
$$

Since gyrobarycentric coordinates are homogeneous, a nonzero common factor of a system of gyrobarycentric coordinates is irrelevant. Hence, it follows from (8.53) that convenient gyrobarycentric coordinates for the point T_3 in (8.47a) are

$$
\begin{aligned}
m_1 &= \tan\frac{\alpha_2}{2}, \\
m_2 &= \tan\frac{\alpha_1}{2}
\end{aligned}
\tag{8.54}
$$

so that, by (8.47a), we have

$$
T_3 = \frac{\tan\frac{\alpha_2}{2}\gamma_{A_1} A_1 + \tan\frac{\alpha_1}{2}\gamma_{A_2} A_2}{\tan\frac{\alpha_2}{2}\gamma_{A_1} + \tan\frac{\alpha_1}{2}\gamma_{A_2}}.
\tag{8.55}
$$

We have thus obtained the following theorem:

Theorem 8.7 *Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $\mathbb{R}^n_s$ and let T_k, $k = 1, 2, 3$, be the point where the A_k-exgyrocircle of the gyrotriangle meets the opposite side of A_k, Fig. 8.4. Gyrotrigonometric gyrobarycentric coordinate representations of the points T_k are*:

$$
\begin{aligned}
T_1 &= \frac{\tan\frac{\alpha_3}{2}\gamma_{A_2} A_2 + \tan\frac{\alpha_2}{2}\gamma_{A_3} A_3}{\tan\frac{\alpha_3}{2}\gamma_{A_2} + \tan\frac{\alpha_2}{2}\gamma_{A_3}}, \\
T_2 &= \frac{\tan\frac{\alpha_3}{2}\gamma_{A_1} A_1 + \tan\frac{\alpha_1}{2}\gamma_{A_3} A_3}{\tan\frac{\alpha_3}{2}\gamma_{A_1} + \tan\frac{\alpha_1}{2}\gamma_{A_3}}, \\
T_3 &= \frac{\tan\frac{\alpha_2}{2}\gamma_{A_1} A_1 + \tan\frac{\alpha_1}{2}\gamma_{A_2} A_2}{\tan\frac{\alpha_2}{2}\gamma_{A_1} + \tan\frac{\alpha_1}{2}\gamma_{A_2}}.
\end{aligned}
\tag{8.56}
$$

Proof The proof of the third equation in (8.56) is given by (8.47a)–(8.55). The proof of the first and the second equation in (8.56) is obtained from the first by vertex permutations. □

The three points T_k, $k = 1, 2, 3$, of Theorem 8.7 are shown in Fig. 8.4, where they are determined by the results of the Theorem. As Fig. 8.4 indicates, the three gyrolines A_1T_1, A_2T_2 and A_3T_3 are concurrent. We show in Sect. 8.12 that this is indeed the case, giving rise to the Nagel gyropoint.

8.10 Excircle Points of Tangency, Part I

The gyrotrigonometric gyrobarycentric coordinates in (8.56) remain invariant in form under the Euclidean limit $s \to \infty$, resulting in the following corollary of Theorem 8.7:

Corollary 8.8 *Let $A_1A_2A_3$ be a triangle in a Euclidean vector space $\mathbb{R}^n$ and let T_k, $k = 1, 2, 3$, be the point where the A_k-excircle of the triangle meets the opposite side of A_k. Trigonometric barycentric coordinate representations of the points T_k are*:

$$
\begin{aligned}
T_1 &= \frac{\tan\frac{\alpha_3}{2}A_2 + \tan\frac{\alpha_2}{2}A_3}{\tan\frac{\alpha_3}{2} + \tan\frac{\alpha_2}{2}},\\
T_2 &= \frac{\tan\frac{\alpha_3}{2}A_1 + \tan\frac{\alpha_1}{2}A_3}{\tan\frac{\alpha_3}{2} + \tan\frac{\alpha_1}{2}},\\
T_3 &= \frac{\tan\frac{\alpha_2}{2}A_1 + \tan\frac{\alpha_1}{2}A_2}{\tan\frac{\alpha_2}{2} + \tan\frac{\alpha_1}{2}}.
\end{aligned}
\tag{8.57}
$$

8.11 Left Gyrotranslated Exgyrocircles

Left gyrotranslating gyrotriangle $A_1A_2A_3$ by $\ominus A_1$, the gyrotriangle becomes gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$, where $O = \ominus A_1 \oplus A_1$ is the arbitrarily selected origin of the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $O = (0, 0, \dots, 0)$. The gyrotriangle exgyrocenters E_k and tangent points T_k, $k = 1, 2, 3$, are left gyrotranslated as well, becoming, respectively, $\ominus A_1 \oplus E_k$ and $\ominus A_1 \oplus T_k$.

Applying to (8.20), p. 229, the Gyrobarycentric Coordinate Representation Gyrocovariance Theorem 4.6, p. 90, we have from Identity (4.29a), p. 91, with $X = \ominus A_1$, using the standard gyrotriangle index notation, shown in Fig. 8.4, in Fig. 6.1, p. 128, and in (6.1), p. 127:

$$
\begin{aligned}
&\ominus A_1 \oplus E_1\\
&= \ominus A_1 \oplus \frac{-\gamma_{23}a_{23}\gamma_{A_1}A_1 + \gamma_{13}a_{13}\gamma_{A_2}A_2 + \gamma_{12}a_{12}\gamma_{A_3}A_3}{-\gamma_{23}a_{23}\gamma_{A_1} + \gamma_{13}a_{13}\gamma_{A_2} + \gamma_{12}a_{12}\gamma_{A_3}}\\
&= \frac{-\gamma_{23}a_{23}\gamma_{\ominus A_1 \oplus A_1}(\ominus A_1 \oplus A_1) + \gamma_{13}a_{13}\gamma_{\ominus A_1 \oplus A_2}(\ominus A_1 \oplus A_2) + \gamma_{12}a_{12}\gamma_{\ominus A_1 \oplus A_3}(\ominus A_1 \oplus A_3)}{-\gamma_{23}a_{23}\gamma_{\ominus A_1 \oplus A_1} + \gamma_{13}a_{13}\gamma_{\ominus A_1 \oplus A_2} + \gamma_{12}a_{12}\gamma_{\ominus A_1 \oplus A_3}}
\end{aligned}
$$

$$= \frac{\gamma_{13}a_{13}\gamma_{12}\mathbf{a}_{12} + \gamma_{12}a_{12}\gamma_{13}\mathbf{a}_{13}}{-\gamma_{23}a_{23} + \gamma_{13}a_{13}\gamma_{12} + \gamma_{12}a_{12}\gamma_{13}}$$

$$= \frac{\sin\alpha_2\gamma_{12}\mathbf{a}_{12} + \sin\alpha_3\gamma_{13}\mathbf{a}_{13}}{-\sin\alpha_1 + \sin\alpha_2\gamma_{12} + \sin\alpha_3\gamma_{13}}. \tag{8.58a}$$

The last equation in (8.58a) is obtained by substitutions from (7.143).

Similarly,

$$\ominus A_1 \oplus E_2 = \ominus A_1 \oplus \frac{\gamma_{23}a_{23}\gamma_{A_1}A_1 - \gamma_{13}a_{13}\gamma_{A_2}A_2 + \gamma_{12}a_{12}\gamma_{A_3}A_3}{\gamma_{23}a_{23}\gamma_{A_1} - \gamma_{13}a_{13}\gamma_{A_2} + \gamma_{12}a_{12}\gamma_{A_3}}$$

$$= \frac{-\gamma_{13}a_{13}\gamma_{12}\mathbf{a}_{12} + \gamma_{12}a_{12}\gamma_{13}\mathbf{a}_{13}}{\gamma_{23}a_{23} - \gamma_{13}a_{13}\gamma_{12} + \gamma_{12}a_{12}\gamma_{13}}$$

$$= \frac{-\sin\alpha_2\gamma_{12}\mathbf{a}_{12} + \sin\alpha_3\gamma_{13}\mathbf{a}_{13}}{\sin\alpha_1 - \sin\alpha_2\gamma_{12} + \sin\alpha_3\gamma_{13}} \tag{8.58b}$$

and

$$\ominus A_1 \oplus E_3 = \ominus A_1 \oplus \frac{\gamma_{23}a_{23}\gamma_{A_1}A_1 + \gamma_{13}a_{13}\gamma_{A_2}A_2 - \gamma_{12}a_{12}\gamma_{A_3}A_3}{\gamma_{23}a_{23}\gamma_{A_1} + \gamma_{13}a_{13}\gamma_{A_2} - \gamma_{12}a_{12}\gamma_{A_3}}$$

$$= \frac{\gamma_{13}a_{13}\gamma_{12}\mathbf{a}_{12} - \gamma_{12}a_{12}\gamma_{13}\mathbf{a}_{13}}{\gamma_{23}a_{23} + \gamma_{13}a_{13}\gamma_{12} - \gamma_{12}a_{12}\gamma_{13}}$$

$$= \frac{\sin\alpha_2\gamma_{12}\mathbf{a}_{12} - \sin\alpha_3\gamma_{13}\mathbf{a}_{13}}{\sin\alpha_1 + \sin\alpha_2\gamma_{12} - \sin\alpha_3\gamma_{13}}. \tag{8.58c}$$

Note that, by Definition 4.5, p. 89, the set of points $S = \{A_1, A_2, A_3\}$ is pointwise independent in the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$. Hence, the two gyrovectors $\mathbf{a}_{12} = \ominus A_1 \oplus A_2$ and $\mathbf{a}_{13} = \ominus A_1 \oplus A_3$ in $\mathbb{R}^n_s \subset \mathbb{R}^n$ in (8.58a), (8.58b), (8.58c) are linearly independent in $\mathbb{R}^n$.

Similarly, under a left gyrotranslation by $\ominus A_1$, the tangent points T_k, $k = 1, 2, 3$, in Fig. 8.4 and in (8.56) become:

$$\ominus A_1 \oplus T_1 = \ominus A_1 \oplus \frac{\tan\frac{\alpha_3}{2}\gamma_{A_2}A_2 + \tan\frac{\alpha_2}{2}\gamma_{A_3}A_3}{\tan\frac{\alpha_3}{2}\gamma_{A_2} + \tan\frac{\alpha_2}{2}\gamma_{A_3}}$$

$$= \frac{\tan\frac{\alpha_3}{2}\gamma_{12}\mathbf{a}_{12} + \tan\frac{\alpha_2}{2}\gamma_{13}\mathbf{a}_{13}}{\tan\frac{\alpha_3}{2}\gamma_{12} + \tan\frac{\alpha_2}{2}\gamma_{13}}, \tag{8.59a}$$

$$\ominus A_1 \oplus T_2 = \ominus A_1 \oplus \frac{\tan\frac{\alpha_3}{2}\gamma_{A_1} A_1 + \tan\frac{\alpha_1}{2}\gamma_{A_3} A_3}{\tan\frac{\alpha_3}{2}\gamma_{A_1} + \tan\frac{\alpha_1}{2}\gamma_{A_3}}$$

$$= \frac{\tan\frac{\alpha_1}{2}\gamma_{13}\mathbf{a}_{13}}{\tan\frac{\alpha_3}{2} + \tan\frac{\alpha_1}{2}\gamma_{13}} \tag{8.59b}$$

and

$$\ominus A_1 \oplus T_3 = \ominus A_1 \oplus \frac{\tan\frac{\alpha_2}{2}\gamma_{A_1} A_1 + \tan\frac{\alpha_1}{2}\gamma_{A_2} A_2}{\tan\frac{\alpha_2}{2}\gamma_{A_1} + \tan\frac{\alpha_1}{2}\gamma_{A_2}}$$

$$= \frac{\tan\frac{\alpha_1}{2}\gamma_{12}\mathbf{a}_{12}}{\tan\frac{\alpha_2}{2} + \tan\frac{\alpha_1}{2}\gamma_{12}}. \tag{8.59c}$$

8.12 Nagel Gyropoint

Definition 8.9 (Nagel Gyropoint) Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ and let T_k, $k = 1, 2, 3$, be the tangent point in which the A_k-excircle of the gyrotriangle meets the gyrotriangle side opposite to A_k, Fig. 8.4, p. 237. The gyrotriangle vertices A_k and the gyrotriangle points of tangency T_k form the three gyrolines A_kT_k that are concurrent. Owing to analogies with Euclidean geometry, this point of concurrency is called the Nagel gyropoint, N_a of the gyrotriangle.

Let the Nagel gyropoint N of gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, Fig. 8.4, be given by its gyrobarycentric coordinate representation with respect to the set $S = \{A_1, A_2, A_3\}$ of the gyrotriangle,

$$N = \frac{m_1\gamma_{A_1} A_1 + m_2\gamma_{A_2} A_2 + m_3\gamma_{A_3} A_3}{m_1\gamma_{A_1} + m_2\gamma_{A_2} + m_3\gamma_{A_3}}, \tag{8.60}$$

where the gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ of N in (8.60) are to be determined.

Left gyrotranslating gyrotriangle $A_1A_2A_3$ by $\ominus A_1$, the gyrotriangle becomes gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$, where $O = \ominus A_1 \oplus A_1$ is the arbitrarily selected origin of the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $O = (0, 0, \ldots, 0)$. The gyrotriangle tangent points T_k, $k = 1, 2, 3$, are left gyrotranslated as well, becoming, respectively, $\ominus A_1 \oplus T_k$, which are given by (8.59a), (8.59b), (8.59c).

Similarly, the gyrotriangle Nagel gyropoint of the left gyrotranslated gyrotriangle becomes $P = \ominus A_1 \oplus N$, given by

$$
\begin{aligned}
P &= \ominus A_1 \oplus N \\
&= \frac{m_2 \gamma_{\ominus A_1 \oplus A_2} (\ominus A_1 \oplus A_2) + m_3 \gamma_{\ominus A_1 \oplus A_3} (\ominus A_1 \oplus A_3)}{m_1 + m_2 \gamma_{\ominus A_1 \oplus A_2} + m_3 \gamma_{\ominus A_1 \oplus A_3}} \\
&= \frac{m_2 \gamma_{12} \mathbf{a}_{12} + m_3 \gamma_{13} \mathbf{a}_{13}}{m_1 + m_2 \gamma_{12} + m_3 \gamma_{13}}.
\end{aligned}
\tag{8.61}
$$

1. The tangent point $\ominus A_1 \oplus T_1$ and the vertex $O = \ominus A_1 \oplus A_1 = (0, 0, \ldots, 0)$ of the left gyrotranslated gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$ form the Euclidean line

$$
L_1 = (\ominus A_1 \oplus T_1) t_1 = \frac{\tan \frac{\alpha_3}{2} \gamma_{12} \mathbf{a}_{12} + \tan \frac{\alpha_2}{2} \gamma_{13} \mathbf{a}_{13}}{\tan \frac{\alpha_3}{2} \gamma_{12} + \tan \frac{\alpha_2}{2} \gamma_{13}} t_1 \tag{8.62a}
$$

   as we see from (8.59a), where $t_1 \in \mathbb{R}$ is the line parameter.
2. The tangent point $\ominus A_1 \oplus T_2$ and the vertex $\mathbf{a}_{12} = \ominus A_1 \oplus A_2$ of the left gyrotranslated gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$ form the Euclidean line

$$
\begin{aligned}
L_2 &= (\ominus A_1 \oplus A_2) + \big(-(\ominus A_1 \oplus A_2) + (\ominus A_1 \oplus T_2)\big) t_2 \\
&= \mathbf{a}_{12} + \left(-\mathbf{a}_{12} + \frac{\tan \frac{\alpha_1}{2} \gamma_{13} \mathbf{a}_{13}}{\tan \frac{\alpha_3}{2} + \tan \frac{\alpha_1}{2} \gamma_{13}}\right) t_2
\end{aligned}
\tag{8.62b}
$$

   as we see from (8.59b), where $t_2 \in \mathbb{R}$ is the line parameter.
3. The tangent point $\ominus A_1 \oplus T_3$ and the vertex $\mathbf{a}_{13} = \ominus A_1 \oplus A_3$ of the left gyrotranslated gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$ form the Euclidean line

$$
\begin{aligned}
L_3 &= (\ominus A_1 \oplus A_3) + \big(-(\ominus A_1 \oplus A_3) + (\ominus A_1 \oplus T_3)\big) t_3 \\
&= \mathbf{a}_{13} + \left(-\mathbf{a}_{13} + \frac{\tan \frac{\alpha_1}{2} \gamma_{12} \mathbf{a}_{12}}{\tan \frac{\alpha_2}{2} + \tan \frac{\alpha_1}{2} \gamma_{12}}\right) t_3
\end{aligned}
\tag{8.62c}
$$

   as we see from (8.59c), where $t_3 \in \mathbb{R}$ is the line parameter.

Since the point P lies on each of the three lines L_k, $k = 1, 2, 3$, there exist values $t_{k,0}$ of the line parameters t_k, $k = 1, 2, 3$, respectively, such that

$$
\begin{aligned}
P - \frac{\tan\frac{\alpha_3}{2}\gamma_{12}\mathbf{a}_{12} + \tan\frac{\alpha_2}{2}\gamma_{13}\mathbf{a}_{13}}{\tan\frac{\alpha_3}{2}\gamma_{12} + \tan\frac{\alpha_2}{2}\gamma_{13}} t_{1,0} &= 0, \\
P - \mathbf{a}_{12} - \left(-\mathbf{a}_{12} + \frac{\tan\frac{\alpha_1}{2}\gamma_{13}\mathbf{a}_{13}}{\tan\frac{\alpha_3}{2} + \tan\frac{\alpha_1}{2}\gamma_{13}}\right) t_{2,0} &= 0, \\
P - \mathbf{a}_{13} - \left(-\mathbf{a}_{13} + \frac{\tan\frac{\alpha_1}{2}\gamma_{12}\mathbf{a}_{12}}{\tan\frac{\alpha_2}{2} + \tan\frac{\alpha_1}{2}\gamma_{12}}\right) t_{3,0} &= 0,
\end{aligned}
\tag{8.63}
$$

where P is given by (8.61).

The system of equations (8.63) was obtained by methods of gyroalgebra, and will be solved below by a common method of linear algebra.

Substituting P from (8.61) into (8.63), and rewriting each equation in (8.63) as a linear combination of $\mathbf{a}_{12}$ and $\mathbf{a}_{13}$ equals zero, we obtain the following homogeneous linear system of three gyrovector equations

$$
\begin{aligned}
c_{11}\mathbf{a}_{12} + c_{12}\mathbf{a}_{13} &= \mathbf{0}, \\
c_{21}\mathbf{a}_{12} + c_{22}\mathbf{a}_{13} &= \mathbf{0}, \\
c_{31}\mathbf{a}_{12} + c_{32}\mathbf{a}_{13} &= \mathbf{0},
\end{aligned}
\tag{8.64}
$$

where each coefficient c_{ij}, $i = 1, 2, 3$, $j = 1, 2$, is a function of the gyrotriangle parameters $\gamma_{12}, \gamma_{13}, \gamma_{23}$ and α_k, and the six unknowns $t_{k,0}$ and m_k, $k = 1, 2, 3$.

Since the set $S = \{A_1, A_2, A_3\}$ is pointwise independent, the two gyrovectors $\mathbf{a}_{12} = \ominus A_1 \oplus A_2$ and $\mathbf{a}_{13} = \ominus A_1 \oplus A_3$ in $\mathbb{R}^n_s$, considered as vectors in $\mathbb{R}^n$, are linearly independent in $\mathbb{R}^n$. Hence, each coefficient c_{ij} in (8.64) equals zero. Accordingly, the three gyrovector equations in (8.64) are equivalent to the following six scalar equations,

$$
c_{11} = c_{12} = c_{21} = c_{22} = c_{31} = c_{32} = 0 \tag{8.65}
$$

for the six unknowns $t_{k,0}$ and m_k, $k = 1, 2, 3$.

An explicit presentation of the resulting system (8.65) reveals that it is slightly nonlinear. Like the system (7.221), p. 208, however, it is linear in the unknowns $t_{1,0}, t_{2,0}, t_{3,0}$. Solving three equations of the system for $t_{1,0}, t_{2,0}, t_{3,0}$, and substituting these into the remaining equations of the system we obtain a system that determines the ratios m_1/m_3 and m_2/m_3 uniquely, from which convenient (homogeneous) gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ are obtained.

The unique determination of m_1/m_3 and m_2/m_3 turns out to be

$$\begin{aligned} \frac{m_1}{m_3} &= \frac{\tan\frac{\alpha_3}{2}}{\tan\frac{\alpha_1}{2}}, \\ \frac{m_2}{m_3} &= \frac{\tan\frac{\alpha_3}{2}}{\tan\frac{\alpha_2}{2}} \end{aligned} \tag{8.66}$$

from which two convenient gyrobarycentric coordinates result. These are:

$$(m_1 : m_2 : m_3) = \left(\tan\frac{\alpha_2}{2}\tan\frac{\alpha_3}{2} : \tan\frac{\alpha_1}{2}\tan\frac{\alpha_3}{2} : \tan\frac{\alpha_1}{2}\tan\frac{\alpha_2}{2}\right) \tag{8.67}$$

and, equivalently,

$$(m_1 : m_2 : m_3) = \left(\cot\frac{\alpha_1}{2} : \cot\frac{\alpha_2}{2} : \cot\frac{\alpha_3}{2}\right). \tag{8.68}$$

Formalizing the main result of this section, we have the following theorem:

Theorem 8.10 (Nagel Gyropoint) *Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$. A gyrotrigonometric gyrobarycentric coordinate representation of the gyrotriangle Nagel gyropoint N_a, Fig. 8.4, p. 237, is given by*

$$N_a = \frac{\cot\frac{\alpha_1}{2}\gamma_{A_1}A_1 + \cot\frac{\alpha_2}{2}\gamma_{A_2}A_2 + \cot\frac{\alpha_3}{2}\gamma_{A_3}A_3}{\cot\frac{\alpha_1}{2}\gamma_{A_1} + \cot\frac{\alpha_2}{2}\gamma_{A_2} + \cot\frac{\alpha_3}{2}\gamma_{A_3}}. \tag{8.69}$$

Proof The proof follows immediately from (8.60) and (8.68). □

The gyrotrigonometric gyrobarycentric coordinates of Nagel gyropoint in (8.69) remain invariant in form under the Euclidean limit $s \to \infty$, resulting in the following corollary of Theorem 8.10:

Corollary 8.11 (Nagel Point) *Let $A_1A_2A_3$ be a triangle in a Euclidean vector space $\mathbb{R}^n$. A trigonometric barycentric coordinate representation of the triangle Nagel point N_a is given by*

$$N_a = \frac{\cot\frac{\alpha_1}{2}A_1 + \cot\frac{\alpha_2}{2}A_2 + \cot\frac{\alpha_3}{2}A_3}{\cot\frac{\alpha_1}{2} + \cot\frac{\alpha_2}{2} + \cot\frac{\alpha_3}{2}}. \tag{8.70}$$

The remarkable similarity in form between the trigonometric barycentric coordinate representation (8.70) of Nagel point and the gyrotrigonometric gyrobarycentric coordinate representation (8.69) of Nagel gyropoint demonstrates the way gyrotrigonometric gyrobarycentric coordinate representations capture analogies that Euclidean and hyperbolic geometry share.

8.13 Exgyrocircle Points of Tangency, Part II

Let us consider the point of tangency T_{12} where the A_1 excircle of a gyrotriangle $A_1A_2A_3$ meets the gyrotriangle side A_1A_2, shown in Fig. 8.4. It is the perpendicular foot of the gyrotriangle exgyrocenter E_1 on the gyroline A_1A_2. Accordingly, T_{12} is the gyroaltitude foot of gyrotriangle $A_1A_2E_1$, drawn from E_1, as shown in Fig. 8.4.

Hence, by Theorem 7.16, p. 183, the gyroaltitude foot T_{12} possesses the gyrobarycentric coordinate representation

$$T_{12} = \frac{m_1\gamma_{A_1}A_1 + m_2\gamma_{A_2}A_2}{m_1\gamma_{A_1} + m_2\gamma_{A_2}} \tag{8.71a}$$

with respect to the set $\{A_1, A_2\}$, with gyrobarycentric coordinates

$$\begin{aligned} m_1 &= \gamma_{\ominus A_1\oplus A_2}\gamma_{\ominus A_2\oplus E_1} - \gamma_{\ominus A_1\oplus E_1} = \gamma_{12}\gamma_{\ominus A_2\oplus E_1} - \gamma_{\ominus A_1\oplus E_1}, \\ m_2 &= \gamma_{\ominus A_1\oplus A_2}\gamma_{\ominus A_1\oplus E_1} - \gamma_{\ominus A_2\oplus E_1} = \gamma_{12}\gamma_{\ominus A_1\oplus E_1} - \gamma_{\ominus A_2\oplus E_1}. \end{aligned} \tag{8.71b}$$

The gyrobarycentric coordinates m_1 and m_2 in (8.47b) involve the gamma factors $\gamma_{\ominus A_1\oplus E_1}$ and $\gamma_{\ominus A_2\oplus E_1}$, which we calculate below.

Being the A_1-exgyrocenter of gyrotriangle $A_1A_2A_3$, E_1 is given by, (8.14a), p. 225,

$$E_1 = \frac{-\gamma_{23}a_{23}\gamma_{A_1}A_1 + \gamma_{13}a_{13}\gamma_{A_2}A_2 + \gamma_{12}a_{12}\gamma_{A_3}A_3}{-\gamma_{23}a_{23}\gamma_{A_1} + \gamma_{13}a_{13}\gamma_{A_2} + \gamma_{12}a_{12}\gamma_{A_3}}. \tag{8.72}$$

Hence, by Theorem 4.6, p. 90,

$$\gamma_{X\oplus E_1} = \frac{-\gamma_{23}a_{23}\gamma_{X\oplus A_1} + \gamma_{13}a_{13}\gamma_{X\oplus A_2} + \gamma_{12}a_{12}\gamma_{X\oplus A_3}}{m_0} \tag{8.73}$$

for all $X \in \mathbb{R}^n_s$, where $m_0 > 0$ is the constant of the gyrobarycentric coordinate representation of E_1 in (8.72). This constant need not be specified as we will see below in the transition from (8.75) to (8.76).

Following (8.73) with $X = \ominus A_k$, $k = 1, 2, 3$, we have, respectively,

$$\begin{aligned} \gamma_{\ominus A_1\oplus E_1} &= \frac{-\gamma_{23}a_{23} + \gamma_{13}a_{13}\gamma_{12} + \gamma_{12}a_{12}\gamma_{13}}{m_0}, \\ \gamma_{\ominus A_2\oplus E_1} &= \frac{-\gamma_{23}a_{23}\gamma_{12} + \gamma_{13}a_{13} + \gamma_{12}a_{12}\gamma_{23}}{m_0}, \\ \gamma_{\ominus A_3\oplus E_1} &= \frac{-\gamma_{23}a_{23}\gamma_{13} + \gamma_{13}a_{13}\gamma_{23} + \gamma_{12}a_{12}}{m_0}. \end{aligned} \tag{8.74}$$

Substituting from (8.74) into (8.71b), we have

$$m_1 = \frac{-(\gamma_{12}^2 - 1)\gamma_{23}a_{23} + (\gamma_{12}\gamma_{23} - \gamma_{13})\gamma_{12}a_{12}}{m_0},$$
$$m_2 = \frac{(\gamma_{12}^2 - 1)\gamma_{13}a_{13} + (\gamma_{12}\gamma_{13} - \gamma_{23})\gamma_{12}a_{12}}{m_0}. \tag{8.75}$$

Being homogeneous, a common nonzero factor of gyrobarycentric coordinates is irrelevant, so that convenient gyrobarycentric coordinates m_1 and m_2 of the point T_3 in (8.71a) are obtained from (8.75):

$$m_1 = -(\gamma_{12}^2 - 1)\gamma_{23}a_{23} + (\gamma_{12}\gamma_{23} - \gamma_{13})\gamma_{12}a_{12},$$
$$m_2 = (\gamma_{12}^2 - 1)\gamma_{13}a_{13} + (\gamma_{12}\gamma_{13} - \gamma_{23})\gamma_{12}a_{12}. \tag{8.76}$$

Substituting from (7.143)–(7.147), p. 187, into (8.76), along with the abbreviation $F = F(\alpha_1, \alpha_2, \alpha_3)$, we have

$$m_1 = \frac{8F^{3/2}}{\sin^2\alpha_1 \sin^2\alpha_2 \sin\alpha_3}\frac{\cos\alpha_2 - 1}{\sin\alpha_2} = -\frac{8F^{3/2}}{\sin^2\alpha_1 \sin^2\alpha_2 \sin\alpha_3}\tan\frac{\alpha_2}{2},$$
$$m_2 = \frac{8F^{3/2}}{\sin^2\alpha_1 \sin^2\alpha_2 \sin\alpha_3}\frac{\cos\alpha_1 + 1}{\sin\alpha_1} = \frac{8F^{3/2}}{\sin^2\alpha_1 \sin^2\alpha_2 \sin\alpha_3}\cot\frac{\alpha_1}{2}. \tag{8.77}$$

Since gyrobarycentric coordinates are homogeneous, a nonzero common factor of a system of gyrobarycentric coordinates is irrelevant. Hence, it follows from (8.77) that convenient gyrobarycentric coordinates for the point T_{12} in (8.71a) are

$$m_1 = \tan\frac{\alpha_2}{2},$$
$$m_2 = -\cot\frac{\alpha_1}{2} \tag{8.78}$$

so that, by (8.71a), we have

$$T_{12} = \frac{\tan\frac{\alpha_2}{2}\gamma_{A_1}A_1 - \cot\frac{\alpha_1}{2}\gamma_{A_2}A_2}{\tan\frac{\alpha_2}{2}\gamma_{A_1} - \cot\frac{\alpha_1}{2}\gamma_{A_2}}. \tag{8.79}$$

We have thus obtained the following theorem:

Theorem 8.12 (The Exgyrocircle Points of Tangency Theorem) *Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, let T_{ij} be the point of tangency where the A_i-exgyrocircle meets the extension of the gyrotriangle side A_iA_j, and let T_i be the point where the A_i-exgyrocircle of the gyrotriangle meets the opposite side of A_i, Fig.* 8.4, *p.* 237.

Then, the gyrotrigonometric gyrobarycentric coordinate representations of the points of tangency on the A_k-exgyrocircle, $k = 1, 2, 3$, *with respect to the set* $S = \{A_1, A_2, A_3\}$ *are given by the equations listed below.*

For $k = 1$,

$$\begin{aligned} T_{12} &= \frac{\tan\frac{\alpha_2}{2}\gamma_{A_1} A_1 - \cot\frac{\alpha_1}{2}\gamma_{A_2} A_2}{\tan\frac{\alpha_2}{2}\gamma_{A_1} - \cot\frac{\alpha_1}{2}\gamma_{A_2}}, \\ T_{13} &= \frac{\tan\frac{\alpha_3}{2}\gamma_{A_1} A_1 - \cot\frac{\alpha_1}{2}\gamma_{A_3} A_3}{\tan\frac{\alpha_3}{2}\gamma_{A_1} - \cot\frac{\alpha_1}{2}\gamma_{A_3}}, \\ T_1 &= \frac{\tan\frac{\alpha_3}{2}\gamma_{A_2} A_2 + \tan\frac{\alpha_2}{2}\gamma_{A_3} A_3}{\tan\frac{\alpha_3}{2}\gamma_{A_2} + \tan\frac{\alpha_2}{2}\gamma_{A_3}}. \end{aligned} \tag{8.80a}$$

For $k = 2$,

$$\begin{aligned} T_{21} &= \frac{-\cot\frac{\alpha_2}{2}\gamma_{A_1} A_1 + \tan\frac{\alpha_1}{2}\gamma_{A_2} A_2}{-\cot\frac{\alpha_2}{2}\gamma_{A_1} + \tan\frac{\alpha_1}{2}\gamma_{A_2}}, \\ T_{23} &= \frac{\tan\frac{\alpha_3}{2}\gamma_{A_2} A_2 - \cot\frac{\alpha_2}{2}\gamma_{A_3} A_3}{\tan\frac{\alpha_3}{2}\gamma_{A_2} - \cot\frac{\alpha_2}{2}\gamma_{A_3}}, \\ T_2 &= \frac{\tan\frac{\alpha_3}{2}\gamma_{A_1} A_1 + \tan\frac{\alpha_1}{2}\gamma_{A_3} A_3}{\tan\frac{\alpha_3}{2}\gamma_{A_1} + \tan\frac{\alpha_1}{2}\gamma_{A_3}}. \end{aligned} \tag{8.80b}$$

And for $k = 3$,

$$\begin{aligned} T_{31} &= \frac{-\cot\frac{\alpha_3}{2}\gamma_{A_1} A_1 + \tan\frac{\alpha_1}{2}\gamma_{A_3} A_3}{-\cot\frac{\alpha_3}{2}\gamma_{A_1} + \tan\frac{\alpha_1}{2}\gamma_{A_3}}, \\ T_{32} &= \frac{-\cot\frac{\alpha_3}{2}\gamma_{A_2} A_2 + \tan\frac{\alpha_2}{2}\gamma_{A_3} A_3}{-\cot\frac{\alpha_3}{2}\gamma_{A_2} + \tan\frac{\alpha_2}{2}\gamma_{A_3}}, \\ T_3 &= \frac{\tan\frac{\alpha_2}{2}\gamma_{A_1} A_1 + \tan\frac{\alpha_1}{2}\gamma_{A_2} A_2}{\tan\frac{\alpha_2}{2}\gamma_{A_1} + \tan\frac{\alpha_1}{2}\gamma_{A_2}}. \end{aligned} \tag{8.80c}$$

Proof The proof of the first equation in (8.80a) is established in (8.79). The proof of the second equation in (8.80a) is obtained from the first by interchanging vertices A_1 and A_2. The third equation in (8.80a) is the result (8.56) of Theorem 8.7, p. 239. Finally, (8.80b) and (8.80c) are obtained from (8.80a) by cyclic permutations of the gyrotriangle vertices. □

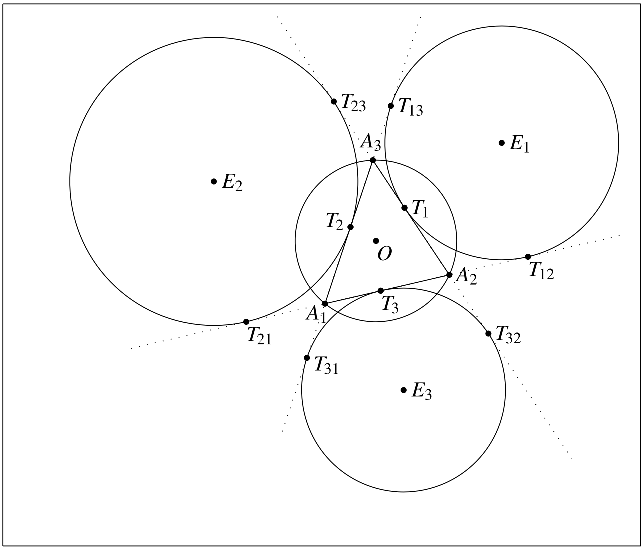


Fig. 8.5 Excircle points of tangency. The points of tangency T_1, etc., are given by Corollary 8.8, p. 240, of Theorem 8.7, and the points of tangency T_{12}, etc., are given by Corollary 8.13, p. 249, of Theorem 8.12. The triangle circumcenter is O, so that its circumradius is $R = \|-A_k + O\|$, $k = 1, 2, 3$

8.14 Excircle Points of Tangency, Part II

The way gyrotrigonometric gyrobarycentric coordinate representations in Einstein gyrovector spaces capture analogies between Euclidean and hyperbolic geometry, shown in Fig. 8.4, p. 237, and in Fig. 8.5, is now demonstrated.

The gyrotrigonometric gyrobarycentric coordinates of tangency points in Theorem 8.12 remain invariant in form under the Euclidean limit $s \to \infty$, resulting in the following corollary of Theorem 8.12:

Corollary 8.13 (The Excircle Points of Tangency Theorem) *Let $A_1A_2A_3$ be a triangle in a Euclidean vector space $\mathbb{R}^n$, let T_{ij} be the point of tangency where the A_i-excircle meets the extension of the triangle side A_iA_j, and let T_i be the point where the A_i-excircle of the triangle meets the opposite side of A_i. 8.4, p. 237, as shown in Fig. 8.5.*

Then, trigonometric barycentric coordinate representations of points of tangency on the A_k-excircle, $k = 1, 2, 3$, are as listed below.

For $k=1$,

$$\begin{aligned} T_{12} &= \frac{\tan\frac{\alpha_2}{2}A_1 - \cot\frac{\alpha_1}{2}A_2}{\tan\frac{\alpha_2}{2} - \cot\frac{\alpha_1}{2}}, \\ T_{13} &= \frac{\tan\frac{\alpha_3}{2}A_1 - \cot\frac{\alpha_1}{2}A_3}{\tan\frac{\alpha_3}{2} - \cot\frac{\alpha_1}{2}}, \\ T_1 &= \frac{\tan\frac{\alpha_3}{2}A_2 + \tan\frac{\alpha_2}{2}A_3}{\tan\frac{\alpha_3}{2} + \tan\frac{\alpha_2}{2}}. \end{aligned} \tag{8.81a}$$

For $k=2$,

$$\begin{aligned} T_{21} &= \frac{-\cot\frac{\alpha_2}{2}A_1 + \tan\frac{\alpha_1}{2}A_2}{-\cot\frac{\alpha_2}{2} + \tan\frac{\alpha_1}{2}}, \\ T_{23} &= \frac{\tan\frac{\alpha_3}{2}A_2 - \cot\frac{\alpha_2}{2}A_3}{\tan\frac{\alpha_3}{2} - \cot\frac{\alpha_2}{2}}, \\ T_2 &= \frac{\tan\frac{\alpha_3}{2}A_1 + \tan\frac{\alpha_1}{2}A_3}{\tan\frac{\alpha_3}{2} + \tan\frac{\alpha_1}{2}}. \end{aligned} \tag{8.81b}$$

And for $k=3$,

$$\begin{aligned} T_{31} &= \frac{-\cot\frac{\alpha_3}{2}A_1 + \tan\frac{\alpha_1}{2}A_3}{-\cot\frac{\alpha_3}{2} + \tan\frac{\alpha_1}{2}}, \\ T_{32} &= \frac{-\cot\frac{\alpha_3}{2}A_2 + \tan\frac{\alpha_2}{2}A_3}{-\cot\frac{\alpha_3}{2} + \tan\frac{\alpha_2}{2}}, \\ T_3 &= \frac{\tan\frac{\alpha_2}{2}A_1 + \tan\frac{\alpha_1}{2}A_2}{\tan\frac{\alpha_2}{2} + \tan\frac{\alpha_1}{2}}. \end{aligned} \tag{8.81c}$$

8.15 Gyrodistance Between Gyrotriangle Tangency Points

In subsections of this section, we determine the gyrodistances between various tangency points of exgyrocircles. These, in turn, will be used to calculate the measures of some gyroangles that exgyrocircles generate.

8.15.1 The Gyrodistance Between T_{12} and T_{13}

Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, and let T_{12} and T_{13} be the tangency points where the gyrotriangle A_1-exgyrocircle

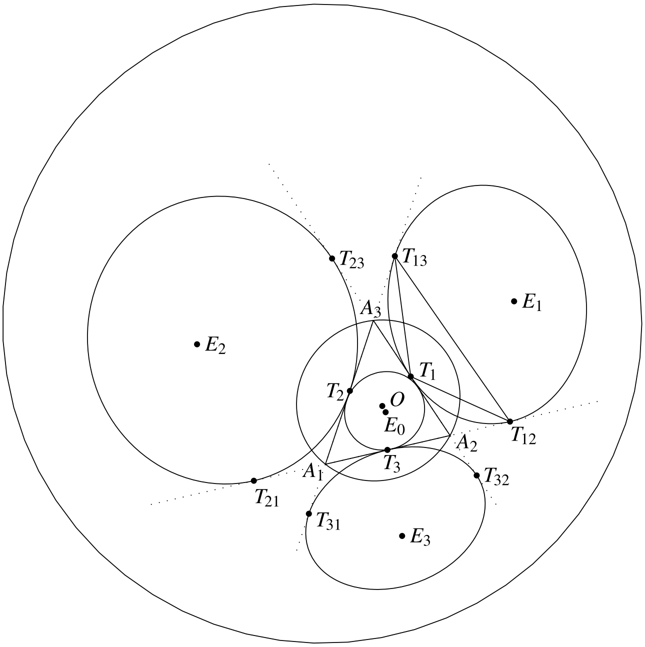


Fig. 8.6 Exgyrocircle points of tangency. The points of tangency T_1, etc., are given by Theorem 8.7, p. 239, and the points of tangency T_{12}, etc., are given by Theorem 8.12, p. 247. The gyroangle $\beta_1 = \angle T_{12}T_1T_{13}$ that the tangency points of the A_1-exgyrocircle generates is determined in terms of the gyroangles α_k, $k = 1, 2, 3$, of the reference gyrotriangle $A_1A_2A_3$ in Sect. 8.16, p. 258

meets the extensions of sides A_1A_2 and A_1A_2, as shown in Fig. 8.6. Their gyrobarycentric coordinate representations with respect to the pointwise independent set $S = \{A_1, A_2, A_3\}$, as given by Theorem 8.12, are:

$$T_{12} = \frac{m'_1\gamma_{A_1}A_1 + m'_2\gamma_{A_2}A_2}{m'_1\gamma_{A_1} + m'_2\gamma_{A_2}}, \tag{8.82a}$$

where the gyrobarycentric coordinates m'_k, $k = 1, 2$, are given by

$$\begin{aligned} m'_1 &= \tan\frac{\alpha_2}{2}, \\ m'_2 &= -\cot\frac{\alpha_1}{2}, \end{aligned} \tag{8.82b}$$

and

$$T_{13} = \frac{m_1\gamma_{A_1}A_1 + m_3\gamma_{A_3}A_3}{m_1\gamma_{A_1} + m_3\gamma_{A_3}}, \tag{8.83a}$$

where the gyrobarycentric coordinates m_k, $k = 1, 3$, are given by

$$m_1 = \tan\frac{\alpha_3}{2}, \qquad m_3 = -\cot\frac{\alpha_1}{2}. \tag{8.83b}$$

Hence, by (4.121), p. 113,

$$\gamma_{\ominus T_{12}\oplus T_{13}} = \frac{1}{m_0 m_0'}\left\{m_1 m_2' \gamma_{12} + m_1' m_3 \gamma_{13} + m_2' m_3 \gamma_{23} + m_1 m_1'\right\}, \tag{8.84}$$

where, by (4.118b) and (4.119b), p. 112, $m_0 > 0$ and $m_0' > 0$ are given by

$$\begin{aligned} m_0^2 &= m_1^2 + m_3^2 + 2m_1 m_3 \gamma_{13}, \\ \left(m_0'\right)^2 &= \left(m_1'\right)^2 + \left(m_2'\right)^2 + 2m_1' m_2' \gamma_{12}. \end{aligned} \tag{8.85}$$

Substituting (8.82b) and (8.83b) into (8.84) (and squaring, but subsequently taking a square root), we obtain the gamma factor $\gamma_{\ominus T_{12}\oplus T_{13}}$ expressed in terms of the gyrotriangle gyroangles α_k, $k = 1, 2, 3$,

$$\begin{aligned} \gamma_{\ominus T_{12}\oplus T_{13}} = \frac{1}{16}\frac{1}{\sin^2\frac{\alpha_2}{2}\sin^2\frac{\alpha_3}{2}}\big\{&2\cos^2\alpha_1 + 2(\cos\alpha_2 + \cos\alpha_3 - 1)^2 \\ &+ \cos(\alpha_1 + \alpha_2 + \alpha_3) + \cos(-\alpha_1 + \alpha_2 + \alpha_3) \\ &+ \cos(\alpha_1 - \alpha_2 + \alpha_3) + \cos(\alpha_1 + \alpha_2 - \alpha_3)\big\}. \end{aligned} \tag{8.86}$$

Substituting (8.86) into the identity, (1.9), p. 5,

$$\|\ominus T_{12}\oplus T_{13}\|^2 = s^2\frac{\gamma^2_{\ominus T_{12}\oplus T_{13}} - 1}{\gamma^2_{\ominus T_{12}\oplus T_{13}}}, \tag{8.87}$$

we obtain the desired gyrodistance,

$$\begin{aligned} \|\ominus T_{12}\oplus T_{13}\|^2 = {} & \frac{8s^2\cos\frac{\alpha_1+\alpha_2+\alpha_3}{2}\cos\frac{-\alpha_1+\alpha_2+\alpha_3}{2}\cos\frac{\alpha_1-\alpha_2+\alpha_3}{2}\cos\frac{\alpha_1+\alpha_2-\alpha_3}{2}}{\{2(1-\cos\alpha_2-\cos\alpha_3)^2 + 2\cos^2\alpha_1 + E(\alpha_1,\alpha_2,\alpha_3)\}^2} \\ & \times\big\{2 + 4(1-\cos\alpha_2-\cos\alpha_3)^2 + 2\big(\cos^2\alpha_1 \\ & - \cos^2\alpha_2 - \cos^2\alpha_3\big) + E(\alpha_1,\alpha_2,\alpha_3)\big\}, \end{aligned} \tag{8.88}$$

where

$$\begin{aligned} E(\alpha_1,\alpha_2,\alpha_3) = {} & \cos(\alpha_1 + \alpha_2 + \alpha_3) + \cos(-\alpha_1 + \alpha_2 + \alpha_3) \\ & + \cos(\alpha_1 - \alpha_2 + \alpha_3) + \cos(\alpha_1 + \alpha_2 - \alpha_3). \end{aligned} \tag{8.89}$$

Eliminating the factor $s^2 \cos\frac{\alpha_1+\alpha_2+\alpha_3}{2}$ between (8.88) and (7.35), p. 163, and taking a square root of both sides of the resulting equation, we obtain the following equation:

$$\|\ominus T_{12}\oplus T_{13}\| = \frac{2\sqrt{2}R\cos\frac{-\alpha_1+\alpha_2+\alpha_3}{2}\cos\frac{\alpha_1-\alpha_2+\alpha_3}{2}\cos\frac{\alpha_1+\alpha_2-\alpha_3}{2}}{2(\cos\alpha_2+\cos\alpha_3-1)^2+2\cos^2\alpha_1+E(\alpha_1,\alpha_2,\alpha_3)} \times\sqrt{2+4(1-\cos\alpha_2-\cos\alpha_3)^2+2(\cos^2\alpha_1-\cos^2\alpha_2-\cos^2\alpha_3)+E(\alpha_1,\alpha_2,\alpha_3)}, \tag{8.90}$$

where R is the circumgyroradius of the reference gyrotriangle $A_1A_2A_3$.

8.15.2 The Gyrodistance Between T_1 and T_{12}, T_{13}

As in Sect. 8.15.1, we calculate here the gyrodistance between gyrotriangle tangency points.

Following (8.80a), p. 248, we have

$$T_{12} = \frac{m'_1\gamma_{A_1}A_1 + m'_2\gamma_{A_2}A_2}{m'_1\gamma_{A_1} + m'_2\gamma_{A_2}}, \tag{8.91a}$$

where the gyrobarycentric coordinates m'_k, $k = 1, 2$, are given by

$$\begin{aligned} m'_1 &= \tan\frac{\alpha_2}{2}, \\ m'_2 &= -\cot\frac{\alpha_1}{2}, \end{aligned} \tag{8.91b}$$

and

$$T_1 = \frac{m_2\gamma_{A_2}A_2 + m_3\gamma_{A_3}A_3}{m_2\gamma_{A_2} + m_3\gamma_{A_3}}, \tag{8.92a}$$

where the gyrobarycentric coordinates m_k, $k = 2, 3$, are given by

$$\begin{aligned} m_2 &= \tan\frac{\alpha_3}{2}, \\ m_3 &= \tan\frac{\alpha_2}{2}. \end{aligned} \tag{8.92b}$$

Hence, by (4.121), p. 113,

$$\gamma_{\ominus T_1\oplus T_{12}} = \frac{1}{m_0m'_0}\{m'_1m_2\gamma_{12} + m'_1m_3\gamma_{13} + m'_2m_3\gamma_{23} + m_2m'_2\}, \tag{8.93}$$

where, by (4.118b) and (4.119b), p. 112, $m_0 > 0$ and $m'_0 > 0$ are given by

$$m_0^2 = m_2^2 + m_3^2 + 2m_1 m_3 \gamma_{23},$$
$$\big(m'_0\big)^2 = \big(m'_1\big)^2 + \big(m'_2\big)^2 + 2m'_1 m'_2 \gamma_{12}. \tag{8.94}$$

Substituting (8.91b) and (8.92b) into (8.93), we obtain the gamma factor $\gamma_{\ominus T_1 \oplus T_{12}}$ expressed in terms of the gyrotriangle gyroangles α_k, $k = 1, 2, 3$,

$$\begin{aligned}\gamma_{\ominus T_1 \oplus T_{12}} &= \frac{1}{16} \frac{1}{\cos^2 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_3}{2}} \big\{2\cos^2 \alpha_2 + 2(1 + \cos\alpha_1 - \cos\alpha_3)^2 \\ &\quad + \cos(\alpha_1 + \alpha_2 + \alpha_3) + \cos(-\alpha_1 + \alpha_2 + \alpha_3) \\ &\quad + \cos(\alpha_1 - \alpha_2 + \alpha_3) + \cos(\alpha_1 + \alpha_2 - \alpha_3)\big\}. \end{aligned} \tag{8.95}$$

Substituting (8.95) into the identity, (1.9), p. 5,

$$\|\ominus T_1 \oplus T_{12}\|^2 = s^2 \frac{\gamma^2_{\ominus T_{12} \oplus T_{13}} - 1}{\gamma^2_{\ominus T_{12} \oplus T_{13}}}, \tag{8.96}$$

we obtain the desired gyrodistance,

$$\begin{aligned}\|\ominus T_1 \oplus T_{12}\|^2 &= \frac{8s^2 \cos \frac{\alpha_1+\alpha_2+\alpha_3}{2} \cos \frac{-\alpha_1+\alpha_2+\alpha_3}{2} \cos \frac{\alpha_1-\alpha_2+\alpha_3}{2} \cos \frac{\alpha_1+\alpha_2-\alpha_3}{2}}{\{2(1 + \cos\alpha_1 - \cos\alpha_3)^2 + 2\cos^2\alpha_2 + E(\alpha_1, \alpha_2, \alpha_3)\}^2} \\ &\quad \times \big\{2 + 4(1 + \cos\alpha_1 - \cos\alpha_3)^2 + 2\big(\cos^2\alpha_1 - \cos^2\alpha_2 + \cos^2\alpha_3\big) \\ &\quad + E(\alpha_1, \alpha_2, \alpha_3)\big\}, \end{aligned} \tag{8.97}$$

where $E(\alpha_1, \alpha_2, \alpha_3)$ is given by (8.89), p. 252.

Eliminating the factor $s^2 \cos \frac{\alpha_1+\alpha_2+\alpha_3}{2}$ between (8.97) and (7.35), p. 163, and taking a square root of both sides of the resulting equation, we obtain the following equation:

$$\begin{aligned}&\|\ominus T_1 \oplus T_{12}\| \\ &\quad = \frac{2\sqrt{2}R \cos \frac{-\alpha_1+\alpha_2+\alpha_3}{2} \cos \frac{\alpha_1-\alpha_2+\alpha_3}{2} \cos \frac{\alpha_1+\alpha_2-\alpha_3}{2}}{2(1 + \cos\alpha_1 - \cos\alpha_3)^2 + 2\cos^2\alpha_2 + E(\alpha_1, \alpha_2, \alpha_3)} \\ &\qquad \times \sqrt{2 + 4(1 + \cos\alpha_1 - \cos\alpha_3)^2 + 2(-\cos^2\alpha_1 + \cos^2\alpha_2 - \cos^2\alpha_3) + E(\alpha_1, \alpha_2, \alpha_3)}, \end{aligned} \tag{8.98}$$

where R is the circumgyroradius of the reference gyrotriangle $A_1A_2A_3$.

8.15.3 Resulting Gyrodistances Between Tangency Points

Formalizing the results of Sects. 8.15.1–8.15.2, we obtain the following theorem in which gyrodistances between various tangency points are related to the gyroangles of the reference gyrotriangle and its circumgyroradius.

Theorem 8.14 (Gyrodistances Between Exgyrocircle Points of Tangency) *Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, let T_{ij} be the point of tangency where the A_i-exgyrocircle meets the extension of the gyrotriangle side A_iA_j, and let T_i be the point where the A_i-exgyrocircle of the gyrotriangle meets the opposite side of A_i, Fig. 8.6, p. 251. Furthermore, let α_k, $k = 1, 2, 3$, and R be the gyrotriangle gyroangles and its circumgyroradius.*

Then, the gyrodistances between tangency points of the A_1-exgyrocircle are:

$$\begin{aligned}\|\ominus T_{12}\oplus T_{13}\| &= \frac{2\sqrt{2}R\cos\frac{-\alpha_1+\alpha_2+\alpha_3}{2}\cos\frac{\alpha_1-\alpha_2+\alpha_3}{2}\cos\frac{\alpha_1+\alpha_2-\alpha_3}{2}}{2\cos^2\alpha_1+2(1-\cos\alpha_2-\cos\alpha_3)^2+E(\alpha_1,\alpha_2,\alpha_3)} \\ &\quad\times\sqrt{2+2(\cos^2\alpha_1-\cos^2\alpha_2-\cos^2\alpha_3)+4(1-\cos\alpha_2-\cos\alpha_3)^2+E(\alpha_1,\alpha_2,\alpha_3)},\end{aligned} \tag{8.99a}$$

$$\begin{aligned}\|\ominus T_1\oplus T_{12}\| &= \frac{2\sqrt{2}R\cos\frac{-\alpha_1+\alpha_2+\alpha_3}{2}\cos\frac{\alpha_1-\alpha_2+\alpha_3}{2}\cos\frac{\alpha_1+\alpha_2-\alpha_3}{2}}{2\cos^2\alpha_2+2(1+\cos\alpha_1-\cos\alpha_3)^2+E(\alpha_1,\alpha_2,\alpha_3)} \\ &\quad\times\sqrt{2+2(-\cos^2\alpha_1+\cos^2\alpha_2-\cos^2\alpha_3)+4(1+\cos\alpha_1-\cos\alpha_3)^2+E(\alpha_1,\alpha_2,\alpha_3)},\end{aligned} \tag{8.99b}$$

$$\begin{aligned}\|\ominus T_1\oplus T_{13}\| &= \frac{2\sqrt{2}R\cos\frac{-\alpha_1+\alpha_2+\alpha_3}{2}\cos\frac{\alpha_1-\alpha_2+\alpha_3}{2}\cos\frac{\alpha_1+\alpha_2-\alpha_3}{2}}{2\cos^2\alpha_3+2(1+\cos\alpha_1-\cos\alpha_2)^2+E(\alpha_1,\alpha_2,\alpha_3)} \\ &\quad\times\sqrt{2+2(-\cos^2\alpha_1-\cos^2\alpha_2+\cos^2\alpha_3)+4(1+\cos\alpha_1-\cos\alpha_2)^2+E(\alpha_1,\alpha_2,\alpha_3)},\end{aligned} \tag{8.99c}$$

where $E(\alpha_1, \alpha_2, \alpha_3)$ is given by (8.89), i.e.,

$$\begin{aligned}E(\alpha_1,\alpha_2,\alpha_3) &= \cos(\alpha_1+\alpha_2+\alpha_3)+\cos(-\alpha_1+\alpha_2+\alpha_3) \\ &\quad+\cos(\alpha_1-\alpha_2+\alpha_3)+\cos(\alpha_1+\alpha_2-\alpha_3),\end{aligned} \tag{8.99d}$$

and where R is the circumgyroradius of the reference gyrotriangle $A_1A_2A_3$.

Furthermore, the gamma factors of these gyrodistances are:

$$\gamma_{\ominus T_{12}\oplus T_{13}} = \frac{1}{16}\frac{1}{\sin^2\frac{\alpha_2}{2}\sin^2\frac{\alpha_3}{2}}\left\{2\cos^2\alpha_1+2(1-\cos\alpha_2-\cos\alpha_3)^2+E(\alpha_1,\alpha_2,\alpha_3)\right\}, \tag{8.100a}$$

$$\gamma_{\ominus T_1 \oplus T_{12}} = \frac{1}{16} \frac{1}{\cos^2 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_3}{2}} \left\{ 2\cos^2 \alpha_2 + 2(1 + \cos\alpha_1 - \cos\alpha_3)^2 + E(\alpha_1, \alpha_2, \alpha_3) \right\}, \tag{8.100b}$$

$$\gamma_{\ominus T_1 \oplus T_{13}} = \frac{1}{16} \frac{1}{\cos^2 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_2}{2}} \left\{ 2\cos^2 \alpha_3 + 2(1 + \cos\alpha_1 - \cos\alpha_2)^2 + E(\alpha_1, \alpha_2, \alpha_3) \right\}. \tag{8.100c}$$

Proof Equations (8.99a) and (8.100a) are established in (8.90) and (8.86), respectively. Equations (8.99b) and (8.100b) are established in (8.98) and (8.95), respectively. Finally, (8.99c) and (8.100c) are obtained, respectively, from (8.99b) and (8.100b) by interchanging the vertices A_2 and A_3 of the reference gyrotriangle $A_1A_2A_3$. □

Interestingly, (8.99a), (8.99b), (8.99c), (8.99d) remain invariant in form under the Euclidean limit $s \to \infty$, so that the equations are valid in Euclidean geometry as well. However, for application in Euclidean geometry (8.99a), (8.99b), (8.99c), (8.99d) can be simplified owing to the fact that triangle angle sum in π.

Indeed, under the condition

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi, \tag{8.101a}$$

we have the trigonometric identities similar to (7.22b), p. 159,

$$\begin{aligned} \cos\frac{-\alpha_1 + \alpha_2 + \alpha_3}{2} &= \sin\alpha_1, \\ \cos\frac{\alpha_1 - \alpha_2 + \alpha_3}{2} &= \sin\alpha_2, \\ \cos\frac{\alpha_1 + \alpha_2 - \alpha_3}{2} &= \sin\alpha_3, \end{aligned} \tag{8.101b}$$

and

$$\begin{aligned} \cos(\alpha_1 + \alpha_2 + \alpha_3) &= -1, \\ \cos(-\alpha_1 + \alpha_2 + \alpha_3) &= -\cos 2\alpha_1, \\ \cos(\alpha_1 - \alpha_2 + \alpha_3) &= -\cos 2\alpha_2, \\ \cos(\alpha_1 + \alpha_2 - \alpha_3) &= -\cos 2\alpha_3. \end{aligned} \tag{8.101c}$$

Hence, we obtain the following corollary of Theorem 8.14:

Corollary 8.15 (Distances Between Excircle Points of Tangency, I) *Let $A_1A_2A_3$ be a triangle in a Euclidean vector space $\mathbb{R}^n$, let T_{ij} be the point of tangency where the A_i-excircle meets the extension of the triangle side A_iA_j, and let T_i be the point* *where the A_i-excircle of the triangle meets the opposite side of A_i, Fig. 8.5, p. 249. Furthermore, let α_k, $k = 1, 2, 3$, and R be the triangle angles and its circumradius.*

Then, the distances between tangency points of the A_1-excircle are:

$$\begin{aligned}\|-T_{12} + T_{13}\| &= \frac{2\sqrt{2}R\sin\alpha_1\sin\alpha_2\sin\alpha_3}{2\cos^2\alpha_1 + 2(1 - \cos\alpha_2 - \cos\alpha_3)^2 + E(\alpha_1,\alpha_2,\alpha_3)} \\ &\quad\times\sqrt{2 + 2(\cos^2\alpha_1 - \cos^2\alpha_2 - \cos^2\alpha_3) + 4(1 - \cos\alpha_2 - \cos\alpha_3)^2 + E(\alpha_1,\alpha_2,\alpha_3)},\end{aligned} \tag{8.102a}$$

$$\begin{aligned}\|-T_{1} + T_{12}\| &= \frac{2\sqrt{2}R\sin\alpha_1\sin\alpha_2\sin\alpha_3}{2\cos^2\alpha_2 + 2(1 + \cos\alpha_1 - \cos\alpha_3)^2 + E(\alpha_1,\alpha_2,\alpha_3)} \\ &\quad\times\sqrt{2 + 2(-\cos^2\alpha_1 + \cos^2\alpha_2 - \cos^2\alpha_3) + 4(1 + \cos\alpha_1 - \cos\alpha_3)^2 + E(\alpha_1,\alpha_2,\alpha_3)},\end{aligned} \tag{8.102b}$$

and

$$\begin{aligned}\|-T_{1} + T_{13}\| &= \frac{2\sqrt{2}R\sin\alpha_1\sin\alpha_2\sin\alpha_3}{2\cos^2\alpha_3 + 2(1 + \cos\alpha_1 - \cos\alpha_2)^2 + E(\alpha_1,\alpha_2,\alpha_3)} \\ &\quad\times\sqrt{2 + 2(-\cos^2\alpha_1 - \cos^2\alpha_2 + \cos^2\alpha_3) + 4(1 + \cos\alpha_1 - \cos\alpha_2)^2 + E(\alpha_1,\alpha_2,\alpha_3)},\end{aligned} \tag{8.102c}$$

where $E(\alpha_1, \alpha_2, \alpha_3)$ is given by

$$E(\alpha_1,\alpha_2,\alpha_3) = -(1 + \cos 2\alpha_1 + \cos 2\alpha_2 + \cos 2\alpha_3) \tag{8.102d}$$

and where R is the circumradius of the reference triangle $A_1A_2A_3$.

Equations (8.99a), (8.99b), (8.99c), (8.99d) of Theorem 8.14 about gyrodistances in hyperbolic geometry remain valid in Euclidean geometry as well, where they form equations about corresponding distances in Euclidean geometry, in which Einstein addition becomes vector addition. However, when Theorem 8.14 is considered in Euclidean geometry, its results can be simplified owing to Condition (8.101a) that triangle angles obey. The trigonometric simplifications that Condition (8.101a) offers, (8.101b)–(8.101c), thus give rise to Corollary 8.15.

Interestingly, Corollary 8.15 can be further simplified by employing rather involved trigonometric simplifications that Theorem 8.14 uncovers, as follows:

Theorem 8.14 remains invariant in form under the Euclidean limit, $s \to \infty$. Hence, it survives unimpaired in the transition from hyperbolic to Euclidean geometry. But in that Euclidean limit, gamma factors tend to 1. Hence, in particular, in the application of Theorem 8.14 in Euclidean geometry the gamma factors in (8.100a), (8.100b), (8.100c) are 1, resulting in the following trigonometric simplifications that become available under Condition (8.101a):

$$\begin{aligned}
16\sin^2\frac{\alpha_2}{2}\sin^2\frac{\alpha_3}{2} &= 2\cos^2\alpha_1 + 2(1-\cos\alpha_2-\cos\alpha_3)^2 + E(\alpha_1,\alpha_2,\alpha_3),\\
16\cos^2\frac{\alpha_1}{2}\sin^2\frac{\alpha_3}{2} &= 2\cos^2\alpha_2 + 2(1+\cos\alpha_1-\cos\alpha_3)^2 + E(\alpha_1,\alpha_2,\alpha_3),\\
16\cos^2\frac{\alpha_1}{2}\sin^2\frac{\alpha_2}{2} &= 2\cos^2\alpha_3 + 2(1+\cos\alpha_1-\cos\alpha_2)^2 + E(\alpha_1,\alpha_2,\alpha_3),\\
\alpha_1+\alpha_2+\alpha_3 &= \pi.
\end{aligned} \tag{8.103}$$

Owing to Identities (8.103) that triangle angles possess, Corollary 8.15 can be simplified as follows:

Corollary 8.16 (Distances Between Excircle Points of Tangency, II) *Let $A_1A_2A_3$ be a triangle in a Euclidean vector space $\mathbb{R}^n$, let T_{ij} be the point of tangency where the A_i-excircle meets the extension of the triangle side A_iA_j, and let T_i be the point where the A_i-excircle of the triangle meets the opposite side of A_i, Fig. 8.5, p. 249.*

Then, the distances between tangency points of the A_1-excircle are:

$$\begin{aligned}
\|-T_{12}+T_{13}\| &= 4R\sin\alpha_1\cos\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2},\\
\|-T_1+T_{12}\| &= 4R\sin\frac{\alpha_1}{2}\sin\alpha_2\cos\frac{\alpha_3}{2},\\
\|-T_1+T_{13}\| &= 4R\sin\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\sin\alpha_3,
\end{aligned} \tag{8.104}$$

where R is the circumradius of the reference triangle $A_1A_2A_3$.

Distances between tangency points of the A_2- and the A_3-excircle of triangle $A_1A_2A_3$ can be obtained from (8.104) by cyclic vertex permutations.

8.16 Exgyrocircle Gyroangles

Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ that possesses an A_1-exgyrocircle. The gyroangle

$$\beta_1 = \angle T_{12}T_1T_{13} \tag{8.105}$$

generated by the tangency points T_1, T_{12} and T_{13} of the A_1-exgyrocircle of the gyrotriangle, as shown in Fig. 8.7, is called the A_1-exgyrocircle gyroangle.

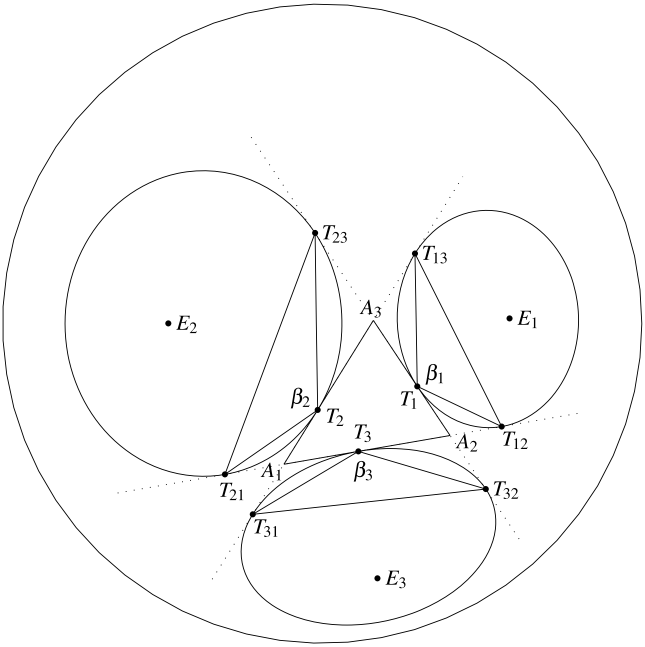


Fig. 8.7 Gyroangles β_k, $k = 1, 2, 3$, generated by exgyrocircle points of tangency

In this section, we face the task of determining exgyrocircle gyroangles β_k, $k = 1, 2, 3$, in terms of the gyroangles of their reference gyrotriangle $A_1A_2A_3$.

Following the *SSS* to *AAA* conversion law in Theorem 6.2, (6.22), p. 135, and (6.20), p. 134, the gyrosine and gyrocosine of gyroangle β_1 are given by the equations

$$\sin^2\beta_1 = \frac{1 + 2\gamma_{\ominus T_1\oplus T_{12}}\gamma_{\ominus T_1\oplus T_{13}}\gamma_{\ominus T_{12}\oplus T_{13}} - \gamma^2_{\ominus T_1\oplus T_{12}} - \gamma^2_{\ominus T_1\oplus T_{13}} - \gamma^2_{\ominus T_{12}\oplus T_{13}}}{(\gamma^2_{\ominus T_1\oplus T_{12}} - 1)(\gamma^2_{\ominus T_1\oplus T_{13}} - 1)},$$

$$\cos^2\beta_1 = \frac{(\gamma_{\ominus T_1\oplus T_{12}}\gamma_{\ominus T_1\oplus T_{13}} - \gamma_{\ominus T_{12}\oplus T_{13}})^2}{(\gamma^2_{\ominus T_1\oplus T_{12}} - 1)(\gamma^2_{\ominus T_1\oplus T_{13}} - 1)}. \tag{8.106a}$$

Substituting the gamma factors from (8.100a), p. 255, into (8.106a) and simplifying the results gyrotrigonometrically, we obtain the equations

$$\sin\beta_1 = \frac{f_s}{\sqrt{f_s^2 + f_c^2}},$$

$$\cos\beta_1 = \frac{f_c}{\sqrt{f_s^2 + f_c^2}}, \tag{8.106b}$$

where

$$f_s = 32\cos^3\tfrac{\alpha_1}{2}\sin\tfrac{\alpha_2}{2}\sin\tfrac{\alpha_3}{2} > 0,$$
$$f_c = 2\big(\sin^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3\big) - 8\cos^2\tfrac{\alpha_1}{2}(\cos\alpha_2 + \cos\alpha_3) + E(\alpha_1,\alpha_2,\alpha_3) < 0, \tag{8.106c}$$

where, as in (8.99d), p. 255,

$$E(\alpha_1,\alpha_2,\alpha_3) = \cos(\alpha_1+\alpha_2+\alpha_3) + \cos(-\alpha_1+\alpha_2+\alpha_3) + \cos(\alpha_1-\alpha_2+\alpha_3) + \cos(\alpha_1+\alpha_2-\alpha_3). \tag{8.106d}$$

Interestingly, the sum $f_s^2 + f_c^2$ of squares can be written as a product,

$$f_s^2 + f_c^2 = f_{123} f_{132}, \tag{8.107a}$$

where

$$f_{123} = 2\big(\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3\big) + 32\cos^2\frac{\alpha_1}{2}\sin^2\frac{\alpha_2}{2} + E(\alpha_1,\alpha_2,\alpha_3) - 2,$$
$$f_{132} = 2\big(\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3\big) + 32\cos^2\frac{\alpha_1}{2}\sin^2\frac{\alpha_3}{2} + E(\alpha_1,\alpha_2,\alpha_3) - 2. \tag{8.107b}$$

Formalizing the main result of this section, we obtain the following theorem:

Theorem 8.17 (Gyroangles Generated by Exgyrocircle Points of Tangency) *Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ with gyroangles α_k, $k = 1, 2, 3$, and let $E(\alpha_1,\alpha_2,\alpha_3)$ be given by the equation*

$$E(\alpha_1,\alpha_2,\alpha_3) = \cos(\alpha_1+\alpha_2+\alpha_3) + \cos(-\alpha_1+\alpha_2+\alpha_3) + \cos(\alpha_1-\alpha_2+\alpha_3) + \cos(\alpha_1+\alpha_2-\alpha_3). \tag{8.108}$$

Furthermore, let T_{ij} be the point of tangency where the A_i-exgyrocircle meets the extension of the gyrotriangle side A_iA_j, and let T_i be the point where the A_i-exgyrocircle of the gyrotriangle meets the opposite side of A_i, Fig. 8.7. Then, the measures of gyroangles

$$\beta_1 = \angle T_{12}T_1T_{12},$$
$$\beta_2 = \angle T_{21}T_2T_{23}, \tag{8.109}$$
$$\beta_3 = \angle T_{31}T_2T_{32}$$

that the exgyrocircle tangency points generate are determined by the following equations:

$$\sin\beta_1 = \frac{f_{s1}}{\sqrt{f_{s1}^2 + f_{c1}^2}}, \qquad \cos\beta_1 = \frac{f_{c1}}{\sqrt{f_{s1}^2 + f_{c1}^2}}, \tag{8.110a}$$

where

$$\begin{aligned} f_{s1} &= 32\cos^3\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2} > 0, \\ f_{c1} &= 2\bigl(\sin^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3\bigr) \\ &\quad - 8\cos^2\frac{\alpha_1}{2}(\cos\alpha_2 + \cos\alpha_3) + E(\alpha_1, \alpha_2, \alpha_3) \\ &< 0, \end{aligned} \tag{8.110b}$$

$$\sin\beta_2 = \frac{f_{s2}}{\sqrt{f_{s2}^2 + f_{c2}^2}}, \qquad \cos\beta_2 = \frac{f_{c2}}{\sqrt{f_{s2}^2 + f_{c2}^2}}, \tag{8.111a}$$

where

$$\begin{aligned} f_{s2} &= 32\sin\frac{\alpha_1}{2}\cos^3\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2} > 0, \\ f_{c2} &= 2\bigl(\cos^2\alpha_1 + \sin^2\alpha_2 + \cos^2\alpha_3\bigr) \\ &\quad - 8\cos^2\frac{\alpha_2}{2}(\cos\alpha_1 + \cos\alpha_3) + E(\alpha_1, \alpha_2, \alpha_3) \\ &< 0, \end{aligned} \tag{8.111b}$$

$$\sin\beta_3 = \frac{f_{s3}}{\sqrt{f_{s2}^2 + f_{c3}^2}}, \qquad \cos\beta_3 = \frac{f_{c3}}{\sqrt{f_{s3}^2 + f_{c3}^2}}, \tag{8.112a}$$

where

$$\begin{aligned} f_{s3} &= 32\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos^3\frac{\alpha_3}{2} > 0, \\ f_{c3} &= 2\bigl(\cos^2\alpha_1 + \cos^2\alpha_2 + \sin^2\alpha_3\bigr) \\ &\quad - 8\cos^2\frac{\alpha_3}{2}(\cos\alpha_1 + \cos\alpha_2) + E(\alpha_1, \alpha_2, \alpha_3) \\ &< 0. \end{aligned} \tag{8.112b}$$

Proof The determination of gyroangle β_1 in (8.110a), (8.110b) is established in (8.106a), (8.106b), (8.106c), (8.106d). The determination of gyroangles β_2 and β_3 follows from that of β_1 in (8.106a), (8.106b), (8.106c), (8.106d) by cyclic permutations of the vertices of the reference gyrotriangle $A_1A_2A_3$. □

8.17 Exgyrocircle Gyroangle Sum

Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ that possesses the A_k-exgyrocircle, $k = 1, 2, 3$. The gyrotriangle exgyrocircles, in turn, generate the exgyrocircle gyroangles β_k, $k = 1, 2, 3$, the sum of which, $\beta_1 + \beta_2 + \beta_3$, is the gyrotriangle exgyrocircle gyroangle sum.

In this section, we face the task of determining the gyrotriangle exgyrocircle gyroangle sum in terms of the gyroangles of the reference gyrotriangle.

Let $A_1A_2A_3$ be a gyrotriangle in the Einstein gyrovector space that possesses three exgyrocircles and let β_k, $k = 1, 2, 3$, be the resulting gyrotriangle exgyrocircle gyroangles. The gyrosine and gyrocosine of the gyrotriangle exgyrocircle gyroangle sum are

$$\begin{aligned}\sin(\beta_1 + \beta_2 + \beta_3) &= -\sin\beta_1 \sin\beta_2 \sin\beta_3 + \sin\beta_1 \cos\beta_2 \cos\beta_3 \\ &\quad + \cos\beta_1 \sin\beta_2 \cos\beta_3 + \cos\beta_1 \cos\beta_2 \sin\beta_3, \\ \cos(\beta_1 + \beta_2 + \beta_3) &= \cos\beta_1 \cos\beta_2 \cos\beta_3 - \cos\beta_1 \sin\beta_2 \sin\beta_3 \\ &\quad - \sin\beta_1 \cos\beta_2 \sin\beta_3 - \sin\beta_1 \sin\beta_2 \cos\beta_3.\end{aligned} \tag{8.113}$$

Substituting $\sin\beta_k$ and $\cos\beta_k$, $k = 1, 2, 3$, from (8.110a), (8.110b), (8.111a), (8.111b), (8.112a), (8.112b) into (8.113), we obtain the equations

$$\begin{aligned}\sin(\beta_1 + \beta_2 + \beta_3) &= \cos\frac{\alpha_1 + \alpha_2 + \alpha_3}{2} f_1(\alpha_1, \alpha_2, \alpha_3), \\ \cos(\beta_1 + \beta_2 + \beta_3) &= f_2(\alpha_1, \alpha_2, \alpha_3),\end{aligned} \tag{8.114}$$

where the functions f_1 and f_2 of α_k, $k = 1, 2, 3$, are too involved and hence are not presented here explicitly.

The factor $\cos\frac{\alpha_1+\alpha_2+\alpha_3}{2}$ on the right-hand side of the first equation in (8.114) enables an interesting result of Euclidean geometry to be uncovered. In the Euclidean limit, $s \to \infty$, a gyrotriangle gyroangle sum tends to a corresponding triangle angle sum, which is π. Hence, in that limit, the factor tends to $\cos\frac{\pi}{2} = 0$, implying that $\sin(\beta_1 + \beta_2 + \beta_3) = 0$ in Euclidean geometry. The latter, in turn, implies that in Euclidean geometry the excircle angle sum is $\beta_1 + \beta_2 + \beta_3 = 2\pi$.

8.18 Exgyrocenter-Point-of-Tangency Gyrocenter

Definition 8.18 (Exgyrocenter-Point-of-Tangency Gyrocenter, P) Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ along with its exgyrocenters E_k and points of tangency T_k, $k = 1, 2, 3$, Fig. 8.8. The points E_k and T_k form the three gyrolines E_kT_k that are concurrent. This point of concurrency is called the exgyrocenter-point-of-tangency Gyrocenter, P, Fig. 8.8, of the gyrotriangle.

Let the gyrocenter P of gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ be given by its gyrobarycentric coordinate representation with respect to the set $S = \{A_1, A_2, A_3\}$ of the gyrotriangle vertices,

$$P = \frac{m_1\gamma_{A_1} A_1 + m_2\gamma_{A_2} A_2 + m_3\gamma_{A_3} A_3}{m_1\gamma_{A_1} + m_2\gamma_{A_2} + m_3\gamma_{A_3}}, \tag{8.115}$$

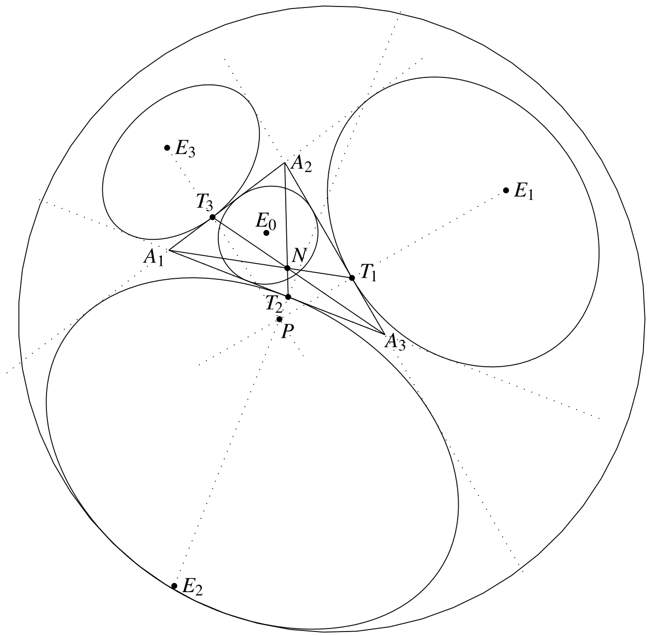


Fig. 8.8 The Exgyrocenter-Point-of-Tangency Gyrocenter, P, of a gyrotriangle $A_1A_2A_3$. This gyrotriangle gyrocenter, P, is the point of concurrency of the three gyrolines E_kT_k, $k = 1, 2, 3$, where E_k are the gyrotriangle exgyrocenters and T_k are the gyrotriangle points of tangency with its exgyrocircles. Gyrotrigonometric gyrobarycentric coordinates of the gyrotriangle gyrocenter P are given in Theorem 8.19. Note that the three points A_k, E_0 and E_k for each k, $k = 1, 2, 3$, are gyrocollinear

where the gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ of P in (8.115) are to be determined.

Similarly to (8.58a), (8.58b), (8.58c), (8.59a), (8.59b) and (8.59c), under the left gyrotranslation by $\ominus A_1$ the gyrotriangle gyrocenter P in (8.115) becomes

$$\begin{aligned}\ominus A_1 \oplus P &= \frac{m_2\gamma_{\ominus A_1\oplus A_2}(\ominus A_1\oplus A_2) + m_3\gamma_{\ominus A_1\oplus A_3}(\ominus A_1\oplus A_3)}{m_1 + m_2\gamma_{\ominus A_1\oplus A_2} + m_3\gamma_{\ominus A_1\oplus A_3}} \\ &= \frac{m_2\gamma_{12}\mathbf{a}_{12} + m_3\gamma_{13}\mathbf{a}_{13}}{m_1 + m_2\gamma_{12} + m_3\gamma_{13}}.\end{aligned} \tag{8.116}$$

1. The exgyrocenter $\ominus A_1 \oplus E_1$ and the tangent point $\ominus A_1 \oplus T_1$ of the left gyrotranslated gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$, as calculated in (8.58a) and (8.59a),

are:

$$\begin{aligned} \ominus A_1 \oplus E_1 &= \frac{\sin\alpha_2 \gamma_{12} \mathbf{a}_{12} + \sin\alpha_3 \gamma_{13} \mathbf{a}_{13}}{-\sin\alpha_1 + \sin\alpha_2 \gamma_{12} + \sin\alpha_3 \gamma_{13}}, \\ \ominus A_1 \oplus T_1 &= \frac{\tan\frac{\alpha_3}{2} \gamma_{12} \mathbf{a}_{12} + \tan\frac{\alpha_2}{2} \gamma_{13} \mathbf{a}_{13}}{\tan\frac{\alpha_3}{2} \gamma_{12} + \tan\frac{\alpha_2}{2} \gamma_{13}}. \end{aligned} \tag{8.117a}$$

These points are contained in the Euclidean line

$$L_1 = (\ominus A_1 \oplus T_1) + \big(-(\ominus A_1 \oplus T_1) + (\ominus A_1 \oplus E_1)\big) t_1, \tag{8.117b}$$

where $t_1 \in \mathbb{R}$ is the line parameter. This line passes through the point $(\ominus A_1 \oplus T_1) \in \mathbb{R}^n_s \subset \mathbb{R}^n$ when $t_1 = 0$, and it passes through the point $(\ominus A_1 \oplus E_1) \in \mathbb{R}^n_s \subset \mathbb{R}^n$ when $t_1 = 1$.

2. Similarly to (8.117a), (8.117b), The exgyrocenter $\ominus A_1 \oplus E_2$ and the tangent point $\ominus A_1 \oplus T_2$ of the left gyrotranslated gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$, as calculated in (8.58b) and (8.59b), are:

$$\begin{aligned} \ominus A_1 \oplus E_2 &= \frac{-\sin\alpha_2 \gamma_{12} \mathbf{a}_{12} + \sin\alpha_3 \gamma_{13} \mathbf{a}_{13}}{\sin\alpha_1 - \sin\alpha_2 \gamma_{12} + \sin\alpha_3 \gamma_{13}}, \\ \ominus A_1 \oplus T_2 &= \frac{\tan\frac{\alpha_1}{2} \gamma_{13} \mathbf{a}_{13}}{\tan\frac{\alpha_3}{2} + \tan\frac{\alpha_1}{2} \gamma_{13}}. \end{aligned} \tag{8.118a}$$

These points are contained in the Euclidean line

$$L_2 = (\ominus A_1 \oplus T_2) + \big(-(\ominus A_1 \oplus T_2) + (\ominus A_1 \oplus E_2)\big) t_2 \tag{8.118b}$$

where $t_2 \in \mathbb{R}$ is the line parameter. This line passes through the point $(\ominus A_1 \oplus T_2) \in \mathbb{R}^n_s \subset \mathbb{R}^n$ when $t_2 = 0$, and it passes through the point $(\ominus A_1 \oplus E_2) \in \mathbb{R}^n_s \subset \mathbb{R}^n$ when $t_2 = 1$.

3. Similarly to (8.117a), (8.117b), (8.118a) and (8.118b), The exgyrocenter $\ominus A_1 \oplus E_3$ and the tangent point $\ominus A_1 \oplus T_3$ of the left gyrotranslated gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$, as calculated in (8.58c) and (8.59c), are:

$$\begin{aligned} \ominus A_1 \oplus E_3 &= \frac{\sin\alpha_2 \gamma_{12} \mathbf{a}_{12} - \sin\alpha_3 \gamma_{13} \mathbf{a}_{13}}{\sin\alpha_1 + \sin\alpha_2 \gamma_{12} - \sin\alpha_3 \gamma_{13}}, \\ \ominus A_1 \oplus T_3 &= \frac{\tan\frac{\alpha_1}{2} \gamma_{12} \mathbf{a}_{12}}{\tan\frac{\alpha_2}{2} + \tan\frac{\alpha_1}{2} \gamma_{12}}. \end{aligned} \tag{8.119a}$$

These points are contained in the Euclidean line

$$L_3 = (\ominus A_1 \oplus T_3) + \big(-(\ominus A_1 \oplus T_3) + (\ominus A_1 \oplus E_3)\big) t_3, \tag{8.119b}$$

where $t_3 \in \mathbb{R}$ is the line parameter. This line passes through the point $(\ominus A_1 \oplus T_3) \in \mathbb{R}_s^n \subset \mathbb{R}^n$ when $t_3 = 0$, and it passes through the point $(\ominus A_1 \oplus E_3) \in \mathbb{R}_s^n \subset \mathbb{R}^n$ when $t_3 = 1$.

If the Euclidean lines L_k, $k = 1, 2, 3$, in (8.117a), (8.117b), (8.118a), (8.118b), (8.119a), (8.119b) are concurrent, then their concurrency point is the gyrocenter P of the gyrotranslated gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$, given by (8.116).

As in (7.219), p. 207, and in (8.64), p. 244, the condition that gyrocenter $\ominus A_1 \oplus P$ lies on each of the three Euclidean lines L_k, $k = 1, 2, 3$, along with the linear independence of the gyrovectors $\mathbf{a}_{12}$ and $\mathbf{a}_{13}$ in $\mathbb{R}_s^n$, when considered as vectors in $\mathbb{R}^n$, gives rise to a system of six homogeneous linear equations for the six unknowns t_k and m_k, $k = 1, 2, 3$. Here the three unknowns t_k are the line parameters that determine the gyrocenter P on each of the three lines L_k, and the three unknowns m_k are the gyrobarycentric coordinates of the gyrocenter P of gyrotriangle $A_1A_2A_3$ with respect to the set $S = \{A_1, A_2, A_3\}$ of the gyrotriangle vertices, in (8.115) and (8.116).

The resulting six homogeneous linear equations for the six unknowns t_1, t_2, t_3 and m_1, m_2, m_3 are not linearly independent. Indeed, they determine uniquely the five unknowns t_1, t_2, t_3 and $m_1/m_3, m_2/m_3$, resulting in

$$
\begin{aligned}
\frac{m_1}{m_3} &= \frac{\sin\alpha_1(1+\cos\alpha_1-\cos\alpha_2-\cos\alpha_3)}{\sin\alpha_3(1-\cos\alpha_1-\cos\alpha_2+\cos\alpha_3)}, \\
\frac{m_2}{m_3} &= \frac{\sin\alpha_1(1-\cos\alpha_1+\cos\alpha_2-\cos\alpha_3)}{\sin\alpha_3(1-\cos\alpha_1-\cos\alpha_2+\cos\alpha_3)}.
\end{aligned}
\tag{8.120}
$$

Gyrobarycentric coordinates are homogeneous, so that a common nonzero factor is irrelevant. Hence, (8.120) gives rise to the following theorem:

Theorem 8.19 (The Exgyrocenter-Point-of-Tangency Gyrocenter Theorem) *Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, and let P be the exgyrocenter-point-of-tangency gyrocenter of the gyrotriangle, Fig. 8.8.*

Then, in the standard gyrotriangle index notation, the gyrotriangle gyrocenter P is given by its gyrobarycentric coordinate representation,

$$
P = \frac{m_1\gamma_{A_1}A_1 + m_2\gamma_{A_2}A_2 + m_3\gamma_{A_3}A_3}{m_1\gamma_{A_1} + m_2\gamma_{A_2} + m_3\gamma_{A_3}},
\tag{8.121}
$$

where gyrotrigonometric gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ of P in (8.121) are

$$
\begin{aligned}
m_1 &= \sin\alpha_1(1+\cos\alpha_1-\cos\alpha_2-\cos\alpha_3), \\
m_2 &= \sin\alpha_2(1-\cos\alpha_1+\cos\alpha_2-\cos\alpha_3), \\
m_3 &= \sin\alpha_3(1-\cos\alpha_1-\cos\alpha_2+\cos\alpha_3).
\end{aligned}
\tag{8.122}
$$

8.19 Problems

Problem 8.1 The Gyrotriangle Exgyroradius:
Derive the system (8.7), p. 223, of three equations by substituting successively (8.4a), (8.4b), (8.4c) and (8.2) into (8.5).

Problem 8.2 The Constant of the Gyrotrigonometric Gyrobarycentric Coordinate Representation of the Gyrotriangle Excenters and Incenter:
Derive the constants m_0^2 in (8.39a), (8.39b), (8.39c), (8.39d), p. 235, of each of the gyrotrigonometric gyrobarycentric coordinate representation of E_k, $k = 1, 2, 3, 0$, from (8.15), p. 226, and (8.37a), (8.37b), (8.37c), (8.37d), (8.38a), (8.38b), (8.38c), (8.38d), p. 234.

Problem 8.3 A Gyrotriangle Gyroangle Inequality:
Prove Inequality (8.41), p. 235, for the gyroangles α_k, $k = 1, 2, 3$ of any gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space.

Problem 8.4 A Substitution:
Derive (8.53), p. 239, by substitutions from (7.143) and (7.147) into (8.52).

Problem 8.5 Gyrotriangle Gyrotrigonometric Substitutions:
Complete the proof of Theorem 8.3, p. 230, in detail by substituting gyrotriangle gyrotrigonometric identities from Sect. 7.12, p. 187, into (8.21)–(8.22) and employing the result of Theorem 7.4, p. 163.

Problem 8.6 In-Exgyroradii Relation:
Derive (8.26) and (8.27) from (8.25), p. 230, by using the software *Mathematica* for computer algebra to manipulate trigonometric functions by commands like `TrigToExp`, `ExpToTrig`, `TrigReduce` and `TrigFactor`.

Problem 8.7 Gyrotrigonometric Substitutions:
Derive the gamma factor (8.86), p. 252, by substituting (8.82b) and (8.83b) into (8.84) (and squaring, but subsequently taking a square root).

Problem 8.8 Gyrotrigonometric Substitutions:
Derive (8.88), p. 252, by substituting (8.86) in Identity (8.87), which is a special case of Identity (1.9), p. 5.

Problem 8.9 Gyrotrigonometric Substitutions:
Derive the gamma factor (8.95), p. 254, by substituting (8.91b) and (8.92b) into (8.93).

Problem 8.10 Gyrotrigonometric Substitutions:
Derive (8.97), p. 254, by substituting (8.95) into Identity (8.96), which is a special case of Identity (1.9), p. 5.

Problem 8.11 Trigonometric Substitutions:
Simplify the results of Corollary 8.15, p. 256, into those of Corollary 8.16, p. 258, in Euclidean geometry by substituting Identities (8.103) that triangle angles possess and employing well-known, elementary trigonometric identities.

Problem 8.12 The *SSS* to *AAA* conversion law:
Apply the *SSS* to *AAA* conversion law, Theorem 6.2, p. 134, to obtain the equations in (8.106a), p. 259.

Problem 8.13 Gyrotrigonometric Substitutions:
Substitute the gamma factors from (8.100a), p. 255, into (8.106a), p. 259, and simplify the results gyrotrigonometrically (= trigonometrically) to obtain (8.106b).

Problem 8.14 A Gyrotrigonometric/Trigonometric Identity:
Prove the gyrotrigonometric/trigonometric Identity (8.107a), (8.107b), p. 260.

Problem 8.15 Gyrotrigonometric Substitutions:
Substitute the $\sin\beta_k$ and $\cos\beta_k$, $k = 1, 2, 3$, from (8.110a), (8.110b), (8.111a), (8.111b), (8.112a), (8.112b) into (8.113) to obtain (8.114), p. 262, explicitly, and deduce that

$$\lim_{s\to\infty} \sin(\beta_1 + \beta_2 + \beta_3) = 0. \tag{8.123}$$

Furthermore, show that (8.123) implies that in Euclidean geometry, the excircle angle sum is $\beta_1 + \beta_2 + \beta_3 = 2\pi$.

Problem 8.16 Gyrocollinearity:
Explain why the three points A_k, E_0 and E_k for each k, $k = 1, 2, 3$, in Fig. 8.8, p. 1263, are gyrocollinear.